Perceptual Error Optimization for Monte Carlo Rendering

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Realistic image synthesis involves computing high-dimensional light transport integrals which in practice are numerically estimated using Monte Carlo integration. The error of this estimation manifests itself in the image as visually displeasing artifacts in the image, unless care is taken to control the correlation of the samples used to obtain the individual pixel estimates. A standard approach is to decorrelate these estimates by randomly assigning sample sets to pixels, turning potential structured artifacts into white noise.

In digital halftoning, the error induced by quantizing continuous tone images has been studied extensively. These studies have shown that a blue-noise distribution of the quantization error is perceptually optimal [Ulichney 1987] and can achieve substantially higher fidelity than a white-noise distribution. Recent works have proposed ways to transfer these ideas empirically to image synthesis [Georgiev and Fajardo 2016; Heitz and Belcour 2019; Heitz et al. 2019]. This is achieved by carefully re-introducing negative pixel correlation, exploiting the local smoothness present in typical images.

We propose a theoretical formulation of perceptual error for rendering which unifies previous methods in a common theoretical framework and justifies the need for blue-noise error distribution. We start by extending the comparatively simpler problem of digital
halftoning [Sullivan et al. 1991; Analoui and Allebach 1992; Pappas and Neuhoff 1999] to the substantially more complex one of MC image synthesis. Our formulation bridges the gap between halftoning and rendering by interpreting the error distribution problem as a general extension of non-uniform multitone energy minimization halftoning, where the MC estimates are taken to be the admissible quantization levels in the halftoning setting. Through this insight virtually any halftoning method can be adapted to work with MC rendering. We demonstrate this by adapting representative methods from the three main classes of halftoning algorithms: dither matrix halftoning, error diffusion halftoning, iterative energy minimization halftoning.

Previous methods [Georgiev and Fajardo 2016; Heitz and Belcour 2019; Heitz et al. 2019] work by distributing MC error via target error masks which are produced by minimizing an energy involving a pre-chosen kernel. The kernel, typically a Gaussian, can be interpreted as an approximation to the human visual system’s (HVS) point spread function (PSF) [Daly 1987; Pappas and Neuhoff 1999]. We revisit the kernel-based perceptual model from halftoning [Sullivan et al. 1991; Analoui and Allebach 1992; Pappas and Neuhoff 1999] and reformulate it for MC rendering. The resulting energy can be optimized for MC error distribution without any explicit mask. By providing a formal analysis, we make the hidden assumptions of previous methods explicit and quantify their limitations. Notably, all previous methods can be seen as variants of dither matrix halftoning introducing varying degrees of bias. In summary:

- Our formulation unifies both a-priori and a-posteriori methods. This removes the necessity of using empirically motivated energies, making all assumptions explicit, and prescribing general guidelines for devising new a-priori and a-posteriori methods in a principled manner.
- All a-posteriori methods in our framework can be seen as methods, which can be used to decrease the bias of a rendered biased image. This provides a continuous spectrum between the fully unbiased estimates (white noise error distribution) and highly biased such (high quality blue noise distribution). This insight makes the limitations of a-posteriori methods quantifiable.
- We demonstrate our theoretical results through several experiments that showcase that we can substantially improve the error distribution in rendering compared to all previous state of the art methods (Section 7) which aim to redistribute the error as blue noise. Notably, this is achieved while using the exact same rendering data as previous methods. These results include but are not limited to: suppressing fireflies, removing visually objectionable hallucinated artifacts from a denoiser, producing images closer to the ground truth after convolution, achieving a high quality blue noise error distribution in animations even in the presence of large temporal discontinuities.

Following an overview of relevant prior work (Section 2), we present necessary background and motivation in Section 3.1. We then devise our perceptual error formulation for image synthesis (Section 3.2) and present our algorithms for solving the proposed optimization problem (Section 4). We discuss extensions to our base model in Section 6, and demonstrate the benefits on several realistic scenes in Section 7. We conclude with an outline of promising directions for future research (Section 8).

2 RELATED WORK

Our work draws from diverse research in imaging, rendering, and perception. We begin our exposition with a structured survey of relevant prior work in these fields.

2.1 Digital halftoning


Motivated by Ulichney’s [1993] approach, various structure-aware halftoning algorithms were developed in graphics [Ostromoukhov 2001; Pang et al. 2008; Chang et al. 2009]. In this work, we leverage the kernel-based model [Analoui and Allebach 1992; Pappas and Neuhoff 1999] in the context of Monte Carlo rendering [Kajiya 1986].

2.2 Quantitative error assessment in rendering

In rendering, the light transport integral estimation error ($\mathcal{L}_1$, MSE, or RMSE) is usually reported as a single value evaluated over the whole image or a set of images. Since these metrics often do not accurately reflect the visual quality, equal-time visual comparisons are also commonly reported. Various theoretical frameworks have been developed in the spatial [Niederreiter 1992; Kuipers and Niederreiter 1974] and Fourier [Singh et al. 2019] domains to understand the error reported through these metrics. Numerous variance reduction algorithms like multiple importance sampling [Veach 1998], and control variates [Loh 1995; Glasserman 2004] improve the error convergence of light transport renderings. Recently, Celarek et al. [2019] proposed error spectrum ensemble (ESE), a tool for measuring the distribution of error over frequencies. ESE reveals correlation between pixels and can be used to detect outliers, which offset the amount of error substantially.

Many denoising methods [Zwicker et al. 2015] employ these metrics to optimize over noisy images to get noise-free renderings. Even if the most advanced denoising techniques driven by such metrics can efficiently steer adaptive sampling [Chaitanya et al. 2017; Kuznetsov et al. 2018; Kaplanyan et al. 2019], they emphasize...
on the local sample density, and are incapable of indicating the optimal sample layout in screen space.

Contrary to aforementioned metrics, we develop a perceptual model for rendered images that focuses on the perceptually optimal distribution of error, i.e., with respect to the human visual system’s (HVS) modulation transfer function (MTF) [Daly 1987; Sullivan et al. 1991]. Our theoretical framework argues that the screen-space sample layout is crucial for perceptual fidelity, which the most commonly used error metrics do not capture.

2.3 Perceptual error assessment in rendering

Although substantial progress in the study of HVS is continuously made, well understood are mostly the early stages of the visual pathways from the eye optics, through the retina to the visual cortex. This limits the scope of existing computational models of the HVS that are used in imaging and computer graphics. Such models should additionally be computationally efficient and generalize over simplistic stimuli that have been used in their derivation through psychophysical experiments.

**Contrast sensitivity function.** The contrast sensitivity function (CSF) is one of the core HVS models that fulfills those conditions and comprehensively characterizes overall optical [Westheimer 1986; Deele et al. 1991] and neural processes [Souza et al. 2011] in detecting contrast visibility as a function of spatial frequency. While originally it is modeled as a band-pass filter [Barten 1999; Daly 1993], its shape changes towards a low-pass filter with retinal eccentricity [Robson and Graham 1981; Peli et al. 1991] and reduced luminance adaptation in scotopic and mesopic levels [Wuerger et al. 2020]. Low-pass characteristics are also inherent for chromatic CSFs [Mullen 1985; Wuerger et al. 2020; Bolin and Meyer 1998]. In many practical imaging applications, e.g., JPEG compression [Rashid et al. 2005], rendering [Ramasubramanian et al. 1999], or halftoning [Pappas and Neuhoff 1999], the CSF is modeled as a low-pass filter, which also allows for a better control of image intensity. By normalizing such a CSF by the maximum contrast sensitivity value, a unitless function akin to the MTF can be derived [Daly 1987; Mannos and Sakrison 1974; Mantiu et al. 2005; Sullivan et al. 1991; Souza et al. 2011] that after transforming from frequency to spatial domain results in the Point Spread Function (PSF) [Analoui and Allebach 1992; Pappas and Neuhoff 1999]. Following Pappas and Neuhoff [1999] we approximate such a PSF by a suitable Gaussian filter, where the error of such approximation becomes practically negligible for sample density of 300 dpi and the viewer distance to the screen beyond 60 cm.

**Advanced quality metrics.** More costly, and often less robust, modeling of the HVS beyond the CSF is performed at advanced quality metrics [Lubin 1995; Daly 1993; Mantiu et al. 2011] that have been adapted to rendering where they guide the computation to the image regions where the visual error is most perceived [Bolin and Meyer 1995, 1998; Ramasubramanian et al. 1999; Ferwerda et al. 1996; Myszkowski 1998; Volevich et al. 2000]. An important application is visible noise reduction in ray and path tracing by content adaptive sample density control [Bolin and Meyer 1995, 1998; Ramasubramanian et al. 1999]. Our framework enables further significant reduction of the noise visibility for the same sample budget.

**Sampling in rendering.** Sample correlations [Singh et al. 2019] directly affect the error in rendering. Quasi-Monte Carlo samplers [Halton 1964; Sobol 1967] preserve correlations (i.e., stratification) well in higher dimensions, which makes them a perfect candidate for rendering problems [Keller 2013]. However, imposing perceptual control over these samplers is not well studied. On the other hand, stochastic samplers are shown to have direct resemblance to the cone layout in the human eye [Yellott 1983]. This has inspired the development of various stochastic sample correlations in rendering [Cook 1986; Dippé and Wold 1985; Mitchell 1991], e.g., blue noise. In this work, we do not focus on construction of blue-noise point sets, but develop a theoretical framework to obtain perceptually pleasing distribution of error in screen space for rendering purposes.

**Blue-noise error distribution.** Mitchell [1991] observed that blue noise is a desirable property for spectrally optimal ray tracing. Georgiev and Fajardo [2016] were the first to apply results from halftoning literature to screen space-error redistribution for path tracing. The resulting perceptual quality improvements are substantial for smooth enough integrands.

Motivated by the results of Georgiev and Fajardo [2016], Heitz and Belcour [2019] devised a technique aiming to directly optimize the error distribution instead of operating on sample distributions. Their pixel permutation strategy fits the initially white-noise pixel intensities to a prescribed blue-noise mask. This approach scales well with sample count and dimensionality, though the error distribution quality is limited by the fitting to a specific mask and degrades to white noise near geometry discontinuities, unlike the methods of Georgiev and Fajardo [2016] and Heitz et al. [2019].

We propose a perceptual error framework based on an expressive model that unifies these prior methods, providing insight into their (implicit) assumptions and guidelines to alleviate some of their drawbacks. Our general perceptual error formulation does not rely on a target (blue-noise) mask.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>(\langle \cdot , \cdot \rangle)</td>
<td>inner product, element-wise product, convolution</td>
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<td>(</td>
<td>\cdot</td>
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<tr>
<td>(g, \hat{g}, g^2)</td>
<td>convolution kernel, its Fourier and power spectra</td>
</tr>
<tr>
<td>(I, Q, e = Q - I)</td>
<td>reference image, estimated image, error image</td>
</tr>
<tr>
<td>(S_i, S)</td>
<td>sample set for pixel (i), sample sets for all pixels</td>
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3 PERCEPTUAL ERROR MODEL

Our goal is to produce Monte Carlo renderings that, at a fixed sampling rate, are perceptually as close to the ground truth as possible. This requires formalizing the perceptual image error along with an optimization problem that minimizes it. In this section, we build a perceptual model upon the extensive studies done in halftoning.
that approximates the HVS’s modulation transfer function with a kernel $g_{\text{MTF}}$ [Daly 1987]. Analoui and Allebach [1992] used a similar approach is to take the error image distribution with a Gaussian kernel. Increasing the kernel’s standard deviation corresponds to lower perceptual sensitivity to error, even though $e_\text{w}$ has the same amplitude as it is obtained by randomly permuting the pixels of $e_\text{b}$.

3.1 Motivation

Several metrics have been proposed in halftoning literature to capture the human perception of image error. Such metrics model the processing done by the HVS as a convolution of the error image $e$ with a kernel $g$ (see notation in Table 1):

$$E = |g * e|^2 = |\hat{g} \odot \hat{e}|^2 = \langle \hat{g}^2, |\hat{e}|^2 \rangle.$$  

The convolution is equivalent to the element-wise product of the corresponding Fourier spectra $\hat{g}$ and $\hat{e}$, whose 2-norm is in turn equal to the inner product of the power spectra images $|\hat{g}|^2$ and $|\hat{e}|^2$. Sullivan et al. [1991] minimized the error (2) w.r.t. a kernel $g$ that approximates the HVS’s modulation transfer function $g_{\text{MTF}}$ [Daly 1987]. Analoui and Allebach [1992] used a similar model in the spatial domain where the kernel approximates the PSF of the human eye.

Convolving the error image with a kernel incorporates both the magnitude and the distribution of the error into the resulting metric $E$. In general, the kernel $g$ can have arbitrary form and characteristics; we assume it represents the HVS PSF. As we discuss in Section 2.3, the HVS sensitivity to a spatial signal can be well approximated by a low-pass filter. Optimizing the error image $e$ to minimize the cost (2) w.r.t. a low-pass kernel would then naturally yield a blue-noise\(^1\) error distribution (see Fig. 2). Consequently, such a distribution can be seen as a byproduct of such optimization, which pushes the spectral components of the error to frequencies least visible to the human eye. To better understand the spatial aspects of contrast sensitivity in the HVS, the MTF (magnitude of the PSF Fourier transform) is usually modeled over a range of viewing distances [Daly 1993]. This is done to account that with increasing viewer distance, spatial frequencies in the image are projected into higher spatial frequencies in the retina. These eventually become invisible, filtered out by the PSF that expands its corresponding kernel in image space. We recreate this experiment to understand the impact of distance over the nature of the error distribution (blue vs. white noise). In Fig. 3, we convolve white- and blue-noise distributions with a Gaussian kernel. Increasing the kernel’s standard deviation corresponds to increasing the distance between observer and screen. The blue-noise distribution reaches a homogeneous state (where the image tone is indiscernible) faster compared to white noise. In the context of rendering, this is equivalent to blue-noise error becoming indiscernible at smaller distances (or smaller denoising low-pass filters) compared to white-noise error. Consequently, a smaller kernel (i.e., viewing distance) allows preserving more image details while removing the noise, for an optimized error distribution.

3.2 Our model

In rendering, the value of each pixel $i$ is estimated via point sampling. Its signed error is thus a function of the sample set $S_i$ used for its estimation: $e_i(S_i) = Q_i(S_i) - I_i$, where $Q_i$ is the pixel estimate and $I_i$ is the reference (i.e., ground-truth) pixel value. The error of the entire image can be written in matrix form, similarly to Eq. (1):

$$e(S) = Q(S) - I.$$  

where $S = \{S_1, \ldots, S_N\}$ is an “image” containing the sample set for each of the $N$ pixels. With these definitions, we can express the perceptual error in Eq. (2) for the case of Monte Carlo rendering as a function of the sample-set image $S$:

$$E(S) = \|g * e(S)\|^2_2,$$  

where $g$ is a given kernel, e.g., the PSF of the HVS.

Our goal is to minimize the energy in Eq. (4). We formulate it as an optimization problem:

\(^1\)The term “blue noise” is often used loosely to refer to any spectrum with minimal low-frequency content and no concentrated energy spikes.
The solution $\mathbf{S}$ produces an image estimate $\mathbf{Q}(\mathbf{S})$ that is closest to the reference $\mathbf{I}$ w.r.t. the kernel $\mathbf{g}$. The search space $\mathbf{\Omega}$ is the set of all possible locations for every sample of every pixel.

Note that the classical MSE metric corresponds to using a zero-width (i.e., one-pixel) kernel $\mathbf{g}$ in Eq. (4). However, the MSE accounts only for the magnitude of the error $\mathbf{e}$, while using wider kernels (such as the PSF) accounts for both magnitude and distribution. Consequently, while the MSE can be minimized by optimizing pixels independently, minimizing the perceptual error requires spatial coordination. In the following section, we devise practical strategies for solving this optimization problem.

4 DISCRETE OPTIMIZATION

The search space in Eq. (5) for each sample set of every pixel is a high-dimensional unit hypercube. Each point in this so-called primary sample space maps to a light transport path in the scene [Pharr et al. 2016]. Optimizing for the sample-set image $\mathbf{S}$ thus entails evaluating the contributions $\mathbf{Q}(\mathbf{S})$ of all corresponding paths. This evaluation is costly, and for any non-trivial scene, $\mathbf{Q}$ is an unpredictable function with many discontinuities. This precludes us from studying all (uncountably infinite) sample locations in practice.

To make the problem tractable, we restrict the search in each pixel to a finite number of (pre-defined) sample sets.¹ We devise two variants of the resulting discrete optimization problem, which differ in their definition of the search space $\mathbf{\Omega}$. In the first variant, each pixel has a separate list of sample sets to choose from (“vertical” search space). The setting is similar to that of (multi-tone) halftoning [Lau and Arce 2007], which allows us to import classical optimization techniques from that field, such as mask-based dithering, error diffusion, and iterative minimization. In the second variant, each pixel has one associated sample set, and the search space comprises permutations of these assignments (“horizontal” search space). We develop a greedy iterative optimization method for this second variant.

In contrast to halftoning, in our setting the ground-truth image $\mathbf{I}$ required to compute the error $\mathbf{e}$ during optimization, is not readily available. We describe our algorithms assuming the ground truth is available and then discuss how to substitute it with a surrogate to make the algorithms practical.

4.1 Vertical search space

Our first variant considers a vertical search space where the sample set for each of the $N$ image pixels is one of $M$ given sets:

$$\mathbf{\Omega} = \{\mathbf{S} = \{S_1, \ldots, S_N\} : S_i \in \{S_{i,1}, \ldots, S_{i,M}\}\}. \quad (6)$$

The objective is to find a sample set for every pixel such that all resulting pixel estimates together minimize the perceptual error (4). This is equivalent to directly optimizing over the $M$ possible estimates $Q_{i,1}, \ldots, Q_{i,M}$ for each pixel $i$, with $Q_{i,j} = Q_i(S_{i,j})$. These estimates can be obtained by, e.g., rendering a stack of $M$ images $\mathbf{Q}_j = \{Q_{1,j}, \ldots, Q_{N,j}\}$, with $j = 1..M$. The resulting minimization problem reads

$$\arg \min_{\mathbf{\Omega}} |\mathbf{g} \ast (\mathbf{Q} - \mathbf{I})|^2. \quad (7)$$

The problem in Eq. (7) is almost identical to that in multi-tone halftoning. The only difference is that in our setting the “quantization levels”, i.e., the pixel estimates, are non-uniform and differ from pixel to pixel as they are not pre-defined but are the result of point-sampling a spatially varying light-transport integral. This similarity allows us to directly apply existing optimization techniques from halftoning [Lau and Arce 2007]. We consider three such methods, which we outline in Alg. 1 and describe next.

Iterative minimization. Some halftoning methods aim to solve the problem (7) directly via iterative greedy minimization [Analoui and Allebach 1992; Pappas and Neuhoff 1999]. After initializing each pixel to a random quantization level, our implementation traverses the image in serpentine order (as is standard practice in halftoning) and for each pixel picks the level that minimizes the energy. Several full-image iterations are performed; in our experiments convergence to a local minimum is achieved within 10–20 iterations. As a further improvement, the optimization can be terminated when the perceptual error reduction rate falls below a certain threshold or when no estimates are updated within one full iteration. Random pixel traversal order allows terminating at any point but converges slightly slower.

Error diffusion. A classical halftoning algorithm, error diffusion scans the image pixel by pixel, snapping each to the closest quantization level and distributing the resulting error to yet unprocessed nearby pixels according to a given diffusion kernel $\mathbf{k}$ [Floyd and Steinberg 1976]. In our setting, this can be interpreted as using a kernel (x) that is learned over a specific class of functions, such that the result of error diffusion approximately implies minimizing Eq. (7) [Hoeve and Niger 2008]. For color images, the Euclidean distance in a relevant color space can be used to find the closest quantization level.

Dither masks. An efficient halftoning approach is to quantize pixel values using thresholds stored in a pre-computed dither mask (or matrix) [Spaulding et al. 1997]. For each pixel, the closest lower and higher (in terms of brightness) quantization levels to the reference value are found, and one of the two is chosen based on the threshold associated with the pixel. This aims to transfer the spectral characteristics of the mask to the error distribution of the dithered image. In a blue-noise mask, neighboring pixels have very different thresholds, leading to a visually pleasant high-frequency output error distribution. Following Eq. (7), here the minimization involves two parts. First, the pre-processing step (performed offline) that involves minimizing arg min $|\mathbf{g} \ast \mathbf{B}|^2$ to obtain a mask $\mathbf{B}$ using $\mathbf{g}$, which is equivalent to the problem in Eq. (7) with a linear integrand and $\mathbf{I} = 0$. Secondly, the simplified energy $|\mathbf{O} - \mathbf{I} - f(\mathbf{B})|$ is minimized, which can be done efficiently through sorting. Here $f$
Algorithm 1. Three algorithms to (approximately) solve the problem associated with a vertical search space (7), producing an output image \( O = \{ O_1, \ldots, O_N \} \) given a reference image \( I \) and a stack of initial estimates \( Q_1, \ldots, Q_M \). Our iterative minimization scheme updates pixels repeatedly, for each picking the estimate that minimizes the perceptual error (4). Error diffusion quantizes each pixel to the closest estimate, distributing the error to its neighbours based on a kernel \( k \). Dithering quantizes each pixel in \( I \) based on thresholds in a dither mask \( B \) (optimized w.r.t. the kernel \( g \)).

\[
\begin{align*}
\text{Algorithm 1. Three algorithms to solve the problem.} \\
1: & \text{function IterativeMinimization}(g, I, Q_1, \ldots, Q_M, O, T) \\
2: & O ← \{ Q_{1,\text{rand}}, \ldots, Q_{N,\text{rand}} \} \quad \text{← Init each pixel to random estimate} \\
3: & \text{for } T \text{ iterations do} \\
4: & \quad \text{for pixel } i = 1..N \text{ do} \\
5: & \quad \quad \text{for estimate } Q_{i,j} ∈ \{ Q_{1,\ldots,M} \} \text{ do} \\
6: & \quad \quad \quad \text{if } O_i = Q_{i,j} \text{ reduces } |g * (O - I)|^2 \text{ then} \\
7: & \quad \quad \quad \quad O_i ← Q_{i,j} \quad \text{← Update estimate} \\
8: & \quad \text{end} \\
9: & \text{end} \\
10: & \text{end} \\
11: & \text{end} \\
\end{align*}
\]

Algorithm 2. Given a convolution kernel \( g \), a reference image \( I \), an initial pixel sample-set assignment \( S \), and an image estimated with that assignment \( Q(S) \), our greedy sample relocation algorithm iteratively swaps pixel assignment to minimize the perceptual error \( E_\Delta(10) \), producing a permutation \( \pi \) of the sample-set assignment.

\[
\begin{align*}
\text{Algorithm 2. Relocation algorithm.} \\
1: & \text{function IterativeMinimization}(g, I, S, Q(S), T, R, \pi) \\
2: & \pi ← \text{identity permutation} \quad \text{← Initialize solution permutation} \\
3: & \text{for } T \text{ iterations do} \\
4: & \quad \text{for pixel } i = 1..N \text{ do} \\
5: & \quad \quad \pi' ← \text{Find best pixel in neighborhood to swap with} \\
6: & \quad \quad \text{if } E_\Delta(\pi_{i_{\text{new}}}(S)) < E_\Delta(\pi_i(S)) \text{ then} \\
7: & \quad \quad \quad \pi_i ← \pi_{i_{\text{new}}} \quad \text{← Eq. (10)} \\
8: & \quad \quad \text{π′ ← π′} \\
9: & \quad \text{End for} \\
10: & \text{End for} \\
11: & \text{End for} \\
\end{align*}
\]

We can explore the permutation space \( \Pi(S) \) by swapping the sample-set assignments between pixels. The minimization requires updating the image estimate \( Q(\pi(S)) \) for each permutation \( \pi(S) \), i.e., after each swap. Such updates are costly as they involve multiple ray-tracing operations for each of potentially millions of swaps. We need to eliminate these extra rendering invocations during the optimization to make it practical. To that end, we observe that for pixels solving similar light-transport integrals, swapping their sample sets gives a similar result to swapping their estimates. We therefore restrict the search space to permutations that can be generated through swaps between such pixels. This enables efficient optimization by directly swapping the pixel estimates of an initial rendering \( Q(S) \).

Error decomposition. Formally, we express the estimate produced by a sample-set permutation in terms of permuting the pixels of the initial rendering: \( Q(\pi(S)) = \pi(Q(S)) + \Delta \). The error \( \Delta \) is zero when the swapped pixels solve the same integral(s). Substituting into Eq. (9), we can approximate the perceptual error by (see derivation in Appendix A)

\[
E(\pi(S)) = |g * (\pi(Q(S))) - I + \Delta)|^2 \approx |g * (\pi(Q(S))) - I|^2 + |g|^2 \sum_i d(i, \pi(i)) = E_\Delta(\pi(S)).
\]

In the approximation \( E_\Delta \), the term \( d(i, \pi(i)) \) measures the dissimilarity between pixel \( i \) and the pixel \( \pi(i) \) it is relocated to by the permutation. The purpose of this metric is to predict how different we expect the result of re-estimating the pixels after relocating their sample sets to be compared to directly relocating their initial estimates. It can be constructed based on assumptions or knowledge about the image, e.g., coming from auxiliary buffers (depth, normals, etc).

Local similarity assumption. In our implementation we use a simple binary dissimilarity function returning zero when \( i \) and \( \pi(i) \) are within some distance \( r \) and infinity otherwise. We set \( r \) between 1 and 3; it should ideally be locally adapted to the image smoothness/regularity. This allows us to restrict the search space \( \Pi(S) \) only to permutations that swap adjacent pixels where it is more likely that \( \Delta \) is small.
Iterative minimization. We devise a greedy iterative minimization scheme for this variant similar in spirit to the iterative minimization in Alg. 1. Given an initial image estimate $Q(S)$ produced by randomly assigning a sample set to every pixel, our algorithm iterates over all pixels and for each considers swaps within a $(2R+1)^2$ neighborhood; we use $R=1$. The swap that brings the largest reduction in the perceptual error $E_P$ is accepted. Several full-image iterations are performed. Algorithm 2 provides pseudo code. We use $T = 10$ iterations in our experiments. The algorithm could be terminated based on the perceptual error or swap reduction rate. Additionally, the algorithm can be sped up considerably by using optimizations applicable to our energy (see supplemental Section 5).

The parameter $R$ controls the trade-off between the cost of one iteration and the amount of exploration it can do. Note that this parameter is different from the maximal relocation distance $r$ in the dissimilarity metric, and also $R \leq r$.

Due to the pixel (dis)similarity assumptions, the optimization can produce some mispredictions, i.e., it may swap the estimates of pixels for which swapping the sample sets produces a significantly different result. Thus $\pi(Q(S))$ cannot be used as a final image estimate. We therefore re-render the image with the optimized permutation $\pi$ to obtain the final estimate $Q(\pi(S))$.

4.3 Discussion

Practical algorithms can be classified based on the choice of search space (horizontal versus vertical) and optimization strategy (iterative, error diffusion, dithering). The above choices affect the overall quality and speed of algorithms.

Search space. Discretizing the search space $\Omega$ makes the optimization problem (5) tractable. To make it truly practical, it is necessary to avoid repeated image estimation (i.e., $Q(S)$ evaluation) during the search for the solution $S$. Our vertical (7) and horizontal (9) optimization variants are formulated specifically with this goal in mind. All methods in Algs. 1 and 2 operate on pre-generated image estimates that constitute the solution search space.

Our vertical formulation takes a set of $M$ input estimates $\{Q_{i,j} = Q_i(S_{i,j})\}_{j=1}^M$ for every pixel $i$, one for each sample set $S_{i,j}$. Noting that $Q_{i,j}$ are MC estimates of the true pixel value, this set can be cheaply expanded to a size as large as $2^M - 1$ by also considering the average of the estimates in each of its subsets (excluding the empty subset). In practice only a fraction of these subsets can be used, since the size of the power set grows exponentially with $M$.

In contrast, our horizontal formulation builds a search space given just a single input estimate $Q_i$ per pixel $i$. We consciously restrict the space to permutations between nearby pixels, so as to leverage local pixel similarity and avoid repeated pixel evaluation. The disadvantage of this approach is that it requires re-rendering the image after optimization, with unpredictable results (mispredictions) that can lead to local degradation of image quality. Additionally, while some halftoning methods (iterative energy minimization and versions of dither matrix halftoning) can be adapted to this search space, it is non-trivial to find straightforward reformulations for other algorithms (e.g., error diffusion).

It is also possible to combine our two formulations into a hybrid one that takes one input estimate per pixel but considers a separate search space for each pixel, constructed by borrowing neighboring estimates.

Optimization strategy. Another important design decision is the choice of optimization method. For the vertical formulation, iterative methods provide the most flexibility, control, and quality, but are most computationally expensive. In contrast, error diffusion and dither-mask halftoning can be seen as only approximately solving the optimization problem (7), yielding results of a lower quality. An important difference between classical halftoning and our extension is the fact that quantization levels differ between pixels. This makes the gap in quality between image-adaptive (iterative, error diffusion) and non-adaptive (dither-mask halftoning) methods even larger. Additionally, in the classical MC rendering setting, iterative and error-diffusion methods handle color natively, while dither-mask halftoning requires grayscale input. While this can be mitigated by modifying the renderer or by extending the dithering through algorithms such as bipartite Euclidean matching, this comes at both a performance and quality cost compared to iterative and error diffusion algorithms. The main advantage of dither-mask halftoning over error diffusion in our setting is that the former involves the kernel $g$ explicitly, while error diffusion relies on a diffusion kernel $k$ that cannot be related directly to $g$.

Surrogate for reference image. All our formulations depend on the reference image $I$ which is unknown, unlike in halftoning. To that end, the reference can be substituted with an approximation, i.e., surrogate image, $I'$, which can be obtained, e.g., via (ideally feature-preserving) filtering of an estimate $Q$. We will discuss surrogates in more detail in Sections 7.1 and 7.3, but it is important to point out that all existing methods rely on a surrogate whether explicitly or implicitly.

5 A-PERIORI & A-PRIORI METHODS

In our framework, the histogram sampling approach of Heitz and Belcour [2019] classifies as a vertical method, and their permutation approach as a horizontal method. Both these methods employ variants of dither-mask halftoning. We next show that in both cases, their algorithms rely on implicitly constructed surrogates.

5.1 Implicit surrogates in a-posteriori approaches

All existing error distribution methods rely on an implicit surrogate. For the histogram method of Heitz and Belcour [2019], since the mean value of the blue-noise mask maps to the median of the estimates, the implicit surrogate is the median of the sorted estimates in each pixel. Furthermore, the histogram method can pick any of the estimates within a pixel—unlike classical halftoning that samples the closest lower and higher quantization level [Spaulding et al. 1997]—which results in a higher error amplitude. Consequently, pixels for which the mask value deviates strongly from the mean value could end up with outliers leading to fireflies in the rendered image, even if those were not present in the averaged image (Fig. 7).

The permutation method of Heitz and Belcour [2019], on the other hand, has a piecewise constant implicit surrogate. We show this formally in the supplemental document (supplemental, Section 7). In words, the permutation method uses a sorting pass which minimizes
the $L_2$ distance between the pixel values and the dither mask within each tile. This transfers the blue-noise characteristics of the dither mask to the signal. However, the actual goal is to transfer these spectral characteristics to the error, not the signal. For the two to be equivalent, the signal within each tile needs to be assumed constant. And indeed, a constant offset does not change the minimizer of the energy which sorting minimizes. This implies that the implicit surrogate in this case is a piecewise constant function within each tile. Our framework, however, uses an explicit surrogate that allows full control over the quality of error distribution.

### 5.2 A-priori methods

By making simplifying assumptions about $I$ and $Q$, our sample relocation algorithm could be used with no actual knowledge about the image. For example, assuming that every pixel solves the same integral turns $I$ into a constant image and makes the error term $\Delta$ in Eq. (10) vanish. This allows us to swap any two pixels and simplifies the perceptual error to $E(\pi(S)) = |g \ast \pi(e(S))|_2^2$. Further assuming a simple analytic shape for the pixel integrand, the per-pixel error image $e(S)$ can be quickly rendered once and the perceptual optimization performed by simply swapping its pixels. This is the case in all examples with a direct swap. In this formulation, prior methods correspond to using a (family of) step function(s) [Heitz et al. 2019] or a class of functions mapping far away samples to far away values [Georgiev and Fajardo 2016] for the integrand, respectively. This approach, coined “a-priori” by Heitz and Belcour [2019], is practical but its output fidelity is limited by lack of adaptivity resulting from its strong assumptions.

Our framework unifies the a-priori and a-posteriori approaches, showing that the two lie on a continuum (supplemental Section 2). In our framework, the “a-priori” method of Heitz et al. [2019] tries to simultaneously minimize for a large number of simple integrands while the search space is restricted to the ranking and scrambling keys of a Sobol sequence. This roughly corresponds to a vertical and horizontal search space, respectively. At the same time, we note that the energy that is used is empirically motivated, but it can be made to agree with our perceptually motivated energy by simply absorbing the Gaussian convolution within the $L_2$ norm.

Similarly, the method of Georgiev and Fajardo [2016] consisting of toroidally shifting a sequence by a mask, can also be cast in our framework provided that the mask is optimized w.r.t. our energy with an appropriate set of integrands. Even if motivated purely empirically, the energy in the paper can be related to our energy if a whole class of integrands is considered where faraway samples are mapped into faraway values. We also note that the toroidal shift degrades the quality of the optimized mask and consequently the quality of the blue noise (see supplemental Section 8).

### 6 EXTENSIONS

Our perceptual error formulation (4) considers some image distortions due to the CSF but not all. The HVS applies additional, localized and non-linear processing to the input signal. We analyze different aspects of the model and devise three extensions that capture more effects.

**Kernels and PSFs.** In Fig. 5a, we compare different PSF models (i.e., low-pass kernels) from halftoning literature [Näsänen 1984; González et al. 2006]. In our experiments, we use a binomial kernel that approximates a Gaussian kernel [Papoulis and Pillai 2002]; it is cheap to evaluate and performs on par with these state-of-the-art PSFs.

We further analyze kernels with band-stop, high-pass, band-pass, and anisotropic spectral characteristics in Fig. 4. Starting from a white-noise error distribution (i.e., with a uniform random value in $[-0.5, 0.5]$ assigned to each pixel), our horizontal iterative minimization algorithm is able to optimize the shape of the noise to produce the inverse behaviour to the kernel in the spectral domain. The rightmost image in Fig. 4 illustrates the result from using such a spatially varying kernel produced from the convex combination of a binomial and an anisotropic high-pass kernel, with the interpolation parameter varying horizontally across the image. Our algorithm can well adapt the noise shape based on the kernel.
Tone reproduction. Considering that the rendered image will be viewed on a medium with limited dynamic range (e.g., screen or paper), we can incorporate a tone mapping operator $T$ into the perceptual error:

$$E(S) = |g \ast (T(Q(S)) - T(I))|_2^2.$$  \hspace{1cm} (11)

Doing this also bounds the per-pixel error: $e(S) = T(Q(S)) - T(I)$, suppressing pixel outliers and making the optimization more robust in scenes with high dynamic range.

Chromatic noise. While the HVS reacts more strongly to luminance compared to color, ignoring chromaticity entirely can have a negative effect on the distribution of color noise in the image. To mitigate this, one can penalize each color channel $c$ separately:

$$E(S) = \sum_{c} \lambda_c |g_c \ast (Q_c(S) - I_c)|_2^2. \hspace{1cm} (12)$$

where $\lambda_c$ is a weight parameter. In our experiments (Section 7), we set $\lambda_c = 1$ and use the same kernel $g_c$ for every channel. It is, however, possible to consider a more appropriate color space than RGB (e.g., YCbCr). In such case, per-channel kernels may differ [Sullivan et al. 1991].

Observer-screen distance. Being based on the perceptual models of the HVS [Sullivan et al. 1991; Analoui and Allebach 1992], our formulation (4) assumes that the estimate $\hat{Q}$ and the reference $I$ are viewed from the same (range of) distance(s). The viewing distances can be decoupled by applying different kernels to $\hat{Q}$ and $I$:

$$E(S) = |g \ast \hat{Q}(S) - h \ast I|_2^2. \hspace{1cm} (13)$$

Minimizing this error makes $\hat{Q}$ appear, from some distance $d_h$, similar to $I$ seen from a different distance $d_h$. The special case of using a Kronecker delta kernel $h = \delta$ (i.e., with the reference $I$ seen from up close) yields $E(S) = |g \ast \hat{Q}(S) - I|_2^2$, which has been shown to have an edge enhancing effect in halftoning [Anastassiou 1989; Pappas and Neuhoff 1999]. We demonstrate this effect in Fig. 5b. We use $h = \delta$ for all our experiments in Section 7.

7 EXPERIMENTS

In this section, we detail our testing methodology and present empirical results employing our proof-of-concept algorithms. We render several scenes using our iterative energy minimization, error diffusion, and dither matrix halftoning approaches and compare against the histogram and permutation methods of Heitz and Belcour [2019]. The quantitative analysis is performed in Table 2 over all the methods, and the runtime for the optimization can be found in Table 3.

7.1 Setup

Energy formulation. We use the perceptual model $\|g \ast \hat{Q}(S) - h \ast I\|_2^2$ with $h = \delta$ from Eq. (13). In our implementation, we tone map (Eq. (11)) the images to match the display and consider all (RGB) channels (Eq. (12)) for both iterative and error diffusion methods. The dithering approach, however, requires solving a bipartite Euclidean matching problem to take into account different channels, thus we consider it only in the greyscale setting, since this allows using sorting for the optimization.

Kernel $g$. In all our experiments (except where explicitly mentioned), we use a $3 \times 3$ binomial kernel $g$ which approximates a Gaussian kernel [Lindeberg 1990] with standard deviation $\sigma = 1/\sqrt{2}$. More sophisticated kernels may be used if desired, however, we found that our Gaussian approximation gives satisfactory results (Fig. 5a) in all experiments. The optimal viewing distance corresponding to our kernel is approximately 90cm for a screen with 218 DPI (136 pixels per visual degree).

Rendering setup. All scenes are rendered in PBRTv3 [Pharr et al. 2016] using the path integrator and, occasionally, the bidirectional path tracing integrator. Our methods do not depend on the dimensionality of the problem, however, in order to keep the rendering time reasonable we set the maximum path depth to 5 for all scenes. The reference images are generated using a Sobol sampler with sample counts greater than 1024 samples per pixel. All other images are rendered using a random sampler. In all scenes (except in the animations), we shoot primary rays through the center of the pixel in order to separate the geometry integration from the light transport integration. We also provide animation results in the supplementary data on the following scenes: Utah Teapot (360 frames), Modern Hall (120 frames), San Miguel (120 frames), Bathroom (120 frames).

Surrogate construction. To construct a surrogate for our methods, we denoise the averaged image with the Intel Open Image Denoiser [int 2018]. The denoiser’s input is the averaged image, a normal buffer, and a buffer containing the base color of materials where applicable. Heitz and Belcour’s methods use implicit surrogates. We analyze the effect of various surrogates in Fig. 10, and showcase the surrogates generated from the Intel Denoiser in Fig. 11.

Perceptual error metrics. We also evaluate the quality of our results using perceptual metrics like S-CIELAB [Zhang and Wandell 1997] (see supplemental) and HDR-VDP-2 [Mantiuk et al. 2011], with parameters matching our kernel. Among these, HDR-VDP-2 quantifies best the considerable visual improvement due to a-posteriori methods. We also include a metric, dubbed perceptual MSE (pMSE). It is derived by normalizing our perceptual error Eq. (4) by the pixel count multiplied by the number of channels (with $h = \delta$). Additionally, to analyze the local blue-noise quality we provide tiled Fourier power spectra (Fig. 1, see supplemental results) of the signed error $\hat{Q}(\pi(S)) - I$ in order to showcase the quality achieved for the distribution of the error as blue noise. The image is split into tiles of size $32 \times 32$ and the power spectrum is computed for each tile. For visualization purposes, a logarithmic transform is applied within each tile and the resulting values are normalized so that the highest value is $(1, 1, 1)$. Note that the power spectrum is computed for the error w.r.t. the reference image $I$ and not the surrogate, which quantifies the blue noise distribution objectively.

7.2 Rendering comparisons

Vertical approaches. We compare all three different class of vertical methods (Alg. 1): dithering-based, error diffusion using Floyd-Steinberg kernel $k$ [Floyd and Steinberg 1976] and the iterative minimization approach using our kernel $g$.

We first compare our iterative minimization approach against the histogram sampling method [Heitz and Belcour 2019] in Fig. 6.
The histogram sampling approach performs suboptimally compared to our iterative minimization approach even when it is provided with 16 times more samples. This goes to illustrate the fact that it is neither theoretically nor practically optimal, mainly because its dithering and implicit surrogate are suboptimal (see Section 5.1, and supplemental Section 7). In Fig. 7, we perform an equal sample comparison for all vertical methods. All our vertical methods consistently outperform the histogram sampling method. This also seems to match the probability of detection map from HDR-VDP-2 shown in the bottom row of the figure.

In Fig. 1, we illustrate that denoising artifacts can be removed through our optimization. A white noise image denoised with the Open Intel Image Denoiser results in objectionable artifacts, while performing our optimization subsequently and then denoising, removes those artifacts entirely. In Fig. 1, the denoised version of the averaged image results in conspicuous unpleasant artifacts. Using this image as a surrogate for our method, the optimization finds a subset of unbiased estimates that match the surrogate well in terms of pMSE. Clearly the new image even if constructed from previously unbiased estimates is not fully unbiased, however, the introduced bias allows for more regularity. This regularity pays off since the subsequent denoising does not produce the artifacts which were inherent to the denoised results using the averaged image (where the error distribution was white noise).

Overall, vertical approaches generally rank in the following order of increasing visual quality: histogram sampling, our dithering, our error diffusion, our iterative minimization with a stack of 4 estimates, and our iterative minimization with the powerset of the estimates (Table 2). For more details and full-size image comparisons we refer the reader to our supplementary HTML material.

Horizontal approaches. We consider two horizontal approaches: our iterative energy minimization approach (Alg. 2) and Heitz and Belcour’s permutation approach. For our relocation method, we use a search radius $R = 1$ (Alg. 2) and we allow pixels to travel within a disk neighbourhood centered around the initial pixel location. We set the disk radius $r$ to 1 pixel (Alg. 2), which approximately corresponds to tile size 3 for Heitz and Belcour’s permutation method. For the permutation approach we consider tiles sizes 2 and 8.

Comparisons on the Modern Hall, White Room, and the Bathroom scenes can be found in Fig. 8. The scenes are rendered over 16 frames, and retargeting is applied for Heitz and Belcour’s approach. The scenes are intentionally kept static in order to achieve optimal results for [Heitz and Belcour 2019], nevertheless, the permutation approach’s results remain substantially closer to white noise in distribution. For non-static animation results we refer the reader to the supplementary videos where we have provided several animations. We have tested the permutation approach at tile sizes 2 and 8 and we always include the better image. In all cases the pMSE is 20 to 40 percent lower for our approach compared to the permutation approach (see Table 2), which also agrees with our perceptual evaluation of the images. The error in our images is a lot closer in distribution to high quality blue noise, while in the permutation approach it has stronger white noise characteristics.

In Fig. 9, we perform an equal sample comparison of our vertical methods with Heitz and Belcour’s permutation approach on the scenes: Living Room, Classroom, and Grey Room. Our vertical methods consistently outperform the permutation approach both in terms of perceptual metrics (Table 2) and our subjective perceptual evaluation.
Perceptual Error Optimization for Monte Carlo Rendering

Fig. 7. Vertical method comparisons where for each pixel one primary estimate is picked from 4 primary estimates. We compare our three variants: dithering (c), error-diffusion (d) and iterative approach (e) with Histogram sampling method [Heitz and Belcour 2019] (f). Our iterative approach in (e) gives very high quality blue-noise error distribution, marked by the lower pMSE. Our error-diffusion based approach in (d) is computationally fast and has quality comparable to the iterative approach in (e). Our dithering approach (c) shows minor improvement but is still better than the histogram sampling method in (f). Note that most of our variants also reduce overall MSE compared to the baseline averaged image (4spp) even though they do not directly optimize for it. Bottom row shows the detection probability [Mantiuk et al. 2011] which indicates how likely it is for a human observer to notice the difference w.r.t. the reference image (blue refers to lower error) when viewed from the appropriate distance.

Timings. Optimization timings for all approaches can be found in Table 3. Here we discuss some peculiarities of the approaches which affect the runtime. Our error diffusion approach (Alg. 1) is comparable in terms of optimization runtime to dither matrix halftoning based approaches like Heitz and Belcour’s histogram sampling and permutation approaches. In the absence of parallelism, it can often outperform those, since no sorting is required. There are also parallel variants of error diffusion in the literature [Metaxas 2003]. At the same time, the quality of the approach is often comparable to the quality of our iterative energy minimization method.

The methods that provide the best quality and control are also the slowest: our iterative energy minimization approaches. For iterative vertical methods we have chosen a maximum of 100 iterations, while usually those converge in about 20. The iterative horizontal approach generally cannot fully converge in less than a hundred iterations thus we limit the iterations to $T = 10$ (see the supplemental for an insight on this difference in convergence). This limit does not affect the quality greatly as mispredictions dominate the error for horizontal methods. Both methods can be made several orders of magnitude faster if additional optimisations are implemented [Analoui and Allebach 1992; Koge et al. 2014], and we have derived
such optimizations in our supplemental (supplemental Section 5). We also note that the runtime scales linearly in the number of pixels of the image multiplied by the number of pixels of the kernel. On the other hand, it doesn’t increase based on the number of samples or the dimensionality of the integrand.

Except for the time required for optimizing the sample sets assignment, our methods rely on an explicit surrogate constructed through the Open Intel Image Denoiser in most of the considered experiments, which for $512 \times 512$ images takes about half a second. The surrogate can be constructed through other means as we discuss next.

7.3 Bias analysis
In order to achieve a blue noise error distribution, a-posteriori methods introduce varying degrees of bias. For instance, Heitz and Belcour’s approaches can be fully unbiased only if a white noise mask is used. However, the resulting error distribution, in that case, would also be white (MC) noise. Making the mask correlated (or non-random) allows for a better error distribution at the cost of introducing bias in the end result. Due to the implicit surrogate, these approaches do not allow for explicit control over the bias, other than modifying the mask to include varying degrees of randomness. Conversely, our approach relies on an explicit surrogate which allows full control over the bias.

We propose different ways to control the amount of bias by: modifying the energy (Appendix B), restricting the search space and limiting the number of iterations for iterative methods. Our iterative energy minimization approaches are able to fit very closely to a surrogate, however, if that surrogate is suboptimal this may be undesirable. In Fig. 10, we perform experiments with several surrogates and extend the energy of Alg. 1 to provide control over the bias (see Appendix B). We can observe that perceptually optimal results are achieved for differing amounts of bias to the surrogate depending on how it was constructed. Notably, with the reference the best result is achieved when maximum bias is allowed. On the other hand, our method with the piecewise tile-constant surrogate requires the amount of bias to be limited in order to remove artifacts due to the surrogate, as can be seen from the third row.
Fig. 9. Quality comparison at equal sample count between our three variants of vertical methods (1/4spp) against Heitz and Belcour’s permutation approach with retargeting (4spp). Since their permutation approach improves visual quality over a sequence of frames, we show the 16-th frame from the sequence. To get better blue-noise error distribution we consider 8 × 8 tile size for their method. Our vertical methods, however, show improvements from the first frame as shown here. Particularly, our error-diffusion method outperforms their approach in both quality and speed (see Table 2 and Table 3). Bottom row shows the baseline averaged (4spp) image.

Fig. 10. Bias analysis. We design different surrogates to analyze their impact on the error distribution. Each column uses a different surrogate. From top to bottom, we vary the bias control parameter: $C = 0$ implies perfect fit to a surrogate whereas $C = 1$ implies otherwise. This showcases that, for poor surrogates, our methods can avoid fitting too closely. We place Heitz and Belcour’s histogram method (in the middle) next to the image that approximately resembles its visual quality. All other images are rendered using one out of the 4 primary estimates per pixel using the iterative vertical method. Zoom-in recommended to see the quality difference.
We transfer a perceptual kernel-based model from halftoning and varying denoising kernels. It also provides valuable insights and introduced.

Our optimization is robust and can also be used to create artistic noise distributions with custom target Fourier power spectra, here showing the SIGGRAPH logo as an example on the right. To obtain the corresponding spatial kernel, we make the logo (grayscale, downscaled) image conjugate symmetric, which ensures that its inverse Fourier transform contains only real values. That inverse, divided by its maximum value, is our optimization kernel $g$.

Our method is also applicable for progressive and adaptive perceptually optimal rendering, which has only been explored in an a-priori context [Heitz et al. 2019]. Particularly, vertical methods can trivially handle a varying number of samples per pixel. Additionally, if samples are sequentially added to the various sample sets from which vertical methods choose, then progressive rendering only requires running the optimization every time new samples are introduced.

One promising avenue for future research is incorporating more complex perceptual effects into our error formulation (4), such as visual masking [Bolin and Meyer 1998; Ramasubramanian et al. 1999], as well as more robust metrics than the squared $L_2$ norm of a convolution (possibly including nonlinear relationships).

It is also interesting to study the various interactions between our formulation and denoising methods. Ideally, the shape of the (spatially varying) denoising kernels and the error distribution (w.r.t. these kernels) would be optimized simultaneously in a loop, to achieve better image reconstruction for a given sample budget (Fig. 1).

The discrete nature of optimization methods prevents them from exploring the entire, continuous sample space. This seems possible for a-priori methods, which could potentially optimize the samples w.r.t. a suitably chosen smooth pixel integrand, e.g., via gradient descent. The same can be done for a-posteriori methods if assumptions on the integrands are made, or approximation of the integrands are constructed based on point samples and a class of reconstruction functions. In an a-posteriori setting it would also be interesting to investigate the use of differentiable rendering [Loubet et al. 2019; Zhang et al. 2020].

Finally, we have discussed: energies, search spaces, and optimization strategies for a-priori methods as an extension of our formulation for a-posteriori methods. We have not evaluated those experimentally however. A promising direction for future research is learning a set of integrands typical for a specific class of scenes. Then those integrands can be used directly in our proposed a-priori energy to optimize sequences with desirable integration and spectral properties.

REFERENCES

Simona Daly. 1987. Subroutine for the generation of a two dimensional human visual contrast sensitivity function.
We approximate the terms involving $Q$ with a surrogate (or equivalently the quality of the surrogate). For greater control $c$ may be defined per-pixel, in which case it can be multiplied component-wise with the vector inside the norm $\|\cdot\|_2$.

\[
\sqrt{E_c(S') \approx c \| g \|_1 \| Q(S') - Q(S) \|_2^2 + (1 - c) \| g \cdot Q(S') - h \cdot I' \|_2^2.}
\]  

Our original formulation is retrieved for $c = 0$, while the initial image is enforced for $c = 1$, and the parameter can be varied in $[0,1]$ to achieve different levels of bias. The parameter $1 - c$ is a smoothness/bias parameter describing how much one trusts the surrogate (or equivalently the quality of the surrogate). For greater control $c$ may be defined per-pixel, in which case it can be multiplied component-wise with the vector inside the norm $\|\cdot\|_2$.

**A ERROR DECOMPOSITION**

For pixels solving similar light transport integrals, swapping their samples gives a similar result to swapping their estimates: $Q(Q(S)) = \pi(Q(S)) + \Delta$. The error $\Delta$ is zero when the swapped pixels solve the same integral. Substituting into Eq. (9), we can bound the perceptual error using the triangle inequality, the discrete Young convolution inequality [Hewitt and Ross 1994], and the Cauchy–Schwarz inequality:

\[
E(\pi(S)) = \| g \ast (\pi(Q(S)) - I + \Delta) \|_2^2 \tag{14a}
\]

\[
= \| g \ast (\pi(Q(S)) - I) \|_2^2 + \| g \ast \Delta \|_2^2 + 2\langle g \ast (\pi(Q(S)) - I), g \ast \Delta \rangle \tag{14b}
\]

\[
\leq \| g \ast (\pi(Q(S)) - I) \|_2^2 + \| g \|_2^2 \| \Delta \|_2^2 + 2\| g \|_2^2 \| \pi(Q(S)) - I \|_2 \| \Delta \|_2. \tag{14c}
\]

The first summand in Eq. (14b) involves pixel permutations in the readily available estimated image $Q(S)$. In the second and third summands we ideally want to use an approximation for the terms involving $\Delta$ that forgoes rendering invocations:

\[
E(\pi(S)) \approx \| g \ast (\pi(Q(S)) - I) \|_2^2 + \| g \|_2^2 \sum_i d(i, \pi(i)) = E_\Delta(\pi, S). \tag{15}
\]

We approximate the terms involving $\| \Delta \|_2$ with a dissimilarity metric $d(i,j)$ between any two pixels $i$ and $j$. The purpose of this metric is to predict how different we expect the result of swapping the sample sets of $i$ and $j$ to be, compared to only swapping their pixel values $Q_i$ and $Q_j$. It can be constructed based on assumptions or prior knowledge about the image coming from, e.g., auxiliary buffers (depth, normals, etc).

In our implementation we use a simple binary dissimilarity function returning zero when $i$ and $j$ are within distance $r$ and infinity otherwise;\(^3\) we set $r$ between 1 and 3. This allows us to restrict the search space $\Pi(S)$ only to permutations that swap adjacent pixels such that it is more likely that $\Delta$ is small. More elaborate heuristics can be devised in the future to better account for pixel dissimilarity.

**B HANDLING BIAS TOWARDS SURROGATE**

The trade-off between white noise and bias towards a surrogate can be controlled by modifying the original energy to also include a data term that penalizes large deviations from the initial unbiased white noise image $Q(S)$ for a derivation see Section 3 in the

\(^3\)The radius should ideally depend on the image smoothness/regularity, and can be locally adapted.

\[\text{supplemental):} \]
Table 2. MSE and pMSE metrics for various methods on different scenes rendered at 4 samples per pixel. All methods aim to minimize the pMSE. We note that our methods consistently outperform current state of the art [Heitz and Belcour 2019]. Notably, every method is superior to the histogram sampling approach, and all of our methods except for dithering are better that the permutation approach. The occasional better performance of horizontal methods can be explained by the fact that they always choose from 4spp estimates, while vertical methods choose from 1spp estimates (except for $2^N$ which uses estimates with 1-4spp). The scene for horizontal approaches is kept static and the metrics for the 16th frame are presented. Thus this is the ideal scenario for such methods, so metrics there may even have an unfair advantage over vertical ones, since generally the scene cannot be assumed to be static in an animation sequence.

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<th>pMSE</th>
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<td>Permutation [2019] (frame 16)</td>
<td>1.40</td>
<td>2.79</td>
<td>3.15</td>
<td>7.25</td>
<td>7.90</td>
<td>2.84</td>
<td>3.38</td>
<td>3.14</td>
<td>5.21</td>
<td>1.51</td>
<td>3.59</td>
<td>8.51</td>
</tr>
</tbody>
</table>

Table 3. Timings for the minimization of the energy for various methods on different scenes rendered at 16 samples per pixel. Note that the timings do not include the construction of the surrogate. We see that our error diffusion approach is consistently the fastest method, since it doesn’t require any sorting. Our dithering approach is comparable to Heitz and Belcour’s approaches in terms of speed, since those use a similar minimization strategy relying on sorting. Finally, our iterative minimization methods are the slowest. We note that those were not optimized, and based on [Analoui and Allebach 1992; Koge et al. 2014] a speed-up of 300× can be expected (for larger kernels and images the speed-up can be even larger). Additional optimization strategies are discussed in the supplement.

<table>
<thead>
<tr>
<th>Method</th>
<th>Bathroom</th>
<th>Classroom</th>
<th>Grey Room</th>
<th>Living Room</th>
<th>Modern Hall</th>
<th>San Miguel</th>
<th>Staircase</th>
<th>White Room</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MSE</td>
<td>pMSE</td>
<td>MSE</td>
<td>pMSE</td>
<td>MSE</td>
<td>pMSE</td>
<td>MSE</td>
<td>pMSE</td>
</tr>
<tr>
<td></td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Dither (Ours)</td>
<td>0.07</td>
<td>0.08</td>
<td>0.07</td>
<td>0.07</td>
<td>0.02</td>
<td>0.12</td>
<td>0.09</td>
<td>0.07</td>
</tr>
<tr>
<td>Error diffusion (Ours)</td>
<td>0.04</td>
<td>0.03</td>
<td>0.04</td>
<td>0.04</td>
<td>0.01</td>
<td>0.06</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>Iterative vertical (Ours)</td>
<td>18.44</td>
<td>111.41</td>
<td>12.82</td>
<td>15.26</td>
<td>5.43</td>
<td>29.09</td>
<td>15.21</td>
<td>19.45</td>
</tr>
<tr>
<td>iterative vertical $2^N$ (Ours)</td>
<td>95.09</td>
<td>404.12</td>
<td>59.69</td>
<td>83.41</td>
<td>23.93</td>
<td>137.89</td>
<td>35.39</td>
<td>102.05</td>
</tr>
<tr>
<td>Histogram [2019]</td>
<td>0.06</td>
<td>0.07</td>
<td>0.11</td>
<td>0.06</td>
<td>0.02</td>
<td>0.09</td>
<td>0.08</td>
<td>0.06</td>
</tr>
<tr>
<td>Relocation (Ours, frame 16)</td>
<td>23.04</td>
<td>21.57</td>
<td>22.00</td>
<td>30.08</td>
<td>8.48</td>
<td>36.36</td>
<td>23.78</td>
<td>22.76</td>
</tr>
<tr>
<td>Permutation [2019] (frame 16)</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.11</td>
<td>0.03</td>
<td>0.21</td>
<td>0.10</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Table 4. MSE and pMSE metrics for various methods on different scenes rendered at 16 samples per pixel. All methods aim to minimize the pMSE. We note that our methods consistently outperform current state of the art [Heitz and Belcour 2019]. Notably, every method is superior to the histogram sampling approach.

<table>
<thead>
<tr>
<th>Method</th>
<th>Bathroom</th>
<th>Classroom</th>
<th>Grey Room</th>
<th>Living Room</th>
<th>Modern Hall</th>
<th>San Miguel</th>
<th>Staircase</th>
<th>White Room</th>
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</thead>
<tbody>
<tr>
<td></td>
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<td>pMSE</td>
<td>MSE</td>
<td>pMSE</td>
<td>MSE</td>
<td>pMSE</td>
<td>MSE</td>
<td>pMSE</td>
</tr>
<tr>
<td></td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>nonoptimized 1 spp</td>
<td>14.03</td>
<td>3.13</td>
<td>3.15</td>
<td>7.92</td>
<td>7.88</td>
<td>3.02</td>
<td>3.38</td>
<td>5.62</td>
</tr>
<tr>
<td>nonoptimized 4 spp</td>
<td>4.94</td>
<td>1.47</td>
<td>1.55</td>
<td>4.89</td>
<td>3.77</td>
<td>1.04</td>
<td>1.23</td>
<td>2.18</td>
</tr>
<tr>
<td>Dither (Ours)</td>
<td>4.97</td>
<td>1.52</td>
<td>1.15</td>
<td>4.69</td>
<td>4.12</td>
<td>1.36</td>
<td>1.09</td>
<td>1.82</td>
</tr>
<tr>
<td>Error diffusion (Ours)</td>
<td>4.07</td>
<td>1.20</td>
<td>0.94</td>
<td>3.85</td>
<td>4.00</td>
<td>0.87</td>
<td>0.86</td>
<td>1.07</td>
</tr>
<tr>
<td>Iterative vertical (Ours)</td>
<td>9.03</td>
<td>1.10</td>
<td>2.03</td>
<td>3.35</td>
<td>5.17</td>
<td>0.84</td>
<td>2.30</td>
<td>0.84</td>
</tr>
</tbody>
</table>
Perceptual Error Optimization for Monte Carlo Rendering:
Supplemental document

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Additional Key Words and Phrases: path tracing, perceptual error, blue noise, sampling

1 OVERVIEW

In the current supplemental we discuss various details related to our general formulation from the main paper. We start with a description of the extension of our framework to the a-priori setting (Section 2). Then we show how the reference image substitution with a surrogate can be translated to a more general problem (Section 3), which also allows optimizing how close the result fits to the surrogate. In Section 4 we describe a way in which textures can be accounted for in our horizontal approach, so that mispredictions due to multiplicative (and additive) factors are eliminated. In Section 5 we describe ways in which the runtime of iterative energy minimization methods can be improved considerably. Notably, an expression is derived allowing the energy difference due to trial swaps to be evaluated in constant time (no scaling with image size or kernel size). In the remaining sections we analyze how current a-posteriori [Heitz and Belcour 2019] and a-priori [Georgiev and Fajardo 2016; Heitz et al. 2019] state of the art approaches can be related to our framework. Interpretations are discussed, major sources of error are identified, and the assumptions of the algorithms are made explicit.

2 OUR A-PRIORI APPROACH

We extend our theory to the a-priori setting and discuss the main factors affecting the quality. The quality of a-priori approaches is determined mainly by three factors: the energy, the search space, and the optimization strategy. We discuss each of those briefly in the following paragraphs.

Our energy. We extend the a-posteriori energy from the main paper in order to handle multiple estimators involving different integrands: \( Q_1, \ldots, Q_T \), with associated weights \( w_1, \ldots, w_T \):

\[
E(S) = \sum_{t=1}^{T} w_t \| g \ast Q_t(S) - I_t \|^2. 
\]  

(1)

In the above \( g \) would typically be a low-pass kernel (e.g., Gaussian), and \( I_t \) is the integral of the function used in the estimator \( Q_t \). Through this energy a whole set of functions can be optimized for, in order for the sequence to be more robust to different scenes and estimators, that do not fit any of the considered integrands exactly.

We note that the derived optimization in Section 5 is also applicable to the minimization of the proposed energy.

Search space. The search space plays an important role towards the qualities which the optimized sequences exhibit. A more restricted search space provides more robustness and may help avoid overfitting to the considered set of integrands.

For instance, sample sets may be generated randomly within each pixel. Then, their assignment to pixels may be optimized over the space of all possible permutations. This is the setting of horizontal methods. If additionally this assignment is done within each dimension separately it allows for an even better fit to the integrands in the energy. The scrambling keys’ search space in [Heitz et al. 2019] is a special case of the latter applied for the Sobol sequence.

Constraining the search space to points generated from low-discrepancy sequences provides further robustness and guarantees desirable integration properties of the considered sequences. Similarly to [Heitz et al. 2019], we can consider a search space of Sobol scrambling keys in order for the optimized sequence to have a low discrepancy.

Ideally, such integration properties should arise directly from the energy. However, in practice the scene integrand cannot be expected to exactly match the set of considered integrands, thus extra robustness is gained through the restriction. Additionally, optimizing for many dimensions at the same time is costly as noted in [Heitz et al. 2019], thus imposing low-discrepancy properties also helps in that regard.

Finally, by imposing strict search space constraints a severe restriction on the distribution of the error is imposed. This can be alleviated by imposing the restrictions through soft penalty terms in the energy. This can allow for a trade-off between blue noise distribution and integration quality for example.

Progressive rendering. In order to make the sequence applicable in progressive rendering, subsets of samples should be considered in the optimization. Given a sample set \( S_i \) for pixel \( i \) we can decompose it in sample sets of \( 1, \ldots, N \) samples: \( S_{i,1} \subset \ldots \subset S_{i,N} \equiv S_i \). We denote the respective images of sample sets \( S_1, \ldots, S_N \).

Then an energy that also optimizes for the distribution of the error at each sample count is:

\[
E(S) = \sum_{t=1}^{T} \sum_{k=1}^{N} w_{t,k} \| g \ast Q_t(S_k) - I_t \|^2. 
\]  

(2)

If \( w_{i,k} \) are set to zero for \( k < N \) then the original formulation is recovered. The more general formulation imposes additional constraints on the samples, thus the quality at the full sample count...
may be compromised if we also require a good quality at lower sample counts.

Choosing samples from \( S_i \) for \( S_{1,1}, \ldots, S_{N-1} \) (in each dimension) constitutes a vertical search space analogous to the one discussed in the main paper for \( a\text{-posteriori} \) methods. The ranking keys’ optimisation in [Heitz et al. 2019] is a special case of this search space for the Sobol sequence.

Adaptive sampling can be handled by allowing a varying number of samples per pixel, and a corresponding energy derived from the one above. Note that this poses further restrictions on the achievable distribution.

**Optimization strategies.** Typically the energies for \( a\text{-priori} \) methods have been optimized through simulated annealing [Georgiev and Fajardo 2016; Heitz et al. 2019]. Metaheuristics can lead to very good minima especially if the runtime is not of great concern, which is the case since the sequences are precomputed. Nevertheless, the computation still needs to be tractable. The energies in previous works are generally not cheap to evaluate. On the other hand, our energies, especially if the optimizations in Section 5 are considered, can be evaluated very efficiently. This is beneficial for keeping the runtime of metaheuristics manageable, allowing for more complex search spaces to be considered.

**Implementation details.** Implementation decisions for a renderer, such as how samples are consumed, or how those are mapped to the hemisphere and light sources affect the estimator \( Q \). This is important, especially when choosing \( Q \) for the described energies to optimize a sequence. It is possible that very small implementation changes make a previously ideal sequence useless for a specific renderer. It is important to keep this in mind when optimising sequences by using the proposed energies and when those are used in a renderer.

### 3 SURROGATE SUBSTITUTION

For practical algorithms we substitute the reference image \( I \) with a surrogate \( I' \). It is instructive to consider error bounds in relation to that, and subsequently extend the formulation to one that allows control over the amount of bias towards the surrogate. We denote \( Q_0 = Q(S) \), and \( Q' = Q(S') \), where we want to optimize for the latter, and \( Q_0 \) remains fixed.

\[
\|g * Q' - h * I\|_2 \\
\|g * Q' - h * I' + h * (I' - I)\|_2 \leq \\
\|g * Q' - h * I'\|_2 + \|h * (I' - I)\|_2
\]

The above bound results in the original formulation when considering only the first term. We can also bound the error through a term involving \( Q_0 \):

\[
\|g * Q' - h * I\|_2 \\
\|g * (Q' - Q_0) + (g * Q_0 - h * I)\|_2 \leq \\
\|g * (Q' - Q_0)\|_2 + \|g * Q_0 - h * I\|_2
\]

Combining the two with weights \( c \) and \( (1 - c) \), for \( c \in [0, 1] \) yields:

\[
\|g * Q' - h * I\|_2 \leq \\
c\|g * Q' - Q_0\|_2 + (1 - c)\|g * Q' - h * I'\|_2 + \\
(1 - c)\|g * (Q_0 - h * I')\|_2
\]

We assume \( g, h, I, I', Q_0, \) and \( c \) to be fixed, and our optimization variable to be \( Q' \). Then the optimization problem simplifies to:

\[
\min_{Q'} c\|g * Q' - Q_0\|_2 + (1 - c)\|g * Q' - h * I'\|_2
\]

The standard energy that we use is retrieved for \( \Delta \) to be zero, while originally both \( \alpha \) and \( \beta \) play a role in \( \Delta \) becoming zero. Intuitively this means that screen space integrand differences due to additive and multiplicative factors do not result in mispredictions with the new formulation, if the integrand is assume to be the same (locally) in screen space.

**Comparison to demodulation.** In the method of Heitz and Belcour the permutation is applied on the albedo demodulated image. This preserves the property that the global minimum of the implicit energy can be found through sorting. Translated to our framework
this can be formulated as (B is a blue noise mask optimized for a kernel g):

\[ E_{HEBP}(\pi) = \| \pi(Q(S')) - I' - B \|_2^2 \approx \| g * \pi(Q(S)) - g * I' \|_2^2. \]  (14)

We have assumed that \( \beta \) is zero, but we can also extend the method to handle an additive term \( \beta \) as in our case. The more important distinction is that while the albedo demodulated image \( Q' \) is used in the permutation, it is never remodulated (\( a \odot \cdot \) is missing). Thus, this does not allow for proper handling of textures, even if it allows for modest improvements in practice. An example of a fail case consists of an image \( a \) that is close to white noise. Then the error distribution will also be close to white noise due to the missing \( a \odot \cdot \) factor. More precisely, even if \( \pi(Q(S')) - I' \) is distributed as \( B \), this does not imply that \( a \odot \pi(Q(S')) - I' \) will be distributed similarly. Dropping \( a \odot \cdot \) is, however, a reasonable option if one is restricted to sorting as an optimisation strategy.

We propose a modification of the original approach (and energy) such that not only the demodulated estimator values are used, but the blue noise mask \( B \) is also demodulated. To better understand how it is derived (and how \( \beta \) may be integrated) we study a bound based on the assumption that \( a_i \in [0, 1] \), and \( \Delta' = 0 \)

\[ E(\pi) = \| g * (a \odot \pi(Q(S')) + \beta) - g * I' \|_2^2 = \]  (15)

\[ |a \odot \pi(Q(S')) + \beta - I'| \|_2 \]

\[ \sum_i a_i^2 \left( \frac{\| \pi(Q(S)) \|_2^2 + \| B \|_2^2}{\| \Delta' \|_2^2} \right)^2 \leq \]  (17)

\[ \| \pi(Q(S)) + \frac{\beta - I'}{a} \|_2^2. \]  (18)

The global minimum of the last energy (w.r.t. \( \pi \)) can be found through sorting also, since there is no spatially varying multiplicative factor \( a \) in front of the permutation.

Sinusoidal Textures. To demonstrate texture handling (multiplicative term \( a \)), in the top row of Fig. 1, a white-noise texture \( W \) is multiplied to a sine-wave input signal: \( f(x, y) = 0.5 * (1.0 + \sin(x + y)) \) \( W(x, y) \). The reference is a constant image at 0.5. Heitz and Belcour proposed to handle such textures by applying their method on the albedo-demodulated image. While this strategy may lead to a modest improvement, it ignores the fact that the image is produced by remodulating the albedo, which can negate that improvement. Instead, our horizontal iterative minimization algorithm can incorporate the albedo explicitly using the discussed energy.

The bottom row demonstrates the effect of a non-flat signal on the error distribution (additive term \( \beta \)). Here \( W \) is added to a sine-wave input signal: \( f(x, y) = 0.5 * (1.0 + \sin(x + y)) + W(x, y) \). The reference image is \( 0.5 * (1 + \sin(x + y)) \). Our optimization is closer to the reference suggesting that our method can greatly outperform the current state of the art by properly accounting for auxiliary information, especially in regions with high-frequency textures.

Dimensional decomposition. The additive factor \( \beta \) can be used to motivate splitting the optimization over several dimensions, since the Liouville–Neumann expansion of the rendering equation is additive [Kajiya 1986]. If some dimensions are smooth (e. g., lower dimensions), then a screen space local integrand similarity assumption can be encoded in \( d(\cdot, \cdot) \) and it will approximate \( \Delta \) better for smoother dimensions. If the optimization is applied over all dimensions at the same time, this may result in many mispredictions due to the assumption being violated for dimensions in which the integrand is less smooth in screen space (e. g., higher dimensions). We propose splitting the optimization problem starting from lower dimensions and sequentially optimizing higher dimensions while encoding a local smoothness (in screen space) assumption on the integrand in \( d(\cdot, \cdot) \) (e. g., swaps limited to a small neighbourhood around the pixel). This requires solving several optimization problems, but potentially reduces the amount of mispredictions. Note that it does not require more rendering operations than usual.

5 ITERATIVE METHODS OPTIMIZATION

The main cost of iterative minimization methods is computing the energy for each trial swap, more specifically the required convolution and the subsequent norm computation. In the work of Analoui and Allebach an optimisation has been proposed to efficiently evaluate such trial swaps, without recomputing a convolution or norm at each step, yielding a speed up of more than 10 times. The optimisation relies on the assumption that the kernel \( g \) is the same in screen space (the above optimization is not applicable for spatially varying kernels). We extend the described optimisation to a more general case, also including spatially varying kernels. We also note some details not mentioned in the original paper.

5.1 Horizontal swaps

We will assume the most general case: instead of just swapping pixels, we consider swapping sample sets from which values are generated through \( Q \). It subsumes both swapping pixel values and swapping pixel values in the presence of a multiplicative factor \( a \).

Single swap. The main goal is to evaluate the change of the energy \( \delta \) due to a swap between the sample sets of some pixels \( a, b \). More precisely, if the original sample set image is \( S \) then the new sample set image is \( S' \) such that \( S_x = S_y = S_a \), and \( S_y = S_b \) everywhere else. This corresponds to images \( Q = Q(S) \) and \( Q' = Q(S') \). The
two images differ only in the pixels with indices $a$ and $b$. Let:

$$
\delta_a = Q'_a - Q_a = Q_a(S_b) - Q_a(S_a)
$$

(19)

$$
\delta_b = Q'_b - Q_b = Q_b(S_a) - Q_b(S_b).
$$

(20)

We will also denote the convolved images $\tilde{Q} = g \ast Q$ and $\tilde{Q}' = g \ast Q'$, and also $\epsilon = \tilde{Q} - I$. Specifically:

$$
\tilde{Q}_I = \sum_{j \in \mathbb{Z}} Q_{j}g_{i-j}, \quad \tilde{Q}'_I = \tilde{Q}_I + \delta_a g_{i-a} + \delta_b g_{i-b}.
$$

(21)

We want to be able to efficiently evaluate $\delta = \|\tilde{Q}' - I\|^2 - \|\tilde{Q} - I\|^2$, since in the iterative minimization algorithm the candidate with the minimum $\delta$ is kept. Using the above expressions for $\tilde{Q}'_I$ we rewrite $\delta$ as:

$$
\delta = \|\tilde{Q}' - I\|^2 - \|\tilde{Q} - I\|^2 = \sum_{i \in \mathbb{Z}} \|\tilde{Q}_I - I_i + \delta_a g_{i-a} + \delta_b g_{i-b}\|^2 - \|\tilde{Q} - I\|^2 = 2 \sum_{i \in \mathbb{Z}} \|\tilde{Q}_I - I_i + \delta_a g_{i-a} + \delta_b g_{i-b}\|^2 + 2(\delta_a \sum_{i \in \mathbb{Z}} g_{i-a} - \sum_{i \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} \delta_b g_{i-b} + 2(\delta_a \sum_{i \in \mathbb{Z}} g_{i-a} - \sum_{i \in \mathbb{Z}} g_{i-b} - g_{i-b} + 2(\delta_a \sum_{i \in \mathbb{Z}} g_{i-a} - \sum_{i \in \mathbb{Z}} g_{i-b}) = 2(\delta_a C_{ge}(a) + 2(\delta_b C_{ge}(b)) + ((\delta_a^2 + \delta_b^2), C_{gg}(0)) + 2(\delta_a C_{ge}(a) + (\delta_a^2) C_{gg}(b - a)),
$$

(26)

where $C_{fh}(x) = \sum_{i \in \mathbb{Z}} f(i - x)h(i)$ is the cross-correlation of $f$ and $h$. We have reduced the computation of $\delta$ to the sum of only 4 terms. Assuming that $C_{ge}$ is known (it can be recomputed once for a known kernel) and that $C_{hh}$ is known (it can be recomputed after a sufficient amount of swaps have been accepted), then evaluating a trial swap takes constant time (it does not scale in the size of the image or the size of the kernel).

**Multiple accepted swaps.** It may be desirable to avoid recomputing $C_{ge}$ even upon accepting a trial swap. For that purpose we extend the strategy from [Analoui and Allebach 1992] for computing $C_{ge}$, where $e^n$ is the error image after $n$ swaps have been accepted:

$$
(\delta_a^0, \delta_b^0), \ldots, (\delta_a^n, \delta_b^n).
$$

(27)

This implies: $\tilde{Q}_I^n = \tilde{Q} + \sum_{k=1}^n (\delta_a g_{i-a} + \delta_b g_{i-b})$. Consequently:

$$
C_{ge}(x) = \sum_{i \in \mathbb{Z}} (\delta_a g_{i-a} + \delta_b g_{i-b}) g_{i-x} = \sum_{k=1}^n (\delta_a C_{gg}(x - a^k) + \delta_b C_{gg}(x - b^k)),
$$

(28)

This allows avoiding the recomputation of $C_{ge}$ after every accepted swap, and instead, the delta on the $n+1$-st swap with trial differences $\delta_a, \delta_b$ is:

$$
\delta^{n+1} = \|\tilde{Q}'^{n+1} - I\|^2 - \|\tilde{Q}^{n+1} - I\|^2 = 2(\delta_a C_{ge}(a)) + 2(\delta_b C_{ge}(b)) + ((\delta_a^2 + \delta_b^2), C_{gg}(0)) + 2(\delta_a C_{ge}(a) + (\delta_a^2) C_{gg}(b - a)),
$$

(31)

where $C_{ge}$ is computed from $C_{ge}$ and $C_{gg}$ as derived in Eq. (22). This computation scales only in the number of accepted swaps since the last recomputation of $C_{ge}$ was done. We also note that $C_{gg}(x - y)$ evaluates to zero if $x - y$ is outside of the support of $C_{gg}$. Additional optimisations have been devised due to this fact [Analoui and Allebach 1992].

5.2 **Vertical swaps.** In the vertical setting swaps happen only within the pixel itself, that is: $\delta_a = Q_a(S'_a) - Q_a(S_a)$. Consequently, $\tilde{Q}_I = \tilde{Q}_I + \delta_a g_{i-a}$. Computing the difference in the energies for the $n + 1$-st swap:

$$
\delta^{n+1} = \|\tilde{Q}'^{n+1} - I\|^2 - \|\tilde{Q}^{n+1} - I\|^2 = 2(\delta_a C_{ge}(a)) + 2(\delta_b C_{ge}(b)) + ((\delta_a^2 + \delta_b^2), C_{gg}(0)) + 2(\delta_a C_{ge}(a) + (\delta_a^2) C_{gg}(b - a)),
$$

(32)

The corresponding expression for $C_{ge}$ is:

$$
C_{ge}(x) = C_{ge}(x) + \sum_{k=1}^n \delta_a C_{gg}(x - a^k)
$$

(33)

5.3 **Multiple simultaneous updates.** If the search space is ignored and the formulation is analyzed in an abstract setting it becomes obvious that the vertical approach corresponds to an update of a single pixel, while the horizontal approach corresponds to an update of two pixels at the same time. This can be generalized further. Let $N$ different pixels be updated per trial, and let there be $n$ trials that have been accepted since $C_{ge}$ has been updated. Let the pixels to be updated in the current trial be: $a_1^{n+1}, \ldots, a_N^{n+1}$, and the accepted update at step $k$ be at pixels: $a_k^0, \ldots, a_k^{N}$. Let $Q^F = Q$ be the original image. We define the sequence of images: $Q^k = Q^{k-1}, i \in \{a_1^k, \ldots, a_N^k\}$ and otherwise let $Q^k = Q^0$ be given. Let $Q^k = Q^k_{a_i^k} - Q^k_{a_i^0}$. Using the above notation we arrive at an expression for $C_{ge}$:

$$
C_{ge}(x) = C_{ge}(x) + \sum_{k=1}^N \delta_a C_{gg}(x - a_k^k).
$$

(34)

The change in the energy due to the $n + 1$-st update is:
\[ \delta^{n+1} = \|Q^{n+1} - I\|^2 - \|Q^n - I\|^2 \]  
\[ 2 \sum_{i \in \mathbb{Z}^2} (\|Q_i^n - I_i + \sum_{j=1}^{N} \delta_j^{n+1} g_{i-a_j^{n+1}}\|^2 - \|Q_i^n - I_i\|^2 = \]  
\[ 2 \sum_{j=1}^{N} \sum_{i \in \mathbb{Z}^2} \|\delta_j^{n+1} + \sum_{i \in \mathbb{Z}^2} e_i^j g_{i-a_j^{n+1}}\|^2 = \]  
\[ \sum_{i \in \mathbb{Z}^2} \sum_{j=1}^{N} (\delta_j^{n+1} + \sum_{i \in \mathbb{Z}^2} e_i^j g_{i-a_j^{n+1}}) + \sum_{i \in \mathbb{Z}^2} (\sum_{j=1}^{N} \delta_j^{n+1} + \sum_{i \in \mathbb{Z}^2} e_i^j g_{i-a_j^{n+1}}) = \]  
\[ \sum_{i \in \mathbb{Z}^2} \sum_{j=1}^{N} (\delta_j^{n+1} + \sum_{i \in \mathbb{Z}^2} e_i^j g_{i-a_j^{n+1}} + \sum_{i \in \mathbb{Z}^2} (\sum_{j=1}^{N} \delta_j^{n+1} + \sum_{i \in \mathbb{Z}^2} e_i^j g_{i-a_j^{n+1}}) = \]  
5.4 Implementation details

Leaky energy. Similar to the original paper [Analoui and Allebach 1992], in our extension \( \delta \) was computed for a "leaky energy" which extended the support of the image by convolution. That is reflected in the fact that the sums are over \( \mathbb{Z}^2 \). To rectify this, the sum needs to be limited to the support of \( I \). This would require clamped sums of the cross-correlation to be evaluated, which can also be precomputed but requires extra memory. The same holds for the cross-correlation with \( e \), where clamped terms are required near the image boundary.

Reflecting boundary conditions. Another desirable property may be a convolution such that it acts on the image extended to be reflected at the boundaries - this avoids artifacts near the borders. This can be achieved by including the relevant terms including pixels for which the kernel is partially outside of the support of \( I \). Care must be taken when expressing \( \hat{Q}_i \), however, since it may include the same updated pixel numerous times (especially if it is near the border). The same ideas apply for a toroidally extended convolution.

Further optimisations. Various other strategies have been proposed in the literature for improving the runtime of iterative error minimization approaches for halftoning.

In our algorithms we usually use a randomized initial state, however, it is possible to initialize the algorithms with the result of a dither matrix halftoning algorithm or error diffusion algorithm which would result in faster convergence [Analoui and Allebach 1992].

Another strategy involves partitioning the image in blocks. Instead of updating the pixels in raster or serpentine order, the blocks are updated simultaneously by keeping only the best update per block in each iteration. This has been reported to run 10+ times faster [Lieberman and Allebach 1997]. In the same paper [Lieberman and Allebach 1997], approximating the kernel with box functions has been proposed yielding a speed up of 6 times. Similarly, if the kernel is separable or can be approximated by a separable kernel, the convolution can also be made considerably faster. A speed-up of an additional 30 times has been reported in [Koge et al. 2014] through the usage of a GPU.

Finally, several heuristics related to the the order in which pixels are iterated over have been proposed in [Bhatt et al. 2006].

5.5 Spatially varying kernels

We propose an optimisation for spatially varying kernels also. Let kernel \( g_i \) be associated with pixel \( i \). Let pixel \( a \) be updated to a new value \( \hat{Q}_a \), while everywhere else the images match: \( Q_i = \hat{Q}_i \), and \( \delta_a = Q_a - \hat{Q_a} \). We denote \( \hat{Q}_i = \{g_i, \hat{Q}_i \}, \hat{Q}'_i = \{g_i, \hat{Q}'_i \} = \hat{Q}_i + g_i \delta_a \).

Our goal is to evaluate the change in the energy due to the update:

\[ \delta = \|\hat{Q} - I\|^2 - \|Q - I\|^2 = \]  
\[ 2 \sum_{i \in \mathbb{Z}^2} (\delta_i + \sum_{j \in \mathbb{Z}^2} e_i^j g_{i-a_j} + \sum_{i \in \mathbb{Z}^2} (\delta_i + \sum_{j \in \mathbb{Z}^2} e_i^j g_{i-a_j}) = \]  
\[ 2(\delta_i + \sum_{j \in \mathbb{Z}^2} e_i^j g_{i-a_j} + \sum_{i \in \mathbb{Z}^2} (\delta_i + \sum_{j \in \mathbb{Z}^2} e_i^j g_{i-a_j} = \]  

In the above \( C_g(a) = \sum_{i \in \mathbb{Z}^2} g_{i-a} \) may be precomputed for every \( a \), which yields a function with support \( \text{supp}(C_g) = \cup \text{supp}(g_i) \), and \( C_{ge}(a) = \sum_{i \in \mathbb{Z}^2} e_i^j g_{i-a} \) can also be recomputed after enough updates have been accepted.

Multiple accepted updates. Let a set of accepted updates results in the differences: \( \{\delta_a^1, \ldots, \delta_a^n\} \). And let \( e_i^k \) be the error image after the updates. We derive an expression for the efficient evaluation of \( C_{ge}^n \):

\[ C_{ge}^n(x) = \sum_{i \in \mathbb{Z}^2} e_i^n g_{i-x} + \sum_{i \in \mathbb{Z}^2} (\delta_a^k + \sum_{i \in \mathbb{Z}^2} g_{i-a^k} g_{i-x}. \]  

An efficient computation of \( C_{ge}^n \) can then be achieved if the function \( C_{ge}(x, y) = \sum_{i \in \mathbb{Z}^2} g_{i-x} g_{i-y} \) is precomputed. Then, at step \( n + 1 \) the change in energy is:

\[ \delta^{n+1} = \|Q^{n+1} - I\|^2 - \|Q^n - I\|^2 = \]  
\[ 2(\delta^{n+1} + C_{ge}^n(a^{n+1}) + (\delta^{n+1} + C_{ge}^n(a^{n+1}). \]  

Multiple simultaneous updates. We derive an expression where an update consists of changing \( N \) pixels simultaneously, and we assume that \( n \) such updates have been accepted previously. We denote the differences of the pixels in update \( k: \{\delta_{a^k}^1, \ldots, \delta_{a^k}^N\} \). The expression for the change in the energy is given as:

\[ \delta^{n+1} = \|Q^{n+1} - I\|^2 - \|Q^n - I\|^2 = \]  
\[ \sum_{i \in \mathbb{Z}^2} (\delta_i^{n+1} + \sum_{j=1}^{N} C_{ge}^n(a^{n+1}) + \sum_{i \in \mathbb{Z}^2} (\delta_i^{n+1} + \sum_{j=1}^{N} C_{ge}^n(a^{n+1}), a^{n+1}. \]  

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Where $C_{gg}(x, y) = \sum_{i \in \mathbb{Z}^2} g_i, x g_i, y$ is assumed to be precomputed, and $C_{ge^{x}}$ can be computed as:

$$C_{ge^{x}}(x) = C_{ge}(x) + \sum_{k=1}^{n} \sum_{j=1}^{N} \delta_{a^k, x} C_{gg}(a^k_j, x). \tag{55}$$

6 RELATIONSHIP TO PREVIOUS WORK

We show that the recent publications [Georgiev and Fajardo 2016; Heitz et al. 2019; Heitz and Belcour 2019] on blue noise error distribution for path tracing, can be seen as special cases in our framework. This allows for a novel analysis and interpretation of the results in the aforementioned works. We also state the necessary assumptions and approximations necessary to get from our general formulation to the algorithms presented in the papers.

Classification. The proposed techniques can be divided into a-priori [Georgiev and Fajardo 2016; Heitz et al. 2019] and a-posteriori [Heitz and Belcour 2019]. The main difference is that for a-priori techniques broad assumptions are made on the integrand without relying on information from renderings of the current scene. The cited a-priori approaches describe ways for constructing offline optimized point sets/sequences. We denote the method in [Georgiev and Fajardo 2016] as BNDS (blue-noise dithered sampling), the method in [Heitz et al. 2019] as HBS (Heitz-Belcour Sobol), and the histogram and permutation method in [Heitz and Belcour 2019] as HBH and HBP respectively (Heitz-Belcour histogram/permutation).

Energy. HBH/HBP both rely on a blue noise dither matrix optimised while using a Gaussian kernel (through void-and-cluster [Ulichney 1993]). This kernel corresponds to the kernel in our framework $g$. The optimisation of this dither matrix happens offline unlike in our iterative energy minimization algorithms. This imposes multiple restrictions while allowing for a lower runtime. On the other hand, the dither matrices in HBS and BNDS are optimized with respect to empirically motivated energies that cannot be related directly to what is used as energy in HBH and HBP. In the case of BNDS the energy does not even introduce an implicit integrand, and instead it is devised to represent a whole class of integrands. We propose to substitute those empirically motivated energies with a modified version of our energy. This allows an intuitive interpretation and relating a-posteriori approaches to a-priori approaches.

Search space. Another notable difference constitute the search spaces on which the different approaches operate. HBH selects a subset from a set of precomputed samples in each pixel, HBP permutes the assignment of sample sets to pixels, BNDS directly modifies the set of samples in each pixel, and HBS considers a search space made up of scrambling and ranking keys for a Sobol sequence. Working on the space of scrambling and ranking keys guarantees the preservation of the desirable integration qualities of the Sobol sequence used, and it should be clear that other methods can also be restricted to such a space. Clearly, a search space restriction diminishes the achievable blue noise quality. On the other hand, it makes sequences more robust to integrands for which those were not optimized.

7 A-PREORI APPROACHES

In this section we analyze the permutation based approach (HBP) and the histogram sampling approach (HBH) proposed in [Heitz and Belcour 2019]. The two methods can be classified as dither matrix halftoning methods in our framework, that operate on a horizontal and vertical search space respectively. We make the approximations and assumptions necessary to get from our general formulation to HBP/HBH explicit.

We also note that a-posteriori methods lead to solutions that adapt to the current render by exploiting known information (e.g. previously rendered data, auxiliary buffers). They can generally produce better results than a-priori methods.

Both HBP and HBH rely on a blue noise dither matrix $B$. Let $B$ be the optimized blue noise dither matrix resulting from the minimization of $E(B) = \| g \ast B \|_2^2$ over a suitable search space. The kernel $g$ is the one used to generate the blue noise images for HBP/HBH. That is, the Gaussian kernel in the void-and-cluster method [Ulichney 1993]. Our analysis does not rely on the kernel being a Gaussian, or on the void-and-cluster optimization, this is simply the setting of the HBP/HBH method. In the more general setting any kernel is admissible.

7.1 Sorting step for the permutation approach

The permutation approach [Heitz and Belcour 2019] consists of two main parts: sorting (optimization), and retargeting (correcting for mispredictions). The sorting step in HBP can be interpreted as minimizing the energy:

$$E_{HBP}(\pi) = \| \pi(Q) - f_2(B) \|_2^2, \forall f_2 : a < b \implies f_2(a) < f_2(b). \tag{56}$$

A global minimum of the above energy is achieved for a permutation $\pi$ that matches the order statistics of $Q$ and $B$. Thus our goal would be to get from the minimization of:

$$E(\pi) = \| g \ast (Q(\pi(S)) - I) \|_2^2 = \| g \ast (\pi(S)) \|_2^2. \tag{57}$$

to the minimization of Eq. (56) over a suitable search space (in practice it is limited to permutations within tiles).

We successively bound the error, while introducing the assumptions implicit to the HBP method. The bounds are not tight, however, the different error terms that we consider illustrate the major sources of error due to the approximation of the more general energy (Eq. (57)) with a simpler one (Eq. (56)). The substitution of the kernel convolution $g \ast -$ by a difference with a blue noise mask $B$ restricts the many possible blue noise error distributions towards which $e(\pi(S))$ can go with a single one: $B$. A global minimizer of the new simplified energy can thus be found by just sorting.

The closer the distributions of $e(\pi(S))$ and $\alpha B, \alpha > 0$ are locally, the lower this restriction error can be made. Notably, for a close to linear relationship between the samples and the integrand, and sufficiently many pixels, $e(\pi(S))$ and $\alpha B$ can be matched closely in practice. A different way to reduce the approximation error is to introduce a sufficient amount of different blue noise images and pick the one that minimizes the error. We start with the original energy (Eq. (57)) and bound it through terms that capture the main assumptions on which the model relies:
We factor out a multiplicative scaling factor \( \alpha \). Then the term discussed in the main paper, this is achieved by introducing a difference \( Q_k(\pi_k(S_k)) - \pi_k(Q_k(S_k)) \). We note that the number of considered pixels depends on the tile size in HBP, and the practical significance of this has been demonstrated through a canonical experiment in the main paper.

7.1.2 Second error term. The second term reflects the error introduced by substituting a large search space (many local minima) with a small search space. It introduces the first implicit assumption of HBP by relating the first and third error terms (by using \( f_2 \) and \( \alpha \) respectively) through the second error term. The assumption is that there exists a permutation for which \( \epsilon(\pi(S)) \) can be made close to \( \alpha B \), which would make the second term small. This holds in practice if the pixel-wise error is zero on average (unbiased estimator within each pixel), and we have a sufficiently large resolution/tiles: which results in a higher probability that pixels from \( \epsilon(\pi(S)) \) can match \( B \) well. Then the term \( |g|_1 f_2 - aB |_2 \) can be made small. We note that this is a generalization of the third optimality condition in [Heitz and Belcour 2019] (correlation-preserving integrand) since an integrand linear in the samples can also better match \( B \) provided enough pixels. For a linear integrand the optimal \( f_2 \) is also a linear function (ideal correlation between samples and integrand). The main difference between a linear integrand and a nonlinear/discontinuous one, is the amount of sample sets/pixels necessary to match \( f_2(B) \) well, given an initial white noise samples’ distribution. So in practice there are 4 factors directly affecting the magnitude of the second term: the number of considered blue noise images, the size of the tiles, the correlation between samples and integrand (accounted for by \( f_2 \)), the bias/consistency of the estimators.

We note that the number of considered pixels depends on the tile size in HBP, and the practical significance of this has been demonstrated through a canonical experiment in the main paper.

7.1.3 First error term. Before we proceed we need to further bound the first error term by substituting \( Q(\pi(S)) \) by \( \pi(Q(S)) \). As discussed in the main paper, this is achieved by introducing a difference term \( \Delta(\pi) = Q(\pi(S)) - \pi(Q(S)) \), and then \( \sqrt{E_{\text{HBP}}} \) is recovered.

The error there can be made arbitrarily small through \( f_2 \) (it is accounted for in the second term). Thus we only need to study the remaining error due to \( \Delta \). In the case of HBP, \( \Delta \) is approximated by non-overlapping characteristic functions in each tile \( d(x, y) = \infty \), for \( x, y \) in different tiles). This means that the approximation error is zero within each tile if the integrands are the same within the tile and permutations act only within the tile, since \( \Delta(\pi) = 0 \). On the other hand, if this assumption is violated, mispredictions occur, usually resulting in white noise.

7.1.4 \( \Delta \) term. HBP partitions screen space into a several tiles \( \mathcal{R}_1, \ldots, \mathcal{R}_K \), and permutations are only over the pixel values in a tile. Having the partition induced by the tiling we can bound the first term:

\[
\| \epsilon(\pi(S)) - f_2(B) \|_2 \leq \sum_{k=1}^{K} \| \epsilon_k(\pi_k(S_k)) - f_2(B) \|_2.
\]

Since additionally the permutations are optimized for the pixel values instead of the sample sets (which saves re-rendering operations), then there is an assumption that within each tile \( \mathcal{R}_k \) the following holds (we denote \( A_k = A_{\mathcal{R}_k} \)):

\[
Q_k(\pi_k(S_k)) = \pi_k(Q_k(S_k)).
\]

Consequently it follows that \( f_k = I_k \), \( \forall i \in \mathcal{R}_k \).

This assumption can be identified with the 4-th optimality proposed in [Heitz and Belcour 2019]: screen-space coherence. As discussed, the search space restriction to the tiles corresponds to an approximation of the \( \Delta \) term in our framework by characteristic functions: \( d_k(x, y) = \infty \), \( x, y \in \mathcal{R}_k \) and \( d_k(x, y) = 0 \), \( x, y \notin \mathcal{R}_k \). To account for the actual error when the assumption is violated we introduce an additional error term per tile: \( \Delta_k = Q_k(\pi_k(S_k)) - \pi_k(Q_k(S_k)) \), then we have the bound:

\[
\| \epsilon_k(\pi_k(S_k)) - f_2(B_k) \|_2 \\
\leq \| \pi_k(Q_k(S_k)) - I_k - f_2(B_k) + \Delta_k \|_2 \leq \| \pi_k(Q_k(S_k)) - I_k - f_2(B_k) \|_2 + \| \Delta_k \|_2.
\]

This means that even if all of the previous error terms are made small, including \( \| \pi_k(\epsilon_k(S_k)) - f_2(B_k) \|_2 \), the error may still be large due to \( \| \Delta \|_2 \). We refer to a large error due to the delta term as misprediction - that is, a mismatch between the predicted error distribution from the minimization of \( \| \pi_k(\epsilon_k(S_k)) - f_2(B_k) \|_2 \) and the actual error distribution resulting from the above permutation applied to \( \epsilon_k(\pi_k(S_k)) \). The best way to identify mispredictions is to compare the predicted image \( \pi_k(Q_k(S_k)) \) and the image rendered with the same permutation for the sample sets \( Q_k(S_k) \). A misprediction occurring means that the assumption made to approximate \( \Delta \) was incorrect (\( \Delta_k \neq 0 \) for some tile \( \mathcal{R}_k \)), equivalently the optimality condition of screen-space coherence is not satisfied.

Avoiding mispredictions. In practice mispredictions often occur for larger tile sizes, since it is hard to guarantee that the integrand remains similar over each tile. On the other hand, larger tiles allow for a better blue noise as long as \( \Delta_k = 0 \) in each tile, thus larger tiles are desirable. The method fails even more often near edges, since even for small tile sizes it allows swapping pixels over an edge. A straightforward improvement involves partitioning the domain by
We also demonstrate the effect of a larger search neighbourhood \( R \) in our optimization (Alg. 2). For our case, we consider disk neighbourhoods so that they are contained within Heitz and Belcour’s tiles in terms of size, but they can also overlap due to our formulation. From left-to-right, the input white noise texture is optimized using our relocation algorithm. The last two columns are from Heitz and Belcour’s [2019] method. The corresponding power spectra of these optimized images (128 × 128) are also shown.

respecting edges. More involved methods may take into account normals, depth, textures, etc.

7.1.5 \( E_{HBP} \) error term. The final step involves the minimization of the energy in Eq. (61). Since different tiles do not affect each other the minimization can be performed per tile (we adopt the assumption from HBP \( \Delta_k = 0 \)):

\[
\pi_k^* \in \arg \min_{\pi_k} \| \pi_k(Q_k(S_k)) - I_k - f_2(B_k) \|_2 = \arg \min_{\pi_k} \| \pi_k(Q_k(S_k)) - f_2(B_k) \|_2^2. \tag{62}
\]

We have dropped the term \( I_k \) since it doesn’t affect the set of minimizers (\( I_k \) is assumed constant in each tile). As discussed in Eq. (56), a global minimum is given by matching the order statistics of \( Q_k \) to the order statistics of \( f_2(B) \) (note that the order statistics of \( B_k \) do not change from the application of \( f_2 \) since it is a strictly increasing function). This is equivalent to performing the sorting pass described in [Heitz and Belcour 2019]. A minor optimization would be to pre-sort \( B \) and instead store the sorted indices.

Tiling effect. In Fig. 2 we compare the effect of the tile size. In our approach, the “tiles” can be defined per pixel, can have arbitrary shapes, and are overlapping, the last being crucial for achieving a good blue noise distribution. We consider white-noise with mean 0.5 (which is an ideal scenario for Heitz and Belcour’s method) and compare various tile sizes. For a fair comparison, our tile radius \( r \) corresponds similar tile-size in the permutation [2019] approach. The power-spectrum profiles confirm the better performance of our method. Retargeting [2019] cannot improve the quality of the permutation approach either, since no misprediction can occur (\( \Delta = 0 \)). The adverse effect of tiling is exacerbated in practice since, for images which are not smooth enough in screen space, tiles of smaller sizes need to be considered.

Custom surrogate. The \( I_k \) term doesn’t need to be assumed constant in fact. If it is assumed constant, that is equivalent to picking a tile-constant surrogate, however, a custom surrogate may be provided instead. Then one would simply minimize the energy:

\[
\| \pi_k(Q_k(S_k)) - (B_k + I_k) \|_2^2.
\]  

The energy has a different minimizer than the original HBP energy, but the global minimum can be found efficiently through sorting once again.

7.2 HBP retargeting

The retargeting pass in HBP achieves two things. It introduces new possible target solutions through new blue noise images, and it corrects for mispredictions. The first is not so much a result of the retargeting, as it is of varying the blue noise image every frame. Ideally several blue noise images would be considered in a single frame, and the best image would be chosen per tile (in that case one must make sure that there are no discontinuities between the blue noise images’ tiles) in order to minimize the second term in Eq. (58). Instead, in HBP this is amortized over several frames.

The more important effect of retargeting is correcting for mispredictions, by transferring the recomputed correspondence between sample set and pixel value (achieved through rerendering) to the next frame. This allows reducing the error due to the approximation of \( \Delta \) (when the piecewise-tile constancy assumption on the integrand is violated). Note however, that this is inappropriate if there is a large temporal discontinuity between the two frames.

Implementation details. Retargeting requires a permutation that transforms the blue noise image in the current frame into the blue noise image of the next frame [Heitz and Belcour 2019]. This permutation is applied on the optimized seeds to transfer the learned correspondence between sample sets and pixel values to the next frame. Implicitly, this transforming permutation also relies on a screen space integrand similarity assumption, since there is no guarantee that the corresponding values from the swap will match, possibly incurring a misprediction once again (it can be modeled by an additional \( \Delta \) term). In HBP [Heitz and Belcour 2019] the maximum radius of travel of each pixel in the permutation is set to 6 pixels. This has a direct effect on the approximation of \( \Delta \), as the travel distance of a pixel is allowed to extend beyond the original tile bounds. In the worst case scenario a pixel may allow to travel a distance of \( \sqrt{r_x^2 + r_y^2} + 6 \) pixels, where \( r_x, r_y \) are the dimensions of the tiles. An additional error is introduced since the retargeting pass does not produce the exact blue noise image used in the next frame, but some image that is close to it [Heitz and Belcour 2019]. This seems to be done purely from memory considerations since it allows one blue noise image to be reused by translating it toroidally each frame to produce the blue noise image for the next frame.

Relationship to our horizontal approach. Our horizontal approach does not require a retargeting pass. It can directly continue with the optimized sample sets and pixel values from last frame. There is also no additional travel distance for a matching permutation as
in retargeting, which further minimizes the probability of misprediction. Thus, it inherently and automatically produces all of the advantages of retargeting while retaining none of its disadvantages.

7.3 Histogram sampling approach
The histogram sampling approach from Heitz and Belcour can be interpreted as both a dithering and a sampling method. We study the dithering aspect to better understand the quality of blue noise achievable by the method.

**Algorithm analysis.** The sampling of an estimate in each pixel by using the corresponding mask value to the pixel can be interpreted as performing a mapping of the mask’s range and then quantizing to the closest estimate. In HBH each estimate is equally likely to be sampled (if a random mask is used), which implies a transformation that maps equal parts of the range to each estimate. Let $Q_k, \ldots, Q_N$ be the greyscale estimates in pixel $k$ sorted in ascending order. Let the range of the blue noise mask be in $[0,1]$. Then the range is split into $N$ equal subintervals: $[0, \frac{1}{N}], \ldots, [\frac{N-1}{N}, 1]$ which respectively map to $[Q_1, Q_1+Q_2], \ldots, [Q_1+Q_2 + Q_3, Q_1+Q_2+Q_3], \ldots, [Q_{N-1} + Q_N, Q_N]$. If the quantization rounds to the closest estimate, then the above mapping guarantees the desired behaviour. We note that since the estimates in each pixel can have different values, the mapping for each pixel may be different. We will denote the above mapping through $f$. Then the mapping plus quantization problem in a pixel $k$ may be formulated as:

$$\min_{i \in \{1, \ldots, N\}} |Q_{k,i} - f_k(B_k)|.$$  

(64)

Note that the minimization in each pixel is independent, and it aims to minimize the distance between the estimates and the remapped value from the blue noise mask. If the set of estimates are assumed to be the same across pixels, and are also assumed to be spaced regularly, then $f$ is only a linear remapping, which effectively transfers the spectral properties of $B$ onto the optimized image. Notably, the former is the *screen-space coherence* assumption from HBP, while the latter is the *correlation-preserving integrand* assumption. Thus we have seen that for optimal results the HBB method relies on exactly the same assumption as the HBF method (while our *vertical iterative minimization approach lifts both assumptions*).

**Disadvantages.** One of the key points is that the error distribution and not the signal itself ought to ideally be shaped as $B$. This is actually the case even in the above energy. From the way $f$ was chosen it follows that the surrogate is equivalent to $f(0.5)$ which can be identified as the image made of the median of the sorted estimates within each pixel. This is the case since if the target surrogate of $B$ (during the offline optimization) was assumed to be 0.5, then after the mapping it is $f(0.5)$. Generally, this is a very bad surrogate in the context of rendering, and it generally increases the error compared to the averaged image, making the method impractical.

Another notable disadvantage is that all estimates are considered with an equal weight. This means that outliers are as likely to be picked as estimates closer to the surrogate. This results in fireflies appearing even when those were not present in the averaged image. Compared to classical halftoning, where only the closest lower and upper quantization levels are considered, HBB does not minimize the magnitude of the error to the surrogate.

Finally, the two assumptions of: *screen-space coherence* and *correlation-preserving integrand*, generally do not hold in practice. Estimates cannot be assumed to match between pixels (especially if samples are taken at random), and they cannot be assumed to be uniformly distributed, which implies that $f$ is not linear. This greatly impacts the quality of the result, especially if it is compared to *adaptive* approaches such as our *vertical* error diffusion approach and our iterative minimization techniques (see the experiments in the main paper).

**Generalization.** The method can be generalized to take a custom surrogate instead of the one constructed by the median of the estimates within each pixel. This is achieved by splitting the per pixel set of estimates into two parts: (greyscale) estimates greater than the value of the (greyscale) surrogate in the current pixel, and estimates lower than it. Then the mapping $f_k$ for the current pixel $k$ maps values in $[0,0.5)$ to the lower set, and values in $[0.5,1]$ to the higher set, such that $f_k(0.5) = k$. The original method is recovered if the surrogate is chosen to be the implicit one for the original histogram sampling method and if the appropriate corresponding mapping $f$ is kept.

The approach can be extended further by setting different probabilities for the different estimates. The original histogram sampling method correspond to setting the same probability for sampling every estimate, equivalently: equal sized sub-intervals from $[0,1]$ map to each estimate. Classical dither matrix halftoning can be interpreted as setting an equal probability for the closest to the surrogate upper and lower estimates, while every other estimate gets a zero probability. Equivalently: equal sub-intervals from $[0,1]$ map to the two aforementioned estimates while no part of the interval maps to the remaining estimates. Generally a custom probability can be assigned to each estimate: $p_1, \ldots, p_N$, by having the intervals $[0, p_1), \ldots, [\sum_{k=1}^{N-1} p_k, 1]$ map to $Q_1, \ldots, Q_N$ (after quantization). We note that an unbiased image can be recovered only if there is a map to every estimate.

8 A-PRIORI APPROACHES
We discuss current state of the art *a-priori* approaches [Georgiev and Fajardo 2016; Heitz et al. 2019] and their relation to our framework, as well as insights regarding those.

8.1 HBS
In Heitz et al.’s work, a scrambling energy and a ranking energy have been proposed (note that those energies are maximized and not minimized):
\[ E_s = \sum_{i,j} \exp\left(-\frac{||i - j||^2}{2\sigma^2}\right) ||E_i - E_j||_2 \]  
(65)

\[ E_r = \sum_{i,j} \exp\left(-\frac{||i - j||^2}{2\sigma^2}\right) \left(||E_i^1 - E_j^1||_2 + ||E_i^2 - E_j^2||_2\right) \]  
(66)

\[ E_i = (e_{i,1}, \ldots, e_{i,T}) \]  
(67)

\[ e_{t,i}(S_t) = \frac{1}{|S_i|} \sum_{k=1}^{|S_i|} f_t(p_{l,k}) - \int_{[0,1]^D} f_t(x) \, dx \]  
(68)

\[ S_i = \{p_{l,1}, \ldots, p_{l,M}\}. \]  
(69)

The upper indices in \( E_s^1, E_r^1 \) indicate that the two energies are evaluated with different subsets of the sample set \( S_i \) in the pixel \( i \). The \( f_t \) are taken from an arbitrary set of functions (in the original paper those are random Heaviside functions). The described form of the energies has been partially motivated by the energy in [Georgiev and Fajardo 2016]. This doesn't allow for a straightforward interpretation or a direct relation to the (implicit) energy used for \textit{a-posteriori} approaches in [Heitz and Belcour 2019].

\textbf{Scrambling energy.} We modify \( E_s \) in order to relate it to the energy in our framework and to provide a meaningful interpretation:

\[ E'_s = \sum_{i=1}^T w_i \|g \ast Q_t(S_i) - I_t\|_2^2, \]  
(70)

\[ Q_{t,i}(S_i) = \frac{1}{|S_i|} \sum_{k=1}^{|S_i|} f_t(p_{l,k}), \quad I_{t,i} = \int_{[0,1]^D} f_t(x) \, dx. \]  
(71)

We have relaxed the Gaussian kernel to an arbitrary kernel \( g \) and absorbed it into the norm. More importantly, we have removed the heuristic dependence of error terms on their neighbours, and instead the coupling happens through the kernel itself. Finally, we have introduced weights \( w_1, \ldots, w_T \) that allow assigning different importance to different integrands. Thus, this is a weighted average of our original energy applied to several different integrands, matching our \textit{a-priori} approach (Eq. (1)). Through this formulation a direct relationship to the \textit{a-posteriori} methods can be established, and it can be motivated in the context of both the human visual system and denoising. Particularly, the scrambling energy \( E'_s \) is over the space of scrambling keys, which allow permitting the assignment of sample sets. This is in fact the horizontal setting from our formulation in the main paper. The space can be extended further if the scrambling keys in each dimension are different (as in HBS). The same can be done in \textit{a-posteriori} methods, if the optimization is performed in each dimension as discussed in Section 4.

\textbf{Ranking energy.} The ranking keys in HBS describe the order in which samples are consumed. This is useful for constructing progressive \textit{a-priori} methods. Notably, the order in which samples will be introduced can be optimized. Having a sequence of sample sets in each pixel: \( S_{t,1} \subset \ldots \subset S_{t,M} \equiv S_t \) and respectively the images formed by those: \( S_{t,1}, \ldots, S_{t,M} \), the progressive energy may be constructed as:

\[ E'_r = \sum_{k=1}^M w_k \|g \ast Q(S_k) - I_t\|_2^2. \]  
(72)

The quality at a specific sample count corresponding to \( S_k \) is controlled through the weight \( w_k \). The original energy maximizing the quality of the full set is retrieved for \((w_1, \ldots, w_{M-1}, w_M) = (0, \ldots, 0, 1)\). Since the sample sets \( S_1, \ldots, S_{i,M} \) are optimized by choosing samples from \( S_i \) this can be seen as a \textit{vertical} method. Finally, the ranking keys can also be defined per dimension, which can be related to \textit{a-posteriori} methods through the suggested dimensional decomposition in Section 4.

8.2 Blue-noise dithered sampling energy

In Georgiev and Fajardo’s work, in order to get an optimized (multi-channel) blue noise mask, the following energy has been proposed:

\[ E(p_1, \ldots, p_N) = \sum_{i=1}^N \exp\left(-\frac{||i - j||^2}{\sigma^2}\right) \exp\left(-\frac{||p_i - p_j||^{d/2}}{\sigma^2}\right), \]  
(73)

which bears some similarity to the weights of a bilateral filter. In the above \( i, j \) are pixel coordinates, and \( p_i, p_j \) are \( d \)-dimensional vectors associated with \( i, j \), let the image formed by those vectors be \( S \). The energy aims to make samples \( p_i, p_j \) distant (\( ||p_i - p_j|| \) must be large) if they are associated with pixels which are close (\( ||i - j|| \) is small).

\textbf{Relation to our framework.} Even though the energy is heuristically motivated, we can very roughly relate it to our framework. The above energy implicitly assumes classes of integrands \( Q_1, \ldots, Q_T \), such that close samples \( p_i, p_j \) are mapped to close values \( Q_{i,t}(p_i), Q_{j,t}(p_j) \), and distant samples are mapped to distant values. Notably, the form of the energy doesn’t change over screen-space, so the same can be implied about the integrands. One such class is the class of bi-Lipschitz functions. The bound can be used to relate a modified version of the original energy, to an energy of the form:

\[ E_Q = \sum_{i \neq j} \exp\left(-\frac{||i - j||^2}{\sigma^2}\right) \exp\left(-\frac{C||Q_{i,t}(p_i) - Q_{j,t}(p_j)||^{d/2}}{\sigma^2}\right). \]  
(74)

Thus, the original energy can indeed be interpreted as reasonable for a whole class of sufficiently smooth integrands, instead of an energy that works very well with one specific integrand.

A similar thing can be achieved in our framework, if the weighted energy is considered:

\[ E'(S) = \sum_{t=1}^T w_t \|g \ast Q_t(S) - I_t\|_2^2. \]  
(75)

The kernel \( g \) can be a Gaussian with standard deviation \( \sigma \), as in the original energy, or it can be relaxed to an arbitrary desired kernel. \( Q_1, \ldots, Q_T \) are representative integrands that satisfy the discussed smoothness requirements, and \( w_t \) are associated weights assigning different importance to the integrands. Finally, the reference images are given by the integrals \( I_t = \int_{[0,1]^D} Q_t(x) \, dx \).
It should be clear that this is a weighted average constructed from the standard error in our framework applied to a set of integrands. There are a number of benefits of such an explicit formulation. Most importantly, it allows for a-priori methods to be studied in the same framework as a-posteriori approaches. Additionally, explicit control is provided over the set of integrands and the kernel in a manner that allows for a straightforward interpretation.

8.3 Blue-noise dithered sampling algorithm

The second contribution of Georgiev and Fajardo’s work is a sampler which relies on an image optimized with Eq. (73) and uses it to achieve a blue noise distribution of the rendering error. We summarize the algorithm and discuss some details related to it.

Algorithm. Let $B$ be an image (with $d$-channels) optimized by minimizing Eq. (73) over a suitable search space. Let $P = \{p_1, \ldots, p_N\}$ be a sequence of $d$-dimensional points. Within each pixel $i$ the sample set $S_i$ is constructed, such that:

$$p_j \in P \implies p_{i,j} \in S_i : p_{i,j} = (p_j + B_j^*) \mod 1. \quad (76)$$

The sequence $P$ can be constructed by using various samplers (e.g., random, low-discrepancy, blue-noise, etc.). The construction of the new points for pixel $i$ can be interpreted either as toroidally shifting the sequence $P$ by $B_i$ or equivalently as toroidally shifting the sequence $\{B_{i,j} \}$ by $P$.

The sequences constructed within each pixel are used to estimate the integral in the usual manner. Since a finite number of dimensions $d$ are optimized the suggestion is to distribute the constructed sequences over smoother dimensions, while other dimensions may use a standard sampler.

Effect of the toroidal shift. Let us consider a linear one-dimensional integrand $f(q) = aq + \beta$ that does not vary in screen space, and a sequence $P$ with a single point $p$. Furthermore, if we assume $p = 0$, then the error is given by:

$$Q(B) - I = aB + \beta - I. \quad (77)$$

Since $Q$ does not vary in screen space, then $I$ also doesn’t. Then the power spectrum of the error (excluding the DC) matches the power spectra of $B$ up to the multiplicative factor $a^2$. Then, under the assumption that the integrand is linear, does not vary in screen space, and there is no toroidal shift, the power spectral properties of $B$ are transferred ideally to the error.

On the other hand, if $p$ is chosen to be non-zero, then the spectral characteristics of the image $(B + p) \mod 1$ will be transferred instead. We have empirically verified that even with a very good quality blue noise image $B$ the toroidal shift degrades its quality due to the introduced discontinuities. Thus, even in the ideal case of a constant in screen space linear 1-D integrand, toroidal shifts degrade the quality.

Effect of using multiple samples. Let us consider the same integrand $f(q) = aq + \beta$, which we have identified as being ideal for transferring the spectral characteristics of $B$ to the error. And let us assume that we are given several samples: $P = \{p_1, ..., p_N\}$, and we have constructed the sample set image $S$ through toroidal shifts with $B$. Then the error is:

$$Q_i(S_i) - I_i = \frac{\alpha}{N} \sum_{k=1}^{N} P_{k,i} + \beta - I_i. \quad (78)$$

The power spectrum of the error thus matches the power spectrum of the image $A_i = \sum_{k=1}^{N} P_{k,i}$ (excluding the DC) up to a multiplicative factor. For a random point sequence $P$ the more points are considered, the closer to white noise $A$ becomes. This is further exacerbated by the discussed discontinuities introduced by the toroidal shifts.

Extension. We have argued that both toroidal shifts and increasing the number of samples has a negative effect on transferring the spectral properties of $B$ even in an ideal scenario. Naturally the question arises whether this can be improved. Our proposal is the direct optimisation of point sets without the application of a toroidal shift.
For the discussed example this entails constructing a sequence of $N$ images $B_1, \ldots, B_N$ such that $A_k = \sum_{j=1}^{k} B_j$ is a blue noise image. Then the error has the (blue noise) spectral characteristics of $A_k$ at each sample count:

$$Q_i(B_{1,i}, \ldots, B_{k,i}) - I_i = \frac{\alpha}{k} \sum_{j=1}^{k} B_{j,i} + \beta - I_i.$$  (79)

9 EXTRA RESULTS

REFERENCES


Fig. 4. We compare Heitz and Belcour’s histogram sampling method to our vertical iterative minimization on additional scenes. We provide 4 times and 16 times more samples for the histogram method, where each method picks one out of all of its allocated samples in each pixel. Despite the fact that our method may use 16 times less samples it still outperforms the histogram sampling approach. This is mainly due to the implicit surrogate and the suboptimal dithering inherent to the histogram sampling approach.

Fig. 5. In the main paper we compare all vertical methods on The Wooden Staircase scene. All of our methods achieve a better pMSE than the baseline (the averaged image), while the histogram sampling method increases the error both in terms of MSE and pMSE. The tiled error power spectra images confirm the pMSE ranking and provide a visualization of the local pMSE distribution. We also show S-CIELAB error visualizations which suggest that pointwise error is heavily weighted in S-CIELAB, which doesn’t make it a very good predictor for the perceptual quality related to the noise distribution, unlike HDR-VDP-2.
Fig. 6. Additional scenes comparing our vertical methods against Heitz and Belcour’s permutation approach. The images are rendered at 4 samples per pixel. The permutation approach uses 16 (static) frames with retargeting to improve mispredictions, however this still does not eliminate all of the mispredictions. This is especially evident in the Bathroom scene.

Fig. 7. All vertical methods from the main paper are compared at 4 samples per pixel. The ranking based on our error metrics is (best last): histogram sampling, dithering, error diffusion, iterative minimization. This also matches our perceptual evaluation.
Fig. 8. Additional comparisons for horizontal approaches at 4 samples per pixel. The left side of the images are using Heitz and Belcour’s permutation approach, while the right side is using our horizontal iterative minimization approach. The achieved blue noise distribution for our approach is consistently better.
Fig. 9. The textured Cornell box scene is compared over different samples per pixel using our texture handling approach in Section 4. The improvements are visible for all spp over Heitz and Belcour’s permutation approach using demodulation. This is especially apparent when comparing the tiled error spectra. Note that the spectra are not normalized to 1.