FAST DETERMINATION OF PLASMA PARAMETERS THROUGH FUNCTION PARAMETRIZATION

B.J. Braams*, W. Jilge, and P. Martin
Max-Planck-Institut für Plasmaphysik, EURATOM Association,
D-8046 Garching bei München, Fed. Rep. Germany.
*F.O.M. Instituut voor Plasmafysica,
Postbus 1207, 3430 BE Nieuwegein, The Netherlands.

Introduction. In the interpretation of tokamak diagnostics the amount of experimental information that is utilized is often not limited by the rate at which measurements can be made, but more by the rate at which the raw diagnostics can be interpreted. Clearly then, efficient methods of data analysis are highly desirable. A very efficient procedure, function parametrization, was developed by H. Wind (CERN) for the purpose of fast momentum determination from spark chamber data [1], [2]. Although this method had not previously been noticed outside the high energy physics field, it has a much wider range of applicability, and can be considered whenever many measurements are to be made with the same diagnostic setup. Its utility for tokamak physics applications, and in particular for the determination of characteristic equilibrium parameters from magnetic measurements, was proposed in [3], and is demonstrated in the present paper.

Function parametrization relies on an analysis of a large data set of simulated experiments, aiming to obtain an optimal representation of some simple functional form for intrinsic physical parameters of a system in terms of the values of the measurements. Statistical techniques for dimension reduction and multiple regression are used in the analysis. The resulting function may be chosen to involve only low-order polynomials in only a few linear combinations of the original measurements; this function can therefore be evaluated very rapidly, and needs only minimal hardware facilities.

Three steps have to be made for experimental data evaluation based on function parametrization. (1) A numerical model of the experiment is used to generate a data base of simulated states of the physical system, in which each state is represented by the values of the relevant physical parameters and of the associated measurements. (2) This data base is made the object of a statistical analysis, with the aim to provide a relatively simple function that expresses the physical parameters in terms of the measurements. (3) The resulting function is then employed for the fast interpretation of real measurements.

Determination of characteristic parameters of a magnetically confined plasma is a particularly indicated application. The MHD equilibrium model provides a well defined and generally accepted connection between the unknown intrinsic plasma parameters, the externally applied fields, and the magnetic measurements. Identification of the position and the profile of the plasma column is the basis for interpretation of practically all other diagnostics, and requires an efficient algorithm. Ultimate aim of our work is in fact on-line real-time analysis, with use of the derived information for feedback plasma control.

Mathematical description. A classical physical system is considered, of which P denotes a typical state. The system may have any number of degrees of freedom, but interest will be restricted to a (partial) characterization by n intrinsic real parameters, represented collectively by a point $\mathbf{p} \in \mathbf{R}^n$. In the experimental situation $\mathbf{p}$ is to be estimated from the readings of m measurements, represented by a point $\mathbf{q} \in \mathbf{R}^m$.

The aim of the function parametrization is to obtain some relatively simple function $\mathbf{F} : \mathbf{R}^m \to \mathbf{R}^n$, such that for any state $\mathcal{P}$ the associated $\mathbf{p}(\mathcal{P})$ and $\mathbf{q}(\mathcal{P})$ satisfy $\mathbf{p} = \mathbf{F}(\mathbf{q}) + \mathbf{e}$ for a sufficiently small error term $\mathbf{e}$. The functional form of $\mathbf{F}$ may typically be chosen as a low-order polynomial in only a few linear combinations of the components of $\mathbf{q}$. The unknown coefficients in $\mathbf{F}$ are then determined by analysis of a data base containing the values of the parameters $\mathbf{p}_\alpha$ and of the measurements $\mathbf{q}_\alpha$ corresponding to N simulated states $\mathcal{P}_\alpha$ $(1 \le \alpha \le N)$. This is a problem for which techniques from multivariate statistical analysis are appropriate.

Since the dimensionality m of the space of the measurements lies between several tens and several hundred in many cases, the dimensionality of the space of trial functions with which the parameters are to be fitted can be very large. A polynomial model of degree l in all variables, for instance, has $\sim m^l/l!$ degrees of freedom for each physical parameter. It is therefore desirable to first reduce the number of independent variables (the components of $\mathbf{q}$) by means of a transformation to a lower-dimensional space. A second, and also very important, aim for this transformation of variables must be to eliminate or reduce multicollinearity (near linear dependencies) between the data points, and thus to improve the conditioning of the regression problem. Multicollinearity is expected to be present whenever the number of measurements is much larger than the number of independently determinable physical parameters.

A common statistical technique to find a lower-dimensional space in which to represent the measurements is based on principal component analysis. From the N suitably scaled pseudo measurements, $\{\mathbf{q}_\alpha\}_{1\le\alpha\le N}$, the sample mean, $\bar{\mathbf{q}} = N^{-1}\sum_\alpha \mathbf{q}_\alpha$, and the $m \times m$ sample dispersion matrix,

$$S = \frac{1}{N}\sum_{\alpha=1}^{N}(\mathbf{q}_\alpha - \bar{\mathbf{q}})(\mathbf{q}_\alpha - \bar{\mathbf{q}})^T,$$

are calculated. S is symmetric and positive semi-definite. An eigenanalysis will yield m eigenvalues, $\lambda_1^2 \ge \ldots \ge \lambda_m^2 \ge 0$, with corresponding orthonormal eigenvectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$. Any measurement vector $\mathbf{q}$ may be resolved along these eigenvectors to obtain a set of transformed measurements, $x_i = \mathbf{a}_i \cdot (\mathbf{q} - \bar{\mathbf{q}})$. The sample variance of the component x_i is given by λ_i. Now the assumption is made that the most significant information will be contained in those transformed measurements that show the largest variation over the simulated data, viz. in the components $(x_i)_{1\le i\le s}$ where $s \le m$, and preferably $s \ll m$. These s components are called the 'significant components', and the associated first s eigenvectors $\mathbf{a}_i$ are the 'significant variables'. The desired dimension reduction is thus achieved through the transformation $\mathbf{R}^m \to \mathbf{R}^s$ defined by $\mathbf{x} = \mathbf{A}^T \cdot (\mathbf{q} - \bar{\mathbf{q}})$, where $\mathbf{A}$ is the matrix that has columns $\mathbf{a}_i$ $(1 \le i \le s)$.

Having obtained the preliminary linear transformation $\mathbf{q} \to \mathbf{x}$ it is next necessary to face the task of fitting the, in general nonlinear, relation between $\mathbf{x}$ and $\mathbf{p}$. It is desired to find for each component p_j $(1 \le j \le n)$ a regression, $p_j = f_j(\mathbf{x}) + \epsilon_j$, to fit the data $(\mathbf{x}_\alpha, \mathbf{p}_\alpha)_{1\le\alpha\le N}$. A polynomial model, of the form

$$p_j = \sum_{\mathbf{k}} c_{\mathbf{k}j} \cdot \prod_{i=1}^{s} \phi_{k_i}(x_i/r_i) + \epsilon_j, \tag{1}$$

is suitable. Here, the multi-index $\mathbf{k}$ has s components $k_1, \ldots, k_s$ in the nonnegative integers, the $c_{\mathbf{k}j}$ are the unknown regression coefficients, which are determined by a least-squares fitting procedure over the data base, $(\phi_\ell)_{\ell \geq 0}$ is some family of polynomials (Chebyshev, Hermite, Legendre, or monomials), r_i is a suitable scaling factor for the component x_i, and ϵ_j is the error term. An upper bound on some norm of $\mathbf{k}$ must be supplied in order to make the model finite, and in addition it is possible to employ with the above model some form of subset regression, the objective being to retain in the final expression only the terms which make a significant contribution to the goodness-of-fit.

Function parametrization thus leads to simple explicit approximations for the physical parameters in terms of the measurements. Although a significant effort may be involved in generating and analyzing the data base, the evaluation of the final function — and this is the operation that has to be performed many times — is almost trivial.

Application to magnetic data analysis. As an initial study we applied function parametrization to the determination of a limited set of characteristic equilibrium parameters for the ASDEX experiment, using only magnetic signals measured outside the plasma. The relevant measurements consist of three differential flux measurements, four field measurements, the current through the multipole shaping coils, and the plasma current. However, the plasma current can be scaled out of the problem, so that 8 independent measurements remain. The physical parameters to be determined include the position of the magnetic axis, the geometric center of the cross section, the current center, the horizontal and vertical minor radius, $\beta_p + l_i/2$, a normalized q-value at the separatrix, the flux difference between the separatrix and the vacuum vessel, the position of the lower and upper saddle points, and the point of intersection of the separatrix with each of the four divertor plates.

The data base was generated using the Garching equilibrium code [4], which has recently been much optimized, and computes an equilibrium on our 64×128 grid in ~ 0.6 sec on the Cray 1. The free parameters of the code were randomly varied in order to cover the operating regime of the ASDEX experiment in the divertor mode, and for each of the ~ 4000 calculated equilibria the corresponding magnetic measurements and physical parameters were recorded.

Results. A principal component analysis of the correlation matrix obtained from the simulated measurements gave the following sequence of eigenvalues: 3.67, 1.91, 1.39, 0.99, 2.01×10^{-2}, 1.12×10^{-2}, 1.90×10^{-3}, 8.93×10^{-4}. It thus appeared that 4 significant variables should be retained for the regression analysis. Of these four, the first two are even under up-down reflection of the equilibrium, and together already provide a measure of the radial position and of $\beta_p + l_i/2$. The third significant variable is odd under reflection, and is related to the vertical position of the plasma, and the fourth one (again even under reflection) is essentially the multipole current.

Further analysis showed that the measurement of the multipole current is of little overall importance, but is mainly relevant for the determination of the intersection of the separatrix with the divertor plates. After experimenting with various possible regression models we selected a model that is second order in the first three significant variables, and first order in the fourth: For each of the physical parameters p,

$$\begin{aligned} p = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 + c_5 H_2(x_1) \\ + c_6 x_1 x_2 + c_7 x_1 x_3 + c_8 H_2(x_2) + c_9 x_2 x_3 + c_{10} H_2(x_3) \end{aligned} \tag{2}$$

where $H_2(x) = (x^2 - 1)/\sqrt{2}$. In order to overcome a first-derivative discontinuity in the physics of the system, separate fits were used for the two cases when the separatrix x-point is located in the lower, resp. in the upper half plane. The results obtained with this model for some representative physical parameters are shown in the following Table.

parameter	aver	min	max	variance	δ ($\epsilon = 0.0$)	δ ($\epsilon = 0.1$)
r_{axis}	1.72	1.63	1.83	5.6×10^{-2}	2.2×10^{-3}	4.1×10^{-3}
r_{curr}	1.71	1.61	1.82	5.6×10^{-2}	3.1×10^{-4}	3.4×10^{-3}
z_{axis}	0	−0.10	0.10	5.4×10^{-2}	2.3×10^{-3}	5.8×10^{-3}
z_{curr}	0	−0.10	0.10	5.4×10^{-2}	2.5×10^{-3}	6.0×10^{-3}
a	0.367	0.290	0.463	3.0×10^{-2}	3.8×10^{-3}	6.2×10^{-3}
b	0.358	0.295	0.438	2.5×10^{-2}	1.9×10^{-3}	4.7×10^{-3}
$\beta_p + l_i/2$	1.79	0.56	3.43	0.63	1.2×10^{-2}	4.4×10^{-2}

For each of the parameters the Table shows first the average, minimum, and maximum values occurring in the data base, and the standard deviation about the average. The last two columns show the standard error, δ, of the model in Eq (2), first for exact measurements ($\epsilon = 0.0$), and then for measurements which have been randomly perturbed by a term coming from a normal distribution with average 0 and width equal to a fraction $\epsilon = 0.1$ of the variance of the measurements.

Conclusion. The application described above establishes function parametrization as a straightforward and effective way in which to obtain numerical approximations for a variety of characteristic equilibrium parameters in terms of the magnetic measurements. These approximations are not only extremely easy to evaluate, but are also more accurate than the analytic approximations that are now in common use. The procedure does not require very specific assumptions about the MHD equilibrium, and is also well suited to a consistent analysis of a system consisting of several different diagnostics. We expect that in the future function parametrization will have an important rôle both for on-line data analysis and for real-time plasma control.

Acknowledgements. The authors wish to acknowledge many valuable discussions with Dr. K. Lackner. This work was performed under the Euratom-IPP and Euratom-FOM association agreements with financial support from ZWO and Euratom.

References.

1. H. Wind, *Function Parametrization*, in "Proceedings of the 1972 CERN Computing and Data Processing School", CERN 72-21, 1972, pp. 53–106.
2. H. Wind, (a) *Principal Component Analysis and its Application to Track Finding,* (b) *Interpolation and Function Representation*, in "Formulae and Methods in Experimental Data Evaluation", Vol. 3, European Physical Society, Geneva, 1984.
3. B.J. Braams and K. Lackner, *A Proposed Method for Fast Determination of Plasma Parameters*, Report IPP 1/228, Max-Planck-Institut für Plasmaphysik, 1984.
4. K. Lackner, *Computation of Ideal MHD Equilibria*, Comput. Phys. Commun. **12** (1976), 33–44.