Spin-Wave Doppler Shift by Magnon Drag in Magnetic Insulators: Supplemental Material

Tao Yu,[1] Chen Wang,[2] Michael A. Sentef,[1] and Gerrit E. W. Bauer[3]

[1] *Max Planck Institute for the Structure and Dynamics of Matter, Luruper Chaussee 149, 22761 Hamburg, Germany*

[2] *Center for Joint Quantum Studies and Department of Physics, School of Science, Tianjin University, Tianjin 300350, China*

[3] *Institute for Materials Research and WPI-AIMR and CSRN, Tohoku University, Sendai 980-8577, Japan*

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I. NONLINEAR HAMILTONIAN

We consider an in-plane magnetized YIG film with surface normal along the $\hat{\mathbf{x}}$-direction and a static magnetic field $\mathbf{H}_{\rm app}$ applied along the $\hat{\mathbf{z}}'$-direction (see Fig. 1 in the main text). We adopt the following Hamiltonian for the YIG film magnetization

$$\hat{H} = \mu_0 \int \left[\frac{\alpha_{\rm ex}}{2} \left(\nabla \hat{\mathbf{M}} \right)^2 - \hat{\mathbf{M}} \cdot \frac{\hat{\mathbf{H}}_{\rm dip}}{2} - \hat{\mathbf{M}} \cdot \mathbf{H}_{\rm app} \right] d\mathbf{r}, \tag{S1}$$

where μ_0 is the vacuum permeability, $\alpha_{\rm ex}$ is the exchange stiffness, and $\mathbf{H}_{\rm dip}$ is the dipolar field. We recover the spin operators by the replacement $\hat{\mathbf{M}} = -\gamma\hbar\hat{\mathbf{S}}$ with $-\gamma$ being the electron gyromagnetic ratio, leading to

$$\hat{H} = \mu_0 \int \left[\frac{\gamma\hbar}{2} \mathbf{S}(\mathbf{r}) \cdot \hat{\mathbf{H}}_{\rm dip}(\mathbf{r}) + \frac{\gamma^2\hbar^2\alpha_{\rm ex}}{2} \nabla \mathbf{S} \cdot \nabla \mathbf{S} + \gamma\hbar \mathbf{S} \cdot \mathbf{H}_{\rm ext} \right] d\mathbf{r}. \tag{S2}$$

We may disregard the dipolar interaction in ultrathin magnetic films that are excited by narrow striplines. In terms of the magnon field operator $\hat{\Theta}(\mathbf{r})$, the Holstein-Primakoff transformation reads

$$\begin{aligned} \hat{S}_x(\mathbf{r}) + i\hat{S}_{y'}(\mathbf{r}) &= \hat{\Theta}^\dagger(\mathbf{r}) \sqrt{2S - \hat{\Theta}^\dagger(\mathbf{r})\hat{\Theta}(\mathbf{r})}, \\ \hat{S}_x(\mathbf{r}) - i\hat{S}_{y'}(\mathbf{r}) &= \sqrt{2S - \hat{\Theta}^\dagger(\mathbf{r})\hat{\Theta}(\mathbf{r})}\,\hat{\Theta}(\mathbf{r}), \\ \hat{S}_{z'}(\mathbf{r}) &= -S + \hat{\Theta}^\dagger(\mathbf{r})\hat{\Theta}(\mathbf{r}). \end{aligned} \tag{S3}$$

In a thin magnetic film wave interference leads to standing waves normal to the interfaces. Assuming free boundary conditions, the magnon operator can be expanded as

$$\hat{\Theta}(\mathbf{r}) = \sum_l \sqrt{\frac{2}{1+\delta_{l0}}} \frac{1}{\sqrt{s}} \cos\left(\frac{l\pi}{s} x \right) \hat{\Psi}_l(\boldsymbol{\rho}), \tag{S4}$$

where l is the subband index and s the film thickness. In terms of $\hat{\Psi}_l(\boldsymbol{\rho})$, the in-plane magnon field operators for subband l, the Zeeman Hamiltonian reads

$$\begin{aligned} \hat{H}_{\rm Z} &= \mu_0\gamma\hbar \int \hat{S}_z H_{\rm app} d\mathbf{r} = \mu_0\gamma\hbar H_{\rm app} \int \hat{\Theta}^\dagger \hat{\Theta} d\mathbf{r} \\ &\to \mu_0\gamma\hbar H_{\rm app} \sum_l \int \hat{\Psi}_l^\dagger(\boldsymbol{\rho}) \hat{\Psi}_l(\boldsymbol{\rho}) d\boldsymbol{\rho}. \end{aligned} \tag{S5}$$

The linear exchange Hamiltonian

$$\begin{aligned} \hat{H}_{\rm ex}^{\rm L} &= \mu_0\gamma\hbar M_s \alpha_{\rm ex} \int \nabla\hat{\Theta}^\dagger \cdot \nabla\hat{\Theta} d\mathbf{r} \\ &\to \mu_0\gamma\hbar M_s\alpha_{\rm ex} \sum_{l\geq 1} \left(\frac{l\pi}{s} \right)^2 \int \hat{\Psi}_l^\dagger(\boldsymbol{\rho}) \hat{\Psi}_l(\boldsymbol{\rho}) d\boldsymbol{\rho} + \mu_0\gamma\hbar M_s\alpha_{\rm ex} \sum_l \int \nabla_{\boldsymbol{\rho}} \hat{\Psi}_l^\dagger(\boldsymbol{\rho}) \cdot \nabla_{\boldsymbol{\rho}} \hat{\Psi}_l(\boldsymbol{\rho}) d\boldsymbol{\rho}. \end{aligned} \tag{S6}$$

The subbands edges of the magnon dispersion are therefore at

$$E_l = \mu_0\gamma\hbar H_{\rm app} + \mu_0\gamma\hbar M_s \alpha_{\rm ex} (l\pi/s)^2. \tag{S7}$$

In a YIG film with thickness below 10 nm and at temperatures $T \lesssim 300$ K, only the lowest three bands $l = \{0, 1, 2\}$ are significantly populated.

The leading non-linear exchange Hamiltonian is given by

$$\begin{aligned}
\hat{H}_{\rm ex}^{\rm NL} &= \frac{\mu_0\gamma^2\hbar^2\alpha_{\rm ex}}{4}\int\left(\hat{\Theta}^\dagger(\mathbf{r})\hat{\Theta}^\dagger(\mathbf{r})\nabla\Theta\cdot\nabla\Theta+\text{H.c.}\right)d\mathbf{r}\\
&=\sum_{l_1l_2l_3l_4}\mathcal{U}_{l_1l_2l_3l_4}\int\hat{\Psi}^\dagger_{l_1}(\boldsymbol{\rho})\hat{\Psi}^\dagger_{l_2}(\boldsymbol{\rho})\hat{\Psi}_{l_3}(\boldsymbol{\rho})\hat{\Psi}_{l_4}(\boldsymbol{\rho})d\boldsymbol{\rho}\\
&+\sum_{l_1l_2l_3l_4}\mathcal{V}_{l_1l_2l_3l_4}\int\hat{\Psi}^\dagger_{l_1}(\boldsymbol{\rho})\hat{\Psi}^\dagger_{l_2}(\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}\hat{\Psi}_{l_3}(\boldsymbol{\rho})\cdot\nabla_{\boldsymbol{\rho}}\hat{\Psi}_{l_4}(\boldsymbol{\rho})d\boldsymbol{\rho}+\text{H.c.},
\end{aligned}\tag{S8}$$

in which

$$\begin{aligned}
\mathcal{U}_{l_1l_2l_3l_4} &= \frac{\mu_0\gamma^2\hbar^2\alpha_{\rm ex}}{4s}\frac{4}{\sqrt{(1+\delta_{l_10})(1+\delta_{l_20})(1+\delta_{l_30})(1+\delta_{l_40})}}\frac{l_3\pi}{s}\frac{l_4\pi}{s}\mathcal{A}_{l_1l_2l_3l_4},\\
\mathcal{V}_{l_1l_2l_3l_4} &= \frac{\mu_0\gamma^2\hbar^2\alpha_{\rm ex}}{4s}\frac{4}{\sqrt{(1+\delta_{l_10})(1+\delta_{l_20})(1+\delta_{l_30})(1+\delta_{l_40})}}\mathcal{B}_{l_1l_2l_3l_4},
\end{aligned}\tag{S9}$$

with form factors

$$\begin{aligned}
\mathcal{A}_{l_1l_2l_3l_4} &= \frac{1}{s}\int_{-s}^{0}\cos\left(\frac{l_1\pi}{s}x\right)\cos\left(\frac{l_2\pi}{s}x\right)\sin\left(\frac{l_3\pi}{s}x\right)\sin\left(\frac{l_4\pi}{s}x\right)dx\\
&=\frac{1}{8}\left(\delta_{l_1+l_2+l_3,l_4}+\delta_{l_1+l_2+l_4,l_3}+\delta_{l_1+l_3,l_2+l_4}+\delta_{l_1+l_4,l_2+l_3}\right.\\
&\left.-\delta_{l_1+l_2+l_3+l_4,0}-\delta_{l_1+l_2,l_3+l_4}-\delta_{l_1+l_3+l_4,l_2}-\delta_{l_2+l_3+l_4,l_1}\right),\\
\mathcal{B}_{l_1l_2l_3l_4} &= \frac{1}{s}\int_{-s}^{0}\cos\left(\frac{l_1\pi}{s}x\right)\cos\left(\frac{l_2\pi}{s}x\right)\cos\left(\frac{l_3\pi}{s}x\right)\cos\left(\frac{l_4\pi}{s}x\right)dx\\
&=\frac{1}{8}\left(\delta_{l_1+l_2+l_3,l_4}+\delta_{l_1+l_2+l_4,l_3}+\delta_{l_1+l_3,l_2+l_4}+\delta_{l_1+l_4,l_2+l_3}+\delta_{l_1+l_2+l_3+l_4,0}\right.\\
&\left.+\delta_{l_1+l_2,l_3+l_4}+\delta_{l_1+l_3+l_4,l_2}+\delta_{l_2+l_3+l_4,l_1}\right).
\end{aligned}\tag{S10}$$

The interaction strength increases with decreasing film thickness. Here we focus on the interaction of $\langle\hat{\Psi}_{l=0}(\boldsymbol{\rho})\rangle$, i.e. the coherent magnons in the lowest band $l_1=0$, and those in the thermally populated higher bands. $\mathcal{A}_{0l_2l_3l_4}$-processes with $\{l_3,l_4\}\neq 0$ are governed by the selection rules

$$\begin{aligned}
\mathcal{A}_{0l_2l_3l_4} &= \frac{1}{4}\left(\delta_{l_2+l_3,l_4}+\delta_{l_2+l_4,l_3}-\delta_{l_3+l_4,l_2}\right),\\
\mathcal{B}_{0l_2l_3l_4} &= \frac{1}{4}\left(\delta_{l_2+l_3,l_4}+\delta_{l_2+l_4,l_3}+\delta_{l_2+l_3+l_4,0}+\delta_{l_3+l_4,l_2}\right).
\end{aligned}\tag{S11}$$

With $[\hat{\Psi}_{l'}(\boldsymbol{\rho}'),\hat{\Psi}^\dagger_l(\boldsymbol{\rho})]=\delta_{ll'}\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')$ and the Heisenberg equation $i\hbar\partial_t\hat{\Psi}_{l'}(\boldsymbol{\rho}')=[\hat{\Psi}_{l'}(\boldsymbol{\rho}'),\hat{H}]$, the dynamics of the coherent magnons in the lowest band ($l=0$) obeys

$$\begin{aligned}
i\hbar\frac{\partial\langle\hat{\Psi}_{l=0}(\boldsymbol{\rho})\rangle}{\partial t} &= E_{l=0}\langle\hat{\Psi}_{l=0}(\boldsymbol{\rho})\rangle-\hbar\omega_M\alpha_{\rm ex}\nabla^2\langle\hat{\Psi}_{l=0}(\boldsymbol{\rho})\rangle\\
&+2\sum_{l_2l_3l_4}\mathcal{U}_{0l_2l_3l_4}\langle\hat{\Psi}^\dagger_{l_2}(\boldsymbol{\rho})\hat{\Psi}_{l_3}(\boldsymbol{\rho})\hat{\Psi}_{l_4}(\boldsymbol{\rho})\rangle+2\sum_{l_1l_2l_3}\mathcal{U}_{l_1l_2l_30}\langle\hat{\Psi}_{l_1}(\boldsymbol{\rho})\hat{\Psi}_{l_2}(\boldsymbol{\rho})\hat{\Psi}^\dagger_{l_3}(\boldsymbol{\rho})\rangle\\
&+2\sum_{l_2l_3l_4}\mathcal{V}_{0l_2l_3l_4}\left(\langle\hat{\Psi}^\dagger_{l_2}\nabla_{\boldsymbol{\rho}}\hat{\Psi}_{l_3}\cdot\nabla_{\boldsymbol{\rho}}\hat{\Psi}_{l_4}\rangle-\nabla_{\boldsymbol{\rho}}\cdot\langle\hat{\Psi}_{l_2}\hat{\Psi}_{l_3}\nabla_{\boldsymbol{\rho}}\hat{\Psi}^\dagger_{l_4}\rangle\right),
\end{aligned}\tag{S12}$$

where $\omega_H=\mu_0\gamma H_{\rm app}$ and $\omega_M=\mu_0\gamma M_s$. The terms involving $\mathcal{U}$ vanish in the mean-field approximation of the 3-magnon amplitudes when $\langle\hat{\Psi}_{l\neq 0}(\boldsymbol{\rho})\rangle=0$. The last term in Eq. (S12) when transformed into momentum space reduces to

$$-\frac{4}{\hbar}\sum_{\mathbf{k}}\sum_{l'}\mathcal{V}_{ll'l'}e^{i\mathbf{k}\cdot\boldsymbol{\rho}}\mathbf{k}\cdot\left(\sum_{\mathbf{k}'}\hbar\mathbf{k}'\left(\langle\hat{\Psi}_{l'}(\mathbf{k}')\hat{\Psi}^{\dagger}_{l'}(\mathbf{k}')\rangle+\langle\hat{\Psi}^{\dagger}_{l'}(\mathbf{k}')\hat{\Psi}_{l'}(\mathbf{k}')\rangle\right)\right)\langle\Psi_l(\mathbf{k})\rangle$$
$$=\frac{8i}{\hbar}\sum_{l'}\mathcal{V}_{ll'l'}\nabla_{\boldsymbol{\rho}}\langle\hat{\Psi}_l(\boldsymbol{\rho})\rangle\cdot\frac{\hbar}{2i}\left(\langle\hat{\Psi}^{\dagger}_{l'}(\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}\hat{\Psi}_{l'}(\boldsymbol{\rho})\rangle-\langle\hat{\Psi}_{l'}(\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}\hat{\Psi}^{\dagger}_{l'}(\boldsymbol{\rho})\rangle\right).\tag{S13}$$

We recognize the magnon current density in subband l' (in units of an angular momentum current J/m^2)

$$\mathbf{J}_{l'}(\boldsymbol{\rho})=\frac{\hbar}{2i}\left(\langle\hat{\Psi}^{\dagger}_{l'}(\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}\hat{\Psi}_{l'}(\boldsymbol{\rho})\rangle-\langle\hat{\Psi}_{l'}(\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}\hat{\Psi}^{\dagger}_{l'}(\boldsymbol{\rho})\rangle\right),\tag{S14}$$

which can be integrated to an expression in terms of magnon occupation numbers

$$\boldsymbol{\mathcal{J}}_{l'}=\int\mathbf{J}_{l'}(\boldsymbol{\rho})d\boldsymbol{\rho}=\frac{1}{2}\sum_{\mathbf{k}'}\hbar\mathbf{k}'\left(\langle\hat{\Psi}^{\dagger}_{l'}(\mathbf{k}')\hat{\Psi}_{l'}(\mathbf{k}')\rangle+\langle\hat{\Psi}_{l'}(\mathbf{k}')\hat{\Psi}^{\dagger}_{l'}(\mathbf{k}')\rangle\right).\tag{S15}$$

$\mathbf{J}$ is a spin current since the magnons carry spin $\hbar$. The expressions are consistent with the magnon density current $\tilde{\mathbf{J}}_l$ defined by the Heisenberg equation and the magnon conservation law (in the absence of damping)

$$\frac{\partial\hat{\rho}^l_m}{\partial t}=\frac{1}{i\hbar}[\hat{\rho}^l_m,\hat{H}^{\mathrm{L}}_{\mathrm{ex}}]=-\nabla\cdot\tilde{\mathbf{J}}_l,\tag{S16}$$

which leads to

$$\left\langle\tilde{\mathbf{J}}_l(\boldsymbol{\rho})\right\rangle=\omega_M\alpha_{\mathrm{ex}}\frac{1}{i}\left(\langle\hat{\Psi}^{\dagger}_{l}(\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}\hat{\Psi}_{l}(\boldsymbol{\rho})\rangle-\langle\hat{\Psi}_{l}(\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}\hat{\Psi}^{\dagger}_{l}(\boldsymbol{\rho})\rangle\right),\tag{S17}$$

which equals $\mathbf{J}_l$ divided by the constant magnon mass. The coherent magnons in the lowest band thereby obey

$$i\hbar\frac{\partial\langle\hat{\Psi}_0(\boldsymbol{\rho})\rangle}{\partial t}=E_{l=0}\langle\hat{\Psi}_0(\boldsymbol{\rho})\rangle-\hbar\omega_M\alpha_{\mathrm{ex}}\nabla^2\langle\hat{\Psi}_0(\boldsymbol{\rho})\rangle$$
$$+\frac{8i}{\hbar}\sum_{l'}\mathcal{V}_{00l'l'}\nabla_{\boldsymbol{\rho}}\langle\hat{\Psi}_0(\boldsymbol{\rho})\rangle\cdot\mathbf{J}_{l'}(\boldsymbol{\rho}).\tag{S18}$$

Both incoherent and coherent magnons contribute to the current density $\mathbf{J}_{l'}$.

A magnon current carried by incoherent or thermal magnons can be driven by a magnon chemical potential or temperature gradients. These can be created either by the spin-Hall effect in, or Ohmic heating of, current-biased Pt contacts. We can estimate the latter (spin Seebeck) effect by the linearized Boltzmann equation in the relaxation-time approximation. Assuming that the drag term in the collision integral is small

$$-\mathbf{v}_{\mathbf{k},l}\cdot\nabla T\frac{\partial f_{\mathbf{k},l}}{\partial T}=-\frac{f_{\mathbf{k},l}-f^{(0)}_{\mathbf{k},l}}{\tau_{\mathbf{k},l}},\tag{S19}$$

where $\mathbf{v}_{\mathbf{k},l}=(1/\hbar)\partial E_l(\mathbf{k})/\partial\mathbf{k}=2\omega_M\alpha_{\mathrm{ex}}\mathbf{k}$ is the magnon group velocity, $f_{\mathbf{k},l}=\langle\hat{\Psi}^{\dagger}_l(\mathbf{k})\hat{\Psi}_l(\mathbf{k})\rangle$ is the magnon distribution in the l-th subband, $f^{(0)}_{\mathbf{k},l}=1/\{\exp[E_{\mathbf{k},l}/(k_BT)]-1\}$ is the equilibrium Planck distribution at temperature T, and $\tau_{\mathbf{k},l}$ is the magnon relaxation time. A uniform $\nabla T=\mathcal{E}_y\hat{\mathbf{y}}$ then generates a magnon momentum current

$$\mathbf{J}_l=\hbar\omega_M\alpha_{\mathrm{ex}}\mathcal{E}_y\hat{\mathbf{y}}\int\frac{dk_ydk_z}{2\pi^2}k_y^2\tau_{\mathbf{k},l}\frac{\partial f^{(0)}_{\mathbf{k},l}}{\partial T},\tag{S20}$$

which affects the coherent magnon amplitude by substitution into Eq. (S18).

To complete the dynamic equation, we need to account for the scattering between magnons which requires a treatment beyond the mean-field approximation, which we address in the next section.

II. ABSENCE OF EXCHANGE SCATTERING OF LOW-ENERGY MAGNONS

Here we describe the scattering processes between the coherent magnon of the lowest band and the incoherent thermal cloud by dividing the magnon operator of the lowest band into a coherent number amplitude Ξ and incoherent operator $\hat{\psi}$

$$\begin{aligned}\hat{\Psi}_0(\boldsymbol{\rho},t) &= \Xi(\boldsymbol{\rho},t)+\hat{\psi}_0(\boldsymbol{\rho},t)\approx \Xi(t)+\hat{\psi}_0(\boldsymbol{\rho},t),\\ \hat{\Psi}_{l>0}(\boldsymbol{\rho},t) &= \hat{\psi}_l(\boldsymbol{\rho},t).\end{aligned} \tag{S21}$$

The first interaction term

$$H_{\rm int}^{(1)} \to 2\sum_{l_2l_3l_4}\mathcal{U}_{0l_2l_3l_4}\Xi^*(t)\int \hat{\psi}_{l_2}^{\dagger}(\boldsymbol{\rho})\hat{\psi}_{l_3}(\boldsymbol{\rho})\hat{\psi}_{l_4}(\boldsymbol{\rho})d\boldsymbol{\rho}+\text{H.c.}, \tag{S22}$$

with selection rules $\delta_{l_2+l_3,l_4}$, $\delta_{l_2+l_4,l_3}$, and $\delta_{l_3+l_4,l_2}$; the second interaction term

$$\hat{H}_{\rm int}^{(2)} \to 2\sum_{l_2l_3l_4}\mathcal{V}_{0l_2l_3l_4}\Xi^*(t)\int \hat{\psi}_{l_2}^{\dagger}(\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}\hat{\psi}_{l_3}(\boldsymbol{\rho})\cdot\nabla_{\boldsymbol{\rho}}\hat{\psi}_{l_4}(\boldsymbol{\rho})d\boldsymbol{\rho}+\text{H.c.}, \tag{S23}$$

with selection rules $\delta_{l_2+l_3,l_4}$, $\delta_{l_2+l_4,l_3}$, $\delta_{l_3+l_4,l_2}$, and $\delta_{l_2+l_3+l_4,0}$. The processes in Eqs. (S22) and (S23) describe the confluence of two low-energy magnons into a single one and *vice versa*, with an efficiency determined by the coherent-magnon amplitude.

In the lowest band, $H_{\rm int}^{(1)}$ vanishes because $\mathcal{U}_{0000}=0$. Energy-conserving scattering processes in

$$\begin{aligned}\hat{H}_{\rm int}^{(2)} &\to 2V_0\Xi^*(t)\int \hat{\psi}_0^{\dagger}(\boldsymbol{\rho})\nabla_{\boldsymbol{\rho}}\hat{\psi}_0(\boldsymbol{\rho})\cdot\nabla_{\boldsymbol{\rho}}\hat{\psi}_0(\boldsymbol{\rho})d\boldsymbol{\rho}+\text{H.c.}\\ &= -2V_0\Xi^*(t)\sum_{\mathbf{kq}}(\mathbf{k}-\mathbf{q})\cdot\mathbf{q}\hat{\psi}_{0,\mathbf{k}}^{\dagger}\hat{\psi}_{0,\mathbf{k}-\mathbf{q}}\hat{\psi}_{0,\mathbf{q}}+\text{H.c.}\end{aligned} \tag{S24}$$

require that $k^2\approx|\mathbf{k}-\mathbf{q}|^2+q^2$ or $(\mathbf{k}-\mathbf{q})\cdot\mathbf{q}\approx 0$. We therefore may disregard interaction effects within the lowest band and focus on the interband scatterings.

Most intriguing is the up-conversion of two magnons into nearly empty states, creating a hot magnon out of two cold ones. The leading order scattering processes conserve energy:

$$\left(\frac{l_3\pi}{s}\right)^2+\left(\frac{l_4\pi}{s}\right)^2+(\mathbf{k}-\mathbf{q})^2+q^2\approx\left(\frac{l_2\pi}{s}\right)^2+k^2, \tag{S25}$$

so

$$(\mathbf{k}-\mathbf{q})\cdot\mathbf{q}\approx\frac{l_3^2+l_4^2-l_2^2}{2}\frac{\pi^2}{s^2}. \tag{S26}$$

The interaction Hamiltonian then reduces to

$$\hat{H}_{\rm int}=\sum_{l_2l_3l_4}\sum_{\mathbf{k},\mathbf{q}}\mathcal{W}_{l_2l_3l_4}\Xi^*(t)\hat{\psi}_{l_2,\mathbf{k}}^{\dagger}\hat{\psi}_{l_3,\mathbf{k}-\mathbf{q}}\hat{\psi}_{l_4,\mathbf{q}}+\text{H.c.}, \tag{S27}$$

where

$$\begin{aligned}\mathcal{W}_{l_2l_3l_4} &= 2\left[\mathcal{U}_{0l_2l_3l_4}-V_{0l_2l_3l_4}(\mathbf{k}-\mathbf{q})\cdot\mathbf{q}\right]\\ &\approx 2\left(\mathcal{U}_{0l_2l_3l_4}+\mathcal{V}_{0l_2l_3l_4}\frac{l_2^2-l_3^2-l_4^2}{2}\frac{\pi^2}{s^2}\right)\end{aligned} \tag{S28}$$

is a contact potential since it does not depend on the wave number. Because of the selection rules $\delta_{l_2+l_3,l_4}$, $\delta_{l_2+l_4,l_3}$, and $\delta_{l_3+l_4,l_2}$, the scattering potential vanishes by a cancellation of the $\mathcal{U}$ and $\mathcal{V}$ terms: when $l_4 = l_2+l_3$ and $l_3 = l_2 + l_4$, $(l_2^2 - l_3^2 - l_4^2)/2 = -l_3 l_4$ while when $l_2 = l_3 + l_4$, $(l_2^2 - l_3^2 - l_4^2)/2 = l_3 l_4$. In conclusion, the coherent magnon amplitude in the lowest subbands interacts only with coherent magnons in the other subbands within the leading nonlinearity. We therefore may use a mean-field approximation to describe the dynamics of low-energy coherent magnons in the nonlinear regime.

III. PARAMETER-DEPENDENCE OF SPIN CURRENTS AND SPIN-ORBIT INTERACTION

Here we discuss the maximal spin current excited by stripline microwaves as a function of material parameters. For small stripline currents $I < I_c$, the coherently excited spin currents shown in Fig. S1 turns out to be proportional to I, rather thant I^2 as expected for non-interacting magnons. We oberve saturation at a critical drive I_c, and suppression for $I > I_c$ by high-order magnon interactions that are implicitly included in the numerical solutions of the LLG equation. The maximum spin current $\mathbf{J}_y^{(c)} \approx 10^{-7}$ kg/(m.s) is smaller but always close to that required for the Doppler-shift–induced spin-wave instability calculated by our mean-field theory, which turns out to be a good estimate for the maximal spin current.

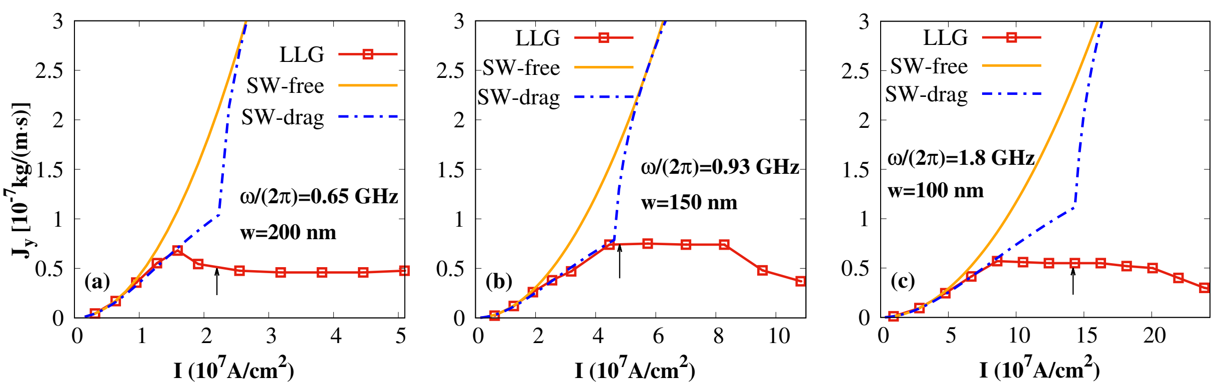

FIG. S1. Coherently pumped magnon current $\mathbf{J}_y$ as a function of the applied electric current density I in the stripline from numerical LLG calculations ("LLG"), non-interacting spin-wave theory ("SW-free"), and spin-wave theory including the drag effect ("SW-drag"). The black arrows indicate the critical current I_c that causes a spin-wave instability in the mean-field theory.

In the presence of an interfacial Dzyaloshinskii-Moriya spin-orbit interaction [1]

$$\hat{H}_{\rm DMI} = -\frac{D}{2\mu_0 M_s^2}\int d\mathbf{r}\left[\hat{\mathbf{y}}\cdot(\mathbf{M}\times\partial_z\mathbf{M}) - \hat{\mathbf{z}}\cdot(\mathbf{M}\times\partial_y\mathbf{M})\right], \tag{S29}$$

where D is the DMI constant, the magnon dispersion [Eq. (11) in the main text] to leading order reads

$$\omega_{k_y} = \mu_0\gamma H_{\rm app} + \omega_M \alpha_{\rm ex} k_y^2 - [\gamma D/M_s + (8/\hbar^2)\mathcal{V}_0 k_y \overline{\mathbf{J}}_y]k_y. \tag{S30}$$

Figure S2 illustrates that a $D = 7.5\times10^{-5}$ J/m^2 reduces the critical spin current to a value $\overline{\mathbf{J}}_y^{(c)} \approx 0.5\times 10^{-7}$ kg/(m s) that according to the LLG equation augmented by the effective magnetic field from Eq. (S29), can be excited by a stripline. A sufficiently large spin-orbit interaction can therefore assist the generation of a Doppler-shift–induced spin-wave instability of the ground state magnetization.

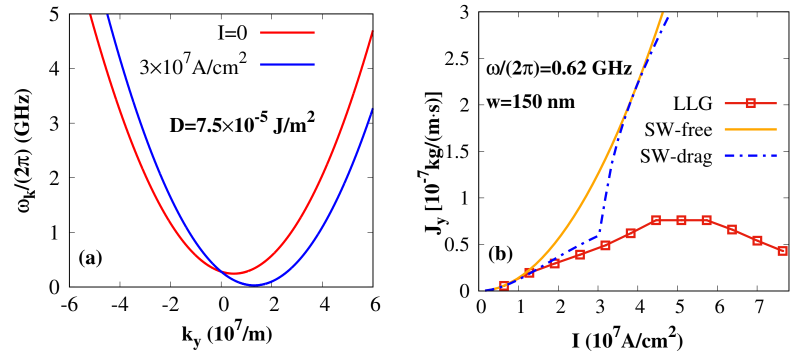

FIG. S2. Magnon dispersion [(a)] and coherently pumped spin current density $\mathbf{J}_y$ as a function of the applied electric current density I in the stripline [(b)] in the presence of a DMI spin-orbit interaction. The curves in (b) are obtained by numerically solving the LLG equation ("LLG"), by non-interacting spin-wave theory ("SW-free"), and by spin-wave theory including the drag effect ("SW-drag").

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[1] J.-H. Moon, S.-M. Seo, K.-J. Lee, K.-W. Kim, J. Ryu, H.-W. Lee, R. D. McMichael, and M. D. Stiles, Phys. Rev. B **88**, 184404 (2013).