

Incompressibility of H -free edge modification problems: Towards a dichotomy

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Abstract

Given a graph G and an integer k , the H -FREE EDGE EDITING problem is to find whether there exists at most k pairs of vertices in G such that changing the adjacency of the pairs in G results in a graph without any induced copy of H . The existence of polynomial kernels for H -FREE EDGE EDITING (that is, whether it is possible to reduce the size of the instance to $k^{O(1)}$ in polynomial time) received significant attention in the parameterized complexity literature. Nontrivial polynomial kernels are known to exist for some graphs H with at most 4 vertices (e.g., path on 3 or 4 vertices, diamond, paw), but starting from 5 vertices, polynomial kernels are known only if H is either complete or empty. This suggests the conjecture that there is no other H with at least 5 vertices where H -FREE EDGE EDITING admits a polynomial kernel. Towards this goal, we obtain a set \mathcal{H} of nine 5-vertex graphs such that if for every $H \in \mathcal{H}$, H -FREE EDGE EDITING is incompressible and the complexity assumption $\text{NP} \not\subseteq \text{coNP/poly}$ holds, then H -FREE EDGE EDITING is incompressible for every graph H with at least five vertices that is neither complete nor empty. That is, proving incompressibility for these nine graphs would give a complete classification of the kernelization complexity of H -FREE EDGE EDITING for every H with at least 5 vertices.

We obtain similar result also for H -FREE EDGE DELETION. Here the picture is more complicated due to the existence of another infinite family of graphs H where the problem is trivial (graphs with exactly one edge). We obtain a larger set \mathcal{H} of nineteen graphs whose incompressibility would give a complete classification of the kernelization complexity of H -FREE EDGE DELETION for every graph H with at least 5 vertices. Analogous results follow also for the H -FREE EDGE COMPLETION problem by simple complementation.

1 Introduction

In a typical graph modification problem, the input is a graph G and an integer k , and the task is to make at most k allowed editing operations on G to make it belong to a certain graph class or satisfy a certain property. For example, VERTEX COVER (remove k vertices to make the graph edgeless), FEEDBACK VERTEX SET (remove k vertices to make the graph acyclic), ODD CYCLE TRANSVERSAL (remove k edges/vertices to make the graph bipartite), and MINIMUM FILL-IN (add k edges to make the graph chordal) are particularly well-studied members of this problem family. Most natural graph modification problems are known to be NP-hard, in fact, there are general hardness results proving hardness for many problems [20, 22, 23]. On the other hand, most of these problems are fixed-parameter tractable (FPT) parameterized by k : it can be solved in time $f(k)n^{O(1)}$, where f is a computable function depending only on k [4, 9, 18, 19]. Looking at the parameterized complexity literature, one can observe that, even though there are certain recurring approaches and techniques, these FPT results are highly problem specific, and often rely on a very detailed understanding of the graph classes at hand.

A class of problems that can be treated somewhat more uniformly is H -FREE EDGE EDITING. This is a separate problem for every fixed graph H : given a graph G and an integer k , the task is to find whether there exists at most k pairs of vertices in G such that changing the adjacency of the pairs in G results in a graph without any induced copy of H . Aravind et al. [2] proved that H -FREE EDGE EDITING is NP-hard for every graph H with at least 3 vertices. However, a simple application of the technique of bounded-depth search trees shows that H -FREE EDGE EDITING is FPT parameterized by k for every fixed H [4].

Graph modification problems were explored also from the viewpoint of polynomial kernelization: is there a polynomial-time preprocessing algorithm that does not necessarily solve the problem, but at least reduces

the size of the an instance to be bounded by a polynomial of k ? The existence of a polynomial kernelization immediately implies that the problem is FPT (after the preprocessing, one can solve the reduced instance by brute force or any exact method). Therefore, one can view polynomial kernelization as a special type of FPT result that tries to formalize the question whether the problem can be efficiently preprocessed in a way that helps exhaustive search methods. There is a wide literature on algorithms for kernelization (see, e.g., [15]). Conversely, incompressibility results can show, typically under the complexity assumption $\text{NP} \not\subseteq \text{coNP/poly}$, that a parameterized problem has no polynomial kernelization.

Most of the highly nontrivial FPT algorithms for graph modification problems do not give kernelization results and, in many cases, it required significant amount of additional work to obtain kernelization algorithms. In particular, the FPT algorithm for H -FREE EDGE EDITING based on the technique of bounded-depth search trees does not give polynomial kernels. For the specific case when $H = K_r$ is a complete graph, it is easy to see that there is a solution using only deletions. Now the problem essentially becomes a HITTING SET problem with sets of bounded size: we have to select at least one edge from the edge set of each copy of K_r . Therefore, known kernelization results for HITTING SET can be used to show that K_r -FREE EDGE EDITING has a polynomial kernel for every fixed r . A similar argument works if H is an empty graph on r vertices.

Besides cliques and empty graphs, it is known for certain graphs H of at most 4 vertices (diamond [6, 10], path [7, 16, 17], paw [8, 14], and their complements) that H -FREE EDGE EDITING has a polynomial kernel, but these algorithms use very specific arguments exploiting the structure of H -free graphs. As there is a very deep known structure theory of claw-free (i.e, $K_{1,3}$ -free) graphs, it might be possible to obtain a polynomial kernel for CLAW-FREE EDGE EDITING, but this is currently a major open question [5, 11, 13]. However, besides cliques and empty graphs, no H with at least 5 vertices is known where H -FREE EDGE EDITING has a polynomial kernel and there is no obvious candidate H for which one would expect a kernel. This suggests the following conjecture:

Conjecture 1. *If H is a graph with at least 5 vertices, then H -FREE EDGE EDITING has a polynomial kernel if and only if H is a complete or empty graph.*

We are not able to resolve this conjecture, but make substantial progress towards it by showing that only a finite number of key cases needs to be understood. Our main result for H -FREE EDGE EDITING is the following.

Theorem 1.1. *There exists a set \mathcal{H}^E of nine graphs, each with five vertices such that if H -FREE EDGE EDITING is incompressible for every $H \in \mathcal{H}^E$, then for a graph H with at least five vertices H -FREE EDGE EDITING is incompressible if and only if H is neither complete nor empty, where the incompressibility assumes $\text{NP} \not\subseteq \text{coNP/poly}$.*

The set \mathcal{H}^E of nine graphs are shown in Figure 1. Note that a simple reduction by complementation shows that H -FREE EDGE EDITING and \overline{H} -FREE EDGE EDITING have the same complexity. Therefore, for each of these nine graphs, we could put either it or its complement into the set \mathcal{H}^E . As it will be apparent later, we made significant efforts to reduce the size of \mathcal{H}^E as much as possible. However, the known techniques for proving incompressibility do not seem to work for these graphs. Let us observe that most of these graphs are very close to the known cases that admit a polynomial kernel: for example, they can be seen as a path, paw, or diamond with an extra isolated vertex or with an extra degree-1 vertex attached. Thus resolving the kernelization complexity of H -FREE EDGE EDITING for any of these remaining graphs seems to be a particularly good research question: either one needs to extend in a nontrivial way the known kernelization results, or a significant new ideas are needed for proving hardness.

The reader might not be convinced of the validity of Conjecture 1 and may wonder about the value of Theorem 1.1 when the conjecture is false. However, we can argue that Theorem 1.1 is meaningful even in this case. It shows that if there is any H violating Conjecture 1, then one of the 9 graphs in \mathcal{H}^E also violates it. That is, if we believe that there are kernelization results violating the conjecture, then we should focus on the 9 graphs in \mathcal{H}^E , as these are the easiest cases where we may have a kernelization result. In other words, Theorem 1.1 precisely shows the frontier where new algorithmic results are most likely to exist.

H -FREE EDGE DELETION is the variant of H -FREE EDGE EDITING where only edge removal is allowed. For the same fixed graph H , it seems that H -FREE EDGE DELETION should be a simpler problem than H -FREE EDGE EDITING, but we want to emphasize that H -FREE EDGE DELETION is *not* a special case of H -FREE EDGE EDITING. There is no known general reduction from the former to the latter, although the technique of completion enforcers (see Section 5 and [5]) can be used for many specific graphs H . There is a known case where H -FREE EDGE DELETION seems to be strictly easier: if H has at most one edge, then there is only one way of destroying a copy of an induced H by edge removal, making the problem polynomial-time solvable. Aravind et al. [1] showed that having at most one edge is the only condition that makes H -FREE EDGE DELETION polynomial-time solvable: if H has at least two edges, then the problem is NP-hard. Therefore, the counterpart of Conjecture 1 for H -FREE EDGE DELETION should take this case also into account.

Conjecture 2. *If H is a graph with at least 5 vertices, then H -FREE EDGE DELETION has a polynomial kernel if and only if H is a complete graph or has at most one edge.*

#	H	\overline{H}
1		
2		
3		

#	H	\overline{H}
4		
5		
6		

#	H	\overline{H}
7		
8		
9		same

Figure 1: The set \mathcal{H} of graphs

#	A	\overline{A}
1		
2		
3		

#	A	\overline{A}
4		same
5		same
6		

#	A	\overline{A}
7		
8		
9		

Figure 2: The set \mathcal{A} of graphs

#	D	\overline{D}
1		
2		

#	B	\overline{B}
1		
2		

#	B	\overline{B}
3		

Figure 3: The sets \mathcal{D} and \mathcal{B} of graphs

Working toward this conjecture, we show that only a finite number of cases needs to be shown incompressible.

Theorem 1.2. *There exists a set \mathcal{H}^D of nineteen graphs, each with either five or six vertices such that if H -FREE EDGE DELETION is incompressible for every $H \in \mathcal{H}^D$ then for a graph H with at least five vertices, H -FREE EDGE DELETION is incompressible if and only if H is a graph with at least two edges but not complete, where the incompressibility assumes $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.*

The set \mathcal{H}^D contains the graphs in set \mathcal{H}^E , as well their complements. This seems reasonable and hard to avoid: if we do not have an incompressibility result for H -FREE EDGE EDITING for some $H \in \mathcal{H}^E$, then it is unlikely that we can find such a result for H -FREE EDGE DELETION (even though, as discussed above, there is no formal justification for this). Together with these 17 graphs (note that H_9 is the same as its complement), we need to include into \mathcal{H}^D the two graphs D_1 and D_2 shown in Figure 3. In the case of editing, we can prove incompressibility for these two graphs by a reduction from H -FREE EDGE EDITING where H is the graph with 5 vertices and one edge. However, H -FREE EDGE DELETION for this H is polynomial-time solvable.

Finally, let us consider the H -FREE EDGE COMPLETION problem, where we have to make G induced H -free by adding at most k edges. As H -FREE EDGE COMPLETION is essentially the same problem as \overline{H} -FREE EDGE DELETION, we can obtain a counterpart of Theorem 1.2 by simple complementation:

Theorem 1.3. *There exists a set \mathcal{H}^C of nineteen graphs, each with either five or six vertices such that if H -FREE EDGE COMPLETION is incompressible for every $H \in \mathcal{H}^C$ then for a graph H with at least five vertices, H -FREE EDGE EDITING is incompressible if and only if H is a graph with at least two nonedges but not empty, where the incompressibility assumes $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.*

Our techniques. We crucially use two earlier results. First, Cai and Cai [5] proved that H -FREE EDGE EDITING is incompressible (assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$) when H or \overline{H} is a cycle or a path of length at least 4, or 3-connected but not complete. While these result handle many graphs and prove to be very useful for our proofs, they do not come close to a complete classification. Second, we use a key tool in the polynomial-time dichotomy result of Aravind et al. [1]: if V_ℓ is the set of lowest degree vertices of H , then $(H - V_\ell)$ -FREE EDGE EDITING can be reduced to H -FREE EDGE EDITING. The same statement holds for the set V_h of highest degree vertices.

Our proofs of Theorems 1.1–1.3 introduce new incompressibility results and new reductions, which we put together to obtain an almost complete classification by a graph-theoretic analysis. Additionally, to make the arguments simpler, we handle small graphs by an exhaustive computer search. In the following, we highlight some of the main ideas that appear in the paper.

- **Analysis of graphs.** Our goal is to prove Theorem 1.1 by induction on the size of H . First we handle the case when H is regular: we show that this typically implies that either H or \overline{H} is 3-connected, and the result of Cai and Cai [5] can be used. If H is not regular, then the graphs $H - V_\ell$ and $H - V_h$ are nonempty and have strictly fewer vertices than H . If one of them, say $H - V_\ell$, has at least 5 vertices and is neither complete nor empty, then the induction hypothesis gives an incompressibility result for $(H - V_\ell)$ -FREE EDGE EDITING, which gives an incompressibility result for H -FREE EDGE EDITING by the reduction of Aravind et al. [1]. Therefore, we only need to handle those graphs H where it is true for both $H - V_\ell$ and $H - V_h$ that they are either small, complete, or empty. But we can obtain a good structural understanding of H in each of these cases, which allows us to show that either H or \overline{H} is 3-connected, or H has some very well defined structure. With these arguments, we can reduce the problem to the incompressibility of H -FREE EDGE EDITING for a few dozen specific graphs H and for a few well-structured infinite families (such as $K_{2,t}$).

For H -FREE EDGE DELETION, we have the additional complication that one or both of $H - V_\ell$ and $H - V_h$ can be near-empty (i.e., has exactly one edge), which is not an incompressible case for this problem. We need additional case analysis to cover such graphs, but the spirit of the proof remains the same.

- **Computer search.** Our analysis of graphs becomes considerably simpler if we assume that H is not too small. In this case, we can assume that at least one of $H - V_\ell$ and $H - V_h$ is a complete or empty graph of certain minimum size, which is a very helpful starting point for proving the 3-connectivity of H or \overline{H} , respectively. Therefore, we handle every graph with at most 9 vertices using an exhaustive computer search and assume in the proof that H has at least 10 vertices. The list provided by McKay [21] shows that there are 288266 different graphs with at most 9 vertices, which is feasible for a computer search. In principle, it would be possible to extend our case analysis to avoid this computer search, but it would significantly complicate the proof and is not clear what additional insight it would give.
- **Reductions.** We investigate different reductions that allow us to reduce H' -FREE EDGE EDITING to H -FREE EDGE EDITING when H' is an induced subgraph of H satisfying certain conditions. With extensive use of such reductions, we can reduce the remaining cases of H -FREE EDGE EDITING that needs to be handled to a smaller finite set.

- **Incompressibility results.** We carefully revisit the proof of Cai and Cai [5] showing the incompressibility of H -FREE EDGE EDITING when H is 3-connected, and observe that, with additional ideas, it can be made to work also for certain 2-connected graphs that are not 3-connected. This allows us to handle every graph, except those finite sets that are mentioned in Theorem 1.1–1.3. A key step in many of these incompressibility results is to establish first incompressibility for the RESTRICTED H -FREE EDGE DELETION problem, which is the generalization of H -FREE EDGE DELETION where some of the edges of G are marked as forbidden in the input, and the solution is not allowed to delete forbidden edges. Then we use deletion and completion enforcer gadgets specific to H to reduce RESTRICTED H -FREE EDGE DELETION to H -FREE EDGE EDITING.

The paper is organized as follows. Preliminaries are in Section 2. Section 3 presents the churning procedure, our main technical tool in the analysis of graphs, and shows that it reduces the problem to a finite number of graphs, plus a few well-defined infinite families. Section 4 presents reductions (old and new) that allow us to further reduce the number of graphs we need to handle. Finally, in Section 5, we give new incompressibility results, showing that only the cases stated in Theorems 1.1–1.3 need to be proved incompressible to complete the exploration of the complexity landscape of the problems.

2 Preliminaries

Graph-theoretic notation and terminology. For a graph G , $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G respectively. For a set $V' \subseteq V(G)$, $G - V'$ denotes the graph obtained by removing all vertices in V' and their incident edges from G . For a set F of pairs of vertices and a graph G , $G \Delta F$ denotes the graph G' such that $V(G') = V(G)$ and $E(G') = \{(u, v) \mid ((u, v) \in E(G) \text{ and } (u, v) \notin F) \text{ or } (u, v \in V(G), (u, v) \notin E(G), \text{ and } (u, v) \in F)\}$. Whenever we say that a set of (non)edges F is a solution of an instance (G, k) of a problem, we refer to a subset of F containing all (non)edges where both the end vertices are in $V(G)$. A graph is *empty* if it does not have any edges. A graph is *near-empty* if it has exactly one edge. A graph is *complete* if it has no nonedges. A component of a graph is a *largest component* if it has maximum number of vertices among all components of the graph. Similarly, a component of a graph is a *smallest component* if it has minimum number of vertices among all components of the graph. For a graph H which is not complete, the vertex connectivity of H is the minimum integer c such that there exists a set $S \subseteq V(H)$ such that $|S| = c$ and $H - S$ is disconnected. For a graph H with vertex connectivity 1, a vertex v in H is known as a cut vertex if $H - v$ is disconnected. A graph is k -connected, if the vertex connectivity of it is at least k . An induced subgraph H' of H is known as a 2-connected component if H' is a maximal 2-connected induced subgraph of H . The adjectives ‘largest’ and ‘smallest’ can be applied to 2-connected components as done for components. A *twin-star* graph T_{ℓ_1, ℓ_2} for $\ell_1, \ell_2 \geq 0$ is defined as the tree with two adjacent vertices u and v such that $|N(u) \setminus \{v\}| = \ell_1$, $|N(v) \setminus \{u\}| = \ell_2$, and every vertex in $N(u) \cup N(v) \setminus \{u, v\}$ has degree 1. A graph G is H -free if G does not contain any induced copy of H . For two graphs G_1 and G_2 , the disjoint union of G_1 and G_2 denoted by $G_1 \cup G_2$ (or $G_2 \cup G_1$) is the graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. For two graphs G_1 and G_2 , the join of G_1 and G_2 denoted by $G_1 \boxtimes G_2$ (or $G_2 \boxtimes G_1$), is the graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{(x, y) \mid x \in V(G_1), y \in V(G_2)\}$. A complete graph, a cycle, and a path with t vertices are denoted by K_t , C_t , and P_t respectively. By $K_t - e$, we denote the graph obtained by deleting an edge from a complete graph on t vertices. We call a graph *non-regular* if it is not regular. A *modular decomposition* \mathcal{M} of a graph G is a partitioning of its vertices into maximal sets, known as *modules*, such that for every set $M \in \mathcal{M}$, every vertex in M has the same neighborhood outside M . Let $\mathcal{M}' \subseteq \mathcal{M}$. Let $V' = \bigcup_{M \in \mathcal{M}'} M$. Then we say that \mathcal{M}' corresponds to V' . For a set \mathcal{S} of graphs, by $\overline{\mathcal{S}}$ we denote the set of complements of graphs in \mathcal{S} . Figure 4 shows all graphs with at most four vertices which are neither empty nor complete.

For $t \geq 3$, let J_t be the graph obtained from $K_2 \boxtimes tK_1$ and C_4 by identifying an edge of C_4 with the edge between the highest degree vertices in $K_2 \boxtimes tK_1$. Let Q_t be the graph graph obtained from $K_{2,t}$, for some $t \geq 3$, by adding a path of length three between the highest degree vertices in $K_{2,t}$. Let \mathcal{H} , \mathcal{A} , \mathcal{D} , \mathcal{B} , \mathcal{S} denote the graphs (H , A , D , B , S respectively) shown in Figures 1, 2, 3, and 5. Let \mathcal{F} be the union of graphs in the classes of graphs shown in column \mathcal{F} of Figure 6. For all these classes of graphs, we use subscripts to identify each graph/graph class. For example H_1 is $P_3 \cup 2K_1$ and \mathcal{F}_1 is the class of graphs $K_{1,t}$. Let \mathcal{W} be the set $\mathcal{H} \cup \overline{\mathcal{H}} \cup \mathcal{A} \cup \overline{\mathcal{A}} \cup \mathcal{D} \cup \overline{\mathcal{D}} \cup \mathcal{B} \cup \overline{\mathcal{B}} \cup \mathcal{S} \cup \overline{\mathcal{S}} \cup \mathcal{F} \cup \overline{\mathcal{F}}$. We observe that $\overline{\mathcal{W}} = \mathcal{W}$.

Parameterized problems and transformations. A parameterized problem is a classical problem with an additional integer input known as the parameter. A parameterized problem admits a polynomial kernel if there is a polynomial-time algorithm which takes as input an instance (I, k) of the problem and outputs an instance (I', k') of the same problem, where $|I'|, k' \leq p(k)$, where $p(k)$ is a polynomial in k , such that (I, k) is a yes-instance if and only if (I', k') is a yes-instance. A parameterized problem is incompressible if it does not admit a polynomial kernel. A *Polynomial Parameter Transformation* (PPT) from one parameterized problem Q to another parameterized problem Q' is a polynomial-time algorithm which takes as input an instance (I, k)

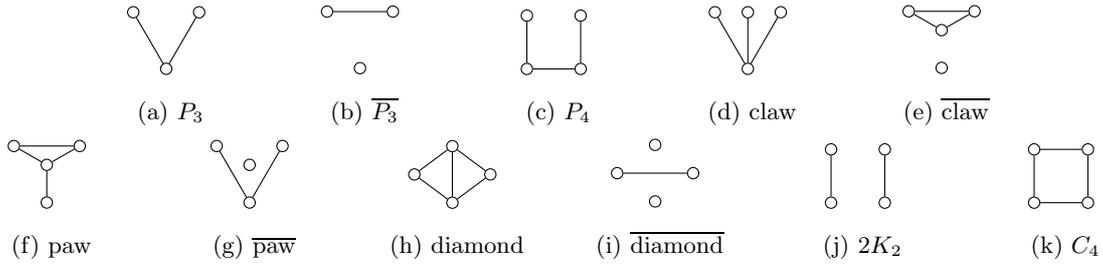


Figure 4: All non-empty and non-complete graphs with at most four vertices

of Q and produces an instance (I', k') of Q' such that (I, k) is a yes-instance of Q if and only if (I', k') is a yes-instance of Q' , and $k' \leq p(k)$, for some polynomial $p(\cdot)$. It is known that if there is a PPT from Q to Q' , then if Q is incompressible, then so is Q' . We refer to the book [12] for various concepts in parameterized algorithms and complexity.

The parameterized problems we deal with in this paper are listed below.

H -FREE EDGE EDITING: Given a graph G and an integer k , do there exist at most k edges such that editing (adding or deleting) them in G results in an H -free graph?	Parameter: k
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H -FREE EDGE DELETION: Given a graph G and an integer k , do there exist at most k edges such that, deleting them from G results in an H -free graph?	Parameter: k
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H -FREE EDGE COMPLETION: Given a graph G and an integer k , do there exist at most k edges such that, adding them in G results in an H -free graph?	Parameter: k
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Basic results. Proposition 2.1 follows from the observations that (G, k) is a yes-instance of H -FREE EDGE EDITING (DELETION) if and only if (\overline{G}, k) is a yes-instance of \overline{H} -FREE EDGE EDITING (COMPLETION). It enables us to focus only on H -FREE EDGE EDITING and H -FREE EDGE DELETION.

Proposition 2.1 (folklore). *Let H be any graph. Then H -FREE EDGE DELETION is incompressible if and only if \overline{H} -FREE EDGE COMPLETION is incompressible. Similarly, H -FREE EDGE EDITING is incompressible if and only if \overline{H} -FREE EDGE EDITING is incompressible.*

For graphs H and H' , by “ H simulates H' ” and by “ H' is simulated by H ”, we mean that, there is a PPT from H' -FREE EDGE EDITING to H -FREE EDGE EDITING, there is a PPT from H' -FREE EDGE DELETION to H -FREE EDGE DELETION, and there is a PPT from H' -FREE EDGE COMPLETION to H -FREE EDGE COMPLETION. We observe that this is transitive, i.e., if H simulates H' and H' simulates H'' , then H simulates H'' . A set of graphs \mathcal{H} is called a *base* for a set \mathcal{G} of graphs if for every graph $H \in \mathcal{G}$ there is a graph $H' \in \mathcal{H}$ such that H simulates H' . The objective of the rest of the paper is to find, for each of the problems, a base $\mathcal{H} \cup \mathcal{X}$ for all graphs with at least five vertices, except the trivial cases, such that the following conditions are satisfied: (i) \mathcal{H} is finite and the incompressibility is not known for any graph in it; (ii) for every graph in \mathcal{X} , the problem is known to be incompressible.

Proposition 2.1 implies Corollary 2.2 and Proposition 2.3 can be deduced directly from the definitions.

Corollary 2.2. *Let H and H' be graphs such that H simulates H' . Then \overline{H} simulates \overline{H}' .*

Proposition 2.3. *Let \mathcal{H} be a base for a set \mathcal{G} of graphs. Assume that for every graph $H' \in \mathcal{H}$, H' -FREE EDGE EDITING (DELETION) is incompressible. Then for every graph $H \in \mathcal{G}$, H -FREE EDGE EDITING (DELETION) is incompressible.*

Intuitively, if H' is an induced subgraph of H , then H -FREE EDGE EDITING (DELETION) seems harder than H' -FREE EDGE EDITING (DELETION). However, there is no general argument why this should be true: there does not seem to be a completely general reduction that would reduce H' -FREE EDGE EDITING (DELETION) to H -FREE EDGE EDITING (DELETION). There is, however, a fairly natural idea for trying to do such a reduction: we extend the graph by attaching copies of $H - H'$ at every place where a copy of H' can potentially appear. The following construction is essentially the same as the main construction used in [2].

Construction 1 (see [2]). *Let (G', k, H, V') be an input to the construction, where G' and H are graphs, k is a positive integer and V' is a subset of vertices of H . We construct a graph G from G' as follows. For every injective function $f : V' \rightarrow V(G')$, do the following:*

- Introduce $k + 1$ sets of vertices V_1, V_2, \dots, V_{k+1} , each of size $|V(H) \setminus V'|$, and $k + 1$ bijective functions $g_i : V(H) \rightarrow (f(V') \cup V_i)$, for $1 \leq i \leq k + 1$, such that $g(v') = f(v')$ for every $v' \in V'$;

- For each set V_i , introduce an edge set $E_i = \{(u, v) \mid u \in (f(V') \cup V_i), v \in V_i, (g_i^{-1}(u), g_i^{-1}(v)) \in E(H)\}$.

This completes the construction. Let the constructed graph be G .

For convenience, we call every set V_i of vertices introduced in the construction as a *satellite* and the vertices in it as *satellite vertices*. This reduction works correctly in one direction: it ensures that the operations that make the new graph G H -free should ensure that the copy of G' inside G is H' -free.

Proposition 2.4 (see Lemma 2.6 in [2]). *Let G be obtained by Construction 1 on the input (G', k, H, V') , where G' and H are graphs, k is a positive integer and $V' \subseteq V(H)$. Then, if (G, k) is a yes-instance of H -FREE EDGE EDITING (DELETION), then (G', k) is a yes-instance of H' -FREE EDGE EDITING (DELETION), where H' is $H[V']$.*

However, the other direction of the correctness of the reduction does not hold in general (this is easy to see for example for $H = K_{1,2}$ and $H' = K_2$). As we shall see, there are particular cases where we can prove the converse of Proposition 2.4, for example, when $H - H'$ consists of exactly the highest- or lowest-degree vertices. Application of such arguments will be our main tool in reducing the complexity of H -FREE EDGE EDITING (DELETION) to simpler cases. Propositions 2.5 to 2.8 summarize the major results on the incompressibility of H -free edge modification problems known so far.

Proposition 2.5 ([5]). *Assuming $\text{NP} \not\subseteq \text{coNP/poly}$, H -FREE EDGE EDITING, H -FREE EDGE DELETION, and H -FREE EDGE COMPLETION are incompressible if H is either of the following graphs.*

- (i) C_ℓ for any $\ell \geq 4$;
- (ii) P_ℓ for any $\ell \geq 5$;
- (iii) $2K_2$.

Proposition 2.6 ([5]). *Assuming $\text{NP} \not\subseteq \text{coNP/poly}$, for 3-connected graphs H , H -FREE EDGE EDITING and H -FREE EDGE DELETION are incompressible if H is not complete and H -FREE EDGE COMPLETION is incompressible if H has at least two nonedges.*

Proposition 2.7 ([5], folklore). *If H is a complete or empty graph, then H -FREE EDGE EDITING admits polynomial kernelization. If H is complete or has at most one edge then H -FREE EDGE DELETION admits polynomial kernelization. If H is an empty graph or has at most one nonedge then H -FREE EDGE COMPLETION admits polynomial kernelization.*

Proposition 2.8. *H -FREE EDGE EDITING, H -FREE EDGE DELETION, and H -FREE EDGE COMPLETION admit polynomial kernels when H is a P_3 [7, 16], P_4 [17], paw [8, 14], or a diamond [6, 10].*

3 Churning

In this section, we introduce and analyze the churning procedure. The main result of the section is that assuming incompressibility for the class \mathcal{W} of graphs defined in the previous section explains incompressibility for every graph with at least five vertices, except the trivial cases. Recall that \mathcal{W} is not finite, as it contains the infinite families shown in Figure 6. In Sections 4 and 5, we will further reduce \mathcal{W} to a finite set.

Lemma 3.1. *If H -FREE EDGE EDITING is incompressible for every $H \in \mathcal{W}$, then H -FREE EDGE EDITING is incompressible for every H having at least five vertices but is neither complete nor empty, where the incompressibility assumes $\text{NP} \not\subseteq \text{coNP/poly}$.*

Lemma 3.2. *If H -FREE EDGE DELETION is incompressible for every $H \in \mathcal{W}$, then H -FREE EDGE DELETION is incompressible for every H having at least five vertices and at least two edges but not complete, where the incompressibility assumes $\text{NP} \not\subseteq \text{coNP/poly}$.*

Corollary 3.3 follows from Lemma 3.2, Proposition 2.1 and from the fact that $\overline{\overline{\mathcal{W}}} = \mathcal{W}$.

Corollary 3.3. *If H -FREE EDGE COMPLETION is incompressible for every $H \in \mathcal{W}$, then H -FREE EDGE COMPLETION is incompressible for every H having at least five vertices and at least two nonedges but not empty, where the incompressibility assumes $\text{NP} \not\subseteq \text{coNP/poly}$.*

By \mathcal{X}_E we denote the set of all graphs (and their complements) listed in Proposition 2.5, Proposition 2.6, and Theorem 3.7 for which the incompressibility is known (assuming $\text{NP} \not\subseteq \text{coNP/poly}$) for H -FREE EDGE EDITING. By \mathcal{Y}_E , we denote the set of all graphs (and their complements) listed in Proposition 2.7 and 2.8 for which there exist polynomial kernels for H -FREE EDGE EDITING; additionally, we include into \mathcal{Y}_E the claw and its

#	S	\bar{S}	#	S	\bar{S}	#	S	\bar{S}	#	S	\bar{S}
1			10			19			28		
2			11			20			29		
3			12		same	21			30		
4			13			22			31		
5			14			23			32		
6			15			24		same	33		
7			16			25			34		
8			17			26			35		
9			18			27			36		

Figure 5: The set \mathcal{S} of graphs

#	\mathcal{F}	$\bar{\mathcal{F}}$	Comment	#	\mathcal{F}	$\bar{\mathcal{F}}$	Comment
1	$K_{2,t}$	$K_t \cup K_2$	$4 \leq t$	6	$\overline{(K_t - e) \cup K_2}$	$(K_t - e) \cup K_2$	$4 \leq t$
2	$K_{1,t}$	$K_t \cup K_1$	$5 \leq t$	7	$K_{1,t} \cup K_2$	$\overline{K_{1,t} \cup K_2}$	$4 \leq t$
3	$K_2 \boxtimes tK_1$	$K_t \cup 2K_1$	$4 \leq t$	8	$\overline{(K_t - e) \cup K_1}$	$(K_t - e) \cup K_1$	$6 \leq t$
4	$T_{t,1}$	$\overline{T_{t,1}}$	$4 \leq t$	9	J_t	\bar{J}_t	$3 \leq t$
5	$\overline{(K_t - e) \cup 2K_1}$	$(K_t - e) \cup 2K_1$	$4 \leq t$	10	Q_t	\bar{Q}_t	$3 \leq t$

Figure 6: The set \mathcal{F} of infinite sets of graphs

complement (as we do not want to conjecture the incompressibility for these cases). Similarly, we define the set \mathcal{X}_D of “hard” and the set \mathcal{Y}_D of “nonhard” cases for H -FREE EDGE DELETION. More formally,

$$\begin{aligned}
\mathcal{X}_D &= \{C_\ell, \overline{C}_\ell \text{ for all } \ell \geq 4, \\
&\quad P_\ell, \overline{P}_\ell \text{ for all } \ell \geq 5, \\
&\quad H \text{ such that } H \text{ is regular but is neither complete nor empty,} \\
&\quad H \text{ such that either } H \text{ is 3-connected but not complete} \\
&\quad \text{or } \overline{H} \text{ is 3-connected with at least two nonedges}\} \\
\mathcal{X}_E &= \mathcal{X}_D \cup \{H \text{ such that } H \text{ has at most one edge and at least five vertices}\} \\
\mathcal{Y}_E &= \{K_t, \overline{K}_t \text{ for all } t \geq 1, \\
&\quad P_3, \overline{P}_3, P_4, \\
&\quad \text{diamond, } \overline{\text{diamond}}, \text{paw, } \overline{\text{paw}}, \text{claw, } \overline{\text{claw}}\} \\
\mathcal{Y}_D &= \mathcal{Y}_E \cup \{H \text{ such that } H \text{ has at most one edge and at least five vertices}\}
\end{aligned}$$

Additionally we define $\mathcal{Y}' = \{P_3, \overline{P}_3, P_4, \text{claw, } \overline{\text{claw}}, \text{paw, } \overline{\text{paw}}, \text{diamond, } \overline{\text{diamond}}\}$. We observe that $\mathcal{Y}' \subseteq \mathcal{Y}_E \cap \mathcal{Y}_D$ and the set of graphs with at most four vertices is a subset of $\mathcal{X}_E \cup \mathcal{Y}_E$ and $\mathcal{X}_D \cup \mathcal{Y}_D$. Further, we observe that near-empty graphs with at least five vertices are in \mathcal{Y}_D but their complements are 3-connected and are in \mathcal{X}_D . We also note that both these graphs and their complements are in \mathcal{X}_E .

The main technical result of the section is the following lemma. It states that if a graph is not in the set \mathcal{Y}_D of “easy” graphs, then it simulates a “hard” graph in \mathcal{X}_D or \mathcal{W} (and there is a similar result for \mathcal{X}_E and \mathcal{Y}_E).

Lemma 3.4. *If $H \notin \mathcal{Y}_D$, then H simulates a graph in $\mathcal{X}_D \cup \mathcal{W}$. If $H \notin \mathcal{Y}_E$, then H simulates a graph in $\mathcal{X}_E \cup \mathcal{W}$.*

In the rest of the paper, integer ℓ and set V_ℓ denote the lowest degree and the set of lowest degree vertices in H respectively; integer h and set V_h denote the highest degree and the set of highest degree vertices in H respectively; and set V_m denotes the set $V(H) \setminus (V_\ell \cup V_h)$. By h^* we denote the degree of vertices of V_h in \overline{H} , i.e., $h^* = |V(H)| - h - 1$.

Now we introduce a procedure (Churn) which is similar to the one used to obtain dichotomy results on the polynomial-time solvable and NP-hard cases of these problems (see Section 5 in [2]). The basic observation is that H can simulate the graphs $H - V_\ell$ and $H - V_h$. This follows from proving that Construction 1 gives a PPT in these cases.

Proposition 3.5 (Corollary 2.9 in [2]). *Let H' be $H - V_\ell$ or $H - V_h$. Then H simulates H' .*

To deal with both H -FREE EDGE EDITING and H -FREE EDGE DELETION in a uniform way, we define $\mathcal{X} = \mathcal{X}_E$ and $\mathcal{Y} = \mathcal{Y}_D$. We observe that $\overline{\mathcal{X} \cup \mathcal{Y}} = \mathcal{X} \cup \mathcal{Y}$ and $\mathcal{X} \cup \mathcal{Y} = \mathcal{X}_E \cup \mathcal{Y}_E = \mathcal{X}_D \cup \mathcal{Y}_D$.

Churn(H):

Step 1: If H is regular, then return H .

Step 2: If $H - V_\ell \notin \mathcal{Y}$, then return Churn($H - V_\ell$).

Step 3: If $H - V_h \notin \mathcal{Y}$, then return Churn($H - V_h$).

Step 4: Return H .

Proposition 3.5 implies Corollary 3.6.

Corollary 3.6. *Let H' be the output of Churn(H). Then H simulates H' .*

We prove Lemma 3.4 by analyzing Churn() and showing that the graph returned by it always satisfies the requirements of the lemma. The procedure first handles the case when H is regular. In Section 3.1, we show that if H is regular, then it is safe to return H , as it is already in $\mathcal{X}_D \subseteq \mathcal{X}_E$. If H is not regular, then $H - V_\ell$ and $H - V_h$ are both defined. If one of these two graphs is not in \mathcal{Y} , then Proposition 3.5 allows us to proceed by recursion on that graph. Step 4 is reached when both $H - V_\ell$ and $H - V_h$ are in \mathcal{Y} . However, at this point the conditions on $H - V_\ell$ and $H - V_h$ give us important structural information about the graph H , which can be exploited to show that it is in $\mathcal{X}_D \cup \mathcal{W}$. Recall that \mathcal{Y} is the union of complete, empty, near-empty, and the finite graphs in \mathcal{Y}' . This means we can split the problem into $4 \cdot 4$ different cases, with very strict structural restrictions on H in each case. These cases are analysed in a sequence of lemmas/corollaries (Lemma 3.8 to Lemma 3.23 in Sections 3.2–3.5).

3.1 Regular graphs

In this section, we handle the case when H is regular.

Theorem 3.7. *Let H be a regular graph. Then H -FREE EDGE DELETION, H -FREE EDGE COMPLETION, and H -FREE EDGE EDITING are incompressible if and only if H is neither complete nor empty, where the incompressibility assumes $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.*

Proof. Let H be an r -regular graph. If H is either empty or complete, then by Proposition 2.7, the problems admit polynomial kernels. To prove the other direction, assume that H is an r -regular graph, which is neither complete nor empty. It can be easily verified that if H has exactly one nonedge then H must be a $2K_1$, an empty graph. Similarly, if H has exactly one edge then H is a K_2 , a complete graph. Therefore, assume that both H and \overline{H} has at least two edges and two nonedges. Now, it is sufficient to prove that either H or \overline{H} is 3-connected or a cycle with at least four vertices (see Propositions 2.5 and 2.6).

Suppose that $4 \leq r \leq |V(H)| - 5$. Assume that H is not 3-connected. Then there exists a set $S \subseteq V(H)$ such that $H - S$ is disconnected and $|S| \leq 2$. Let A be the set of vertices of any component in $H - S$. Since $r \geq 4$, we obtain that $|A| \geq 3$. Therefore $\overline{H - S}$ is a 3-connected graph. Since $r \leq |V(H)| - 5$, every vertex in S has at least three neighbors outside S in \overline{H} . Therefore, \overline{H} is 3-connected. Suppose that $r > |V(H)| - 5$ (the case $r < 4$ can be handled by considering \overline{H}). Let $|V(H)| \geq 9$. Then every pair of non-adjacent vertices has at least three common neighbors. Therefore, H is 3-connected. By using a computer search, we verified that if $r > |V(H)| - 5$ and $|V(H)| \leq 8$, then H or \overline{H} is either 3-connected, or H is a $2K_2$, or a C_4 , or a C_5 . \square

3.2 Small graphs

If both $H - V_\ell$ and $H - V_h$ are in the finite set \mathcal{Y}' of graphs, then H has bounded size. An exhaustive computer search showed the correctness of the procedure in this case.

Lemma 3.8. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that both $H - V_\ell$ and $H - V_h$ are in \mathcal{Y}' . Then $H \in \mathcal{W}$.*

Proof. Since every graph in \mathcal{Y}' has only at most four vertices, H has only at most eight vertices. By a computer search we found that $H \in \mathcal{W}$. \square

3.3 Cliques and empty graphs

In this section, we consider the cases when both $H - V_\ell$ and $H - V_h$ are cliques or empty graphs. In this case, the structure of H is very limited. In principle, we need to consider four cases separately depending on the type of $H - V_\ell$ and $H - V_h$. However, a simple complementation argument shows that the case when both of them are cliques is equivalent to the case when both of them are empty.

Lemma 3.9. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that both $H - V_\ell$ and $H - V_h$ are complete graphs. Then $H \in \mathcal{W}$.*

Proof. If H has only at most nine vertices, by a computer search we found that $H \in \mathcal{W}$. Assume that H has at least ten vertices and $H \notin \mathcal{W}$. We claim that either H or \overline{H} is 3-connected, which is a contradiction.

Since $H - V_\ell$ and $H - V_h$ are complete graphs, both $V_m \cup V_\ell$ and $V_m \cup V_h$ induce complete graphs. This implies that every vertex in V_m is universal and hence is having the highest degree, which is a contradiction. Therefore $V_m = \emptyset$. Assume that there exists at least one edge between V_ℓ and V_h . Then every vertex in V_ℓ has a neighbor in V_h and every vertex in V_h has a neighbor in V_ℓ . Then it can be easily verified that H is 3-connected. Therefore, assume that there is no edge between V_ℓ and V_h . Then H is $K_t \cup K_s$ where $t > s$. If $s = 1$, then $H \in \overline{\mathcal{F}_2}$, a contradiction. If $s = 2$, then $H \in \overline{\mathcal{F}_1}$, a contradiction. Therefore, $s \geq 3$. Then \overline{H} is 3-connected. \square

Corollaries in this section and in Sections 3.4 and 3.5 use the facts that various sets we consider are self-complementary, i.e., $\mathcal{X} \cup \mathcal{Y} = \overline{\mathcal{X} \cup \mathcal{Y}}$, $\mathcal{W} = \overline{\mathcal{W}}$, $\mathcal{Y}' = \overline{\mathcal{Y}'}$.

Corollary 3.10. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that both $H - V_\ell$ and $H - V_h$ are empty graphs. Then $H \in \mathcal{W}$.*

Lemma 3.11. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that $H - V_\ell$ is a complete graph and $H - V_h$ is an empty graph. Then $H \in \mathcal{W}$.*

Proof. If H has at most nine vertices, by a computer search we verified that $H \in \mathcal{W}$. Assume that H has at least ten vertices. For a contradiction, assume that $H \notin \mathcal{W}$. Then we will show that either H or \overline{H} is 3-connected, which is a contradiction. If $\ell \geq 3$, then H is 3-connected. Therefore, $\ell \leq 2$. If $h^* = |V(H)| - h - 1 \geq 3$, then \overline{H} is 3-connected. Therefore, $h^* \leq 2$, which implies that $h \geq 10 - 3 = 7$, as H has at least ten vertices. We observe that every vertex in V_h has same number of neighbors, say t , in V_ℓ . Since $V_m \cup V_\ell$ is an independent set and $V_m \cup V_h$ is a clique, we obtain that $|V_m| \leq 1$. Further, by symmetry, we can assume that $|V_\ell| \geq |V_h|$ (otherwise we can consider \overline{H}).

Case 1: $\ell = 0$. Then the graph is $K_t \cup sK_1$ (for $s \geq t$). If $s \geq 3$, then \overline{H} is 3-connected. If $s \leq 2$, then H has only at most five vertices, which is a contradiction.

Case 2: $\ell = 1$. Since H is not $K_{1,t}$ ($t \geq 9, \in \mathcal{F}_2$), we obtain that $|V_h| \geq 2$. If $|V_h| = 2$ and $t \geq 3$, then $h^* \geq 3$, a contradiction. If $|V_h| = 2$ and $t \leq 2$, then $|V(H)| \leq 7$, a contradiction. If $|V_h| = 3$ and $t \geq 2$, then $h^* \geq 3$, a contradiction. If $|V_h| = 3$ and $t \leq 1$, then H has only at most seven vertices, which is a contradiction. If $|V_h| \geq 4$, then $h^* \geq 3$, a contradiction.

Case 3: $\ell = 2$. Clearly, $|V_h| \geq 2$. If $|V_h| = 2$, then H is $K_2 \boxtimes sK_1$ (for $s \geq 8$), a contradiction as it is in \mathcal{F}_3 . We have that $2|V_\ell| = |V_h|(|V_\ell| - h^*)$. This implies that $|V_\ell| = (h^*|V_h|)/(|V_h| - 2)$. Hence if $|V_h| \geq 3$, then $|V_\ell| \leq 6$ as $h^* \leq 2$. Therefore, if $|V_h| = 3$, then $|V_\ell| = 6$ (as $|V(H)| \geq 10$ and $|V_m| \leq 1$) and H is S_{28} , a contradiction. If $|V_h| = 4$, then $|V_\ell| \leq 4$, which contradicts with the assumption that H has at least ten vertices. If $|V_h| \geq 5$, then $|V_\ell| < |V_h|$, a contradiction. \square

Our last case is when $H - V_\ell$ is empty and $H - V_h$ is complete. Let us observe that this case *does not* follow from Lemma 3.11 by complementation. If \overline{V}_ℓ and \overline{V}_h are the lowest- and highest-degree vertices in \overline{H} , then $\overline{V}_\ell = V_h, \overline{V}_h = V_\ell$ and hence $\overline{H} - \overline{V}_\ell$ is empty and $\overline{H} - \overline{V}_h$ is a clique, that is, we have the same condition as for H . Fortunately, this last case is very simple to handle.

Lemma 3.12. *There exists no graph $H \notin \mathcal{X} \cup \mathcal{Y}$ such that $H - V_\ell$ is an empty graph and $H - V_h$ is a complete graph.*

Proof. The constraints imply that H is a split graph with a partitioning $\{V_\ell \cup V_m, V_h\}$, where $V_\ell \cup V_m$ forms a clique and V_h forms an independent set. Then we obtain that the degree of a vertex in $V_\ell \cup V_m$ is greater than that of a vertex in V_h , which is a contradiction. \square

3.4 Cliques/empty graphs plus small graphs

Next we consider the cases when one of $H - V_\ell$ or $H - V_h$ is a clique or an empty graph, while the other is a graph from the finite set \mathcal{Y}' . Assuming that H is not too small, this means that H is essentially a clique or an empty graph, and intuitively it should follow that H or \overline{H} is 3-connected, respectively. However, this requires a detailed proof considering several cases.

Lemma 3.13. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that $H - V_\ell \in \mathcal{Y}'$ and $H - V_h$ is a complete graph. Then $H \in \mathcal{W}$.*

Proof. If H has only at most nine vertices, by using a computer search we verified that $H \in \mathcal{W}$. Assume that H has at least ten vertices. For a contradiction, assume that $H \notin \mathcal{W}$. We will prove that H is 3-connected, which is a contradiction. Since every graph in \mathcal{Y}' has at most four vertices, $H - V_h$, which is a complete graph, has at least six vertices. This implies that a vertex in $V_\ell \cup V_m$ has degree at least five and hence $h \geq 6$. Since the maximum degree of every graph in \mathcal{Y}' is at most 3, every vertex in V_h has at least three neighbors in $V_\ell \cup V_m$. Hence H is a 3-connected graph. \square

Corollary 3.14. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that $H - V_\ell$ is an empty graph and $H - V_h \in \mathcal{Y}'$. Then $H \in \mathcal{W}$.*

Lemma 3.15. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that $H - V_\ell \in \mathcal{Y}'$ and $H - V_h$ is an empty graph. Then $H \in \mathcal{W}$.*

Proof. If H has only at most nine vertices, by using a computer search we verified that $H \in \mathcal{W}$. Let H has at least ten vertices. For a contradiction, assume that $H \notin \mathcal{W}$. We will show that either H or \overline{H} is 3-connected, which is a contradiction. We observe that $|V_\ell| \geq 6$. Therefore, if $\ell = 0$, then \overline{H} is 3-connected. Therefore, $\ell \geq 1$. If every vertex in V_ℓ is adjacent to every vertex in $V_h \cup V_m$, then \overline{H} is 3-connected. Therefore, assume that $\ell \leq |V_h \cup V_m| - 1 \leq 3$.

Case 1: $\ell = 1$. Clearly, every vertex in V_ℓ has only one neighbor in V_h and has no neighbors in V_m . Since there are no triple of mutually nonadjacent vertices each with degree at least two in the graphs in \mathcal{Y}' , we obtain that $|V_m| \leq 2$. Since degrees of vertices in V_m is at least two, we get that $|V_h| \geq 2$ (if $V_m = \emptyset$, then $|V_h| \geq 3$ as graphs in \mathcal{Y}' has at least three vertices). Therefore, if every vertex in V_h has at least $5 - |V_h|$ neighbors in V_ℓ , then every vertex in V_h has at least $(|V_h| - 1)(5 - |V_h|) \geq 3$ non-neighbors in V_ℓ . Therefore, \overline{H} is 3-connected. If a vertex in V_h has only at most $4 - |V_h|$ neighbors in V_ℓ , then $h^* \geq 6 - (4 - |V_h|) = |V_h| + 2$. If $|V_h| \geq 3$, then every vertex in V_h has at least three neighbors outside V_h in \overline{H} (due to the fact that the difference in degrees of vertices in graphs in \mathcal{Y}' is at most two). Then, since the vertices outside V_h (i.e., $V_\ell \cup V_m$) forms an independent set, \overline{H} is 3-connected. Let $|V_h| = 2$. Then if the difference in degrees of vertices in V_h is at most one, then every vertex in V_h has at least three neighbors outside V_h in \overline{H} and hence \overline{H} is 3-connected. The difference in degrees of vertices in V_h is two in $H[V_h \cup V_m]$ only if one vertex in V_h is adjacent to all vertices in V_m and the other vertex in V_h is nonadjacent to both the vertices in V_m . Then every vertex in V_h has at least three neighbors outside V_h in \overline{H} and hence \overline{H} is 3-connected.

Case 2: $\ell = 2$. Since there are no two nonadjacent vertices having degree at least three in the graphs in \mathcal{Y}' , $|V_m| = 0$ or 1. Therefore, $|V_h| = 3$ or 4.

Suppose $|V_h| = 3$. Since $H - V_\ell$ is not a clique and every vertex in V_ℓ is nonadjacent to exactly one vertex in V_h , we obtain that the sum of degrees of vertices of V_h in \overline{H} is at least $|V_\ell| + 2 \geq 8$. This means that there is a vertex in V_h whose degree is at least 3 in \overline{H} , which is then true for every vertex in V_h . If every vertex of V_h has degree at least 4 in \overline{H} , then it is easy to see that \overline{H} is 3-connected. If every vertex in V_h has degree 3 in \overline{H} , then we have to rule out the case that $x, y \in V_h$ can be separated from the rest by deleting two vertices. However, this means that $\{x, y\}$ is adjacent to at most two vertices of V_ℓ in \overline{H} , hence the third vertex in V_h is adjacent to at least $|V_\ell| - 2 \geq 4$ vertices of V_h in \overline{H} , contradicting the assumption that it has degree 3 in \overline{H} .

Suppose that $|V_h| = 4$. Since there are $2|V_\ell|$ edges between V_ℓ and V_h , at least one vertex in V_h has at most $2|V_\ell|/4$ neighbors in V_ℓ . Therefore, there is a vertex in V_h that is adjacent to at least $|V_\ell|/2 \geq 3$ vertices of V_ℓ in \overline{H} . So the degree of every vertex in V_h is at least three in \overline{H} . If it is at least four, then \overline{H} is 3-connected (note that it is not possible to separate 3 vertices of V_h from the rest of the graph by deleting at most two vertices, as any set $\{x, y, z\} \subseteq V_h$ is adjacent to every vertex in V_ℓ). If degree of every vertex in V_h is exactly 3 in \overline{H} , then suppose that $\{x, y\}$ is separated from the rest of the graph by the deletion of two vertices. This means that $\{x, y\}$ see at most two vertices of V_ℓ in \overline{H} . The remaining at least 4 vertices of V_ℓ are adjacent to both of the remaining two vertices of V_h in \overline{H} , making the degree of those two vertices at least 4.

Case 3: $\ell = 3$. Since $\ell \leq |V_h \cup V_m| - 1$, $|V_h \cup V_m| = 4$. Since there is no vertex in \mathcal{Y}' with degree more than three, we obtain that $V_m = \emptyset$. Let $V_h = \{w, x, y, z\}$. We claim that H is 3-connected. For a contradiction, assume that the vertex-connectivity of H is at most two. It can be verified easily that by deleting at most one vertex from V_ℓ and at most one vertex from V_h we get a connected graph. Assume that by deleting at most two vertices in V_ℓ from H , we get a disconnected graph. This implies that every other vertex in V_ℓ is adjacent to exactly a set of same three vertices, say $\{x, y, z\}$, in $V_h \cup V_m$. Therefore, w has at most two neighbors in V_ℓ and at least one vertex in $\{x, y, z\}$ has at least six neighbors in V_ℓ . This is a contradiction as the difference in degrees of vertices in $H[V_h]$ is at most two. Now, assume that, by deleting at most two vertices in V_h (say x, y) from H , we get a disconnected graph. Then every vertex in V_ℓ is adjacent to both x and y . Then there is a vertex in V_h which has at most $|V_\ell|/2$ neighbors in V_ℓ . Since $V_\ell \geq 6$, we get a contradiction, as in the previous case. \square

Corollary 3.16. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that $H - V_\ell$ is a complete graph and $H - V_h \in \mathcal{Y}'$. Then $H \in \mathcal{W}$.*

3.5 Near-empty graphs

Finally, we consider the cases when one of $H - V_\ell$ or $H - H_h$ is near empty. These cases are similar to the corresponding ones for empty graphs, but more technical and a higher number of corner cases need to be handled. Let us remark that this part of the proof is needed only for the H -FREE EDGE DELETION problem: near-empty graphs are not in \mathcal{Y}_E , hence if our goal is to prove Theorem 1.1 for H -FREE EDGE EDITING, then the churning procedure can recurse on such graphs.

Lemma 3.17. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that $H - V_\ell$ is a complete graph and $H - V_h$ is a near-empty graph. Then $H \in \mathcal{W}$.*

Proof. If H has at most nine vertices, then by a computer search we found that $H \in \mathcal{W}$. Therefore, let H has at least ten vertices. For a contradiction, assume that $H \notin \mathcal{W}$. We will prove that either H or \overline{H} is 3-connected, a contradiction.

If $h^* = |V_h| - h - 1 \geq 3$, then \overline{H} is 3-connected. Therefore, assume that $h^* \leq 2$. If $|V_\ell \cup V_m| \leq 4$, then $V_\ell \cup V_m$ induces either a K_2 or a graph in \mathcal{Y}' . Then the statement follows from Lemma 3.9 and Corollary 3.16. Therefore, assume that $|V_\ell \cup V_m| \geq 5$. The conditions imply that $|V_m| \leq 2$. Therefore, $|V_\ell| \geq 3$. Clearly, every vertex in V_h has the same number $t \geq 0$ of neighbors in V_ℓ . If $t = 0$ then V_ℓ induces a K_2 , a contradiction. Therefore, $t \geq 1$. Hence $\ell \geq 1$. If $\ell \geq 4$, then H is 3-connected. Therefore, $1 \leq \ell \leq 3$. Every vertex in V_ℓ is adjacent to at least one vertex in V_h except possibly for two vertices (due to the single edge in $H - V_h$). Let uv be the edge in $H[V_\ell \cup V_m]$ and let $x = |V_\ell \cap \{u, v\}|$. We observe that the number of edges between V_h and V_ℓ is $t|V_h| = \ell|V_\ell| - x$. Since $h^* \leq 2$, $t \geq |V_\ell| - 2$. Thus we obtain that

$$|V_\ell| \leq \frac{2|V_h| - x}{|V_h| - \ell} \quad (1)$$

Case 1: $\ell = 1$. Let $|V_h| \geq 2$. Then by (1), $|V_\ell| \leq 2|V_h|/(|V_h| - 1)$. Since H has at least ten vertices and V_m has at most two vertices, we obtain that $|V_h| \geq 6$. Then $|V_\ell| \leq 2$, a contradiction. Therefore, $|V_h| = 1$. If $V_m = \emptyset$, then H is $K_{1,t} \cup K_2$ ($\in \mathcal{F}_7$), a contradiction. If $|V_m| = 1$, then H is $T_{t,1}$ ($\in \mathcal{F}_4$), a contradiction. If $|V_m| = 2$, then H is $(K_{t+2} - e) \cup K_1$ ($\in \mathcal{F}_8$), a contradiction.

Case 2: $\ell = 2$. Clearly, $|V_h| \geq 2$. If $|V_h| \geq 7$, then by (1), $|V_\ell| \leq 2$, a contradiction. Therefore, $2 \leq |V_h| \leq 6$.

Case 2i: $|V_h| = 2$. If $V_m = \emptyset$, then H is J_t ($\in \mathcal{F}_9$), a contradiction. It can be easily verified that there is no H with $|V_m| = 1$. If $|V_m| = 2$, then H is $(K_{t+2} - e) \cup 2K_1$ ($\in \mathcal{F}_5$), a contradiction.

Case 2ii: $|V_h| = 3$. If $x = 2$, then by (1), $|V_\ell| \leq 4$. Since H has at least ten vertices, $|V_m| \geq 3$, a contradiction. If $x = 1$, then by (1), $|V_\ell| \leq 5$ and hence $|V_m| \geq 2$. Therefore $|V_m| = 2$, which is a contradiction, as then both u and v are in V_m . If $x = 0$, then by (1), $|V_\ell| \leq 6$ and hence $|V_m| \geq 1$. Since $x = 0$, both u, v must be in V_m . Therefore, $|V_m| = 2$. Then H is S_{29} .

Case 2iii: $|V_h| = 4$. If $x = 2$ or 1 , then by (1), $|V_\ell| \leq 3$. Since $|V(H)| \geq 10$, we obtain that $|V_m| \geq 3$, a contradiction. If $x = 0$, then by (1), $|V_\ell| \leq 4$. Then, $|V_\ell| = 4$ and $|V_m| = 2$ and H is either S_{30} or S_{31} , which are contradictions.

Case 2iv: $|V_h| = 5$. If $x = 2$, then by (1), $|V_\ell| \leq 2$, a contradiction. If $x = 1$, then by (1), $|V_\ell| \leq 3$. Since $|V(H)| \geq 10$, $|V_m| \geq 2$. Then we get a contradiction as then both u and v are in V_m . If $x = 0$, then by (1), $|V_\ell| \leq 3$. Then $|V_\ell| = 3$ and $|V_m| = 2$. Then the number of edges between V_ℓ and V_h is 6, which is not a multiple of $|V_h|$, a contradiction.

Case 2v: $|V_h| = 6$. If $x = 2$ or 1 , then by (1), $|V_\ell| \leq 2$, a contradiction. If $x = 0$, then by (1), $|V_\ell| \leq 3$. Therefore, $|V_\ell| = 3$ and $|V_m| = 2$. Then H is S_{32} , a contradiction.

Case 3: $\ell = 3$. Clearly, $|V_h| \geq 3$. It is easy to verify that H is 3-connected unless both u and v are in V_ℓ (i.e., $x = 2$). Therefore, $x = 2$ and $|V_m| \leq 1$. Then by (1),

$$|V_\ell| \leq \frac{2|V_h| - 2}{|V_h| - 3} \quad (2)$$

If $|V_h| \geq 8$, we obtain from (2) that $|V_\ell| \leq 2$, a contradiction. Therefore, $3 \leq |V_h| \leq 7$.

Case 3i: $|V_h| = 3$. Then the number of edges between V_ℓ and V_h is $3|V_\ell| - 2$ which is not a multiple of $|V_h|$, a contradiction.

Case 3ii: $|V_h| = 4$. Then by (2), $|V_\ell| \leq 6$. Since $|V_m| \leq 1$ and $|V(H)| \geq 10$, we obtain that $|V_\ell| \geq 5$. We observe that the number of edges between V_ℓ and V_h is $3|V_\ell| - 2$. Since it must be a multiple of $|V_h|$, we obtain that $|V_\ell| = 6$. Then H is S_{33} (when $|V_m| = 0$) or S_{34} (when $|V_m| = 1$).

Case 3iii: $|V_h| = 5$. Then by (2), $|V_\ell| \leq 4$. Since $|V(H)| \geq 10$, we obtain that $|V_\ell| = 4$ and $|V_m| = 1$. Then H is S_{36} , a contradiction.

Case 3iv: $|V_h| = 6$. Then by (2), $|V_\ell| \leq 3$. Then $|V_\ell| = 3$ and $|V_m| = 1$. We observe that the number of edges between V_ℓ and V_h is 7, which is a contradiction, as it is not a multiple of $|V_h|$.

Case 3v: $|V_h| = 7$. Then by (2), $|V_\ell| \leq 3$. Then $|V_\ell| = 3$ as it cannot be less than 3. Then the number of edges between V_ℓ and V_h is $3 \cdot 3 - 2 = 7$. This implies that $t = 1$. This is a contradiction as two vertices in V_h has two neighbors (u and v) each in V_ℓ . \square

Lemma 3.18. *There exists no $H \notin \mathcal{X} \cup \mathcal{Y}$ such that $H - V_\ell$ is a near-empty graph and $H - V_h$ is a complete graph.*

Proof. By a computer search, we found that there exists no such graph H with at most nine vertices. For a contradiction, assume that there exists such a graph H with at least ten vertices. Let $H - V_h$ has s vertices. Since it forms a clique, $\ell \geq s - 1$. Therefore, $h \geq s$. This implies that a vertex in V_h is adjacent to at least $s - 1$ vertices in $H - V_h$ (recall that $H[V_h]$ has only at most one edge). Therefore, a vertex in $V_m \cup V_\ell$ has degree at least s and hence $h \geq s + 1$. Therefore, every vertex in V_h has degree 1 in $H[V_h]$ and is adjacent to every vertex in $V_m \cup V_\ell$. This implies that $\ell = h - 1$ and hence $V_m = \emptyset$. This implies that H is K_{s+2} , which is a contradiction. \square

Lemma 3.19. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ such that $H - V_\ell$ is an empty graph and $H - V_h$ is a near-empty graph. Then $H \in \mathcal{W}$.*

Proof. If H has only at most nine vertices, by a computer search we found that $H \in \mathcal{W}$. Assume that H has at least ten vertices. For a contradiction, assume that $H \notin \mathcal{W}$. Then we will show that either H or \overline{H} is 3-connected, a contradiction.

If $H - V_h$ is a K_2 , then by Lemma 3.12, H does not exist. Therefore, let $|V_m \cup V_\ell| \geq 3$. If $\ell = 0$, then $H[V_m]$ contains an edge, which is a contradiction as $V_h \cup V_m$ is an independent set. Therefore, $\ell \geq 1$. Since $H - V_\ell$ has no edge and $H - V_h$ has exactly one edge, a vertex in V_m has degree at most 1, which is a contradiction as it is not more than ℓ . Therefore, $V_m = \emptyset$. Hence $|V_\ell| \geq 3$. Further, $|V_h| \geq \ell$ and $|V_\ell| \geq h$. If a vertex in V_h is adjacent to all vertices in V_ℓ , then every vertex in V_h is adjacent to all vertices in V_ℓ . Then there will be a discrepancy in the degrees of vertices in V_ℓ (due to the single edge in $H[V_\ell]$ and the fact that $|V_\ell| \geq 3$). Therefore, $h < |V_\ell|$. Let $|V_h| \geq 3$. Then it can be easily verified that $\ell < |V_h|$. Therefore, every vertex in V_h has a neighbor in V_ℓ and every vertex in V_ℓ has a neighbor in V_h in \overline{H} . Hence \overline{H} is 3-connected. Therefore, $|V_h| \leq 2$. Hence $\ell = 1$ or 2 . Let $\ell = 1$. Then H is $sK_{1,h} \cup K_2$. If $s > 1$, then \overline{H} is 3-connected. When $s = 1$, H is $K_{1,h} \cup K_2 \in \mathcal{F}_7$, a contradiction. Let $\ell = 2$. Then $|V_h| = 2$ and H is $Q_t \in \mathcal{F}_{10}$, a contradiction. \square

Lemma 3.20. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ such that $H - V_\ell$ is a near-empty graph and $H - V_h$ is an empty graph. Then $H \in \mathcal{W}$.*

Proof. If H has only at most nine vertices, then by a computer search we found that $H \in \mathcal{W}$. Let H has at least ten vertices. For a contradiction, assume that $H \notin \mathcal{W}$. We will show that H or \overline{H} is 3-connected, a contradiction.

If $H - V_\ell$ is a K_2 , then the statement follows from Lemma 3.11. Therefore, let $|V_m \cup V_h| \geq 3$. If $\ell = 0$, then H is $K_2 \cup |V_\ell|K_1$ ($\in \mathcal{X} \cup \mathcal{Y}$), a contradiction. Therefore, $\ell \geq 1$. Since $H - V_h$ has no edge and $H - V_\ell$ has exactly one edge, a vertex in V_m has degree at most 1, which is a contradiction as it is not more than ℓ . Therefore, $V_m = \emptyset$. Hence $|V_h| \geq 3$. Further, $|V_h| \geq \ell$ and $|V_\ell| \geq h$. If a vertex in V_ℓ is adjacent to all vertices in V_h , then every vertex in V_ℓ is adjacent to all vertices in V_h . Then there will be a discrepancy in the degrees of vertices in V_h (due to the single edge in $H[V_h]$ and the fact that $|V_h| \geq 3$). Therefore, $\ell < |V_h|$. Let $|V_\ell| \geq 3$. Then it can be easily verified that $h < |V_\ell|$. Therefore, every vertex in V_h has a neighbor in V_ℓ and every vertex in V_ℓ has a neighbor in V_h in \overline{H} . Hence \overline{H} is 3-connected. Therefore, $|V_\ell| \leq 2$. Hence $h = 1$ or 2 . Since $\ell \geq 1$, $h = 2$. Then $\ell = 1$. Then H is $P_4 \cup sP_3$, for $s \geq 2$ (as $|V(H)| \geq 10$). Then \overline{H} is 3-connected. \square

Lemma 3.21. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that both $H - V_\ell$ and $H - V_h$ are near-empty graphs. Then $H \in \mathcal{W}$.*

Proof. If H has only at most nine vertices, by a computer search we found that $H \in \mathcal{W}$. Assume that H has at least ten vertices. For a contradiction, assume that $H \notin \mathcal{W}$. Then we will show that either H or \overline{H} is 3-connected, a contradiction.

If $H - V_\ell$ is a K_2 , then the statement follows from Lemma 3.11. Therefore, let $|V_m \cup V_h| \geq 3$. If $H - V_h$ is a K_2 , then by Lemma 3.18, H does not exist. Therefore, let $|V_\ell \cup V_m| \geq 3$. Clearly, $\ell \geq 1$. Since $H - V_h$ and $H - V_\ell$ have exactly one edge each, a vertex in V_m has degree at most 2. Therefore, if $V_m \neq \emptyset$, then $|V_m| = 1$, $\ell = 1$, and $h \geq 3$. Then H is $T_{h-1,1} \cup sK_{1,h}$, for $s \geq 1$ (recall that $|V_h \cup V_m| \geq 3$). Then \overline{H} is 3-connected. Therefore, $V_m = \emptyset$. Further, $|V_h| > \ell$ and $|V_\ell| > h$. Therefore, every vertex in V_h has a neighbor in V_ℓ and every vertex in V_ℓ has a neighbor in V_h in \overline{H} . Hence \overline{H} is 3-connected. \square

Lemma 3.22. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that $H - V_\ell$ is a near-empty graph and $H - V_h \in \mathcal{Y}'$. Then $H \in \mathcal{W}$.*

Proof. If H has only at most nine vertices, by using a computer search we verified that $H \in \mathcal{W}$. Let H has at least ten vertices. Since every graph in \mathcal{Y}' has only at most four vertices, V_h has at least six vertices. Since $H[V_h]$ has at most one edge and $H[V_\ell \cup V_m]$ has at least one edge, a simple degree counting gives us a contradiction. \square

Lemma 3.23. *Let $H \notin \mathcal{X} \cup \mathcal{Y}$ be such that $H - V_\ell \in \mathcal{Y}'$ and $H - V_h$ is a near-empty graph. Then $H \in \mathcal{W}$.*

Proof. If H has only at most nine vertices, by using a computer search we verified that $H \in \mathcal{W}$. Assume that H has at least ten vertices. For a contradiction, assume that $H \notin \mathcal{W}$. We will show that either H or \overline{H} is 3-connected, a contradiction.

We observe that $|V_\ell| \geq 6$ if $|V_h \cup V_m| = 4$, and $|V_\ell| \geq 7$ if $|V_h \cup V_m| = 3$ (recall that a graph in \mathcal{Y}' has either 3 or 4 vertices). Therefore, if $\ell = 0$, then \overline{H} is 3-connected. Hence assume that $\ell \geq 1$. If a vertex in V_ℓ is adjacent to all vertices in $V_h \cup V_m$, then every degree-0 vertex in $H[V_\ell]$ is adjacent to every vertex in $V_h \cup V_m$. Since there are at least three vertices in V_ℓ adjacent to all vertices in $V_h \cup V_m$, H is 3-connected. Therefore, assume that none of the vertices in V_ℓ is adjacent to all vertices in $V_h \cup V_m$. Therefore, $\ell \leq |V_h \cup V_m| - 1$. Let $H[V_\ell \cup V_m]$ be $K_2 \cup tK_1$, for $t \geq 4$. If $|V_m| = 3$, then the single edge in $H[V_\ell \cup V_m]$ and the edges from the single vertex in V_h cannot give degree at least two for every vertex in V_m . Therefore, $|V_m| \leq 2$. Further, since $H[V_\ell \cup V_m]$ induces a $K_{t+2} - e$ in \overline{H} , if every vertex in V_h has at least three non-neighbors in $V_\ell \cup V_m$ in H , then \overline{H} is 3-connected. Therefore, assume that there exists a vertex in V_h with at most two non-neighbors in $V_\ell \cup V_m$. Since the difference in degrees of vertices in graphs in \mathcal{Y}' is at most two, we obtain that every vertex in V_h has only at most four non-neighbors in $V_\ell \cup V_m$.

Case 1: $\ell = 1$. Since there is a vertex in V_h with at most two non-neighbors in V_ℓ and every vertex in V_h has only at most four non-neighbors in V_ℓ , we obtain that if $|V_h| \geq 3$, then there is a vertex in V_ℓ with degree at least 2, a contradiction. Therefore, $|V_h| \leq 2$. Now, we have the following cases.

Case 1a: $|V_h| = 1, |V_m| = 2$. Since both the vertices in V_m has degree at least two, both must be mutually adjacent and adjacent to the vertex in V_h . Then $H[V_h \cup V_m]$ is a K_3 , a contradiction.

Case 1b: $|V_h| = 2, |V_m| = 1$. Since $H[V_h \cup V_m]$ is a graph in \mathcal{Y}' with three vertices, the difference in degrees of vertices in V_h in $H[V_h \cup V_m]$ is at most one. Let v_1, v_2 be the vertices in V_h . Let v_1 be a vertex with at most two non-neighbors in V_ℓ . Then v_2 must be having only at most three non-neighbors in V_ℓ . Since $\ell = 1$, this implies that $|V_\ell| \leq 5$. This contradicts with the fact that H has at least ten vertices.

Case 1c: $|V_h| = 2, |V_m| = 2$. Due to the bounds on the non-neighbors in V_ℓ of vertices in V_h , we obtain that $|V_\ell| = 6$. Then one vertex in V_h is adjacent to four vertices in V_ℓ and non-adjacent to two vertices in V_m and the other vertex in V_h is adjacent to two vertices in V_ℓ and two vertices in V_m . Then the two vertices in V_m are adjacent. Then H can be one of the two graphs obtained based on whether the two vertices in V_h are adjacent or not. Then both the vertices in V_h are adjacent to four vertices in $V_\ell \cup V_m$ in \overline{H} and hence \overline{H} is 3-connected.

Case 2: $\ell = 3$. Since $\ell \leq |V_h \cup V_m| - 1$, we obtain that $|V_h \cup V_m| = 4$. Since there is no vertex in \mathcal{Y}' with degree more than three, we obtain that either $V_m = \emptyset$ or $|V_m| = 1$ and the vertex in V_m has exactly three neighbors in V_h and one neighbor in V_ℓ .

Case 2a: $V_m = \emptyset$. Let u, v be the two adjacent vertices in $H[V_\ell]$. We observe that V_ℓ induces $K_{|V_\ell|} - e$ in \overline{H} . Since $\ell = 3$ and $|V_h| = 4$, u and v has exactly two neighbors each in V_h in \overline{H} . Further, every other vertex in V_ℓ has exactly one neighbor each in V_h in \overline{H} . Therefore, there are $2 \cdot 2 + (|V_\ell| - 2) = |V_\ell| + 2$ edges between V_ℓ and V_h in \overline{H} . Let p be the number of edges in the graph induced by V_h in \overline{H} . We observe that $1 \leq p \leq 5$. Then we have

$$|V_\ell| + 2 = 4h^* - 2p \quad (3)$$

Recall that $|V_\ell| \geq 6$. Therefore, if $h^* \leq 2$, we get a contradiction by (3). Let $h^* = 3$. Since \overline{H} is not 3-connected, we obtain that there are two adjacent (in \overline{H}) vertices in V_h adjacent (in \overline{H}) to both u and v . By (3), $V_\ell = 6$ or 8. If $V_\ell = 6$ then $p = 2$. Then V_h induces a $2K_2$, a contradiction. Let $|V_\ell| = 8$. Then H is S_{35} , a contradiction. If $h^* \geq 4$, it can be easily verified that \overline{H} is 3-connected.

Case 2b: $|V_m| = 1$ and the vertex, say v_m , in V_m has exactly three neighbors in V_h and one neighbor, say w , in V_ℓ . Further, V_ℓ forms an independent set of at least six vertices. Since $\ell = 3$, every vertex in V_ℓ , except w is adjacent to all the three vertices in V_h . Then w must be adjacent to exactly two vertices in V_h . Then this gives a contradiction as degrees of vertices in V_h are not the same.

Case 3: $\ell = 2$.

Case 3a: $|V_h \cup V_m| = 3$. Since a vertex in V_m has degree at least three, we obtain that either $V_m = \emptyset$ or $|V_m| = 1$ and the vertex in V_m is adjacent to both the vertices in V_h and adjacent to exactly one vertex in V_ℓ .

Case 3ai: $V_m = \emptyset$. Clearly V_ℓ induces $K_2 \cup tK_1$, for $t \geq 5$. As claimed earlier, there is a vertex in V_h non-adjacent to only at most two vertices in V_ℓ . Since the difference in degrees of vertices in $H[V_h]$ is only at most one, we obtain that the other two vertices in V_h is adjacent to at least $t - 1$ vertices in V_ℓ . Then a degree-counting gives a contradiction as $3t - 2 > 2t + 2$.

Case 3aii: $|V_m| = 1$ and the vertex in V_m is adjacent to both the vertices in V_h and adjacent to exactly one vertex, say v , in V_ℓ . Then all the vertices in V_ℓ , except v , is adjacent to both the vertices in V_h . Then v must be adjacent to exactly one vertex in V_h , which is a contradiction.

Case 3b: $|V_h \cup V_m| = 4$.

Case 3bi: $V_m = \emptyset$. Since there is a vertex in V_h adjacent to at least t vertices in V_ℓ and the difference in degrees of vertices in every graph in \mathcal{Y}' is at most two, we obtain that there are at least $t + 3(t - 2) = 4t - 6$ edges between V_h and V_ℓ . Since $\ell = 2$ and $H[V_\ell]$ induces $K_2 \cup tK_1$, we obtain that there are exactly $2t + 2$ edges between V_h and V_ℓ . Therefore, $t \leq 4$. Since $t \geq 4$, we have $t = 4$. Then we obtain that a vertex in V_h is adjacent to exactly t vertices in V_ℓ and the other three vertices in V_h is adjacent to exactly $t - 2$ vertices in V_ℓ . Therefore, $H[V_h]$ is a claw. Then it can be verified that \overline{H} is 3-connected.

Case 3bii: $|V_m| = 1$. Let v_m be the vertex in V_m . Assume that v_m is not adjacent to V_ℓ . Since v_m has degree at least three, v_m must be adjacent to all the three vertices in V_h . Recall that a vertex in V_h is adjacent to at least t vertices in V_ℓ . Since the difference in degrees of vertices in V_h in $H[V_h \cup V_m]$ is at most one, there are at least $t + 2(t - 1) = 3t - 2$ edges between V_h and V_ℓ . Since $H[V_\ell]$ induces $K_2 \cup tK_1$, there are exactly $2t + 2$ edges between V_ℓ and V_h . Therefore, $2t + 2 \geq 3t - 2$. Then $t \leq 4$. Since $t \geq 4$, we obtain that $t = 4$. Then one vertex in V_h is adjacent to exactly four vertices in V_ℓ and the other two vertices in V_h are adjacent to exactly three vertices each in V_ℓ . Therefore, $H[V_h]$ induces $K_1 \cup K_2$. Then it can be verified that \overline{H} is 3-connected. Assume that v_m is adjacent to a vertex, say w , in V_ℓ . Then $H[V_\ell]$ is an independent set of at least $t + 1 \geq 6$ vertices. Further, v_m is adjacent to at least two vertices in V_h . Then in \overline{H} , w is adjacent to exactly two vertices in V_h and every vertex in $V_\ell \setminus \{w\}$ has exactly one neighbor in V_h . Therefore, if \overline{H} is not 3-connected, there is a vertex in V_h adjacent to exactly two vertices in V_ℓ and is not adjacent to any other vertex in V_h . Therefore, $h^* = 2$ and hence $h = |V(H)| - 3$. Then, every vertex in V_h is non-adjacent to only at most two vertices in V_ℓ in H . Then there are at least $3(t - 1)$ edges between V_ℓ and V_h . But there are exactly $2t + 1$ edges between V_h and V_ℓ . Therefore, $t \leq 4$, a contradiction.

Case 3biii: $|V_m| = 2$. Let $x, y \in V_m$. Assume that x and y are adjacent. Then V_ℓ is an independent set of size at least $t \geq 6$ vertices. Since vertices in V_m has degree at least 3, both x and y are adjacent to both the vertices in V_h and $H[V_h \cup V_m]$ is a diamond. Further, every vertex in V_ℓ is adjacent to both the vertices in V_h . Then the graph is $(K_{t+2} - e) \cup K_2 \in \mathcal{F}_6$, a contradiction. Assume that x and y are nonadjacent. Then either x or y cannot have degree at least three, a contradiction. \square

3.6 Putting it together

Now we are able to formally prove the main results of the section.

Proof of Lemma 3.4. We prove by induction on the number of vertices in H . To prove the first statement, let $H \notin \mathcal{Y}_D$. The base case is when $H \in \mathcal{X}_D \cup \mathcal{W}$. Then the statement is trivial, as H simulates H . Assume that $H \notin \mathcal{X}_D \cup \mathcal{W}$. Hence $H \notin \mathcal{Y}_D \cup \mathcal{X}_D = \mathcal{X} \cup \mathcal{Y}$. If $H - V_\ell \notin \mathcal{Y}$, then by the inductive hypothesis (as $H - V_\ell \notin \mathcal{Y}_D$),

$H - V_\ell$ simulates a graph in $\mathcal{X}_D \cup \mathcal{W}$. By Proposition 3.5, H simulates $H - V_\ell$. Therefore, by transitivity, H simulates a graph in $\mathcal{X}_D \cup \mathcal{W}$. The same arguments apply when $H - V_h \notin \mathcal{Y}$. Therefore, assume that both $H - V_\ell$ and $H - V_h$ are in \mathcal{Y} . Then by Lemma 3.9 to Lemma 3.23, we have that $H \in \mathcal{W}$, which is a contradiction.

To prove the second statement, let $H \notin \mathcal{Y}_E$. The base case is when $H \in \mathcal{X}_E \cup \mathcal{W}$, then the statement is trivially true. Assume that $H \notin \mathcal{X}_E \cup \mathcal{W}$. Therefore, $H \notin \mathcal{X}_E \cup \mathcal{Y}_E = \mathcal{X} \cup \mathcal{Y}$. If $H - V_\ell \in \mathcal{Y} \setminus \mathcal{Y}_E$, then $H - V_\ell$ is a graph with exactly one edge and at least five vertices. Therefore, $H - V_\ell \in \mathcal{X}_E$. Then by Proposition 3.5, H simulates a graph in \mathcal{X}_E . Assume that $H - V_\ell \notin \mathcal{Y}_E$. Then we are done by induction hypothesis and Proposition 3.5. Therefore, $H - V_\ell \notin \mathcal{Y}$. Similar arguments apply when $H - V_h \in \mathcal{Y}$. Therefore, assume that both $H - V_\ell$ and $H - V_h$ are not in \mathcal{Y} . Then by Lemma 3.9 to Lemma 3.23, we obtain that $H \in \mathcal{W}$, which is a contradiction. \square

Proof of Lemma 3.1. Assume that for every $H \in \mathcal{W}$, H -FREE EDGE EDITING is incompressible. Let H be a graph with at least five vertices but neither complete nor empty. That is $H \notin \mathcal{Y}_E$. Then by Lemma 3.4, H simulates a graph in $\mathcal{X}_E \cup \mathcal{W}$. Then the statement follows from Proposition 2.3. \square

Proof of Lemma 3.2. Assume that for every $H \in \mathcal{W}$, H -FREE EDGE DELETION is incompressible. Let H be a graph with at least five and at least two edges but not complete. That is $H \notin \mathcal{Y}_D$. Then by Lemma 3.4, H simulates a graph in $\mathcal{X}_D \cup \mathcal{W}$. Then the statement follows from Proposition 2.3. \square

4 Reductions

Recall that we defined $\mathcal{W} = \mathcal{H} \cup \overline{\mathcal{H}} \cup \mathcal{A} \cup \overline{\mathcal{A}} \cup \mathcal{D} \cup \overline{\mathcal{D}} \cup \mathcal{B} \cup \overline{\mathcal{B}} \cup \mathcal{S} \cup \overline{\mathcal{S}} \cup \mathcal{F} \cup \overline{\mathcal{F}}$ and Section 3 reduced our main questions to assuming incompressibility for the set \mathcal{W} . In this section, we further refine the result and show that incompressibility needs to be assumed only for the finite set $\mathcal{W}' = \mathcal{H} \cup \overline{\mathcal{H}} \cup \mathcal{A} \cup \overline{\mathcal{A}} \cup \mathcal{B} \cup \mathcal{D}$. That is, we recall and introduce some further simple reductions and use them to prove that every graph in $\mathcal{W} \setminus \mathcal{W}'$ simulates a graph in $\mathcal{W}' \cup \mathcal{X}_D$. To begin with, we observe that deleting the lowest degree vertices in the graphs in $\overline{\mathcal{B}} \cup \overline{\mathcal{D}}$ results in 3-connected graphs which are not complete. Then by Proposition 3.5, we have:

Proposition 4.1. *If $H \in \overline{\mathcal{B}} \cup \overline{\mathcal{D}}$, then H -FREE EDGE EDITING and H -FREE EDGE DELETION are incompressible, assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.*

Proofs in the rest of this section are written only for EDITING. The proofs can be replicated for DELETION and COMPLETION by replacing ‘EDITING’ with ‘DELETION’ and ‘COMPLETION’ respectively. The reductions are based on Construction 1 and a few other similar constructions.

4.1 Reductions based on Construction 1

The following lemma can be proved using a straight-forward application of Construction 1.

Lemma 4.2. *Let H be $J \cup K_t$, for some graph J and integer $t \geq 1$, where the K_t is induced by V_ℓ . Let V' be $V(H) \setminus \{v\}$, where v is any vertex in the K_t . Let H' be $H[V']$. Then H simulates H' . In particular, H simulates $J \cup K_1$.*

Proof. Let (G', k) be an instance of H' -FREE EDGE EDITING. Apply Construction 1 on (G', k, H, V') to obtain G . We claim that (G', k) is a yes-instance if and only if (G, k) is a yes-instance of H -FREE EDGE EDITING. Proposition 2.4 proves the backward direction. For the forward direction, let (G', k) be a yes-instance and F' be a solution of (G', k) of size at most k . Since $H[V_\ell]$ is K_t and every satellite vertex has degree ℓ , it can be part of only an induced K_t (in an induced H in $G \triangle F'$). Then if $G \triangle F'$ has an induced H , then $G' \triangle F'$ has an induced H' , which is a contradiction. \square

Corollary 4.3. (i) *Let H be $K_t \cup K_2$, for $t \geq 4$ ($\in \overline{\mathcal{F}}_1$). Then H simulates $K_t \cup K_1$ ($\in \{\overline{\mathcal{H}}_5\} \cup \overline{\mathcal{F}}_2$).*

(ii) *Let H be $(K_t - e) \cup K_2$, for $t \geq 4$ ($\in \overline{\mathcal{F}}_6$). Then H simulates $(K_t - e) \cup K_1$ ($\in \{\overline{\mathcal{H}}_8, \overline{\mathcal{D}}_2\} \cup \overline{\mathcal{F}}_8$).*

Next we consider the removal of a path of degree-2 vertices. We can prove the correctness of the reduction only under a certain uniqueness condition on the path.

Lemma 4.4. *Let H be a graph with minimum degree two and let $p \geq 2$ be an integer such that there is a unique induced path P of length p with the property that all the internal vertices of the path are having degree exactly two in H . Let H' be obtained from H by removing all internal vertices of P . Then H simulates H' .*

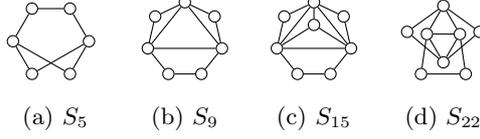


Figure 7: Graphs handled by Corollary 4.7

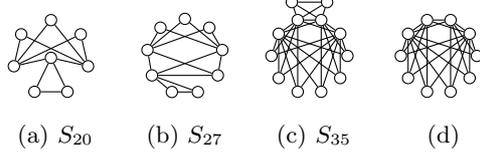


Figure 8: Graphs handled by Corollary 4.9 (8a, 8b) and Lemma 4.10 (8c, 8d)

Proof. Let (G', k) be an instance of H' -FREE EDGE EDITING. We apply Construction 1 on (G', k, H, V') to obtain G , where V' is the set of vertices inducing H' in H . We claim that (G', k) is a yes-instance of H' -FREE EDGE EDITING if and only if (G, k) is a yes-instance of H -FREE EDGE EDITING.

Before proving the claim, we note that there exist no induced path in H with length more than p such that every internal vertex has degree two. Proposition 2.4 proves the backward direction of the claim. For the forward direction, let (G', k) be a yes-instance and let F' be a solution. For a contradiction, assume that $G \triangle F'$ has an induced H with a vertex set V'' . If there is no satellite vertex in V'' , then clearly, $G' \triangle F'$ has an induced H' , a contradiction. Therefore, V'' contains at least one satellite vertex $v_j \in V_j$ for some satellite V_j . Since the minimum degree of H is two and the path P (with length p and having degree two for all internal vertices) is unique in H , all vertices in V_j must be in V'' and forms the internal vertices of the path P in H induced by V'' . Then $G' \triangle F'$ has an induced H' , a contradiction. \square

Corollary 4.5. *Let H be J_t , for some $t \geq 3$ ($\in \mathcal{F}_9$). Then H simulates $K_2 \boxtimes tK_1$ ($\in \{\overline{H_3}\} \cup \mathcal{F}_3$).*

Corollary 4.6. *Let H be Q_t , for some $t \geq 3$ ($\in \mathcal{F}_{10}$). Then H simulates $K_{2,t}$ ($\in \{S_1\} \cup \mathcal{F}_1$).*

Corollary 4.7. (i) S_5 simulates C_4 .

(ii) S_9 simulates $\overline{H_4}$.

(iii) Let H be S_{15} . Then H simulates $\overline{H_9 \cup K_1}$. Further H simulates H_9 (Proposition 3.5).

(iv) S_{22} simulates $\overline{\text{diamond} \cup K_2}$ ($\in \overline{\mathcal{F}_6}$).

Figure 7 shows the graphs handled by Corollary 4.7. Lemma 4.8 essentially says the following: If H has vertex connectivity 1 and has a unique smallest 2-connected component which is a ‘leaf’ in the tree formed by the 2-connected components, then H simulates a graph obtained by removing all vertices in the 2-connected component except the cut vertex.

Lemma 4.8. *Let H be a graph with vertex connectivity 1 and be not a complete graph. Let \mathcal{C} be the set of all 2-connected components of H having exactly one cut vertex of H . Assume that there exists a unique smallest (among \mathcal{C}) 2-connected component J in \mathcal{C} . Let v be the cut vertex of H in J . Let H' be $H - \{J \setminus \{v\}\}$. Then H' is simulated by H .*

Proof. Let (G, k) be an instance of H' -FREE EDGE EDITING. Let G be obtained by applying Construction 1 on $(G', k, H, V(H'))$. We claim that (G', k) is a yes-instance H' -FREE EDGE EDITING if and only if (G, k) is a yes-instance. Proposition 2.4 proves one direction. For the other direction, assume that (G', k) is a yes-instance and let F' be a solution. For a contradiction, assume that $G \triangle F'$ has an H induced by U . Since every satellite corresponds to a unique smallest 2-connected component (with exactly one cut vertex of H) sans the cut vertex, if a satellite has nonempty intersection with U then every vertex in the satellite is in U and no other satellite vertices can be in U . Then $G' - F'$ has an induced H' , which is a contradiction. \square

Corollary 4.9. (i) S_{20} simulates $K_{2,4}$ ($\in \mathcal{F}_1$). (ii) S_{27} simulates $\overline{(K_5 - e) \cup K_2}$ ($\in \mathcal{F}_6$).

Lemma 4.10. *Let H be S_{35} . Then H simulates a 3-connected graph, which is not complete ($\in \mathcal{X}_D$).*

Proof. We will show that H simulates H' , where H' is the graph shown in Figure 8d. We observe that H' is 3-connected.

Let (G', k) be an instance of H' -FREE EDGE EDITING. Let G be obtained from (G', k, H, V') by applying Construction 1, where V' is the set all vertices of H except the two adjacent degree-3 vertices. We claim that (G', k) is a yes-instance of H' -FREE EDGE EDITING if and only if (G, k) is a yes-instance of H -FREE EDGE EDITING. Proposition 2.4 proves the backward direction. For the forward direction, let (G', k) be a yes-instance of H' -FREE EDGE EDITING and let F' be a solution. For a contradiction, assume that $G \Delta F'$ has an H induced by U . Clearly, U contains at least one satellite vertex. Since every satellite has a pair of adjacent degree-3 vertices, if a satellite vertex is in U then the other vertex in the satellite must be in U . Since there is a unique pair of adjacent degree-3 vertices in H , the rest of the vertices in U must be from the copy of G' in G . Then $G \Delta F'$ has an induced H' , a contradiction. \square

Figure 8 shows the graphs handled by Corollary 4.9 and Lemma 4.10.

4.2 Reductions based on Construction 2

The following is a simplified version of Construction 1.

Construction 2. Let (G', k, ℓ) be an input to the construction, where G' is a graph and k and ℓ are positive integers. For every set S of ℓ vertices in G' introduce a clique C of $k + 1$ vertices and make all the vertices in C adjacent to all the vertices in S .

As before, we call every clique C introduced during the construction as a *satellite* and the vertices in it as *satellite vertices*. Lemma 4.11 can be proved using a straight-forward application of Construction 2. It says that if H satisfies some properties, then H simulates H' where H' is obtained by removing one vertex from each module of H contained within V_ℓ .

Lemma 4.11. Let H be a non-regular graph such that the following conditions hold true:

- (i) $1 \leq \ell \leq 2, |V(H)| \geq 5$;
- (ii) V_ℓ is an independent set, $V_h \cup V_m$ induces a connected graph, and every vertex in V_h is adjacent to at least one vertex in V_ℓ ;
- (iii) Every vertex in V_m has at least $\ell + 1$ neighbors outside V_m or there exists no pair u, v of adjacent vertices in V_m such that $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.

Consider a modular decomposition \mathcal{M} of H . Let $\mathcal{M}' \subseteq \mathcal{M}$ corresponds to V_ℓ . Let H' be the graph obtained from H by removing one vertex from each module in \mathcal{M}' . Then H simulates H' .

Proof. Let (G', k) be an instance of H' -FREE EDGE EDITING. We apply Construction 2 on (G', k, ℓ) to obtain G . We claim that (G', k) is a yes-instance of H' -FREE EDGE EDITING if and only if (G, k) is a yes-instance of H -FREE EDGE EDITING.

For the backward direction, let (G, k) be a yes-instance of H -FREE EDGE EDITING and let F be a solution. We claim that $G \Delta F$ is H' -free. For a contradiction, assume that $G \Delta F$ has an H' induced by V' . Since every set S of ℓ vertices in G' has a corresponding clique C of $k + 1$ vertices, every vertex of which is adjacent to every vertex of S , we obtain that $V' \cup V''$ induces an H in $G \Delta F$, which is a contradiction, where V'' is a carefully chosen subset of all satellite vertices.

To prove the forward direction, let (G', k) be a yes-instance and let F' be a solution. We claim that $G \Delta F'$ is H -free. For a contradiction, assume that $G \Delta F'$ has an induced H with a vertex set V' . Let C be any satellite. Since V_ℓ is an independent set, $|V_\ell \cap C| \leq 1$ for the H induced by V' . Therefore, for every module $M' \in \mathcal{M}'$ of V_ℓ , M' contains at most one satellite vertex in a satellite. Since every pair of satellite has distinct neighborhood, this implies that M' contains at most one satellite vertex. Therefore, if $(V_h \cup V_m) \cap C = \emptyset$ for every satellite C , then $G \Delta F'$ contains an induced H' , a contradiction. Therefore, assume that $|(V_h \cup V_m) \cap C| \geq 1$ for at least one satellite C .

Case 1: $|V_h \cap C| \geq 1$ for some satellite C : Let $v \in V_h$ be in the H induced by V' in $G \Delta F'$, where v is in C which was introduced for a set S of ℓ vertices in G' . Since every vertex in $C \cap V(H)$ has the same degree in the H , $C \cap V(H) \subseteq V_h$. If $\ell = 1$, then $|S| = 1$ and $S \subseteq V_\ell$ as v must be adjacent to some vertex in V_ℓ . This gives a contradiction as the vertex in S has degree at least that of v in the induced H . Let $\ell = 2$. Then $h \geq 3$. As $|S| = 2$, we note that v cannot be adjacent to more than two vertices in V_ℓ . If v is adjacent to two vertices in V_ℓ , then S is exactly the set of those two vertices. Then H is $K_2 \boxtimes 2K_1$ which is a contradiction, as H has at least five vertices. So, v is adjacent to exactly one vertex in V_ℓ . Then one vertex, say $s_1 \in S$ is in V_ℓ and the other vertex $s_2 \in S$ is in $V_h \cup V_m$ (if $s_2 \notin V_h \cup V_m$ then s_1 will have degree at least that of v in the H). If $|V_h \cap C| \geq 3$, then s_1 has degree at least three in the H , which is a contradiction. Therefore, since $h \geq 3$,

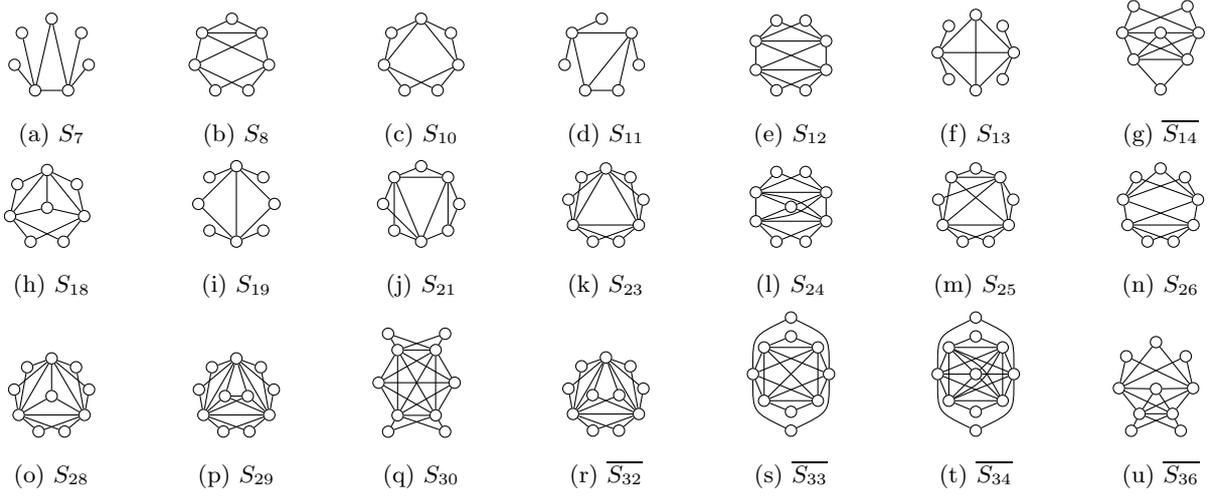


Figure 9: Graphs handled by Corollary 4.12

$|V_h \cap C| = 2$. Hence $h = 3$. Since $\ell = 2$ and $h = 3$, we obtain that $V_m = \emptyset$. Therefore, $s_2 \in V_h$. We observe that s_1 and s_2 are nonadjacent in $G \Delta F'$ as otherwise the degree of s_1 becomes at least h , which is a contradiction. Since every vertex in V_h is adjacent to at least one vertex in V_ℓ , s_2 is adjacent to a vertex u (which is not s_1) in V_ℓ . Now, s_2 cannot have any other neighbors in H (other than u , and two vertices in $V_h \cap C$). Since $V_h \cup V_m$ induces a connected graph, $|V_h \cup V_m| = |V_h| = 3$ and u has degree at most one, which is a contradiction.

Case 2: $|V_m \cap C| \geq 1$: Since every vertex in $V' \cap C$ has the same degree in the induced H , $V' \cap C \subseteq V_m$. Consider condition (iii). Assume that every vertex in V_m has at least $\ell + 1$ neighbors outside V_m . Then we get a contradiction, as a vertex in C has only at most ℓ neighbors outside C . Now assume that there is no pair u, v of adjacent vertices in V_m such that their neighbors outside them are same. Then $|V_m \cap C| = 1$. Then the vertex in $V_m \cap C$ has degree only at most ℓ in the H , a contradiction. \square

The following corollary lists many graphs (see Figure 9) that can be handled by Lemma 4.11.

- Corollary 4.12.** (i) S_7 simulates H_9 . (viii) S_{18} simulates $\overline{A_1}$. (xv) S_{28} simulates $\overline{A_9}$.
(ii) S_8 simulates $\overline{A_1}$. (ix) S_{19} simulates $\overline{S_3}$. (xvi) S_{29} simulates S_{17} .
(iii) S_{10} simulates C_4 . (x) S_{21} simulates $\overline{H_4}$. (xvii) S_{30} simulates $\overline{S_{19}}$.
(iv) S_{11} simulates $\overline{H_7}$. (xi) S_{23} simulates $\overline{A_7}$. (xviii) $\overline{S_{32}}$ simulates $\overline{S_{16}}$.
(v) S_{12} simulates S_2 . (xii) S_{24} simulates $\overline{S_7}$. (xix) $\overline{S_{33}}$ simulates $\overline{K_{1,4} \cup K_2} \in \overline{\mathcal{F}_7}$.
(vi) S_{13} simulates S_3 . (xiii) S_{25} simulates $\overline{H_1}$. (xx) $\overline{S_{34}}$ simulates $\overline{K_{1,5} \cup K_2} \in \overline{\mathcal{F}_7}$.
(vii) $\overline{S_{14}}$ simulates $\overline{B_1}$. (xiv) S_{26} simulates S_8 . (xxi) $\overline{S_{36}}$ simulates $\overline{S_{14}}$.

The following three corollaries are obtained by application of Lemma 4.11: they show that in certain families of graphs, every member simulates the simplest member. Corollary 4.13 deals with star graphs ($K_{1,t}$). For every graph H in this class, V_ℓ is a single module of the graph and H simulates a graph H' , where H' is obtained by removing one vertex from V_ℓ . Corollary 4.14 handles $K_2 \boxtimes sK_1$, where V_ℓ forms a single module of the graph. As in the previous case, H' is obtained by removing one vertex from V_ℓ . Corollary 4.15 deals with the set of twin-star graphs (T_{t_1, t_2}). For every graph H in this class, there are two modules of H in V_ℓ : t_1 vertices adjacent to one vertex in $H - V_\ell$ and t_2 vertices adjacent to the other vertex in $H - V_\ell$. Then H simulates a graph H' , where H' is obtained by removing one vertex each from the two modules.

Corollary 4.13 (see Lemma 6.4 in [2] for a partial result). *Let H be $K_{1,t}$, for any $t \geq 5$ ($\in \mathcal{F}_2$). Let H' be $K_{1,t-1}$. Then H simulates H' . Furthermore, H simulates H_5 ($K_{1,4}$).*

Corollary 4.14 (see Lemma 4.5 in [1] for a partial result). *Let H be $K_2 \boxtimes sK_1$, for any $s \geq 4$ ($\in \mathcal{F}_3$) and let H' be $K_2 \boxtimes (s-1)K_1$. Then H simulates H' . Furthermore, H simulates $\overline{H_3}$ ($K_2 \boxtimes 3K_1$).*

Corollary 4.15 (see Lemma 6.6 in [2] for a partial result). *Let H be a twin-star graph T_{t_1, t_2} , such that $t_1, t_2 \geq 1$. Let H' be T_{t_1-1, t_2-1} . Then H simulates H' . In particular, if H is $T_{t,1}$, for some $t \geq 4$ ($\in \mathcal{F}_4$), then H simulates $K_{1,t}$ ($\in \{H_5\} \cup \mathcal{F}_2$).*

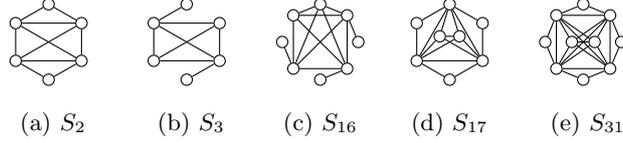


Figure 10: Graphs handled by Corollary 4.18

Lemma 4.16. $K_{2,3}$ ($= S_1$) simulates C_4 .

Proof. Let H be $K_{2,3}$. Let H' be C_4 . Let (G', k) be an instance of H -FREE EDGE EDITING. Let G be constructed from (G', k, ℓ) by applying Construction 2. We claim that (G', k) is a yes-instance of H' -FREE EDGE EDITING if and only if (G, k) is a yes-instance of H -FREE EDGE EDITING.

Let (G', k) be a yes-instance of H' -FREE EDGE EDITING and let F' be a solution. For a contradiction, assume that $G \Delta F'$ has a $K_{2,3}$ induced by V' . Clearly, $V' \cap C \neq \emptyset$ for some clique C introduced during the construction for some set S of two vertices in G' . Since $V' \cap C$ forms a clique and has the same neighborhood outside $V' \cap C$ in the H , $|V' \cap C| = 1$. Therefore, the vertex in $V' \cap C$ must be a degree-2 vertex in the H . Then the two vertices in S act as the two highest degree vertices in the H . Then the other two degree-2 vertices must be from the copy of G' in G (there is no other constructed clique adjacent to both the vertices in S) and hence $G' \Delta F'$ contains an induced C_4 , a contradiction.

For the other direction, assume that (G, k) is a yes-instance and let F be a solution. For a contradiction, let $G' \Delta F$ has a C_4 induced by $\{v_1, v_2, v_3, v_4\}$, where v_1 and v_3 are nonadjacent. Since there are $k + 1$ vertices in a clique C adjacent to both v_1 and v_3 (due to the construction), there is at least one vertex v in C adjacent to both v_1 and v_3 and not adjacent to v_2 and v_4 in $G \Delta F$. Then $G \Delta F$ has an induced $K_{2,3}$, a contradiction. \square

4.3 Reductions based on Construction 3

Now we give another construction that will be used in a few reductions.

Construction 3. Let (G', k, t) be an input to the construction, where G' is a graph and k and t are positive integers. For every set S of t vertices in G' introduce an independent set I_S of $k + 2$ vertices such that every vertex in I_S is adjacent to every vertex in G' except those in S . Let $\bigcup_{S \subseteq V(G'), |S|=t} I_S = I$. Let the resultant graph be G .

Lemma 4.17. Let H be a graph such that V_h forms a clique and for every pair of vertices $u, v \in V_h$, $H - u$ is isomorphic to $H - v$. Further assume that there exists no independent set S of size $s \geq 2$ where each vertex in S has degree at least $h - s + 1$ in H . Then H simulates $H - u$, where u is any vertex in V_h .

Proof. Let H' be obtained from H by deleting a vertex in V_h . Let (G', k) be an instance of H' -FREE EDGE EDITING. Apply Construction 3 on $(G', k, h^* = |V(H)| - h - 1)$ to obtain G . We claim that (G', k) is a yes-instance of H' -FREE EDGE EDITING if and only if (G, k) is a yes-instance of H -FREE EDGE EDITING.

Let (G', k) be a yes-instance of H' -FREE EDGE EDITING and let F' be a solution. We claim that $G \Delta F'$ is H -free. For a contradiction, assume that $U \subseteq V(G)$ induces H in $G \Delta F'$. Clearly, $I \cap U \neq \emptyset$. Let $|U \cap I| = 1$. Since a vertex in I is nonadjacent to only h^* vertices in the copy of G' in G , we obtain that the vertex in $I \cap U$ must be a vertex in V_h in the H . Therefore, $G' \Delta F'$ has an induced H' , a contradiction. Let $|U \cap I| = s > 1$. Since $U \cap I$ is an independent set and each vertex in it is nonadjacent to only at most h^* vertices in the copy of G' in G , we get that each vertex in $U \cap I$ has degree at least $|V(H)| - 1 - (s - 1) - h^*$ in the H induced by U in $G \Delta F'$. Since $h^* = |V(H)| - h - 1$, we obtain that each vertex in $U \cap I$ has degree at least $h - s + 1$ in the H , which is a contradiction.

Let (G, k) be a yes-instance and let F be a solution. For a contradiction, assume that $G' \Delta F$ has an H' induced by U . Let $U' \subseteq U$ be such that $|U'| = h^*$ and introducing a new vertex u and making it adjacent to every vertex in $U \setminus U'$ of $(G' \Delta F')[U]$ results in H . Since there are at least $k + 1$ vertices adjacent to every vertex, except those in U' in the copy of G' in G , we obtain that $G \Delta F$ has an induced H , a contradiction. \square

Figure 10 shows the graphs handled by Corollary 4.18.

Additional results used by Corollary 4.18 are shown in parenthesis.

- Corollary 4.18.** (i) S_2 simulates $\overline{H_7}$. (iv) S_{17} simulates $\overline{H_1}$ (Proposition 3.5).
(ii) S_3 simulates H_6 . (v) S_{31} simulates S_3 (Proposition 3.5).
(iii) S_{16} simulates S_3 (Proposition 3.5).

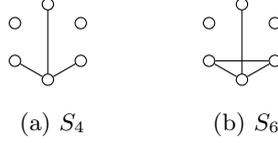


Figure 11: Two of the graphs handled by Corollary 4.23

Lemma 4.19. *Let H be $(K_t - e) \cup K_1$ for $t \geq 6$ ($\overline{\mathcal{F}_8}$). Let H' be $K_{t-2} \cup K_1$ ($\in \{\overline{H_5}\} \cup \overline{\mathcal{F}_2}$). Then H simulates H' .*

Proof. We observe that H' is obtained by removing the two vertices with degree $t - 2$ from H . Let (G', k) be an instance of H' -FREE EDGE EDITING. Let G be obtained by applying Construction 3 on $(G', k, 1)$. We claim that (G', k) is a yes-instance of H' -FREE EDGE EDITING if and only if (G, k) is a yes-instance of H -FREE EDGE EDITING.

For the forward direction, let (G', k) be a yes-instance of H' -FREE EDGE EDITING. Let F' be a solution of it. For a contradiction, assume that $G \Delta F'$ has an H induced by U . Clearly $U \cap I \neq \emptyset$. Since an isolated vertex in H is not adjacent to $t \geq 6$ vertices in H , a vertex in I cannot be the isolated vertex in the H induced by U . Therefore, $|U \cap I| \leq 2$. Let $|U \cap I = \{u\}| = 1$. Since u is adjacent to all except one vertex in the copy of G' in G , u must be a vertex with degree $t - 1$ in the induced H . Then $G' \Delta F'$ contains an induced $(K_{t-1} - e) \cup K_1$ and hence a $K_{t-2} \cup K_1$, which is a contradiction (we note that $K_{t-2} \cup K_1$ is an induced subgraph of $(K_{t-1} - e) \cup K_1$). Let $|U \cap I = \{u, u'\}| = 2$. Clearly, $u, u' \in I_{\{v\}}$ for some vertex v in the copy of G' in G . Then u and u' are the vertices with degree $t - 2$ in the H . Therefore, $G' \Delta F'$ has an induced $K_{t-2} \cup K_1$, which is a contradiction.

For the other direction, let (G, k) be a yes-instance of H -FREE EDGE EDITING. Let F be a solution of it. For a contradiction, assume that $G' \Delta F$ contains an H' induced by U . Let v be the isolated vertex in the induced H . Since there are $k + 2$ vertices adjacent to all vertices, except v , in the copy of G' in G , at least two of them along with U induces H in $G \Delta F$, which is a contradiction. \square

4.4 Other reductions

To resolve graphs in \mathcal{F}_7 ($= K_{1,t} \cup K_2$), we resort to a known reduction. There is a PPT in [2] from H' -FREE EDGE EDITING to H -FREE EDGE EDITING, where H' is a largest component in H . It is a composition of two reductions: one from H' -FREE EDGE EDITING to H'' -FREE EDGE EDITING and another from H'' -FREE EDGE EDITING to H -FREE EDGE EDITING, where H'' is the union of all components in H isomorphic to H' . The first reduction uses a simple construction (take a disjoint union of the input graph and join of $k + 1$ copies of H') and the second reduction uses Construction 1.

Proposition 4.20 (see Lemma 3.5 in [2]). *Let H' be a largest component of H . Then H simulates H' .*

Corollary 4.21. *Let H be $K_{1,t} \cup K_2$, for $t \geq 4$ ($\in \mathcal{F}_7$). Then H simulates $K_{1,t}$ ($\in \{H_5\} \cup \mathcal{F}_2$).*

The following statement consider reduction that involve the removal of independent vertices.

Lemma 4.22. *Let H be $J \cup tK_1$, for any $t \geq 2$ such that J has no component which is a clique. Let H' be $J \cup (t - 1)K_1$. Then H simulates H' . In particular, H simulates $J \cup K_1$.*

Proof. Let (G', k) be an instance of H' -FREE EDGE EDITING. Let G be $G' \cup K$, where K is K_{k+1} . We claim that (G', k) is a yes-instance of H' -FREE EDGE EDITING if and only if (G, k) is a yes-instance of H -FREE EDGE EDITING.

Let (G', k) be a yes-instance. Let F' be a solution of size at most k . For a contradiction assume that $G \Delta F'$ has an induced $J \cup tK_1$ with a vertex set V' . Since $G' \Delta F'$ is H' -free, $V' \cap K \neq \emptyset$. Since $V' \cap K$ induces a clique component in $G \Delta F'$ and J does not have a clique component, $V' \cap K$ is a singleton set and induces K_1 . Therefore $G' \Delta F'$ induces $J \cup (t - 1)K_1$, which is a contradiction. For the other direction, let (G, k) be a yes-instance and let F be a solution. We claim that $G' \Delta F$ is H' -free. For a contradiction, assume that $G' \Delta F$ has an induced H' with a vertex set V' . Then clearly, $V' \cup \{v\}$ induces H in $G \Delta F$, which is a contradiction, where v is any vertex in K . \square

Corollary 4.23. (i) S_4 simulates H_2 .

(ii) S_6 simulates H_6 .

(iii) Let H be $(K_t - e) \cup 2K_1$, for $t \geq 4$ ($\in \overline{\mathcal{F}_5}$). Then H simulates $(K_t - e) \cup K_1$ ($\in \{\overline{H_8}, \overline{D_2}\} \cup \overline{\mathcal{F}_8}$).

H	Simulates	By
S_1	C_4	Lemma 4.16
S_2	$\overline{H_7}$	Corollary 4.18
S_3	H_6	Corollary 4.18
S_4	H_2	Corollary 4.23
S_5	C_4	Corollary 4.7
S_6	H_6	Corollary 4.23
S_7	H_9	Corollary 4.12
S_8	$\overline{A_1}$	Corollary 4.12
S_9	$\overline{H_4}$	Corollary 4.7
S_{10}	C_4	Corollary 4.12
S_{11}	$\overline{H_7}$	Corollary 4.12
S_{12}	S_2	Corollary 4.12

H	Simulates	By
S_{13}	S_3	Corollary 4.12
S_{14}	B_1	Corollary 4.12
S_{15}	H_9	Corollary 4.7
S_{16}	S_3	Corollary 4.18
S_{17}	$\overline{H_1}$	Corollary 4.18
S_{18}	$\overline{A_1}$	Corollary 4.12
S_{19}	$\overline{S_3}$	Corollary 4.12
S_{20}	a graph in \mathcal{F}_1	Corollary 4.9
S_{21}	$\overline{H_4}$	Corollary 4.12
S_{22}	a graph in $\overline{\mathcal{F}_6}$	Corollary 4.7
S_{23}	$\overline{A_7}$	Corollary 4.12
S_{24}	$\overline{S_7}$	Corollary 4.12

H	Simulates	By
S_{25}	$\overline{H_1}$	Corollary 4.12
S_{26}	S_8	Corollary 4.12
S_{27}	a graph in \mathcal{F}_6	Corollary 4.9
S_{28}	$\overline{A_9}$	Corollary 4.12
S_{29}	S_{17}	Corollary 4.12
S_{30}	$\overline{S_{19}}$	Corollary 4.12
S_{31}	S_3	Corollary 4.18
S_{32}	S_{16}	Corollary 4.12
S_{33}	a graph in \mathcal{F}_7	Corollary 4.12
S_{34}	a graph in \mathcal{F}_7	Corollary 4.12
S_{35}	a graph in \mathcal{X}_D	Lemma 4.10
S_{36}	S_{14}	Corollary 4.12

Figure 12: Summary of results in Section 4 handling graphs in \mathcal{S}

$H \in$	Simulates a graph in	By
\mathcal{F}_1	$\{H_5\} \cup \mathcal{F}_2$	Corollary 4.3
\mathcal{F}_2	$\{H_5\}$	Corollary 4.13
\mathcal{F}_3	$\{\overline{H_3}\}$	Corollary 4.14
\mathcal{F}_4	$\{H_5\} \cup \mathcal{F}_2$	Corollary 4.15
\mathcal{F}_5	$\{H_8, D_2\} \cup \mathcal{F}_8$	Corollary 4.23

$H \in$	Simulates a graph in	By
\mathcal{F}_6	$\{H_8, D_2\} \cup \mathcal{F}_8$	Corollary 4.3
\mathcal{F}_7	$\{H_5\} \cup \mathcal{F}_2$	Corollary 4.21
\mathcal{F}_8	$\{H_5\} \cup \mathcal{F}_2$	Lemma 4.19
\mathcal{F}_9	$\{\overline{H_3}\} \cup \mathcal{F}_3$	Corollary 4.5
\mathcal{F}_{10}	$\{S_1\} \cup \mathcal{F}_1$	Corollary 4.6

Figure 13: Summary of results in Section 4 handling graphs in \mathcal{F}

Some graphs handled by Corollary 4.23 is shown in Figure 11.

Summary of results in this section handling graphs in \mathcal{S} and \mathcal{F} are given in Figure 12 and 13 respectively.

Lemma 4.24 follows from Corollary 2.2, Proposition 4.1, the transitivity of PPTs, and other results in this section (see Figures 12 and 13) for details.

Lemma 4.24. *Let $H \in \mathcal{W} \setminus \mathcal{W}'$. Then H simulates a graph in $\mathcal{W}' \cup \mathcal{X}_D$.*

5 Incompressibility results for the graphs in \mathcal{A} and \mathcal{B}

In this section, we prove that for every graph $H \in \mathcal{A} \cup \overline{\mathcal{A}}$, all three problems H -FREE EDGE EDITING, H -FREE EDGE DELETION, and H -FREE EDGE COMPLETION are incompressible, assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$. With the same assumption, we prove that H -FREE EDGE DELETION is incompressible for every graph $H \in \mathcal{B}$; Proposition 2.1 then implies incompressibility of H -FREE EDGE COMPLETION for every $H \in \overline{\mathcal{B}}$.

Theorem 5.1. *Assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$:*

- (i) *Let $H \in \mathcal{A}$. Then H -FREE EDGE EDITING is incompressible.*
- (ii) *Let $H \in \mathcal{A} \cup \overline{\mathcal{A}} \cup \mathcal{B}$. Then H -FREE EDGE DELETION is incompressible.*
- (iii) *Let $H \in \mathcal{A} \cup \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$. Then H -FREE EDGE COMPLETION is incompressible.*

We apply the technique used by Cai and Cai [5] by which they obtained a complete dichotomy on the incompressibility of H -free edge modification problems on 3-connected graphs H . We will give a self-contained summary of their proof technique, with only a few references to proofs of formal statements. The reader is referred to [5] for a more detailed exposition of terminology and concepts discussed in this section.

The first step in the proof is to establish incompressibility for the *restricted* versions of H -FREE EDGE DELETION and H -FREE EDGE COMPLETION, where only allowed edges can be deleted/added. Then deletion and completion *enforcer gadgets* can be used to reduce the restricted problems to the original versions. Cai and Cai [5] presented constructions that were proved to work correctly when H is 3-connected. We show, by careful inspection, that the same technique works for certain graphs H that are not 3-connected. For certain graphs H , we can prove incompressibility of the restricted problem, but enforcer gadgets of the required form provably do not exist. In these cases, we use ad hoc ideas to reduce the restricted version to the original one. In yet further cases, we need even trickier reductions, where we reduce H' -FREE EDGE DELETION to H -FREE EDGE DELETION for some $H' \neq H$.

5.1 Incompressibility results for the restricted problems

A graph is called *edge-restricted* if a subset of its edges are marked as *forbidden*. All edges other than forbidden are *allowed*. A graph is called *nonedge-restricted* if a subset of its nonedges are marked as *forbidden*. All nonedges other than forbidden are *allowed*.

RESTRICTED H -FREE EDGE DELETION: Given a graph G , an integer k , and a set R of edges of G , do there exist at most k edges disjoint from R such that deleting them from G results in an H -free graph?

Parameter: k

RESTRICTED H -FREE EDGE COMPLETION: Given a graph G , an integer k , and a set R of nonedges of G , do there exist at most k nonedges disjoint from R such that adding them in G results in an H -free graph?

Parameter: k

Propagational formula satisfiability. A ternary Boolean function $f(x, y, z)$ (where x, y , and z are either Boolean variables or constants 0 or 1) is *propagational* if $f(1, 0, 0) = 0, f(0, 0, 0) = f(1, 0, 1) = f(1, 1, 0) = f(1, 1, 1) = 1$. This has the meaning: if x is true then either y is true or z is true.

PROPAGATIONAL- f SATISFIABILITY: Given a conjunctive formula φ of a propagational ternary function f with distinct variables in each clause of φ , find whether there exists a satisfying truth assignment with weight at most k . The parameter we consider is k .

Proposition 5.2 (Theorem 3.4 in [5]). *For any propagational ternary Boolean function f , PROPAGATIONAL- f SATISFIABILITY on 3-regular conjunctive formulas (every variable appears exactly three times) admits no polynomial kernel, assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.*

Satisfaction-testing components. For H -FREE EDGE DELETION, a *satisfaction-testing component* $S_D(x, y, z)$ is a constant-size edge-restricted H -free graph with exactly three allowed edges $\{x, y, z\}$ such that there is a propagational Boolean function $f(x, y, z)$ such that $f(x, y, z) = 1$ if and only if the graph obtained from $S_D(x, y, z)$ by deleting edges in $\{x, y, z\}$ with value 1 is H -free. For H -FREE EDGE COMPLETION, a *satisfaction-testing*

component $S_C(x, y, z)$ is a constant-size nonedge-restricted H -free graph with exactly three allowed nonedges $\{x, y, z\}$ such that there is a propagational Boolean function $f(x, y, z)$ such that $f(x, y, z) = 1$ if and only if the graph obtained from $S_C(x, y, z)$ by adding edges in $\{x, y, z\}$ with value 1 is H -free.

There is an easy construction (Lemma 4.3 in [5]) showing that $S_D(x, y, z)$ exists for every connected graph H with at least four vertices but not complete and $S_C\{x, y, z\}$ exists for every connected graph with at least four vertices and at least two nonedges. The construction for this is as follows. $S_D\{x, y, z\}$: Let x be a nonedge, and y and z be two edges in H . Then $H + x$ is a $S_D(x, y, z)$ where x, y, z are the only allowed edges. $S_C\{x, y, z\}$: Let x be an edge, and y and z be two nonedges in H . Then $H - x$ is a $S_C(x, y, z)$ where x, y, z are the only allowed nonedges.

Truth-setting components. For H -FREE EDGE DELETION, a *truth-setting component* ($T_D(u)$) is a constant-sized, edge-restricted H -free graph such that it contains at least three allowed edges x, y, z without a common vertex and admits exactly two deletion sets \emptyset and the set of all allowed edges. For H -FREE EDGE COMPLETION, a *truth-setting component* ($T_C(u)$) is a constant-sized, nonedge-restricted H -free graph such that it contains at least three allowed nonedges x, y, z without a common vertex and admits exactly two completion sets \emptyset and the set of all allowed nonedges.

There is a construction given in [5] for $T_D(u)$ and $T_C(u)$ when H is 3-connected but not complete. The constructions are given below.

Construction of $T_D(u)$: Let e', e be a nonedge and an edge sharing no common vertex in H . Let the *basic unit* $U = H + e'$ and set all edges except e and e' in U as forbidden. Let p be the number of vertices in H . Take p copies U_1, U_2, \dots, U_p of U . Identify the edge e of U_i with the edge e' of U_{i+1} to form a chain of U 's. This is a *basic chain* $B(u)$. Let us call the unidentified edge e' of U_1 as the *left-most allowed edge* of $B(u)$ and unidentified edge of U_p as the *right-most allowed edge* of $B(u)$. Take three basic chains B_0, B_1 , and B_2 . Attach them in a cyclic fashion: Identify the right-most allowed edge of B_i with the left-most allowed edge of B_{i+1} , where indices are taken mod 3. This is the claimed truth-setting component $T_D(u)$. Let us call the allowed edges thus identified as *variable edges*. We note that there are exactly three variable edges in $T_D(u)$.

It is easy to see that, for every H , there are only two possible deletion sets in $T_D(u)$: the empty set and the set of all allowed edges. To see this, observe that if we remove any of the allowed edges, then it creates a copy of H in one of the units, forcing us to remove the next allowed edge as well. However, it is not clear if these two deletion sets really make the graph H free. As Cai and Cai [5] show, this construction for $T_D(u)$ works correctly for 3-connected graphs H : Since the ‘‘cycle’’ of basic units is long enough, every subgraph having vertices from different basic units and having at most $|V(H)|$ vertices has vertex connectivity at most 2. In general, the construction may not give correct truth-setting components for 2-connected graphs H . But, as we shall see later, by carefully choosing e and e' in these constructions, we can obtain truth-setting components for many 2-connected graphs H .

Construction of $T_C(u)$: Let e', e be a nonedge and an edge sharing no common vertex in H . Let the *basic unit* $U = H - e$ and set all nonedges except e and e' in U as forbidden. Let p be the number of vertices in H . Take p copies U_1, U_2, \dots, U_p of U . Identify the nonedge e of U_i with the nonedge e' of U_{i+1} to form a chain of U 's. This is a *basic chain* $B(u)$. Let us call the unidentified nonedge e' of U_1 as the *left-most allowed nonedge* of $B(u)$ and unidentified nonedge of U_p as the *right-most allowed nonedge* of $B(u)$. Take three basic chains B_0, B_1 , and B_2 . Attach them in a cyclic fashion: Identify the right-most allowed nonedge of B_i with the left-most allowed nonedge of B_{i+1} , where indices are taken mod 3. This is the claimed truth-setting component $T_C(u)$. Let us call the allowed nonedges thus identified as *variable nonedges*. We note that there are exactly three variable nonedges in $T_C(u)$. Similarly to $T_D(u)$, we can argue that for any H , there are only two potential completion sets (the empty set and the set of all allowed nonedges), and for 3-connected H , these two sets are indeed completion sets.

The following is the construction used in the reduction from PROPAGATIONAL- f SATISFIABILITY to RESTRICTED H -FREE EDGE DELETION (COMPLETION).

Construction 4. Let (φ, k, H) be an input to the construction, where φ is a 3-regular conjunctive formula on a propagational ternary Boolean function f , and k is a positive integer. The construction gives a graph G_φ , an integer k' , and a set of restricted (non)edges in G_φ .

- For every clause in φ , introduce a satisfaction-testing component $S_D(x, y, z)$ ($S_C(x, y, z)$) for H -FREE EDGE DELETION (COMPLETION).
- If $c \in \{x, y, z\}$ is 1, then the corresponding allowed (non)edge is deleted(added) and if $c = 0$ then the corresponding allowed (non)edge is set as forbidden.
- For every variable u in f , introduce a truth-setting component $T_D(u)$ ($T_C(u)$) for H -FREE EDGE DELETION (COMPLETION)
- For every variable u , identify each of the variable (non)edges in $T_D(u)$ ($T_C(u)$) with an allowed (non)edge

in a satisfaction-testing component corresponds to a different clause in which u appears—since φ is 3-regular, u appears in exactly three clauses.

Let the graph obtained be G_φ and let $k' = 3|V(H)|k$. For the deletion problem the set R of forbidden edges is all the edges in G_φ except the allowed edges in the units. For the completion problem, the set R of forbidden nonedges contains every nonedge of G_φ except the allowed nonedges in the units.

Let H be a graph and (φ, k) be an instance of a PROPAGATIONAL- f SATISFIABILITY problem. Let (G_φ, k', R) be the output of the Construction 4 applied on (φ, k, H) . The construction works correctly in one direction: If (G_φ, k', R) is a yes-instance of RESTRICTED H -FREE EDGE DELETION (COMPLETION), then (φ, k) is a yes-instance of PROPAGATIONAL- f SATISFIABILITY. To see this, let F be a solution of (G_φ, k', R) . By the definition of $T_D(u)$ ($T_C(u)$), if an allowed (non)edge is in F then so is every allowed (non)edge in it. Therefore, since $|F| \leq k' = 3k|V(H)|$ and every truth-setting component has exactly $3|V(H)|$ many allowed (non)edges, only at most k (non)edges of satisfaction-testing components are in F . By the definition of $S_D(x, y, z)$ ($S_C(x, y, z)$), if $x \in F$ then either y or z is in F , otherwise there is an induced H in $G_\varphi + F$. Therefore, setting the variables to 1 corresponding to the (non)edges, which are part of F , in satisfaction-testing components, we obtain that (φ, k) is a yes-instance of PROPAGATIONAL- f SATISFIABILITY. Thus we have the following Proposition.

Proposition 5.3 (see Lemma 5.1 in [5]). *Let (φ, k) be an instance of PROPAGATIONAL- f SATISFIABILITY. For a graph H , let (G_φ, k', R) be obtained by applying Construction 4 on (φ, k, H) . Then, if (G_φ, k', R) is a yes-instance of RESTRICTED H -FREE EDGE DELETION (COMPLETION) then (φ, k) is a yes-instance of PROPAGATIONAL- f SATISFIABILITY.*

We remark that the proof of Proposition 5.3 works even if we use a gadget for the truth-setting component which satisfies only a weak property: it has at most two deletion (completion) sets, the \emptyset and the set of all allowed (non)edges. As we have seen, the construction of $T_D(u)$ and $T_C(u)$ discussed above satisfies this weak property.

To prove the other direction, one needs to show that there is no induced H in the “vicinity” of a satisfaction-testing component after deleting (adding) the (non)edges corresponding to the variables being set to 1 in φ . This can be done very easily for 3-connected graphs H . Proving this direction for 2-connected graphs H (if provable) requires careful structural analysis of the constructed graph G_φ .

In Figure 14, we give various gadgets required for the proofs of this section. We use *unit* as a general term to refer to a satisfaction-testing component or a basic unit.

Lemma 5.4. *Let $H \in \{\overline{A_1}, \overline{A_2}, A_3, \overline{A_3}, A_4, A_5, \overline{A_7}, \overline{A_9}\}$. Then RESTRICTED H -FREE EDGE DELETION is incompressible, assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.*

Proof. By definitions, the gadget shown in the corresponding cell in the column ‘ $S_D(x, y, z)$ ’ (where x is the edge added to H , and y and z are the other two darkened edges) of Figure 14 is a satisfaction-testing component for H -FREE EDGE DELETION and the gadget shown in the corresponding cell in the column ‘Basic unit’ (under DELETION) (with two distinguished edges which are darkened) is a basic unit for H -FREE EDGE DELETION. It can be easily verified that the $T_D(u)$ obtained by the construction for truth-setting component using the basic unit for H -FREE EDGE DELETION (given in Figure 14) has the property that if an allowed edge is deleted then the component has an induced H unless all allowed edges are deleted. We use these $S_D(x, y, z)$ and $T_D(u)$ for the reduction given below.

We give a PPT from PROPAGATIONAL- f SATISFIABILITY to RESTRICTED H -FREE EDGE DELETION. Then the statement follows from the incompressibility of the source problem (Proposition 5.2). Let (φ, k) be an instance of PROPAGATIONAL- f SATISFIABILITY such that every variable appears exactly three times in φ . We apply Construction 4 on (φ, k, H) to obtain (G_φ, k', R) . We claim that (φ, k) is a yes-instance of PROPAGATIONAL- f SATISFIABILITY if and only if $(G_\varphi, k' = 3|V(H)|k, R)$ is a yes-instance of RESTRICTED H -FREE EDGE DELETION. One direction is proved by Proposition 5.3. For the other direction, let (φ, k) be a yes-instance with a satisfying truth assignment with weight at most k . Let F contains all allowed edges of $T_D(u)$ for every true variable u . Clearly, $|F| \leq 3|V(H)|k$. We claim that $G'_\varphi = G_\varphi - F$ is H -free. For a contradiction, assume that there is an H induced by Z in G'_φ . Clearly, the vertices in Z cannot be from a single unit. Since there is a chain of $|V(H)|$ many basic units between every pair of satisfaction-testing components, we obtain that the vertices of an allowed edge act as a 2-separator in the H induced by Z . We list down the arguments which lead to contradiction for each graph H .

$\overline{A_1}, A_3, A_4, A_5, \overline{A_9}$: The vertices of every 2-separator of H has more number of mutually adjacent common neighbors than that of vertices of every allowed edge in a unit.

$\overline{A_2}$: The vertices u, v of every 2-separator has two common neighbors x, y such that x and y have a common neighbor non-adjacent to both u and v . Therefore, the vertices u, v, x, y must be from a unit. That is, the four vertices with degree at least three and form a diamond in the H must be from a single unit, say

Graph		DELETION			COMPLETION		
		$S(x, y, z)$	Basic unit	Enforcer	$S(x, y, z)$	Basic unit	Enforcer
$\overline{A_1}$							
$\overline{A_2}$							
A_3							
$\overline{A_3}$							
A_4							
A_5							
$\overline{A_6}$							
$\overline{A_7}$							
$\overline{A_8}$							
$\overline{A_9}$							
$\overline{B_1}$							
$\overline{B_2}$							
$\overline{B_3}$							

Figure 14: Various gadgets used in the proofs of this section. In a satisfaction-testing component $S_D(x, y, z)$ ($S_C(x, y, z)$), x is the darkened (non)edge added (deleted) in H to obtain the gadget, and y and z are the other two darkened (non)edges. Both the allowed (non)edges in basic units are darkened. The distinguished edge in a deletion enforcer and the distinguished nonedge in a completion enforcer are darkened.

U . Now it can be seen that for every diamond in $U - F$ neither the middle edge of the diamond nor the nonedge of the diamond is an allowed edge in U .

$\overline{A_3}$: Clearly, the induced paw in the H must be from a single unit, say U . But none of the induced paw in $U - F$ has a nonedge which is an allowed edge in U .

$\overline{A_7}$: Every 2-separator in H induces a K_2 . In a unit, we note that if uv is an allowed edge and x a common neighbor of u and v , then neither ux nor vx is an allowed edge. Therefore, if uv acts as a 2-separator in the induced H , then there must be vertices x, y, z in a unit, say U containing u and v such that, in $U - F$, it must be the case that $x \in N(u) \cap N(v)$, $y \in (N(u) \cap N(x)) \setminus N(v)$, $z \in (N(v) \cap N(x)) \setminus N(u)$, and y, z are nonadjacent. This is not the case with vertices of any of the allowed edges in a unit. □

The following corollary follows from the fact that there is no subgraph isomorphic to a C_4 where all edges are allowed in the graph G_φ constructed in the proof of Lemma 5.4 for RESTRICTED H -FREE EDGE DELETION, when H is a $\overline{A_1}$. We will be using this result later to handle $\overline{A_7}$ (Lemma 5.14).

Corollary 5.5. *Let H be $\overline{A_1}$. Then, assuming $\text{NP} \not\subseteq \text{coNP/poly}$, RESTRICTED H -FREE EDGE DELETION is incompressible even if input graphs does not contain a subgraph (not necessarily induced) isomorphic to H where all the side-edges of the diamond (a side-edge of a diamond is an edge incident to a degree-2 vertex in the diamond) in the subgraph are allowed.*

Lemma 5.6. *Let $H \in \{\overline{A_2}, \overline{A_7}, \overline{A_8}, \overline{A_9}, \overline{B_1}, \overline{B_2}, \overline{B_3}\}$. Then RESTRICTED H -FREE EDGE COMPLETION is incompressible, assuming $\text{NP} \not\subseteq \text{coNP/poly}$.*

Proof. By definitions, the gadget shown in the corresponding cell in the column ‘ $S_C(x, y, z)$ ’ (where x is the nonedge added to H , and y and z are the other two dashed nonedges) of Figure 14 is a satisfaction-testing component for H -FREE EDGE COMPLETION and the gadget shown in the corresponding cell in the column ‘Basic unit’ (under COMPLETION) (with two distinguished nonedges which are dashed) is a basic unit for H -FREE EDGE COMPLETION. It can be easily verified that the $T_C(u)$ obtained by the construction for truth-setting component using the basic unit for H -FREE EDGE COMPLETION (given in Figure 14) has the property that if an allowed nonedge is added then the component has an induced H unless all allowed nonedges are added. We use these $S_C(x, y, z)$ and $T_C(u)$ for the reduction given below.

We give a PPT from PROPAGATIONAL- f SATISFIABILITY to RESTRICTED H -FREE EDGE COMPLETION. Then the statement follows from the incompressibility of the source problem (Proposition 5.2). Let (φ, k) be an instance of PROPAGATIONAL- f SATISFIABILITY such that every variable appears exactly three times in φ . We apply Construction 4 on (φ, k, H) to obtain $(G_\varphi, k' = 3|V(H)|k, R)$. We claim that (φ, k) is a yes-instance of PROPAGATIONAL- f SATISFIABILITY if and only if (G_φ, k', R) is a yes-instance of RESTRICTED H -FREE EDGE COMPLETION. One direction is proved by Proposition 5.3. For the other direction, let (φ, k) be a yes-instance with a satisfying truth assignment with weight at most k . Let F contains all allowed nonedges of $T_C(u)$ for every true variable u . Clearly, $|F| \leq 3|V(H)|k$. We claim that $G'_\varphi = G_\varphi + F$ is H -free. For a contradiction, assume that there is an H induced by Z in G'_φ . Clearly, the vertices in Z cannot be from a single unit. Since there is a chain of $|V(H)|$ many basic units between every pair of satisfaction-testing components, we obtain that the vertices of an allowed edge act as a 2-separator in the H induced by Z . We list down the arguments which lead to contradiction for each graph H .

$\overline{A_2}$: The vertices u, v of every 2-separator has two common neighbors x, y such that x and y have a common neighbor non-adjacent to both u and v . Therefore, the vertices u, v, x, y must be from a unit. That is, the four vertices with degree at least three and form a diamond in the H must be from a single unit, say U . Now it can be seen that for every diamond in $U + F$ the nonedge in the diamond is not an allowed edge and if the middle edge of the diamond is an allowed nonedge in U then the nonedge in the diamond does not have a common vertex nonadjacent to the middle edge.

$\overline{A_7}$: Every 2-separator in H induces a K_2 . In a unit, we note that, if uv is an allowed nonedge and x a common neighbor of u and v , then neither ux nor vx is an allowed nonedge. Therefore, if uv acts as a 2-separator in the induced H then there must be vertices x, y, z in a unit, say U containing u and v such that, in $U + F$, it must be the case that $x \in N(u) \cap N(v)$, $y \in (N(u) \cap N(x)) \setminus N(v)$, $z \in (N(v) \cap N(x)) \setminus N(u)$, and y, z are nonadjacent. This is not the case with vertices of any of the allowed nonedges in a unit.

$\overline{A_8}$: It must be the case that either the middle edge of an induced diamond in $U + F$ is an allowed nonedge in U or the nonedge between a degree-1 vertex and a degree-3 vertex of an induced H' in $U + F$ is an allowed nonedge in U , where H' is obtained by deleting a degree-2 vertex from H . These do not hold true for a unit.

$\overline{A_9}, \overline{B_2}, \overline{B_3}$: The vertices of every 2-separator of H has more number of mutually adjacent common neighbors than that of vertices of every allowed nonedge in a unit, even if all allowed nonedges are added to the unit.

$\overline{B_1}$: Clearly, the $K_5 - e$ in the H must be from a single unit, say U . Further the nonedge in the $K_5 - e$ must be an allowed nonedge. But for every $K_5 - e$ in $U + F$, the nonedge is not an allowed nonedge. □

5.2 Using enforcers to reduce to the unrestricted problems

If we want to reduce RESTRICTED H -FREE EDGE DELETION to H -FREE EDGE DELETION, then there is a fairly natural idea to try: for each restricted edge $e' = x'y'$, we introduce a copy of H on set U of new vertices and identify $x'y'$ with xy , where $x, y \in U$ are nonadjacent vertices. Now U induces a copy of H plus an extra edge, but as soon as e' is deleted, it becomes a copy of H , effectively preventing the deletion of e' .

There are two problems with this approach. First, the solution could delete other edges from the new copy of H , and then it is not necessarily true that the removal of e' automatically creates an induced copy of H . However, this problem is easy to avoid by repeating this gadget construction $k + 1$ times: a solution of size at most k cannot interfere with all $k + 1$ gadgets. The second problem is more serious: it is possible that attaching the new vertices creates a copy of H , even when e is not deleted. For certain graphs H , with a careful choice of x and y we can ensure that this does not happen: no induced copy of H can go through the separator x, y .

An H -free deletion enforcer (X, e) consists of an H -free graph X and a distinguished edge e in X such that (a) $X - e$ contains an induced H , and (b) for any graph G vertex disjoint with X , and any edge e' of G , all induced copies of H in the graph obtained by attaching X to G through identifying e with e' reside entirely inside G . Similarly, an H -free completion enforcer (X, e) consists of an H -free graph X and a distinguished nonedge e such that (a) $X + e$ contains an induced H , and (b) for any graph G vertex disjoint with X , and any nonedge e' in G , all induced copies of H in the graph obtained by attaching X to G through identifying e with e' reside entirely inside G . It can be shown that if we can come up with enforcer gadgets satisfying these conditions, then the ideas sketched above can be made to work, and we obtain a reduction from the restricted problem to the unrestricted version.

Proposition 5.7 (See Lemma 6.5 in [5]). *For a graph H :*

- (i) *If RESTRICTED H -FREE EDGE DELETION is incompressible and there exists an H -free deletion enforcer, then H -FREE EDGE DELETION is incompressible.*
- (ii) *If RESTRICTED H -FREE EDGE COMPLETION is incompressible and there exists an H -free completion enforcer, then H -FREE EDGE COMPLETION is incompressible.*
- (iii) *If H -FREE EDGE DELETION is incompressible and there exists an H -free completion enforcer, then H -FREE EDGE EDITING is incompressible.*

In the rest of the section, we establish the existence of enforcer gadgets for certain graphs H .

Lemma 5.8. *Let $H \in \{\overline{A_1}, \overline{A_2}, A_3, \overline{A_3}, A_4, A_5\}$. Then the gadget X with a distinguished edge e shown in the corresponding cell in the column 'Enforcer' (under DELETION) in Figure 14 is an H -free deletion enforcer.*

Proof. Clearly, X is H -free and $X - e$ contains an induced H as required by the definition. Let G be a graph vertex-disjoint with X . Let G' be obtained from G and X by identifying e and any edge e' of G . Let u, v be the vertices in G' obtained by the identification of e and e' . We need to prove that every induced H in G' is induced by a subset of vertices in G . For a contradiction, assume that there is an H in G' induced by Z where Z has vertices from $V(X) \setminus \{u, v\}$ and from $V(G) \setminus \{u, v\}$. Since H is 2-connected, $\{u, v\} \subseteq Z$ and $\{u, v\}$ must act as a 2-separator which induces a K_2 in the induced H . We list down arguments which lead to contradiction with the assumption for each graph H .

$\overline{A_1}, \overline{A_3}$: None of the 2-separators in H induces a K_2 .

$\overline{A_2}, A_4, A_5$: Every 2-separator xy in H has at least one common neighbor in every component obtained after deleting x and y from H . But the vertices of e do not have a common neighbor.

A_3 : Since H is 2-connected, Z induces a graph containing an induced C_4 with the vertices in X . But H does not have an induced C_4 . □

Lemma 5.9. *Let $H \in \{\overline{A_1}, \overline{A_2}, A_3, A_4, A_5, \overline{A_6}, \overline{A_7}, \overline{A_8}, \overline{A_9}, \overline{B_1}, \overline{B_2}, \overline{B_3}\}$. Then the gadget X with a distinguished nonedge e shown in the corresponding cell in the column ‘Enforcer’ (under COMPLETION) in Figure 14 is an H -free completion enforcer.*

Proof. Clearly, X is H -free and $X + e$ contains an induced H as required by the definition. Let G be a graph vertex-disjoint with X . Let G' be obtained from G and X by identifying e and any nonedge e' of G . Let u, v be the vertices in G' obtained by the identification of e and e' . We need to prove that every induced H in G' is induced by a subset of vertices in G . For a contradiction, assume that there is an H in G' induced by Z where Z has vertices from $V(X) \setminus \{u, v\}$ and from $V(G) \setminus \{u, v\}$. Since H is 2-connected, $\{u, v\} \subseteq Z$ and $\{u, v\}$ must act as a 2-separator which induces a $2K_1$ in the induced H . We list down arguments which lead to contradiction with the assumption for each graph H .

$\overline{A_1}, \overline{A_2}, \overline{A_6}, \overline{A_8}, \overline{B_1}$: Every 2-separator xy in H which induces a $2K_1$ has at least one common neighbor in every component obtained after deleting x and y from H . But the vertices of e does not have a common neighbor.

$A_4, A_5, \overline{A_7}, \overline{A_9}, \overline{B_2}, \overline{B_3}$: There is no 2-separator in H inducing a $2K_1$.

A_3 : Since H is 2-connected, Z induces a graph containing an induced P_5 with the vertices in X . But H does not have an induced P_5 between vertices of a 2-separator inducing $2K_1$.

□

Lemma 5.10. *Let $H \in \{\overline{A_1}, \overline{A_2}, A_3, \overline{A_3}, A_4, A_5\}$. Then H -FREE EDGE DELETION and H -FREE EDGE EDITING are incompressible, assuming $\text{NP} \not\subseteq \text{coNP/poly}$.*

Proof. By Lemma 5.4, RESTRICTED H -FREE EDGE DELETION is incompressible. By Lemma 5.8 and 5.9, we have H -free deletion enforcer and H -free completion enforcer. Then by Proposition 5.7, H -FREE EDGE DELETION and H -FREE EDGE EDITING are incompressible, assuming $\text{NP} \not\subseteq \text{coNP/poly}$. □

Similarly, we can prove Lemma 5.11. The cases of H being A_4 or A_5 follows from the fact that H and \overline{H} are isomorphic (see Proposition 2.1).

Lemma 5.11. *Let $H \in \{\overline{A_2}, A_4, A_5, \overline{A_7}, \overline{A_8}, \overline{A_9}, \overline{B_1}, \overline{B_2}, \overline{B_3}\}$. Then H -FREE EDGE COMPLETION is incompressible, assuming $\text{NP} \not\subseteq \text{coNP/poly}$.*

5.3 Further tricky reductions

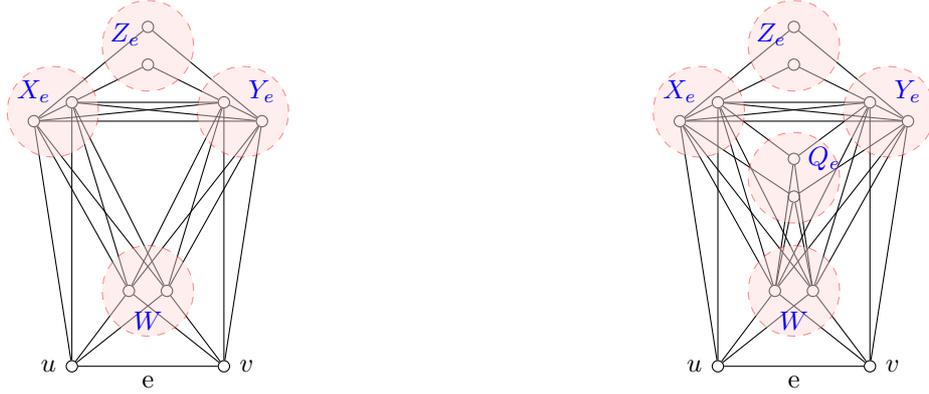
There are graphs for which we can show that no completion/deletion enforcers, as defined in the previous section, exist (this can be checked by going through every pair x, y of (non)adjacent vertices). For some of these graphs, we can find a different way of enforcing that certain edges are forbidden; typically, we introduce some vertices that are used globally by every enforcer gadget. Furthermore, there are graphs H , where we were unable to obtain a reduction from RESTRICTED H -FREE EDGE DELETION (COMPLETION), but could choose an induced subgraphs $H' \subseteq H$ and obtain a reduction from RESTRICTED H' -FREE EDGE DELETION (COMPLETION), whose incompressibility was established earlier.

Lemma 5.12. *Let $H = \overline{A_7}$. Then H -FREE EDGE DELETION and H -FREE EDGE EDITING are incompressible, assuming $\text{NP} \not\subseteq \text{coNP/poly}$.*

Proof. By Lemma 5.4, RESTRICTED H -FREE EDGE DELETION is incompressible. We give a PPT from RESTRICTED H -FREE EDGE DELETION to H -FREE EDGE DELETION. Then the incompressibility of H -FREE EDGE EDITING follows from the existence of H -free completion enforcer (Lemma 5.9).

Let (G', k, R) be an instance of RESTRICTED H -FREE EDGE DELETION. We obtain a graph G from G' as follows. Introduce a set W of k independent vertices such that all of them are adjacent to all vertices in G' . For every forbidden edge $e = uv \in R$, introduce three sets X_e, Y_e, Z_e of k vertices each such that $ux, vy, wx, wy \in E(G)$ for every $x \in X_e$, and $y \in Y_e$. Further, for every i such that $1 \leq i \leq k$, $x_i z_i, y_i z_i \in E(G)$, where $x_i \in X_e, y_i \in Y_e$, and $z_i \in Z_e$ (assuming a labelling of vertices in sets X_e, Y_e , and Z_e). Let $X = \bigcup_{e \in R} X_e$, $Y = \bigcup_{e \in R} Y_e$, and $Z = \bigcup_{e \in R} Z_e$. The set $C = X \cup Y$ forms a clique and Z forms an independent set. This completes the construction and let the resultant graph be G (see Figure 15a). We claim that (G', k, R) is a yes-instance of RESTRICTED H -FREE EDGE DELETION if and only if (G, k) is a yes-instance of H -FREE EDGE DELETION.

Let (G, k) be a yes-instance. Let F be a solution. Since G' is an induced subgraph of G , $G' - F$ is H -free. Let F contains a forbidden edge $e = uv \in R$. Then there are at least k edge disjoint H due to the vertices in the sets W, X_e, Y_e and Z_e . Since $|F| \leq k$, this cannot happen. Therefore F does not contain any forbidden edge and hence F is a solution for (G', k, R) .



(a) The gadget used to handle $\overline{A_7}$ in Lemma 5.16

(b) The gadget used to handle $\overline{A_9}$ in Lemma 5.13

Figure 15

Let (G', k, R) be a yes-instance with a solution F' . We claim that F' is a solution for (G, k) . For a contradiction, let U induces an H in $G - F'$. Then U must contain at least one vertex newly introduced in G . Since there is no pair of vertices with the same neighborhood in H , $|U \cap W| \leq 1$. If U contains no vertex from W , and if U contains at least one vertex from $V(G')$, then U induces a graph with either a cut vertex or an induced C_4 , which is a contradiction. If U contains no vertex from W and $V(G')$, then U is a subgraph of a split graph where every vertex in the clique (formed by $X \cup Y$) is adjacent to exactly one vertex of degree two (a vertex in Z). Therefore, U cannot induce H . Hence $|U \cap W| = 1$. Let $U \cap W = \{w\}$. Let $e = uv$ be any forbidden edge in G' . For every vertex $x \in X_e$, ux is part of exactly one triangle (disregarding all vertices in W except w) in $G - F'$ and so is the case with every edge vy for every $y \in Y_e$. Therefore, neither ux nor vy can be part of the central triangle (triangle formed by the degree-4 vertices) of the induced H .

Case 1: The vertex w is a degree-4 vertex in the induced H . Then the other two vertices in the triangle formed by the degree-4 vertices in the H must be either u, v or x, y for some forbidden edge $e = uv$, $x \in X_e$, and $y \in Y_e$. Let the degree-4 vertices in the H are w, u , and v . Since every common neighbor, other than those in W , of u and v is adjacent to w , U does not induce an H . Therefore, assume that the degree-4 vertices in H are w, x , and y . Since w is adjacent to every vertex in C , U contains only x and y from the set C . Then, since the unique common neighbor (u) of w and x not in C and the unique common neighbor (v) of w and y not in C are adjacent in $G - F'$, U cannot induce an H , which is a contradiction.

Case 2: The vertex w is a degree-2 vertex in the induced H . The other two vertices part of the triangle in which w is a part of must be either u, v or x, y for some forbidden edge $e = uv$, $x \in X_e$, and $y \in Y_e$. Since every common neighbor, other than those in W , of u and v is adjacent to w in $G - F'$, both u and v cannot be in U . Therefore, let x and y be the degree-4 neighbors of w in H . Since w is adjacent to every vertex in C , $U \cap X_e = \{x, y\}$. Since x and y have no other common neighbors with degree at least 4, (other than vertices in $W \cup C$), U cannot induce an H , a contradiction. \square

We can handle $\overline{A_9}$ in a similar way.

Lemma 5.13. *Let $H = \overline{A_9}$. Then H -FREE EDGE DELETION and H -FREE EDGE EDITING are incompressible, assuming $\text{NP} \not\subseteq \text{coNP/poly}$.*

Proof. By Lemma 5.4, RESTRICTED H -FREE EDGE DELETION is incompressible. We give a PPT from RESTRICTED H -FREE EDGE DELETION to H -FREE EDGE DELETION. Then the incompressibility of H -FREE EDGE EDITING follows from the existence of H -free completion enforcer (Lemma 5.9).

Let (G', k, R) be an instance of RESTRICTED H -FREE EDGE DELETION. We obtain a graph G from G' as follows. Introduce a set W of k independent vertices such that all of them are adjacent to all vertices in G' . For every forbidden edge $e = uv \in R$, introduce four sets Q_e, X_e, Y_e, Z_e of k vertices each such that $ux, vy, wq, wx, wy \in E(G)$ for every $x \in X_e$, and $y \in Y_e$. Further, for every i such that $1 \leq i \leq k$, $x_i z_i, y_i z_i, x_i q_i, y_i q_i \in E(G)$, where $q_i \in Q_e, x_i \in X_e, y_i \in Y_e$, and $z_i \in Z_e$ (assuming a labelling of vertices in sets Q_e, X_e, Y_e , and Z_e). Let $Q = \bigcup_{e \in R} Q_e$, $X = \bigcup_{e \in R} X_e$, $Y = \bigcup_{e \in R} Y_e$, and $Z = \bigcup_{e \in R} Z_e$. The set $C = X \cup Y$ forms a clique and $Q \cup Z$ forms an independent set. This completes the construction and let the resultant graph be G (see Figure 15b). We claim that (G', k, R) is a yes-instance of RESTRICTED H -FREE EDGE DELETION if and only if (G, k) is a yes-instance of H -FREE EDGE DELETION.

Let (G, k) be a yes-instance. Let F be a solution. Since G' is an induced subgraph of G , $G' - F$ is H -free. Let F contains a forbidden edge $e = uv \in R$. Then there are at least k edge disjoint H due to the vertices in

the sets W, Q_e, X_e, Y_e and Z_e . Since $|F| \leq k$, this cannot happen. Therefore F does not contain any forbidden edge and hence F is a solution for (G', k, R) .

Let (G', k, R) be a yes-instance with a solution F' . We claim that F' is a solution for (G, k) . For a contradiction, let U induces an H in $G - F'$. Then U must contain at least one vertex newly introduced in G . Since there is no pair of vertices with the same neighborhood in H , $|U \cap W| \leq 1$. If U contains no vertex from W , and if U contains at least one vertex from $V(G')$, then U induces a graph with either a cut vertex or an induced C_4 , which is a contradiction. If U contains no vertex from $W \cup V(G')$, then U is a subgraph of a split graph where every pair of vertices in the clique (formed by $X \cup Y$) has exactly two common neighbors in $Q \cup Z$ or has no common neighbors in $Q \cup Z$. Then it can be verified that U cannot induce H . Hence $|U \cap W| = 1$. Let $U \cap W = \{w\}$. Let $e = uv$ be any forbidden edge in G' . For every vertex $x \in X_e$, ux is part of exactly one triangle in $G - F'$, disregarding all vertices in W except w ; and so is the case with every edge vy for every $y \in Y_e$. Therefore, neither ux nor vy can be part of any of the edges in the K_4 in the H . Further, a vertex in Q cannot be a degree-5 vertex in the H as every vertex in Q , disregarding the vertices in W other than w , has adjacent to only three vertices.

Case 1: The vertex w is a degree-5 vertex in the induced H . Then the other two vertices in the triangle formed by the degree-5 vertices in the H must be either u, v or x, y for some forbidden edge $e = uv$, $x \in X_e$, and $y \in Y_e$. Let the degree-5 vertices in the H are w, u , and v . Since every common neighbor, other than those in W , of u and v is adjacent to w , U does not induce an H . Therefore, assume that the degree-5 vertices in H are w, x , and y . Since the unique common neighbor (u) of w and x which is not in C and not adjacent to y , and the unique common neighbor (v) of w and y , which is not in C and not adjacent to x , are adjacent in $G - F'$, U cannot induce an H , which is a contradiction.

Case 2: The vertex w is a degree-2 vertex in the induced H . The two degree-5 neighbors of w in the H must be either u, v or x, y for some forbidden edge $e = uv$, $x \in X_e$, and $y \in Y_e$. Since every common neighbor, other than those in W , of u and v is adjacent to w in $G - F'$, both u and v cannot be in U . Therefore, let x and y be the degree-5 neighbors of w in H . Since x and y have no other common neighbors with degree at least 5 and nonadjacent to w , (other than the vertices in W), U cannot induce an H , a contradiction.

Case 3: The vertex w is a degree-3 vertex in the induced H . Then the three degree-5 neighbors of w in the H must be from C (recall that a vertex in Q cannot be a degree-5 vertex). Then all the three degree-2 vertices in the H must be from Z as they must be nonadjacent with w (and not belong to W). This cannot happen as every vertex in C has only one neighbor in Z . \square

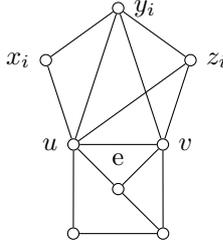
Let us observe that $\overline{A_1}$ can be obtained from $\overline{A_6}$ by removing a degree-2 vertex. We can reduce RESTRICTED $\overline{A_1}$ -FREE EDGE DELETION to $\overline{A_6}$ -FREE EDGE DELETION, but we need the additional assumption of Corollary 5.5 to make this reduction work.

Lemma 5.14. *Let H be $\overline{A_6}$. Then H -FREE EDGE DELETION and H -FREE EDGE EDITING are incompressible, assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.*

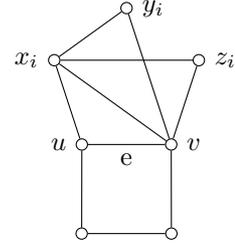
Proof. We give a PPT from RESTRICTED H' -FREE EDGE DELETION to H -FREE EDGE DELETION, where H' is $\overline{A_1}$. Then it follows that H -FREE EDGE DELETION is incompressible (Lemma 5.4) and H -FREE EDGE EDITING is incompressible by Proposition 5.7, and by the existence of H -free completion enforcer (Lemma 5.9), assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.

By Corollary 5.5, RESTRICTED H' -FREE EDGE DELETION is incompressible (assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$) even if at least one side-edge of a diamond (an edge incident to the degree-2 vertex of a diamond) in every subgraph isomorphic to H' in the input graph is forbidden. Let (G', k, R) be such an instance of RESTRICTED H' -FREE EDGE DELETION. We construct a graph G as follows. For every forbidden edge $e = uv \in R$, introduce three sets X_e, Y_e, Z_e of $k + 1$ independent vertices each such that u is adjacent to all vertices in $X_e \cup Y_e \cup Z_e$ and v is adjacent to all vertices in $Y \cup Z$. Further, for $1 \leq i \leq k + 1$, x_i is adjacent to y_i and y_i is adjacent to z_i , where $x_i \in X_e, y_i \in Y_e, z_i \in Z_e$ (assuming a labelling of the vertices in X_e, Y_e , and Z_e). This completes the construction (see Figure 16a). Let the constructed graph be G .

We claim that (G', k, R) is a yes-instance of RESTRICTED H' -FREE EDGE DELETION if and only if (G, k) is a yes-instance of H -FREE EDGE DELETION. Let (G, k) be a yes-instance of H -FREE EDGE DELETION. Let F be a solution of it. Assume that F contains some forbidden edge $e = uv$ in G' . Since $|F| \leq k$, there exists integers $i \neq j$ such that u and v along with $\{x_i, y_i, z_i\}$ and z_j induce an H in $G - F$, which is a contradiction. Therefore, F does not contain any forbidden edge in G' . For a contradiction, let $G' - F$ contains an H induced by U . Then by the assumption on G' , at least one side-edge of a diamond in the H is a forbidden edge, say $e = uv$. Since $|F| \leq k$, there exists at least one integer i such that U along with z_i induces an H in $G - F$, which is a contradiction. For the other direction, let (G', k, R) be a yes-instance and let F' be a solution of it. For a contradiction, assume that $G - F'$ contains an H induced by a set U of vertices. It is straight-forward to verify that none of the vertices in $X_e \cup Y_e \cup Z_e$ can be part of an induced C_4 in $G - F'$. Therefore, all the vertices in both the induced C_4 in the H must be in G' . Then $G' - F'$ has an induced H' , which is a contradiction. \square



(a) The gadget used to handle $\overline{A_6}$ in Lemma 5.14



(b) The gadget used to handle $\overline{A_8}$ in Lemma 5.16

Figure 16

Graph $\overline{A_8}$ is handled in a similar way, by noting that C_4 can be obtained by removing two degree-2 vertices. For the reduction, we need to observe that an additional assumption can be made in the incompressibility proof for RESTRICTED C_4 -FREE EDGE DELETION given in [5].

Observation 5.15. *Let H be C_4 . Then RESTRICTED H -FREE EDGE DELETION is incompressible (assuming $\text{NP} \not\subseteq \text{coNP/poly}$) even if the input graph does not contain any subgraph (not necessarily induced) C_4 such that all its edges are allowed.*

Lemma 5.16. *Let H be $\overline{A_8}$. Then H -FREE EDGE DELETION and H -FREE EDGE EDITING are incompressible, assuming $\text{NP} \not\subseteq \text{coNP/poly}$.*

Proof. We give a PPT from RESTRICTED H' -FREE EDGE DELETION to H -FREE EDGE DELETION where H' is C_4 . Then the completion enforcer given by Lemma 5.9 implies the incompressibility for H -FREE EDGE EDITING (by Proposition 5.7).

By Observation 5.15, RESTRICTED H' -FREE EDGE DELETION is incompressible even if the input graph does not contain a C_4 (not necessarily induced) having only allowed edges. Let (G', k, R) be an instance of RESTRICTED H' -FREE EDGE DELETION such that every subgraph C_4 in G' has a forbidden edge. For every forbidden edge $e = uv$ in G' , introduce three sets X_e, Y_e, Z_e of $k + 2$ independent vertices each such that u is adjacent to every vertex in X_e , and v is adjacent to every vertex in $Y_e \cup Z_e$. Further, for $1 \leq i \leq k$, $x_i y_i$ and $x_i z_i$ are edges in the graph, for $x_i \in X_e$, $y_i \in Y_e$, and $z_i \in Z_e$ (assuming a labelling of vertices in X_e, Y_e , and Z_e). This completes the construction (see Figure 16b). Let the resultant graph be G .

We claim that (G', k, R) is a yes-instance of RESTRICTED H' -FREE EDGE DELETION if and only if (G, k, R) is a yes-instance of H -FREE EDGE DELETION. Let (G, k) be a yes-instance of H -FREE EDGE DELETION. Let F be a solution of it. If F contains a forbidden edge $e = uv$ in G' , then there exists integers $i \neq j$ such that $\{u, v, w_i, x_i, x_j, y_i, z_i\}$ ($w_i \in W_e, x_i, x_j \in X_e, y_i \in Y_e, z_i \in Z_e$) induces an H in $G - F$, which is a contradiction. Therefore, F can contain none of the forbidden edges in G' . Now let $G' - F$ contains a C_4 induced by a set U . Then, at least one of the edge in the C_4 must be a forbidden edge, say $e = uv$. Since $|F| \leq k$, there exists at least two vertices x_i, x_j ($i \neq j$) in X_e such that the U along with x_i and x_j induces an H in $G - F$, which is a contradiction. For the other direction, let (G', k, R) be a yes-instance of RESTRICTED H' -FREE EDGE DELETION. Let F' be a solution of it. We claim that $G - F'$ is H -free. For a contradiction, let there be an induced H in $G - F'$. It is straight-forward to verify that all the vertices of the induced C_4 in the H must be in G' . Therefore, $G' - F'$ has an induced C_4 , which is a contradiction. \square

As C_4 can be obtained from $\overline{A_1}$ by removing a degree-3 vertex, we can reduce RESTRICTED C_4 -FREE EDGE COMPLETION to $\overline{A_1}$ -FREE EDGE COMPLETION using the following observation on the proof of incompressibility of C_4 -FREE EDGE COMPLETION in [5].

Observation 5.17. *Let H be C_4 . Then H -FREE EDGE COMPLETION is incompressible (assuming $\text{NP} \not\subseteq \text{coNP/poly}$) for inputs (G, R, k) even if the following conditions are satisfied:*

- (i) *For every forbidden nonedge xy , x and y have only at most two common neighbors in the graph obtained by adding all allowed nonedges to G ;*
- (ii) *Let S be a subset of the set of all allowed edges in G . If $G + S$ has an induced C_4 , then the following conditions are satisfied:*
 - *Let e, e' be the two edges of an induced P_3 in the C_4 . Then at least one of them is in G .*
 - *If only at most one edge of the C_4 is an allowed nonedge in G , then one of the two nonedges in the C_4 is forbidden in G .*

- If $G + S$ has an induced C_4 where two nonadjacent edges in the C_4 are allowed nonedges in G , then there is an induced C_4 in $G + S$ where only at most one edge in the C_4 is an allowed nonedge in G .

Lemma 5.18. *Let H be $\overline{A_1}$. Then H -FREE EDGE COMPLETION is incompressible, assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.*

Proof. We will prove that RESTRICTED H -FREE EDGE COMPLETION is incompressible (assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$), then the statement follows from the existence of completion enforcer (see Lemma 5.9 and Proposition 5.7). We give a PPT from RESTRICTED H' -FREE EDGE COMPLETION where H' is a C_4 .

Let (G', R', k) be an instance of RESTRICTED H' -FREE EDGE COMPLETION where (G', R') satisfies the properties given in Observation 5.17. Initialize G to be G' and R to be R' . For every $xy \in R'$ such that x and y are the end vertices of an induced P_3 in G' , introduce a vertex v in G adjacent to x, y, z , where z is the middle vertex of an induced P_3 in G' , where x and y are the end vertices of the P_3 . Add all nonedges incident to v to R . Let I be the set of all newly introduced vertices. Add all nonedges among vertices in I to R . We claim that (G', R', k) is a yes-instance of RESTRICTED H' -FREE EDGE COMPLETION if and only if (G, R, k) is a yes-instance of RESTRICTED H -FREE EDGE COMPLETION.

Let (G', R', k) be a yes-instance of RESTRICTED H' -FREE EDGE COMPLETION. Let F' be a solution of it. For a contradiction, assume that $G + F'$ has an H induced by U . Clearly, U contains at least one vertex, say v , in I . Let x, y, z be the neighbors of v , where z is the middle vertex of the P_3 induced by $\{x, y, z\}$ in G' . Since the neighborhood of v forms an induced P_3 in G' , v cannot be a degree-3 vertex adjacent to the degree-2 vertex in the H . Assume that v is a degree-2 vertex in the H . Then the nonedge in the diamond in the H must be xy . Since there are only at most two common neighbors of x, y in $G' + F'$ (see condition (i) in Observation 5.17), one of the degree-3 vertex nonadjacent to v in the H must be from I (recall that one of the common neighbors of x and y is adjacent to v). This is a contradiction, as there is only one vertex in I adjacent to both x and y . Now, assume that v is a degree-3 vertex nonadjacent to a degree-2 vertex in H . Since there is no other vertex in I (other than v) adjacent to both x and y , the remaining vertices in the H must be from the copy of G' in G . Therefore, $G' + F'$ has an induced C_4 , a contradiction.

For the other direction, let (G, R, k) be a yes-instance and let F be a solution. For a contradiction, assume that $G' + F$ has a C_4 induced by U . If the C_4 contains only at most one allowed nonedge in G' , then by condition (ii) of Observation 5.17, one of the nonedge xy in the C_4 is forbidden in G' . By condition (i), x and y do not have any other common neighbors other than the other two vertices in the C_4 . Then there is an induced P_3 formed by three vertices of the C_4 such that a vertex in I is adjacent to all vertices in the P_3 . Hence there is an induced H in $G + F$, a contradiction. If two edges in the C_4 are allowed nonedges in G' , then they must be nonadjacent edges of the C_4 (condition (ii)). Then by condition (ii), there is an induced C_4 in $G' + F$ where only at most one edge of the C_4 is allowed. Then the above arguments give a contradiction. \square

$\overline{A_6}$ can be handled in a similar way.

Lemma 5.19. *Let H be $\overline{A_6}$. Then H -FREE EDGE COMPLETION is incompressible, assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$.*

Proof. We will prove that RESTRICTED H -FREE EDGE COMPLETION is incompressible (assuming $\text{NP} \not\subseteq \text{coNP}/\text{poly}$), then the statement follows from the existence of completion enforcer (see Lemma 5.9 and Proposition 5.7). We give a PPT from RESTRICTED H' -FREE EDGE COMPLETION where H' is a C_4 .

Let (G', R', k) be an instance of RESTRICTED H' -FREE EDGE COMPLETION where (G', R') satisfies the properties given in Observation 5.17. Initialize G to be G' and R to be R' . For every $xy \in R'$ such that x and y are the end vertices of an induced P_3 in G' , introduce two vertices u, v in G such that v is adjacent to x, y, z , where z is the middle vertex of an induced P_3 in G' , where x and y are the end vertices of the P_3 . Further, u is adjacent to y . Add all nonedges incident to u and v to R . Let I be the set of all newly introduced vertices. Add all nonedges among vertices in I to R . We claim that (G', R', k) is a yes-instance of RESTRICTED H' -FREE EDGE COMPLETION if and only if (G, R, k) is a yes-instance of RESTRICTED H -FREE EDGE COMPLETION.

Let (G', R', k) be a yes-instance of RESTRICTED H' -FREE EDGE COMPLETION. Let F' be a solution of it. For a contradiction, assume that $G + F'$ has an H induced by U . Clearly, a degree-2 vertex in I can act as only a degree-2 vertex in the H whose neighborhood induces a K_2 . Let U contains a vertex, say v having degree 4, from I . Let x, y, z be the neighbors of v in G' , where z is the middle vertex of the P_3 induced by $\{x, y, z\}$ in G' . Since the neighborhood of v in G' forms an induced P_3 in G' , v cannot be a degree-3 or degree-4 vertex adjacent to the degree-2 vertex (whose neighborhood induces a $2K_1$) in the H . Assume that v is a degree-2 vertex in the H . Then the nonedge in the diamond in the H formed by deleting the degree-2 vertices must be xy . Since there are only at most two common neighbors of x, y in $G' + F'$ (see condition (i) in Observation 5.17), one of the degree-3 or degree-4 vertex nonadjacent to v in the H must be from I (recall that one of the common neighbors of x and y is adjacent to v). This is a contradiction, as there is only one vertex in I adjacent to both x and y . Now, assume that v is a degree-3 or degree-4 vertex nonadjacent to a degree-2 vertex (whose neighborhood induces a $2K_1$) in H . Since there is no other vertex in I (other than v) adjacent to both x and y , $G' + F'$ has an induced C_4 , a contradiction.

For the other direction, let (G, R, k) be a yes-instance and let F be a solution. For a contradiction, assume that $G' + F$ has a C_4 induced by U . If the C_4 contains only at most one allowed nonedge in G' , then by condition (ii) of Observation 5.17, one of the nonedge xy in the C_4 is forbidden in G' . By condition (i), x and y do not have any other common neighbors other than the other two vertices in the C_4 . Then there is an induced P_3 formed by three vertices of the C_4 such that a vertex v in I is adjacent to all vertices in the P_3 and a vertex $u \in I$ is adjacent to v and y (one of the end-vertices of the forbidden edge xy). Hence there is an induced H in $G + F$, a contradiction. If two edges in the C_4 are allowed nonedges in G' , then they must be nonadjacent edges of the C_4 (condition (ii)). Then by condition (ii), there is an induced C_4 in $G' + F$ where only at most one edge of the C_4 is allowed. Then the above arguments give a contradiction. \square

Now, Theorem 5.1(i) follows from Lemma 5.10, 5.14, 5.12, 5.16, 5.13, and Proposition 2.1. Similarly, Theorem 5.1(ii) follows from Lemma 5.10, 5.14, 5.12, 5.16, 5.13, 5.11, 5.18, 5.19, and Proposition 2.1. Theorem 5.1(iii) follows from Theorem 5.1(ii) and Proposition 2.1. Theorem 1.1 follows from Lemma 3.1, 4.24, Theorem 5.1(i), and Proposition 2.3. Similarly, Theorem 1.2 follows from Lemma 3.2, 4.24, Theorem 5.1(ii), and Proposition 2.3.

6 Concluding Remarks

We obtained a set \mathcal{H}^E of nine 5-vertex graphs such that proving the incompressibility of H -FREE EDGE EDITING for every $H \in \mathcal{H}^E$ will lead to a complete dichotomy of the incompressibility for H -FREE EDGE EDITING for graphs H with at least five vertices. We obtained similar sets \mathcal{H}^D and $\mathcal{H}^C (= \overline{\mathcal{H}^D})$ of nineteen graphs each for H -FREE EDGE DELETION and H -FREE EDGE COMPLETION respectively. Thus we have the following future problems.

- Prove incompressibility or obtain polynomial kernel for H -FREE EDGE EDITING for every graph $H \in \mathcal{H}^E$.
- Prove incompressibility or obtain polynomial kernel for H -FREE EDGE DELETION for every graph $H \in \mathcal{H}^D$.

As remarked in the introduction, these sets \mathcal{H}^E and \mathcal{H}^D give the frontier where the possibility of existence of polynomial kernels is the highest. For some graph H in these sets, if the problem admits polynomial kernel, then one needs to include H in \mathcal{Y} and has to analyze the few extra cases arising out of it to obtain a possibly larger base of graphs.

There is a curious case still unresolved when H has at most four vertices—the claw. It is known that CLAW-FREE EDGE DELETION admits a polynomial kernel when the input graphs does not contain a clique of size t , for any fixed positive integer t [3]. It is also known that {CLAW, DIAMOND}-FREE EDGE DELETION admits a polynomial kernel [13].

- Does claw-free edge modification problems admit polynomial kernels?

All these efforts can be seen as steps toward two larger goals: for finite sets \mathcal{H} of graphs

- Obtain a dichotomy on polynomial-time solvable and NP-hard cases for \mathcal{H} -free edge modification problems.
- Obtain a dichotomy on the incompressibility of \mathcal{H} -free edge modification problems.

As a next step towards these larger goals one may look at the case when \mathcal{H} contains exactly two graphs. We hope that the reductions we introduced in this paper can be of help to obtain various hardness results in this and related settings.

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