TOROIDAL CONFINEMENT (THEORY)

Stable Magnetohydrostatic Equilibria without Rotational Transform

by

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Abstract: The general class of vacuum fields $\underline{B} = \nabla\phi$ with closed lines of force in the neighborhood of a given closed curve C was investigated, where $\underline{B}$ is tangential on C and $q = \oint |\underline{B}|^{-1} dl$ has a maximum on C.

The condition that a toroidal MHD equilibrium have closed lines of force and vanishing longitudinal current can be met by the following symmetry:

(A) The current density $\underline{j}$ has reflexional symmetry to a plane and $\underline{B}$ the respective antisymmetry such that $\underline{j}$ is tangential and $\underline{B}$ is normal on the symmetry plane.

Such equilibria with symmetry (A) have three major advantages:

a) The existence of a toroidal finite β equilibrium can be proved [1] (in contrast to a stellarator configuration)
b) No interior conductors are necessary (in contrast to multipoles)
c) Sufficient stability criteria can be satisfied (in contrast to axisymmetric configurations without interior conductors).

It is well known [2], [3] that for the stability of closed line configurations without a longitudinal current it is sufficient to construct a vacuum field

$$(1) \qquad \underline{B} = \nabla\phi = \nabla\Psi \times \nabla\chi$$

such that the function

$$q(\Psi(\underline{x}), \chi(\underline{x})) = \oint |\underline{B}|^{-1} dl$$

posseses a local maximum. Several special configurations of this kind have already been discussed [4], [5], [6], [7].

In this paper the general conditions are derived that in the neighborhood of a given closed curve C there exists a vacuum field with symmetry (A) which is tangential on C and such that q has a maximum on C. If the independent variables $\underline{x}$ are interchanged with the dependent variables ϕ, Ψ, χ then eq. (1) is equivalent to

$$(2) \qquad \frac{\partial \underline{x}}{\partial\phi} = \frac{\partial \underline{x}}{\partial\Psi} \times \frac{\partial \underline{x}}{\partial\chi}$$

A solution of eq. (2) has the advantage that the field lines are explicitly known. If the curve C is described by $\underline{x}(\phi, 0, 0)$ one can find a systematic expansion of $\underline{x}(\phi, \Psi, \chi)$ with respect to Ψ and χ. The n-th order derivative of eq. (2) represents a system of 3 (n+1) inhomogeneous linear algebraic equations for the 3 (n+2) (n+1)-order derivatives

$$\frac{\partial^{n+1}\underline{x}}{\partial\Psi^{n-\tau+1}\partial\chi^{\tau}}, \quad (\tau = 0, 1, 2, \ldots, n+1)$$

on C, the inhomogeneities depending on the lower order derivatives. It is found that the rank of the system is 2 n+3 if $|\underline{B}|$ is everywhere finite on C. So there are n solubility conditions for the right-hand sides. If the field has the symmetry (A), then these conditions can always be satisfied and in each order two functions of ϕ can be arbitrarily prescribed.

If the curve C with curvature $\varkappa$ lies in the plane $\Psi = 0$ and $\underline{B}$ has reflexional symmetry to this plane, then the condition that q have a maximum on C veduces to

$$\partial q/\partial\chi = -2\int_0^{2\pi} h'\, dl = 0, \quad h' = \varkappa f$$
$$\partial^2 q/\partial\Psi^2 = 2\int_0^{2\pi} f^{-5}(f'^2 f - f h g' - 2 f' h' h)\, dl < 0$$
$$\partial^2 q/\partial\chi^2 = 2\int_0^{2\pi} (f'^2 + 2h'^2 + h g')\, dl < 0$$

where $|\underline{B}| = 1$ has been chosen on C. $f, g, \varkappa$ are symmetric periodic functions of the arc length l with period 2π and $f > 0$. If $\underline{t}$ is the tangential unit vector, $\underline{n}$ the unit normal on C, and $\underline{b} = \underline{t} \times \underline{n}$, then

$$\partial\underline{x}/\partial\chi = f\underline{n}, \quad \partial\underline{x}/\partial\Psi = -f^{-1}\underline{b}, \quad \partial^2\underline{x}/\partial\Psi^2 = U\underline{t} + V\underline{n},$$
$$\partial^2\underline{x}/\partial\chi^2 = u\underline{t} + v\underline{n}, \quad \partial^2\underline{x}/\partial\Psi\partial\chi = f^{-1}g\underline{b}, \quad U = f^{-3}f', \quad u = -f'f,$$
$$v = gf - \varkappa f^2, \quad f^4(f^{-1}V)' + g' + 2\varkappa f' = 0.$$

A configuration of this kind has been plotted in the figure: a) half of the curve C $(0 \le l \le 8\pi)$, b) - g) cross sections of the surfaces q = const perpendicular to C for the values $l = k\pi/5$ $(k = 0, 1, \ldots, 5)$. The figures b) – g) are magnified by a factor of 100 relative to figure a). Ψ, χ has been restricted in such a way that the neglected terms cubic in Ψ, χ are absolutely less than 5 % of the linear terms. This results in $\delta q/q \sim 1.5\,\%$.

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