Separable electromagnetic perturbations of rotating black holes

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We identify a set of Hertz potentials for solutions to the vector wave equation on black hole spacetimes. The Hertz potentials yield Lorenz gauge electromagnetic vector potentials that represent physical solutions to the Maxwell equations, satisfy the Teukolsky equation, and are related to the Maxwell scalars by straightforward and separable inversion relations. Our construction, based on the GHP formalism, avoids the need for a mode ansatz and leads to potentials that represent both static and non-static solutions. As an explicit example, we specialise the procedure to mode-decomposed perturbations of Kerr spacetime and in the process make connections with previous results.

I. INTRODUCTION

In a seminal work, Teukolsky [1] showed that the equations governing perturbations of rotating black holes can be recast into a form where they are given by decoupled equations. These equations further had the remarkable property of being separable, reducing the problem to the solution of a set of uncoupled ordinary differential equations. In the electromagnetic case, Teukolsky's results yield solutions for the spin-weight ±1 components of the Faraday tensor, but do not give a method for obtaining a corresponding vector potential. Subsequent results (and their corresponding equivalents for gravitational perturbations) [2–6] derived a method for reconstructing a vector potential from a Hertz potential, which in turn can be obtained from the spin-weight ±1 components of the Faraday tensor. These were initially restricted to the radiation gauge, but have recently been extended to the Lorenz gauge case [7–13].

In this work, we reformulate the Lorenz gauge Hertz potential of Dolan [12] using the Geroch-Held-Penrose (GHP) formalism, which allows us to derive his results without requiring a mode decomposition. Furthermore, our method has allowed us to identify additional Lorenz gauge Hertz potentials which are more generally applicable. For example, Dolan's result involved division by the frequency ω, which fails in the static $\omega = 0$ case, whereas our potentials do not have this limitation.

The layout of this paper is as follows: in Sec. II we review some relevant background material, including details on the Maxwell equations in curved spacetime, the GHP formalism, the Teukolsky equations and Teukolsky-Starobinsky identities, and radiation gauge reconstruction of the vector potential; in Sec. III we describe methods for reconstructing the vector potential in Lorenz gauge; in Sec. IV we give coordinate expressions for our results decomposed into spin-weighted spheroidal harmonic modes. Finally, we provide some concluding remarks in V.

Throughout this work we follow the conventions of Misner, Thorne and Wheeler [14]: a "mostly positive" metric signature, $(-,+,+,+)$, is used for the spacetime metric; the connection coefficients are defined by $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$; the Riemann tensor is $R^{\tau}{}_{\lambda\mu\nu} = \Gamma^{\tau}_{\lambda\nu,\mu} - \Gamma^{\tau}_{\lambda\mu,\nu} + \Gamma^{\tau}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\tau}_{\sigma\nu}\Gamma^{\sigma}_{\lambda\mu}$, the Ricci tensor and scalar are $R_{\mu\nu} = R^{\tau}{}_{\mu\tau\nu}$ and $R = R_{\mu}{}^{\mu}$, and the Einstein equations are $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$. Standard geometrised units are used, with $c = G = 1$. We use Greek letters for spacetime indices, denote symmetrisation of indices using round brackets [e.g. $T_{(\alpha\beta)} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha})$] and anti-symmetrisation using square brackets [e.g. $T_{[\alpha\beta]} = \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})$], and exclude indices from symmetrisation by surrounding them by vertical bars [e.g. $T_{(\alpha|\beta|\gamma)} = \frac{1}{2}(T_{\alpha\beta\gamma} + T_{\gamma\beta\alpha})$].

II. ELECTROMAGNETIC PERTURBATIONS OF TYPE-D SPACETIMES

A. Maxwell equations

The Faraday tensor may be written in terms of the anti-symmetrised derivative of a vector potential

$$F_{\alpha\beta} = 2\nabla_{[\alpha}A_{\beta]}. \tag{1}$$

In terms of the vector potential, the Maxwell equations, $\nabla_\alpha F^{\alpha\beta} = J^\beta$ are given by

$$(\mathcal{E}A)^\beta \equiv 2\nabla_\alpha\nabla^{[\alpha}A^{\beta]} = J^\beta. \tag{2}$$

B. Geroch-Held-Penrose formalism

In this work, we make use of the formalism of Geroch, Held and Penrose (GHP) [15], which provides a compact means of working with tensor equations when a tetrad based on a pair of null directions is available. Here we provide a concise review of the key features needed for this work, see Refs. [16–18] for detailed reviews, and Ref. [19] for a review using conventions and notation consistent with ours.

The GHP formalism prioritises the concepts of spin- and boost-weights; within the GHP formalism, everything has a well-defined type $\{p, q\}$, which is related to its spin-weight $s = (p - q)/2$ and its boost-weight $b = (p + q)/2$. Only objects of the same type can be added together, providing a useful consistency check on any equations. Multiplication of two quantities yields a resulting object with type given by the sum of the types of its constituents.

The formalism relies on the introduction of a null tetrad $(l, n, m, \bar{m})$ with normalisation

$$l^\alpha n_\alpha = -1, \quad m^\alpha \bar{m}_\alpha = 1, \tag{3}$$

and with all other inner products vanishing. In terms of the tetrad vectors, the metric may be written as

$$g_{\alpha\beta} = -2l_{(\alpha} n_{\beta)} + 2m_{(\alpha}\bar{m}_{\beta)}. \tag{4}$$

There are three discrete transformations that reflect the inherent symmetry in the GHP formalism, corresponding to simultaneous interchange of the tetrad vectors:

1. $'$: $l^\alpha \leftrightarrow n^\alpha$ and $m^\alpha \leftrightarrow \bar{m}^\alpha$, $\{p, q\} \to \{-p, -q\}$;
2. $^-$: $m^\alpha \leftrightarrow \bar{m}^\alpha$, $\{p, q\} \to \{q, p\}$;
3. $*$: $l^\alpha \to m^\alpha$, $n^\alpha \to -\bar{m}^\alpha$, $m^\alpha \to -l^\alpha$, $\bar{m}^\alpha \to \bar{n}^\alpha$.

The 8 spin coefficients of well defined GHP type are defined as the directional derivatives of the tetrad vectors,

$$\kappa = -l^\mu m^\nu \nabla_\mu l_\nu, \quad \sigma = -m^\mu m^\nu \nabla_\mu l_\nu,$$
$$\rho = -\bar{m}^\mu m^\nu \nabla_\mu l_\nu, \quad \tau = -n^\mu m^\nu \nabla_\mu l_\nu, \tag{5}$$

along with their primed variants, κ', σ', ρ' and τ'. These have GHP type given by

$$\kappa : \{3, 1\}, \quad \sigma : \{3, -1\}, \quad \rho : \{1, 1\}, \quad \tau : \{1, -1\}. \tag{6}$$

The remaining spin coefficients are used to define the GHP derivative operators (that depend on the GHP type of the object on which they are acting),

$$Þ \equiv (l^\alpha \nabla_\alpha - p\epsilon - q\bar{\epsilon}), \qquad Þ' \equiv (n^\alpha \nabla_\alpha + p\epsilon' + q\bar{\epsilon}'),$$
$$ð \equiv (m^\alpha \nabla_\alpha - p\beta + q\bar{\beta}'), \quad ð' \equiv (\bar{m}^\alpha \nabla_\alpha + p\beta' - q\bar{\beta}), \tag{7}$$

where the spin coefficients are given by

$$\beta = \frac{1}{2}(m^\mu \bar{m}^\nu \nabla_\mu m_\nu - m^\mu n^\nu \nabla_\mu l_\nu), \tag{8a}$$
$$\epsilon = \frac{1}{2}(l^\mu \bar{m}^\nu \nabla_\mu m_\nu - l^\mu n^\nu \nabla_\mu l_\nu), \tag{8b}$$

along with their primed variants, β' and ϵ'. The action of a GHP derivative causes the type to change by an amount $\{p, q\} \to \{p + r, q + s\}$ where $\{r, s\}$ for each of the operators is given by

$$Þ : \{1, 1\}, \; Þ' : \{-1, -1\}, \; ð : \{1, -1\}, \; ð' : \{-1, 1\}. \tag{9}$$

The adjoints of the GHP operators are given by

$$Þ^\dagger \equiv -(Þ - \rho - \bar{\rho}), \quad Þ'^\dagger \equiv -(Þ' - \rho' - \bar{\rho}'),$$
$$ð^\dagger \equiv -(ð - \tau - \bar{\tau}'), \quad ð'^\dagger \equiv -(ð' - \tau' - \bar{\tau}), \tag{10}$$

and may also be written concisely as

$$\mathcal{D}^\dagger = -(\psi_2\bar{\psi}_2)^{1/3}\mathcal{D}(\psi_2\bar{\psi}_2)^{-1/3}, \; \mathcal{D} \in \{Þ, Þ', ð, ð'\}. \tag{11}$$

The Weyl scalars are defined to be the tetrad components of the Weyl tensor,

$$\psi_0 = C_{lmlm}, \quad \psi_1 = C_{lnlm}, \quad \psi_2 = C_{lm\bar{m}n},$$
$$\psi_3 = C_{ln\bar{m}n}, \quad \psi_4 = C_{n\bar{m}n\bar{m}}. \tag{12}$$

These have types inherited from the tetrad vectors that appear in their definition,

$$\psi_0 : \{4, 0\}, \quad \psi_1 : \{2, 0\}, \quad \psi_2 : \{0, 0\},$$
$$\psi_3 : \{-2, 0\}, \quad \psi_4 : \{-4, 0\}. \tag{13}$$

Many of the results that follow will be specialised to type-D spacetimes with l^μ and n^μ aligned to the two principal null directions, in which case the Goldberg-Sachs theorem implies that 4 of the of the spin coefficients vanish,

$$\kappa = \kappa' = \sigma = \sigma' = 0, \tag{14}$$

and also that most of the Weyl scalars vanish

$$\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0. \tag{15}$$

The GHP equations give relations between the Weyl scalars and the directional derivatives of the spin coefficients. Similarly the commutator of any pair of directional derivatives can be written in terms of a linear combination of spin coefficients multiplying single directional derivatives. Specialising to type-D spacetimes, the GHP equations are

$$Þ\rho = \rho^2, \qquad Þ\tau = \rho(\tau - \bar{\tau}'),$$
$$ð\rho = \tau(\rho - \bar{\rho}), \quad ð\tau = \tau^2,$$
$$Þ'\rho = \rho\bar{\rho}' - \tau\bar{\tau} \; -\psi_2 + ð'\tau, \tag{16}$$

the Bianchi identities are

$$Þ\psi_2 = 3\rho\psi_2, \quad ð\psi_2 = 3\tau\psi_2, \tag{17}$$

and the commutators of the GHP operators are

$$[Þ, Þ'] = (\bar{\tau} - \tau')ð + (\tau - \bar{\tau}')ð' - p(\psi_2 - \tau\tau') - q(\bar{\psi}_2 - \bar{\tau}\bar{\tau}'), \tag{18a}$$
$$[Þ, ð] = \bar{\rho}ð - \bar{\tau}'Þ + q\bar{\rho}\bar{\tau}', \tag{18b}$$
$$[ð, ð'] = (\bar{\rho}' - \rho')Þ + (\rho - \bar{\rho})Þ' + p(\psi_2 + \rho\rho') - q(\bar{\rho}\bar{\rho}' + \bar{\psi}_2), \tag{18c}$$

along with the conjugate, prime, and prime conjugate of these.

If we further restrict to spacetimes that admit a Killing tensor, $K_{\alpha\beta} = K_{(\alpha\beta)}$, that satisfies $\nabla_{(\alpha}K_{\beta\gamma)} = 0$, the associated symmetries lead to three additional identities relating the spin coefficients,

$$\frac{\rho}{\bar{\rho}} = \frac{\rho'}{\bar{\rho}'} = -\frac{\tau}{\bar{\tau}'} = -\frac{\tau'}{\bar{\tau}} = \frac{\bar{C}^{1/3}}{C^{1/3}}\frac{\psi_2^{1/3}}{\bar{\psi}_2^{1/3}}, \tag{19}$$

for some complex constant C.[1] These identities can be used along with the GHP equations to obtain a complementary set of identities,

$$Þ\tau' = 2\rho\tau' = ð'\rho, \tag{20a}$$

$$Þ'\rho = \rho\rho' + \tau'(\tau - \bar{\tau}') - \frac{1}{2}\psi_2 - \frac{\rho}{2\bar{\rho}}\bar{\psi}_2, \tag{20b}$$

$$ð'\tau = \tau\tau' + \rho(\rho' - \bar{\rho}') + \frac{1}{2}\psi_2 - \frac{\rho}{2\bar{\rho}}\bar{\psi}_2, \tag{20c}$$

along with the conjugate, prime, and prime conjugate of these equations. Introducing the Killing spinor coefficient

$$\zeta = -C^{1/3}\psi_2^{-1/3}, \tag{21}$$

a consequence of these additional relations is that there is an operator

$$\pounds_\xi = -\zeta\big(-\rho'Þ + \rhoÞ' + \tau'ð - \tauð'\big) - \frac{p}{2}\zeta\psi_2 - \frac{q}{2}\bar{\zeta}\bar{\psi}_2, \tag{22}$$

associated with the Killing vector

$$\xi^\alpha = -\zeta(-\rho' l^\alpha + \rho n^\alpha + \tau' m^\alpha - \tau\bar{m}^\alpha). \tag{23}$$

There is a second operator (specialised to the case where C is real, i.e. spacetimes with zero NUT charge)

$$\begin{aligned}\pounds_\eta = &-\tfrac{\zeta}{4}\big[(\zeta - \bar{\zeta})^2(\rho'Þ - \rhoÞ') - (\zeta + \bar{\zeta})^2(\tau'ð - \tauð')\big] \\ &+ p\,{}_\eta h_1 + q\,{}_\eta\bar{h}_1,\end{aligned} \tag{24}$$

where

$$\begin{aligned}{}_\eta h_1 = &\tfrac{1}{8}\zeta(\zeta^2 + \bar{\zeta}^2)\psi_2 - \tfrac{1}{4}\zeta\bar{\zeta}^2\bar{\psi}_2 \\ &+ \tfrac{1}{2}\rho\rho'\zeta^2(\bar{\zeta} - \zeta) + \tfrac{1}{2}\tau\tau'\zeta^2(\bar{\zeta} + \zeta).\end{aligned} \tag{25}$$

This is associated with the second Killing vector

$$\eta^\alpha = -\tfrac{\zeta}{4}\big[(\zeta - \bar{\zeta})^2(\rho' l^\alpha - \rho n^\alpha) - (\zeta + \bar{\zeta})^2(\tau' m^\alpha - \tau\bar{m}^\alpha)\big]. \tag{26}$$

Both $\pounds_\xi$ and $\pounds_\eta$ commute with all of the GHP operators and annihilate all of the spin coefficients and ψ_2.

C. Teukolsky equations

The components of the Faraday tensor may be written in GHP form as

$$\phi_0 \equiv F_{lm} = (Þ - \bar{\rho})A_m - (ð - \bar{\tau}')A_l \equiv \mathcal{T}_0 A, \tag{27a}$$

$$\begin{aligned}\phi_1 &\equiv \tfrac{1}{2}\big(F_{ln} - F_{m\bar{m}}\big) \\ &= \frac{1}{2}\Big[(Þ + \rho - \bar{\rho})A_n - (Þ' + \rho' - \bar{\rho}')A_l \\ &\quad + (ð' + \tau' - \bar{\tau})A_m - (ð + \tau - \bar{\tau}')A_{\bar{m}}\Big] \equiv \mathcal{T}_1 A,\end{aligned} \tag{27b}$$

$$\phi_2 \equiv F_{\bar{m}n} = -(Þ' - \bar{\rho}')A_{\bar{m}} + (ð' - \bar{\tau})A_n \equiv \mathcal{T}_2 A, \tag{27c}$$

These have types inherited from the tetrad vectors that appear in their definition,

$$\phi_0 : \{2, 0\}, \quad \phi_1 : \{0, 0\}, \quad \phi_2 : \{-2, 0\}. \tag{28}$$

The scalars ϕ_0 and ϕ_2 satisfy the Teukolsky equations,

$$\mathcal{O}\phi_0 = (\mathcal{S}_0 J), \qquad \mathcal{O}'\phi_2 = (\mathcal{S}_2 J), \tag{29}$$

where[2]

$$\begin{aligned}\mathcal{O} \equiv &\big(Þ - 2\,s\,\rho - \bar{\rho}\big)\big(Þ' - \rho'\big) - \big(ð - 2\,s\,\tau - \bar{\tau}'\big)\big(ð' - \tau'\big) \\ &+ \tfrac{1}{2}\big[(6s - 2) - 4s^2\big]\psi_2.\end{aligned} \tag{30}$$

and

$$(\mathcal{S}_0 J) = (ð - 2\tau - \bar{\tau}')J_l - (Þ - 2\rho - \bar{\rho})J_m, \tag{31a}$$

$$(\mathcal{S}_2 J) = (ð' - 2\tau' - \bar{\tau})J_n - (Þ' - 2\rho' - \bar{\rho}')J_{\bar{m}}. \tag{31b}$$

It is worth noting that $\mathcal{O}'\phi_2 = \zeta^{-2}\mathcal{O}\zeta^2\phi_2$ and $\mathcal{O}'\phi_0 = \zeta^2\mathcal{O}\zeta^{-2}\phi_0$, and also that the Teukolsky and vector wave operators can be written in the simple forms $\mathcal{O} = -\mathcal{T}_{1-s}\zeta^2\mathcal{T}^\dagger_{1-s}$ and $\mathcal{E} = -(\mathcal{T}_0^\dagger\mathcal{T}_0 + \mathcal{T}_1^\dagger\mathcal{T}_1 + \mathcal{T}_2^\dagger\mathcal{T}_2)$, respectively.

In vacuum Kerr-NUT spacetimes, the Teukolsky equations may be written in manifestly separable form by rewriting them in terms of the commuting symmetry operators [17]

$$\mathcal{R} \equiv \zeta\bar{\zeta}(Þ - \rho - \bar{\rho})(Þ' - 2b\rho') + \frac{2b - 1}{2}(\zeta + \bar{\zeta})\pounds_\xi, \tag{32}$$

and

$$\mathcal{S} \equiv \zeta\bar{\zeta}(ð - \tau - \bar{\tau}')(ð' - 2s\tau') + \frac{2s - 1}{2}(\zeta - \bar{\zeta})\pounds_\xi. \tag{33}$$

Then, Teukolsky operator is given by

$$\zeta\bar{\zeta}\mathcal{O} = \mathcal{R} - \mathcal{S}. \tag{34}$$

The new operators satisfy the commutation relations $\big[\mathcal{R}, \mathcal{S}\big] = 0$. When written as a coordinate expression in Boyer-Lindquist coordinates in Kerr spacetime the operators $\mathcal{R}$ and $\mathcal{S}$ reduce to the radial Teukolsky and spin-weighted spheroidal operators (with a common eigenvalue), while $\pounds_\xi$ reduces to ∂_t.

D. Teukolsky-Starobinsky identities

In the homogeneous case, $(\mathcal{S}_0 J) = 0 = (\mathcal{S}_2 J)$, the Teukolsky-Starobinsky identities that relate ϕ_0 to ϕ_2 are given in GHP form by

$$Þ^2\zeta^2\phi_2 = ð'^2\zeta^2\phi_0, \tag{35a}$$

$$Þ'^2\zeta^2\phi_0 = ð^2\zeta^2\phi_2, \tag{35b}$$

$$[Þ'ð' + \bar{\tau}Þ']\zeta^2\phi_0 = [Þð + \bar{\tau}'Þ]\zeta^2\phi_2. \tag{35c}$$

[1] In the case of Kerr spacetime $C = M$, the mass of the spacetime.

[2] Some authors (e.g. [4]) define $\mathcal{O}$ to be the operator with $s = +1$. Then, the operator for the negative s fields is its adjoint.

From these, we can also derive fourth-order Teukolsky-Starobinsky identities,

$$Þ^2\bar{\zeta}^2Þ'^2\zeta^2\phi_0 = ð'^2\bar{\zeta}^2ð^2\zeta^2\phi_0, \quad (36a)$$
$$Þ'^2\bar{\zeta}^2Þ^2\zeta^2\phi_2 = ð^2\bar{\zeta}^2ð'^2\zeta^2\phi_2. \quad (36b)$$

This latter form can be rewritten in terms of the symmetry operators,

$$Þ^2\bar{\zeta}^2Þ'^2\zeta^2\phi_0 = \big[\mathscr{R}^2 + \mathscr{L}_\eta\mathscr{L}_\xi\big]\phi_0, \quad (37a)$$
$$ð'^2\bar{\zeta}^2ð^2\zeta^2\phi_0 = \big[\mathscr{S}^2 + \mathscr{L}_\eta\mathscr{L}_\xi\big]\phi_0, \quad (37b)$$
$$Þ'^2\bar{\zeta}^2Þ^2\zeta^2\phi_2 = \big[\mathscr{R}'^2 + \mathscr{L}_\eta\mathscr{L}_\xi\big]\phi_2, \quad (37c)$$
$$ð^2\bar{\zeta}^2ð'^2\zeta^2\phi_2 = \big[\mathscr{S}'^2 + \mathscr{L}_\eta\mathscr{L}_\xi\big]\phi_2. \quad (37d)$$

E. Reconstruction of a vector potential in radiation gauge

In the ingoing radiation gauge (IRG), the vector potential may be reconstructed by applying a first-order differential operator to a type $\{-2,0\}$ scalar (i.e. the same type as ϕ_2), Φ^{IRG}, called the Hertz potential. In terms of the Hertz potential, the IRG vector potential is given explicitly by

$$A^{\text{IRG}}_\alpha = \Re\big[m_\alpha(Þ+\rho)\Phi^{\text{IRG}} - l_\alpha(ð+\tau)\Phi^{\text{IRG}}\big] \equiv \Re[(\mathcal{S}_0^\dagger\Phi^{\text{IRG}})_\alpha]. \quad (38)$$

This IRG vector potential manifestly satisfies the gauge condition $A_\alpha l^\alpha = 0$. Computing the Maxwell scalars from the IRG vector potential, we find

$$\phi_0 = Þ^2\overline{\Phi^{\text{IRG}}}, \quad (39a)$$
$$\phi_2 = ð'^2\overline{\Phi^{\text{IRG}}} - \mathcal{O}\Phi^{\text{IRG}}. \quad (39b)$$

Acting on the Maxwell scalars with the Teukolsky operator and commuting operators, we find

$$\mathcal{O}\phi_0 = \Big(Þ-\rho-\bar{\rho}\Big)^2\Big(\overline{\mathcal{O}\Phi^{\text{IRG}}}\Big), \quad (40a)$$
$$\mathcal{O}'\phi_2 = \Big(ð'-\tau'-\bar{\tau}\Big)^2\Big(\overline{\mathcal{O}\Phi^{\text{IRG}}}\Big) - \mathcal{O}'\mathcal{O}\Phi^{\text{IRG}}. \quad (40b)$$

Thus, the Maxwell scalars satisfy the homogeneous Teukolsky equation if $\mathcal{O}\Phi^{\text{IRG}} = 0 = \overline{\mathcal{O}\Phi^{\text{IRG}}}$, i.e. Φ^{IRG} is a homogeneous solution of the equation satisfied by $\zeta^2\phi_2$ (equivalently, the adjoint of the equation satisfied by ϕ_0).

Similarly, in the outgoing radiation gauge (ORG) the vector potential is given by the prime of the IRG vector potential,

$$A^{\text{ORG}}_\mu = \Re\big[\bar{m}_\mu(Þ'-\mu)\Phi^{\text{ORG}} - n_\mu(ð'-\varpi)\Phi^{\text{ORG}}\big] \equiv \Re[(\mathcal{S}_2^\dagger\Phi^{\text{ORG}})_\alpha], \quad (41)$$

where the ORG Hertz potential, Φ^{ORG}, is of type $\{2,0\}$ (i.e. the same as ϕ_0). This ORG vector potential manifestly satisfies the gauge condition $A_\alpha n^\alpha = 0$. Computing the Maxwell scalars from the ORG vector potential, we find

$$\phi_0 = -ð^2\overline{\Phi^{\text{ORG}}} + \mathcal{O}'\Phi^{\text{ORG}}, \quad (42a)$$
$$\phi_2 = -Þ'^2\overline{\Phi^{\text{ORG}}}. \quad (42b)$$

Acting on these with the Teukolsky operator and commuting operators, we find

$$\mathcal{O}\phi_0 = -\Big(ð-\tau-\bar{\tau}'\Big)^2\Big(\overline{\mathcal{O}'\Phi^{\text{ORG}}}\Big) + \mathcal{O}\mathcal{O}'\Phi^{\text{ORG}}, \quad (43a)$$
$$\mathcal{O}'\phi_2 = -\Big(Þ'-\rho'-\bar{\rho}'\Big)^2\Big(\overline{\mathcal{O}'\Phi^{\text{ORG}}}\Big). \quad (43b)$$

Thus, the Maxwell scalars also satisfy the appropriate Teukolsky equation if $\mathcal{O}'\Psi^{\text{ORG}} = 0 = \overline{\mathcal{O}'\Psi^{\text{ORG}}}$, i.e. $\zeta^{-2}\Psi^{\text{ORG}}$ is a homogenous solution of the equation satisfied by ϕ_0 (equivalently, the adjoint of the equation satisfied by $\zeta^2\phi_2$).

The fact that these potentials are solutions of the homogeneous Maxwell equations were succinctly summarised by Wald [4] using the method of adjoints: since the operators satisfy the identity $\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$, by taking the adjoint and using the fact that $\mathcal{E}$ is self-adjoint we find that $\mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger$ so we have a valid solution provided our Hertz potential satisfies the (adjoint) Teukolsky equation.

III. LORENZ GAUGE HERTZ POTENTIALS

In Lorenz gauge the vector potential satisfies the vector wave equation

$$(\mathcal{L}A)^\alpha \equiv \Box A^\alpha - R^\alpha{}_\beta A^\beta = J^\alpha, \quad (44)$$

along with the Lorenz gauge condition

$$\nabla^\alpha A_\alpha = 0. \quad (45)$$

In a vacuum type-D spacetime, these may be written in GHP form as

$$(\mathcal{L}A)_l = 2\big[(ð-\bar{\tau}')(ð'-\tau') - (Þ-\bar{\rho})(Þ'-\rho') + \bar{\rho}\bar{\rho}'\big]A_l + 2\rho\bar{\rho}A_n + 2\big[\bar{\rho}ð' - \bar{\tau}Þ\big]A_m + 2\big[\rhoð - \tauÞ\big]A_{\bar{m}}, \quad (46a)$$
$$(\mathcal{L}A)_m = 2\big[(ð-\bar{\tau}')(ð'-\tau') - (Þ-\bar{\rho})(Þ'-\rho') - \bar{\tau}\bar{\tau}'\big]A_m - 2\tau\bar{\tau}'A_{\bar{m}} + 2\big[\bar{\rho}'ð - \bar{\tau}'Þ'\big]A_l + 2\big[\rhoð - \tauÞ\big]A_n, \quad (46b)$$
$$(\mathcal{L}A)_n = \overline{(\mathcal{L}A)'_l}, \quad (46c)$$
$$(\mathcal{L}A)_{\bar{m}} = \overline{(\mathcal{L}A)_m} \quad (46d)$$

and

$$\nabla^\alpha A_\alpha = (ð'-\tau'-\bar{\tau})A_m + (ð-\tau-\bar{\tau}'A_{\bar{m}} - (Þ'-\rho'-\bar{\rho}')A_l - (Þ-\rho-\bar{\rho})A_n = 0. \quad (47)$$

A. Hertz potential derived from a two-form

Start with a real tensor with the same symmetries as the Faraday tensor (i.e. a two-form): $H_{\alpha\beta} = H_{[\alpha\beta]}$. This can be decomposed onto a null tetrad,

$$H_{\alpha\beta} = 2\Big[(\Phi_1^{\mathcal{L}1} + \bar{\Phi}_1^{\mathcal{L}1}) n_{[\alpha} l_{\beta]} + (\Phi_1^{\mathcal{L}1} - \bar{\Phi}_1^{\mathcal{L}1}) m_{[\alpha}\bar{m}_{\beta]} + \Phi_0^{\mathcal{L}1}\bar{m}_{[\alpha}n_{\beta]} + \bar{\Phi}_0^{\mathcal{L}1} m_{[\alpha}n_{\beta]} + \Phi_2^{\mathcal{L}1} l_{[\alpha}m_{\beta]} + \bar{\Phi}_2^{\mathcal{L}1} l_{[\alpha}\bar{m}_{\beta]}\Big], \tag{48}$$

where the GHP type of the complex scalars is the same as that of the Maxwell scalars:

$$\begin{aligned} &\Phi_0^{\mathcal{L}1} : \{2,0\}, \quad \Phi_1^{\mathcal{L}1} : \{0,0\}, \quad \Phi_2^{\mathcal{L}1} : \{-2,0\}, \\ &\bar{\Phi}_0^{\mathcal{L}1} : \{0,2\}, \quad \bar{\Phi}_1^{\mathcal{L}1} : \{0,0\}, \quad \bar{\Phi}_2^{\mathcal{L}1} : \{0,-2\}. \end{aligned} \tag{49}$$

We will allow this tensor to only have maximum spin-weight components, i.e. $\Phi_1^{\mathcal{L}1} = 0 = \bar{\Phi}_1^{\mathcal{L}1}$ and use it to construct a complex anti-self-dual tensor

$$\mathcal{H}_{\alpha\beta} = \frac{1}{2}(H_{\alpha\beta} - \frac{i}{2}{}^*H_{\alpha\beta}), \tag{50}$$

where ${}^*H_{\alpha\beta} = \epsilon_{\alpha\beta}{}^{\gamma\delta}H_{\gamma\delta}$ is the Hodge dual of $H_{\alpha\beta}$ and the anti-self-dual property means that ${}^*\mathcal{H}_{\alpha\beta} = -i\mathcal{H}_{\alpha\beta}$. The anti-self-dual tensor depends only on $\Phi_0^{\mathcal{L}1}$ and $\Phi_2^{\mathcal{L}1}$ and its conjugate depends only on $\bar{\Phi}_0^{\mathcal{L}1}$ and $\bar{\Phi}_2^{\mathcal{L}1}$,

$$\mathcal{H}_{\alpha\beta} = 2\Big[\Phi_0^{\mathcal{L}1}\bar{m}_{[\alpha}n_{\beta]} + \Phi_2^{\mathcal{L}1} l_{[\alpha}m_{\beta]}\Big], \tag{51a}$$

$$\bar{\mathcal{H}}_{\alpha\beta} = 2\Big[\bar{\Phi}_0^{\mathcal{L}1} m_{[\alpha}n_{\beta]} + \bar{\Phi}_2^{\mathcal{L}1} l_{[\alpha}\bar{m}_{\beta]}\Big]. \tag{51b}$$

Now construct a complex vector[3] by taking the divergence of $\zeta\mathcal{H}_{ab}$

$$A_\alpha^{\mathcal{L}1} = \nabla^\beta(\zeta\mathcal{H}_{\beta\alpha}). \tag{52}$$

This vector has tetrad components

$$A_l^{\mathcal{L}1} = -\zeta(\eth' - 2\tau')\Phi_0^{\mathcal{L}1}, \tag{53a}$$

$$A_n^{\mathcal{L}1} = \zeta(\eth - 2\tau)\Phi_2^{\mathcal{L}1}, \tag{53b}$$

$$A_m^{\mathcal{L}1} = -\zeta(\text{Þ}' - 2\rho')\Phi_0^{\mathcal{L}1}, \tag{53c}$$

$$A_{\bar{m}}^{\mathcal{L}1} = \zeta(\text{Þ} - 2\rho)\Phi_2^{\mathcal{L}1}. \tag{53d}$$

It is straightforward to check that this vector satisfies the Lorenz gauge condition as a consequence of the GHP equations and the GHP commutators without assuming anything further about the scalars.

If we now compute the Maxwell scalars from this vector potential, we find

$$\phi_0^{\mathcal{L}1} = (-\zeta\mathcal{O} + \pounds_\xi)\Phi_0^{\mathcal{L}1}, \tag{54a}$$

$$\phi_2^{\mathcal{L}1} = (-\zeta\mathcal{O}' - \pounds_\xi)\Phi_2^{\mathcal{L}1}. \tag{54b}$$

Acting on both sides with the appropriate Teukolsky operator and commuting the operators on the right hand side (recalling that $\pounds_\xi$ commutes with everything), then we find

$$\mathcal{O}\phi_0^{\mathcal{L}1} = (-\mathcal{O}\zeta + \pounds_\xi)\mathcal{O}\Phi_0^{\mathcal{L}1}, \tag{55a}$$

$$\mathcal{O}'\phi_2^{\mathcal{L}1} = (-\mathcal{O}'\zeta - \pounds_\xi)\mathcal{O}'\Phi_2^{\mathcal{L}1}, \tag{55b}$$

so ϕ_0 will satisfy the homogeneous Teukolsky equation provided $\mathcal{O}\Phi_0^{\mathcal{L}1}$ is in the kernel of $(-\mathcal{O}\zeta + \pounds_\xi)$, and similarly for ϕ_2. If we make the particular choice $\mathcal{O}\Phi_0^{\mathcal{L}1} = 0 = \mathcal{O}'\Phi_2^{\mathcal{L}1}$ then the the relationship between our potentials and the Maxwell scalars simplifies to

$$\phi_0^{\mathcal{L}1} = \pounds_\xi\Phi_0^{\mathcal{L}1}, \tag{56a}$$

$$\phi_2^{\mathcal{L}1} = -\pounds_\xi\Phi_2^{\mathcal{L}1}. \tag{56b}$$

Since $\pounds_\xi$ is a commuting operator, this implies that the scalars must also satisfy a set of Teukolsky-Starobinsky-like identities (up to terms annihilated by $\pounds_\xi$),

$$-\text{Þ}^2\zeta^2\Phi_2 = \eth'^2\zeta^2\Phi_0, \tag{57a}$$

$$\text{Þ}'^2\zeta^2\Phi_0 = -\eth^2\zeta^2\Phi_2, \tag{57b}$$

$$[\text{Þ}'\eth' + \bar{\tau}\text{Þ}']\zeta^2\Phi_0 = -[\text{Þ}\eth + \bar{\tau}'\text{Þ}]\zeta^2\Phi_2. \tag{57c}$$

This also implies that our complex Hertz potential does not contribute to the complex conjugate of the Maxwell scalars, since

$$\overline{\phi_0^{\mathcal{L}1}} = \zeta^{-1}(\text{Þ}^2\zeta^2\Phi_2^{\mathcal{L}1} + \eth'^2\zeta^2\Phi_0^{\mathcal{L}1}) = 0, \tag{58a}$$

$$\overline{\phi_2^{\mathcal{L}1}} = \zeta^{-1}(\text{Þ}'^2\zeta^2\Phi_0^{\mathcal{L}1} + \eth^2\zeta^2\Phi_2^{\mathcal{L}1}) = 0. \tag{58b}$$

Instead, it is the conjugate potential that produces the expected conjugate Maxwell scalars.

Finally, it is straightforward to verify that this vector potential satisfies the homogeneous Lorenz gauge field equations by direct substitution in combination with the homogeneous Teukolsky equation, the Teukolsky-Starobinsky identities and the GHP equations and commutators. An explicit derivation of this final result is given in Appendix A.

When decomposed into modes in Kerr spacetime, the potential $A_\alpha^{\mathcal{L}1}$ is the same as the one identified by Dolan [12], and thus so far we have merely reproduced his derivation in GHP form without relying on a mode decomposition.

B. A second Lorenz-gauge Hertz potential

We can arrive at the vector potential $A_\alpha^{\mathcal{L}1}$ by a different means. If we look for a vector potential that is constructed from at most a first-order GHP operator acting on spin-weight ±1 scalars (i.e. scalars of type $\{2,0\}$, $\{0,2\}$, $\{-2,0\}$ and $\{0,-2\}$) then the most general pos-

[3] It is, of course, possible to obtain a real vector from this complex potential and its complex conjugate.

sibility is the complex potential

$$A_l = (c_{l_1}\eth' + c_{l_2}\tau' + c_{l_3}\bar{\tau})\Phi_0^{\mathcal{L}1}, \tag{59a}$$

$$A_n = (c_{n_1}\eth + c_{n_2}\tau + c_{n_3}\bar{\tau}')\Phi_2^{\mathcal{L}1}, \tag{59b}$$

$$A_m = (c_{m_1}\text{Þ}' + c_{m_2}\rho' + c_{m_3}\bar{\rho}')\Phi_0^{\mathcal{L}1}, \tag{59c}$$

$$A_{\bar{m}} = (c_{\bar{m}_1}\text{Þ} + c_{\bar{m}_2}\rho + c_{\bar{m}_3}\bar{\rho})\Phi_2^{\mathcal{L}1}, \tag{59d}$$

and its complex conjugate. The coefficients here must be of GHP type $\{0,0\}$ (in particular, the may be constructed from numeric constants and functions of ζ and $\bar{\zeta}$). If we restrict to the case that the coefficients are linear functions of ζ and $\bar{\zeta}$ with arbitrary numeric constants, we find that the only possibility that satisfies the gauge condition and field equations is $A_a^{\mathcal{L}1}$.[4]

This approach also allows us to identify a second Hertz potential by considering alternative forms for the coefficients. In particular, if we consider coefficients that are polynomials in ζ and $\bar{\zeta}$ we find a second complex potential,

$$A_l^{\mathcal{L}2} = \left[\tfrac{1}{2}\zeta(\zeta-\bar{\zeta})\eth' + \zeta\bar{\zeta}\tau'\right]\Phi_0^{\mathcal{L}2}, \tag{60a}$$

$$A_n^{\mathcal{L}2} = \left[\tfrac{1}{2}\zeta(\zeta-\bar{\zeta})\eth + \zeta\bar{\zeta}\tau\right]\Phi_2^{\mathcal{L}2}, \tag{60b}$$

$$A_m^{\mathcal{L}2} = \left[\tfrac{1}{2}\zeta(\zeta+\bar{\zeta})\text{Þ}' - \zeta\bar{\zeta}\rho'\right]\Phi_0^{\mathcal{L}2}, \tag{60c}$$

$$A_{\bar{m}}^{\mathcal{L}2} = \left[\tfrac{1}{2}\zeta(\zeta+\bar{\zeta})\text{Þ} - \zeta\bar{\zeta}\rho\right]\Phi_2^{\mathcal{L}2}, \tag{60d}$$

that satisfies the Lorenz gauge condition and homogeneous Lorenz gauge equations provided the scalars satisfy the homogeneous Teukolsky equation and the Teukolsky-Starobinsky type identities given in Eq. (57). This second vector potential can also be written in tensor notation as

$$A_\alpha^{\mathcal{L}2} = h^{\beta\gamma}\nabla_\beta(\zeta\mathcal{H}_{\alpha\gamma}), \tag{61}$$

where

$$h_{\alpha\beta} = (\zeta+\bar{\zeta})n_{[\alpha}l_{\beta]} - (\zeta-\bar{\zeta})\bar{m}_{[\alpha}m_{\beta]} \tag{62}$$

is the conformal Killing-Yano tensor.

Computing the Maxwell scalars, we find

$$\phi_0^{\mathcal{L}2} = \frac{1}{2}\left(\zeta^2\mathcal{O} + \mathscr{R} + \mathcal{S}\right)\Phi_0^{\mathcal{L}2}, \tag{63a}$$

$$\phi_2^{\mathcal{L}2} = -\frac{1}{2}\left(\zeta^2\mathcal{O}' + \mathscr{R}' + \mathcal{S}'\right)\Phi_2^{\mathcal{L}2}. \tag{63b}$$

Acting on both sides with the appropriate Teukolsky operator and commuting the operators on the right hand side, then we find

$$\mathcal{O}\phi_0^{\mathcal{L}2} = \frac{1}{2}\left[\mathcal{O}\zeta^2 + (\zeta\bar{\zeta})^{-1}(\mathscr{R}+\mathcal{S})(\zeta\bar{\zeta})\right]\mathcal{O}\Phi_0^{\mathcal{L}1}, \tag{64a}$$

$$\mathcal{O}'\phi_2^{\mathcal{L}2} = -\frac{1}{2}\left[\mathcal{O}'\zeta^2 + (\zeta\bar{\zeta})^{-1}(\mathscr{R}'+\mathcal{S}')(\zeta\bar{\zeta})\right]\mathcal{O}'\Phi_2^{\mathcal{L}1}, \tag{64b}$$

[4] Note that the gauge condition is satisfied unconditionally, but the field equations are only satisfied if the scalars satisfy the Teukolsky equation and Teukolsky-Starobinsky-like identities.

so ϕ_0 will satisfy the homogeneous Teukolsky equation provided $\mathcal{O}\Phi_0^{\mathcal{L}2}$ is in the kernel of $\frac{1}{2}\left[\mathcal{O}\zeta^2 + (\zeta\bar{\zeta})^{-1}(\mathscr{R}+\mathcal{S})\zeta\bar{\zeta}\right]$, and similarly for ϕ_2. If we make the particular choice $\mathcal{O}\Phi_0^{\mathcal{L}2} = 0 = \mathcal{O}'\Phi_2^{\mathcal{L}2}$ then the the relationship between our potentials and the Maxwell scalars simplifies to

$$\phi_0^{\mathcal{L}2} = \frac{1}{2}(\mathscr{R}+\mathcal{S})\Phi_0^{\mathcal{L}2}, \tag{65a}$$

$$\phi_2^{\mathcal{L}2} = -\frac{1}{2}(\mathscr{R}'+\mathcal{S}')\Phi_2^{\mathcal{L}2}. \tag{65b}$$

We recognise these as the radial and angular Teukolsky operators, meaning that when decomposed into modes in Kerr spacetime the potentials are simply the Maxwell scalars multiplied by the separation constant (eigenvalue).

It is straightforward to verify that this vector potential satisfies the Lorenz gauge condition and the homogeneous Lorenz gauge field equations by direct substitution in combination with the homogeneous Teukolsky equation, the Teukolsky-Starobinsky identities and the GHP equations and commutators.

C. Higher order Lorenz gauge potentials

Starting from the two potentials, $A_\alpha^{\mathcal{L}1}$ and $A_\alpha^{\mathcal{L}2}$, we can use the symmetry operators $\pounds_\xi$, $\pounds_\eta$, $\mathscr{R}$ and $\mathcal{S}$ to construct further, higher order potentials. For example, the potential

$$\begin{aligned}A_l^{\mathcal{L}3} &= -\zeta(\eth' - 2\tau')\pounds_\eta\Phi_0^{\mathcal{L}3}\\ &\quad + \left[\tfrac{1}{2}\zeta(\zeta-\bar{\zeta})\eth' + \zeta\bar{\zeta}\tau'\right]\mathscr{R}\Phi_0^{\mathcal{L}3},\end{aligned} \tag{66a}$$

$$\begin{aligned}A_n^{\mathcal{L}3} &= \zeta(\eth - 2\tau)\pounds_\eta\Phi_2^{\mathcal{L}3}\\ &\quad + \left[\tfrac{1}{2}\zeta(\zeta-\bar{\zeta})\eth + \zeta\bar{\zeta}\tau\right]\mathscr{R}'\Phi_2^{\mathcal{L}3},\end{aligned} \tag{66b}$$

$$\begin{aligned}A_m^{\mathcal{L}3} &= -\zeta(\text{Þ}' - 2\rho')\pounds_\eta\Phi_0^{\mathcal{L}3}\\ &\quad + \left[\tfrac{1}{2}\zeta(\zeta+\bar{\zeta})\text{Þ}' + \zeta\bar{\zeta}\rho'\right]\mathscr{R}\Phi_0^{\mathcal{L}3},\end{aligned} \tag{66c}$$

$$\begin{aligned}A_{\bar{m}}^{\mathcal{L}3} &= \zeta(\text{Þ} - 2\rho)\pounds_\eta\Phi_2^{\mathcal{L}3}\\ &\quad + \left[\tfrac{1}{2}\zeta(\zeta+\bar{\zeta})\text{Þ} + \zeta\bar{\zeta}\rho\right]\mathscr{R}'\Phi_2^{\mathcal{L}3},\end{aligned} \tag{66d}$$

is also a homogeneous solution of the Lorenz gauge equations and satisfies the Lorenz gauge condition. Note that this is constructed from a third-order operator acting on spin-weight ± 1 scalars, so we are free to add terms involving a first order operator acting on the Teukolsky equation while still satisfying the Lorenz gauge equations.

Computing the corresponding Maxwell scalars we find[5]

$$\phi_0^{\pounds 3} = \frac{1}{2}\zeta\big[(\zeta-\bar\zeta)\mathcal{O}\mathfrak{R} - \zeta\mathcal{O}\pounds_\eta\big]\Phi_0^{\pounds 3} + \text{Þ}^2\bar\zeta^2\text{Þ}'^2\zeta^2\Phi_0^{\pounds 3}, \tag{68a}$$

$$\phi_2^{\pounds 3} = -\frac{1}{2}\zeta\big[(\zeta-\bar\zeta)\mathcal{O}'\mathfrak{R}' - \zeta\mathcal{O}'\pounds_\eta\big]\Phi_2^{\pounds 3} - \text{Þ}'^2\bar\zeta^2\text{Þ}^2\zeta^2\Phi_2^{\pounds 3}. \tag{68b}$$

In the homogeneous case we recognise these as the fourth-order Teukolsky-Starobinsky operators, meaning that in modes the potentials are simply the Maxwell scalars divided by the Teukolsky-Starobinsky constant.

IV. MODE DECOMPOSED EQUATIONS IN KERR SPACETIME

The metric of Kerr spacetime in Boyer-Lindquist coordinates (t, r, θ, φ) is given by

$$ds^2 = -\left[1 - \frac{2Mr}{\Sigma}\right]dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dt\,d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left[\Delta + \frac{2Mr(r^2+a^2)}{\Sigma}\right]\sin^2\theta d\varphi^2, \tag{69}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$.

A. Null tetrads in Kerr spacetime

A null tetrad proposed by Kinnersley [20] is a common choice when dealing with perturbations of Kerr spacetime. Indeed, it formed a crucial part of Teukolsky's separability result for perturbations of the Weyl tensor [1]. However, the Kinnersley tetrad has two unfortunate features that make it less than ideal for elucidating the symmetric structure of perturbations of Kerr spacetime: (i) it violates the $\{t, \varphi\} \to \{-t, -\varphi\}$ symmetry; and (ii) it destroys a symmetry in $\{r, \theta\}$. An alternative tetrad proposed by Carter [21] does not suffer from either of these deficiencies. In terms of Boyer-Lindquist coordinates, Carter's tetrad has components[6]

$$l^\alpha = \frac{1}{\sqrt{2\Delta\Sigma}}\Big[r^2 + a^2, \Delta, 0, a\Big], \tag{70a}$$

$$n^\alpha = \frac{1}{\sqrt{2\Delta\Sigma}}\Big[r^2 + a^2, -\Delta, 0, a\Big], \tag{70b}$$

$$m^\alpha = \frac{1}{\sqrt{2\Sigma}}\Big[ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\Big], \tag{70c}$$

$$\bar m^\alpha = \frac{1}{\sqrt{2\Sigma}}\Big[-ia\sin\theta, 0, 1, -\frac{i}{\sin\theta}\Big]. \tag{70d}$$

Under $\{t, \varphi\} \to \{-t, -\varphi\}$ the Carter tetrad transforms as $l \leftrightarrow -n$, $m \leftrightarrow \bar m$ (note the minus sign means that this does not correspond to the GHP prime operation).

For the Carter tetrad the non-zero spin coefficients have particularly symmetric form

$$\rho = -\rho' = -\frac{1}{\zeta}\sqrt{\frac{\Delta}{2\Sigma}}, \tag{71a}$$

$$\tau = \tau' = -\frac{ia\sin\theta}{\zeta\sqrt{2\Sigma}}, \tag{71b}$$

$$\beta = -\beta' = -\frac{i}{\zeta}\frac{a + ir\cos\theta}{2\sin\theta\sqrt{2\Sigma}}, \tag{71c}$$

where $\zeta = r - ia\cos\theta$. The commuting GHP operators have the same form in both the Carter and Kinnersley tetrads, and are given by

$$\pounds_\xi = \partial_t, \tag{72a}$$

$$\pounds_\eta = a^2\partial_t + a\partial_\varphi. \tag{72b}$$

B. Mode decomposed Teukolsky equation

When working with Carter's tetrad, the Teukolsky equation separates in the form[7] $\Psi = \Delta^{s/2}\zeta^2 R(r)S(\theta)e^{i(m\varphi-\omega t)}$ with $R(r)$ and $S(\theta)$ satisfying the standard Teukolsky radial and spin-

[5] To derive these, it is useful to use the identities

$$\bar\zeta^{-n}\text{Þ}^n\bar\zeta^{2n}\text{Þ}^n\bar\zeta^n = \text{Þ}^{2n}\bar\zeta^{2n},$$
$$\bar\zeta^{-n}\text{Þ}'^n\bar\zeta^{2n}\text{Þ}'^n\bar\zeta^n = \text{Þ}'^{2n}\bar\zeta^{2n}.$$

[6] Carter's original tetrad had interchanged $l^\mu \leftrightarrow n^\mu$ and $m^\mu \leftrightarrow \bar m^\mu$. We deviate from that here and keep with the convention of having $\boldsymbol{l}$ point outwards. Using this convention, Carter's canonical tetrad corresponds to a simple rescaling of Kinnersley's tetrad (denoted with a K subscript): $\boldsymbol{l} = \sqrt{\Delta/2\Sigma}\,\boldsymbol{l}_{\rm K}$, $\boldsymbol{n} = \sqrt{2\Sigma/\Delta}\,\boldsymbol{n}_{\rm K}$, $\boldsymbol{m} = \bar\zeta/\sqrt{\Sigma}\boldsymbol{m}_{\rm K}$, $\bar{\boldsymbol{m}} = \zeta/\sqrt{\Sigma}\bar{\boldsymbol{m}}_{\rm K}$.

[7] The factor of $\Delta^{s/2}$ is only to make $R(r)$ consistent with Teukolsky's function, it is not required to obtain a separable equation.

weighted spheroidal harmonic equations,

$$\frac{d}{d\chi}\left[(1-\chi^2)\frac{dS}{d\chi}\right] + \left[a^2\omega^2\chi^2 - \frac{(m+s\chi)^2}{1-\chi^2} - 2as\omega\chi + s + A\right]S = 0, \tag{73a}$$

$$\Delta^{-s}\frac{d}{dr}\left[\Delta^{s+1}\frac{dR}{dr}\right] + \left[\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - {}_s\lambda_{\ell m\omega}\right]R = 0, \tag{73b}$$

with $A \equiv {}_s\lambda_{\ell m\omega} + 2am\omega - a^2\omega^2$ and $K \equiv (r^2+a^2)\omega - am$.

C. Lorenz gauge Hertz potentials

The mode decomposed version of our Lorenz-gauge Hertz potentials is given by

$$\Phi_0^{\mathcal{L}1} = -\frac{\phi_0}{i\omega}, \quad \Phi_0^{\mathcal{L}2} = \frac{2\phi_0}{{}_{|s|}\lambda_{\ell m\omega}}, \quad \Phi_0^{\mathcal{L}3} = \frac{\phi_0}{(\mathcal{C}_{\ell m\omega})^2},$$
$$\Phi_2^{\mathcal{L}1} = \frac{\phi_2}{i\omega}, \quad \Phi_2^{\mathcal{L}2} = -\frac{2\phi_2}{{}_{|s|}\lambda_{\ell m\omega}}, \quad \Phi_2^{\mathcal{L}3} = -\frac{\phi_2}{(\mathcal{C}_{\ell m\omega})^2}. \tag{74}$$

where $(\mathcal{C}_{\ell m\omega})^2 = {}_{|s|}\lambda^2_{\ell m\omega} + 4\omega a(m - a\omega)$ is the (squared) Teukolsky-Starobinsky constant and ${}_{|s|}\lambda_{\ell m\omega} = {}_s\lambda_{\ell m\omega} + s^2 + s$ is Chandrasekhar's separation constant that is independent of the sign of s.

V. CONCLUSIONS

In this paper we have reviewed and extended recent results for a Hertz potential for the Lorenz gauge Maxwell (vector wave) equation. Dolan [12] previously found a mode version of the potential $\mathcal{L}1$, but our derivation is the first time it has been given without relying on a mode decomposition. The other two potentials, $\mathcal{L}2$ and $\mathcal{L}3$ appear to be new, and have not previously been given in the literature.

There are several important directions for future study. Our work so far has been restricted to the homogenous case (as was all previous work on Lorenz gauge Hertz potentials). It would be desirable to extend this to allow for sourced perturbations. It is quite likely that this would be possible with our method by restoring terms involving the Teukolsky equation that we have eliminated in the process of our derivation. It may also be necessary to apply the "corrector tensor" method of Green, Hollands and Zimmerman [22] in order to obtain the most general sourced perturbations. We leave this for future work.

A more challenging goal is to extend our work to the gravitational case, where we are interested in solving the Lorenz gauge equation for metric perturbations. Many of the identities (including a Teukolsky equation and Teukolsky-Starobinsky identities) we have used in the electromagnetic case have analogues in the gravitational case. One possible complication is that the Teukolsky-Starobinsky identities for the perturbed Weyl scalars mix not only the scalars ψ_0 and ψ_4, but also their complex conjugates. We would therefore not expect the simplification we found using an anti-self-dual bivector to apply in the gravitational case. Nonetheless, there is reason for optimism that the gravitational case is solvable given the otherwise strong similarities to the electromagnetic case.

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Appendix A: Lorenz gauge field equations with Lorenz gauge Hertz potential

Here we give an explicit derivation showing that our first potential, $A_\alpha^{\mathcal{L}1}$, satisfies the homogeneous Lorenz gauge field equations (for brevity, we omit similar derivation for the other potentials). Substituting the expression for our potential in terms of $\mathcal{H}_{\alpha\beta}$, Eq. (52), into the vacuum vector wave equation, Eq. (44) with $R_{\alpha\beta} = 0$, and commuting the wave operator with the covariant derivative, we find that $\mathcal{H}_{ab}$ must satisfy

$$\nabla^\nu\left[\Box\zeta\mathcal{H}_{\mu\nu} + 2R_\mu{}^\alpha{}_\nu{}^\beta\zeta\mathcal{H}_{\alpha\beta}\right] + \nabla^\gamma R_{\mu\alpha\beta\gamma}\zeta\mathcal{H}^{\alpha\beta} = 0. \tag{A1}$$

In type D spacetimes the final term vanishes and the condition reduces to the requirement that $H_{\mu\nu}$ satisfies the divergence a the tensor wave equation. The tetrad components of $A_{\mu\nu} \equiv \Box\zeta\mathcal{H}_{\mu\nu} + 2R_\mu{}^\alpha{}_\nu{}^\beta\zeta\mathcal{H}_{\alpha\beta}$ are given in GHP form by

$$A_{ln} - A_{m\bar{m}} = 4(\varpi Þ' - 4\mu ð')\zeta\Phi_0^{\mathcal{L}1} - 4(\tau Þ - 4\rho ð)\zeta\Phi_2^{\mathcal{L}1}, \tag{A2a}$$

$$A_{lm} = \Big[2ð'ð - 2Þ'Þ - 2\bar{\mu}Þ + 2\rho Þ' - 2\tau ð' - 2\bar{\tau}ð - 4\mu\rho + 6\psi_2 + 4\tau\varpi\Big]\zeta\Phi_0^{\mathcal{L}1}, \tag{A2b}$$

with all other components given by symmetries, $\overline{A_{ln} + A_{m\bar{m}}} = \overline{A_{ln} - A_{m\bar{m}}}$, $A_{l\bar{m}} = \overline{A_{lm}}$, $A_{nm} = \overline{A'_{lm}}$ and $A_{n\bar{m}} = A'_{lm}$ along with the identification $\Phi_0^{\mathcal{L}1} = -(\Phi_2^{\mathcal{L}1})'$. If we impose the Teukolsky equation then the second of these equations simplifies to

$$A_{lm} = -2\pounds_\xi\Phi_0^{\mathcal{L}1}, \tag{A3}$$

and similarly the prime of this equation is

$$A_{n\bar{m}} = -2\pounds_\xi \Phi_2^{\mathcal{L}1}. \tag{A4}$$

We now wish to show that the Lorenz gauge field equation for the vector potential is satisfied. Using the homogeneous Teukolsky equation, the equation for $\nabla^\mu A_{\mu\nu}$ reduces to a vector given by

$$A(\rho\, n_\mu - \rho' l_\mu + \tau \bar{m}_\mu - \tau' m_\mu) + \text{c.c.}, \tag{A5}$$

where $A = 0$ by virtue of the third Teukolsky-Starobinsky identity, Eq. (57c). Note that this does not rely on any cancellations between $\Phi_0^{\mathcal{L}1}$ and $\Phi_2^{\mathcal{L}1}$ and their complex conjugates; the field equations are satisfied by $A_a^{\mathcal{L}1}$ and its complex conjugate independently.

The above conclusion for the Lorenz gauge field equation can also be obtained by an alternative approach. Requiring that Eq. (A1) is satisfied means that, for a suitable choice of "gauge vector of third kind", **X**, we should be able to solve

$$\Box \zeta \mathcal{H}_{\mu\nu} + 2R_\mu{}^\alpha{}_\nu{}^\beta \zeta \mathcal{H}_{\alpha\beta} + \epsilon_{\alpha\beta}{}^{\gamma\delta} X_{\gamma;\delta} = 0. \tag{A6}$$

To solve for X_μ for our given $\mathcal{H}_{\mu\nu}$ we need only consider the two components

$$\epsilon_{lm}{}^{\gamma\delta} X_{\gamma;\delta} = i\big[(\text{Þ} - \bar{\rho})X_m - (\eth + \bar{\varpi})X_l\big], \tag{A7a}$$
$$\epsilon_{n\bar{m}}{}^{\gamma\delta} X_{\gamma;\delta} = i\big[(\text{Þ}' + \bar{\mu})X_{\bar{m}} - (\eth' + \bar{\tau})X_n\big], \tag{A7b}$$

the second of which is the prime of the first. It is clear by inspection that these would cancel A_{lm} and $A_{n\bar{m}}$ by virtue of the Teukolsky equation if

$$X_l = -2i\zeta^{-1}\eth'\zeta^2\Phi_0^{\mathcal{L}1} \tag{A8a}$$
$$X_n = 2i\zeta^{-1}\eth\zeta^2\Phi_2^{\mathcal{L}1} \tag{A8b}$$
$$X_m = -2i\zeta^{-1}\text{Þ}'\zeta^2\Phi_0^{\mathcal{L}1} \tag{A8c}$$
$$X_{\bar{m}} = 2i\zeta^{-1}\text{Þ}\zeta^2\Phi_2^{\mathcal{L}1}, \tag{A8d}$$

where the second and fourth lines are the prime of the first and third lines, respectively. The remaining equation for $A_{ln} + A_{m\bar{m}}$ with this **X** reduces to

$$A_{ln} + A_{m\bar{m}} = 2A, \tag{A9}$$

which we have already established vanishes as a consequence of the third Teukolsky-Starobinsky identity.

Appendix B: Gauge transformation between radiation and Lorenz gauges

As shown by Dolan, we can find a gauge transformation between radiation and Lorenz gauges by introducing a scalar χ such that $\nabla_\alpha \chi = \Re(A_\alpha^{\mathcal{L}1}) - (A_\mu^{\text{IRG}} + A_\mu^{\text{ORG}})$. Substituting in the GHP expression for the vector potentials, Eqs. (38), (41) and (53), we find four conditions which must be satisfied,

$$\begin{aligned}\text{Þ}\chi = &\Big[(\eth + \bar{\varpi})\bar{\Phi}_0^{\mathcal{L}1} + (\eth' + \varpi)\Phi_0^{\mathcal{L}1}\Big] \\ &- \Big[(\eth' - \varpi)\Psi^{\text{ORG}} + (\eth - \bar{\varpi})\bar{\Psi}^{\text{ORG}}\Big],\end{aligned} \tag{B1a}$$

$$\begin{aligned}\text{Þ}'\chi = &\Big[-(\eth' - \bar{\tau})\bar{\Phi}_2^{\mathcal{L}1} - (\eth - \tau)\Phi_2^{\mathcal{L}1}\Big] \\ &- \Big[(\eth + \tau)\Psi^{\text{IRG}} + (\eth' + \bar{\tau})\bar{\Psi}^{\text{IRG}}\Big],\end{aligned} \tag{B1b}$$

$$\begin{aligned}\eth\chi = &\Big[-(\text{Þ} - \bar{\rho})\bar{\Phi}_2^{\mathcal{L}1} + (\text{Þ}' + \mu)\Phi_0^{\mathcal{L}1}\Big] \\ &- \Big[(\text{Þ} + \bar{\rho})\bar{\Psi}^{\text{IRG}} + (\text{Þ}' - \mu)\Psi^{\text{ORG}}\Big],\end{aligned} \tag{B1c}$$

$$\begin{aligned}\eth'\chi = &\Big[(\text{Þ}' + \bar{\mu})\bar{\Phi}_0^{\mathcal{L}1} - (\text{Þ} - \rho)\Phi_2^{\mathcal{L}1}\Big] \\ &- \Big[(\text{Þ}' - \bar{\mu})\bar{\Psi}^{\text{ORG}} + (\text{Þ} + \rho)\Psi^{\text{IRG}}\Big].\end{aligned} \tag{B1d}$$

Let us now follow Dolan [12] and look for a transformation from IRG to Lorenz. Working with the first of the above equations, this gives

$$\text{Þ}\chi = (\eth' + \varpi)\Phi_0^{\mathcal{L}1} + \text{c.c.} \tag{B2}$$

Dolan then uses the radial part of the separated Teukolsky-Starobinsky identities to replace $\Phi_0^{\mathcal{L}1}$ on the right hand side here with the second radial derivative of $\Phi_2^{\mathcal{L}1}$ and then "peels off" a Þ from both sides to obtain an expression for χ. We can do similarly, by acting on Eq. (B2) with $\eth'$, using the Teukolsky-Starobinsky identities to rewrite the right hand side in terms of $\Phi_2^{\mathcal{L}1}$, and peeling off a Þ from both sides. This produces a result consistent with Dolan's.

[1] Saul A. Teukolsky, "Rotating black holes - separable wave equations for gravitational and electromagnetic perturbations," Phys. Rev. Lett. **29**, 1114–1118 (1972).

[2] P. L. Chrzanowski, "Vector Potential and Metric Perturbations of a Rotating Black Hole," Phys. Rev. **D11**, 2042–2062 (1975).

[3] L. S. Kegeles and J. M. Cohen, "Constructive procedure for perturbations of space-times," Phys. Rev. **D19**, 1641–1664 (1979).

[4] Robert M. Wald, "Construction of Solutions of Gravitational, Electromagnetic, Or Other Perturbation Equations from Solutions of Decoupled Equations," Phys. Rev. Lett. **41**, 203–206 (1978).

[5] B.F. Whiting and L.R. Price, "Metric reconstruction from Weyl scalars," Class. Quant. Grav. **22**, S589–S604 (2005).

[6] Adam Pound, Cesar Merlin, and Leor Barack, "Gravitational self-force from radiation-gauge metric perturbations," Phys. Rev. **D89**, 024009 (2014), arXiv:1310.1513 [gr-qc].

[7] Oleg Lunin, "Maxwell's equations in the Myers-Perry geometry," JHEP **12**, 138 (2017), arXiv:1708.06766 [hep-th].

[8] Valeri P. Frolov, Pavel Krtouš, and David Kubizňák, "Separation of variables in Maxwell equations in Plebański-Demiański spacetime," Phys. Rev. D **97**, 101701 (2018), arXiv:1802.09491 [hep-th].

[9] Pavel Krtouš, Valeri P. Frolov, and David Kubizňák, "Separation of Maxwell equations in Kerr–NUT–(A)dS spacetimes," Nucl. Phys. B **934**, 7–38 (2018), arXiv:1803.02485 [hep-th].

[10] Valeri P. Frolov, Pavel Krtouš, David Kubizňák, and Jorge E. Santos, "Massive Vector Fields in Rotating Black-Hole Spacetimes: Separability and Quasinormal Modes," Phys. Rev. Lett. **120**, 231103 (2018), arXiv:1804.00030 [hep-th].

[11] Sam R. Dolan, "Instability of the Proca field on Kerr spacetime," Phys. Rev. D **98**, 104006 (2018), arXiv:1806.01604 [gr-qc].

[12] Sam R. Dolan, "Electromagnetic fields on Kerr spacetime, Hertz potentials and Lorenz gauge," Phys. Rev. D **100**, 044044 (2019), arXiv:1906.04808 [gr-qc].

[13] Tsuyoshi Houri, Norihiro Tanahashi, and Yukinori Yasui, "On symmetry operators for the Maxwell equation on the Kerr-NUT-(A)dS spacetime," Class. Quant. Grav. **37**, 015011 (2020), arXiv:1908.10250 [gr-qc].

[14] Charles W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation (Freeman, San Francisco, 1973).

[15] Robert P. Geroch, A. Held, and R. Penrose, "A space-time calculus based on pairs of null directions," J. Math. Phys. **14**, 874–881 (1973).

[16] Larry Price, Ph.D. thesis, University of Florida (2007).

[17] Steffen Aksteiner, Ph.D. thesis, Leibniz U., Hannover (2014).

[18] Roger Penrose and Wolfgang Rindler, Spinors and Space-Time, Cambridge Monographs on Mathematical Physics (Cambridge Univ. Press, Cambridge, UK, 2011).

[19] Adam Pound and Barry Wardell, "Black hole perturbation theory and gravitational self-force," in Handbook of Gravitational Wave Astronomy, Springer Reference, edited by Cosimo Bambi, Stavros Katsanevas, and Kostantinos D. Kokkotas (Springer International Publishing, 2020).

[20] William Kinnersley, "Type D Vacuum Metrics," J. Math. Phys. **10**, 1195–1203 (1969).

[21] B. Carter and J. B. Hartle, "GRAVITATION IN ASTROPHYSICS," NATO Sci. Ser. B **156**, pp.1–399 (1987).

[22] Stephen R. Green, Stefan Hollands, and Peter Zimmerman, "Teukolsky formalism for nonlinear Kerr perturbations," Class. Quant. Grav. **37**, 075001 (2020), arXiv:1908.09095 [gr-qc].