Nematicity Arising from a Chiral Superconducting Ground State in Magic-Angle Twisted Bilayer Graphene under In-Plane Magnetic Fields: Supplemental Material

Tao Yu,[1] Dante M. Kennes,[2,1] Angel Rubio,[1,3,4] and Michael A. Sentef[1]

[1] *Max Planck Institute for the Structure and Dynamics of Matter, Luruper Chaussee 149, 22761 Hamburg, Germany*

[2] *Institut für Theorie der Statistischen Physik, RWTH Aachen, 52056 Aachen, Germany*

[3] *Center for Computational Quantum Physics (CCQ), The Flatiron Institute, 162 Fifth Avenue, New York, New York 10010, USA*

[4] *Nano-Bio Spectroscopy Group, Departamento de Física de Materiales, Universidad del País Vasco, 20018 San Sebastian, Spain*

(Dated: August 10, 2021)

I. SYMMETRY CONSTRUCTION OF GINZBURG-LANDAU LAGRANGIAN

In this part, we construct the superconducting Ginzburg-Landau (GL) Lagrangian by the symmetry analysis [1–3]. Generally, when there are several Bose fields $\Phi_{l=1,\cdots,N}$ to describe the superconductivity, the superconducting energy density

$$f[\Phi] = \alpha_{ij}\Phi_i^*\Phi_j + \beta_{ijkl}\Phi_i^*\Phi_j^*\Phi_k\Phi_l + K_{ijkl}\partial_i\Phi_j^*\partial_k\Phi_l + \cdots \tag{S1}$$

is invariant under the overall $U(1)$ gauge transformation $\Phi_i \to e^{i\phi}\Phi_i$, where α_{ij}, β_{ijkl}, and K_{ijkl} are coefficients, and repeated indices imply the summation. The free energy is invariant under the point group G_0 and time-reversal symmetry $\hat{T}$. Under the time-reversal symmetry, $\hat{T}\Phi_i = \Phi_i^*$. Under the operation $g \in G_0$, the components of the Bose field transform as

$$\hat{O}_g\Phi_i = R_{ij}(g)\Phi_j, \tag{S2}$$

where $R_{ij}(g)$ is the representation matrix of elements g that can always be chosen to be Hermitian. Thus, the first term in the free energy transforms as

$$\alpha_{ij}\Phi_i^*\Phi_j \to \alpha_{ij}R_{ik}^*(g_0)\Phi_k^*R_{jl}(g_0)\Phi_l = R_{ki}^\dagger(g_0)\alpha_{ij}R_{jl}(g_0)\Phi_k^*\Phi_l. \tag{S3}$$

Therefore, the coefficient α_{ij} satisfies the relation

$$\alpha_{ij} = R_{ik}^\dagger(g_0)\alpha_{kl}R_{lj}(g_0), \tag{S4}$$

implying that the matrix $[\alpha_{ij}]$ is Hermitian since $[\alpha] = R^\dagger[\alpha]R$ leads to $[\alpha]^\dagger = [\alpha]$. The Schur's first and second lemma then imply that when the representations $R(g)$ are brought into the irreducible representations that are block-diagonalized in dimensions $\{g_1, g_2, \cdots, g_N\}$, the matrix $[\alpha_{ij}]$ is diagonalized into blocks [1]

$$[\alpha_{ij}] = \{\alpha_1 I_{g_1g_1}, \alpha_2 I_{g_2g_2}, ..., \alpha_N I_{g_Ng_N}\}. \tag{S5}$$

Noting that $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_N)^T$ has transformation $\hat{O}_g\Phi = R(g)\Phi$ for all $g \in G_0$, similar to that of the basis of representation, say $\mathbf{x}^{(j)}$, the Bose field can be expanded by $\mathbf{x}^{(j)}$ as

$$\Phi = \sum_j \psi_j \mathbf{x}^{(j)}, \tag{S6}$$

where ψ_j represents the superconducting order parameters [1, 2]. When the basis $\mathbf{x}^{(j)}$ is changed in terms of new basis $\tilde{\mathbf{x}}^{(k)}$, i.e.,

$$\mathbf{x}^{(j)} = \sum_k S_{jk}\tilde{\mathbf{x}}^{(k)}, \tag{S7}$$

the Bose field $\Phi = \sum_k \sum_j \psi_j S_{jk}\tilde{\mathbf{x}}^{(k)} = \sum_k \tilde{\psi}_k \tilde{\mathbf{x}}^{(k)}$, via which the order parameters transform as

$$\tilde{\psi}_k = \sum_j \psi_j S_{jk}. \tag{S8}$$

The order parameter also has a transformation law under the operation of the point group. By the expansion (S6), one finds

$$(\psi_1, \psi_2, \cdots, \psi_N)\hat{O}_g(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(N)})^T = (\psi_1, \psi_2, \cdots, \psi_N)R(g)(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(N)})^T. \tag{S9}$$

So we can interpret that the operation $\hat{O}_g$ can transform the order parameter without changing the basis by

$$\hat{O}_g(\psi_1, \psi_2, \cdots, \psi_N)^T = [R(g)]^T(\psi_1, \psi_2, \cdots, \psi_N)^T. \tag{S10}$$

Finally, under the time-reversal symmetry $\hat{T}\sum_j \psi_j \mathbf{x}_i^{(j)} = \sum_j \psi_j^*[\mathbf{x}_i^{(j)}]^*$. If the basis is real, i.e., $[\mathbf{x}_i^{(j)}]^* = \mathbf{x}_i^{(j)}$, we have $\hat{T}\psi_j \to \psi_j^*$; if $[\mathbf{x}_i^{(j)}]^* = \mathbf{x}_i^{(k)}$, we have $\hat{T}\psi_j \to \psi_k^*$.

Now we are interested in one particular superconducting state that corresponds to one irreducible representation of dimension M in Eq. (S5), and derive the corresponding Lagrangian functional of $\psi_{i=\{1,2,\cdots,M\}}$. For the first term

$$\alpha_{ij}\Phi_i^*\Phi_j = \Phi^\dagger[\alpha]\Phi = \sum_{l_1 l_2}\left[\mathbf{x}^{(l_1)}\right]^\dagger[\alpha]\mathbf{x}^{(l_2)}\psi_{l_1}^*\psi_{l_2} = \alpha\sum_l |\psi_l|^2, \tag{S11}$$

where we have used $[\alpha] = \alpha\delta_{l_1 l_2}$ for the irreducible representation. For the second term,

$$\sum_{ijkl}\beta_{ijkl}\Phi_i^*\Phi_j^*\Phi_k\Phi_l = \sum_{l_1 l_2 l_3 l_4}\left(\sum_{ijkl}\beta_{ijkl}\mathbf{x}_i^{(l_1)*}\mathbf{x}_j^{(l_2)*}\mathbf{x}_k^{(l_3)}\mathbf{x}_l^{(l_4)}\right)\psi_{l_1}^*\psi_{l_2}^*\psi_{l_3}\psi_{l_4} = \sum_{l_1 l_2 l_3 l_4}\tilde{\beta}_{l_1 l_2 l_3 l_4}\psi_{l_1}^*\psi_{l_2}^*\psi_{l_3}\psi_{l_4}, \tag{S12}$$

where $\tilde{\beta}_{l_1 l_2 l_3 l_4} = \sum_{ijkl}\beta_{ijkl}\mathbf{x}_i^{(l_1)*}\mathbf{x}_j^{(l_2)*}\mathbf{x}_k^{(l_3)}\mathbf{x}_l^{(l_4)}$. For the third term,

$$\sum_{ijkl}K_{ijkl}\partial_i\Phi_j^*\partial_k\Phi_l = \sum_{l_1 l_2}\sum_{ik}\left(\sum_{jl}K_{ijkl}\mathbf{x}_j^{(l_1)*}\mathbf{x}_l^{(l_2)}\right)\partial_i\psi_{l_1}^*\partial_k\psi_{l_2} = \sum_{l_1 l_2}\sum_{ik}\tilde{K}_{il_1 kl_2}\partial_i\psi_{l_1}^*\partial_k\psi_{l_2}, \tag{S13}$$

where $\tilde{K}_{il_1 kl_2} = \sum_{jl}K_{ijkl}\mathbf{x}_j^{(l_1)*}\mathbf{x}_l^{(l_2)}$. Thus, we obtain the Lagrangian functional for the order parameters

$$f[\psi] = \alpha\sum_i |\psi_i|^2 + \sum_{ijkl}\tilde{\beta}_{ijkl}\psi_i^*\psi_j^*\psi_k\psi_l + \sum_{ijkl}\tilde{K}_{ijkl}\partial_i\psi_j^*\partial_k\psi_l. \tag{S14}$$

Under the symmetry transformation, the form of $f[\psi]$ is invariant.

Now we build the invariant polynomials of ψ_i [3] for the twisted bilayer graphene of D_6 group, with the characteristics table given by Table I. The D_6 group is isomorphic to $Z_3 \times Z_2$, where $Z_3 = \{E, C_{3z}, C_{3z}^{-1}\}$

TABLE I. Characteristics table of D_6 group.

D_6	E	$2C_{6z}$	$2C_{3z}$	C_{2z}	$3C_2'$	$3C_2$"	linear	quadratic
A_1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	-	x^2+y^2, z^2
A_2	$+1$	$+1$	$+1$	$+1$	-1	-1	z, R_z	-
B_1	$+1$	-1	$+1$	-1	$+1$	-1	-	-
B_2	$+1$	-1	$+1$	-1	-1	$+1$	-	-
E_1	$+2$	$+1$	-1	-2	0	0	(x,y), (R_x,R_y)	(xz,yz)
E_2	$+2$	-1	-1	$+2$	0	0	-	(x^2-y^2,xy)

and $Z_2 = \{E, C_{2x}\}$. So only C_{3z} and C_{2x} operations need to be considered when constructing the invariant polynomials [3]. So the results below are the same for D_3 symmetry. We are interested in the E_2 representation that supports the chiral d-wave superconductivity. From the basis function (x^2-y^2, xy), we construct the basis function for the $d+id$ and $d-id$ superconductivity:

$$\xi_1 = (k_x^2 - k_y^2 + 2ik_xk_y)/\sqrt{2} = k_+^2/\sqrt{2},$$
$$\xi_2 = (k_x^2 - k_y^2 - 2ik_xk_y)/\sqrt{2} = k_-^2/\sqrt{2}, \tag{S15}$$

where $k_\pm = k_x \pm ik_y$. The transformation matrices for them are

$$C_{3z} = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} \omega^* & 0 \\ 0 & \omega \end{pmatrix}, \quad C_{2x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\omega = e^{i2\pi/3}$. We first construct the quartic term in form of $\psi_1^{N_1}(\psi_1^*)^{N_1'}\psi_2^{N_2}(\psi_2^*)^{N_2'}$, where $\{N_1, N_1', N_2, N_2'\}$ are non-negative integers that have to satisfy several conditions

- **Condition 1** by quartic term: $N_1 + N_1' + N_2 + N_2' = 4$.
- **Condition 2** by $U(1)$ symmetry: $N_1 + N_2 = N_1' + N_2'$.
- **Condition 3** by C_{3z} operation: $N_1 - N_2 - N_1' + N_2' = 0 \pmod 3$.
- **Condition 4** by C_{2x} operation: Lagragian is invariant under operation $\psi_1 \leftrightarrow \psi_2$.
- **Condition 5** by the time reversal symmetry: Lagragian is invariant under operation $\psi_1 \leftrightarrow \psi_2^*$.

From Conditions 1, 2, 3, we find several suitable solutions

- $N_2 = N_2' = 0$ and $N_1 = N_1' = 2 \Longrightarrow |\psi_1|^4$ is allowed;
- $N_2 = N_2' = 1$ and $N_1 = N_1' = 1 \Longrightarrow |\psi_1|^2|\psi_2|^2$ is allowed;
- $N_2 = N_2' = 2$ and $N_1 = N_1' = 0 \Longrightarrow |\psi_2|^4$ is allowed.

Then from Condition 4 or 5, the coefficients of terms $|\psi_1|^4$ and $|\psi_2|^4$ are equal. We thereby obtain the quartic terms

$$\mathcal{L}_q = \lambda_1(|\psi_1|^2 + |\psi_2|^2)^2 + \lambda_2(|\psi_1|^2 - |\psi_2|^2)^2 \tag{S16}$$

in terms of real number $\{\lambda_1, \lambda_2\}$. Similarly, we can construct the gradient terms. In terms of $\{\partial_+, \partial_-\} = \{\partial_x + i\partial_y, \partial_x - i\partial_y\}$ that transform under the operation by

$$C_{3z} = \begin{pmatrix} \omega & 0 \\ 0 & \omega^* \end{pmatrix}, \quad C_{2x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{S17}$$

there are twelve possibilities for the gradient terms

$$\begin{aligned}
&①, \quad \partial_+\psi_1\partial_+\psi_1^* + \text{H.c.}, \quad &&②, \quad \partial_+\psi_1\partial_+\psi_2^* + \text{H.c.},\\
&③, \quad \partial_+\psi_2\partial_+\psi_1^* + \text{H.c.}, \quad &&④, \quad \partial_+\psi_2\partial_+\psi_2^* + \text{H.c.},\\
&⑤, \quad \partial_+\psi_1\partial_-\psi_1^* + \text{H.c.}, \quad &&⑥, \quad \partial_+\psi_1\partial_-\psi_2^* + \text{H.c.},\\
&⑦, \quad \partial_+\psi_2\partial_-\psi_1^* + \text{H.c.}, \quad &&⑧, \quad \partial_+\psi_2\partial_-\psi_2^* + \text{H.c.},\\
&⑨, \quad \partial_+\psi_1^*\partial_-\psi_1 + \text{H.c.}, \quad &&⑩, \quad \partial_+\psi_1^*\partial_-\psi_2 + \text{H.c.},\\
&⑪, \quad \partial_+\psi_2^*\partial_-\psi_1 + \text{H.c.}, \quad &&⑫, \quad \partial_+\psi_2^*\partial_-\psi_2 + \text{H.c.}.
\end{aligned}$$

By the C_{3z}-symmetry, only the terms ②, ⑤, ⑧, ⑨, and ⑫ are allowed. Thus, one can generally construct the gradient terms to be

$$\mathcal{L}_g = (a\partial_+\psi_1\partial_+\psi_2^* + \text{H.c.}) + b\partial_+\psi_1\partial_-\psi_1^* + c\partial_+\psi_2\partial_-\psi_2^* + d\partial_+\psi_1^*\partial_-\psi_1 + e\partial_+\psi_2^*\partial_-\psi_2, \tag{S18}$$

where the coefficients $\{b, c, d, e\}$ are real. From Eq. (S17), under C_{2x} operation, $\partial_+ \leftrightarrow -\partial_-$ and $\psi_1 \leftrightarrow \psi_2$. Therefore, the first term in Eq. (S18) transforms as

$$a\partial_+\psi_1\partial_+\psi_2^* + a^*\partial_-\psi_1^*\partial_-\psi_2 \Longrightarrow a\partial_-\psi_2\partial_-\psi_1^* + a^*\partial_+\psi_2^*\partial_+\psi_1, \tag{S19}$$

which is invariant when $a = a^* \equiv \gamma$ is real. The invariance of the second to fifth terms under C_{2x} operation leads to $b = e \equiv \beta_1$ and $c = d \equiv \beta_2$. Thus, Eq. (S18) becomes

$$\mathcal{L}_g = \gamma\left(\partial_+\psi_1\partial_+\psi_2^* + \text{H.c.}\right) + \beta_1\left(\partial_+\psi_1\partial_-\psi_1^* + \partial_-\psi_2\partial_+\psi_2^*\right) + \beta_2\left(\partial_-\psi_1\partial_+\psi_1^* + \partial_+\psi_2\partial_-\psi_2^*\right). \tag{S20}$$

The Lagragian is invariant under the time-reversal symmetry $\psi_1 \leftrightarrow \psi_2^*$.

II. EFFECTIVE LAGRANGIAN

We employ the Yuan-Fu model [4] to describe the flat band in MATBG. Figure S1 gives a brief description of the model and defines the symbols. The moiré honeycomb lattice has two sites in a unit cell, i.e., A site and B site, labeled, respectively, by "i" and "j" below. At every site, there are two p-orbitals $\{p_x, p_y\}$. The two primitive vectors of the lattice are $\mathbf{a}_1 = \mathbf{c}_1 - \mathbf{c}_3$ and $\mathbf{a}_2 = \mathbf{c}_2 - \mathbf{c}_3$ in terms of three nearest bonding vectors $\mathbf{c}_{1,2,3}$; $\mathbf{d}_{\mu=\{1,2,3\}}$ are three fifth neighboring bonding vectors that connect two A sites or two B sites. Yuan-Fu model, with a suppression of spin index, gives

$$\begin{aligned}\hat{H}_{\text{TB}} =& -\mu\sum_i \hat{\mathbf{a}}_i^\dagger\cdot\hat{\mathbf{a}}_i - \mu\sum_j \hat{\mathbf{b}}_j^\dagger\cdot\hat{\mathbf{b}}_j \\ &+ t_1\sum_{\langle ij\rangle}(\hat{\mathbf{a}}_i^\dagger\cdot\hat{\mathbf{b}}_j + \text{H.c.}) + t_2\sum_{\langle ii'\rangle_5}(\hat{\mathbf{a}}_i^\dagger\cdot\hat{\mathbf{a}}_{i'} + \text{H.c.}) + t_2\sum_{\langle jj'\rangle_5}(\hat{\mathbf{b}}_j^\dagger\cdot\hat{\mathbf{b}}_{j'} + \text{H.c.}) \\ &+ t_3\sum_{\langle ii'\rangle_5}\left[(\hat{\mathbf{a}}_i^\dagger\times\hat{\mathbf{a}}_{i'})_z + \text{H.c.}\right] + t_3\sum_{\langle jj'\rangle_5}\left[(\hat{\mathbf{b}}_j^\dagger\times\hat{\mathbf{b}}_{j'})_z + \text{H.c.}\right],\end{aligned} \tag{S21}$$

where $\hat{\mathbf{a}}_i = (\hat{a}_{ix}, \hat{a}_{iy})$ and $\hat{\mathbf{b}}_j = (\hat{b}_{jx}, \hat{b}_{jy})$ are the operators of different sites and orbitals, μ is the chemical potential and $t_{i=1,2,3}$ are the hopping parameters. Particularly, only t_3-term mixes the x- and y-orbitals.

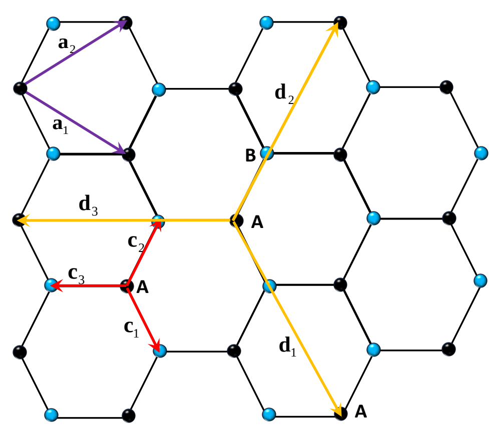

FIG. S1. Honeycomb moiré lattice. Parameters are defined in the text.

This model can be solved analytically. In the momentum space with $\hat{\mathbf{a}}_i = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_i}\hat{\mathbf{a}}_{\mathbf{k}}$ and $\hat{\mathbf{b}}_j =$

$\frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_j}\hat{\mathbf{b}}_{\mathbf{k}}$, the Hamiltonian

$$\begin{aligned}\hat{H}_{\rm TB} = &-\mu\sum_{\mathbf{k}}\hat{\mathbf{a}}^\dagger_{\mathbf{k}}\cdot\hat{\mathbf{a}}_{\mathbf{k}} - \mu\sum_{\mathbf{k}}\hat{\mathbf{b}}^\dagger_{\mathbf{k}}\cdot\hat{\mathbf{b}}_{\mathbf{k}} \\ &+ t_1\sum_{\mathbf{k}} f(\mathbf{k})\hat{\mathbf{a}}^\dagger_{\mathbf{k}}\cdot\hat{\mathbf{b}}_{\mathbf{k}} + t_2\sum_{\mathbf{k}} g(\mathbf{k})(\hat{\mathbf{a}}^\dagger_{\mathbf{k}}\cdot\hat{\mathbf{a}}_{\mathbf{k}} + \hat{\mathbf{b}}^\dagger_{\mathbf{k}}\cdot\hat{\mathbf{b}}_{\mathbf{k}}) + t_3\sum_{\mathbf{k}} g(\mathbf{k})\left[(\hat{\mathbf{a}}^\dagger_{\mathbf{k}}\times\hat{\mathbf{a}}_{\mathbf{k}})_z + (\hat{\mathbf{b}}^\dagger_{\mathbf{k}}\times\hat{\mathbf{b}}_{\mathbf{k}})_z\right] + \text{H.c.},\end{aligned} \tag{S22}$$

where $f(\mathbf{k}) = \sum_{\delta=1,2,3} e^{i\mathbf{k}\cdot\mathbf{c}_\delta}$ and $g(\mathbf{k}) = \sum_{\delta=1,2,3} e^{i\mathbf{k}\cdot\mathbf{d}_\delta}$, which, under the basis $\hat{\Psi}_{\mathbf{k}} = (\hat{a}_{\mathbf{k}x}, \hat{a}_{\mathbf{k}y}, \hat{b}_{\mathbf{k}x}, \hat{b}_{\mathbf{k}y})^T$, reads $\mathcal{H}(\mathbf{k}) = -\mu + t_2(g_{\mathbf{k}} + g_{-\mathbf{k}}) + \begin{pmatrix} 0 & t_3(g_{\mathbf{k}} - g_{-\mathbf{k}}) & t_1 f_{\mathbf{k}} & 0 \\ t_3(g_{-\mathbf{k}} - g_{\mathbf{k}}) & 0 & 0 & t_1 f_{\mathbf{k}} \\ t_1 f_{-\mathbf{k}} & 0 & 0 & t_3(g_{\mathbf{k}} - g_{-\mathbf{k}}) \\ 0 & t_1 f_{-\mathbf{k}} & t_3(g_{-\mathbf{k}} - g_{\mathbf{k}}) & 0\end{pmatrix}$. This matrix is brought to be block diagonal under the chiral basis

$$\begin{pmatrix}\hat{a}_{\mathbf{k}x}\\ \hat{a}_{\mathbf{k}y}\\ \hat{b}_{\mathbf{k}x}\\ \hat{b}_{\mathbf{k}y}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0\\ -i & i & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & -i & i\end{pmatrix}\begin{pmatrix}\hat{a}_{\mathbf{k}+}\\ \hat{a}_{\mathbf{k}-}\\ \hat{b}_{\mathbf{k}+}\\ \hat{b}_{\mathbf{k}-}\end{pmatrix}, \tag{S23}$$

where " $\pm$ " holds angular momentum ± 1, respectively. With basis $(\hat{a}_{\mathbf{k}\pm}, \hat{b}_{\mathbf{k}\pm})^T$, the subspace is explicitly written by

$$h_\pm(\mathbf{k}) = -\mu + t_2(g_{\mathbf{k}} + g_{-\mathbf{k}}) \pm i t_3(g_{-\mathbf{k}} - g_{\mathbf{k}}) + \begin{pmatrix} 0 & t_1 f_{\mathbf{k}} \\ t_1 f_{-\mathbf{k}} & 0, \end{pmatrix}. \tag{S24}$$

For the pairing interaction, we look into the Hamiltonian of the form [2, 5, 6]

$$\hat{H}_{\rm int} \simeq \sum_{\alpha=\pm}\sum_{\mathbf{k}\mathbf{k}'} V_\alpha(\mathbf{k}-\mathbf{k}')\left[\hat{a}^\dagger_{\alpha\uparrow}(\mathbf{k})\hat{b}^\dagger_{\alpha\downarrow}(-\mathbf{k}) - \hat{a}^\dagger_{\alpha\downarrow}(\mathbf{k})\hat{b}^\dagger_{\alpha\uparrow}(-\mathbf{k})\right]\left[\hat{b}_{\alpha\downarrow}(-\mathbf{k}')\hat{a}_{\alpha\uparrow}(\mathbf{k}') - \hat{b}_{\alpha\uparrow}(-\mathbf{k}')\hat{a}_{\alpha\downarrow}(\mathbf{k}')\right], \tag{S25}$$

where we express the pairing potential

$$V_\alpha(\mathbf{k}-\mathbf{k}') = \frac{V_\alpha}{N}\sum_{\mu=\{1,2,3\}} e^{i(\mathbf{k}-\mathbf{k}')\cdot\tilde{\mathbf{c}}_\mu}. \tag{S26}$$

To construct the Lagrangian of the superconducting order parameter, we use a continuous description that is equivalent to the above lattice model via the continuous field operators $\hat{a}_\alpha(\mathbf{r}) = \frac{1}{\sqrt{S}}\sum_{\mathbf{k}}\hat{a}_\alpha(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$ and $\hat{b}_\alpha(\mathbf{r}) = \frac{1}{\sqrt{S}}\sum_{\mathbf{k}}\hat{b}_\alpha(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$, where $S = N\Omega$ is the area of the honeycomb lattices with $\Omega = \sqrt{3}a^2/2 = 3\sqrt{3}c^2/2$ being the area of one unit cell in which $a = |\mathbf{a}_1| = |\mathbf{a}_2|$ and $c = |\mathbf{c}_\delta|$. Then interaction Eq. (S25) corresponds to

$$\hat{H}_{\rm int} \simeq \Omega\sum_\mu\sum_\alpha V_\alpha\int d\mathbf{r}\left[\hat{b}^\dagger_{\alpha\uparrow}(\mathbf{r}_\mu)\hat{a}^\dagger_{\alpha\downarrow}(\mathbf{r}) - \hat{b}^\dagger_{\alpha\downarrow}(\mathbf{r}_\mu)\hat{a}^\dagger_{\alpha\uparrow}(\mathbf{r})\right]\left[\hat{a}_{\alpha\downarrow}(\mathbf{r})\hat{b}_{\alpha\uparrow}(\mathbf{r}_\mu) - \hat{a}_{\alpha\uparrow}(\mathbf{r})\hat{b}_{\alpha\downarrow}(\mathbf{r}_\mu)\right], \tag{S27}$$

where $\mathbf{r}_\mu \equiv \mathbf{r} + \tilde{\mathbf{c}}_\mu$. The subspace now is independent that allows us to omit the index "α" for simplicity below. With the field operator $\hat{\Psi}(\mathbf{r}) = (\hat{a}_\uparrow(\mathbf{r}), \hat{b}_\uparrow(\mathbf{r}), \hat{a}^\dagger_\downarrow(\mathbf{r}), \hat{b}^\dagger_\downarrow(\mathbf{r}))^T$, the free part of the Hamiltonian becomes

$$\hat{H}_0 = \int d\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r})\begin{pmatrix} h(\hat{\mathbf{k}}) & O \\ O & -h^\dagger(-\hat{\mathbf{k}})\end{pmatrix}\hat{\Psi}(\mathbf{r}). \tag{S28}$$

Now we define two 4×4 matrix $\tau_1 = \mathcal{P}_{23}$ and $\tau_2 = \mathcal{P}_{14}$ via relations $\hat{\Psi}^\dagger(\mathbf{r}_\mu)\tau_1\hat{\Psi}(\mathbf{r}) = \hat{b}^\dagger_\uparrow(\mathbf{r}_\mu)\hat{a}^\dagger_\downarrow(\mathbf{r})$ and $\hat{\Psi}^\dagger(\mathbf{r})\tau_2\hat{\Psi}(\mathbf{r}_\mu) = -\hat{b}^\dagger_\downarrow(\mathbf{r}_\mu)\hat{a}^\dagger_\uparrow(\mathbf{r}) = \hat{a}^\dagger_\uparrow(\mathbf{r})\hat{b}^\dagger_\downarrow(\mathbf{r}_\mu)$, where $\mathcal{P}_{ij}$ represents a matrix with only one non-zero element $(i, j) = 1$, with which the interaction Hamiltonian becomes

$$\hat{H}_{\text{int}} = \Omega \sum_\mu V \int d\mathbf{r} \left[\hat{\Psi}^\dagger(\mathbf{r}_\mu)\tau_1\hat{\Psi}(\mathbf{r}) + \hat{\Psi}^\dagger(\mathbf{r})\tau_2\hat{\Psi}(\mathbf{r}_\mu)\right] \left[\hat{\Psi}^\dagger(\mathbf{r})\tau_1^T\hat{\Psi}(\mathbf{r}_\mu) + \hat{\Psi}^\dagger(\mathbf{r}_\mu)\tau_2^T\hat{\Psi}(\mathbf{r})\right]. \tag{S29}$$

With the Grassman field $\overline{\Psi}(\mathbf{r}, t) = \left(\overline{a}_\uparrow(\mathbf{r}), \overline{b}_\uparrow(\mathbf{r}), a_\downarrow(\mathbf{r}), b_\downarrow(\mathbf{r})\right)$, the action is expressed as

$$\begin{aligned} S =& \int_0^\beta d\tau d\mathbf{r}\overline{\Psi}(\mathbf{r},\tau)\partial_\tau\Psi(\mathbf{r},\tau) + \int_0^\beta d\tau d\mathbf{r}\overline{\Psi}(\mathbf{r},\tau)H_0(\hat{\mathbf{k}})\Psi(\mathbf{r},\tau) \\ &- \Omega\sum_\mu V\int_0^\beta d\tau \int d\mathbf{r}\left[\overline{\Psi}(\mathbf{r}_\mu)\tau_1\Psi(\mathbf{r}) + \overline{\Psi}(\mathbf{r})\tau_2\Psi(\mathbf{r}_\mu)\right]\left[\overline{\Psi}(\mathbf{r})\tau_1^T\Psi(\mathbf{r}_\mu) + \overline{\Psi}(\mathbf{r}_\mu)\tau_2^T\Psi(\mathbf{r})\right]. \end{aligned} \tag{S30}$$

We introduce the complex Bose field $\phi(\mathbf{r}, \mathbf{r}_\mu)$ by Hubbard-Stratonovich transformation

$$1 = \int \mathcal{D}\phi(\mathbf{r},\mathbf{r}_\mu)\mathcal{D}\overline{\phi}(\mathbf{r},\mathbf{r}_\mu)\exp\left(-\int_0^\beta d\tau\sum_\mu\int d\mathbf{r}\overline{\phi}(\mathbf{r},\mathbf{r}_\mu)\frac{1}{V\Omega}\phi(\mathbf{r},\mathbf{r}_\mu)\right),$$

with which the action

$$\begin{aligned} S =& \int_0^\beta d\tau d\mathbf{r}\overline{\Psi}(\mathbf{r},\tau)\left(\partial_\tau + H_0(\hat{\mathbf{k}})\right)\Psi(\mathbf{r},\tau) + \sum_\mu\int_0^\beta d\tau d\mathbf{r}\overline{\phi}(\mathbf{r},\mathbf{r}_\mu)\frac{1}{V\Omega}\phi(\mathbf{r},\mathbf{r}_\mu) \\ &+ \sum_\mu\int_0^\beta d\tau d\mathbf{r}\overline{\phi}(\mathbf{r},\mathbf{r}_\mu)\left(\overline{\Psi}(\mathbf{r})\tau_1^T\Psi(\mathbf{r}_\mu) + \overline{\Psi}(\mathbf{r}_\mu)\tau_2^T\Psi(\mathbf{r})\right) + \sum_\mu\int_0^\beta d\tau d\mathbf{r}\left(\overline{\Psi}(\mathbf{r}_\mu)\tau_1\Psi(\mathbf{r}) + \overline{\Psi}(\mathbf{r})\tau_2\Psi(\mathbf{r}_\mu)\right)\phi(\mathbf{r},\mathbf{r}_\mu). \end{aligned}$$

We integrate out the fermion degree of freedom in momentum-frequency space with the fermion field $\Psi(\tau, \mathbf{r}) = \frac{1}{\sqrt{\beta}}\frac{1}{\sqrt{S}}\sum_\mathbf{k}\sum_{\omega_n}\Psi(\omega_n, \mathbf{k})e^{-i\omega_n\tau}e^{i\mathbf{k}\cdot\mathbf{r}}$ and the boson field, with respect to the *center-of-mass* coordinate $\mathbf{r}+\mathbf{r}_\mu/2$, $\phi(\tau, \mathbf{r}, \mathbf{r}+\tilde{\mathbf{c}}_\mu) = \frac{1}{\sqrt{\beta}}\frac{1}{\sqrt{S}}\sum_\mathbf{k}\sum_{\omega_m}\phi(\omega_m, \mathbf{k})e^{-i\omega_n\tau}e^{i\mathbf{k}\cdot(\mathbf{r}+\tilde{\mathbf{c}}_\mu/2)}$. With $k = \{\omega_n, \mathbf{k}\}$ and $q = \{\omega_m, \mathbf{q}\}$, the action

$$\begin{aligned} S =& \sum_\mu\sum_q\overline{\phi}_\mu(q)\frac{1}{\tilde{V}}\phi_\mu(q) \\ &+ \sum_{kk'}\overline{\Psi}_k\Big\{\big[-i\omega_n + H_0(\mathbf{k})\big]\delta_{kk'} + \frac{1}{\sqrt{S\beta}}\sum_\mu\big[\overline{\phi}_\mu(k'-k)(e^{i\boldsymbol{\kappa}\cdot\tilde{\mathbf{c}}_\mu}\tau_1^T + e^{-i\boldsymbol{\kappa}\cdot\tilde{\mathbf{c}}_\mu}\tau_2^T) \\ &+ \phi_\mu(k-k')(e^{-i\boldsymbol{\kappa}\cdot\tilde{\mathbf{c}}_\mu}\tau_1 + e^{i\boldsymbol{\kappa}\cdot\tilde{\mathbf{c}}_\mu}\tau_2)\big]\Big\}\Psi_{k'}, \end{aligned} \tag{S31}$$

where $\tilde{V} \equiv V/N$ and $\boldsymbol{\kappa} \equiv (\mathbf{k}+\mathbf{k}')/2$ denotes the center-of-mass momentum. For convenience, we define the operator $\hat{Q}$ via matrix elements

$$\langle k|\hat{Q}|k'\rangle = \frac{1}{\sqrt{S\beta}}\sum_\mu\left[\overline{\phi}_\mu(k'-k)\left(e^{i\boldsymbol{\kappa}\cdot\tilde{\mathbf{c}}_\mu}\tau_1^T + e^{-i\boldsymbol{\kappa}\cdot\tilde{\mathbf{c}}_\mu}\tau_2^T\right) + \phi_\mu(k-k')\left(e^{-i\boldsymbol{\kappa}\cdot\tilde{\mathbf{c}}_\mu}\tau_1 + e^{i\boldsymbol{\kappa}\cdot\tilde{\mathbf{c}}_\mu}\tau_2\right)\right],$$

and the operator $\hat{G}_0^{-1}[\phi] = i\hat{\omega} - \hat{H}_0$ with elements $\langle k|(i\hat{\omega} - \hat{H}_0)|k'\rangle = (i\omega_n - \hat{H}_0(\mathbf{k}))\delta_{kk'}$. By integrating out the Fermion field, we arrive at the partition function

$$Z = \int\mathcal{D}\overline{\phi}_\mu(q)\mathcal{D}\phi_\mu(q)\exp\left(-\sum_\mu\sum_q\overline{\phi}_\mu(q)\frac{1}{\tilde{V}}\phi_\mu(q) + \operatorname{tr}\ln\left((-i\hat{\omega} + \hat{H}_0) + \hat{Q}\right)\right),$$

with $\operatorname{tr}(...) = \sum_k\langle k|...|k\rangle$. The effective action becomes

$$S_{\text{eff}}[\overline{\phi}, \phi] = \sum_\mu\sum_q\overline{\phi}_\mu(q)\frac{1}{\tilde{V}}\phi_\mu(q) - \operatorname{Tr}\ln\left(-\hat{G}_0^{-1}(1 - \hat{G}_0\hat{Q})\right). \tag{S32}$$

When the order parameter is small, we expand

$$-\mathrm{Tr}\ln\left(-\hat{G}_0^{-1}(1-\hat{G}_0\hat{Q})\right)\to\frac{1}{2}\mathrm{Tr}(\hat{G}_0\hat{Q}\hat{G}_0\hat{Q})=\frac{1}{2}\sum_{kk'}\mathrm{tr}\left(G^0_kQ_{k,k'}G^0_{k'}Q_{k',k}\right),$$

and arrive at the effective action

$$S_{\rm eff}[\overline{\phi},\phi]=\sum_{\mu q}\overline{\phi}_\mu(q)\frac{1}{\tilde{V}}\phi_\mu(q)+\frac{1}{\beta}\frac{1}{S}\sum_{kq}\sum_{\mu\mu'}B_{\mu\mu'}\left(k-\frac{q}{2},k+\frac{q}{2}\right)\overline{\phi}_\mu(q)\phi_{\mu'}(q)+O(\phi^4),\tag{S33}$$

where

$$B_{\mu\mu'}\left(k-\frac{q}{2},k+\frac{q}{2}\right)=\mathrm{tr}\left(G^0_{k-\frac{q}{2}}(e^{i\mathbf{k}\cdot\tilde{\mathbf{c}}_\mu}\tau_1^T+e^{-i\mathbf{k}\cdot\tilde{\mathbf{c}}_\mu}\tau_2^T)G^0_{k+\frac{q}{2}}(e^{-i\mathbf{k}\cdot\tilde{\mathbf{c}}_{\mu'}}\tau_1+e^{i\mathbf{k}\cdot\tilde{\mathbf{c}}_{\mu'}}\tau_2)\right).\tag{S34}$$

Assuming a small $\mathbf{q}$, we define

$$\begin{aligned}&\frac{1}{S}\frac{1}{\beta}\sum_k B^{\mu\mu'}_{k,k}+\frac{1}{V\Omega}\delta_{\mu\mu'}\equiv\mathcal{M}^{\mu\mu'}(q=0),\\&\frac{1}{S}\frac{1}{\beta}\sum_k\left(B^{\mu\mu'}_{k-q/2,k+q/2}-B^{\mu\mu'}_{k,k}\right)\Big|_{\omega_q\to0,\mathbf{q}\to0}\longrightarrow\sum_{\delta\gamma}\mathcal{T}^{\mu\mu'}_{\delta\gamma}\mathbf{q}_\delta\mathbf{q}_\gamma,\end{aligned}\tag{S35}$$

where $\mathcal{T}^{\mu\mu'}_{\delta\gamma}=\frac{1}{2}\frac{\partial^2}{\partial\mathbf{q}_\delta\partial\mathbf{q}_\gamma}\left(\frac{1}{S}\frac{1}{\beta}\sum_k B^{\mu\mu'}_{k-\frac{q}{2},k+\frac{q}{2}}\right)\Big|_{\omega_q\to0,\mathbf{q}\to0}$. Finally, with inverse Fourier transformation $\phi_\mu(q)=\frac{1}{S}\frac{1}{\sqrt{\beta}}\int d\tau d\mathbf{r}e^{i\omega_m\tau}e^{-i\mathbf{q}\cdot(\mathbf{r}+\tilde{\mathbf{c}}_\mu/2)}\phi_\mu(\tau,\mathbf{r})$ the effective action for the Boson fields in the real space reads

$$S_{\rm eff}[\overline{\phi},\phi]=\sum_{\mu\mu'}\int_0^\beta d\tau d\mathbf{r}\overline{\phi}_\mu(\mathbf{r},\tau)\mathcal{M}^{\mu\mu'}\phi_{\mu'}(\mathbf{r},\tau)+\sum_{\mu\mu'}\sum_{\delta\gamma}\mathcal{T}^{\mu\mu'}_{\delta\gamma}\int_0^\beta d\tau d\mathbf{r}\partial_\delta\phi^*_\mu(\mathbf{r},\tau)\partial_\gamma\phi_{\mu'}(\mathbf{r},\tau)+O(\phi^4).\tag{S36}$$

The mass term $\mathcal{M}_{\mu\mu'}$ determines the equilibrium configuration via the gap equation $\delta S_{\rm eff}[\phi,\overline{\phi}]/\delta\overline{\phi}=0$, while $\mathcal{T}^{\mu\mu'}_{\delta\gamma}$ controls the spatial fluctuation. Without spatial fluctuation and near the transition temperature, the linearization of the gap equation yields an eigenvalue equation $1/(V\Omega)\vec{\Phi}=\boldsymbol{\mathcal{M}}\vec{\Phi}$ [5], where Ω is the area of the moiré unit cell and the matrix $\boldsymbol{\mathcal{M}}$ is given by the components $\mathcal{M}_{\mu\mu'}$. The Eigenvectors $\xi_s=(1,1,1)^T/\sqrt{3}$, $\xi_1=-\left(e^{-i\frac{2\pi}{3}},1,e^{i\frac{2\pi}{3}}\right)^T/\sqrt{3}$, $\xi_2=\left(1,e^{i\frac{2\pi}{3}},e^{i\frac{4\pi}{3}}\right)^T/\sqrt{3}$ correspond to extended s-, $(d_{x^2-y^2}+id_{xy})$- and $(d_{x^2-y^2}-id_{xy})$-wave superconductivity, respectively.

III. CALCULATION OF GINZBURG-LANDAU PARAMETERS

Here we derive the coefficients $\{\alpha,\beta,\gamma_1,\lambda_1,\lambda_2\}$ used in the main text. The Green function of the free Hamiltonian in the Nambu space $G_0(\omega_m,\mathbf{k})=\mathrm{diag}\{G_e(\omega_m,\mathbf{k}),G_h(\omega_m,\mathbf{k})\}$, where

$$\begin{aligned}G_e^\pm(\mathbf{k},\omega_m)&=\frac{P_1(\mathbf{k})}{i\omega_m-\varepsilon_1^\pm(\mathbf{k})}+\frac{P_2(\mathbf{k})}{i\omega_m-\varepsilon_2^\pm(\mathbf{k})},\\G_h^\pm(\mathbf{k},\omega_m)&=\frac{P_1(\mathbf{k})}{i\omega_m+\varepsilon_1^\pm(\mathbf{k})}+\frac{P_2(\mathbf{k})}{i\omega_m+\varepsilon_2^\pm(\mathbf{k})},\end{aligned}\tag{S37}$$

are the Green functions in the particle and hole space, respectively. Here,

$$\begin{aligned}\varepsilon_1^\pm(\mathbf{k})&=t_2(g_\mathbf{k}+g_{-\mathbf{k}})\pm it_3(g_{-\mathbf{k}}-g_\mathbf{k})+t_1|f(\mathbf{k})|-\mu,\\\varepsilon_2^\pm(\mathbf{k})&=t_2(g_\mathbf{k}+g_{-\mathbf{k}})\pm it_3(g_{-\mathbf{k}}-g_\mathbf{k})-t_1|f(\mathbf{k})|-\mu,\end{aligned}$$

are the electron and hole dispersions, and

$$P_1(\mathbf{k}) = \frac{1}{2}\begin{pmatrix} 1 & -e^{i\phi_\mathbf{k}} \\ -e^{-i\phi_\mathbf{k}} & 1 \end{pmatrix}, \quad P_2(\mathbf{k}) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\phi_\mathbf{k}} \\ e^{-i\phi_\mathbf{k}} & 1 \end{pmatrix} \tag{S38}$$

are the projection operators for these two bands with $e^{i\phi_\mathbf{k}} = f_\mathbf{k}/|f_\mathbf{k}|$. Below we omit the label "±" for simplicity. From Eq. (S35), both the mass and spatial fluctuation depend on

$$\begin{aligned}
\mathcal{B}^{\mu\mu'}_{\mathbf{q},\pm} &\equiv \frac{1}{S}\frac{1}{\beta}\sum_k B^{\mu\mu'}_{k-\frac{q}{2},k+\frac{q}{2}} \\
&= \frac{1}{2S}\sum_\mathbf{k}\left(\frac{n_F\left(\varepsilon_1(\mathbf{k}-\frac{\mathbf{q}}{2})\right)+n_F(\varepsilon_1\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right))-1}{\varepsilon_1\left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)+\varepsilon_1\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)} + \frac{n_F\left(\varepsilon_2(\mathbf{k}-\frac{\mathbf{q}}{2})\right)+n_F(\varepsilon_2\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right))-1}{\varepsilon_2\left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)+\varepsilon_2\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)}\right) \\
&\times\left(\cos\left(\mathbf{k}\cdot(\tilde{\mathbf{c}}_\mu+\tilde{\mathbf{c}}_{\mu'})-\phi_{\mathbf{k}-\frac{\mathbf{q}}{2}}-\phi_{\mathbf{k}+\frac{\mathbf{q}}{2}}\right)+\cos\mathbf{k}\cdot(\tilde{\mathbf{c}}_\mu-\tilde{\mathbf{c}}_{\mu'})\right) \\
&+\frac{1}{2S}\sum_\mathbf{k}\left(\frac{n_F\left(\varepsilon_1(\mathbf{k}-\frac{\mathbf{q}}{2})\right)+n_F(\varepsilon_2\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right))-1}{\varepsilon_1\left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)+\varepsilon_2\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)} + \frac{n_F\left(\varepsilon_2(\mathbf{k}-\frac{\mathbf{q}}{2})\right)+n_F(\varepsilon_1\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right))-1}{\varepsilon_2\left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)+\varepsilon_1\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)}\right) \\
&\times\left(-\cos\left(\mathbf{k}\cdot(\tilde{\mathbf{c}}_\mu+\tilde{\mathbf{c}}_{\mu'})-\phi_{\mathbf{k}-\frac{\mathbf{q}}{2}}-\phi_{\mathbf{k}+\frac{\mathbf{q}}{2}}\right)+\cos\mathbf{k}\cdot(\tilde{\mathbf{c}}_\mu-\tilde{\mathbf{c}}_{\mu'})\right),
\end{aligned} \tag{S39}$$

which is calculated by the summation over Matsubara frequency. Via Eq. (S39) that is calculated numerically for momentum summation, the coefficients for the mass and spatial fluctuation

$$\begin{aligned}
\mathcal{M}_{\mu\mu'} &= \frac{1}{V\Omega}\delta_{\mu\mu'} + \mathcal{B}^{\mu\mu'}_{\mathbf{q}=0}, \\
\mathcal{T}^{\mu\mu'}_{\delta\gamma} &= \frac{1}{2}\left(\frac{\partial^2}{\partial\mathbf{q}_\delta\partial\mathbf{q}_\gamma}\mathcal{B}^{\mu\mu'}_\mathbf{q}\right)\Big|_{\mathbf{q}\to 0}.
\end{aligned} \tag{S40}$$

These matrices are computed numerically by performing the intergral and derivatives over the momentum, which are then used to calculate the GL parameters. To this end, by substituting $(\phi_1(\mathbf{r}), \phi_2(\mathbf{r}), \phi_3(\mathbf{r}))^T = \psi_1(\mathbf{r})\xi_1 + \psi_2(\mathbf{r})\xi_2$ into Eq. (S36), we obtain the Lagrangian density for the order parameters

$$\mathcal{L}_{\text{eff}} = \sum_{i=1,2} a_i|\psi_i(\mathbf{r})|^2 + \sum_{\delta\gamma}\sum_{ij} c^{ij}_{\delta\gamma}\partial_\delta\psi_i^*(\mathbf{r})\partial_\gamma\psi_j(\mathbf{r}), \tag{S41}$$

where we define and numerically compute

$$\begin{aligned}
a_i &= \xi_i^\dagger \boldsymbol{\mathcal{M}}\xi_i, \\
c^{ij}_{\delta\gamma} &= \xi_i^\dagger \boldsymbol{\mathcal{T}}_{\delta\gamma}\xi_j.
\end{aligned} \tag{S42}$$

With these computation, we find the GL parameters $\{\alpha, \beta, \gamma\}$ used in the main text via relations

$$\begin{aligned}
&a_1 = a_2 = \alpha, \\
&c^{11}_{\delta\gamma} = c^{22}_{\delta\gamma} = \beta\delta_{\delta\gamma}, \\
&c^{21}_{xx} = -c^{21}_{yy} = -ic^{21}_{xy} = \gamma.
\end{aligned} \tag{S43}$$

Finally, we address the calculation of GL parameters $\{\lambda_1, \lambda_2\}$. Expanding the Bose fields to $O(\phi^4)$, the nonlinear part of the action is given by

$$S^{\text{NL}}_{\text{eff}} = \int_0^\beta d\tau d\mathbf{r}\left(\lambda_1\left(|\psi_1|^2+|\psi_2|^2\right)^2 + \lambda_2\left(|\psi_1|^2-|\psi_2|^2\right)^2\right), \tag{S44}$$

in which the coefficients

$$\begin{aligned}\lambda_1+\lambda_2 &= \frac{1}{2\beta^2}\frac{1}{S}\sum_k \mathrm{tr}\left[G^h_k\Lambda_1(\mathbf{k})G^e_k\Lambda_2(\mathbf{k})G^h_k\Lambda_1(\mathbf{k})G^e_k\Lambda_2(\mathbf{k})\right],\\ \lambda_1-\lambda_2 &= \frac{1}{2\beta^2}\frac{1}{S}\sum_k \mathrm{tr}\left[G^h_k\Lambda_1(\mathbf{k})G^e_k\Lambda_1(\mathbf{k})G^h_k\Lambda_2(\mathbf{k})G^e_k\Lambda_2(\mathbf{k})\right]\\ &\quad+\frac{1}{2\beta^2}\frac{1}{S}\sum_k \mathrm{tr}\left[G^h_k\Lambda_1(\mathbf{k})G^e_k\Lambda_2(\mathbf{k})G^h_k\Lambda_2(\mathbf{k})G^e_k\Lambda_1(\mathbf{k})\right],\end{aligned} \tag{S45}$$

where

$$\begin{aligned}\Lambda_1(\mathbf{k}) &= \frac{1}{\sqrt{3}}\begin{pmatrix}0 & e^{i\mathbf{k}\cdot\tilde{\mathbf{c}}_1}+e^{i\frac{4\pi}{3}}e^{i\mathbf{k}\cdot\tilde{\mathbf{c}}_2}+e^{i\frac{2\pi}{3}}e^{i\mathbf{k}\cdot\tilde{\mathbf{c}}_3}\\ e^{-i\mathbf{k}\cdot\tilde{\mathbf{c}}_1}+e^{i\frac{4\pi}{3}}e^{-i\mathbf{k}\cdot\tilde{\mathbf{c}}_2}+e^{i\frac{2\pi}{3}}e^{-i\mathbf{k}\cdot\tilde{\mathbf{c}}_3} & 0\end{pmatrix},\\ \Lambda_2(\mathbf{k}) &= \frac{1}{\sqrt{3}}\begin{pmatrix}0 & e^{i\mathbf{k}\cdot\tilde{\mathbf{c}}_1}+e^{i\frac{2\pi}{3}}e^{i\mathbf{k}\cdot\tilde{\mathbf{c}}_2}+e^{i\frac{4\pi}{3}}e^{i\mathbf{k}\cdot\tilde{\mathbf{c}}_3}\\ e^{-i\mathbf{k}\cdot\tilde{\mathbf{c}}_1}+e^{i\frac{2\pi}{3}}e^{-i\mathbf{k}\cdot\tilde{\mathbf{c}}_2}+e^{i\frac{4\pi}{3}}e^{-i\mathbf{k}\cdot\tilde{\mathbf{c}}_3} & 0\end{pmatrix}.\end{aligned} \tag{S46}$$

Here we focus on the case with a high hole doping with $\mu \lesssim -t_1$, in which case the particle and hole Green functions $G^h_k \approx \frac{P_2(\mathbf{k})}{i\omega_m+\varepsilon_2(\mathbf{k})}$ and $G^e_k \approx \frac{P_2(\mathbf{k})}{i\omega_m-\varepsilon_2(\mathbf{k})}$, with which, e.g.,

$$\lambda_1+\lambda_2 \simeq \frac{1}{2\beta^2}\frac{1}{S}\sum_{\mathbf{k}}\sum_{\omega_m}\mathrm{tr}\left[\frac{P_2(\mathbf{k})\Lambda_1(\mathbf{k})P_2(\mathbf{k})\Lambda_2(\mathbf{k})P_2(\mathbf{k})\Lambda_1(\mathbf{k})P_2(\mathbf{k})\Lambda_2(\mathbf{k})}{(i\omega_m+\varepsilon_2(\mathbf{k}))^2(i\omega_m-\varepsilon_2(\mathbf{k}))^2}\right].$$

We calculate the summation over Matsubara frequency according to the residue theorem and arrive at

$$\begin{aligned}\lambda_1+\lambda_2 &= \frac{1}{2\beta}\frac{1}{S}\sum_{\mathbf{k}}\mathrm{tr}\left[P_2(\mathbf{k})\Lambda_1(\mathbf{k})P_2(\mathbf{k})\Lambda_2(\mathbf{k})P_2(\mathbf{k})\Lambda_1(\mathbf{k})P_2(\mathbf{k})\Lambda_2(\mathbf{k})\right]\\ &\quad\times\left[\frac{1}{4\varepsilon_2^2(\mathbf{k})}\Big(n'_F(\varepsilon_2(\mathbf{k}))+n'_F(-\varepsilon_2(\mathbf{k}))\Big)+\frac{1}{4\varepsilon_2^3(\mathbf{k})}\Big(-n_F(\varepsilon_2(\mathbf{k}))+n_F(-\varepsilon_2(\mathbf{k}))\Big)\right],\end{aligned} \tag{S47}$$

and similarly for $\lambda_1-\lambda_2$. The values of $\lambda_{1,2}$ are obtained via numerical computation of these integrals over momentum.

IV. CALCULATION OF T_c BY BKT TRANSITION

In two-dimensional superconductors, the superconducting critical temperature T_c is often determined by the BKT transition. We now include an estimation of the BKT transition temperature from the calculated superconducting stiffness via $k_BT_c = \pi\beta(T_c)$, and compare with that obtained by the mean-field theory via $\alpha(T_c) = 0$. As shown in Fig. S2, we find that the latter is well below the BKT transition. As a consequence, the mean-field regime correctly describes the physics near the boundary of the superconducting dome around T_c (we note that T_{BKT} does not show a dome-shaped behavior as a function of doping), since the phase fluctuation for the BKT transition costs much higher energy and is therefore sufficiently far away in temperature. The mean-field theory in two-dimensional superconductors thereby does have predictive power in, e.g., the superconducting paraconductivity [7].

V. NEMATICITY OF SUPERCONDUCTING FLUCTUATIONS ABOVE T_c

In the experiments by Cao *et al.*, the nematicity in the transport, under the applied in-plane magnetic field, is observed at temperatures slightly above T_c [8], at which the superconducting order parameter actually

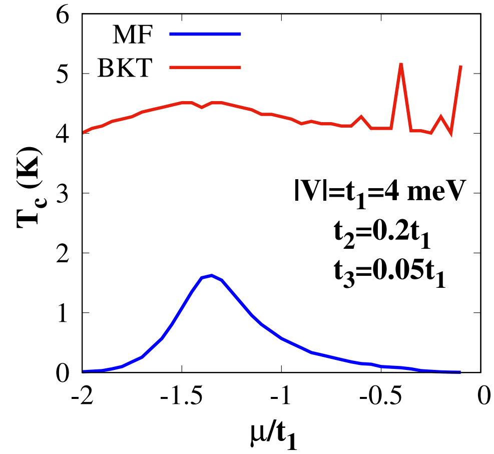

FIG. S2. Comparison of superconducting critical temperature T_c by the mean-field theory ("MF") and BKT transition ("BKT") under different hole dopings.

vanishes. Here we explain in more detail how the nematic behavior can arise in the transport at temperature slightly above T_c in the absence of superconducting orders. This is because the nematicity of the stiffness is carried to the paraconductivity via the spatial correlations of the superconducting order parameter, not the order parameter itself. By the thermal fluctuation above T_c, the superconducting order parameter

$$\psi(\mathbf{q},t) = \frac{1}{\hbar\Gamma}\int_{-\infty}^{t} dt' \exp\left[-\frac{1}{\hbar\Gamma}\int_{t'}^{t} d\tilde{t}\mathcal{H}(\mathbf{q},\tilde{t})\right] f(\mathbf{q},t')$$

is given by the damping parameter Γ, the stiffness $\{\beta,\gamma\}$ in $\mathcal{H}(\mathbf{q},t)$ and the thermal noise $f(\mathbf{q},t)$, which has an average $\langle f(\mathbf{q},t)\rangle = 0$. Therefore, there is no superconducting order at temperature $T > T_c$. However, the electric current carried by the superconducting fluctuation

$$\mathbf{j} = \sum_{\mathbf{q}} \psi^\dagger(\mathbf{q}) \left(-\frac{c\partial\mathcal{H}(\mathbf{q})}{\partial \mathbf{A}_E}\right) \psi(\mathbf{q}) \tag{S48}$$

arises by the correlation of the superconducting order parameter $\langle\psi^\dagger(\mathbf{q})\psi(\mathbf{q})\rangle$, rather than $\langle\psi(\mathbf{q})\rangle$, and the same applies to the paraconductivity [7, 9], which therefore generally exists when $T \gtrsim T_c$.

Since the nematicity is inherited by the paraconductivity via the superconducting stiffness, it is useful here to emphasize its existence at temperatures above T_c. To calculate the paraconductivity at $T \gtrsim T_c$ by the thermal fluctuation of superconducting order parameters [7, 9], one has to determine the correct ground state, which is conveniently done by going to the ordered phase with $T \lesssim T_c$. With both calculation performed at $T \sim T_c$, the stiffness $\{\beta,\gamma\}$ does not change, in contrast to the sign change of α. In other words, the superconducting stiffness $\{\beta,\gamma\}$ vanishes only when the temperature is far above T_c. Thus, the superconducting stiffness $\{\beta,\gamma\}$ obtained at temperatures slightly below T_c can be used to calculate the paraconductivity at temperatures slightly above T_c.

---

[1] J. F. Annett, Adv. Phys. **39**, 83 (1990).

[2] M. Sigrist and K. Ueda, Rev. Mod. Phys. **63**, 239 (1991).

[3] R. E. Rudd and W. E. Pickett, Phys. Rev. B **57**, 557 (1998).

[4] N. F. Q. Yuan and L. Fu, Phys. Rev. B **98**, 045103 (2018).

[5] A. M. Black-Schaffer and C. Honerkamp, J. Phys. Condens. Matter **26**, 423201 (2014).

[6] M. Claassen, D. M. Kennes, M. Zingl, M. A. Sentef, and A. Rubio, Nat. Phys. **15**, 766 (2019).

[7] R. Wakatsuki, Y. Saito, S. Hoshino, Y. M. Itahashi, T. Ideue, M. Ezawa, Y. Iwasa, and N. Nagaosa, Sci. Adv. 3, e1602390 (2017).

[8] Y. Cao, D. R.-Legrain, J. M. Park, F. N. Yuan, K. Watanabe, T. Taniguchi, R. M. Fernandes, L. Fu, and P. J.-Herrero, Science **372**, 264 (2021).

[9] Skocpol and Tinkham, Rep. Prog. Phys. **38**, 1049 (1975)