Detecting and Counting Small Subgraphs, and Evaluating a Parameterized Tutte Polynomial: Lower Bounds via Toroidal Grids and Cayley Graph Expanders

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Abstract
Given a graph property \( \Phi \), we consider the problem \( \text{EdgeSub}(\Phi) \), where the input is a pair of a graph \( G \) and a positive integer \( k \), and the task is to decide whether \( G \) contains a \( k \)-edge subgraph that satisfies \( \Phi \). Specifically, we study the parameterized complexity of \( \text{EdgeSub}(\Phi) \) and of its counting problem \( \#\text{EdgeSub}(\Phi) \) with respect to both approximate and exact counting. We obtain a complete picture for minor-closed properties \( \Phi \): the decision problem \( \text{EdgeSub}(\Phi) \) always admits an FPT (“fixed-parameter tractable”) algorithm and the counting problem \( \#\text{EdgeSub}(\Phi) \) always admits an FPTRAS (“fixed-parameter tractable randomized approximation scheme”).

For exact counting, we present an exhaustive and explicit criterion on the property \( \Phi \) which, if satisfied, yields fixed-parameter tractability and otherwise \( \#W[1] \)-hardness. Additionally, most of our hardness results come with an almost tight conditional lower bound under the so-called Exponential Time Hypothesis, ruling out algorithms for \( \#\text{EdgeSub}(\Phi) \) that run in time \( f(k) \cdot |G|^{o(k/\log k)} \) for any computable function \( f \).

As a main technical result, we gain a complete understanding of the coefficients of toroidal grids and selected Cayley graph expanders in the homomorphism basis of \( \#\text{EdgeSub}(\Phi) \). This allows us to establish hardness of exact counting using the Complexity Monotonicity framework due to Curticapean, Dell and Marx (STOC’17). This approach does not only apply to \( \#\text{EdgeSub}(\Phi) \) but also to the more general problem of computing weighted linear combinations of subgraph counts. As a special case of such a linear combination, we introduce a parameterized variant of the Tutte Polynomial \( T^k_G \) of a graph \( G \), to which many known combinatorial interpretations of values of the (classical) Tutte Polynomial can be extended. As an example, \( T^k_G(2,1) \) corresponds to the number of \( k \)-forests in the graph \( G \). Our techniques allow us to completely understand the parametrized complexity of computing the evaluation of \( T^k_G \) at every pair of rational coordinates \( (x,y) \). In particular, our results give a new proof for the \#W[1]-hardness of the problem of counting \( k \)-forests in a graph.

Keywords and phrases Counting complexity, parametrized complexity, fixed-parameter tractability, graph homomorphisms, Tutte polynomial
Introduction

Be it searching for cliques in social networks or understanding protein-protein interaction networks, many interesting real-life problems boil down to finding (or counting) small patterns in large graphs. Hence, to no surprise, finding (and counting) small patterns in large graphs are among the most well-studied computational problems in the fields of database theory [13, 45, 36, 14, 31], molecular biology and bioinformatics [42, 1, 71, 64], and network science [72, 60, 59]. In fact, already in the 1970s, the relevance of finding patterns became apparent in the context of finding cliques, finding Hamiltonian paths, or finding specific subgraphs in general [21, 20, 75, 13]. However, with the advent of motif counting for the frequency analysis of small structures in complex networks [60, 59], it became evident that detecting the existence of a pattern graph is not enough; we also need to count all of the occurrences of the pattern.

In this work, our patterns are (not necessarily induced) edge subgraphs that satisfy a certain graph property: for instance, given a graph, we want to count all occurrences of edge subgraphs that are are planar or connected.

From a classical point of view, often the problem of finding patterns is already NP-hard: prime examples include the aforementioned problems of finding (maximum) cliques or Hamiltonian paths. However, for the task of network motif counting, the patterns are (almost) always much smaller than the network itself (see [60, 59, 1]). This motivates a parametrized view: can we obtain fast algorithms to compute the number of occurrences of “small” patterns? If we cannot, can we at least obtain fast (randomized) algorithms to compute an estimate of this number? And if we cannot even do this, can we at least obtain fast algorithms to detect an occurrence? In this work, we completely answer all of the above questions for patterns that are specified by minor-closed graph properties (such as planarity) or selected other graph properties (such as connectivity).

As it turns out, the techniques we develop for answering the above questions are quite powerful: they easily generalize to a parametrized version of the Tutte polynomial. Specifically, our techniques allow us to completely understand at which rational points we can evaluate said parametrized Tutte polynomial in reasonable time, and at which rational points this is not feasible. This dichotomy turns out to be similar, but not equal, to the complexity landscape of the classical Tutte polynomial due to Jaeger et al. [47].

Parametrized Counting and Hardness

By now, counting complexity theory is a well established subfield of theoretical computer science. Already in the 1970s, Valiant started a formal study of counting problems when investigating the complexity of the permanent [76, 77]: counting the number of perfect matchings in a graph is #P-complete, and hence harder than any problem in the polynomial-time hierarchy PH by Toda’s Theorem [74]. In contrast, detecting a perfect matching in a graph is much easier and can be done in polynomial time [37]. Hence, counting problems can be much harder than their decision problem counterparts.

As an attempt to overcome the hardness of counting problems in general, the focus shifted to a multivariate or parametrized view on these problems. Consider for example the following problem: given a query $\varphi$ of size $k$ and a database $B$ of size $n$, we want to count the number of answers to $\varphi$ in $B$. If we make the very reasonable assumption that $k$ is much smaller than $n$, then we may consider an algorithm running in time $O(2^k \cdot n)$ as tractable. Note that in particular, such an algorithm may even outperform an algorithm running in time $O(n^2)$. Also consider [13] for a more detailed and formal discussion.

Formally, given a problem $P$ and a parametrization $\kappa$ that maps each instance $I$ of $P$ to a parameter $\kappa(I)$, we say that $P$ is fixed-parameter tractable (FPT) with respect to $\kappa$, if there is an algorithm that solves each instance $I$ of size $n$ in time $f(\kappa(I)) \cdot n^{O(1)}$, for some computable function $f$. This notion was introduced by Downey and Fellows in the early 1990s [33, 34] and has itself spawned a rich body of
literature (see [39, 35, 27]). In the context of the problems of detecting and counting small patterns in large networks, we parameterize by the size of the pattern: given a pattern of size $k$ and a network of size $n$, we aim for algorithms that run in time $f(k) \cdot n^{O(1)}$, for some computable function $f$. However, for some patterns, even this goal is too ambitious: it is widely believed that even finding a clique of size $k$ is not fixed-parameter tractable; in particular, an FPT algorithm for finding a clique of size $k$ would also imply a breakthrough result for the Satisfiability Problem and thereby refute the widely believed Exponential Time Hypothesis [16, 17]. If a problem $P$ is at least as hard as finding a clique (or counting all cliques) of size $k$, we say that $P$ is W[1]-hard (or $\#$W[1]-hard, respectively).

For such a (\#)W[1]-hard problem, the hope is to (significantly) improve upon the naive brute-force algorithm, which runs in time $n^{O(k)}$ for the problems considered in this work. However, in view of the aforementioned reduction from the Satisfiability Problem to the problem of finding cliques of size $k$ [15, 16], we can see that for finding cliques this, too, would require a breakthrough for the Satisfiability Problem, which, again, is believed to be unlikely [46]. In our paper, via suitable reductions from the problem of finding cliques, we establish that exact algorithms faster than the brute-force algorithms are unlikely for the problems we study.

**Parametrized Detection and Counting of Edge Subgraphs**

Vertex-induced subgraphs as patterns are notoriously hard to detect or to count. The long line of research on this problem [52, 19, 48, 49, 58, 50, 25, 69, 32, 70] showed that this holds even if the patterns are significantly smaller than the host graphs, as witnessed by W[1] and $\#$W[1]-hardness results and almost tight conditional lower bounds. In case of exact counting, it is in fact an open question whether there are non-trivial instances of induced subgraph counting that admit efficient algorithms; recent work [70] supports the conjecture that no such instances exist.

In search for fast algorithms, in this work, we hence consider a related, but different version of network-motif counting: for a computable graph property $\Phi$, in the problem $\#\text{EdgeSub}(\Phi)$ we are given a graph $G$ and a positive integer $k$, and the task is to compute the number of (not necessarily induced) edge subgraphs with $k$ edges in $G$ that satisfy $\Phi$. Similarly, we write EdgeSub($\Phi$) for the corresponding decision problem. Then, in contrast to the case of counting vertex-induced subgraphs, for (\#)EdgeSub($\Phi$), we identify non-trivial properties $\Phi$ for which (\#)EdgeSub($\Phi$) is fixed-parameter tractable; we discuss this in more detail later. First, however, let us take a detour to elaborate more on what is known already for (\#)EdgeSub($\Phi$).

If the property $\Phi$ is satisfied by at most a single graph for each value of the parameter $k$, the decision problem EdgeSub($\Phi$) becomes the subgraph isomorphism problem. Hence, naturally there is a vast body of known techniques and results for special properties $\Phi$: for FPT algorithms, think of the Colour-Coding technique by Alon, Yuster and Zwick [3], the “Divide and Colour”-technique [17], narrow sieving [8], representative sets [40], or “extensor-coding” [10] to name but a few. For hardness results, apart from the aforementioned example of detecting a clique, Lin quite recently established that detecting a $k$-biclique is also W[1]-hard [55]. However, a complete understanding of the parameterized decision version of the subgraph isomorphism is one of the major open problems of parameterized complexity theory [35, Chapter 33.1], that is still to be solved.

In the setting of parameterized counting, the situation is much better understood: Flum and Grohe [38] proved $\#\text{EdgeSub}(\Phi)$ to be $\#$W[1]-hard when $\Phi$ is the property of being a cycle, or the property of being a path. Curticapean [23] established the same result for the property of being a matching. In [26],

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1 Recall that an edge subgraph $G'$ of a graph $G$ may have fewer edges than the subgraph of $G$ that is induced by the vertices of $G'$. 
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Curticapean and Marx established a complete classification in case $\Phi$ does not hold on two different graphs with the same number of edges, which is essentially the parameterized subgraph counting problem. In particular, they identified a bound on the matching number as the tractability criterion. In a later work, together with Dell [25], they presented what is now called the framework of Complexity Monotonicity, which can be considered to be one of the most powerful tools in the field of parameterized counting problems. Note that this does not classify the decision version, as $\#W[1]$-hardness for a counting problem does not imply $W[1]$-hardness for the corresponding decision problem.

In contrast to the parameterized subgraph detection/counting problems, the problem $(\#)\text{EdgeSub}(\Phi)$ allows to search for more general patterns. For example, while the (parameterized) complexity of counting all subgraphs of a graph $G$ isomorphic to a fixed connected graph $H$ with $k$ edges is fully understood [26], the case of counting all connected $k$-edge subgraphs of a graph $G$ remained open so far. As one of our main results, we completely understand the problem $(\#)\text{EdgeSub}(\Phi)$ for the property $\Phi = \text{connectivity}$; we discuss our main results next.

Main Results

In a first part, we present our results on $(\#)\text{EdgeSub}(\Phi)$; we continue with a definition and our results for a parametrized Tutte polynomial in a second part.

Our main results on $(\#)\text{EdgeSub}(\Phi)$ can be categorized in roughly three categories: (1) exact algorithms and hardness results for the counting problem; (2) approximation algorithms for the counting problem; and (3) algorithms for the decision problem. For minor-closed properties $\Phi$, we obtain exhaustive results for all three categories, for other (classes of) properties that we study, we obtain partial criteria. For an overview over our results on $(\#)\text{EdgeSub}(\Phi)$, also consider Table 1; we go into more detail in the following.

Complete Classification for Minor-Closed Properties

Let us start with the case where the graph property $\Phi$ is closed under taking minors, that is, if $\Phi$ holds for a graph, then $\Phi$ still holds after removing vertices or edges, or after contracting edges. For minor-closed properties $\Phi$, we obtain a complete picture of the complexity of $(\#)\text{EdgeSub}(\Phi)$ and $\text{EdgeSub}(\Phi)$. In what follows, we say that a property $\Phi$ has bounded matching number if there is a constant bound on the size of a largest matching in graphs satisfying $\Phi$.

Main Theorem 1. Let $\Phi$ denote a minor-closed graph property.

1. Exact Counting: If $\Phi$ is either trivially true or of bounded matching number, then the (exact) counting version $(\#)\text{EdgeSub}(\Phi)$ is fixed-parameter tractable. Otherwise $(\#)\text{EdgeSub}(\Phi)$ is $\#W[1]$-hard. If, additionally, each forbidden minor of $\Phi$ has a vertex of degree at least 3, and the Exponential Time Hypothesis holds, then $(\#)\text{EdgeSub}(\Phi)$ cannot be solved in time $f(k) \cdot |G|^{o(k/\log k)}$, for any function $f$.

2. Approximate Counting: The problem $(\#)\text{EdgeSub}(\Phi)$ always has a fixed-parameter tractable randomised approximation scheme (FPTRAS)\footnote{The formal definition is given in Section 6; intuitively an FPTRAS is the parameterized equivalent of a fully polynomial-time randomised approximation scheme (FPRAS).}

3. Decision: The problem $\text{EdgeSub}(\Phi)$ is always fixed-parameter tractable.

Consider for example the property $\Phi$ of being planar: planar graphs do not have bounded matching number. Additionally, by Kuratowski’s Theorem, the forbidden minors of planar graphs are the 3-biclique $K_{3,3}$ and the 5-clique $K_5$. Since both $K_{3,3}$ and $K_5$ contain a vertex of degree at least 3, we conclude
<table>
<thead>
<tr>
<th>Property $\Phi$</th>
<th>Exact Counting</th>
<th>Approximate Counting</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minor-closed† (e.g. $\Phi = \text{planarity}$)</td>
<td>$#W[1]$-hard not in $f(k) \cdot</td>
<td>G</td>
<td>^{o(k/\log k)}$</td>
</tr>
<tr>
<td>$\Phi = \text{connectivity}$</td>
<td>$#W[1]$-hard not in $f(k) \cdot</td>
<td>G</td>
<td>^{o(k/\log k)}$</td>
</tr>
<tr>
<td>$\Phi = \text{Hamiltonicity}$</td>
<td>$#W[1]$-hard not in $f(k) \cdot</td>
<td>G</td>
<td>^{o(k/\log k)}$</td>
</tr>
<tr>
<td>$\Phi = \text{Eulerianity}$</td>
<td>$#W[1]$-hard not in $f(k) \cdot</td>
<td>G</td>
<td>^{o(k/\log k)}$</td>
</tr>
<tr>
<td>$\Phi = \text{claw-freeness}$</td>
<td>$#W[1]$-hard not in $f(k) \cdot</td>
<td>G</td>
<td>^{o(k/\log k)}$</td>
</tr>
<tr>
<td>Bounded matching number</td>
<td>FPT</td>
<td>FPTRAS</td>
<td>FPT</td>
</tr>
<tr>
<td>Bounded treewidth</td>
<td>mixed†</td>
<td>FPTRAS</td>
<td>FPT</td>
</tr>
<tr>
<td>Matching crit. and star crit.</td>
<td>mixed*</td>
<td>FPTRAS</td>
<td>FPT</td>
</tr>
<tr>
<td>Matching crit. or star crit.</td>
<td>mixed†</td>
<td>mixed†</td>
<td>FPT</td>
</tr>
<tr>
<td>$\Phi = \Psi$ from Definition 7.5</td>
<td>$#W[1]$-hard not in $f(k) \cdot</td>
<td>G</td>
<td>^{o(k/\log k)}$</td>
</tr>
<tr>
<td>$\Phi = \text{Clique}$</td>
<td>$#W[1]$-hard not in $f(k) \cdot</td>
<td>G</td>
<td>^{o(k/\log k)}$</td>
</tr>
</tbody>
</table>

Table 1: An overview of the complexity of $(\#)\text{EDGE}\text{SUB}(\Phi)$ for different classes and examples of properties $\Phi$, with respect to exact counting, approximate counting and decision. All run-time lower bounds rely on the Exponential Time Hypothesis, and the absence of FPTRASes relies on the assumption that $W[1]$ does not coincide with $\text{FPT}$ under randomised parameterized reductions. We write “mixed” whenever the respective classes contain both tractable properties and hard properties. The known results about the clique problem are added for completeness; note that $W[1]$-hardness of decision immediately rules out an FPTRAS for approximate counting under the previous assumptions.

† We assume that the minor-closed property $\Phi$ does not have bounded matching number, is not trivially true and that each forbidden minor has a vertex of degree at least 3.

‡ $\Phi = \text{true}$ and $\Phi = \text{false}$ always yield fixed-parameter tractability of exact counting. $\Phi(H) = 1 \iff H$ is a matching yields $\#W[1]$-hardness of exact counting [29]; note that the latter property is of bounded treewidth and satisfies the matching criterion.

∗ $\Phi = \text{true}$ always yields fixed-parameter tractability of exact counting. $\Phi(H) = 1 \iff (H$ is a matching or a star) yields $\#W[1]$-hardness by Theorem 1.3; note that the latter property satisfies the matching criterion and the star criterion.

§ $\Phi = \text{true}$ always yields an FPTRAS for approximate counting. $\Phi = \Psi$ (from Definition 7.5) does not allow for an FPTRAS while satisfying the matching criterion.
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that computing the number of planar subgraphs with $k$ edges in a graph $G$ is $\#W[1]$-hard and, assuming ETH, cannot be solved in time $f(k) \cdot |G|^{o(k/\log k)}$ for any function $f$. In sharp contrast, approximating the number of planar subgraphs with $k$ edges in a graph, as well as deciding whether there is such a planar subgraph can be done efficiently.

We obtain Main Theorem 1 as a combination of our (more general) results for each of the three settings that we study; we discuss these results next.

Results for Exact Counting

Let us return to the case of arbitrary graph properties $\Phi$. Without any further assumptions on $\Phi$, the naive algorithm for $\#\text{EDGE}\text{SUB}(\Phi)$ on the input $(k, G)$ proceeds by enumerating the $k$-edge subsets of $G$ and counting the number of cases where the corresponding subgraph satisfies $\Phi$. This leads to a running time of the form $f(k) \cdot |V(G)|^{k-o(k)}$. However, at least the linear constant in the exponent can be substantially improved using the currently fastest known algorithm for counting subgraphs with $k$ edges due to Curticapean, Dell and Marx [25]. We show in Section 2.2 that it easily extends to the case of $\#\text{EDGE}\text{SUB}(\Phi)$:

**Proposition 1.1.** Let $\Phi$ denote a computable graph property. Then $\#\text{EDGE}\text{SUB}(\Phi)$ can be solved in time $f(k) \cdot |V(G)|^{0.174k+o(k)}$, where $f$ is some computable function.

On the other hand, it was shown by Curticapean and Marx [26] that for the property $\Phi$ of being a matching, the problem $\#\text{EDGE}\text{SUB}(\Phi)$ cannot be solved in time $f(k) \cdot |V(G)|^{k-o(k/\log k)}$ for any function $f$, unless ETH fails. In other words, asymptotically and up to a factor of $1/\log k$, the exponent of $|V(G)|$ in the running time of $\#\text{EDGE}\text{SUB}(\Phi)$ cannot be improved without posing any restriction on $\Phi$.

The goal is hence to identify properties $\Phi$ for which the algorithm in Proposition 1.1 can be (significantly) improved. In the best possible outcome, we hope to identify the properties for which the exponent of $|V(G)|$ does not depend on $k$; those cases are precisely the fixed-parameter tractable ones. An easy consequence of known results for subgraph counting (see for instance [26]) establishes the following tractability criterion; we include the proof only for the sake of completeness in Section 2.2:

**Main Theorem 2.** Let $\Phi$ denote a computable graph property satisfying that there is $M > 0$ such that for all $k$ either the graphs with $k$ edges satisfying $\Phi$ or the graphs with $k$ edges satisfying $\neg \Phi$ have matching number bounded by $M$. Then $\#\text{EDGE}\text{SUB}(\Phi)$ is fixed-parameter tractable.

Examples of properties satisfying the tractability criterion of Main Theorem 2 include, among others, the property of being a star, or the complement thereof. We conjecture that all remaining properties induce $\#W[1]$-hardness and rule out any algorithm running in time $f(k) \cdot |G|^{o(k/\log k)}$ for any function $f$, unless ETH fails. For the case of minor-closed graph properties, we have seen above that this conjecture holds.

Further, the techniques we develop to prove hardness of $\#\text{EDGE}\text{SUB}(\Phi)$ for minor-closed properties $\Phi$ in Main Theorem 1 can also be applied directly to show hardness for other specific properties $\Phi$. Below, we record several natural examples of such properties which are covered by our methods.

**Main Theorem 3.** Consider the following graph properties.

- $\Phi_C(H) = 1$ if and only if $H$ is connected.
- $\Phi_H(H) = 1$ if and only if $H$ is Hamiltonian.

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3 Note that it does not matter whether we choose $|G|$ or $|V(G)|$ for the size of the large graph since we care about the asymptotic behaviour of the exponent.
\( \Phi_E(H) = 1 \) if and only if \( H \) is Eulerian.

\( \Phi_{CF}(H) = 1 \) if and only if \( H \) is claw-free.

For \( \Phi \in \{ \Phi_C, \Phi_H, \Phi_E, \Phi_{CF} \} \), the problem \#\text{EdgeSub}(\Phi) is \#W[1]-hard. Further, unless ETH fails, the problem \#\text{EdgeSub}(\Phi) cannot be solved in time \( f(k) \cdot |G|^{o(k/\log k)} \) for any function \( f \).

**Results for Approximate Counting and Decision**

Our results on exact counting indicate that we have to relax the problem if we aim for tractability results for a larger variety of properties. One approach is to only ask for an approximate count of the number of \( k \)-edge subgraphs satisfying \( \Phi \). Tractability of approximation in the parameterized setting is given by the notion of a fixed-parameter tractable randomized approximation scheme (FPTRAS) as introduced by Arvind and Raman [4]. While we give the formal definition in Section 6, it suffices for now to think of an FPTRAS as a fixed-parameter tractable algorithm that can compute an arbitrarily good approximation of the answer with high probability. Readers familiar with the classical notions of approximate counting algorithms should think of an FPTRAS as an FPRAS in which we additionally allow a factor of \( f(k) \) in the running time, for any computable function \( f \).

For the statement of our results, we say that a property \( \Phi \) satisfies the **matching criterion** if it is true for all but finitely many matchings, and we say that it satisfies the **star criterion** if it is true for all but finitely many stars. Furthermore, we say that \( \Phi \) has bounded treewidth if there is a constant upper bound on the treewidth of graphs that satisfy \( \Phi \).

**Main Theorem 4.** Let \( \Phi \) denote a computable graph property. If \( \Phi \) satisfies the matching criterion **and** the star criterion, or if \( \Phi \) has bounded treewidth, then \#\text{EdgeSub}(\Phi) admits an FPTRAS.

For example, the property of being planar satisfies both, the star and the matching criterion. Moreover, we can show that every minor-closed graph property \( \Phi \) satisfies at least one of the three criteria, and thus always admits an FPTRAS.

Additionally, if not only exact but also approximate counting is intractable, we ask whether we can at least obtain an efficient algorithm for the decision version \text{EdgeSub}(\Phi). Again, we obtain a tractability criterion; observe the subtle difference in the tractability criterion compared to Main Theorem 4.

**Main Theorem 5.** Let \( \Phi \) denote a computable graph property. If \( \Phi \) satisfies the matching criterion **or** the star criterion, or if \( \Phi \) has bounded treewidth, then \text{EdgeSub}(\Phi) is fixed-parameter tractable.

As an easy corollary, we can conclude that for monotone, that is, subgraph-closed properties \( \Phi \), the problem \text{EdgeSub}(\Phi) is always fixed-parameter tractable.

For many previously studied problems, the complexity analysis of approximate counting and decision were related: often an algorithm solving one setting can be used to solve the other setting [58, 30]. However, in our results Main Theorems 4 and 5 we see an asymmetry between the two settings: it suffices for \( \Phi \) to satisfy only one of the star and the matching criterion to induce tractability of the decision version, but we require satisfaction of both for approximate counting. One might expect that this reflects a shortcoming of our proof methods (and that in fact it suffices to check one of the criteria to have tractability of approximate counting). Interestingly, this is not the case:

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4 Every graph property has either bounded treewidth or unbounded matching number. In the latter case, if the property is additionally monotone, it must satisfy the matching criterion.
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Proposition 1.2. There is a computable graph property \( \Psi \) (see Definition 7.5) that satisfies the matching criterion, but not the star criterion, such that \( \text{EdgeSub}(\Psi) \) is fixed-parameter tractable, but \( \#\text{EdgeSub}(\Psi) \) does not admit an FPTRAS unless \( W[1] \) coincides with FPT (the class of all fixed-parameter tractable decision problems) under randomised parameterized reductions.

Dichotomy for Evaluating a Parametrized Tutte Polynomial

As a final part of the presentation of our main results, let us discuss our results on a parametrized Tutte polynomial.

The classical Tutte polynomial (as well as its specializations like the chromatic, flow or reliability polynomial) have received widespread attention, both from a combinatorial as well as a complexity theoretic perspective \cite{Jaeger1992, Dyer1993, Brightwell1999, Hell1993, Hell1993a, Hell1993b, Hell1993c, Hell1993d}. The classical Tutte polynomial is of special interest from a complexity theoretic perspective \cite{Jerrum1986, Hell1993, Hell1993a, Hell1993b, Hell1993c, Hell1993d}. The classical Tutte polynomial is of special interest from a complexity theoretic perspective \cite{Jerrum1986, Hell1993, Hell1993a, Hell1993b, Hell1993c, Hell1993d}.

Formally, the Tutte polynomial is a bivariate graph polynomial defined as follows (see \cite{Jaeger1992}):

\[
T_G(x, y) := \sum_{A \subseteq E(G)} (x - 1)^{k(A) - k(E(G))} \cdot (y - 1)^{k(A) + \#A - \#V(G)} ,
\]

where \( k(S) \) is the number of connected components of the graph \((V(G), S)\). In the aforementioned result, Jaeger et al. \cite{Jaeger1992} also classified the complexity of evaluating the Tutte Polynomial in every pair of (complex) coordinates, that is, for every pair \((a, b)\), the complexity of computing the function \( G \mapsto T_G(a, b) \) is fully understood.

In this work, we consider the following parameterized version of the Tutte Polynomial by restricting to edge-subsets \( A \) in \( G \) of size \( k \):

\[
T_G^k(x, y) := \sum_{A \in \binom{E(G)}{k}} (x - 1)^{k(A) - k(E(G))} \cdot (y - 1)^{k(A) + k - \#V(G)} .
\]

We observe that the parameterized Tutte polynomial can be seen as a weighted version of counting small \( k \)-edge subgraph patterns by assigning to each \( k \)-edge subset \( A \) of \( G \) the weight

\[
(x - 1)^{k(A) - k(E(G))} \cdot (y - 1)^{k(A) + k - \#V(G)} .
\]

By establishing a so-called deletion-contraction recurrence, we show that \( T_G^k(x, y) \) has similar expressive power as its classical counterpart \( T_G(x, y) \):

Main Theorem 6. For any graph \( G \) and positive integer \( k \), the following graph invariants are encoded in \( T_G^k(x, y) \):

1. \( T_G^k(2, 1) \) is the number of \( k \)-forests in \( G \). In other words \( T_G^k(2, 1) \) corresponds to \( \#\text{EdgeSub}(\Phi) \) for the property \( \Phi \) of being a forest.
2. For each positive integer \( c \), the values of \( T_G^k(1 - c, 0) \) determine the numbers of pairs \((A, \sigma)\), where \( A \) is a \( k \)-edge subset of \( G \), and \( \sigma \) is a proper \( c \)-colouring of \((V(G), A)\).
3. From \( T_G^k(2, 0) \) we can compute the numbers of pairs \((A, \eta)\), where \( A \) is a \( k \)-edge subset of \( G \), and \( \eta \) is an acyclic orientation of \((V(G), A)\).

\[5\] They are equal up to trivial modifications; in particular, their complexities coincide.
4. $T^k_G(2, 0)$ also determines the number of $k$-edge subsets $A$ of $G$, such that $(V(G), A)$ has even Betti Number (we give a formal definition of the Betti number in Section 8.4).

5. $T^k_G(0, 2)$ determines the number of $k$-edge subsets $A$ of $G$, such that $(V(G), A)$ has an even number of components.

Note that, while $\#\text{EdgeSub}(\Phi)$ only allows us to count the number of subgraphs with $k$ edges that satisfy $\Phi$, the parameterized Tutte polynomial allows us to count more intricate objects, such as tuples of an edge-subset and a colouring (or acyclic orientation) on the induced graph.

From a complexity theoretic point of view, we obtain a similar result as [47], albeit only for rational coordinates: for each fixed pair $(x, y)$ of coordinates, we consider the problem receiving as input a graph $G$ and a positive integer $k$ and computing $T^k_G(x, y)$. Following the paradigm of this work, we choose $k$ as a parameter, that is, we consider inputs in which $k$ is significantly smaller than $|G|$.

Main Theorem 7. Let $(x, y)$ denote a pair of rational numbers. The problem of computing $T^k_G(x, y)$ is fixed-parameter tractable if $x = 1$ or $(x - 1)(y - 1) = 1$, and #W[1]-hard otherwise.

Consider Figure 1 for a depiction of the tractable cases. Note that Main Theorem 7 yields #W[1]-hardness for each of the aforementioned problems from Main Theorem 6. Note further, that the tractable cases are similar, but not equal to the classical counterpart [47].

Moreover, our proof uses entirely different tools than [47] and illustrates the power and utility of the method presented in the subsequent discussion of our techniques.

Techniques

Our Main Theorems 4 and 5 are obtained easily: the proof of Main Theorem 4 is a standard application (see for instance [58]) of the Monte-Carlo approach, in combination with Ramsey’s theorem, and Arvind and Raman’s algorithm for approximately counting subgraphs of bounded treewidth [4]. Similarly, the proof of Main Theorem 5 uses a standard parameterized Win-Win approach for graphs of bounded treewidth or bounded degree.

Hence, in this technical discussion, we want to focus on the technique that enables us to prove the #W[1]-hardness results for Main Theorems 1 and 3 and, perhaps surprisingly, also for Main Theorem 7.

As a main component, we use the Complexity Monotonicity framework of Curticapean, Dell and Marx [23]. Given a property $\Phi$ and a positive integer $k$, we write $\#\text{EdgeSub}(\Phi, k \to *)$ for the function that
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maps a graph \( G \) to the number of \( k \)-edge subgraphs of \( G \) that satisfy \( \Phi \). Using a well-known transformation via Möbius inversion \([56, \text{Chapter 5.2}]\), we can show that there are rational numbers \( a_1, \ldots, a_\ell \) and graphs \( H_1, \ldots, H_\ell \) such that for each graph \( G \) we have

\[
\#\text{EdgeSub}(\Phi, k \to G) = \sum_{i=1}^{k} a_i \cdot \#\text{Hom}(H_i \to G),
\]

where \( \#\text{Hom}(H_i \to G) \) is the number of graph homomorphisms from \( H_i \) to \( G \). In other words, we can express \( \#\text{EdgeSub}(\Phi, k \to \star) \) as a finite linear combination of homomorphism counts. Here, we can then apply the Complexity Monotonicity framework \([25]\), which asserts that computing a finite linear combination of homomorphism counts is precisely as hard as its hardest term (among the terms with a non-zero coefficient). However, the complexity of computing the number of homomorphisms from small pattern graphs to large host graphs is very well-understood \([28, 57]\). Roughly speaking, the higher the treewidth of the pattern graph, the harder the problem becomes; we make this formal in Section 2.2.

Instead of our original problem \( \#\text{EdgeSub}(\Phi) \), we can thus consider the problem of computing linear combinations of graph homomorphism counts. In particular, to obtain hardness, it suffices to understand for which of the coefficients in equation (1) we have \( a_i \neq 0 \), depending on \( k \) and \( \Phi \).

Relying on the well-known fact that the Möbius function of the partition lattice alternates in sign, Curticapean, Dell, and Marx \([25]\) observed that non-trivial cancellations cannot occur in equation (1) if, for each \( k \), every \( k \)-edge graph that satisfies \( \Phi \) must have the same number of vertices. Consequently, if the matching number is unbounded, those properties yield \#W[1]-hardness. An example for such a property is the case of \( \Phi(H) = 1 \) if and only if \( H \) is a tree. In contrast, the intractability result for the case of \( \Phi = \) acyclicity (that is, being a forest) turned out to be much harder to show \([12]\), indicated by connections to parameterized counting problems in matroid theory.

In later work, the coefficients \( a_i \) were shown to have even more interesting structure: the coefficients \( a_i \) describe topological and algebraic invariants of the set of pattern graphs. For example, in \([65]\) it was shown that the coefficient of the \( k \)-clique in case of counting vertex-induced subgraphs with property \( \Phi \) is the reduced Euler characteristic of a simplicial complex associated with \( \Phi \) and can thus, if non-zero, be used to establish evasiveness of certain graph properties \([51]\).

In this work, we prove additional insights into said coefficients \( a_i \).\(^6\) For any graph \( H \) we give an explicit formula for its coefficient \( a_H \) in terms of a sum over the fractures on \( H \), an additional combinatorial structure on a graph \( H \) (see Definition 2.7 for details). Our most crucial insight is then that we can drastically simplify the expression of the coefficient \( a_H \) modulo a prime \( \ell \) if \( H \) admits a vertex-transitive action of a group of order given by a power of \( \ell \). In this case, we obtain an action of the group on the set of fractures on \( H \) and in the formula for \( a_H \) all contributions from fractures not fixed by the group cancel out modulo \( \ell \).

In particular, we consider graphs \( H \) which are Cayley graphs of a finite group of prime power order and a symmetric set of generators. Since the Cayley graph of a group always has a natural vertex-transitive action of this group, such Cayley graphs always have the desired symmetry properties. We exploit this by showing that there is a constant number of fractures fixed by the group action. This in turn allows us to write \( (a_H \mod \ell) \) as a finite sum of terms depending on the value of \( \Phi \) on some explicit graphs.

Specifically, the first set of Cayley graphs we consider are the toroidal grids \( \mathbb{G}_\ell \), which are depicted in Figure 2. Since the treewidth of \( \mathbb{G}_\ell \) diverges with \( \ell \), we thus obtain a \#W[1]-hardness result whenever the coefficient \( a_{\mathbb{G}_\ell} \) does not vanish for infinitely many \( \ell \).

\(^6\) For technical reasons, the approach we describe below requires us to consider a coloured version of \( \#\text{EdgeSub}(\Phi) \), which is, however, shown to be interreducible with the uncoloured one.
Figure 2 Two isomorphic representations of the toroidal grid ⊙:\: On the left hand side as a grid with connected endpoints, on the right hand side as a stylized torus.

Writing $M_k$ for the matching of size $k$, $P_2$ for the path consisting of 2 edges, $C_k$ for the cycle of length $k$, $S_k$ for a sun (a cycle with dangling edges) of size $k$, and ⊙\$ for the toroidal grid of size $k$, our first main technical result reads as follows:

**Theorem 1.3 (Simplified version).** Let $\Phi$ denote a computable graph property and assume that infinitely many primes $\ell$ satisfy the equation

$$-6\Phi(M_{2\ell^2}) + 4\Phi(M_{\ell^2} + \ell^2C_\ell) + 8\Phi(\ell^2P_2) - \Phi(2\ellC_\ell) - 2\Phi(\ell^2C_2\ell) - 4\Phi(\ellS_\ell) + \Phi(\bullet\bullet\bullet) \neq 0 \mod \ell. \tag{2}$$

Then $\#\text{EdgeSub}(\Phi)$ is $\#W[1]$-hard.

As a toy example for an application of Theorem 1.3 let us consider the property $\Phi$ of being connected. Observe that among the graphs in (2), only ⊙\$ is connected, and thus the sum is always 1 for $\ell \geq 2$. Thus, indeed the left-hand side of (2) is nonzero, proving that $\#\text{EdgeSub}(\Phi)$ is $\#W[1]$-hard.

Using Theorem 1.3 we can prove most of the $\#W[1]$-hardness results of Main Theorem 1. However, using the toroidal grid ⊙\$ we cannot prove (almost) tight conditional lower bounds: the treewidth grows only with the square-root of the parameter $k$ (that is the number of edges of the graph). To address this problem, we consider a second family of 4-regular Cayley graphs, constructed explicitly by Peyerimhoff and Vdovina [62], which have the additional property of being expander graphs. In particular, for these graphs, the treewidth grows linearly in the number of edges. This allows us to obtain almost tight conditional lower bounds. The variant of Theorem 1.3 for these Cayley graph expanders can be found in Lemma 4.7 (in combination with Lemma 3.8).

The only drawback of the Cayley graphs from [62] is that the corresponding groups always have orders given by powers of 2 (in contrast to having arbitrary primes $\ell$ in Theorem 1.3). Hence, our criterion for hardness is the nonvanishing of some expression modulo 2. Ultimately, this is the reason why for the conditional lower bounds in Main Theorem 1 we need to exclude forbidden minors having a vertex of degree 2 or less.

Finally, to obtain Main Theorem 7, we express the parametrized Tutte polynomial at a rational point $(x, y)$ as a linear combination of (fractures of) toroidal grids; the proof of Theorem 1.3 then essentially shows that this linear combination always contains a graph with unbounded treewidth, yielding $\#W[1]$-hardness.

---

7 We write + for (disjoint) graph union and $\ell H$ for the graph consisting of $\ell$ disjoint copies of $H$. Further, we set $\Phi(H) = 1$ if $H$ satisfies $\Phi$ and $\Phi(H) = 0$ otherwise.
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2 Preliminaries

Given a finite set $S$, we write $|S|$ and $\#S$ for the cardinality of $S$. Further, given a function $f : X \times Y \to Z$ and an element $x \in X$, we write $f(x,*) : B \to C$ for the function $y \mapsto f(x,y)$.

2.1 Graphs and Homomorphisms

Graphs in this work are simple and irreflexive, that is, we do not allow multiple edges or self-loops. Given a graph $G$ and a subset $A$ of $E(G)$, we write $G(A)$ for the graph $(V(G), A)$, and we write $G[A]$ for the graph obtained from $G(A)$ by deleting all isolated vertices.

Given graphs $F$ and $G$, a homomorphism from $F$ to $G$ is a mapping $\varphi : V(F) \to V(G)$ which is edge-preserving, that is, for each edge $\{u, v\} \in E(F)$ we have $\{\varphi(u), \varphi(v)\} \in E(G)$. A homomorphism is called an embedding if it is injective (on the vertices). We write $\text{Hom}(F \to G)$ and $\text{Emb}(F \to G)$ for the set of all homomorphisms and embeddings, respectively, from $F$ to $G$.

An isomorphism between two graphs $F$ and $G$ is a bijective embedding $\varphi$ satisfying the stronger constraint $\{u, v\} \in E(F) \iff \{\varphi(u), \varphi(v)\} \in E(G)$. We say that $F$ and $G$ are isomorphic, denoted by $F \cong G$, if an isomorphism from $F$ to $G$ exists. An isomorphism from a graph $F$ to itself is called an automorphism and we write $\text{Aut}(F)$ for the group formed by such automorphisms (where the group operation is the composition of automorphisms).

A graph $F$ is a minor of a graph $G$ if it can be obtained from $G$ by a sequence of edge- and vertex-deletions and edge-contractions (with multiple edges and self-loops deleted).

A graph $G$ is called $k$-edge-coloured if the edges of $G$ are coloured with (at most) $k$ pairwise different colours. Given a homomorphism $\varphi \in \text{Hom}(G \to H)$ for some graphs $G$ and $H$, we also call $\varphi$ an $H$-colouring. Moreover, an $H$-coloured graph is a pair of a graph $G$ and an $H$-colouring $\varphi$. For ease of readability, we say that a graph $G$ is $H$-coloured if the $H$-colouring is implicit or clear from the context. Observe that an $H$-colouring $\varphi$ of a graph $G$ induces a $\#E(H)$-edge-colouring by mapping an edge $\{u, v\} \in E(G)$ to the colour $\{\varphi(u), \varphi(v)\}$. Throughout this work, we use the following notion of homomorphisms between $H$-coloured graphs:

Definition 2.1. Let $F$ and $G$ denote two $H$-coloured graphs and let $c_F$ and $c_G$ denote their $H$-colourings. A homomorphism $\varphi$ from $F$ to $G$ is called colour-preserving if $c_G(\varphi(v)) = c_F(v)$ for every $v \in V(F)$. We write $\text{Hom}^{\text{cp}}(F \to_H G)$ for the set of all colour-preserving homomorphisms from $F$ to $G$. Colour-preserving embeddings and the set $\text{Emb}^{\text{cp}}(F \to_H G)$ are defined similarly.

Further, we say that two $H$-coloured graphs $F$ and $G$ are isomorphic as $H$-coloured graphs, denoted by $F \cong_H G$, if there is a colour-preserving isomorphism from $F$ to $G$.

Note that, given two $H$-coloured graphs $F$ and $G$, we write $F \cong G$ (rather than $F \cong_H G$) if the underlying uncoloured graphs are isomorphic.

For the treatment of decision and approximate counting, we introduce the following classification criteria on (computable) graph properties. To this end, we write $K_{\ell, r}$ for the biclique with $\ell$ vertices on the left and $r$ vertices on the right side, respectively. In particular, $K_{1, r}$ denotes the star of size $r$.

Definition 2.2. Let $\Phi$ denote a computable graph property. We say that

- $\Phi$ satisfies the matching criterion if $\Phi(M_k) = 1$ for all but finitely many $k$.
- $\Phi$ satisfies the star criterion if $\Phi(K_{1, k}) = 1$ for all but finitely many $k$.
- $\Phi$ has bounded treewidth if there is a constant $B$ such that $\Phi$ is false on all graphs of treewidth at least $B$. 


For example, the properties of being bipartite or planar satisfy both, the matching and the star criterion. Furthermore, the property of being 2-regular has bounded treewidth, while the criterion of just being regular satisfies only the matching criterion. Further, the property of being a tree is an example that satisfies both, the star criterion and is of bounded treewidth, while the property of being a forest satisfies all three criteria.

Expander Graphs

All (almost-tight) conditional lower bounds in this work rely on the existence of certain (families of) expander graphs. Given a positive integer \(d\), a rational \(c > 0\), and a class of graphs \(\mathcal{G} = \{G_1, G_2, \ldots\}\) with \(#V(G_i) = n_i\), we call \(\mathcal{G}\) a family of \((n_i, d, c)\)-expanders if, for all \(i\), the graph \(G_i\) is \(d\)-regular and satisfies

\[
\forall X \subseteq V(G_i) : |S(X)| \geq c \left(1 - \frac{|X|}{n_i}\right)|X|,
\]

where \(S(X)\) is the set of all vertices in \(V(G_i) \setminus X\) that are adjacent to a vertex in \(X\).

While being sparse due to \(d\)-regularity, expander graphs have treewidth\(^8\) linear in the number of vertices (see for instance Proposition 1 in [44] and set \(\alpha = 1/2\)). Furthermore, they admit arbitrarily large clique minors\(^9\). Formally, we have:

**Fact 2.3.** Fix a rational \(c\) and an integer \(d\), and let \(\mathcal{G}\) denote a family of \((n_i, d, c)\)-expanders. Then, \(#E(G_i) \in \Theta(n_i)\) and \(\text{tw}(G_i) \in \Theta(n_i)\). Furthermore, for each positive integer \(k\) there is an index \(j\) such that for all \(i \geq j\), the graph \(G_i\) contains the complete graph on \(k\) vertices as a minor.

2.2 Parameterized Complexity Theory

A parameterized counting problem is a pair of a counting problem \(P : \Sigma^* \to \mathbb{N}\) and a parameterization \(\kappa : \Sigma^* \to \mathbb{N}\). A parameterized decision problem is a pair \((P, \kappa)\) of a decision problem \(P : \Sigma^* \to \{0, 1\}\) and a parameterization \(\kappa\). Consider for example the problems \#CLIQUE and CLIQUE: on input a graph \(G\) and a positive integer \(k\), the task is to either compute the number of \(k\)-cliques in \(G\) or to detect the mere existence of a \(k\)-clique, respectively. The parameterization is given by \(\kappa(G, k) := k\) for both problems.

A parameterized problem \((P, \kappa)\) is called fixed-parameter tractable (FPT) if there is a computable function \(f\) such that \(P\) can be computed in time \(f(\kappa(x)) \cdot |x|^{O(1)}\). For historic reasons, the class of all fixed-parameter tractable decision problems is called \(\text{FPT}\)\(^8\). Furthermore, a parameterized Turing reduction from \((P, \kappa)\) to \((\hat{P}, \hat{\kappa})\) is a Turing reduction from \(P\) to \(\hat{P}\) that, on input \(x\), runs in time \(f(\kappa(x)) \cdot |x|^{O(1)}\) and additionally satisfies \(\hat{\kappa}(y) \leq f(\kappa(x))\) for each oracle query \(y\). Again, \(f\) only needs to be some fixed computable function. We write \((P, \kappa) \leq_{\text{FPT}} (\hat{P}, \hat{\kappa})\) if a parameterized Turing reduction exists.

A parameterized counting problem is \#W[1]-hard if it can be reduced from \#CLIQUE, and, similarly, a parameterized decision problem is \(W[1]\)-hard if it can be reduced from CLIQUE, both with respect to parameterized Turing reductions. Under reasonable assumptions, such as the Exponential Time Hypothesis (ETH)\(^8\), defined below, \#W[1]- and \(W[1]\)-hard problems are not fixed-parameter tractable\(^10\).

---

\(^8\) We use the graph parameter “treewidth” (tw) in a black-box manner only, and refer the reader to, for instance, Chapter 7 of [27] for a detailed exposition.

\(^9\) In some literature FPT is used for both, parameterized decision and counting problems, while some authors write FFPT for the class of all fixed-parameter tractable parameterized counting problems.

\(^10\) In fact, Chen et al. [15, 16] proved the much stronger statement that \#CLIQUE cannot be solved in time \(f(k) \cdot |V(G)|^{o(k)}\) for any function \(f\), unless ETH fails.
Conjecture 2.4 (Exponential Time Hypothesis). The Exponential Time Hypothesis (ETH) asserts that 3-SAT cannot be solved in time $\exp(o(n))$, where $m$ is the number of clauses of the input formula.

Our hardness results in this paper are obtained by reducing from the problem $\#\text{Hom}(H)$. Given a fixed class of graphs $\mathcal{H}$, in the problem $\#\text{Hom}(H)$ the input is a graph $H \in \mathcal{H}$ and an arbitrary graph $G$ and the task is to compute the number of homomorphisms from $H$ to $G$; the parameter is $|H|$. Dalmau and Jonsson [28] established an exhaustive classification for this problem, stating that $\#\text{Hom}(H)$ is fixed-parameter tractable if the treewidth of graphs in $\mathcal{H}$ is bounded by a constant, and $\#\text{W}[1]$-hard, otherwise.

Let $\Phi$ denote a graph property, that is, a function from (isomorphism classes) of graphs to $\{0, 1\}$. Setting

$$\text{EdgeSub}(\Phi, k \to G) := \{ A \subseteq E(G) \mid \#A = k \land \Phi(G[A]) = 1 \},$$

we define $\#\text{EdgeSub}(\Phi)$ as the parameterized counting problem in which on input a graph $G$ and a positive integer $k$, the task is to compute the number $\#\text{EdgeSub}(\Phi, k \to G)$; the parameter is $k$.

In this paper, we often rely on the following important, but easy observation: write $\Phi_k$ for the set of graphs $H$ with $k$ edges and without isolated vertices, that satisfy $\Phi$. Then we have

$$\#\text{EdgeSub}(\Phi, k \to G) = \sum_{H \in \Phi_k} \#\text{Sub}(H \to G),$$

where $\#\text{Sub}(H \to G)$ is the number of subgraphs of $G$ that are isomorphic to $H$.

Using the aforementioned transformation, both, Proposition 1.1 and Main Theorem 2 are easy consequences of known results regarding the subgraph counting problem. We add their proofs only for the sake of completeness:

 Proposition 1.1. Let $\Phi$ denote a computable graph property. Then $\#\text{EdgeSub}(\Phi)$ can be solved in time $f(k) \cdot |V(G)|^{0.174k+o(k)}$, where $f$ is some computable function.

Proof. The fastest known algorithm for computing $\#\text{Sub}(H \to G)$ for a $k$-edge graph $H$ runs in time $k^{O(k)} \cdot |V(G)|^{0.174k+o(k)}$ and is due to Curticapean, Dell and Marx [25]. Now observe that the size of $\Phi_k$ is bounded by a function in $k$, since graphs in $\Phi_k$ have $k$ edges and no isolated vertices and thus have at most $2k$ vertices. Consequently, their algorithm extends to $\#\text{EdgeSub}(\Phi)$ by computing $\#\text{EdgeSub}(\Phi, k \to G)$ as given in Equation (3).

Note that the growth of $f$ in the previous result depends, among other factors, on the complexity of verifying $\Phi$.

For the following, recall that a property $\Phi$ has bounded matching number if there is a constant $c$ such that $\Phi$ is false on all graphs containing a matching of size at least $c$. Furthermore, write $\neg\Phi$ for the complement of $\Phi$.

 Main Theorem 2. Let $\Phi$ denote a computable graph property satisfying that there is $M > 0$ such that for all $k$ either the graphs with $k$ edges satisfying $\Phi$ or the graphs with $k$ edges satisfying $\neg\Phi$ have matching number bounded by $M$. Then $\#\text{EdgeSub}(\Phi)$ is fixed-parameter tractable.

Proof. Applying Equation (3), we observe that counting subgraphs isomorphic to $H$ is fixed-parameter tractable (even polynomial-time solvable) if there is a constant upper bound on the size of the largest matching of $H$ [28]. This allows us to compute $\#\text{EdgeSub}(\Phi, k \to G)$ in the desired running time if the graphs in $\Phi_k$ have matching number bounded by $M$. In case the latter is true for $\neg\Phi_k$ instead, we use the fact that $\#\text{EdgeSub}(\Phi, k \to G) = (\#E(G))^k - \#\text{EdgeSub}(\neg\Phi, k \to G)$ and proceed similarly.
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2.3 Combinatorial Commutative Algebra

We assume familiarity with the notions of basic group theory and refer the reader to for instance [54] for a detailed introduction. Given a positive integer \( \ell \), we write \( \mathbb{Z}_\ell \) for the group of integers modulo \( \ell \), and we write \( \mathbb{Z}_\ell^k \) for the \( k \)-fold direct product of \( \mathbb{Z}_\ell \); recall that the binary operation of the direct product is defined coordinate-wise.

For a prime \( p \), recall that a finite group \( G \) is called a \( p \)-group if the order \( |G| \) of \( G \) is a power of \( p \). Recall that by Lagrange’s theorem, this implies that the order of any subgroup \( H \) of \( G \) is likewise a power of \( p \).

Given a group \( G \) and a subgroup \( H \subseteq G \), we write \( G/H \) for the set of left cosets of \( H \). Formally, a left coset of \( H \) is an equivalence class of the following equivalence relation on \( G \): two elements \( g, g' \in G \) are equivalent if and only if \( g' = gh \) for some \( h \in H \). We write \( gH \) for the equivalence class of \( g \in G \). We define the index of \( H \) in \( G \) as the cardinality \( |G:H| = |G|/|H| \) of the set of left cosets. Note that \( |G:H| \) might be finite even though \( |G| \) is infinite; in fact, we encounter this case when we treat Cayley graph expanders of 2-groups. The index satisfies the basic identity \(|G|=|G:H|\cdot|H|\), and again, with a slight abuse of notation, we observe that the identity remains well-defined in the infinite case: \(|G|=|G:H|\) is infinite if and only if one of \(|G:H|\) or \(|H|\) is infinite. If the subgroup \( H \) is normal in \( G \) (that is for each \( g \in G \) we have that the subset \( gHg^{-1} = \{ghg^{-1} \mid h \in H \} \) of \( G \) is equal to \( H \)), then the set \( G/H \) naturally carries the structure of a group, with the group operation defined by \((g_1H)\cdot(g_2H) = (g_1 \cdot g_2)H\). In this case, we call \( G/H \) a quotient group.

Given an element \( g \in G \) we write

\[
\langle g \rangle = \{g^a : a \in \mathbb{Z} \} \subseteq G
\]

for the subgroup generated by \( g \) (recall that for a negative integer \( a \) we define \( g^a \) as the \( |a| \)-th power of the inverse element of \( g \)). If there is a positive integer \( a \) such that \( g^a \) equals the neutral element of \( G \), we define the order \( \text{ord}_G(g) \) of \( g \) as the smallest such positive integer \( a \) (and set \( \text{ord}_G(g) = \infty \) otherwise). If \( g \) has finite order, the subgroup \( \langle g \rangle \) of \( G \) generated by \( g \) is isomorphic to the cyclic group \( \mathbb{Z}_{\text{ord}_G(g)} \).

Given a finite group \( A \) and a set \( S \) of generators of \( A \), the Cayley graph of \( A \) and \( S \), denoted by \( \Gamma(A, S) \) has as vertices the elements of \( A \), and two vertices \( u \) and \( v \) are adjacent if there is an \( s \in S \) such that \( v = us \). For example, the Cayley graph \( \Gamma(\mathbb{Z}_\ell, \{1, -1\}) \) is isomorphic to the cycle of length \( \ell \).

Möbius Inversion and the Partition Lattice

We follow the notation of the standard textbook of Stanley [73]. Given a finite partially ordered set \( (L, \leq) \), and a function \( f : L \to \mathbb{Q} \), the zeta transformation \( \zeta f : L \to \mathbb{Q} \) is defined as

\[
\zeta f(\sigma) := \sum_{\rho \leq \sigma} f(\rho).
\]

The principle of Möbius inversion allows us to invert a zeta transformation; a proof of the following theorem can be found in [73, Chapt. 3]

\textbf{Theorem 2.5.} Given a partially ordered set \( (L, \leq) \), there is a computable function \( \mu : L \times L \to \mathbb{Z} \), called the Möbius function, such that for all \( f : L \to \mathbb{Q} \) and \( \sigma \in L \) we have

\[
f(\sigma) = \sum_{\rho \geq \sigma} \mu(\sigma, \rho) \cdot \zeta f(\rho).
\]

\[11\]Note that, in some literature, Cayley graphs are coloured and directed. However, we only consider the underlying uncoloured and simple graph.
We use Möbius inversion on the ordering of partitions of finite sets. Given two partitions \( \sigma \) and \( \rho \) of a finite set \( S \), we say that \( \sigma \) refines \( \rho \) if every block of \( \sigma \) is a subset of a block of \( \rho \), and in this case we write \( \sigma \leq \rho \). This induces a partial order, called the partition lattice\(^{12}\) of \( S \). The explicit formula of the Möbius function over the partition lattice is of particular importance in this work:

\[ \mu(\sigma, \top) = (-1)^{|\sigma|-1} \cdot (|\sigma|-1)!, \]

where \( \top = \{ S \} \) denotes the coarsest partition.

**Fractured graphs**

**Definition 2.7 (Fractures).** Let \( H \) denote a graph. A fracture of \( H \) is a tuple \( \rho = (\rho_v)_{v \in V(H)} \), where \( \rho_v \) is a partition of the set of edges \( E_H(v) \) of \( H \) incident to \( v \).

Note that the set of all fractures of \( H \), denoted by \( \mathcal{L}(H) \), is a lattice that is isomorphic to the (point-wise) product of the partition lattices of \( E_H(v) \) for each \( v \in V(H) \). In particular, we write \( \sigma \leq \rho \) if, for each \( v \in V(H) \), the partition \( \sigma_v \) refines the partition \( \rho_v \). Consider Figure 3 for a visualization of a fracture and its refinement.

---

\(^{12}\) A lattice is a partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound. For a formal definition we refer to [73].
Note further that a fracture describes how to split (or fracture) each vertex of a given graph: for each vertex \( v \), create a vertex \( v^B \) for each block \( B \) in the partition \( \rho_v \); edges originally incident to \( v \) are made incident to \( v^B \) if and only if they are contained in \( B \). We call the resulting graph the \emph{fractured graph} \( H \sharp \rho \); a formal definition is given in Definition \ref{fractured-graph}. A visualization is given in Figure \ref{fractured-graph}.

**Definition 2.8 (Fractured Graph \( H \sharp \rho \)).** Given a graph \( H \), consider the matching \( M_H \) containing one edge for each edge of \( H \); formally,

\[
V(M_H) := \bigcup_{e = \{u,v\} \in E(H)} \{e_u, e_v\} \quad \text{and} \quad E(M_H) := \{\{e_u, e_v\} \mid e = \{u, v\} \in E(H)\}.
\]

For a fracture \( \rho \) of \( H \), we define \(^{13}\) the graph \( H \sharp \rho \) as the quotient graph of \( M_H \) under the equivalence relation on \( V(M_H) \) which identifies two vertices \( e_v, f_w \) of \( M_H \) if and only if \( v = w \) and \( e, f \) are in the same block of the partition \( \rho_v \) of \( E_H(v) \). We write \( v^B \) for the vertex of \( H \sharp \rho \) given by the equivalence class of the vertices \( e_v \) (for which \( e \in B \)) of \( M_H \).

The fractured graph \( H \sharp \rho \) comes with a natural \( H \)-colouring. Indeed, the homomorphism \( M_H \to H \) which sends \( e_v \in V(M_H) \) to \( e \in V(H) \) descends to \( H \sharp \rho \) so that we always have a canonical diagram \( M_H \to H \sharp \rho \to H \) of graph homomorphisms. For example, for any graph \( H \), the fracture \( \perp \), with \( \perp_v := \{e \mid e \in E_H(v)\} \), induces the fractured graph \( H \perp H = M_H \); the fracture \( \top \), with \( \top_v := \{E_H(v)\} \), induces the fractured graph \( H \top H = H \). The fractures \( \perp, \top \) are the minimal and maximal elements of the lattice \( \mathcal{L}(H) \), respectively.

Given a graph property \( \Phi \) and a graph \( H \), we write \( \mathcal{L}(\Phi, H) \) for the set of all fractures \( \rho \) of \( H \) that satisfy \( \Phi(H \sharp \rho) = 1 \).

\section{The Colour-Preserving Homomorphism Basis}

It turns out that the analysis of the complexity of \( \#\text{EDGE}_\text{SUB}(\Phi) \) is much easier if a \emph{colourful} version of the problem is considered. For our hardness results, we then show that the colourful version reduces to the uncoloured version. To this end, recall that an \( H \)-colouring of a graph \( G \) is a homomorphism from \( G \) to \( H \), and that a graph \( G \) is \emph{\( H \)-coloured} if \( G \) is equipped with an \( H \)-colouring \( c \). Recall further the implicitly defined \( \#E(H) \)-edge-colouring of \( G \). In the colourful version of \( \#\text{EDGE}_\text{SUB}(\Phi) \), denoted by \( \#\text{COLEdge}_\text{SUB}(\Phi) \), the task is to compute the cardinality of the set

\[
\text{ColEdge}_\text{SUB}(\Phi \to H \ G) := \{A \subseteq E(G) \mid \#A = \#E(H) \land c(A) = E(H) \land \Phi(G[A]) = 1\}.
\]

In particular, we write \( \#\text{ColEdge}_\text{SUB}(\Phi \to H \ast) \) for the function that maps an \( H \)-coloured graph \( G \) to the number \( \#\text{ColEdge}_\text{SUB}(\Phi \to H \ G) \). Note that \( \#A = \#E(H) \land c(A) = E(H) \) implies that the \( A \) contains each of the \( \#E(H) \) colours precisely once. Further, note that \( \Phi(G[A]) = 1 \) if and only if \( \Phi \) holds on the (uncoloured) graph \( G[A] \).

Each element \( A \in \text{ColEdge}_\text{SUB}(\Phi \to H \ G) \) induces a fracture \( \rho \) of \( H \), where for \( v \in V(G) \) two edges \( e, f \in E_H(v) \) are in the same block of \( \rho_v \) if and only if their (unique) preimages \( \widehat{e}, \widehat{f} \in A \) under \( c : A \to E(H) \) are connected to the same endpoint in \( V_v \). From the construction, it immediately follows that \( G[A] \) and \( H \sharp \rho \) are canonically isomorphic as \( H \)-coloured graphs, that is, \( G[A] \cong_H H \sharp \rho \).

\(^{13}\) The notation \( H \sharp \rho \) stems from the fact that the symbol “\( \sharp \)” is commonly used for medical fractures.
Our goal is to express \( \#\text{ColEdgeSub}(\Phi \rightarrow_H \ast) \) as a linear combination of (colour-preserving) homomorphism counts from graphs only depending on \( \Phi \) and \( H \). In case \( H \) is a torus, we establish an explicit criterion sufficient for the term \( \#\text{Hom}_{\text{cp}}(\sigma_r \rightarrow \sigma_t, \ast) \) to survive with a non-zero coefficient in this linear combination. The existence of the linear combination is given by the following lemma:

\[ \text{Lemma 3.1.} \quad \text{Let } H \text{ denote a graph. We have} \]

\[
\#\text{ColEdgeSub}(\Phi \rightarrow_H \ast) = \sum_{\sigma \in \mathcal{L}(\Phi, H)} \sum_{\rho \geq \sigma} \mu(\sigma, \rho) \cdot \#\text{Hom}_{\text{cp}}(H^\ast \rho \rightarrow_H \ast),
\]

where the relation \( \leq \) and the Möbius function \( \mu \) are over the lattice of fractures \( \mathcal{L}(H) \).

\[ \text{Proof.} \quad \text{Let } G \text{ denote an } H-\text{coloured graph. We first partition the elements } A \text{ of } \text{ColEdgeSub}(\Phi \rightarrow_H G) \text{ according to their induced fractures. Writing } \text{ColEdgeSub}(\Phi \rightarrow_H G)[\sigma] \text{ for the set of } A \text{ inducing the fracture } \sigma \in \mathcal{L}(H), \text{ we obtain} \]

\[
\#\text{ColEdgeSub}(\Phi \rightarrow_H G) = \sum_{\sigma \in \mathcal{L}(\Phi, H)} \#\text{ColEdgeSub}(\Phi \rightarrow_H G)[\sigma],
\]

since \( \#\text{ColEdgeSub}(\Phi \rightarrow_H G)[\sigma] = 0 \) for all \( \sigma \notin \mathcal{L}(\Phi, H) \). From the fact that \( G[A] \) is canonically isomorphic to \( H^\ast \sigma \) as an \( H-\)coloured graph, for \( \sigma \) associated to \( A \in \text{ColEdgeSub}(\Phi \rightarrow_H G) \), it follows that

\[
\#\text{ColEdgeSub}(\Phi \rightarrow_H G)[\sigma] = \#\text{Emb}_{\text{cp}}(H^\ast \sigma \rightarrow_H G).
\]

Note that we are using that graphs of the form \( H^\ast \sigma \) can have no non-trivial automorphisms as \( H-\)coloured graphs (since all edges must be fixed). It remains to show that

\[
\#\text{Emb}_{\text{cp}}(H^\ast \sigma \rightarrow_H G) = \sum_{\rho \geq \sigma} \mu(\sigma, \rho) \cdot \#\text{Hom}_{\text{cp}}(H^\ast \rho \rightarrow_H G) \tag{4}
\]

To this end, we establish the following zeta transformation, which should be considered as a colour-preserving version of the standard transformation in the uncoloured case (see e.g. [56, Section 5.2.3]).

\[ \text{Claim 3.2.} \quad \text{For every fracture } \sigma \text{ of } H, \text{ we have} \]

\[
\#\text{Hom}_{\text{cp}}(H^\ast \sigma \rightarrow_H G) = \sum_{\rho \geq \sigma} \#\text{Emb}_{\text{cp}}(H^\ast \rho \rightarrow_H G)
\]

\[ \text{Proof.} \quad \text{Every colour-preserving homomorphism } \varphi \text{ from } H^\ast \sigma \text{ to } G, \text{ induces a fracture } \rho \geq \sigma, \text{ that is, } \rho_v \text{ is a coarsening of } \sigma_v \text{ for every } v \in V(H). \text{ Indeed, recall that the vertices of } H^\ast \sigma \text{ over } v \in V(H) \text{ correspond to the blocks } B \text{ of the partition } \sigma_v \text{ of the edges } E_H(v). \text{ Then the partition } \rho_v \text{ of } E_H(v) \text{ is obtained from } \sigma_v \text{ by joining those blocks } B, B' \text{ whose associated vertices in } H^\ast \sigma \text{ map to the same vertex of } G \text{ under } \varphi. \text{ We have that the subgraph of } G \text{ given by the image of } H^\ast \sigma \text{ under } \varphi \text{ is canonically isomorphic to } H^\ast \rho \text{ as an } H-\text{coloured graph.}

\text{Let us call two homomorphisms in } \text{Hom}_{\text{cp}}(H^\ast \sigma \rightarrow_H G) \text{ equivalent if they induce the same fracture and write } \text{Hom}_{\text{cp}}(H^\ast \sigma \rightarrow_H G)[\rho] \text{ for the equivalence class of all homomorphisms inducing } \rho. \text{ The claim then follows by partitioning the set } \text{Hom}_{\text{cp}}(H^\ast \sigma \rightarrow_H G) \text{ into those equivalence classes and observing that}

\[
\#\text{Hom}_{\text{cp}}(H^\ast \sigma \rightarrow_H G)[\rho] = \#\text{Emb}_{\text{cp}}(H^\ast \rho \rightarrow_H G).
\]

\( \square \)

Equation (4) is now obtained by using Möbius inversion (Theorem 2.5) on the zeta-transformation given by the previous claim. This concludes the proof.
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Let us now collect for the coefficient of the term $\#\text{Hom}_{cp}(H \not\simeq \top \rightarrow_H G)$, where $\top$ is the maximum fracture of $H$ with respect to the ordering $\leq$. In particular, each partition of $\top$ only consists of a single block and thus $H \not\simeq \top \cong H$, where the isomorphism is given by the $H$-colouring of $H \not\simeq \top$.

**Corollary 3.3.** Let $\Phi$ denote a computable graph property and let $H$ denote a graph. There is a unique computable function $a_{\Phi,H} : \mathcal{L}(H) \rightarrow \mathbb{Z}$ such that

$$\#\text{ColEdgeSub}(\Phi \rightarrow_H *) = \sum_{\rho \in \mathcal{L}(H)} a_{\Phi,H}(\rho) \cdot \#\text{Hom}_{cp}(H \not\simeq \top \rightarrow_H *) ,$$

For $\rho = \top$ we have

$$a_{\Phi,H}(\top) = \sum_{\sigma \in \mathcal{L}(\Phi,H)} \prod_{v \in V(H)} (-1)^{|\sigma_v| - 1} \cdot (|\sigma_v| - 1)!$$

Here, $|\sigma_v|$ denotes the number of blocks of partition $\sigma_v$.

**Proof.** The first claim follows immediately from Lemma 3.1 by collecting coefficients; note that $\Phi$ and $\mu$ are computable, and that the image of $\mu$ is a subset of $\mathbb{Z}$. For the second claim, we collect the coefficients of $\#\text{Hom}_{cp}(H \not\simeq \top \rightarrow_H *)$ in Lemma 3.1 and obtain

$$a_{\Phi,H}(\top) = \sum_{\sigma \in \mathcal{L}(\Phi,H)} \mu(\sigma, \top) .$$

Recall that $\mu$ is the Möbius function of $\mathcal{L}(H)$ and that the latter is the product of the partition lattices of $N_H(v)$ for each $v \in V(H)$. Observe that the Möbius function is multiplicative with respect to the product (see for instance Proposition 3.8.2)) and applying the explicit formula for the partition lattice (Theorem 2.6) we obtain the second claim.

In the remainder of the paper, given $\Phi$ and $H$, we refer to the function $a_{\Phi,H}$ from Corollary 3.3 as the coefficient function of $\Phi$ and $H$.

### 3.1 Complexity Monotonicity of Counting Colour-Preserving Homomorphisms

Our next goal is to prove that computing a finite linear combination of colour-preserving homomorphism counts, as given by Corollary 3.3, is precisely as hard as computing its hardest term. While the proof strategy follows the approach used in [25] and [32], we need to adapt to the setting of colour-preserving homomorphisms between fractured graphs.

We rely on the tensor product of $H$-coloured graphs in the following way: let $H$ denote a fixed graph, and let $G$ and $F$ denote $H$-coloured graphs with colourings $c_G$ and $c_F$. The colour-preserving tensor product $G \times_H F$ has vertices $\{(x,y) \in V(G) \times V(F) | c_G(x) = c_F(y)\}$, and two vertices $(x,y)$ and $(x',y')$ are made adjacent in $G \times_H F$ if (and only if) $\{x,x'\} \in E(G)$ and $\{y,y'\} \in E(F)$. Observe that the graph $G \times_H F$ is $H$-coloured as well by the colouring $(x,y) \mapsto c_G(x) = c_F(y)$.

**Lemma 3.4.** Let $H$ denote a graph, and let $F$, $G_1$ and $G_2$ denote $H$-coloured graphs. We have

$$\#\text{Hom}_{cp}(F \rightarrow_H G_1 \times_H G_2) = \#\text{Hom}_{cp}(F \rightarrow_H G_1) \cdot \#\text{Hom}_{cp}(F \rightarrow_H G_2) .$$

**Proof.** The function

$$b : \text{Hom}_{cp}(F \rightarrow_H G_1) \times \text{Hom}_{cp}(F \rightarrow_H G_2) \rightarrow \text{Hom}_{cp}(F \rightarrow_H G_1 \times_H G_2),$$

$$b(\varphi,\psi)(u) := (\varphi(u),\psi(u))$$

for $u \in V(F)$

is the canonical bijection.
The reduction for isolating terms with non-zero coefficient requires to solve a system of linear equations. For the definition of the corresponding matrix, we fix a linear extension $\preceq$ of the order $\leq$ of the $H$-fractures. Recall that $\leq$ is also the order of the product of the partition lattices of the set $E(v)$ for all $v \in V(H)$. In particular, $\sigma \leq \rho$ if and only if $\sigma_v$ refines $\rho_v$ for all $v \in V(H)$. As a consequence, we observe that $\rho \succ \sigma$, that is, $\neg(\rho \preceq \sigma)$, implies the existence of a vertex $v \in V(H)$ such that $\rho_v$ does not refine $\sigma_v$. Now let $M_H$ denote the matrix whose columns and rows are associated with the set of all $H$-fractures, ordered by $\preceq$, and whose entries are given by

$$M_H[\rho, \sigma] := \#\text{Hom}_{cp}(H \times \rho \rightarrow H \times \sigma).$$

Lemma 3.5. For each $H$, the matrix $M_H$ is upper triangular with entries 1 on the diagonal.

Proof. Let us first consider the diagonal. We claim that $\#\text{Hom}_{cp}(H \times \rho \rightarrow H \times \rho) = 1$. Due to the trivial (identity) homomorphism we have $\#\text{Hom}_{cp}(H \times \rho \rightarrow H \times \rho) \geq 1$. On the other hand, the canonical colouring $H \times \rho \rightarrow H$ induces a bijection from the edges of $H \times \rho$ to the edges of $H$ that preserves the colouring. Since a colour-preserving homomorphism $H \times \rho \rightarrow H \times \rho$ must commute with this map, it must act as the identity on all edges of $H \times \rho$ and is thus equal to the identity.

It remains to prove that $M_H[\rho, \sigma] = 0$ for every $\rho \succ \sigma$. Recall that the latter implies the existence of a vertex $v \in V(H)$ such that $\rho_v$ does not refine $\sigma_v$, that is, there is a block $B$ of $\rho_v$, which is not a subset of any block of $\sigma_v$. Thus, there are edges $e, f \in B \subseteq E_H(v)$ such that $e, f$ are in different blocks of $\sigma_v$. Identifying $E(H \times \sigma) = E(H \times \rho) = E(H)$ using the colouring, we see that $e, f$ are adjacent to the same vertex in $H \times \rho$ (corresponding to the block $B$), but to different vertices in $H \times \sigma$. This implies that there cannot be a colour preserving homomorphism $\varphi : H \times \rho \rightarrow H \times \sigma$ since $e, f$ being incident at $v^B$ in $H \times \rho$ would imply that $e = \varphi(e), f = \varphi(f)$ must be incident at $\varphi(v^B)$ in $H \times \rho$. \[\]

We are now able to prove a version of the Complexity Monotonicity principle which is sufficient for the purposes in this work. In what follows, given a graph property $\Phi$, we write $\#\text{ColEdgeSub}(\Phi \rightarrow \ast)$ for the function that expects as input a graph $H$ and an $H$-coloured graph $G$, and outputs $\#\text{ColEdgeSub}(\Phi \rightarrow H \times G)$. 

Lemma 3.6. Let $\Phi$ denote a computable graph property. There is a deterministic algorithm $\&$ which has oracle access to the function $\#\text{ColEdgeSub}(\Phi \rightarrow \ast)$, and computes, on input a graph $H$ and an $H$-coloured graph $G$, the numbers $\#\text{Hom}_{cp}(H \times \rho \rightarrow H \times G)$ for every $H$-fracture $\rho$ satisfying that $a_{\Phi, H}(\rho) \neq 0$, where $a_{\Phi, H}$ is the coefficient function of $\Phi$ and $H$.

Furthermore, there is a computable function $f$ such that $\&$ runs in time $f(|H|) \cdot |G|^{O(1)}$ and every posed oracle query $(H, G)$ satisfies $\hat{H} = H$ and $|G| \leq f(|H|) \cdot |G|$. 

Proof. Given $H$ and $G$, the algorithm $\&$ can obtain the numbers $\#\text{ColEdgeSub}(\Phi \rightarrow H \times H \times (H \times \sigma))$ for all $H$-fractures $\sigma$ via access to its oracle. By Corollary 3.3 and Lemma 3.4, we have

$$\#\text{ColEdgeSub}(\Phi \rightarrow_H G \times_H (H \times \sigma)) = \sum_{\rho \in E(H)} a_{\Phi, H}(\rho) \cdot \#\text{Hom}_{cp}(H \times \rho \rightarrow_H G \times_H (H \times \sigma))$$

$$= \sum_{\rho \in E(H)} a_{\Phi, H}(\rho) \cdot \#\text{Hom}_{cp}(H \times \rho \rightarrow_H G) \cdot \#\text{Hom}_{cp}(H \times \rho \rightarrow_H H \times \sigma)$$

Observe that the latter yields a system of linear equations for the numbers $a_{\Phi, H}(\rho) \cdot \#\text{Hom}_{cp}(H \times \rho \rightarrow_H G)$ with matrix $M_H$ which has full rank according to Lemma 3.5. Thus $\&$ can compute the number $a_{\Phi, H}(\rho) \cdot \#\text{Hom}_{cp}(H \times \rho \rightarrow_H G)$ for each $H$-fracture $\rho$. Consequently, $\#\text{Hom}_{cp}(H \times \rho \rightarrow_H G)$ can be computed whenever $a_{\Phi, H}(\rho) \neq 0$. 

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Observe that $a_{\Phi,H}$, which is computable, only depends on $\Phi$, which is fixed, and $H$. Furthermore $\mathcal{L}(H)$ and all $H\nRightarrow p$ only depend on $H$. Thus the computation of $\#\text{Hom}_{\mathcal{G}}(H \nRightarrow p \rightarrow_H H \nRightarrow \sigma)$, takes time only depending on $H$ as well. Consequently, the system of linear equations can be solved in time $f'(|H|) \cdot |G|^{O(1)}$ for some computable function $f'$. Furthermore, the size of $G \times_H (H \nRightarrow \sigma)$ is bounded by $|H\nRightarrow \sigma| \cdot |G|$. Setting $f(|H|) := \max \{f'(|H|), \max_{\sigma \in \mathcal{L}(H)} |H\nRightarrow \sigma|\}$ concludes the proof; since each fractured graph $H\nRightarrow \sigma$ has only $\#E(H)$ many edges.

3.2 Intractability of Counting Homomorphisms from Tori and Expanders

The final step of this section is to establish $\#W[1]$-hardness of the (uncoloured) problem $\#\text{E}d\text{geSub}(\Phi)$ whenever $a_{\Phi,\mathcal{G},(\top)} \neq 0$ for infinitely many $\ell$. Essentially, we rely on the fact that tori have high treewidth and that the problem of counting (colour-preserving) homomorphisms from high-treewidth graphs is hard. We can proceed similarly in case of expanders, and due to the fact that expanders have higher treewidth and are sparse (see Fact 2.3), we even obtain an almost tight conditional lower bound.

In both cases, we use Complexity Monotonicity, which yields hardness of the (edge-)colourful version of $\#\text{ColEdgeSub}(\Phi)$. Consequently, we need to show that the colourful version reduces to the uncoloured version. This can be achieved by a standard inclusion-exclusion argument:

**Lemma 3.7.** Let $\Phi$ denote a computable graph property. There is a deterministic algorithm $A$, equipped with oracle access to the function

$$(k, \hat{G}) \mapsto \#\text{E}d\text{geSub}(\Phi, k \rightarrow \hat{G}),$$

which expects as input a graph $H$ and an $H$-coloured graph $G$, and computes $\#\text{ColEdgeSub}(\Phi \rightarrow_H G)$ in time $2^{|E(H)|} \cdot |G|^{O(1)}$. Furthermore, every oracle query $(k, \hat{G})$ posed by $A$ satisfies $k = |E(H)|$ and $|\hat{G}| \leq |G|$.

**Proof.** Given $H$ and an $H$-coloured graph $G$, we write $c : E(G) \rightarrow E(H)$ for the induced edge-colouring of $G$. Given a set of edge-colours $J \subseteq E(H)$, we write $G \setminus J$ for the graph obtained from $G$ by deleting all edges $e$ with $c(e) \in J$. Now recall that

\[
\text{E}d\text{geSub}(\Phi, |E(H)| \rightarrow G) = \{A \subseteq E(G) \mid \#A = |E(H)| \land \Phi(G[A]) = 1\}, \text{ and}
\]

\[
\text{ColEdgeSub}(\Phi \rightarrow_H G) = \{A \subseteq E(G) \mid \#A = |E(H)| \land c(A) = E(H) \land \Phi(G[A]) = 1\}.
\]

Next set $k := |E(H)|$, then we have

$\#\text{ColEdgeSub}(\Phi \rightarrow_H G)$

$= \#\text{E}d\text{geSub}(\Phi, k \rightarrow G) - \# \left( \bigcup_{c \in E(H)} \{A \in \text{E}d\text{geSub}(\Phi, k \rightarrow G) \mid e \notin c(A)\} \right)$

$= \#\text{E}d\text{geSub}(\Phi, k \rightarrow G) - \sum_{\emptyset \neq J \subseteq E(H)} (-1)^{|J| + 1} \cdot \# \left( \bigcap_{c \in J} \{A \in \text{E}d\text{geSub}(\Phi, k \rightarrow G) \mid e \notin c(A)\} \right)$

$= \#\text{E}d\text{geSub}(\Phi, k \rightarrow G) - \sum_{\emptyset \neq J \subseteq E(H)} (-1)^{|J| + 1} \cdot \#\text{E}d\text{geSub}(\Phi, k \rightarrow G \setminus J)$

$= \sum_{J \subseteq E(H)} (-1)^{|J|} \cdot \#\text{E}d\text{geSub}(\Phi, k \rightarrow G \setminus J)$

$= \sum_{J \subseteq E(H)} (-1)^{|J|} \cdot \#\text{E}d\text{geSub}(\Phi, k \rightarrow G \setminus J)$
Note that the second equation is due to the inclusion-exclusion principle. We conclude that the desired number \( \# \text{ColEdgeSub}(\Phi \rightarrow H, G) \) can be computed using \( 2^{\ell(H)} \) many oracle calls of the form \( \# \text{EdgeSub}(\Phi, |E(H)| \rightarrow G \setminus J) \). The claim of the lemma follows since \(|G \setminus J| \leq |G|\).

For the formal statement of this section’s main lemma, we define \( \mathcal{H}[\Phi, \emptyset] \) as the set of all \( \emptyset \) such that \( a_{\Phi, \emptyset}(\emptyset) \neq 0 \). Furthermore, given a family \( \mathcal{G} = \{G_1, G_2, \ldots \} \) of \( \langle n, d, c \rangle \)-expanders, we write \( \mathcal{H}[\Phi, \mathcal{G}] \) for the set of all \( G_i \) such that \( a_{\Phi, G_i}(\emptyset) \neq 0 \)

**Lemma 3.8.** Let \( \Phi \) denote a computable graph property, fix a rational \( c \) and an integer \( d \), and let \( \mathcal{G} = \{G_1, G_2, \ldots \} \) denote a family of \( \langle n, d, c \rangle \)-expanders. If at least one of \( \mathcal{H}[\Phi, \emptyset] \) and \( \mathcal{H}[\Phi, \mathcal{G}] \) is infinite, then \( \# \text{EdgeSub}(\Phi) \) is \#W[1]-hard. Moreover, if \( \mathcal{H}[\Phi, \mathcal{G}] \) is infinite, then \( \# \text{EdgeSub}(\Phi) \) cannot be solved in time

\[
f(k) \cdot |G|^{o(k/ \log k)}
\]

for any function \( k \), unless the ETH fails.

**Proof.** We start with the case of \( \mathcal{H}[\Phi, \emptyset] \) being infinite. If the latter is true, then \( \mathcal{H}[\Phi, \emptyset] \) has unbounded treewidth since it contains graphs with arbitrary large grid minors [55]. This allows us to reduce from the problem \( \# \text{Hom}(\mathcal{H}[\Phi, \emptyset]) \) which is known to be \#W[1]-hard since \( \mathcal{H}[\Phi, \emptyset] \) has unbounded treewidth [25]. It is convenient to consider the following intermediate problem: given a graph \( H \in \mathcal{H}[\Phi, \emptyset] \) and an \( H \)-coloured graph \( G \) with colouring \( c \), in the problem \( \# \text{CP-Hom}(\mathcal{H}[\Phi, \emptyset]) \) the task is to compute the number \( \# \text{CP-Hom}(H \rightarrow G) \) of homomorphisms \( \varphi \in \text{Hom}(H \rightarrow G) \) such that \( c(\varphi(v)) = v \) for each vertex \( v \) of \( H \). It is well-known that \( \# \text{Hom}(\mathcal{H}) \) reduces to \( \# \text{CP-Hom}(\mathcal{H}) \) for every class of graphs \( \mathcal{H} \); see for instance [55 52 31 24]—note that, in the latter, the problem is referred to as \( \# \text{PartitionedSub}(\mathcal{H}) \). Thus

\[
\# \text{Hom}(\mathcal{H}[\Phi, \emptyset]) \leq_{\text{fpt}} \# \text{CP-Hom}(\mathcal{H}[\Phi, \emptyset]). \tag{5}
\]

Now observe that \( \# \text{CP-Hom}(H \rightarrow G) = \# \text{Hom}_{\text{cp}}(H \rightarrow_H G) \) for every graph \( H \) and \( H \)-coloured graph \( G \), since \( H \rightarrow_H \emptyset = H \). By definition of \( \mathcal{H}[\Phi, \emptyset] \), we have that \( a_{\Phi, \emptyset}(\emptyset) \neq 0 \) whenever \( \emptyset \in \mathcal{H}[\Phi, \emptyset] \). Thus we can use Complexity Monotonicity (Lemma 3.6) which yields the reduction

\[
\# \text{CP-Hom}(\mathcal{H}[\Phi, \emptyset]) \leq_{\text{fpt}} \# \text{ColEdgeSub}(\Phi). \tag{6}
\]

Finally, we can reduce to the uncoloured version by Lemma 3.7 and obtain

\[
\# \text{ColEdgeSub}(\Phi) \leq_{\text{fpt}} \# \text{EdgeSub}(\Phi). \tag{7}
\]

Consequently, \( \# \text{EdgeSub}(\Phi) \) is \#W[1]-hard by \([5] - [7] \) and \#W[1]-hardness of \( \# \text{Hom}(\mathcal{H}[\Phi, \emptyset]) \).

In case of \( \mathcal{H}[\Phi, \mathcal{G}] \), we reduce from the homomorphism counting problem as well and obtain \#W[1]-hardness analogously. However, for the almost tight conditional lower bound, we rely on a result of Marx [57] implying that for any class \( \mathcal{H} \) of unbounded treewidth, the problem \( \# \text{Hom}(\mathcal{H}) \) cannot be solved in time

\[
f(|H|) \cdot |G|^{o(\text{tw}(H)/ \log \text{tw}(H))}
\]

for any function \( f \), unless ETH fails [14]. Let us use the aforementioned lower bound for the case of \( \mathcal{H} = \mathcal{H}[\Phi, \mathcal{G}] \). We observe that the reduction sequence from \( \# \text{Hom}(\mathcal{H}[\Phi, \mathcal{G}]) \) to \( \# \text{EdgeSub}(\Phi) \) as

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14 Observe that this result follows only implicitly from [57], but we made it explicit in [70].
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illustrated before only leads to a linear blow up of the parameter: given an input \((G_1, G)\) for which we wish to compute \(#\text{Hom}(G_1 \to G)\), we only query the oracle for \(#\text{EDGESUB}(\Phi)\) on instances \((k, G')\) where \(k = \#E(G_1)\) and \(|G'| \leq f''(\#E(G_1)) \cdot |G|\) for some function \(f''\). Since both \(#E(G_i)\) and the treewidth of \(G_i\) are linear in \(|V(G_i)|\) (see Fact 2.3), any algorithm that, for some function \(f'\), solves \(#\text{EDGESUB}(\Phi)\) in time

\[
f'(k) \cdot |G'|^{o(k/\log k)}
\]
yields an algorithm for \(#\text{Hom}(\mathcal{H}[\Phi, G])\) running in time

\[
f(|G_i|) \cdot |G|^{o(\text{tw}(G_i)/\log \text{tw}(G_i))},
\]
for some function \(f\) (depending only on \(f'\) and \(f''\)), contradicting ETH by Marx’ lower bound.

Regarding the previous proof, observe that we cannot obtain a similar conditional lower bound if only \(\mathcal{H}[\Phi, \mathbb{S}]\) is infinite, since in that case the parameter grows quadratically: while \(\mathbb{S}_\ell\) has treewidth \(O(\ell)\), it has \(2\ell^2\) edges.

### 4 Coefficients of Tori and Cayley Graph Expanders

The previous section allows us to establish hardness of \(#\text{EDGESUB}(\Phi)\) by the purely combinatorial problem of determining whether one of the sets \(\mathcal{H}[\Phi, \mathbb{S}]\) and \(\mathcal{H}[\Phi, G]\), for some family of expanders \(G\), is infinite. Still, this is a challenging combinatorial problem and we consider the treatment of the coefficients of the tori and Cayley graph expanders to be our main technical contribution in this work.

Recall from Corollary 3.3 that the coefficient function of \(\Phi\) and \(H\) satisfies

\[
a_{\Phi, H}(T) = \sum_{\sigma \in \mathcal{L}(\Phi, H)} \prod_{v \in V(H)} (-1)^{|\sigma_v| - 1} \cdot (|\sigma_v| - 1)!
\]

In case that \(H\) satisfies certain symmetry properties, we obtain that it suffices to consider only those fractures in the previous sum that are fixed-points under suitable group actions. More formally, we obtain the desired symmetries from the structure of the groups underlying the Cayley graph constructions for tori and expanders as introduced in the subsequent subsections.

#### 4.1 Symmetries of the Torus

As promised, we start with a simple Cayley graph given by the direct product of two cyclic groups:

**Definition 4.1 (The Torus).** Let \(\ell \geq 3\) denote an integer. The **torus**, also called the **toroidal grid**, \(\mathbb{T}_\ell\) of size \(\ell\) is the Cayley graph of \(\mathbb{Z}_\ell^2\) with generators \((\pm 1, 0), (0, \pm 1)\), that is,

\[
\mathbb{T}_\ell := \Gamma \left( \mathbb{Z}_\ell^2, \{(1, 0), (-1, 0), (0, 1), (0, -1)\} \right).
\]

Equivalently, the vertices of \(\mathbb{T}_\ell\) are \(\mathbb{Z}_\ell^2\) and two vertices \((x, y)\) and \((x', y')\) are adjacent if and only if

\[
x = x' \text{ and } y' = y + 1 \mod \ell, \text{ or } y = y' \text{ and } x' = x + 1 \mod \ell.
\]

Consult Figure 2 for a visualization. In the following, for simplicity, we write \(+\) for (point-wise) addition modulo \(\ell\). Our goal is to understand the symmetries of \(\mathbb{T}_\ell\). Consider the following action of \(\mathbb{Z}_\ell^2\) on the vertices of \(\mathbb{T}_\ell\). Let \((i, j) \in \mathbb{Z}_\ell^2\) and let \((x, y) \in V(\mathbb{T}_\ell)\). We set \((i, j) \vdash (x, y) := (i, j) + (x, y)\). The following is immediate:

\[
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\]
The fact above allows us to consider the set of (all possible) \(1\)-shifts, the function \((i,j) \mapsto \star\) is an automorphism of \(G\). We remark that not all automorphisms are given by such shifts, but for our arguments we will not need to consider the full group of automorphisms.

Fractures of the Torus

Recall that a fracture \(\rho\) of a graph \(H\) is a tuple \(\rho = (\rho_v)_{v \in V(H)}\) where \(\rho_v\) is a partition of the set \(E_H(v)\) of edges of \(H\) incident to \(v\). Now given an automorphism \(\varphi : H \to H\) of \(H\), it gives a bijection from the edges \(E_H(v)\) at \(v\) to the edges \(E_H(\varphi(v))\) at \(\varphi(v)\). Thus, given a fracture \(\rho\) of \(H\), we obtain a fracture \(\varphi(\rho)\) of \(H\), such that two edges \(e_1, e_2 \in E_H(\varphi(v))\) are in the same block of \(\varphi(\rho)(\varphi(v))\) if and only if their preimages \(\varphi^{-1}(e_1), \varphi^{-1}(e_2) \in E_H(v)\) are in the same block of \(\rho_v\).

We claim that that the two fractured graphs \(H \hat{\varphi} \rho \cong H \hat{\varphi}(\rho)\) are isomorphic. To see this, note that the automorphism \(\varphi : H \to H\) lifts to an automorphism \(\hat{\varphi} : M_H \to M_H\) of the matching \(M_H\) associated to \(H\), where \(\hat{\varphi}\) lifts the vertex \(e_v\) of \(M_H\) to \(\varphi(e)(\varphi(v))\). The map \(\hat{\varphi}\) sends the equivalence relation on \(M_H\) associated to \(\rho\) (with quotient \(H \hat{\varphi}\rho\)) to the equivalence relation associated to \(\varphi(\rho)\) (with quotient \(H \hat{\varphi}(\rho)\)). Thus \(\hat{\varphi} : M_H \to M_H\) descends to an isomorphism \(H \hat{\varphi}\rho \to H \hat{\varphi}(\rho)\) fitting in a diagram of graph homomorphisms, depicted in Figure 4).

Given a finite group \(G\) acting on the graph \(H\) by graph isomorphisms \(\varphi_g : H \to H\) (for \(g \in G\)), we obtain an action \(\|\) of \(G\) on the lattice \(L(H)\) of fractures on \(H\), where \(g \| \rho = \varphi_g(\rho)\). Clearly, this action respects the order of the lattice (\(\rho \leq \rho'\) if and only if \(g \| \rho \leq g \| \rho'\)) and as seen above, for any \(\rho\) in the \(G\)-orbit of \(\rho\) we have \(H \hat{\varphi}\rho \cong H \hat{\varphi}\rho\).

We now return to the special case when \(H = G\) is a torus. Here, given a vertex \((i,j)\) of \(H\) it is convenient to identify the edges incident to the vertex (connecting it to \((i,j+1), (i,j-1), (i-1,j)\), and \((i+1,j)\)) with the four “directions” \(\triangle, \triangledown, <,\) and \(>\), respectively, so that each \(\rho(i,j)\) is a partition of the set \(\{\triangle, \triangledown, <,>\}\).

We have seen that \(Z^2\) acts transitively on the vertices of \(G\) in such a way that every element of \(Z^2\) induces an automorphism of \(G\). Thus, by the discussion above, we obtain an action \(\|\) of \(Z^2\) on the set of fractures of \(G\). Let us make this action explicit: \((i,j) \| \rho := \hat{\rho}\), where \(\hat{\rho}(i,j)=\rho(i',j')\) for all \((i',j') \in Z^2\).

Analysis of the Fixed-points

We proceed with the fixed-points of the action \(\|\) of \(Z^2\) on the fractures \(G\). Since this action consists of (all possible) \((i,j)\)-shifts, the fixed-points are precisely those fractures \(\rho\) for which all partitions \(\rho(i,j)\) are equal — recall that we assumed every \(\rho(i,j)\) to be a partition of \(\{\triangle, \triangledown, <,>\}\). Fortunately, there are only 15 partitions of the four-element set, and thus we can analyse the fixed-points by hand. Indeed, one special case of our main result, as well as the classification of the parameterized Tutte polynomial, rely on
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the understanding of all of those 15 fixed-points. However, while there are 15 fixed-points $\rho$, we can group those into 7 types according to the isomorphism class of $@_\ell \sharp \rho$; an illustration of all fixed-points is given in Figure 5.

**Observation 4.3.** The fixed-points of the action of $\mathbb{Z}^2$ on the fractures of $@_\ell$ are as follows.

**Matching:** $@_\ell \sharp \rho \cong M_{2\ell^2}$, the matching of size $2\ell^2$.
1. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$, that is, $\rho = \perp$.

**Matching and cycles:** $@_\ell \sharp \rho \cong M_{\ell^2} + \ell C_\ell$, the union of a matching of size $\ell^2$ and $\ell$ disjoint cycles of length $\ell$.
2. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.
3. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.

**Wedge packing:** $@_\ell \sharp \rho \cong \ell^2 P_2$, the union of $\ell^2$ disjoint paths of length 2.
4. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.
5. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.
6. $\rho_{i,j} = \{\{\vee, \{<\}, \{\Delta, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.
7. $\rho_{i,j} = \{\{\vee, \{<\}, \{>\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.

**Cycle packing I:** $@_\ell \sharp \rho \cong \ell^2 C_\ell$, the union of $2\ell$ disjoint cycles of length $\ell$.
8. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.

**Cycle packing II:** $@_\ell \sharp \rho \cong \ell C_{2\ell}$, the union of $\ell$ disjoint cycles of length $2\ell$.
9. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.
10. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.

**Sun packing:** $@_\ell \sharp \rho \cong \ell S_\ell$, the union of $\ell$ sums of size $\ell$. Here a a sum of size $\ell$ is obtained from a cycle of length $\ell$ by adding one “dangling” edge at every vertex of the cycle.
11. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.
12. $\rho_{i,j} = \{\{\vee, \{<\}, \{\Delta, \{\vee, \{<\}, \{>\}\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.
13. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.
14. $\rho_{i,j} = \{\{\vee, \{<\}, \{\Delta, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$.

**Torus:** $@_\ell \sharp \rho \cong @_\ell$, the torus of size $\ell$.
15. $\rho_{i,j} = \{\{\Delta, \{\vee, \{<\}, \{>\}\}\}\} \text{ for all } (i, j) \in \mathbb{Z}^2$, that is $\rho = \top$.

While it might be surprising at first glance, we observe that, for many properties $\Phi$, our analysis of the complexity of $\#\text{EDGESUB}(\Phi)$ only depends on which of the previous 15 fixed-points $\rho$ satisfy that $@_\ell \sharp \rho$ has the property $\Phi$.

### 4.2 Symmetries of Cayley Graph Expanders of 2-groups

For the second family of Cayley graphs, we rely on an explicit construction of 4-regular Cayley graph expanders due to Peyerimhoff and Vdovina [62]. They are constructed from an explicit infinite group $G$ with generators $x_0, x_1$ and a sequence

$$G \supseteq N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$$

of normal subgroups $N_i$ of $G$, such that the indices $|G : N_i| = 2^{i_1} = n_i$ are powers of 2 converging to infinity as $i$ increases. Moreover, writing $K_i$ for the quotient group $G/N_i$, the set of Cayley graphs $G_i := \Gamma(K_i, \{v_0^{1}, v_1^{1}, v_i^{1}\})$ is a family $\mathcal{G}$ of $(n_i, 4, c)$-expanders for some constant $c > 0$. Here, $v_0 = x_0 N_i$ and $v_1 = x_1 N_i$ are generators of $K_i$. 
Figure 5 The 15 fixed points of the action $\triangleleft \gamma$ of $\mathbb{Z}_2^3$ on the fractures of $\otimes_\gamma$. Vertices in the fractured graphs that correspond to the same vertex in the original graph are encircled with a dashed line; the corresponding fixed point is denoted below its graphical representation.
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Similar to the case of the toroidal grid, we obtain an action of the group $K_i$ on the graph $G_i$, where an element $g \in K_i$ acts on a vertex $v \in V(G_i) = K_i$ sending it to $g \cdot v = gv$, where the latter is the product of $g$ and $v$ in the group $K_i$. The action of $g$ defines a graph automorphism of $G_i$ since the four edges $\{v, vv_j^\pm 1\}$ (for $j = 0, 1$) at $v$ are sent to the four edges $\{gv, gvv_j^\pm 1\}$ incident to $gv$.

Fractures of the Cayley Graph Expanders

For $v \in V(G_i) \cong K_i$, the edges adjacent to $v$ connect $v$ to the vertices $vs$ for $s \in S$ and thus can be uniquely labelled

\[
\begin{align*}
\triangledown &= \{v, vv_0\},
\triangleleft &= \{v, vv_0^{-1}\},
\triangle &= \{v, vv_1\},
\nabla &= \{v, vv_1^{-1}\}.
\end{align*}
\]

Thus a fracture $\rho \in \mathcal{L}(G_i)$ is a collection $\rho = (\rho_v)_{v \in V(G_i)}$ of partitions of the set $\{\triangledown, \triangleleft, \triangle, \nabla\}$.

Analysis of the Fixed-points

As seen before, the action $\triangledown$ of $K_i$ on $G_i$ induces an action $\triangledown$ of $K_i$ on the lattice of partitions $\mathcal{L}(G_i)$. A fracture $\rho = (\rho_v)_{v \in V(G_i)}$ is invariant under the action of $K_i$ if and only if $\rho_v$ does not depend on $v$.

Later we want to compute the coefficient $\alpha_{\mathcal{L}(G_i)}(\triangledown)$ modulo two. As before we observe that only fixed points of the action of $K_i$ contribute, and additionally we observe that only such fixed points $\rho$ can contribute where $\rho_v$ has at most two blocks. Thus in the following we consider only such fixed points.

**Lemma 4.4.** Fix $i \geq 2$ and denote $b_0 = \text{ord}_{K_i}(v_0), b_1 = \text{ord}_{K_i}(v_1), b_2 = \text{ord}_{K_i}(v_1^{-1}v_0), b_3 = \text{ord}_{K_i}(v_1v_0)$, and $a_j = \#K_i/b_j$ for $j = 0, \ldots, 3$. Then the fixed-points $\rho = (\rho_v)_{v \in V(G_i)}$ of the action of $K_i$ on the fractures of $G_i$, satisfying that all $\rho_v$ have at most two blocks are as follows:

**Cycle packing I:**
1. $\rho_v = \{\{\triangle, \nabla\}, \{\triangleleft, \triangledown\}\}$ for all $v \in V(G_i)$ and $G_i \sharp \rho \cong a_0 \cdot C_{b_0} + a_1 \cdot C_{b_1}$.

**Cycle packing II:**
2. $\rho_v = \{\{\triangle, \triangledown\}, \{\nabla, \triangleleft\}\}$ for all $v \in V(G_i)$ and $G_i \sharp \rho \cong a_2 \cdot C_{b_2}$.
3. $\rho_v = \{\{\triangle, \triangleleft\}, \{\nabla, \triangledown\}\}$ for all $v \in V(G_i)$ and $G_i \sharp \rho \cong a_3 \cdot C_{b_3}$.

**Sun packing:**
4. $\rho_v = \{\{\triangle\}, \{\nabla, \triangleleft, \triangledown\}\}$ for all $v \in V(G_i)$ and $G_i \sharp \rho \cong a_0 \cdot S_{b_0}$.
5. $\rho_v = \{\{\nabla\}, \{\triangle, \triangleleft, \triangledown\}\}$ for all $v \in V(G_i)$ and $G_i \sharp \rho \cong a_0 \cdot S_{b_0}$.
6. $\rho_v = \{\{\triangleleft\}, \{\triangle, \nabla, \triangledown\}\}$ for all $v \in V(G_i)$ and $G_i \sharp \rho \cong a_1 \cdot S_{b_1}$.
7. $\rho_v = \{\{\triangledown\}, \{\triangle, \nabla, \triangleleft\}\}$ for all $v \in V(G_i)$ and $G_i \sharp \rho \cong a_1 \cdot S_{b_1}$.

**Full graph:**
8. $\rho_v = \{\{\triangle, \nabla, \triangleleft, \triangledown\}\}$ for all $v \in V(G_i)$, that is $\rho = \triangledown$ and $G_i \sharp \rho \cong G_i$.

Moreover, the numbers $a_j, b_j$ are all powers of 2 and $a_j \geq 8$.

**Proof.** In cases 1, 2, 3 it follows from the definition of the fractured graph that $G_i \sharp \rho$ is 2-regular and thus a union of circles. In case 1 the first type of circles (associated to the directions $\triangleleft, \triangledown$) is given by

\[
w_0 \to w_0v_0 \to w_0v_0^2 \to \cdots \to w_0v_0^{b_0-1} \to w_0v_0^{b_0} = w_0
\]  

\[\text{(8)}\]

We use the same notation here as in the section about the torus grid $@_c$. Many of the intuitions that we gained so far are still valid; for instance the edge $\triangledown$ going out from the vertex $v$ to $vv_0$ is equal to the edge $\triangleleft$ associated to the vertex $vv_0$. On the other hand, we also need to be more careful in our proofs, since e.g. going along an edge $\triangledown$ followed by $\triangle$ does not necessarily go to the same vertex as the path $\triangle$ followed by $\triangledown$, since the group $K_i$ is in general not abelian.
and thus isomorphic to $C_{b_0}$, with one circle for each $w_0 \in K_i/\langle v_0 \rangle$ giving a total number of $\#(K_i/\langle v_0 \rangle) = \#K_i/b_i = a_i$. Analogously we obtain $a_1$ copies of $C_{b_1}$ associated to the directions $\triangle, \triangledown$.

In case 2 the circles are of the form

$$w_0 \to w_0 v_1^{-1} \to w_0 v_1^{-1} v_0 \to w_0 v_1^{-1} v_0 v_1^{-1} \to w_0 (v_1^{-1} v_0)^2 \to \cdots \to w_0 (v_1^{-1} v_0)^{b_2-1} v_1^{-1} \to w_0 (v_1^{-1} v_0)^{b_2} = w_0.$$ 

Thus they are isomorphic to $C_{2b_2}$ and since the total number of edges of $G_i \rho$ is equal to $\#E(G_i) = 4 \#K_i/2 = 2 \#K_i$, the number of copies of $C_{2b_2}$ is given by $2 \#K_i/(2b_2) = a_2$. The case 3 is treated analogously.

In case 4, the connected component of a vertex $w_0 \in G_i \rho$ associated to the directions $\triangledown, \triangleleft, \triangleright$ certainly contains the circle $C_{b_0}$ given by (3) and in addition, each vertex $w_0 v_0^i$ is connected to $w_0 v_0^i v_1^{-1}$, which forms a leaf of $G_i \rho$. Thus, these are the only additional vertices connected to the circle and thus the connected component of each vertex in $G_i \rho$ forms a sun $S_{b_0}$. The total number of suns is $\#E(G_i)/\#E(S_{b_0}) = (2 \#K_i)/(2b_0) = a_0$. The cases 5,6,7 are treated completely analogously.

Finally, case 8 follows from the general property $H \star \top \cong H$. The fact that $a_j, b_j$ divide the order of $\#K_i$, together with the property that $K_i$ is a 2-group, imply that $a_j, b_j$ are powers of 2. Finally, we show the inequality $a_j \geq 8$ by induction on $i \geq 2$. Note that in the case $i = 2$ this can be checked by hand. For this one uses the explicit description of the group law of $K_2$ presented in Section 3 and verifies that the orders $b_j$ of elements $v_0, v_1, v_1^{-1} v_0, v_1 v_0$ are precisely 4, so that $a_j = \#K_2/b_j = 8$.

To conclude the general case for $i \geq 2$, denote $V_0 = \langle v_0 \rangle \subseteq K_i$ the subgroup generated by $v_0$, so that $a_0 = [K_i : V_0]$. Recalling the facts from the start of the section, we saw that $K_i = G/N_i$ with $N_2 \supseteq N_i$ for $i \geq 3$. Thus we have a surjective group homomorphism

$$\varphi_i : K_i = G/N_i \to G/N_2 = K_2, ~ x N_i \mapsto x N_2$$

sending $V_0^i \subseteq K_i$ to $V_0^2 \subseteq K_2$ (this follows since the generator $v_0 = x_0 N_i$ of $V_0^i$ maps to the generator $v_0 = x_0 N_2$ of $V_0^2$). As mentioned above, we checked by hand that $V_0^2$ has index 8 in $K_2$. Then by Lemma 4.5 we have that $8 = [K_2 : V_0^2] = [\varphi(K_i) : \varphi(V_0^i)]$ divides $[K_i : V_0^i] = a_i$ and thus $a_0 \geq 8$. The bounds for $a_1, a_2, a_3$ work exactly the same way.

\[\square\]

**Lemma 4.5.** Let $G, G'$ denote finite groups and $\varphi : G \to G'$ a group homomorphism. Then, for any subgroup $H \subseteq G$ we have that $[\varphi(G) : \varphi(H)]$ divides $[G : H]$.

**Proof.** Let $K = \ker(\varphi)$ and $K_H = K \cap H = \ker(\varphi|_H)$, then by the First isomorphism theorem we have $\varphi(G) \cong G/K$ and $\varphi(H) \cong H/K_H$. Using this, we have

$$[G : H] = \frac{\#G}{\#H} = \frac{\#G/\#K}{\#H/\#K_H} = \frac{\#(G/K)}{\#(H/K_H)} = \frac{\#(G/K)}{\#(H/K_H)} \cdot \frac{\#K}{\#K_H} = [\varphi(G) : \varphi(H)] \cdot \frac{\#K}{\#K_H}.$$ 

But $K_H \subseteq K$ is a subgroup, so by Lagrange’s theorem, the order of $K_H$ divides the order of $K$, so that $\#K/\#K_H$ is an integer. Thus the above equality shows that $[\varphi(G) : \varphi(H)]$ divides $[G : H]$. \[\square\]

### 4.3 Analysis of the Coefficient Function via Fixed-points

While the value $a_{\Phi,H}(\top)$ of the coefficient function seems to be very difficult to handle for arbitrary graphs $H$, we now use our observations on the symmetries of the torus and the Cayley graph expanders to prove that the coefficient function does not vanish infinitely often under specific constraints on the behaviour of $\Phi$ on the fixed-points presented in the preceding section.

We start with the case of $a_{\Phi,\mathbb{Z}}(\top)$, which, while being simple, turns out to be required for one of the special cases in our main classification for minor-closed graph properties:
Lemma 4.6. Let \( \ell \) denote a prime and let \( \Phi \) denote a computable graph property. We have that
\[
a_{\Phi, \oplus_2}(T) = -6\Phi(M_{2\ell^2}) + 4\Phi(M_{\ell^2} + \ell C_{\ell}) + 8\Phi(\ell^2 P_2) - \Phi(2\ell C_{\ell}) - 2\Phi(\ell C_{\ell}) - 4\Phi(\ell S_{\ell}) + \Phi(\otimes_{\ell}) \mod \ell.
\]

Proof. By Corollary 3.3 we have
\[
a_{\Phi, \oplus_2}(T) = \sum_{\sigma \in L(\Phi, \oplus_2)} \prod_{v \in V(\oplus_2)} (-1)^{|\sigma_v|-1} \cdot (|\sigma_v|-1)!
\]
Setting \( f(\sigma) := \prod_{v \in V(\oplus_2)} (-1)^{|\sigma_v|-1} \cdot (|\sigma_v|-1)! \), this rewrites to
\[
a_{\Phi, \oplus_2}(T) = \sum_{\sigma \in L(\Phi, \oplus_2)} f(\sigma).
\]

We now use the action \( \mathbb{Z}_{\ell}^2 \) on the subset \( L(\Phi, \oplus_2) \) of \( L(\oplus_2) \), given by permuting the elements of a fracture \( \rho \) according to the coordinate shift induced by an element \((i,j) \in \mathbb{Z}_{\ell}^2\). Restricting this action to \( L(\Phi, \oplus_2) \) is well-defined since the action does not change the isomorphism class\(^{16}\) of \( \oplus_2 \mathbb{Z}_{\ell} \). In particular, we have that \( f(\sigma) = f(\rho) \) whenever \( \sigma \) and \( \rho \) are in the same orbit of the action. This allows us to rewrite as follows; the sum is taken over all orbits \([\sigma] \) of the group action:
\[
a_{\Phi, \oplus_2}(T) = \sum_{[\sigma]} \#[\sigma] \cdot f(\sigma)
\]
Since \( \ell \) is a prime, the group order of \( \mathbb{Z}_{\ell}^2 \) is a power of \( \ell \). As the size of every orbit must divide the group order, we can ignore all orbits which are not fixed-points, that is \( \sigma \) for which \( \#[\sigma] = 1 \), if we take the sum modulo \( \ell \). All 15 fixed-points are explicitly given in Observation 4.3. Let us now compute the coefficients of each collection of fixed-points that induce the same graph, up to isomorphism; we use Fermat’s Little Theorem—recall that \( \ell \) is a prime.

**Matching:** One fixed-point \( \rho \) satisfies \( \oplus_2 \mathbb{Z}_{\ell} \rho \cong M_{2\ell^2} \). The contribution to \( a_{\Phi, \oplus_2}(T) \) is thus
\[
1 \cdot f(\rho) \cdot \Phi(M_{2\ell^2}) = ((-1)^{3-1} \cdot (4-1))! \ell^2 \Phi(M_{2\ell^2}) = -6\Phi(M_{2\ell^2}) \mod \ell.
\]

**Matching and cycles:** Two fixed-points \( \rho \) satisfy \( \oplus_2 \mathbb{Z}_{\ell} \rho \cong M_{\ell^2} + \ell C_{\ell} \). The contribution to \( a_{\Phi, \oplus_2}(T) \) is thus
\[
2 \cdot f(\rho) \cdot \Phi(M_{\ell^2} + \ell C_{\ell}) = 2 \cdot ((-1)^{3-1} \cdot (3-1)! \ell^2 \Phi(M_{\ell^2} + \ell C_{\ell}) = 4\Phi(M_{\ell^2} + \ell C_{\ell}) \mod \ell.
\]

**Wedge packing:** Four fixed-points \( \rho \) satisfy \( \oplus_2 \mathbb{Z}_{\ell} \rho \cong \ell^2 P_2 \). The contribution to \( a_{\Phi, \oplus_2}(T) \) is thus
\[
4 \cdot f(\rho) \cdot \Phi(\ell^2 P_2) = 4 \cdot ((-1)^{3-1} \cdot (3-1)! \ell^2 \Phi(\ell^2 P_2) = 8\Phi(\ell^2 P_2) \mod \ell.
\]

**Cycle packing I:** One fixed-point \( \rho \) satisfies \( \oplus_2 \mathbb{Z}_{\ell} \rho \cong 2\ell C_{\ell} \). The contribution to \( a_{\Phi, \oplus_2}(T) \) is thus
\[
1 \cdot f(\rho) \cdot \Phi(2\ell C_{\ell}) = ((-1)^{2-1} \cdot (2-1)! \ell^2 \Phi(2\ell C_{\ell}) = -2\Phi(2\ell C_{\ell}) \mod \ell.
\]

**Cycle packing II:** Two fixed-points \( \rho \) satisfy \( \oplus_2 \mathbb{Z}_{\ell} \rho \cong \ell C_{2\ell} \). The contribution to \( a_{\Phi, \oplus_2}(T) \) is thus
\[
2 \cdot f(\rho) \cdot \Phi(\ell C_{2\ell}) = 2 \cdot ((-1)^{2-1} \cdot (2-1)! \ell^2 \Phi(\ell C_{2\ell}) = -2\Phi(\ell C_{2\ell}) \mod \ell.
\]

\(^{16}\)Note that while the action can change the isomorphism class as a \( \oplus_2 \)-coloured graph, the property \( \Phi \) only depends on the underlying uncoloured graph, which is unchanged, and thus \( L(\Phi, \oplus_2) \) is indeed invariant under the action.
Sun packing: Four fixed-points $\rho$ satisfy $\otimes_{i=1}^{2} \rho \cong \ell S_{\ell}$. The contribution to $a_{\Phi,\otimes_{i=1}^{2}}(T)$ is thus

$$4 \cdot f(\rho) \cdot \Phi(\ell S_{\ell}) = 4 \cdot ((-1)^{2-1} \cdot (2 - 1)!)^{2} \Phi(\ell S_{\ell}) = -4 \Phi(\ell S_{\ell}) \mod \ell.$$  

Torus: One fixed-point $\rho$ satisfies $\otimes_{i=1}^{2} \rho \cong \otimes_{i=1}^{2}$. The contribution to $a_{\Phi,\otimes_{i=1}^{2}}(T)$ is thus

$$1 \cdot f(\rho) \cdot \Phi(\otimes_{i=1}^{2}) = ((-1)^{1-1} \cdot (1 - 1)!)^{2} \Phi(\otimes_{i=1}^{2}) = \Phi(\otimes_{i=1}^{2}) \mod \ell.$$  

Taking the sum of the previous terms (modulo $\ell$) concludes the proof.

We proceed with a similar lemma for the Cayley graph expanders.

**Lemma 4.7.** Let $G = \{G_1, G_2, \ldots\}$ denote the family of Cayley graph expanders given in Section 4.2 and let $\Phi$ denote a computable graph property. For $i \geq 2$ we have

$$a_{\Phi,G_i}(T) = \Phi(a_0 \cdot C_{b_0} + a_1 \cdot C_{b_1}) + \Phi(a_2 \cdot C_{2b_2}) + \Phi(a_3 \cdot C_{2b_3}) + \Phi(G_i) \mod 2.$$  

Moreover, the numbers $a_j, b_j$ are all powers of 2 and $a_j \geq 8$.

**Proof.** By Corollary 3.3 we have

$$a_{\Phi,G_i}(T) = \sum_{\rho \in \mathcal{L}(\Phi,G_i)} \prod_{v \in V(G_i)} (-1)^{|\rho_v|-1} \cdot (|\rho_v| - 1)!$$

Setting $f(\rho) := \prod_{v \in V(G_i)} (-1)^{|\rho_v|-1} \cdot (|\rho_v| - 1)!$, this rewrites to

$$a_{\Phi,G_i}(T) = \sum_{\rho \in \mathcal{L}(\Phi,G_i)} f(\rho).$$

As before, the action of the 2-group $K_i$ leaves the set $\mathcal{L}(\Phi,G_i)$ invariant and modulo 2 the contribution of all elements $\rho$ not fixed under $K_i$ vanishes. Thus we only consider the fixed points $\rho = (\rho_v)_{v \in V(G_i)}$, for which $\rho_v$ is independent of $v$.

From the formula of $f(\rho)$ it is easy to see that $f(\rho) = 1 \mod 2$ if $\rho$ has at most two blocks and $f(\rho) = 0 \mod 2$ otherwise. Thus only the fractures $\rho$ from cases 1 to 8 of Lemma 4.4 can give a nontrivial contribution to $a_{\Phi,G_i}(T)$. The fixed-point $\rho$ contributes if and only if $\Phi(G_i, \not\exists \rho) = 1$. Finally, since the pairs of cases 4,5 and 6,7 lead to isomorphic graphs $G_i, \not\exists \rho$, any possible contributions from these cancel modulo 2 and we are left with the four summands above.

## 5 Exact Counting of Small Subgraph Patterns

Building upon our analysis of the coefficient function of the torus and the Cayley graph expanders above, we are now able to present the proofs of our results on exact counting.

### 5.1 Hardness for Minor-closed Properties

We present an exhaustive and explicit complexity dichotomy of $\#\text{EDGESUB}(\Phi)$ for properties $\Phi$ that are minor-closed. Recall that a graph $H$ is a minor of a graph $G$ if it can be obtained from $G$ by a sequence of vertex-deletions, edge-deletions and edge-contractions (where multiple edges and self-loops are deleted). A property $\Phi$ is minor-closed if, for all graphs $G$ with $\Phi(G) = 1$, we have that $\Phi$ is true for all minors of $G$ as well.
Given a minor-closed property \( \Phi \), by the celebrated Robertson-Seymour Theorem [60], there is a finite set \( \mathcal{F} \) such that for all graphs \( H \) we have that \( \Phi(H) = 1 \) if and only if no graph in \( \mathcal{F} \) is a minor of \( H \).

Recall that \#\text{EdgeSub}(\Phi) is fixed-parameter tractable if \( \Phi \) has bounded matching number or if \( \Phi \) is trivially true. We show that \#\text{EdgeSub}(\Phi) is \#W[1]-hard in all other cases, given that \( \Phi \) is minor-closed.

It turns out that we need to distinguish\(^{17}\) whether \( \mathcal{F} \) contains a graph of degree at most 2.

**Lemma 5.1.** Let \( \Phi \) denote a minor-closed graph property with unbounded matching number and assume that \( \mathcal{F} \) contains a graph \( F \) of degree at most 2. Then \( \mathcal{H}[\Phi, \emptyset] \) is infinite.

**Proof.** The assumption that \( \Phi \) has unbounded matching number implies that \( \Phi \) is satisfied by graphs containing arbitrarily large matchings. Since \( \Phi \) is closed under taking minors, this implies that \( \Phi(M_k) = 1 \) for all \( k \).

Now observe that since \( F \) has degree at most 2, the graph \( F \) is a union of paths and cycles. Therefore, there is a constant \( c \) (only depending on \( F \)) such that for all \( \ell > c \), the graph \( F \) is a minor of each of the following graphs:

\[
M_{2\ell} + \ell C_{\ell}, 2\ell C_{\ell}, \ell C_{2\ell}, \ell S_{\ell}, \text{ and } \emptyset_{\ell}.
\]

Indeed, any finite union of paths and cycles can be obtained as a minor of sufficiently large cycle packings, sun packings and tori.

Now assume that \( \ell \) is additionally a prime and greater than 3. By Lemma 4.6\(^{2}\), we thus have that

\[
a_{\Phi, \emptyset_{\ell}}(\mathbb{T}) = -6\Phi(M_{2\ell}) + 8\Phi(\ell^2 P_2) = -6 + 8\Phi(\ell^2 P_2) \mod \ell.
\]

The claim follows by observing that \(-6 + 8\Phi(\ell^2 P_2) \neq 0 \mod \ell \) for every prime \( \ell > 3 \), regardless on whether \( \Phi(\ell^2 P_2) = 1 \) or \( \Phi(\ell^2 P_2) = 0 \).

Recall that \( \mathcal{G} \) is the family of Cayley graph expanders introduced in Section 4.2.

**Lemma 5.2.** Let \( \Phi \) denote a minor-closed graph property which is not trivially true, and assume that \( \mathcal{F} \) does not contain a graph \( F \) of degree at most 2. Then \( \mathcal{H}[\Phi, \mathcal{G}] \) is infinite.

**Proof.** Since \( \Phi \) is not trivially true, the set \( \mathcal{F} \) is non-empty. Thus let \( F \) denote an arbitrary graph in \( \mathcal{F} \). By Fact 2.3\(^{3}\), there is an index \( j \) such that for all \( i \geq j \), the graph \( G_i \) contains the complete graph on \( \#V(F) \) vertices (and thus also \( F \)) as a minor. In other words, \( \Phi(G_i) = 0 \) for all \( i \geq j \).

By Lemma 4.7\(^{4}\), we have that for all \( i \geq 2 \)

\[
a_{\Phi, G_i}(\mathbb{T}) = \Phi(a_0 \cdot C_{b_0} + a_1 \cdot C_{b_1}) + \Phi(a_2 \cdot C_{2b_2}) + \Phi(a_3 \cdot C_{2b_3}) + \Phi(G_i) \mod 2.
\]

Hence, for \( i \geq \max\{2, j\} \), we have

\[
a_{\Phi, G_i}(\mathbb{T}) = \Phi(a_0 \cdot C_{b_0} + a_1 \cdot C_{b_1}) + \Phi(a_2 \cdot C_{2b_2}) + \Phi(a_3 \cdot C_{2b_3}) \mod 2.
\]

Finally, we rely on the premise of the lemma, implying that each graph in \( \mathcal{F} \) has a vertex of degree at least 3. Consequently, no graph in \( \mathcal{F} \) can be a minor of a cycle-packing. Thus \( \Phi(a_0 \cdot C_{b_0} + a_1 \cdot C_{b_1}) = \Phi(a_2 \cdot C_{2b_2}) = \Phi(a_3 \cdot C_{2b_3}) = 1 \), and, consequently, \( a_{\Phi, G_i}(\mathbb{T}) = 1 \mod 2 \) for each \( i \geq \max\{2, j\} \).

\(^{17}\)For example, if \( \mathcal{F} \) only contains the path of two edges, then \#\text{EdgeSub}(\Phi) is the problem of counting \( k \)-matchings. If we would be able to use the Cayley graph expanders for this property as given by Lemma 4.7\(^{4}\), then the coefficient function would satisfy \( a_{\Phi, \mathbb{T}}(\mathbb{T}) = 1 \mod 2 \), which could be used to establish that counting \( k \)-matchings modulo 2 is hard, contradicting the fact that the latter problem is known to be polynomial time solvable [9] Section 1.4.\]
We are finally able to prove our main result for exact counting; note that all minor-closed graph properties are (polynomial-time) computable due to the finite set of forbidden minors.

\[ \text{Theorem 5.3.} \quad \text{Let } \Phi \text{ denote a minor-closed graph property. If } \Phi \text{ is either trivially true or of bounded matching number}^{18}, \text{then the (exact) counting version } \#\operatorname{Edgesub}(\Phi) \text{ is fixed-parameter tractable. Otherwise } \#\operatorname{Edgesub}(\Phi) \text{ is } \#W[1]-\text{hard. If, additionally, each forbidden minor of } \Phi \text{ has a vertex of degree at least } 3, \text{then } \#\operatorname{Edgesub}(\Phi) \text{ cannot be solved in time } f(k) \cdot |G|^{o(k/\log k)}, \]

\text{for any function } f, \text{ unless the Exponential Time Hypothesis fails.}

\textbf{Proof.} The (fixed-parameter) tractability part is given by Main Theorem\footnote{We say that a property has bounded matching number if there is a constant bound on the size of a largest matching in graphs satisfying } \Phi. If \Phi \text{ has unbounded matching number, but at least one forbidden minor is of degree at most } 2, \text{ then, by Lemma } 5.1 \text{ the set } \mathcal{H}[\Phi, \emptyset] \text{ is infinite, which implies } \#W[1]-\text{hardness by Lemma } 3.8.

If \Phi \text{ is not trivially true and each forbidden minor has a vertex of degree at least } 3, \text{ then, by Lemma } 5.2 \text{ the set } \mathcal{H}[\Phi, \mathcal{G}] \text{ is infinite. Again by Lemma } 3.8 \text{ this implies both, } \#W[1]-\text{hardness and the conditional lower bound.} \]

5.2 Hardness for Selected Natural Graph Properties

In addition to classifying \#\operatorname{Edgesub}(\Phi) \text{ for minor-closed properties } \Phi, \text{ we can also use the criteria for establishing } \#W[1]-\text{hardness and an almost tight conditional lower bound of } \#\operatorname{Edgesub}(\Phi) \text{ directly to some natural, but non-minor-closed properties. With this we aim to illustrate the simplicity of applying our fixed-points result for Cayley graph expanders (Lemma } 4.7 \text{) to explicitly given graph properties.}

\[ \text{Main Theorem 3. Consider the following graph properties.} \]

\[ = \Phi_{C}(H) = 1 \text{ if and only if } H \text{ is connected.} \]

\[ = \Phi_{H}(H) = 1 \text{ if and only if } H \text{ is Hamiltonian.} \]

\[ = \Phi_{E}(H) = 1 \text{ if and only if } H \text{ is Eulerian.} \]

\[ = \Phi_{CF}(H) = 1 \text{ if and only if } H \text{ is claw-free.} \]

\text{For } \Phi \in \{ \Phi_{C}, \Phi_{H}, \Phi_{E}, \Phi_{CF} \}, \text{ the problem } \#\operatorname{Edgesub}(\Phi) \text{ is } \#W[1]-\text{hard. Further, unless ETH fails, the problem } \#\operatorname{Edgesub}(\Phi) \text{ cannot be solved in time } f(k) \cdot |G|^{o(k/\log k)} \text{ for any function } f. \]

\textbf{Proof.} Our proof proceeds by applying Lemma } 4.7 \text{ to show } a_{\Phi,G_i}(T) \neq 0 \text{ for each of the properties } \Phi, \text{ allowing us to conclude using Lemma } 3.8.

\text{For } \Phi \in \{ \Phi_{C}, \Phi_{H}, \Phi_{E} \}, \text{ observe that the graphs } a_0 \cdot C_{b_0} + a_1 \cdot C_{b_1} + a_2 \cdot C_{b_2}, \text{ and } a_3 \cdot C_{b_3} \text{ are each disconnected (and hence not Hamiltonian, nor Eulerian either) if } a_i \geq 8 \text{ for } i = 1, 2, 3. \text{ Further, the graphs } G_i \text{ are connected since Cayley graphs are connected}\footnote{Recall that our definition of Cayley graphs enforces the set } S \text{ to be a set of generators of the group.}\text{. Thus, the graphs } G_i \text{ are also Eulerian, since they are 4-regular. Moreover, Cayley graphs of } p\text{-groups are Hamiltonian}^{19}. \text{ Thus, by Lemma } 4.7 \text{ we have that for } i \geq 2:}

\[ a_{\Phi,G_i}(T) = \Phi(a_0 \cdot C_{b_0} + a_1 \cdot C_{b_1}) + \Phi(a_2 \cdot C_{b_2}) + \Phi(a_3 \cdot C_{b_3}) + \Phi(G_i) = 1 \mod 2. \]

Consequently, \( \mathcal{H}[\Phi, \mathcal{G}] \) is infinite if \( \Phi \in \{ \Phi_{C}, \Phi_{H}, \Phi_{E} \} \). By Lemma } 3.8 \text{ we obtain both, } \#W[1]-\text{hardness and the conditional lower bound.}
For $\Phi = \Phi_{CF}$ we can perform a similar analysis: observe that cycle-packings are always claw-free. On the other hand, for each $i > 2$, the graphs $G_i$ do contain an (induced) claw. To see this, let $\epsilon_{K_i}$ denote the neutral element of $K_i$ and consider the vertices of $G_i$ associated to $\epsilon_{K_i}$, $v_0$, $v_1$ and $v_1^{-1}$. While $\epsilon_{K_i}$ is adjacent to the remaining three cosets, it is easy to check by hand that $v_0$, $v_1$ and $v_1^{-1}$ constitute an independent set in $G_i$.

Consequently, by Lemma 4.7 we have that for $i > 2$:

$$a_{\Phi,G_i}(\top) = \Phi(a_0 \cdot C_{b_0} + a_1 \cdot C_{b_1}) + \Phi(a_2 \cdot C_{b_2}) + \Phi(a_3 \cdot C_{b_3}) + \Phi(G_i) = 3 + 0 = 1 \mod 2.$$  

Thus, $\mathcal{H}[\Phi_{CF}, G]$ is infinite. By Lemma 3.8 we hence obtain both $\#W[1]$-hardness and the conditional lower bound.

6 Approximate Counting of Small Subgraph Patterns

Recall that we identified $\#\text{Edgesub}(\Phi)$ as an inherently hard problem in case we aim for exactly counting the solutions. In particular, we established $\#W[1]$-hardness for any non-trivial minor-closed property $\Phi$ of unbounded matching number. For this reason, the section below deals with the complexity of approximating the number of solutions. Tractability of approximating the solutions of parameterized counting problems is given by the notion of a fixed-parameter tractable randomized approximation scheme.

**Definition 6.1 (FPTRAS).** Let $(P, \kappa)$ denote a parameterized counting problem. A fixed-parameter tractable randomized approximation scheme “FPTRAS” for $(P, \kappa)$ is a randomized algorithm $A$ that, given $x \in \Sigma^*$ and rational numbers $\epsilon > 0$ and $0 < \delta < 1$ computes an integer $z$ such that

$$\Pr[(1-\epsilon)P(x) \leq z \leq (1+\epsilon)P(x)] \geq 1-\delta.$$  

The running time of $A$ must be bounded by $f(\kappa(x)) \cdot \text{poly}(|x|, 1/\epsilon, \log(1/\delta))$ for some computable function $f$.

Indeed, we can show that $\#\text{Edgesub}(\Phi)$ allows an FPTRAS for every minor-closed property $\Phi$. In fact, we prove the following general criterion, which implies the existence of an FPTRAS for minor-closed properties.

**Main Theorem 4.** Let $\Phi$ denote a computable graph property. If $\Phi$ satisfies the matching criterion and the star criterion, or if $\Phi$ has bounded treewidth, then $\#\text{Edgesub}(\Phi)$ admits an FPTRAS.

We start with the case of $\Phi$ satisfying both the matching and the star criterion. For readers familiar with the meta-theorem of Dell, Lapinskas and Meeks [30], we point out that their method cannot be used to achieve the desired goal in the current setting: the results in [30] Section 1.3 imply that $\#\text{Edgesub}(\Phi)$ admits an FPTRAS whenever the edge-colourful decision version of $\text{Edgesub}(\Phi)$ is fixed-parameter tractable; in the latter, we expect as input a graph $G$ with $k$ different edge-colours and the goal is to decide whether there is a subset $A$ of edges containing each colour exactly once such that $G[A]$ satisfies $\Phi$ (w.r.t. the underlying uncoloured graph). Thus, if we could show that the edge-colourful decision version is fixed-parameter tractable for properties satisfying the matching and the star criterion, Main Theorem 4 would follow.

However, the latter cannot be true (unless $\text{FPT} = \#W[1]$) since the following property $\Phi$ induces a $\#W[1]$-hard colourful decision version, while satisfying both the matching and the star criterion: $\Phi(H) = 1$ if and only if $H$ is either a star, a matching, or the union of a clique and a triangle. $\#W[1]$-hardness follows
from a reduction from finding edge-colourful $k$-cliques in a graph, which is known to be $\text{W}[1]$-hard\textsuperscript{20} The reduction is straightforward: given a graph $G$ with $\binom{k}{2}$ edge colours, we construct a graph $G'$ by adding a triangle with three fresh colours to the graph. Then $G'$ contains a colourful $\binom{k}{2}$ + 3-edge-subset $A$ that satisfies $\Phi$ if and only if $G$ contains an edge-colourful $k$-clique. The latter is true since any colourful $\binom{k}{2}$ + 3-edge-subset must contain the triangle with the three fresh colours and can thus neither induce a star, nor a matching.

Being unable to rely on the colourful decision version, we thus use a different approach using Ramsey’s Theorem, similarly to the one in [58]. More precisely, we use the following consequence:

\begin{lemma}
Let $k \geq 4$ denote a positive integer and let $G$ denote a graph with at least $R(k, k)$ edges. Then $G$ contains either $K_{1,k}$ or $M_k$ as a subgraph.
\end{lemma}

\begin{proof}
We apply Ramsey’s Theorem to the line graph $L(G)$ of $G$: The vertices of $L(G)$ are the edges of $G$, and two vertices $e$ and $e'$ of $L(G)$ are adjacent if and only if $e \cap e' \neq \emptyset$. Since $L(G)$ contains at least $R(k, k)$ vertices, Ramsey’s Theorem implies that $L(G)$ either contains an independent set of a clique of size $k$. Note that a $k$-independent set of $L(G)$ corresponds to a $k$-matching in $G$, and that a $k$-clique in $L(G)$ corresponds to a star $K_{1,k}$ in $G$; the latter requires that $k \geq 4$ since the line graph of a triangle is a triangle (and thus a clique) as well.

The subsequent observation enables our Monte-Carlo algorithm to only rely on “FPT-many” samples:

\begin{lemma}
Let $k \geq 4$ denote a positive integer and let $G$ denote a graph with at least $R(k, k)$ edges. Assume a subset $A$ of $k$ edges is sampled uniformly at random. We have

$$
\Pr[G[A] \cong M_k \lor G[A] \cong K_{1,k}] \geq \left( \frac{R(k, k)}{k} \right)^{-1}.
$$

\end{lemma}

\begin{proof}
Set $m = |E(G)|$ and $r = R(k, k)$. It is convenient to assume that $A$ is sampled as follows: we first choose $r$ edges u.a.r., denote this set by $S$, and afterwards we obtain $A$ by choosing $k$ edges among $S$ u.a.r.; of course, we need to show that this yields a uniform distribution. Let $B$ denote any $k$-edge subset of $G$. By the law of total probability, we have that

$$
\Pr[A = B] = \sum_{T \in \binom{E(G)}{r}} \Pr[S = T] \cdot \Pr[A = B | S = T] = \sum_{T \in \binom{E(G)}{r}} \left( \frac{m}{r} \right)^{-1} \cdot \Pr[A = B | S = T].
$$

Note that $\Pr[A = B | S = T] = \binom{k}{r}^{-1}$ if $B \subseteq T$, and $\Pr[A = B | S = T] = 0$ otherwise. Consequently

$$
\Pr[A = B] = \# \{T \subseteq E(G) | B \subseteq T \wedge |T| = r\} \cdot \left( \frac{m}{r} \right)^{-1} \left( \frac{r - k}{k} \right)^{-1} = \binom{m - k}{r - k} \left( \frac{m}{r} \right)^{-1} \left( \frac{r}{k} \right)^{-1} = \binom{m}{k}^{-1}. \quad \text{(1)}
$$

Now let $\mathcal{E}$ denote the event $G[A] \cong M_k \lor G[A] \cong K_{1,k}$ and note that for every $r$-edge subset $T$ of $G$ we have that $\Pr[\mathcal{E} | S = T] \geq \binom{k}{r}^{-1}$ since, by the previous lemma, $G[T]$ contains either $M_k$ or $K_{1,k}$ as a subgraph. We conclude that

$$
\Pr[\mathcal{E}] = \sum_{T \in \binom{E(G)}{r}} \Pr[S = T] \cdot \Pr[\mathcal{E} | S = T] = \left( \frac{m}{r} \right) \left( \frac{m}{r} \right)^{-1} = \Pr[\mathcal{E} | S = T] \geq \left( \frac{r}{k} \right)^{-1},
$$

which concludes the proof.

\end{proof}

\textsuperscript{20}The vertex-colourful clique problem is $\text{W}[1]$-hard (see Chapter 13 in [27]) and reduces to the edge-colourful clique problem by assigning an edge $\{u, v\}$ the colour $\{c(u), c(v)\}$, where $c(u)$ and $c(v)$ are the vertex-colours of $u$ and $v$.\[1]
Algorithm 1: An FPTRAS for $\#\text{EdgeSub}(\Phi)$ if $\Phi$ satisfies the matching and the star criterion.

1. \textbf{MatchingsAndStarsFPTRAS}(\(G, k, \varepsilon, \delta\))
2. \textbf{if} \(k < c\) \textbf{or} \(|E(G)| \leq R(k, k)\) \textbf{then}
3. \hspace{1em} Solve the problem exactly by brute force.;
4. \textbf{else}
5. \hspace{1em} \(X \leftarrow 0; t \leftarrow \left(\frac{R(k, k)}{k}\right) \cdot \frac{3\ln(2/\delta)}{\varepsilon^2};\)
6. \hspace{1em} \textbf{for} \(i \leftarrow 1 \textbf{ to } t\) \textbf{do}
7. \hspace{2em} \textbf{if} \(\Phi(G[A]) = 1\) \textbf{then} \(X \leftarrow X + 1;\)
8. \hspace{1em} \textbf{return} \(\frac{X}{t} \cdot \left(\frac{|E(G)|}{k}\right);\)

For our FPTRAS, we use the following (consequence of a) Chernoff bound:

\textbf{Theorem 6.4} (see Theorem 11.1 in [61]). Let \(X_1, \ldots, X_t\) denote independent and identically distributed indicator random variables with expectation \(\eta = E[X_i]\), and let \(0 < \varepsilon, \delta < 1\) denote positive rationals.

If \(t \geq \left(3\ln(2/\delta)\right)/\varepsilon^2\), then

\[
\Pr\left[\left|\frac{1}{t} \sum_{i=1}^{t} X_i - \eta\right| < \varepsilon\eta\right] \geq 1 - \delta.
\]

\textbf{Lemma 6.5.} Let \(\Phi\) denote a computable graph property satisfying both, the matching criterion and the star criterion. Then \(\#\text{EdgeSub}(\Phi)\) has an FPTRAS.

\textbf{Proof.} By assumption, there is a constant \(c'\) such that \(\Phi\) is true for all matchings and stars of size at least \(c'\); we set \(c = \max(c', 4)\). Our FPTRAS \(A\) is constructed as follows: If \(k < c\) or if \(|E(G)| \leq R(k, k)\), then we solve the problem (exactly) by the naive brute-force algorithm. Otherwise, we take \(\left(\frac{R(k, k)}{k}\right) \cdot \frac{3\ln(2/\delta)}{\varepsilon^2}\) many independent samples of \(k\)-edge sets \(A\) of \(G\), each taken uniformly at random. Finally, we output the fraction of those samples \(A\) such that \(\Phi(G[A]) = 1\). Consult Algorithm 1 for a visualization as pseudo-code.

Let us first argue about the running time: if \(k < c\) then the brute force algorithm takes time at most \(|G|^c\) \footnote{\(|G|\) rather than \(|E(G)|\) since \(G\) might contain many isolated vertices.} and if \(|E(G)| \leq R(k, k)\) then the brute force algorithm takes time at most \(|G| + R(k, k)^k\).

Otherwise, we iterate through the loop \(t\) times, and each iteration can clearly be done in time \(f'(k) \cdot \text{poly}(|G|)\) for some computable function \(f'\) — note that the factor \(f'(k)\) depends on the complexity of verifying whether \(\Phi(G[A])\) holds, which might require super-polynomial time in \(|G[A]| \in O(k)\). The overall running time is thus bounded by

\[
\max\left\{|G|^c, |G| + R(k, k)^k, \left(\frac{R(k, k)}{k}\right) \cdot \frac{3\ln(2/\delta)}{\varepsilon^2} \cdot f'(k) \cdot \text{poly}(|G|)\right\} \leq f(k) \cdot \text{poly}(|G|, 1/\varepsilon, \log(1/\delta))
\]

for some computable function \(f\).
Next note that correctness is trivial in case the brute force algorithm is executed. Hence assume that \( k \geq c \) and \(|E(G)| > R(k, k)\). To avoid notational clutter, we set \( r := R(k, k) \) and \( m := |E(G)|\). Now let \( X_i \) denote the indicator variable defined to be 1 if the \( i \)-th sample, denoted \( A_i \), satisfies \( \Phi(G[A_i]) = 1 \), and \( X_i = 0 \) otherwise. Observe that \( E[X_i] = \text{EdgeSub}(\Phi, k \to G) \cdot \binom{m}{k}^{-1} \) for all \( i \). In what follows, we thus just set \( \eta := E[X_i] \). Since \( \Phi \) is true for \( M_k \) and \( K_{1,k} \), and by Lemma 6.3 we furthermore have

\[
\eta = \Pr[\Phi(G[A]) = 1] \geq \Pr[G[A] \cong M_k \lor G[A] \cong K_{1,k}] \geq \left( \frac{r}{k} \right)^{-1}.
\]

Consequently, \( t \geq (3 \ln(2/\delta))/(\varepsilon^2 \eta) \). By the previous Chernoff bound, we thus have

\[
\Pr\left[ \left| \frac{1}{t} \sum_{i=1}^{t} X_i - \eta \right| < \varepsilon \eta \right] \geq 1 - \delta.
\]

Finally, recall that \( X = \sum_{i=1}^{t} X_i \) and observe that

\[
\left| \frac{1}{t} \sum_{i=1}^{t} X_i - \eta \right| < \varepsilon \eta \Rightarrow \left| \frac{X}{t} - \frac{\text{EdgeSub}(\Phi, k \to G)}{\binom{m}{k}} \right| < \varepsilon \cdot \frac{\text{EdgeSub}(\Phi, k \to G)}{\binom{m}{k}}
\]

\[
\Rightarrow \left| \frac{X}{t} \cdot \binom{m}{k} - \text{EdgeSub}(\Phi, k \to G) \right| < \varepsilon \cdot \text{EdgeSub}(\Phi, k \to G),
\]

the latter of which implies

\[
(1 - \varepsilon) \cdot \text{EdgeSub}(\Phi, k \to G) \leq \frac{X}{t} \cdot \binom{m}{k} \leq (1 + \varepsilon) \cdot \text{EdgeSub}(\Phi, k \to G).
\]

The proof is thus concluded. \(\square\)

For the case of \( \Phi \) having bounded treewidth, we rely on the following result of Arvind and Raman; to this end, given a fixed positive integer \( T \), let \( \text{SUB}(T) \) denote the problem that, on input a graph \( H \) of treewidth at most \( T \) and an arbitrary graph \( G \), requires to compute \( \text{SUB}(H \to G) \).

\[\text{Theorem 6.6 (4). For each positive integer } T, \text{ there is an FPTRAS for } \text{SUB}(T) \text{ if parameterized by the size of the graph } H.\]

\[\text{Lemma 6.7. Let } \Phi \text{ denote a computable graph property. If } \Phi \text{ has bounded treewidth, then } \text{EdgeSub}(\Phi) \text{ admits an FPTRAS.}\]

\[\text{Proof. By assumption, there is a constant } T \text{ such that the treewidth of each graph } H \text{ with } \Phi(H) = 1 \text{ is at most } T. \text{ Define } g(k) := |\Phi_k| \text{ and observe that } g \text{ is computable as } \Phi \text{ is.}\]

Recall from equation (3) that for each \( G \) and \( k \) we have

\[\text{EdgeSub}(\Phi, k \to G) = \sum_{H \in \Phi_k} \text{Sub}(H \to G)\]

We thus just use the FPTRAS from Theorem 6.6 to approximate (with probability \( 1 - \delta/g(k) \)) each term \( \text{Sub}(H \to G) \) with \( H \in \Phi_k \) and output the sum given by the previous equation.

Observe that approximating each term \( \text{Sub}(H \to G) \) takes time at most

\[f'(|H|) \cdot \text{poly}(|G|, 1/\varepsilon, \log(g(k)/\delta)),\]

for some computable function \( f' \).
Since each \( H \in \Phi_k \) has \( k \) edges, the overall running time is thus clearly bounded by

\[
f(k) \cdot \text{poly}(|G|, 1/\epsilon, \log(\delta))
\]

for some computable function \( f \)—note that \( f \) depends on \( \Phi, f' \) and \( g \), but the latter three are independent of the input. Now let \( r \) denote the output of our algorithm. It remains to show that

\[
\Pr[(1 - \epsilon) \cdot \#\text{EdgeSub}(\Phi, k \to G) \leq r \leq (1 + \epsilon) \cdot \#\text{EdgeSub}(\Phi, k \to G)] \geq 1 - \delta.
\]

Write \( r_H \) for the output of the FPRAS from Theorem 6.6 on input \( G, H, \epsilon, \) and \( \delta/g(k) \). Then

\[
r = \sum_{H \in \Phi_k} r_H,
\]

and the following holds for each \( H \in \Phi_k \)

\[
\Pr[(1 - \epsilon) \cdot \#\text{Sub}(H \to G) \leq r_H \leq (1 + \epsilon) \cdot \#\text{Sub}(H \to G)] \geq 1 - \delta/g(k).
\]

Since the outcomes \( r_H \) are independent and \( g(k) = |\Phi_k| \), we have

\[
\Pr[\forall H \in \Phi_H : (1 - \epsilon) \cdot \#\text{Sub}(H \to G) \leq r_H \leq (1 + \epsilon) \cdot \#\text{Sub}(H \to G)] \geq (1 - \delta/g(k))^{g(k)} \geq (1 - \delta),
\]

where the last step is Bernoulli’s inequality. Consequently, with probability at least \((1 - \delta)\), we have that

\[
(1 - \epsilon) \cdot \#\text{EdgeSub}(\Phi, k \to G) = (1 - \epsilon) \sum_{H \in \Phi_k} \#\text{Sub}(H \to G)
\]

\[
= \sum_{H \in \Phi_k} (1 - \epsilon) \cdot \#\text{Sub}(H \to G)
\]

\[
\leq \sum_{H \in \Phi_k} r_H = r
\]

\[
\leq \sum_{H \in \Phi_k} (1 + \epsilon) \cdot \#\text{Sub}(H \to G)
\]

\[
= (1 + \epsilon) \sum_{H \in \Phi_k} \#\text{Sub}(H \to G)
\]

\[
= (1 + \epsilon) \cdot \#\text{EdgeSub}(\Phi, k \to G),
\]

which concludes the proof.

Proof of Main Theorem 4. Holds by Lemmas 6.5 and 6.7.

7 Detection of Small Subgraph Patterns

In this section, we study the complexity of the decision problem \( \text{EdgeSub}(\Phi) \). As a first observation we observe that \( \text{EdgeSub}(\Phi) \) essentially subsumes the (parameterized) subgraph isomorphism problem: consider for instance the property \( \Phi \) defined as \( \Phi(H) = 1 \) if and only if \( H \cong K_{\ell, \ell} \) for some positive integer \( \ell \). Then \( \text{EdgeSub}(\Phi) \) is equivalent to the problem \( k\text{-BICLIQUE} \) which was only recently shown to be \( \text{W}[1] \)-hard by the seminal result of Lin [55] after being unresolved for at least a decade.
More generally, let $\mathcal{H}$ denote a class of graphs and define $\text{Emb}(\mathcal{H})$ as the problem that asks, given a graph $H \in \mathcal{H}$ and an arbitrary graph $G$, whether there is a subgraph embedding from $H$ to $G$; the parameterization is given by $|H|$. Plehn and Voigt [63] proved $\text{Emb}(\mathcal{H})$ to be fixed-parameter tractable whenever the treewidth of graphs in $\mathcal{H}$ is bounded by a constant. On the other hand, the question whether $\text{Emb}(\mathcal{H})$ is $\mathcal{W}[1]$-hard in all remaining cases is one of the “most infamous” [35, Chapter 33.1] open problems in parameterized complexity. Since $\text{EdgeSub}(\Phi)$ subsumes $\text{Emb}(\mathcal{H})$ as we have seen in case of $k$-BICLIQUE, a complete classification of $\text{EdgeSub}(\Phi)$ seems to be elusive at the moment.

However, we identify the following tractable instances of $\text{EdgeSub}(\Phi)$, which significantly extends the case of bounded treewidth.

**Main Theorem 5.** Let $\Phi$ denote a computable graph property. If $\Phi$ satisfies the matching criterion or the star criterion, or if $\Phi$ has bounded treewidth, then $\text{EdgeSub}(\Phi)$ is fixed-parameter tractable.  

In case of $\Phi$ satisfying the matching or the star criterion, fixed-parameter tractability is obtained by a surprisingly simple Win-Win approach relying on the treewidth and the maximum degree of a graph. Assume, for example, that $\Phi$ is true for all matchings. Now, given a graph $G$ and an integer $k$, we can easily verify whether $G$ contains a maximum matching of size at least $k$. If the latter is true, $G$ contains a subgraph with $k$ edges that satisfies $\Phi$. More interestingly, if $G$ does not contain a maximum matching of size $k$, then the treewidth of $G$ is bounded by $2k$, and we can efficiently use dynamic programming over a tree-decomposition of small width of $G$ to verify whether $\text{EdgeSub}(\Phi, k \rightarrow G) \neq \emptyset$. Formally, the latter can be established by an easy application of Courcelle’s Theorem [22] as shown in the following lemma:

**Lemma 7.1.** Let $\Phi$ denote a computable graph property. There is a computable function $g$ and an algorithm $A$ that, given a graph $G$ and a positive integer $k$, correctly decides whether $\text{EdgeSub}(\Phi, k \rightarrow G) \neq \emptyset$ in time $g(\text{tw}(G), k) \cdot |G|$.

**Proof.** We use Courcelle’s Theorem as stated in [39, Theorem 11.37]. Thus it remains to provide an MSO-sentence $\varphi$ such that $G$ satisfies $\varphi$ if and only if $\text{EdgeSub}(\Phi, k \rightarrow G) \neq \emptyset$. To this end, let $H \in \Phi_k$ and assume that $V(H) = \{1, \ldots, v_H\}$. Consider the following sentence

$$\varphi_H := \exists x_1, \ldots, \exists x_{v_H} : \bigwedge_{i \neq j} x_i \neq x_j \land \bigwedge_{(i,j) \in E(H)} E(x_i, x_j).$$

Observe that $G$ satisfies $\varphi_H$ if and only if $H$ is a subgraph of $G$. Consequently, we set

$$\varphi := \bigvee_{H \in \Phi_k} \varphi_H.$$

Since the length of $\varphi$ only depends on $\Phi$ and $k$, the lemma holds by Courcelle’s Theorem.  

We are now able to establish fixed-parameter tractability of $\text{EdgeSub}(\Phi)$ whenever $\Phi$ satisfies the matching criterion.

**Lemma 7.2.** Let $\Phi$ denote a computable graph property that satisfies the matching criterion. Then the problem $\text{EdgeSub}(\Phi)$ is fixed-parameter tractable.

---

22 To be precise, $\text{EdgeSub}(\Phi)$ subsumes $\text{Emb}(\mathcal{H})$ whenever $\mathcal{H}$ does not contain two graphs with the same number of edges, which is, however, true for most of the natural instances of the subgraph isomorphism problem such as finding cliques, bicliques, cycles, paths and matchings, only to name a few.

23 We refer the reader to e.g. Chapter 4 in [39] for an introduction to Monadic Second Order (MSO) logic.
Detecting and Counting Small Subgraphs, and a Parameterized Tutte Polynomial

**Proof.** Since $\Phi$ satisfies the matching criterion, there is a constant $c$ (only depending on $\Phi$) such that $\Phi(M_k) = 1$ for all $k \geq c$. The FPT algorithm is constructed as follows:

Given a graph $G$ and a positive integer $k$, we first check whether $k < c$. If this is true, then we proceed by brute force, that is, for each subset $A$ of $k$ edges of $G$, we verify whether $\Phi(G[A]) = 1$ and we output 1 if there is at least one $A$ for which $\Phi(G[A]) = 1$, and 0 otherwise; note that the latter can be done in time $f(k)$ for some computable function $f$ since $\Phi$ is computable. The overall running time of the brute-force approach is hence bounded by $f(k) \cdot m^k \leq m^{O(1)}$ since $k < c$.

If $k \geq c$, we compute a maximum matching $M$ of $G$ in polynomial time by, e.g., the Blossom Algorithm [37]. If $|M| \geq k$, then we can output 1, since any $k$-subset $A$ of $M$ satisfies that $\Phi(G[A]) = 1$ by assumption.

In the remaining case, we can thus assume that the matching number of $G$ is bounded by $k$. Consequently, the vertex cover number of $G$ is bounded by $2k$. Since the treewidth of a graph is bounded by its vertex cover number, we conclude that $\text{tw}(G) \leq 2k$. Invoking the algorithm from the previous lemma thus yields an overall running time bounded by

$$m^{O(1)} + g(2k, k) \cdot |G|,$$

which proves fixed-parameter tractability.

We continue with the case of $\Phi$ satisfying the star criterion. To this end, we require the following result, which seems to be folklore; we provide a proof based on the bounded search-tree paradigm for completeness.

**Lemma 7.3.** Let $\Phi$ denote a computable graph property. There is a computable function $g$ and an algorithm $\mathcal{A}$ that, given a graph $G$ and a positive integer $k$, correctly decides whether $\text{EdgeSub}(\Phi, k \rightarrow G) \neq \emptyset$ in time $g(\deg(G), k) \cdot |G|$.

**Proof.** We check for each $H \in \Phi_k$ whether $H$ is a subgraph of $G$ and output 1 if (and only if) at least one of those checks is positive.

Assume for a moment that $H$ is connected. In this case, the strategy is very simple: We guess a vertex $v$ of $G$ and search for a subgraph embedding of $H$ in $G$ that includes $v$. Since $H$ is connected and has $k$ edges, the image of the subgraph embedding can only contain vertices of distance at most $k$ from $v$. This allows us to search for a copy of $H$ in the $\leq k$ neighbourhood of $v$ by brute-force, since the latter contains at most $\deg(G)^k$ vertices. The overall running time of finding a subgraph isomorphic to $H$ in $G$ is thus bounded by $|V(G)| \cdot \deg(G)^k$.

The situation becomes slightly more complicated if $H$ is not connected. We would like to perform the previous strategy for each connected component of $H$, adding an additional factor of $k$ in the worst case. However, since a subgraph embedding needs to be injective, we have to guarantee that we do not construct a solution that uses vertices of $G$ twice. This issue is solved by a standard application of colour-coding: We choose a function $\text{col} : V(G) \to V(H)$ uniformly at random. If $G$ contains a subgraph isomorphic to $H$, then with probability at least $p(k) > 0$ there is a subgraph embedding $\psi : V(H) \to V(G)$ such that additionally $\text{col}(\psi(v)) = v$ for each vertex $v \in V(H)$, and such a subgraph embedding can be found in time $O(k \cdot |V(G)| \cdot \deg(G)^k)$ by adapting the above strategy for every connected component $H$ accordingly. Finally, derandomization can be achieved by perfect hashing as shown in [3] (see also [39] Chapter 13.3).

Let us now establish fixed-parameter tractability of $\text{EdgeSub}(\Phi)$ whenever $\Phi$ satisfies the star criterion.

**Lemma 7.4.** Let $\Phi$ denote a computable graph property satisfying the star criterion. Then the problem $\text{EdgeSub}(\Phi)$ is fixed-parameter tractable.
Proof. Since $\Phi$ satisfies the star criterion, there is a constant $c$ (only depending on $\Phi$) such that $\Phi(K_{1,k}) = 1$ for all $k \geq c$. The FPT algorithm is constructed as follows:

Given a graph $G$ and a positive integer $k$, we first check whether $k < c$. If this is true, then we proceed by brute force, that is, for each subset $A$ of $k$ edges of $G$, we verify whether $\Phi(G[A]) = 1$ and we output 1 if there is at least one $A$ for which $\Phi(G[A]) = 1$, and 0 otherwise; note that the latter can be done in time $f(k)$ for some computable function $f$ since $\Phi$ is computable. The overall running time of the brute-force approach is hence bounded by $f(k) \cdot m^k \leq m^{O(1)}$ since $k < c$.

If $k \geq c$, we check whether $G$ contains a vertex $v$ of degree at least $k$, in which case we can output 1, since any $k$-subset $A$ of the incident edges of $v$ satisfies that $\Phi(G[A]) = 1$ by assumption.

In the remaining case, we can thus assume that $\deg(G) \leq k$. Invoking the algorithm from the previous lemma thus yields an overall running time bounded by $m^{O(1)} + g(k,k) \cdot |G|$, which proves fixed-parameter tractability.

Proof of Main Theorem 5. In case $\Phi$ satisfies the matching criterion or the star criterion, the claim holds by Lemma 7.2 and Lemma 7.4. If $\Phi$ has bounded treewidth, then, given $G$ and $k$, we can use the algorithm of Plehn and Voigt [63] for each $H \in \Phi_k$. Since the size of $\Phi_k$ is bounded by a function in $k$, the overall running time still yields fixed-parameter tractability.

Our main result regarding minor-closed properties is now obtained by the combination of our results in the realms of exact counting, approximate counting, as well as decision:

Main Theorem 1. Let $\Phi$ denote a minor-closed graph property.

1. Exact Counting: If $\Phi$ is either trivially true or of bounded matching number, then the (exact) counting version $\text{#EdgeSub}(\Phi)$ is fixed-parameter tractable. Otherwise $\text{#EdgeSub}(\Phi)$ is $\#W[1]$-hard. If, additionally, each forbidden minor of $\Phi$ has a vertex of degree at least 3, and the Exponential Time Hypothesis holds, then $\text{#EdgeSub}(\Phi)$ cannot be solved in time $f(k) \cdot |G|^{o(k/\log k)}$, for any function $f$.

2. Approximate Counting: The problem $\text{#EdgeSub}(\Phi)$ always has a fixed-parameter tractable randomised approximation scheme (FPTRAS).

3. Decision: The problem $\text{EdgeSub}(\Phi)$ is always fixed-parameter tractable.

Proof. Note that each minor-closed property is computable (even in polynomial time) by the Robertson-Seymour Theorem [66]. The classification of exact counting follows by Theorem 5.3. For approximate counting and decision, we claim that each minor-closed property $\Phi$ either has bounded treewidth or satisfies both, the matching and the star criterion. If the latter holds, then the existence of an FPTRAS for approximate counting follows by Main Theorem 1 and the FPT algorithm for decision follows by Main Theorem 5.

To prove the claim, we assume that $\Phi$ has unbounded treewidth; otherwise we are done. In that case, by the Excluded-Grid-Theorem [65], $\Phi$ holds for a sequence of graphs containing arbitrarily large grids as minors. Since every planar graph (including matchings and stars) is a minor of a grid [67], and $\Phi$ is minor-closed, we conclude that $\Phi$ holds for all matchings and all stars and thus satisfies both the matching and the star criterion.

7.1 Separating Approximate Counting and Decision

Below we establish the existence of a (computable) graph property $\Psi$ such that $\text{EdgeSub}(\Psi)$ is fixed-parameter tractable but $\text{#EdgeSub}(\Psi)$ does not admit an FPTRAS unless $\#W[1]$ coincides with FPT, under randomised parameterized reductions.
We rely on the subgraph isomorphism problem restricted to grids: given a positive integer \( k \), the \( k \)-grid, denoted by \( \square_k \), has vertices \( |k| \times |k| \) and edges
\[
E(\square_k) := \{(i, j), (i', j') \mid |i - i'| + |j - j'| = 1\}.
\]
In other words, \( \square_k \) is obtained from the torus \( \otimes_k \) by removing edges \( \{(0, j), (k-1, j)\} \) and \( \{(i, 0), (i, k-1)\} \) for all \( i, j \in [k] \). Hence observe that \( \#E(\square_k) = 2k(k-1) \) and consider the definition of \( \Psi \):

**Definition 7.5 (Property \( \Psi \)).** Let \( H \) denote a graph. We set
\[
\Psi(H) = 1 :\!\!: \begin{cases}
H \cong \square_k + K_{1,k^2+2k} \lor H \cong M_{3k^2} & \exists k : \#E(H) = 3k^2 \\
H \cong M_{\#E(H)} & \text{otherwise}
\end{cases}
\]
Here, \( \square_k + K_{1,k^2+2k} \) is the (disjoint) union of the \( k \)-grid and the star of size \( k^2 + 2k \). In particular, \( \square_k + K_{1,k^2+2k} \) has precisely \( 2k(k-1) + k^2 + 2k = 3k^2 \) edges.

Observe that \( \Psi \) is clearly computable and satisfies the matching criterion (but not the star criterion). Thus \( \text{EGDGE}_{\text{SUB}}(\Psi) \) is fixed-parameter tractable by Main Theorem 5.

Write \( \boxplus \) for the set of all grids and recall that the problem \( \text{EMB}(\boxplus) \) asks, given as input a grid \( \square_k \) and a graph \( G \), to correctly decide whether there is an embedding from \( \square_k \) to \( G \). Chen, Grohe and Lin [18] proved that this problem is \( \text{W}[1] \)-hard. The following result establishes thus hardness of approximating \( \#\text{EDGE}_{\text{SUB}}(\Psi) \).

**Theorem 7.6.** If \( \#\text{EDGE}_{\text{SUB}}(\Psi) \) admits an FPTRAS, then there is a randomized decision procedure \( \mathbb{P} \) which, given a graph \( G \), a positive integer \( k \), and a rational number \( 0 < \delta < 1 \), satisfies
\[
\Pr[\mathbb{P}(G, k, \delta) = 1 \iff \text{EMB}(\square_k \rightarrow G) \neq \emptyset] \geq (1 - \delta).
\]
Furthermore, the running time of \( \mathbb{P} \) is bounded by \( f(k) \cdot \text{poly}(|G|, \log(1/\delta)) \) for some computable function \( f \).

The previous theorem is an easy consequence of the following lemma.

**Lemma 7.7.** Let \( G \) denote a graph with \( n \) vertices, let \( k \) denote a positive integer, and set \( G' = G + K_{1,n^n} \). We have
\[
\#\text{Edge}_{\text{SUB}}(\Psi, 3k^2 \rightarrow G') \begin{cases}
\geq (3k^2)^{-3k^2} n^{6k^2+12k} & \text{if } \text{EMB}(\square_k \rightarrow G) \neq \emptyset \\
\leq 2n^{6k^2+4} & \text{otherwise}
\end{cases}
\]

**Proof.** We rely on the following well-known bounds on the binomial coefficient:
\[
\frac{n^k}{k^k} \leq \binom{n}{k} \leq n^k.
\]
Let us start with the lower bound; thus assume that \( \text{EMB}(\square_k \rightarrow G) \neq \emptyset \). Consequently, there is a subset \( A \) of \( 2k(k-1) \) many edges in \( G \) such that \( G[A] \cong \square_k \). Observe further that there are
\[
\binom{n^6}{k^2 + 2k} \geq (k^2 + 2k)^{-3k^2} n^{6k^2+12k} \geq (3k^2)^{-3k^2} n^{6k^2+12k}
\]
edge subsets \( A' \) of \( K_{1,n^n} \) that induce \( K_{1,k^2+2k} \). Thus, for any such \( A' \), we have that \( G'[A' \cup A'] \cong \square_k + K_{1,k^2+2k} \). In particular, there are at least \( (3k^2)^{-3k^2} n^{6k^2+12k} \) edge subsets of size \( 3k^2 \) of \( G' \) that induce a graph satisfying \( \Psi \).
For the second case, recall that we wish to upper bound the number of $3k^2$-edge subsets $A$ of $G'$ such that $\Psi(G'[A]) = 1$. By definition of $\Psi$ and $G'$, and under the assumption that $\text{Emb}(\Xi_k \rightarrow G) = \emptyset$, it remains to upper bound the number of $3k^2$-matchings of $G'$. Note that each matching of $G'$ can use at most one edge of the star $K_{1,n^6}$. In particular, this allows us to partition the $3k^2$-matchings in two groups: For the first one, every edge of the matching must be contained in $G$, and for the second one, precisely one edge is contained in $K_{1,n^6}$ (for which there are $n^6$ possibilities), and all remaining edges must be contained in $G$. Since $G$ has less than $n^6$ edges, we can generously bound the number of $3k^2$-matchings of $G'$ as follows:

$$\left(\frac{n^2}{3k^2}\right) + n^6 \cdot \left(\frac{n^2}{3k^2-1}\right) \leq n^{6k^2} + n^6 \cdot n^{6k^2-2} \leq 2n^{6k^2+4}.$$ 

The proof is thus concluded. 

**Proof of Theorem 7.6.** Assume that $A$ is an FPTRAS for $\#\text{EdgeSub}(\Psi)$. Given $G$ with $n$ vertices and $\Xi_k$ (for which we wish to decide whether $\text{Emb}(\Xi_k \rightarrow G) \neq \emptyset$), we first check whether $n \leq 6(3k^2)^{3k^2}$. If this is the case, then $\mathbb{P}$ search for an embedding from $\Xi_k$ to $G$ via brute-force, the running time of which is bounded by $g(k)$ for some computable function $g$ since the size of $G$ is bounded by a function in $k$.

Thus assume that $n > 6(3k^2)^{3k^2}$. Then $\mathbb{P}$ constructs $G'$ as in Lemma 7.7 in time $n^6$ and simulates $A$ on $G'$, $3k^2$, $\delta$ and $\varepsilon = 1/2$. Finally, $\mathbb{P}$ outputs 0 if the output of $A$ is at most $3n^{6k^2+4}$, and $\mathbb{P}$ outputs 1 otherwise.

Since $A$ is an FPTRAS and $\varepsilon = 1/2$, its running time is bounded by $f'(3k^2) \cdot \text{poly}(|G'|, \log(1/\delta))$ for some computable function $f'$. Since $|G'|$ is bounded polynomial in $|G|$, we conclude that $\mathbb{P}$ has the desired running time.

It remains to prove correctness. Given that $A$ is an FPTRAS and $\varepsilon = 1/2$, we note that, with probability at least $(1 - \delta)$, the output $X$ of $A$ satisfies

$$1/2 \cdot \#\text{EdgeSub}(\Psi, 3k^2 \rightarrow G') \leq X \leq 3/2 \cdot \#\text{EdgeSub}(\Psi, 3k^2 \rightarrow G'),$$

Assume first that $\text{Emb}(\Xi_k \rightarrow G) = \emptyset$. Then, by Lemma 7.7 we have

$$X \leq 3/2 \cdot \#\text{EdgeSub}(\Psi, 3k^2 \rightarrow G') \leq 3n^{6k^2+4},$$

and thus the output of $\mathbb{P}$ is correct.

Now assume that $\text{Emb}(\Xi_k \rightarrow G) \neq \emptyset$. We have to show that $X > 3n^{6k^2+4}$ for $\mathbb{P}$ being correct. Using the assumption that $n > 6(3k^2)^{3k^2}$, and relying on Lemma 7.7 once more, we have

$$X \geq 1/2 \cdot \#\text{EdgeSub}(\Psi, 3k^2 \rightarrow G') \geq 1/2 \cdot (3k^2)^{-3k^2} n^{6k^2+12k} > 3n^{6k^2+12k-1} > 3n^{6k^2+4},$$

where the last inequality is trivial since $k > 0$. This concludes the proof. 

Finally, since $\text{Emb}(\Xi)$ is W[1]-hard, the previous theorem yields that the existence of an FPTRAS for $\#\text{EdgeSub}(\Phi)$ would imply that W[1] coincides with FPT under randomized parameterized reductions, which proves Proposition 1.2.
8 A Parameterized Tutte Polynomial

In the last part of the paper, we take a step back and revisit exact counting: Recall that problem \(#\text{EDGE}_{\text{SUB}}(\Phi)\) can be interpreted as the problem of evaluating a linear combination of subgraph counts, given by

\[
\#\text{EdgeSub}(\Phi, k \rightarrow *) = \sum_{H \in \Phi_k} \#\text{Sub}(H \rightarrow G),
\]

where \(\Phi_k\) is the set of all \(k\)-edge graphs that satisfy \(\Phi\). In particular, each coefficient in this linear combination is 0 or 1. We have seen that the values of \(\Phi\) on the fixed-points of certain group actions on (fractures of) Cayley graphs can be used to obtain explicit criteria for (\(#\text{W}[1]\)-)hardness of \(#\text{EdgeSub}(\Phi)\).

In the current section, we show that the aforementioned method applies to the significantly more general problem of computing weighted linear combinations of \(k\)-edge subgraph counts. More precisely, we consider a natural parameterized variant of the Tutte polynomial and obtain an exhaustive classification for the complexity of evaluating it at any rational coordinates.

Recall that the (classical) Tutte polynomial is defined as follows:

\[
T_G(x, y) := \sum_{A \subseteq E(G)} (x - 1)^{\#(A) - k(E(G))} \cdot (y - 1)^{k(A) + \#A - \#V(G)},
\]

where \(k(S)\) is the number of connected components of the graph \((V(G), S)\). In this work, we consider the specialization of the Tutte polynomial to edge-subsets of size \(k\), which we call the parameterized Tutte polynomial:

\[
T^k_G(x, y) := \sum_{A \in \binom{E(G)}{k}} (x - 1)^{\#(A) - k(E(G))} \cdot (y - 1)^{k(A) + k - \#V(G)},
\]

Similarly to the classical counterpart due to Jaeger et al.\,[47], our goal is to understand the parameterized complexity of evaluating \(T^k_G(x, y)\) for any fixed pair of coordinates \((x, y)\), when parameterized by \(k\). Note that at points \((x, y)\) with \(x \neq 1, y \neq 1\) we can write the polynomial as

\[
T^k_G(x, y) = (x - 1)^{-k(E(G))}(y - 1)^{k - \#V(G)} \sum_{A \in \binom{E(G)}{k}} ((x - 1) \cdot (y - 1))^{k(A)}.
\]

So, up to the global factor \((x - 1)^{-k(E(G))}(y - 1)^{k - \#V(G)}\) (which can be computed in linear time in the input size) in this region the polynomial is really just a polynomial in the single variable \(z = (x - 1)/(y - 1)\). Still, we keep the variables \(x, y\) separate in the treatment below. On the one hand, this facilitates comparisons to the classical Tutte polynomial. On the other hand, we see some interesting behaviour of \(T^k_G(x, y)\) at points with \(x = 1\) or \(y = 1\). Indeed, let us start by investigating the expressibility of the parameterized Tutte polynomial in some individual points.

8.1 Interpretation in Individual Points

Recall that, given a graph \(G\) and a subset \(A \subseteq E(G)\) of its edges, we write \(G[A] = (V(G), A)\) for the graph induced by \(A\). We emphasize the difference from the construction \(G[A]\) we saw before: the graph \(G[A]\) is obtained from \(G(A)\) by removing all isolated vertices.

The most immediate information encoded in the parameterized Tutte polynomial is the number of \(k\)-forests in a graph:

\[
\text{EdgeSub}(\Phi, k \rightarrow *) = \sum_{H \in \Phi_k} \#\text{Sub}(H \rightarrow G),
\]
Observation 8.1. The number of forests with \( k \) edges in a graph \( G \) is given by \( T^k_G(2, 1) \).

In particular, evaluating \( T^k_G(2, 1) \) is equivalent to evaluation \( \#\text{IndSub}(\Phi, k \to G) \) for the (minor-closed) property of being acyclic.

For further individual points, it is convenient to consider the following modification.

Definition 8.2. Define the modified Tutte polynomial of a graph \( G \) as

\[
\widetilde{T}_G(x, y) := \sum_{A \subseteq E(G)} (x - 1)^{k(A)} \cdot (y - 1)^{k(A) + \#A},
\]

so that \( \widetilde{T}_G(x, y) = (x - 1)^{k(E(G))} (y - 1)^{\#V(G)} T_G(x, y) \). Similarly we define the parametrized version as

\[
\widetilde{T}_G^k(x, y) := \sum_{A \in (E(G)/k)} (x - 1)^{k(A)} \cdot (y - 1)^{k(A) + \#A}.
\]

As for its classical counter-part, we observe a deletion-contraction recurrence, which enables us establish the properties at individual points. Setting \( \widetilde{T}_G^{-1}(x, y) = 0 \) we obtain:

Lemma 8.3. Given a graph \( G \) and an edge \( e \in E(G) \) we have

\[
\widetilde{T}_G^k(x, y) = \widetilde{T}_G^{k-1}(x, y) + (y - 1) \widetilde{T}_G^{k-1}(x, y),
\]

for any \( k \geq 0 \) and similarly

\[
\widetilde{T}_G(x, y) = \widetilde{T}_G^{k-1}(x, y) + (y - 1) \widetilde{T}_G^{k-1}(x, y).
\]

Proof. In the definition of \( \widetilde{T}_G^k \) we split the sum over \( A \in (E(G)/k) \) as

\[
\widetilde{T}_G^k(x, y) = \sum_{A \in (E(G)/k)} (x - 1)^{k(A)} \cdot (y - 1)^{k(A) + \#A} + \sum_{A \in (E(G)/k)} (x - 1)^{k(A)} \cdot (y - 1)^{k(A) + \#A}. \tag{10}
\]

The subsets \( A \in (E(G)/k) \) with \( e \notin A \) are naturally identified with the subsets \( A \in (E(G)\setminus e)/(k-1) \) and we have \( G(A) = (G \setminus e)(A) \). Thus the first sum in \( \tag{10} \) is equal to \( \widetilde{T}_G^{k-1}(x, y) \). On the other hand, the subsets \( A \in (E(G)/k) \) with \( e \in A \) are naturally identified with the subsets \( A' \in (E(G)/k-1) \) by \( A \mapsto A' = A \setminus \{e\} \) and we have \( k(A) = k(A') \) (in their respective graphs \( G \) and \( G/e \)). Thus the second summand in \( \tag{10} \) equals \( (y - 1)^2 \widetilde{T}_G^{k-1}(x, y) \), with the factor \( y - 1 \) coming from the fact that \( \#A = \#A' + 1 \) in the above correspondence. The deletion-contraction formula for the (unparametrized) modified Tutte polynomial is obtained by summing over all \( k \).

Using the previous recurrence, the following transformation encapsulates the relation between the parameterized and the classical Tutte polynomial.

Proposition 8.4. Given a graph \( G \) and \( k \geq 0 \) we have

\[
\sum_{\ell=0}^{k} \binom{\#E(G) - \ell}{k - \ell} \cdot \widetilde{T}_G^\ell(x, y) = \sum_{A \in (E(G)/k)} \widetilde{T}_G(A)(x, y). \tag{11}
\]
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**Proof.** We prove the statement by induction on the number of edges. For $E(G) = \emptyset$ the two sides are zero for $k \neq 0$ and equal to $\tilde{T}_G^k(x, y) = T_G(x, y)$ for $k = 0$.

We show the induction step using the deletion-contraction relations above. Let $G$ denote a graph with at least one edge $e$. Then we have

$$
\sum_{\ell=0}^k \left( \frac{\#E(G) - \ell}{k - \ell} \right) \cdot \tilde{T}_{G/e}^\ell(x, y) = \sum_{\ell=0}^k \left( \frac{\#E(G) - \ell}{k - \ell} \right) \cdot \tilde{T}_{G/e}^\ell(x, y) + \left( \frac{\#E(G) - \ell}{k - \ell} \right) \cdot (y - 1) \tilde{T}_{G/e}^{\ell-1}(x, y).
$$

(12)

Furthermore, since $\#E(G) = \#E(G \setminus e) + 1$, we can use the usual recursion of binomial coefficients to see

$$
\sum_{\ell=0}^k \left( \frac{\#E(G \setminus e) - \ell}{k - \ell} \right) \cdot \tilde{T}_{G/e}^\ell(x, y) = \sum_{\ell=0}^k \left( \frac{\#E(G \setminus e) - \ell}{k - \ell} \right) \cdot \tilde{T}_{G/e}^\ell(x, y) + \left( \frac{\#E(G \setminus e) - \ell}{k - \ell} \right) \cdot \tilde{T}_{G/e}^{\ell-1}(x, y)
$$

and using the induction step. For the second summand in (12) we make the index shift $\ell' = \ell - 1$ and obtain

$$
\sum_{\ell=0}^k \left( \frac{\#E(G \setminus e) - \ell}{k - \ell} \right) \cdot (y - 1) \tilde{T}_{G/e}^{\ell-1}(x, y) = \sum_{\ell'=0}^{k-1} \left( \frac{\#E(G/e) - \ell'}{(k - 1) - \ell'} \right) \cdot (y - 1) \tilde{T}_{G/e}^{\ell'}(x, y)
$$

Combining the last two equations we can conclude using suitable identifications, for instance identifying the $A \in \binom{E(G)}{k}$ with $e \in A$ with $A' \in \binom{\{E(G \setminus e)\}}{k-1}$ via $A \mapsto A' = A \setminus \{e\}$ and using

$$(G \setminus e)(A') = G(A) \setminus e \text{ and } (G/e)(A') = G(A)/e.$$  

Then we see that (12) equals

$$
\sum_{A \in \binom{E(G \setminus e)}{k}} \tilde{T}_{G \setminus e}(A)(x, y) + \sum_{A' \in \binom{\{E(G \setminus e)\}}{k-1}} \tilde{T}_{G/e}(A')(x, y) + (y - 1) \tilde{T}_{G/e}(A')(x, y)
$$

Using Proposition 8.4 we can now present combinatorial interpretations of the specialisation of $T_G^k(x, y)$ to some individual points.
Chromatic Polynomial

For \( x = 1 - c, y = 0 \) the modified Tutte polynomial \( \tilde{T}_G(x, y) \) specializes to the chromatic polynomial \( \chi_G(c) \), so we see that the \( \tilde{T}_G^k(1 - c, 0) \) (for \( 0 \leq \ell \leq k \)) contain the information of the number of pairs \((A, \sigma)\) with \( A \subseteq E(G) \) with \( \#A = k \) and \( \sigma \) a \( c \)-colouring on \( G(A) \).

\[
\sum_{\ell=0}^{k} \left( \frac{\#E(G)}{k-\ell} \right) \cdot \tilde{T}_G^\ell(1-c,0) = \# \left\{ (A, \sigma) : A \in \binom{E(G)}{k}, \sigma \text{ c-coloring on } G(A) \right\}.
\]

Acyclic Orientations

For \( x = 2, y = 0 \) the Tutte polynomial \( T_G(x, y) \) specializes to the number of acyclic orientations of \( G \). We have \( \tilde{T}_G^k(2, 0) = (-1)^{\#V(G)} T_G^k(2, 0) \). Thus the \( \tilde{T}_G^\ell(2, 0) \) (for \( 0 \leq \ell \leq k \)) contain the information of the number of pairs \((A, \vec{\eta})\) where \( A \subseteq E(G) \) with \( \#A = k \) and \( \vec{\eta} \) is an acyclic orientation on \( G(A) \). Indeed, multiplying \([11]\) with \((-1)^{\#V(G)}\) we obtain

\[
\sum_{\ell=0}^{k} \left( \frac{\#E(G)}{k-\ell} \right) \cdot \tilde{T}_G^\ell(2,0) = \# \left\{ (A, \vec{\eta}) : A \in \binom{E(G)}{k}, \vec{\eta} \text{ acyclic orientation on } G(A) \right\}.
\]

\( k \)-Edge Sets Inducing an Even Number of Components

\[\text{Proposition 8.5.} \quad \text{Given a graph } G \text{ and a positive integer } k, \text{ we have} \]

\[
\frac{1}{2} \left( \frac{\#E(G)}{k} + (-1)^{k(E(G))} T_G^k(0,2) \right) = \# \left\{ A \subseteq E(G) : \#A = k \land k(A) = 0 \mod 2 \right\}.
\]

\[\text{Proof.} \quad \text{We have} \]

\[
\frac{1}{2} \left( \frac{\#E(G)}{k} + (-1)^{k(E(G))} T_G^k(0,2) \right) = \frac{1}{2} \left( \frac{\#E(G)}{k} + (-1)^{k(E(G))} \sum_{A \in \binom{E(G)}{k}} (-1)^{k(A) + k(E(G))} \right)
\]

\[
= \frac{1}{2} \left( \sum_{A \in \binom{E(G)}{k}} 1 + (-1)^{k(A)} \right).
\]

But observe that the summand above is 0 for \( k(A) \) odd and 2 for \( k(A) \) even. Thus after summing and dividing by 2 we count the subsets \( A \) with the graph \( G(A) \) having an even number of components. \( \blacksquare \)

\( k \)-Edge Sets of Even Betti Number

The (first) Betti number\(^{24}\) of a graph is defined as \( b_1(G) = k(E(G)) + \#E(G) - \#V(G) \) (cf. \([5]\) Chapt. 4).

\[\text{Proposition 8.6.} \quad \text{Given a graph } G \text{ and a positive integer } k, \text{ we have} \]

\[
\frac{1}{2} \left( \frac{\#E(G)}{k} + T_G^k(2,0) \right) = \# \left\{ A \subseteq E(G) : \#A = k \land b_1(G(A)) = 0 \mod 2 \right\}.
\]

\(^{24}\)The first Betti number is also called the circuit rank, cyclomatic number, cycle rank, or nullity.
Proof. We have

\[ \frac{1}{2} \left( \binom{\#E(G)}{k} + T^k_G(2,0) \right) = \frac{1}{2} \left( \binom{\#E(G)}{k} + \sum_{A \subseteq (E(G))^k} (-1)^{k(A) + \#A - \#V(G)} \right) \]

where we use \( b_1(G(A)) = k(A) + \#A - \#V(G) \). But observe that the summand above is 0 for \( b_1(G(A)) \) odd and 2 for \( b_1(G(A)) \) even. Thus after summing and dividing by 2 we count the subsets \( A \) with \( G(A) \) having even Betti number.

8.2 Classification for Rational Coordinates

We now classify the complexity of computing \( T^k_G(x,y) \) for each pair of rational coordinates \( x \) and \( y \). Formally, for each such pair, we consider the parameterized problem which expects as input \( G \) and \( k \) and outputs the value \( T^k_G(x,y) \); the parameterization is given by \( k \). Let us start with the following easy fact:

Lemma 8.7. For any \( y \in \mathbb{Q} \), the problem of computing \( T^k_G(1,y) \) is fixed-parameter tractable.

Proof. Observe that \( T^k_G(1,y) = 0 \) unless there is \( A \subseteq E(G) \) of size \( k \) such that \( k(A) = k(E(G)) \). In other words, \( G \) has a spanning subgraph of \( k \) edges. Consequently, \( G \) can have at most \( 2k \) vertices, implying that \( G \) has at most \( \binom{2k}{2} \leq 4k^2 \) many edges. Therefore an algorithm for computing \( T^k_G(1,y) \) is obtained as follows: Given \( G \) and \( k \), first check whether \( |V(G)| > 2k \), and output 0 in that case. Otherwise, obtain \( T^k_G(x,y) \) by naively computing the sum, which takes time

\[ O \left( \binom{4k^2}{k} \cdot |G| \right), \]

concluding the proof.

Next, similarly to the classical counter-part [47], we obtain a trivial algorithm for coordinates \( x \) and \( y \) that lie on the hyperbola \((x-1)(y-1) = 1\):

Lemma 8.8. Let \( x \) and \( y \) denote rational numbers such that \((x-1)(y-1) = 1\). Then the problem of computing \( T^k_G(x,y) \) is solvable in polynomial time (and thus fixed-parameter tractable as well).

Proof. Observe that, given \((x-1)(y-1) = 1\), we have

\[ T^k_G(x,y) = \sum_{A \subseteq (E(G))^k} (x-1)^{k(A)-k(E(G))}(y-1)^{k(A)+\#V(G)} = (x-1)^{-k(E(G))}(y-1)^{k+\#V(G)} \binom{\#E(G)}{k}, \]

which can be computed trivially.

In what follows, we show that computing \( T^k_G(x,y) \) is \#W[1]-hard for all remaining rational coordinates \( x \) and \( y \). First, it is convenient to rewrite the quantity \( k(A) \) as follows: given an edge-subset \( A \) of a graph \( G \), recall that \( G[A] \) is the graph obtained from \((V(G),A)\) by deleting isolated vertices. Let us write \( \text{cc}(H) \) for the number of connected components of a graph \( H \).
**Fact 8.9.** Let $G$ denote a graph and let $A$ denote a subset of edges of $G$. We have

$$k(A) = \text{cc}(G[A]) + \#V(G) - \#V(G[A]).$$

Similarly as in case of $\#\text{EdgeSub}(\Phi)$, our goal is to reduce from a linear combination of (colour-preserving) homomorphism counts. For this reason, we again consider an easy modification by excluding the term $(x-1)\#V(G) - \text{cc}(G)$; more precisely, consider

$$\tilde{T}_G^k(x, y) := \sum_{A \subseteq \binom{E(G)}{k}, \text{cfu}(A)} \left( (x-1)^{\text{cc}(G[A]) - \#V(G[A])} \cdot (y-1)^{\text{cc}(G[A]) - \#V(G[A]) + k} \right),$$

and observe that

$$T_G^k(x, y) = (x-1)^{\#V(G) - \text{cc}(G)} \cdot \tilde{T}_G^k(x, y).$$

In particular, $\tilde{T}_G^k(x, y)$ is trivially interreducible with $T_G^k(x, y)$ if $x \neq 1$. Next we introduce an $(H)$-coloured version of the parameterized Tutte polynomial; given an edge-subset $A$ of a $k$-edge-coloured graph, we write $\text{cfu}(A)$ if $A$ contains each of the $k$ colours precisely once.

**Definition 8.10 (Colourful Parameterized Tutte Polynomial).** Let $G$ denote a $k$-edge-coloured graph. We define

$$\text{col-}T_G^k := \sum_{A \subseteq \binom{E(G)}{k}, \text{cfu}(A)} \left( (x-1)^{\text{cc}(G[A]) - \#V(G[A])} \cdot (y-1)^{\text{cc}(G[A]) - \#V(G[A]) + k} \right)$$

as the colourful Parameterized Tutte Polynomial.

The next lemma allows us to reduce the colourful version to the uncoloured version.

**Lemma 8.11.** Let $G$ denote a $k$-edge-coloured graph and assume that the set of colours is $[k]$. For each pair $(x, y)$ we have

$$\text{col-}T_G^k(x, y) = \sum_{J \subseteq [k]} (-1)^{\#J} \cdot \tilde{T}_{G \setminus J}^k(x, y),$$

where $G \setminus J$ is the graph obtained from $G$ by deleting all edges coloured with an element of $J$.

**Proof.** Follows by the inclusion-exclusion principle (similarly as in Lemma 3.7) and the fact that, given a $k$-edge-subset $A$ of $G$, deleting edges in $E(G) \setminus A$ does not change the quantity

$$(x-1)^{\text{cc}(G[A]) - \#V(G[A])} \cdot (y-1)^{\text{cc}(G[A]) - \#V(G[A]) + k}.$$

Next, we express $\text{col-}T_G^k$ as a linear combination of colour-preserving homomorphisms counts. More precisely, given an $H$-coloured graph $G$ such that $H$ has $k$ edges, we implicitly assume the $k$-edge-colouring of $G$ induced by its $H$-colouring. Further, given a fracture $\rho$ of a graph $H$, we set $r(\sigma) := \text{cc}(H^{\rho}\sigma) - \#V(H^{\rho}\sigma)$.
**Lemma 8.12.** Let $H$ denote a graph with $k$ edges. For every $H$-coloured graph $G$, we have

$$\#\text{col}^k T_G(x,y) = \sum_{\sigma \in \mathcal{L}(H)} (x-1)^{r(\sigma)} \cdot (y-1)^{r(\sigma)+k} \cdot \sum_{\rho \succeq \sigma} \mu(\sigma, \rho) \cdot \#\text{Hom}_{\text{cp}}(H\sharp \rho \rightarrow_H G),$$

where the relation $\leq$ and the Möbius function $\mu$ are over the lattice of fractures $\mathcal{L}(H)$.

**Proof.** Every colourful $k$-edge-subset $A$ of $G$ induces a fracture of $H$, similarly as we have seen in Section 3. In particular, if $A$ and $A'$ induce the same fracture $\sigma$, then $G[A] \cong G[A'] \cong H\sharp \sigma$. Writing $[\sigma]$ for the equivalence class of the induced fracture $\sigma$, we obtain:

$$\#\text{col}^k T_G(x,y) = \sum_{\sigma \in \mathcal{L}(H)} (x-1)^{r(\sigma)} \cdot (y-1)^{r(\sigma)+k} \cdot [\sigma].$$

Next observe that $[\sigma] = \#\text{ColEdgeSub}(H\sharp \sigma \rightarrow_H G)$. Finally, we have already seen in (the proof of) Lemma 3.1 that

$$\#\text{ColEdgeSub}(H\sharp \sigma \rightarrow_H G) = \sum_{\rho \succeq \sigma} \mu(\sigma, \rho) \cdot \#\text{Hom}_{\text{cp}}(H\sharp \rho \rightarrow_H G),$$

which concludes the proof.

The following lemma establishes that the coefficient of the torus does not vanish apart from a few exceptions; which eventually allows us to prove $\#W[1]$-hardness.

**Lemma 8.13.** Let $\ell > 2$ denote a prime and let $(x,y) \in \mathbb{Q}^2$. There is a unique and computable function $a_{(x,y)}^\ell$ from fractures of $\bigotimes \ell$ to rational numbers such that

$$\text{col}^2 T^\ell_\star = \sum_{(x,y) \in \mathbb{Q}^2} a_{(x,y)}^\ell (\rho) \cdot \#\text{Hom}_{\text{cp}}(\bigotimes \ell \rho \rightarrow \bigotimes \star).$$

Moreover, if both the denominators of $x,y$ and the numerators of $x-1$ and $(x-1)(y-1)-1$ are not divisible by $\ell$, then $a_{(x,y)}^\ell (\top) \neq 0$.

**Proof.** The first claim follows immediately from the previous lemma. For the second claim, we rely on the following fact from commutative algebra.

**Fact 8.14.** Let $q \in \mathbb{Z}$ denote a nonzero integer, then the localization $\mathbb{Z}[1/q]$ of $\mathbb{Z}$ at $q$ is the set

$$\mathbb{Z}[1/q] = \left\{ u \in \mathbb{Q} : \exists v \in \mathbb{Z}, m \in \mathbb{N} \text{ with } u = \frac{v}{q^m} \right\}$$

of rational numbers which can be brought to a denominator which is a power of $q$. The subset $\mathbb{Z}[1/q]$ of $\mathbb{Q}$ is closed under addition and multiplication. Let furthermore $\ell$ denote a prime not dividing $q[25]$, implying that $q$ has an inverse $q^{-1}$ mod $\ell$. Then there is a well-defined map

$$\mathbb{Z}[1/q] \rightarrow \mathbb{Z}_\ell, \frac{v}{q^m} \mapsto v \cdot (q^{-1})^m,$$

and this map is compatible with addition and multiplication.

---

[25] In both cases, we refer to denominators and numerators of the corresponding shortened fractions.
Let us now collect the coefficients of $\#\text{Hom}_{\mathbb{C}}(\mathbb{O}_\ell \to \mathbb{O}_\ell) \ast$ in the sum appearing in Lemma 8.12. Completely similar to Corollary 3.3 we obtain

$$a_{(x,y)}(\mathbb{T}) = \sum_{\sigma \in \mathcal{L}(\mathbb{O}_\ell)} (x-1)^{r(\sigma)} (y-1)^{r(\sigma)+2\ell^2} \prod_{v \in \mathbb{V}(\mathbb{O}_\ell)} (-1)^{|\sigma_v|-1}(|\sigma_v|-1)! \in \mathbb{Q}. \quad (13)$$

Note that in this expression we have for the exponents of $x-1$ and $y-1$ that $r(\sigma) \leq 0$ but $r(\sigma)+k \geq 0$. Let $q$ denote the least common multiple of the denominators of $x,y$ and the numerator of $x-1$, then we have that $(x-1)^{\pm 1}$ and $(y-1)$ are elements in $\mathbb{Z}/q[1/q]$. By the assumption that $\ell$ does not divide $q$ together with Fact 8.14 above, we can see these expressions (and thus the entire sum (13)) as an element of $\mathbb{Z}/\ell$. Now recall the 15 fixed-points of the action of $\mathbb{Z}/\ell$ on the fractures of $\mathbb{O}_\ell$ as given by Observation 4.3.

Counting modulo $\ell$ allows us to rely on the same analysis as presented in the proof of Lemma 4.6 which yields

$$a_{(x,y)}(\mathbb{T}) = -6R(M_{2\ell\xi}) + 4R(M_{2\ell + \ell C_\xi}) + 8R((\ell^2P_2) - R(2\ell C_\xi) - 2R(\ell C_\xi) - 4R(\ell S_\xi) + R(\mathbb{O}_\ell) \mod \ell,$n

where $R(H) := (x-1)^{\sigma(H)} - \#V(H). (y-1)^{2\ell^2 + \sigma(H) - \#V(H)}$. Consequently, we have

$$a_{(x,y)}(\mathbb{T}) = -6(x-1)^{2\ell^2-4\ell^2} (y-1)^{2\ell^2+2\ell^2-4\ell^2} + 4(x-1)^{2\ell^2+\ell^2-3\ell^2} (y-1)^{2\ell^2+\ell^2-3\ell^2} + 8(x-1)^{2\ell^2-3\ell^2} (y-1)^{2\ell^2+\ell^2-3\ell^2} - 1(x-1)^{2\ell^2-2\ell^2} (y-1)^{2\ell^2+2\ell^2-2\ell^2} - 2(x-1)^{2\ell^2+\ell^2} (y-1)^{2\ell^2+\ell^2-2\ell^2} - 4(x-1)^{2\ell^2-2\ell^2} (y-1)^{2\ell^2+\ell^2-2\ell^2} + 1(x-1)^{2\ell^2+1-\ell^2} \mod \ell.$$n

The first simplification is obtained by observing that the first and the third term, and the second, fifth and sixth term, respectively, contain the same monomial:

$$a_{(x,y)}(\mathbb{T}) = 2(x-1)^{-2\ell^2} - 2(x-1)^{-2\ell^2+\ell} (y-1)^{\ell} - (x-1)^{-2\ell^2+2\ell} (y-1)^{2\ell} + (x-1)^{1-\ell^2} (y-1)^{2\ell+1} \mod \ell.$$n

Using Fermat’s little theorem, we obtain

$$a_{(x,y)}(\mathbb{T}) = 2(x-1)^{-2} - 2(x-1)^{-2} (y-1)^{\ell} \mod \ell$$

The assumption that the denominator of $x$ (which is the numerator of $(x-1)^{-2}$) and the numerator of $(x-1)(y-1) - 1$ are not divisible by $\ell$ implies that each factor in the product $2(x-1)^{-2} \cdot (1-(x-1) \cdot (y-1))$ gives a nonzero residue class mod $\ell$. Since $\ell$ is a prime, their product is still nonzero in $\mathbb{Z}/\ell$, and thus the original rational number $a_{(x,y)}(\mathbb{T})$ is likewise nonzero, concluding the proof.

We are thus able to rely on Complexity Monotonicity to establish hardness as promised.

Lemma 8.15. Let $(x,y)$ denote a pair of rational numbers such that $x \neq 1$ and $(x-1)(y-1) \neq 1$. Then the problem of computing $T^\ell_{2\ell}(x,y)$ is $\#W[1]$-hard.
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Proof. Let $\mathcal{H}[x, y]$ denote the set of all $\mathcal{G}(x)\mathcal{G}(y)$ such that $\ell$ is prime and both the denominators of $x$ and $y$ as well as the numerators of $x - 1$ and $(x - 1)(y - 1) - 1$ are not divisible by $\ell$. Since $x$ and $y$ are fixed, the latter is true for infinitely many primes $\ell$ and thus $\mathcal{H}[x, y]$ contains tori of unbounded size. In particular, it contains graphs with arbitrary large grid minors and has thus unbounded treewidth [65], and hence, the problem $\#\text{Hom}(\mathcal{H}[x, y])$ is $\#\text{W}[1]$-hard by the classification of Dalmau and Jonsson [28].

Completely analogously to the proof of Lemma 3.8, the problem $\#\text{Hom}(\mathcal{H}[x, y])$ reduces to computing $\text{col}^{-\ell, k}_G(x, y)$ via Complexity Monotonicity (Lemma 3.6), since the coefficients of the tori do not vanish by the previous lemma.

Next, reducing to the uncoloured version $\tilde{T}^k_G(x, y)$ can be done via Lemma 8.11, and, finally, $\tilde{T}^k_G(x, y)$ is trivially interreducible with $T^k_G(x, y)$ whenever $x \neq 1$.

At last, we are able to prove this section’s main dichotomy theorem; consider Figure 1 for a depiction of the tractable cases.

Main Theorem 7. Let $(x, y)$ denote a pair of rational numbers. The problem of computing $T^k_G(x, y)$ is fixed-parameter tractable if $x = 1$ or $(x - 1)(y - 1) = 1$, and $\#\text{W}[1]$-hard otherwise.

Proof. The fixed-parameter tractable cases follow from Lemmas 8.7 and 8.8 and the $\#\text{W}[1]$-hard cases follow from the previous lemma.

As an immediate consequence, the computation of each individual point considered in Section 8.4 is $\#\text{W}[1]$-hard. Moreover, observe that the transformation

$$\sum_{\ell=0}^k \left( \frac{\#E(G)}{k - \ell} \right) \cdot \tilde{T}^\ell_G(x, y) = \sum_{A \in \left\{E(G)\right\}} T^{G(A)}_G(x, y),$$

given by Proposition 8.4, is invertible in the sense that the numbers

$$\sum_{A \in \left\{E(G)\right\}} \tilde{T}^{G(A)}_G(x, y)$$

for $\ell = 0, \ldots, k$ reveal $\tilde{T}^k_G(x, y)$. Consequently, we obtain $\#\text{W}[1]$-hardness of the information encoded in all considered individual points as well.

Corollary 8.16. The following problems are $\#\text{W}[1]$-hard when parameterized by $k$:
- Given $G$ and $k$, compute the number of $k$-edge subsets $A$ of $G$ such that $G(A)$ has an even number of components.
- Given $G$ and $k$, compute the number of pairs $(A, \sigma)$ such that $A$ is a $k$-edge subset of $G$ and $\sigma$ is a $c$-colouring of $G(A)$. Here $c \geq 2$ is a fixed integer.
- Given $G$ and $k$, compute the number of pairs $(A, \vec{\eta})$ such that $A$ is a $k$-edge subset of $G$ and $\vec{\eta}$ is an acyclic orientation of $G(A)$.
- Given $G$ and $k$, compute the number of $k$-edge subsets $A$ of $G$ such that $G(A)$ has even Betti number.

Comparison to the classical dichotomy and real FPT cases

The reader might ask which of the fixed-parameter tractable cases actually allows a polynomial-time algorithm. We can answer this question under the assumption $P \neq \#P$ by considering the classical dichotomy of Jaeger, Vertigan and Welsh [26].

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26 We state their classification only for rational numbers, but point out that the full dichotomy includes all complex pairs.
Theorem 8.17. Given a pair \((x, y)\) of rational numbers, computing \(T_G(x, y)\) is solvable in polynomial time if \((x, y)\) is in \([(1, 1), (-1, -1), (0, -1), (-1, 0)]\) or if \((x - 1)(y - 1) = 1\). In all other cases the problem is \(#\text{P}\)-hard.

First, we observe that the parameterized dichotomy coincides with the classical dichotomy, except for the three points \((-1, -1), (0, -1),\) and \((-1, 0),\) in which the parameterized Tutte polynomial is \(#\text{W}[1]\)-hard to compute, but the non-parameterized one is polynomial-time solvable. The latter indicates that taking the sum only over the \(k\)-edge subsets can, in fact, make the problem harder.

However, the non-parameterized Tutte polynomial always reduces to the parameterized Tutte polynomial via polynomial-time Turing reductions, since we can compute \(T^0_G(x, y) + \cdots + T^\#E(G)(x, y)\) which is equal to \(T_G(x, y)\). Thus any point \((x, y)\) in which the non-parameterized Tutte polynomial is \(#\text{P}\) hard and in which the parameterized Tutte polynomial is fixed-parameter tractable, constitutes a “real” FPT case. In particular, the latter shows that each point on the line \(x = 1\) yields a real FPT case, except for the point \((1, 1),\) which needs special treatment. More precisely, we have to determine whether computing \(T^k_G(1, 1)\) is not only fixed-parameter tractable (see Lemma 8.7), but also polynomial-time solvable. To this end, observe that

\[
T^k_G(1, 1) = \begin{cases} 
T_G(1, 1) & k = \#V(G) - k(E(G)) \\
0 & k \neq \#V(G) - k(E(G)) 
\end{cases}
\]

since, for \(x = y = 1\), we have

\[(x - 1)^{k(A) - k(E(G))}(y - 1)^{k(A) + \#A - \#V(G)} = 0,
\]

unless \#A = \#V(G) - k(E(G)). Thus, in point \((1, 1),\) the parameterized Tutte polynomial can be computed in polynomial time by relying on the algorithm given by Theorem 8.17 in case \(k = \#V(G) - k(E(G))\), and outputting 0, otherwise.

Finally, recall that by Lemma 8.8 the case \((x - 1)(y - 1) = 1\) allows for a polynomial-time algorithm. The complete picture is hence given by the following refined classification:

Corollary 8.18. Let \((x, y)\) denote a pair of rational numbers. The problem of computing \(T^k_G(x, y)\) is solvable in polynomial-time if \(x = y = 1\) or \((x - 1)(y - 1) = 1\), fixed-parameter tractable, but \(#\text{P}\)-hard, if \(x = 1\) and \(y \neq 1\), and \(#\text{W}[1]\)-hard otherwise.

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References


Detecting and Counting Small Subgraphs, and a Parameterized Tutte Polynomial


