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The Weinberg-Witten theorem on massless particles: an essay

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In this essay we deal with the Weinberg-Witten theorem [1] which imposes limitations on *massless* particles. First we motivate a classification of massless particles given by the Poincaré group as the symmetry group of Minkowski spacetime. We then use the fundamental structure of the background in the form of Poincaré covariance to derive restrictions on charged massless particles known as the Weinberg-Witten theorem. We address possible misunderstandings in the proof of this theorem motivated by several papers on this topic. In the last section the consequences of the theorem are discussed. We treat it in the context of known particles and as a constraint for emergent theories.

1 Introduction

In the year 1980, Steven Weinberg and Edward Witten impressed by a short and very elegant proof of a rather general statement [1]. The so-called Weinberg-Witten theorem constrains the spin of massless particles charged under conserved Poincaré covariant currents to values $j \leq 1/2$ or $j \leq 1$, respectively. Even though the theorem is of great importance for several reasons, certainly the most interesting consequence is that it restricts attempts to construct a composed graviton in the context of ordinary quantum field theory. Since the time the theorem was published, it is known as an important constraint for theories containing all kinds of massless particles, first of all a graviton. However, the proof of the theorem heavily relies on Poincaré covariance. Therefore, trying to bypass the Weinberg-Witten theorem, this is the ingredient evaded by most new physical ideas towards a theory of massless particles. Recently the theorem attracted notice in the context of approaches to emergent gravity modifying the natural Poincaré invariant background geometry, Minkowski spacetime. It can be considered as a fundamental reason for why we might have to leave the Minkowski background in order to search for an emergent theory of gravity.

2 What are massless particles?

Concealing the "problem child" gravity, Minkowski spacetime is the fundamental background geometry for all remaining theories in our physical world, at least at the current experimental level. Asking for symmetry transformations which leave this background geometry invariant, the answer is given by the Poincaré group consisting of three boosts, three rotations and four translations, including parity and time reversal. Hence, the Poincaré group is the basic structure of every theory on this background and it suggests itself to connect the definition of elementary particles to this group. In this second section we search for a physically motivated definition of particles according to the symmetry structure of Minkowski spacetime.

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2.1 The Poincaré group

The Poincaré or *inhomogeneous* Lorentz group is the 10-dimensional noncompact Lie group of isometries of Minkowski spacetime. It associates coordinate systems in different inertial frames defined by Einstein's special relativity principle. The corresponding coordinate transformations are given by

$$\eta_{\mu\nu} dx'^{\mu} dx'^{\nu} = \eta_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{2.1}$$

or equivalently

$$\eta_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \eta_{\rho\sigma}, \tag{2.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Hence, these transformations must be of the linear form

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}, \tag{2.3}$$

with constants a^{μ} and $\Lambda^{\mu}{}_{\nu}$ satisfying

$$\Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} \eta^{\rho\sigma} = \eta^{\mu\nu}, \tag{2.4}$$

where the last line defines a Lorentz transformation.

If we denote a general Poincaré transformation by (Λ, a), we find the multiplication rule

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2). \tag{2.5}$$

Setting a to 0 we get a subgroup of the Poincaré group, the *homogeneous* Lorentz group. The group of homogeneous Lorentz transformations is again divided into four disconnected subgoups. The most interesting, which excludes the discrete transformations parity P and time reversal T, is the *proper orthochronous* Lorentz group $SO(3,1)^{\uparrow}$ with $\det(\Lambda) = +1$ and $\Lambda^{0}{}_{0} \geq 1$. The whole homogeneous Lorentz group can be obtained from this subgroup by multiplication with $\Lambda_{\mathrm{T}} = \mathrm{diag}(-1, 1, 1, 1)$ and $\Lambda_{\mathrm{P}} = \mathrm{diag}(1, -1, -1, -1)$. Therefore the analysis of the homogeneous Lorentz group, in the following simply referred to as Lorentz group, reduces to the study of the proper orthochronous subgroup.

2.2 The Poincaré algebra

We now consider the action of the Poincaré group on one-particle state vectors in a physical Hilbert space, i.e. how a Poincaré transformation (Λ, a) maps the state vector from one inertial frame into another.

Searching for a definition of an elementary particle, we first label our particle vectors by $|X\rangle$, where X represents the definition we search for. We assume the representation $U(\Lambda, a)$ of the transformation on the state vectors to be *linear*, as we want to preserve the superposition principle, and to obey the *composition rule* of the underlying symmetry group

$$U(\Lambda, a)\left(\xi|X\rangle + \zeta|Y\rangle\right) = \xi U(\Lambda, a)|X\rangle + \zeta U(\Lambda, a)|Y\rangle, \tag{2.6a}$$

$$U(\Lambda_2, a_2) U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2). \tag{2.6b}$$

This basically represents the mathematical definition of a representation. Furthermore we assume this representation to be *unitary* as we aim for a Poincaré invariant scalar product of our one-particle states.

$$U^{\dagger} U = 1. \tag{2.6c}$$

Now we consider Poincaré transformations near the identity of the form of

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}, \qquad a^{\mu} = \epsilon^{\mu}, \tag{2.7}$$

where $\omega^{\mu}{}_{\nu}$ and ϵ^{μ} are taken to be infinitesimal and $\omega^{\mu\nu}$ has to be antisymmetric in μ and ν to fulfill $\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\eta^{\rho\sigma} = \eta^{\mu\nu}$. This transformation can be represented on the Hilbert space by writing

$$U(1+\omega,\epsilon) = 1 + \frac{1}{2} i\omega_{\rho\sigma} M^{\rho\sigma} - i\epsilon_{\rho} P^{\rho} + \dots . \tag{2.8}$$

If one evaluates the physically motivated assumptions (2.6), one naturally finds *the Poincaré algebra* [2] given by

$$i[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\sigma\mu} M^{\rho\nu} + \eta^{\sigma\nu} M^{\rho\mu}, \tag{2.9a}$$

$$i[P^{\mu}, M^{\rho\sigma}] = \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho}, \tag{2.9b}$$

$$[P^{\mu}, P^{\rho}] = 0. \tag{2.9c}$$

Furthermore (2.6) yields

$$M^{\mu\nu\dagger} = M^{\mu\nu}, \qquad P^{\mu\dagger} = P^{\mu}, \tag{2.10}$$

and from the antisymmetry of $\omega^{\mu\nu}$ we find

$$M^{\mu\nu} = -M^{\nu\mu}. \tag{2.11}$$

For the generators P^{μ} and $M^{\mu\nu}$ we can define the *Pauli-Lubanski vector* by

$$W_{\mu} = \frac{1}{2}\epsilon_{\mu\rho\sigma\nu} M^{\rho\sigma} P^{\nu}, \tag{2.12}$$

and, furthermore,

$$C_1 = P_{\mu} P^{\mu}, \tag{2.13a}$$

$$C_2 = W_{\mu} W^{\mu}. \tag{2.13b}$$

It can be shown that the last two operators commute with all other operators of the Poincaré algebra. They represent the only quadratic *Casimir operators* of the Poincaré algebra. In particular C_1 and C_2 commute with each other and can therefore be simultaneously diagonalized. For this reason their eigenvalues are the best candidates to label our one-particle states

$$C_1 |p^2, w^2, Y\rangle = p^2 |p^2, w^2, Y\rangle, \tag{2.14}$$

$$C_2 |p^2, w^2, Y\rangle = w^2 |p^2, w^2, Y\rangle. \tag{2.15}$$

Here Y stands for possible other labels. Of course, p^2 represents the momentum square whereas the physical significance of w^2 will become clear in due course. Note that at this stage p^2 and w^2 can be negative since they represent the square of a four-vector.

Using the Poincaré generators we may now define a triple of three vector operators by

$$\mathbf{P} = (P^1, P^2, P^3), \qquad \text{(momentum)} \tag{2.16a}$$

$$\mathbf{J} = (M^{23}, M^{31}, M^{12}) \quad \text{or} \quad J_i = \frac{1}{2}\epsilon_{ijk} M_{jk}, \qquad \text{(total angular momentum)} \tag{2.16b}$$

$$\mathbf{K} = (M^{01}, M^{02}, M^{03}) \quad \text{or} \quad K_i = M_{0i}, \qquad \text{(boost)} \tag{2.16c}$$

which obey the following commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \qquad [P_i, P_j] = 0, \qquad [K_i, K_j] = -i\epsilon_{ijk} J_k, \tag{2.17a}$$

$$[J_i, P_j] = i\epsilon_{ijk} P_k, \qquad [J_i, K_j] = i\epsilon_{ijk} K_k, \qquad [K_i, P_j] = -iP^0 \delta_{ij}, \tag{2.17b}$$

$$[J_i, P^0] = 0, \qquad [P_i, P^0] = 0, \qquad [K_i, P^0] = -iP_i. \tag{2.17c}$$

We recognize that J_i and P_i are conserved as expected since they commute with the energy operator P^0. Furthermore we have seen in (2.9c) that the momentum operators also commute with each other and we therefore also choose their eigenvalue p^μ as a label

$$P^\mu |p, p^2, w^2, Z\rangle = p^\mu |p, p^2, w^2, Z\rangle, \tag{2.18}$$

where Z again denotes possible other labels. As K_i is not conserved, its eigenvalues are out of question for labeling particles. The vectors P_i and J_k do not commute with each other and as we have already chosen P^μ as a label we cannot use the total angular momentum J_i any more. Nevertheless the operators J_i are going to play an important role in our classification. We will see that the Poincaré Casimir C_2, the square of the Pauli-Lubanski vector, is the conserved quantity related to the total angular momentum. If we consider the total angular momentum as the sum of spin and orbital angular momentum

$$M^{\mu\nu} = S^{\mu\nu} + X^\mu P^\nu - X^\nu P^\mu, \tag{2.19}$$

we find that the Pauli-Lubanski vector only depends on the spin part $S^{\mu\nu}$. Hence, our considerations are valid for elementary particles without orbital angular momentum as well as for composed particles with angular momentum. In each case the quantum number w^2 is related to the spin of the particle as we will see in due course.

2.3 Representations

Following [2] we now want to find a way to determine the representations $U(\Lambda, a)$. At first we consider translations given by $U(1, a)$. As U is supposed to be a representation, we know that

$$U(1, a_1)U(1, a_2) = U(1, a_1 + a_2), \tag{2.20}$$

and therefore we can write

$$U(1, a) = \left[U(1, \frac{a}{n})\right]^n, \tag{2.21}$$

for an integer n. But since we consider a pure translation, Eq. (2.8) becomes

$$U(1, \epsilon) = 1 - i\epsilon_\rho P^\rho + \dots, \tag{2.22}$$

and hence, keeping only the first order term as a/n is small for large n,

$$U(1, a) = \lim_{n\to\infty} \left[1 - \frac{i}{n} a_\rho P^\rho\right]^n. \tag{2.23}$$

This yields

$$U(1, a) = e^{-ia_\rho P^\rho}, \tag{2.24}$$

and thus we have found the general form of a unitary representation of a translation. It is a general result that in a positive metric space each irreducible representation of an abelian group is one dimensional. If we apply (2.24) to our particle states, we see that a translation is given by a phase shift

$$U(1, a)|p, p^2, w^2, Z\rangle = e^{-ip\cdot a}|p, p^2, w^2, Z\rangle. \tag{2.25}$$

Having treated translations, our problem now is reduced to finding representations of the remaining Lorentz group on these states. We consider

$$\begin{aligned} P^\mu U(\Lambda,0)|p,p^2,w^2,Z\rangle &= U(\Lambda,0)\left[U^{-1}(\Lambda,0)P^\mu U(\Lambda,0)\right]|p,p^2,w^2,Z\rangle \\ &= U(\Lambda,0)\left(\Lambda^{-1\mu}_{\rho}P^\rho\right)|p,p^2,w^2,Z\rangle \\ &= \Lambda^\mu{}_\rho p^\rho U(\Lambda,0)|p,p^2,w^2,Z\rangle, \end{aligned} \tag{2.26}$$

to find that $U(\Lambda,0)|p,p^2,w^2,Z\rangle$ is an eigenstate of P with eigenvalue Λp. Since $|\Lambda p,p^2,w^2,Z\rangle$ spans by definition the eigenspace of ΛP, we find

$$U(\Lambda,0)|p,\alpha\rangle = \sum_\beta C'_{\alpha\beta}(\Lambda,p)|\Lambda p,\beta\rangle, \tag{2.27}$$

where α is a shorthand notation for $(p^2_\alpha, w^2_\alpha, Z_\alpha)$. We want to choose our labels α such that the matrix $C'(\Lambda,p)$ is in block diagonal form. As p^2 and w^2 are Casimir labels and the Casimir operators commute with all Poincaré operators and are therefore invariant under Lorentz transformations, we know that $|p,\alpha\rangle$ and $|\Lambda p,\beta\rangle$ in Eq. (2.27) have the same p^2 and w^2 eigenvalue. Therefore these labels are appropriate for the goal to write C in block diagonal form and we can rewrite Eq. (2.27) as

$$U(\Lambda,0)|p,p^2,w^2,Z\rangle = \sum_{Z'} C_{Z'Z}(\Lambda,p)|\Lambda p,p^2,w^2,Z'\rangle. \tag{2.28}$$

It is now possible to choose the labels Z such that the matrix $C(\Lambda,p)$ is block diagonal. In order to describe our particles, we look for the shortest possible definition which contains all information given by the symmetry transformations. Hence, it suffices to consider one block of $C(\Lambda,p)$ which by itself forms a representation of the Lorentz group. In the following we want to restrict to these irreducible representations.

As p^2 is an eigenvalue of a Casimir operator, it remains invariant under all Lorentz transformations. For each value of p^2 we can choose a reference four momentum k^μ with $k^2 = p^2$ which can be obtained by a Lorentz transformation L

$$p^\mu = L^\mu{}_\nu(p)k^\nu. \tag{2.29}$$

Then we find for the one-particle states

$$|p,p^2,w^2,Z\rangle = N(p)U(L(p),0)|k,p^2,w^2,Z\rangle, \tag{2.30}$$

where $N(p)$ is just a normalization factor. Now we can apply an arbitrary Lorentz transformation on this state to find

$$\begin{aligned} U(\Lambda,0)|p,p^2,w^2,Z\rangle &= N(p)U(\Lambda L(p),0)|k,p^2,w^2,Z\rangle \\ &= N(p)U(L(\Lambda p),0)U(L^{-1}(\Lambda p)\Lambda L(p),0)|k,p^2,w^2,Z\rangle. \end{aligned} \tag{2.31}$$

The transformation $L^{-1}(\Lambda p)\Lambda L(p)$ does actually nothing to the vector k^μ, i.e. it belongs to the *little group* or *stability subgroup* of k^μ given by transformations B with

$$B^\mu{}_\nu k^\nu = k^\mu. \tag{2.32}$$

For any B and $\bar{B}$ in the little group we have

$$U(B,0)|k,p^2,w^2,Z\rangle = \sum_{Z'} D_{Z'Z}(B)|k,p^2,w^2,Z'\rangle, \tag{2.33}$$

and the representation property yields

$$D_{Z'Z}(\bar{B}B) = \sum_{Z''} D_{Z'Z''}(\bar{B}) D_{Z''Z}(B). \tag{2.34}$$

Note that the matrix $D(B)$ on its own furnishes a representation of the little group. In particular we may choose the little group transformation in Eq. (2.31) to define

$$B(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p), \tag{2.35}$$

such that (2.31) becomes

$$U(\Lambda, 0)|p, p^2, w^2, Z\rangle = N(p) \sum_{Z'} D_{Z'Z}(B(\Lambda, p)) U(L(\Lambda p), 0)|k, p^2, w^2, Z'\rangle, \tag{2.36}$$

and by Eq. (2.30) we get

$$U(\Lambda, 0)|p, p^2, w^2, Z\rangle = \left(\frac{N(p)}{N(\Lambda p)}\right) \sum_{Z'} D_{Z'Z}(B(\Lambda, p))|\Lambda p, p^2, w^2, Z'\rangle. \tag{2.37}$$

We see that one can express the representation of the Lorentz group in terms of the matrix D which corresponds to a representation of the little group[1]. Comparison of (2.37) and (2.28) makes it clear that the induced representation of the Lorentz group represented by the matrix C is irreducible if and only if the induced representation of the little group given by the matrix D is irreducible. Hence, the problem of finding the irreducible representations of the Lorentz group is reduced to finding irreducible representations of the little group. This also implies that the little group is sufficient to further categorize a particle. It can be shown [2] that a useful choice of normalization is given by

$$N(p) = \sqrt{\frac{k^0}{p^0}}, \tag{2.38}$$

so that we get a Lorentz invariant scalar product of the form

$$\langle p', \alpha' | p, \alpha\rangle = \delta_{\alpha'\alpha}\delta^3(\mathbf{p}' - \mathbf{p}). \tag{2.39}$$

Note that the Poincaré invariance of this normalization of one-particle states requires unitarity of the representation on the states.

We now want to analyze the Lie algebra of the little group of a reference four-momentum k^μ as it suffices to further classify our particle. A general Lorentz transformation near the identity can be written as

$$U(1 + \omega, 0) = 1 + \frac{1}{2} i\omega_{\rho\sigma} M^{\rho\sigma} + \dots, \tag{2.40}$$

where again ω is assumed to be infinitesimal. This implies that k^μ transforms as

$$\Lambda^\mu{}_\nu k^\nu = (1 + \omega^\mu{}_\nu)k^\nu + \dots. \tag{2.41}$$

Little group transformations are defined to leave k^μ invariant which yields

$$\omega_{\mu\nu} k^\nu = 0. \tag{2.42}$$

[1] This method is called *method of induced representations* and is studied in detail in [3].

Table 1 Representations of the Poincaré group

Gender	Orbit	Reference k^μ	Little group	Representation
$p^2 = -m^2$	Mass-shell	$(\pm m, 0, 0, 0)$	$SO(3)$	Massive
$p^2 = 0$	Light-cone	$(\pm E, 0, 0, E)$	$ISO(2)$	Massless
$p^2 = n^2$	Hyperboloid	$(0, 0, 0, n)$	$SO(2,1)^\uparrow$	Tachyonic
$p^\mu = 0$	Origin	$(0, 0, 0, 0)$	$SO(3,1)^\uparrow$	Vacuum

This gives us the definiton of the elements ω of the Lie algebra of the little group, a Lie subgroup of the Lorentz group. Since $\omega_{\mu\nu}$ is antisymmetric, the general solution of (2.42) can be parametrized by an arbitrary constant vector n^ν, i.e.

$$\omega_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} n^\rho k^\sigma. \tag{2.43}$$

Therefore a generic element of the little group algebra restricted to the eigenspace of P with respect to k^μ can be described by

$$M^{\mu\nu}\omega_{\mu\nu} = n^\rho \epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} k^\sigma, \tag{2.44}$$

where we have just multiplied Eq. (2.43) with the generators $M^{\mu\nu}$. But in this equation we rediscover the Pauli-Lubanski vector as the generator of the little group

$$W_\rho = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} P^\sigma. \tag{2.45}$$

Because of the antisymmetry of the ϵ-symbol and the symmetry of the product $P^\sigma P^\mu$, we see that

$$W_\mu P^\mu = 0, \tag{2.46}$$

which implies that the little group is in general three dimensional.

Writing W^μ in terms of the three vectors (2.16),

$$W^0 = \mathbf{P}\cdot\mathbf{J}, \qquad \mathbf{W} = -\mathbf{J}P^0 - \mathbf{K}\times\mathbf{P}, \tag{2.47}$$

we find that the zero component of the Pauli-Lubanski vector equals the projection of the spin onto the momentum vector. This means that for a single elementary particle, W^0 is proportional to the helicity operator.

Let us look at what we have done on the last pages. We started with a particle state $|X\rangle$ in a Hilbert space and the Poincaré group as the fundmental symmetry group of Minkowski spacetime. We found that the Poincaré algebra has two Casimir operators which can be used to label particles by their eigenvalues. As the four momentum operators P^μ commute with each other and are conserved, their eigenvalues can also be used as a label. At this point we can give the general name $|p, p^2, w^2, Z\rangle$ to the particle where Z is still undetermined. We saw that the representation of a translation is given by $\exp(-ia_\mu P^\mu)$. In search of representations of the remaining Lorentz group we found that for each value of p^2 we can choose a reference momentum k^μ. Represention of the Lorentz group are then equivalent to representations of the little group of k^μ. The little group algebra is generated by the Pauli-Lubanski vector and we can now look at the different categories of p^2 to analyze which algebra W^μ generates in each case [4]. In the following we only consider the massive and massless case. We furthermore restrict ourselves to $p^0 > 0$, i.e. to positive energy.

2.3.1 Massive representations

For a massive particle with $p_\mu p^\mu = -m^2$, we can always find a Lorentz frame in which $k^\mu = (m, 0, 0, 0)$. It is obvious that the little group which leaves this vector invariant is the rotation group $O(3)$ and as we are only interested in a classification by algebra operators, we can restrict to $SO(3)$. On the other hand we can also consider the action of the Pauli-Lubanski vector (2.47) restricted to this momentum class

$$W_0|k, m, w^2, Z\rangle = \mathbf{J} \cdot \mathbf{P}|k, m, w^2, Z\rangle = 0, \tag{2.48a}$$

$$W_i|k, m, w^2, Z\rangle = -J_i P^0|k, m, w^2, Z\rangle = -mJ_i|k, m, w^2, Z\rangle. \tag{2.48b}$$

This shows that the action of W^μ is reduced to the action of the angular momentum operator J_i and hence the generated algebra is indeed the Lie algebra of $SO(3)$ representing the spin of the particle in its rest frame. Thus, we can use the eigenvalues j and j_3 to label our states. The Casimir operator C_2 is now given by

$$C_2 = W_\mu W^\mu = -m^2 \mathbf{J}^2, \tag{2.49}$$

i.e. proportional to the Casimir of the $SO(3)$ algebra. Because of the irreducibility of the representation and because C_2 is a Casimir, it must be proportional to the identity on the whole Hilbert space by Schur's Lemma. Since we know the spectrum of the angular momentum operator, we have

$$C_2 = -m^2 j(j-1) \cdot 1. \tag{2.50}$$

The complete set of labels for the particle which represents a massive irreducible representation of the Poincaré group is therefore given by

$$|X\rangle = |p, m, j, j_3\rangle. \tag{2.51}$$

This can be regarded as the *definition of a massive particle* based on the fundamental symmetry group of the assumed background geometry, namely Minkowsi spacetime.

2.3.2 Massless representations

We now want to consider massless representations with $p_\mu p^\mu = 0$. The natural reference momentum is given by $k^\mu = (E, 0, 0, E)$. However, to determine the little group in this case is not as easy as for the massive representations. The only thing we can say immediately is that $SO(2)$ has to be a part of the little group. Again we compute the action of the Pauli-Lubanski vector on $|k, w^2, Z\rangle = |k, p^2 = 0, w^2, Z\rangle$ given by

$$W_0|k, w^2, Z\rangle = \mathbf{J} \cdot \mathbf{P}|k, w^2, Z\rangle = EJ_3|k, w^2, Z\rangle, \tag{2.52a}$$

$$W_1|k, w^2, Z\rangle = (-\mathbf{J}P^0 - \mathbf{K} \times \mathbf{P})_1|k, w^2, Z\rangle = E(K_2 - J_1)|k, w^2, Z\rangle, \tag{2.52b}$$

$$W_2|k, w^2, Z\rangle = (-\mathbf{J}P^0 - \mathbf{K} \times \mathbf{P})_2|k, w^2, Z\rangle = -E(K_1 + J_2)|k, w^2, Z\rangle, \tag{2.52c}$$

$$W_3|k, w^2, Z\rangle = (-\mathbf{J}P^0 - \mathbf{K} \times \mathbf{P})_3|k, w^2, Z\rangle = -EJ_3|k, w^2, Z\rangle. \tag{2.52d}$$

Motivated by these equations we define

$$A = K_1 + J_2, \tag{2.53a}$$

$$B = K_2 - J_1, \tag{2.53b}$$

and find the commutation relations for the three generators of the little group of a massless particle

$$[A, B] = 0, \tag{2.54a}$$

$$[J_3, A] = iB, \tag{2.54b}$$

$$[J_3, B] = -iA. \tag{2.54c}$$

These commutation relations can be identified with the algebra of the *Euclidean group* $ISO(2)$ consisting of two dimensional rotations and translations. As the operators A and B commute with each other, they can be simultaneously diagonalized and their eigenvalues appear to be useful labels for the particles. However, if we set

$$A|k, w^2, Z\rangle = a|k, w^2, Z\rangle, \tag{2.55}$$

$$B|k, w^2, Z\rangle = b|k, w^2, Z\rangle, \tag{2.56}$$

we can define functions f and g of a parameter $\theta \in \mathbb{R}$ by [5]

$$f(\theta) = e^{-i\theta J_3} A e^{i\theta J_3} |k, w^2, Z\rangle, \tag{2.57a}$$

$$g(\theta) = e^{-i\theta J_3} B e^{i\theta J_3} |k, w^2, Z\rangle. \tag{2.57b}$$

Using the commutation relations (2.54) it can be shown that

$$\frac{df}{d\theta} = g, \tag{2.58a}$$

$$\frac{dg}{d\theta} = -f. \tag{2.58b}$$

Solving this with the initial conditions $f(0) = a|k, w^2, Z\rangle$ and $g(0) = b|k, w^2, Z\rangle$, we find

$$f(\theta) = (a\cos\theta + b\sin\theta)\,|k, w^2, Z\rangle, \tag{2.59a}$$

$$g(\theta) = (b\cos\theta - a\sin\theta)\,|k, w^2, Z\rangle, \tag{2.59b}$$

and hence

$$Ae^{i\theta J_3}|k, w^2, Z\rangle = (a\cos\theta + b\sin\theta)\,e^{i\theta J_3}|k, w^2, Z\rangle, \tag{2.60a}$$

$$Be^{i\theta J_3}|k, w^2, Z\rangle = (b\cos\theta - a\sin\theta)\,e^{i\theta J_3}|k, w^2, Z\rangle. \tag{2.60b}$$

This implies that for all $\theta \in \mathbb{R}$ the state $e^{i\theta J_3}|k, w^2, Z\rangle$ is an eigenstate of A and B. However, we do not observe any continuous degrees of freedom for massless particles which forces us to impose a restriction: We simply limit the physical states to those with $A = B = 0$. This issue arises because the little group of a massless particle $ISO(2)$ is not semi-simple as it has an Abelian subgroup given by the translations

$$ISO(2) = SO(2) \ltimes P(2). \tag{2.61}$$

However, this restriction makes the problem of labeling massless one-particle states much easier. We can now distinguish the state vectors by their eigenvalue of J_3 which generates a $U(1)$ algebra

$$J_3|p, w^2, j_3\rangle = j_3|p, w^2, j_3\rangle. \tag{2.62}$$

Here we have replaced the missing label Z by j_3. If we remember that $W_0 = \mathbf{J}\cdot\mathbf{P} = EJ_3$ in the massless case, we see that the eigenvalue j_3 corresponds to the helicity of the particle. Since $U(1)$ is abelian like the translation group, all representations of the little group are of the form $e^{i\theta J_3}$. The action of the Casimir operator C_2 on the k^μ subspace is now given by

$$C_2 = W_\mu W^\mu = 0, \tag{2.63}$$

and again by Schur's Lemma C_2 and thereby our label w^2 vanish on the whole Hilbert space. Hence, we find the *definition of a massless particle* given by

$$|X\rangle = |p, j_3\rangle. \tag{2.64}$$

Massless particles do not have a full spin representation as their little group contains only two dimensional rotations given by $SO(2)$ and are therefore classified by only the value of their helicity j_3.

Now we can consider how a massless particle state transforms under a Lorentz transformation. By arguments similar to those given for the representations of the translation group, a generic element G of the massless little group can be written in the form

$$U(G, 0) = \exp(i\alpha A + i\beta B) \exp(iJ_3\theta), \tag{2.65}$$

where the last part corresponds to a rotation. If we take into account that we have restricted physical states to those with $A = B = 0$, we find

$$U(G, 0)|k, j_3\rangle = \exp(i\theta j_3)|k, j_3\rangle. \tag{2.66}$$

Comparison to (2.33) yields

$$D_{j_3' j_3}(G) = \exp(i\theta J_3)\delta_{j_3' j_3}, \tag{2.67}$$

where Z was replaced by j_3. The Eqs. (2.37) and (2.38) now show how the helicity representations of a general Lorentz transformation act on the massless state vector

$$U(\Lambda, 0)|p, j_3\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \exp[ij_3\theta(\Lambda, p)]|\Lambda p, j_3\rangle. \tag{2.68}$$

Here $\theta(\Lambda, p)$ is the angle defined by the rotation component of the Lorentz transformation Λ. Note that the helicity of a massless particle is Lorentz invariant in contrast to that of a massive particle. The physical reason for this is that we cannot change the velocity of a massless particle and thereby the projection of its spin onto the momentum by going to a different reference frame.

3 The Weinberg-Witten theorem

We have seen how to classify massless particles by means of the Poincaré group. Furthermore it was shown that it is rather easy to find representations of the abelian translation group and that the more complex structures are hidden in the Lorentz group. As a matter of fact the Lorentz group is a powerful tool. Already in 1965 Weinberg stated that "Maxwell's theory and Einstein's theory are essentially the unique Lorentz invariant theories of massless particles with spin $j = 1$ and $j = 2$" [6]. Note, however, that besides the crucial assumption of Poincaré invariance, Weinberg also uses the fact that far from the source we experience an inverse square law for the electrostatic and the gravitational interactions. In the following theorem we get to know another fundamental constraint on massless particles primarily based on the assumption of Poincaré covariance.

3.1 The theorem

We consider the Weinberg-Witten theorem as presented in the original paper [1]. It is interesting to notice that the first part of this theorem was already shown in 1962 by Case and Gasiorowicz [7].

Weinberg-Witten theorem.

(A) A theory containing a Poincaré covariant conserved four-vector current J^μ forbids massless particles of spin $j > 1/2$ with a non-vanishing eigenvalue of the conserved charge $\int J^0 d^3x$.

(B) A theory containing a Poincaré covariant conserved tensor $T^{\mu\nu}$ forbids massless particles of spin $j > 1$ for which $\int T^{0\nu} d^3x$ is the conserved energy-momentum four-vector.

These statements hold for elementary as well as for composite particles. Note that the structure of both parts of the theorem is the same. If we include the energy momentum four-vector into our notion of charge, we can generally speak of charged particles under Poincaré covariant currents. Requiring that $\int T^{0\mu} d^3x$ represents the energy-momentum four-vector implies that this quantity does not vanish, whereas in the first part of the theorem we claim explicitly that $\int J^0 d^3x$ is non-vanishing.

Table 2 The Weinberg-Witten theorem

Theorem	A theory with a	forbids massless particles of
(A)	Poincaré covariant conserved current J^μ	spin $j > \frac{1}{2}$ and $Q = \int J^0 d^3x \neq 0$.
(B)	Poincaré covariant conserved tensor $T^{\mu\nu}$	spin $j > 1$ and $P^\mu = \int T^{0\mu} d^3x \neq 0$.

3.2 The proof

As both parts of the theorem have the same structure, they can be proved similarly. In principle we follow the elegant presentation of Weinberg and Witten [1]. This procedure is not entirely unquestioned [8–12] and we hope to clarify some misunderstandings at certain points of the proof. These misunderstandings are especially based on a footnote of Weinberg and Witten in which the authors explain how they define charge and four-momentum in terms of limits of matrix elements without requiring *continuity* of these matrix elements. Nevertheless, some of the papers cited above took this footnote as a reason to claim that Weinberg and Witten would require continuity and their proof was even described as "inconclusive" [11]. According to Witten, he and Weinberg did just not want to give an a priori discussion of what kind of possible non-smoothness the matrix elements of T or J might have at zero momentum because this was not necessary[2] [13]. In the following we hope to shed some light on this point related to the difficulties of defining charge in a physical, i.e. experimental sense.

P r o o f. We consider the matrix elements of the given Poincaré covariant quantities J^μ and $T^{\mu\nu}$ between massless one-particle states

$$\langle p', \pm j | J^\mu | p, \pm j \rangle \qquad \text{and} \qquad \langle \mathrm{p}', \pm \mathrm{j} | \mathrm{T}^{\mu\nu} | \mathrm{p}, \pm \mathrm{j} \rangle, \tag{3.1}$$

where we have introduced the one-particle states $|p, \pm j\rangle$ of four momentum p and helicity $\pm j$. The basic idea of the proof is to show that these matrix elements $\boxed{1}$ cannot vanish in the limit $p' \to p$ and that they

[2] The paper [8] can be regarded as an a posteriori discussion of possible non-smoothness of these matrix elements.

2 have to vanish in this limit for spin $j > 1/2$ or $j > 1$ massless particles, respectively. We conclude that a theory based on the given assumptions cannot contain massless particles of higher spin. The key point is that we derive a contradiction between equations for matrix elements defined in the limit $p' \to p$. We are not interested in the case $p' = p$. Note, in particular, that we do not require continuity of the matrix elements at $p' = p$. If the limit exists we arrive at a contradiction.

1 In this first part of the proof we show that the matrix elements (3.1) cannot vanish in the limit $p' \to p$. For this purpose we use our assumption of the existence of nonvanishing charges $Q = \int J^0 d^3x$ in part (A) and the corresponding expression $P^\mu = \int T^{0\mu} d^3x$ in part (B) of the theorem. Following the arguments of Weinberg and Witten, we want to base the definition of charge and four-momentum of a particle on how these quantities are measured in physical experiments. Note that, particularly for massless particles, the translation of this measurement process into mathematical quantities is far from being trivial [14, 15]. As Weinberg and Witten state in the footnote of their paper, charges, energies, and momenta are determined by "measuring the nearly forward scattering caused by exchange of spacelike but nearly lightlike massless vector bosons or gravitons" [1]. This limit of lightlike momentum transfer corresponds to the limit $(p' - p) \to 0$. As the momentum transfer goes to zero, the scattering matrix becomes dominated by its one-particle exchange poles. In this limit only the charge and not the higher multipole moments are measured [16]. The measuring process of charges is therefore represented by approaching lightlike momentum transfer. Hence, we could *define* the charge q and the four-momentum p^μ of the particles like Weinberg and Witten for $p' \to p$ by

$$\langle p', \pm j | J^\mu | p, \pm j \rangle \to \frac{q p^\mu}{E(2\pi)^3}, \tag{3.2a}$$

$$\langle p', \pm j | T^{\mu\nu} | p, \pm j \rangle \to \frac{p^\mu p^\nu}{E(2\pi)^3}. \tag{3.2b}$$

The form of the right hand side of these equations, i.e. the proportionality to p^μ, is dictated by Poincaré covariance. If we define the charges in this way, i.e. based on physical measurement, it does not matter how the matrix elements look like at $p' = p$. In particular we do not require continuity of the matrix elements at $p' = p$. This is what Weinberg and Witten explain in the footnote of their original paper. However, they also state that "a continuity assumption seems entirely plausible" [1]; [8] disproves this assumption giving examples for currents with a discontiuity[3] at $p' = p$.

The mathematical problem behind upcoming misunderstandings seems to be that physicists like to make computations with sharp one-particle states normalized by $\langle p', \pm j | p, \pm j \rangle = \delta^{(3)}(\mathbf{p'} - \mathbf{p})$. Even though these are actually not physical, one might easily forget this fact. We believe this to be a crucial point in the proof of the theorem, as the arguments of Weinberg and Witten are based on physical measurements. The proof was recently revised by Jenkins who works out some of the sketchy comments of Weinberg and Witten [17]. However, the arguments presented in [17] rely on the sharp definition of the one-particle states. This is rather misleading and finally still requires an assumption of continuity in order to end up with a contradiction, in contrast to the proof of Weinberg and Witten.

Weinberg and Witten find Eqs. (3.2) by inspection. We want to try to derive these equations close to the ideas given in [17] but keeping at the back of our mind that we are dealing with physical particle states which are not normalized by an exact delta function.

[3] The continuity assumption is not required in the proof of Weinberg and Witten. The relevance of this important point can be read off from the following quotations referring to the Weinberg and Witten paper [1]:
1) "If these results are taken together with a continuity requirement on the matrix elements [...] one ends with a contradiction which can be removed only by assuming that such particles do not exist" [8].
2) "A first remark is that the continuity assumption of a kernel around the diagonal $p = p'$, which is replaced in a footnote by a plausibility argument based on measurement, is by no means an obvious property" [9].
3) "Weinberg and Witten claim that (3.1) or (3.3) imply the vanishing of these matrix elements also for $(p' - p)^2 = 0$ by continuity. Unfortunately, their statement is incorrect as the continuity with respect to the particle momenta cannot be proven, which was pointed out by Sudarshan and Flato et al." [11] referring to [8, 9].
4) "We complete the proof of the theorem, as that given by Weinberg and Witten it is inconclusive" [sic] [11].

We define the charge and the four-momentum of the massless one-particle states as eigenvalues of Q and P^μ

$$Q|p, \pm j\rangle = q|p, \pm j\rangle, \qquad q \neq 0, \tag{3.3a}$$

$$P^\mu|p, \pm j\rangle = p^\mu|p, \pm j\rangle, \qquad p^\mu \neq 0. \tag{3.3b}$$

In contrast to the sharp unphysical particle states, we want to deal with physical states which are smeared out and hence normalized by

$$\langle p, \pm j|p, \pm j\rangle_{\rm phys} = \delta_a^{(3)}(\mathbf{p'} - \mathbf{p}), \tag{3.4}$$

where $\delta_a^{(3)}(\mathbf{p'} - \mathbf{p})$ is a nascent delta function sharply peaked at $\mathbf{p'} = \mathbf{p}$ with

$$\lim_{a \to 0} \delta_a^{(3)}(\mathbf{p'} - \mathbf{p}) = \delta^{(3)}(\mathbf{p'} - \mathbf{p}). \tag{3.5}$$

The parameter a can be regarded as small but finite. It is defined by the level of sharpness we can achieve in physical experiments.

What we are actually doing here is to consider the physical state vector as a superposition of pure momentum eigenstates in the form of

$$|p, \pm j\rangle_{\rm phys} = \int d^3\tilde{p}\, \delta_a^{(3)}(\tilde{\mathbf{p}} - \mathbf{p})|\tilde{p}, \pm j\rangle. \tag{3.6}$$

Then we find for the normalization

$$\begin{aligned} \langle p_1, \pm j|p_2, \pm j\rangle_{\rm phys} &= \int d^3\tilde{p}_1 d^3\tilde{p}_2\, \delta_{a,1}^{(3)}(\tilde{\mathbf{p}}_1 - \mathbf{p}_1)\delta_{a,2}^{(3)}(\tilde{\mathbf{p}}_2 - \mathbf{p}_2)\delta^{(3)}(\tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2) \\ &= \int d^3\tilde{p}_1\, \delta_{a,1}^{(3)}(\tilde{\mathbf{p}}_1 - \mathbf{p}_1)\delta_{a,2}^{(3)}(\tilde{\mathbf{p}}_1 - \mathbf{p}_2) \\ &= \delta_a^{(3)}(\mathbf{p}_1 - \mathbf{p}_2) \end{aligned} \tag{3.7}$$

where in the last equation we have defined δ_a. Note that if we define a physical state as in (3.6) this state is no longer a pure momentum eigenstate. The eigenmomentum also becomes a superposition of the form

$$P^\mu|p, \pm j\rangle_{\rm phys} = \int d^3\tilde{p}\, \delta_a^{(3)}(\tilde{\mathbf{p}} - \mathbf{p})\tilde{p}^\mu|\tilde{p}, \pm j\rangle. \tag{3.8}$$

However, the momentum eigenvalue can be identified with the eigenvalue p^μ since it is sharply peaked at this momentum. In the following we drop the subscript '$\rm phys$' to keep the notation as simple as possible. The essence of the last few lines is that the normalization of our particle states is given by $\delta_a^{(3)}$ rather than $\delta^{(3)}$.

Continuing the proof, Eqs. (3.3) and (3.4) yield

$$\langle p', \pm j|Q|p, \pm j\rangle = q\delta_a^{(3)}(\mathbf{p} - \mathbf{p'}), \tag{3.9a}$$

$$\langle p', \pm j|P^\mu|p, \pm j\rangle = p^\mu\delta_a^{(3)}(\mathbf{p} - \mathbf{p'}). \tag{3.9b}$$

On the other hand we can evaluate the charge and the four-momentum as a physical integral over a large but finite volume V_a of radius $1/a$. This corresponds to the physical definition of the normalization in terms of the parameter a. We find

$$\langle p', \pm j|Q|p, \pm j\rangle = \int_{V_a} d^3x\, \langle p', \pm j|J^0(t, \mathbf{x})|p, \pm j\rangle$$

$$= \int_{V_a} d^3x \, \langle p', \pm j | e^{i\mathbf{P}\cdot\mathbf{x}} J^0(t,0) e^{-i\mathbf{P}\cdot\mathbf{x}} | p, \pm j\rangle$$
$$= \int_{V_a} d^3x \, e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} \langle p', \pm j | J^0(t,0) | p, \pm j\rangle$$
$$= (2\pi)^3 \delta_a^{(3)}(\mathbf{p}' - \mathbf{p}) \langle p', \pm j | J^0(t,0) | p, \pm j\rangle, \tag{3.10a}$$

and

$$\langle p', \pm j | P^\mu | p, \pm j\rangle = \int_{V_a} d^3x \, \langle p', \pm j | T^{0\mu}(t,\mathbf{x}) | p, \pm j\rangle$$
$$= \int_{V_a} d^3x \, \langle p', \pm j | e^{i\mathbf{P}\cdot\mathbf{x}} T^{0\mu}(t,0) e^{-i\mathbf{P}\cdot\mathbf{x}} | p, \pm j\rangle$$
$$= \int_{V_a} d^3x \, e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} \langle p', \pm j | T^{0\mu}(t,0) | p, \pm j\rangle$$
$$= (2\pi)^3 \delta_a^{(3)}(\mathbf{p}' - \mathbf{p}) \langle p', \pm j | T^{0\mu}(t,0) | p, \pm j\rangle. \tag{3.10b}$$

The last equality in each case is justified by

$$\lim_{a\to 0} \int_{V_a} d^3x \, e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} = \delta^{(3)}(\mathbf{p}' - \mathbf{p}). \tag{3.11}$$

Comparison of (3.9) and (3.10) shows

$$\lim_{p'\to p} \langle p', \pm j | J^0(t,0) | p, \pm j\rangle = \frac{q}{(2\pi)^3}, \tag{3.12a}$$

$$\lim_{p'\to p} \langle p', \pm j | T^{0\mu}(t,0) | p, \pm j\rangle = \frac{p^\mu}{(2\pi)^3}, \tag{3.12b}$$

and from this we get by Lorentz covariance

$$\lim_{p'\to p} \langle p', \pm j | J^\mu | p, \pm j\rangle = \frac{qp^\mu}{E(2\pi)^3} \neq 0, \tag{3.13a}$$

$$\lim_{p'\to p} \langle p', \pm j | T^{\mu\nu} | p, \pm j\rangle = \frac{p^\mu p^\nu}{E(2\pi)^3} \neq 0, \tag{3.13b}$$

i.e. we recover the physical definition of the charges (3.2) given by Weinberg and Witten[4]. If we had not used the physical parameter a we could have only concluded that (3.13) holds for $\mathbf{p}' = \mathbf{p}$, i.e. at the only point where the delta-function in (3.9) and (3.10) would not vanish. In this case we would have required continuity of the matrix elements to extend the equation to the limit $p' \to p$, as it was done by Jenkins [17]. However, Weinberg and Witten explicitly state that they do not want to use this continuity assumption for their proof. That this was a good decision is illustrated by the examples for discontinuous currents given in [8]. Our treatment in terms of physical particle states corresponds to the measurement argument given by Weinberg and Witten.

As we are dealing with massless particles for which

$$p^\mu p_\mu = 0, \tag{3.14}$$

the right hand side of Eqs. (3.13) vanishes, if multiplied by p^μ. Hence, these equations imply that the matrix elements of the divergences of J^μ and $T^{\mu\nu}$ have to vanish. Therefore one might think that this is the only

[4] Note that it was stated in [9] that the definitions in Eq. (3.13) were "guessed" by Weinberg and Witten.

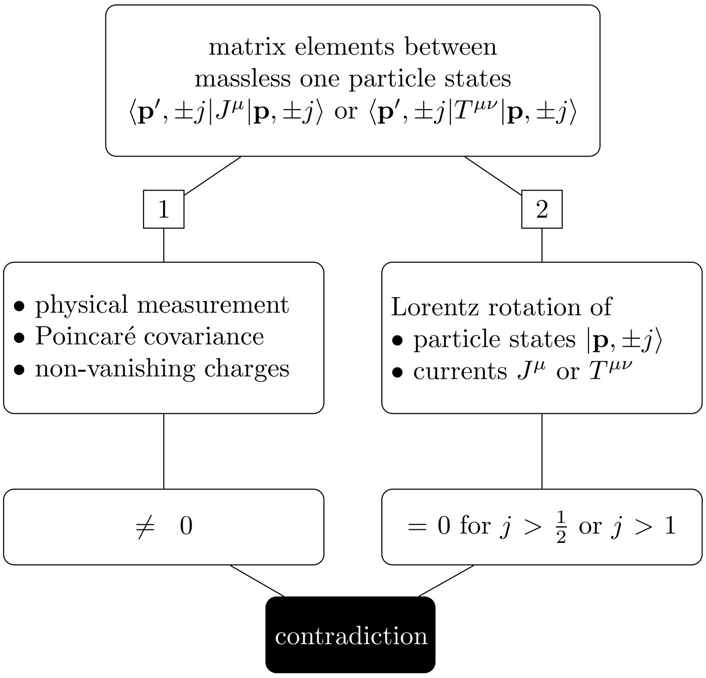

Fig. 1 The proof of the Weinberg-Witten theorem

point of the proof where we need the condition that J^μ and $T^{\mu\nu}$ are conserved tensors[5]. Note, however, that it is not clear, especially in the presence of massless particles, how to find a well-defined charge operator for a non-conserved current [18, 19].

The assumption that the massless particle states are charged under the given currents implies that the right hand sides of Eqs. (3.13) do not vanish and therefore we have shown that the matrix elements of J^μ and $T^{\mu\nu}$ do not vanish in the limit $p' \to p$.

$\boxed{2}$ Now we want to argue in the opposite direction in order to show that these matrix elements have to vanish for massless particles of higher spin. For arbitrary light-like p and p' we have

$$(p' + p)^2 = 2(p' \cdot p) = 2(\mathbf{p}' \cdot \mathbf{p} - |\mathbf{p}'||\mathbf{p}|)$$
$$= 2|\mathbf{p}'||\mathbf{p}|(\cos\phi - 1) \leq 0, \tag{3.15}$$

where ϕ is the angle between the three-momenta $\mathbf{p}$ and $\mathbf{p}'$. We may distinguish the two cases $\phi \neq 0$ and $\phi = 0$. As we are only interested in the limit $p' \to p$, the second case is not relevant for the proof. Nevertheless it is interesting to consider it with regard to the continuity discussion above.

For $\phi \neq 0$ the momentum sum is timelike and by Poincaré covariance we can choose a natural Lorentz frame in which $(p' + p)$ has no spatial components such that in this frame

$$p = (|\mathbf{p}|, \mathbf{p}), \qquad p' = (|\mathbf{p}|, -\mathbf{p}). \tag{3.16}$$

Now we consider a rotation by an angle θ around the $\mathbf{p}$ direction. A rotation by $+\theta$ around $\mathbf{p}$ implies a rotation of $\theta' = -\theta$ around $\mathbf{p}' = -\mathbf{p}$. Note that this is only true for $\phi \neq 0$ where we can choose the spatial momenta to have opposite sign. The induced transformations on the particle states are given by Eq. (2.68),

[5] Sudarshan states in his discussion of the proof of Weinberg and Witten [8]: "In the general case with particles of $j \geq 3/2$ and $j \geq 1$, respectively, we also prove that the divergence of any tensor and any vector vanishes between one-particle states." Furthermore he says about the tensor in part (B) of the theorem: "This tensor need not be the energy-momentum tensor nor need it be conserved." However, we do not see where his statement is proved in [8].

where the normalization factor is equal to 1 since $\Lambda p = p$ under the rotation considered. Hence, we get

$$|p, \pm j\rangle \to e^{\pm i\theta j}|p, \pm j\rangle, \tag{3.17a}$$

$$|p', \pm j\rangle \to e^{\mp i\theta j}|p', \pm j\rangle, \tag{3.17b}$$

which yields

$$\langle p', \pm j|J^{\mu}(t,0)|p, \pm j\rangle \to e^{\pm 2i\theta j}\langle p', \pm j|J^{\mu}(t,0)|p, \pm j\rangle, \tag{3.18a}$$

$$\langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle \to e^{\pm 2i\theta j}\langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle. \tag{3.18b}$$

On the other hand, a rotation can be described by the fundamental representation of the Lorentz group acting on J^{μ} and $T^{\mu\nu}$ making use of the postulated Poincaré covariance of these quantities. The helicities are invariant under Poincaré transformations as seen in Sect. 2.3.2. Equating two different transformation rules for the same rotation, we find

$$e^{\pm 2i\theta j}\langle p', \pm j|J^{\mu}(t,0)|p, \pm j\rangle = \Lambda(\theta)^{\mu}{}_{\nu}\langle p', \pm j|J^{\nu}(t,0)|p, \pm j\rangle, \tag{3.19a}$$

$$e^{\pm 2i\theta j}\langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle = \Lambda(\theta)^{\mu}{}_{\rho}\Lambda(\theta)^{\nu}{}_{\sigma}\langle p', \pm j|T^{\rho\sigma}(t,0)|p, \pm j\rangle, \tag{3.19b}$$

where $\Lambda(\theta)$ represents the Lorentz transformation according to a rotation by θ around the direction of $\mathbf{p}$, i.e. a simple rotation matrix. This can be compared to the idea of active and passive transformations. On the left hand side we transform the basis states while on the right hand side we transform the currents themselves. That $\Lambda(\theta)$ is a rotation matrix implies that the eigenvalues of this matrix can only be $e^{\pm i\theta}$ and 1. Therefore all components of the matrix elements of $J^{\mu}(t,0)$ in our special Lorentz frame have to vanish unless $2j \in \{0,1\}$. Similarly all matrix elements of $T^{\mu\nu}$ have to vanish unless $2j \in \{0,1,2\}$.

As J^{μ} and $T^{\mu\nu}$ are assumed to be Poincaré covariant and the helicities of massless particles are Poincaré invariant, these matrix elements have to vanish in all Lorentz frames. This holds for all momenta with $\phi \neq 0$ and thus also in the limit $p' \to p$. Again, only the limit is important, the value of the matrix element at $p' = p$ does not matter.

Let us look at what we have done. In the first part of the proof we found Eqs. (3.13) stating that

$$\lim_{p' \to p}\langle p', \pm j|J^{\mu}(t,0)|p, \pm j\rangle = \frac{qp^{\mu}}{E(2\pi)^3} \neq 0,$$

$$\lim_{p' \to p}\langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle = \frac{p^{\mu}p^{\nu}}{E(2\pi)^3} \neq 0,$$

whereas the second part yields

$$\lim_{p' \to p}\langle p', \pm j|J^{\mu}(t,0)|p, \pm j\rangle = 0 \quad \text{for} \quad j > \frac{1}{2}, \tag{3.21a}$$

$$\lim_{p' \to p}\langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle = 0 \quad \text{for} \quad j > 1. \tag{3.21b}$$

This is the contradiction we wanted and the end of the proof. Under the given assumptions there cannot be any charged massless particles of spin $j > 1/2$ and $j > 1$, respectively. We have not required that the matrix elements are continuous at $p' = p$.

□

Nevertheless we should not forget that we can distinguish two different cases for the angle ϕ. Even though not necessary for the proof, this shows that the considered matrix elements do not have to be continuous. For $\phi = 0$ we consider the case that $\mathbf{p}'$ and $\mathbf{p}$ are parallel. Hence, the particle states transform as

$$|p, \pm j\rangle \to e^{\pm i\theta j}|p, \pm j\rangle, \tag{3.22a}$$

$$|p', \pm j\rangle \to e^{\pm i\theta j}|p', \pm j\rangle, \tag{3.22b}$$

i.e. equally, in contrast to (3.17), because we have now a rotation of $+\theta$ around both $\mathbf{p}$ and $\mathbf{p}'$. Therefore, under a rotation we have on the one hand

$$\langle p', \pm j|J^\mu(t,0)|p, \pm j\rangle \to \langle p', \pm j|J^\mu(t,0)|p, \pm j\rangle, \tag{3.23a}$$

$$\langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle \to \langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle, \tag{3.23b}$$

i.e. the matrix elements are now invariant if we apply the rotation on the states. Evaluating the rotation as a Lorentz transformation applied to the tensors on the other hand shows that

$$\langle p', \pm j|J^\mu(t,0)|p, \pm j\rangle = \Lambda(\theta)^\mu{}_\nu \langle p', \pm j|J^\nu(t,0)|p, \pm j\rangle, \tag{3.24a}$$

$$\langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle = \Lambda(\theta)^\mu{}_\rho \Lambda(\theta)^\nu{}_\sigma \langle p', \pm j|T^{\rho\sigma}(t,0)|p, \pm j\rangle. \tag{3.24b}$$

Obviously the right hand side depends on θ whereas the left hand side does not. Nevertheless, a spatial rotation only involves two spatial dimensions and therefore only the components of the matrix elements of J^μ and $T^{\mu\nu}$ along these involved dimensions have to vanish. The components of the matrix elements along $\mathbf{p}$ and $\mathbf{p}'$ or the time axis do not have to vanish as their rotation eigenvalues are 1. In the case $\phi \neq 0$, these also had to vanish. Furthermore, for $\phi = 0$, the tensor components involved in the rotation have to vanish for all helicities j, and not only for $j > 1/2$ or $j > 1$, respectively. This shows that discontinuous currents are admitted and as mentioned above, examples for such currents are given in [8].

4 The consequences of the Weinberg-Witten theorem

If we intend to build a theory containing massless particles, a good thing to do would be to first ask the Weinberg-Witten theorem for what we are allowed to do. For this purpose it might be useful to write down the theorem the other way round:

Weinberg-Witten theorem (Construction mode).

A massless particle of spin $j > 1/2$ ($j > 1$) cannot carry a charge (energy-momentum) induced by a conserved Poincaré covariant vector (tensor) current.

This just states that if we want a massless particle of higher spin, we should not try to give it a charge induced by a Poincaré covariant current. But how can we violate the Poincaré covariance of the current and still preserve a consistent theory? One answer is actually pretty close and given by making the massless particle a gauge particle. The corresponding tensors will no longer be Poincaré covariant due to the presence of gauges. We will see that this is one reason for why the success of gauge theories is not restrained by the Weinberg-Witten theorem.

4.1 What about the particles we know?

At this point we want to look at the massless particles predicted by the theories which describe the current picture of our world. Except for the graviton, all of them are experimentally observed and hence the question is how they find their way around the constraints in the last section. Apart from being a gauge particle, the photon does not carry a charge of the electromagnetic current and therefore is allowed to exist. The weak gauge bosons are given a mass by the hopefully realized Higgs-mechanism which manifests their way around the Weinberg-Witten theorem.

Table 3 The particles we know and why they do not violate the Weinberg-Witten theorem

Gauge boson	Spin	El. charge	Col. charge	Mass	Theory
Photon γ	1	0	0	no	gauge
Weak W-boson $W^{\pm}$	1	± 1	0	yes	gauge
Weak Z-boson Z	1	0	0	yes	gauge
Gluons $g_k, k = 1, .., 8$	1	0	1	no	gauge
Graviton	2	0	0	no	*gauge*

4.1.1 The gluon

If we look at the gluons, the problem becomes more interesting. They have spin $j = 1$ and are charged under $SU(3)_{\text{color}}$ in the Standard Model. Hence, at first glance the Weinberg-Witten theorem seems to be applicable. They obey the equations of motion

$$D_\mu G_a^{\mu\nu} = J_a^\nu, \tag{4.1}$$

where

$$G_a^{\mu\nu} = \partial^\mu G_a^\nu - \partial^\nu G_a^\mu + g c_{abc} G^{b\mu} G^{c\nu}, \tag{4.2}$$

$$D_\mu G_a^{\mu\nu} = \partial_\mu G_a^{\mu\nu} + g c_{abc} G_\mu^b G^{c\mu\nu}, \tag{4.3}$$

$$J_a^\mu = \frac{\delta S_{\text{matter}}}{\delta G_\mu^a}, \tag{4.4}$$

the c_{abc} denote the $SU(3)_{\text{color}}$ structure constants, g the coupling and G_a^μ the potential. Note that J_a^μ is not a current we can use in the context of the Weinberg-Witten theorem. This current only includes the quark fields and therefore the gluon is not charged under it

$$\left(\int J_a^0 d^3x\right) |p, \pm j\rangle = 0. \tag{4.5}$$

To avoid the Weinberg-Witten theorem it does not matter that for this current we have $D_\mu J_a^\mu = 0$, rather than $\partial_\mu J_a^\mu = 0$, the current does simply not include the gluon field.

However, we can split the covariant derivative and set the matter current to zero. Then the equations of motion become

$$\partial_\mu G_a^{\mu\nu} = -g c_{abc} G_\mu^b G^{c\mu\nu}. \tag{4.6}$$

Now the right hand side provides a current only consisting of the gluon which is the gauge field appearing in the second part of the covariant derivative. Defining

$$\mathcal{J}_a^\nu = -g c_{abc} G_\mu^b G^{c\mu\nu}, \tag{4.7}$$

the equations of motion furthermore show that

$$\partial_\mu \mathcal{J}_a^\mu = 0. \tag{4.8}$$

This is the Noether current induced by the global symmetry of the matter-free Lagrangian. However, this current is not gauge invariant, as it includes the bare field G_μ^b. To see why this violates the assumptions of the Weinberg-Witten theorem, we have to consider a more general idea:

If one tries to construct a massless spin 1 vector field A^μ in a quantum field theory [2], this inevitably leads to the consequence that the field transforms under Lorentz transformations as

$$U(\Lambda,0)A^\mu(x)U^{-1}(\Lambda,0) = \Lambda^\mu{}_\nu A^\nu(\Lambda x) + \partial^\mu\Omega(x,\Lambda). \tag{4.9}$$

The reason for this non-Lorentz covariant behavior is the redundancy generated by describing a massless field with *two* degrees of freedom by a *four* component vector. This can be regarded as a physical motivation for postulating local gauge invariance and the invariance of the Lagrangian under gauge transformations of the form

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\epsilon(x). \tag{4.10}$$

Postulating that the physics of our theory does not change under the transformation (4.10) means a redefinition of our equations in order to save Lorentz covariance. Even though the Lagrangian does not change under a Lorentz transformation, the field A_μ does and is, in particular, not Lorentz covariant. In this sense gauge invariance and Lorentz covariance are affiliated with each other.

This implies that the non-gauge invariant gluon field G^b_μ is not a four-vector under Lorentz transformations either and thus the current $\mathcal{J}^\mu_a$ is not Lorentz covariant. The gluon is not forbidden by the Weinberg-Witten theorem because it is not charged under a Lorentz covariant current. In the context of the Weinberg-Witten theorem and the arguments above, the gluon can be regarded as a representative for every gauge field and it shows why the theorem is not applicable to all of them[6].

In the case of the gluon there might be a much simpler but also much less understood argument for why the gauge particle of QCD does not violate Weinberg's and Witten's theorem: Physicists believe that a massless gluon does not exist in isolation. The gluon is supposed to occur only in form of so called glueballs, massive particles composed of only gluons. However, this is still pure theory since glueballs have not been experimentally verified yet. In fact, every Yang-Mills theory is supposed to generate such a mass gap, a statement which is not proved yet either. From this point of view no Yang-Mills theory gives rise to massless particles and we do not have to bother about the Weinberg-Witten theorem. Furthermore, there is the unproved idea of confinement, i.e. physicists believe that color charged particles are forbidden in nature. Hence, a charged massless particle called gluon could never be measured in any experiment, and as such does not exist as a single particle. However, the gauge arguments given above become interesting in the light of gravity.

4.1.2 The graviton

We have a beautiful geometric theory of gravitation which does not ask for a particle description at first sight. Nevertheless, what makes a lot of people search for an alternative is that, as Witten says, the "existence of gravity clashes with our description of the rest of physics by quantum fields" [20]. People seem to be pretty sure that if gravitation is mediated by a particle, then this particle has to be massless and of spin 2. Feynman on this: "In any case, the fact is that a spin-two field has this geometrical interpretation; this is not something readily explainable - it is just marvelous" [21]. That the graviton has to have even spin follows from the purely attractive nature of gravity; that it has to be spin 2, from the required coupling to the energy momentum tensor. Fields of higher spin would also have to couple to conserved tensors with three or more indices but there are no such tensors except for Pauli-type currents. Up to total derivatives, the unique conserved tensor with two indices is the energy-momentum tensor. Finally the masslessness of the graviton is dictated by the long range nature of the gravitational interaction [2, 22].

[6] The described relation between gauge invariance and Lorentz covariance made the author of [10] even write: "Although the proof given by Weinberg and Witten (W-W) is very elegant, their theorem is not yet general and clear enough: First the condition of Lorentz covariance of J_μ or $\Theta_{\mu\nu}$ in their theorem is in fact a requirement of gauge invariance of them." The author uses $\Theta_{\mu\nu}$ instead of $T_{\mu\nu}$.

The physical importance of the spin 2 field stems from the fact that it is equivalent to linearized general relativity treated on a Minkowski background. Furthermore it can be shown that if we start with linearized general relativity of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4.11}$$

full general relativity follows from imposing a self consistent coupling to the energy-momentum tensor, i.e. we iteratively add terms containing the graviton $h_{\mu\nu}$ to the energy momentum tensor in order to do justice to the nonlinear character of Einstein's theory [23].

This is the reason why particle physicists generally talk about gravity as the theory of the graviton. Thus one should consider whether this massless particle is actually allowed in the context of the Weinberg-Witten theorem. However, to discuss this point the most important question seems to be in which framework the graviton should be treated. Shall we think about it in the context of linearized general relativity, full general relativity or somehow in terms of an approach to quantum gravity? Like in the case of the gluon, we may start by writing down the gravitational field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}. \tag{4.12}$$

Again, the right hand side, i.e. the matter energy-momentum tensor defined by

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\mathrm{matter}}}{\delta g_{\mu\nu}}, \tag{4.13}$$

is not the tensor to be considered as its matrix elements between one-graviton states vanish identically. The four-momentum of the graviton is not given by $\int T^{0\nu} d^3x$, as $T^{\mu\nu}$ represents all the energy except for gravitational energy and again we can say that the graviton is not charged under this tensor.

What we can do is to write Einstein's field equation in the form [24]

$$R^{(\mathrm{lin})}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R^{(\mathrm{lin})} = -8\pi G [T_{\mu\nu} + \mathcal{T}_{\mu\nu}]. \tag{4.14}$$

This is exactly the analogous equation to (4.12) but we shifted a part from the left to the right hand side so that on the left hand side we now have the linearized Ricci quantities in terms of the fluctuations $h_{\mu\nu}$ over the Minkowski metric $\eta_{\mu\nu}$. This equation has the form one could expect for the wave equation of a spin 2 field. The quantity $\mathcal{T}_{\mu\nu}$ represents the difference between the linerarized and the full field equations, the energy-momentum pseudotensor of the gravitational field. We have

$$\mathcal{T}_{\mu\nu} = \frac{1}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - R^{(\mathrm{lin})}_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} R^{(\mathrm{lin})} \right) \tag{4.15}$$

and

$$R^{(\mathrm{lin})}_{\mu\nu} = \frac{1}{2} \left(\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda} \right). \tag{4.16}$$

The sum of both quantities now obeys

$$\partial^\mu [T_{\mu\nu} + \mathcal{T}_{\mu\nu}] = 0, \tag{4.17}$$

whereas the statement that $T_{\mu\nu}$ is covariantly conserved illustrates the exchange of energy between matter and gravitation. It is well known that Einstein's theory has no local gauge invariant observables. To construct the quantity $\mathcal{T}_{\mu\nu}$ explicitly, we have to choose a specific reference frame, i.e. we have to choose a gauge. However, there are several different ways to construct this pseudotensor [24–26]. In all cases it seems to be clear that constructing the pseudotensor breaks general covariance. The failure in finding a gauge invariant energy momentum tensor for the graviton represents the statement that the energy of the gravitational field is not localizable or in words of Misner, Thorne and Wheeler, the "local gravitational energy momentum tensor has no weight" [27]. The gravitational field can be gauged away locally by introducing Riemannian normal coordiantes, and hence we cannot assign a spacetime position to gravitational energy[7]. However, in [24, 25] it is pointed out that the obtained pseudotensor of the gravitational field is still Lorentz covariant, even though not generally covariant. Thus, at first sight, the Weinberg-Witten theorem appears to be applicable to this tensor. However, the crucial point seems to be again that these pseudotensors are constructed in the context of a classical, i.e. a non-quantum theory.

From our viewpoint the situation should be regarded similarly to the case of the gluon. Like the current $\mathcal{J}^\mu$ for the gluon, the quantity $\mathcal{T}_{\mu\nu}$, or likewise constructed pseudotensors for the graviton, are not gauge invariant, i.e. not generally covariant. Constructing a spin 2 field $h^{\mu\nu}$ in a quantum field theory, i.e. giving birth to the graviton, we expect to be confronted with a behavior under Lorentz transformations similar to that of the massless spin 1 field in (4.9) of the form

$$U(\Lambda,0)h^{\mu\nu}U^{-1}(\Lambda,0) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma h^{\rho\sigma}(\Lambda x) + \partial^\mu\Omega^\nu(x,\Lambda) + \partial^\nu\Omega^\mu(x,\Lambda). \tag{4.18}$$

But this equation has the same structure as those general gauge transformations which leave the linearized field equations invariant

$$h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu\Omega^\nu(x) + \partial^\nu\Omega^\mu(x). \tag{4.19}$$

Hence, the additional terms under Lorentz transformations could be absorbed by these gauge terms. To maintain a Lorentz covariant pseudotensor in the quantum theory would require that $\mathcal{T}_{\mu\nu}$ is generally covariant, i.e. invariant under transformations like (4.19). However, as mentioned above, general covariance is broken already in the classical theory by the procedure to construct the energy-momentum pseudotensor $\mathcal{T}_{\mu\nu}$ for the gravitational field.

Again this shows a close connection between Lorentz covariance and local gauge invariance. One should notice that this connection appears to come up on the way to a quantum theory. The process of constructing the energy-momentum pseudotensor for the gravitational field in a classical theory only breaks general covariance, not Lorentz covariance. Constructing a quantum field then requires additional terms under Lorentz transformations. To maintain Lorentz covariance of the theory, these additional terms would have to be eliminated by gauge transformations requiring general covariance. Like for the gluon, this gauge invariance of the current is not given for the graviton.

Accordingly the graviton is not charged under a Lorentz covariant current and the Weinberg-Witten theorem does not apply. The fact that gravitational energy cannot be localized seems to be the fundamental argument for why the Weinberg-Witten theorem is not applicable to the graviton in any theory. Kugo even says that "the powerful second part of the theorem becomes empty in the presence of gravity" [10]. One way to think about this is that the geometric nature of the spin 2 particle necessarily leads to the fact that we cannot construct a gauge invariant energy-momentum tensor in a consistent theory. This implies that as soon as we suggest the graviton to be of spin 2 or even to have the properties of gravity, we have ruled out the Weinberg-Witten theorem[8].

One should always keep in mind that the arguments given above try to justify the existence of the quantum of the gravitational field. However, we do not have a quantum theory of gravity and hence the framework for this discussion is debatable.

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[7] Note that for every non-abelian gauge theory it can be shown that the gauge field can be transformed to zero at a given point, which illustrates the similarity to the gluon case.

[8] Note in this context that just recently M. Porrati pubished an interesting article in which he presents a proof for an extension of the Weinberg-Witten theorem to energy-momentum tensors which are *not* gauge invariant. His argumentation imposes limitations on massless particles interacting with gravity on a Minkowski background [28]. This result, however, only forbids massless particles of helicity $j > 2$ in contrast to $j > 1$ in case of the result of Weinberg and Witten. Therefore it does not apply to the graviton.

4.2 Minkowski spacetime – no background for emergent gravity

We should remember that the Weinberg-Witten theorem actually was an answer to several papers which had suggested to describe the graviton as a composed state in ordinary field theories on a Minkowski background. The most important consequence of the theorem is that this is not possible. We cannot compose a graviton from massless particles in Minkowski spacetime which are charged under a Lorentz covariant current. If we would, for example, try to compose the spin 2 graviton out of two spin 1 gluons, the gluons would contribute to a Lorentz covariant energy momentum tensor. This implies that their four-momentum can be described by $\int T^{0\nu} d^3x$. Accordingly, the graviton would be charged under this tensor. This possibility for an emerging graviton was ruled out by the Weinberg-Witten theorem.

The question actually is whether two gluons, which are charged under a Lorentz covariant energy-momentum tensor, could provide the crucial property of gravity to have no local gauge invariant degrees of freedom. In the light of the discussion above showing the close relation between Lorentz invariance, gauge invariance and general covariance, the answer given by the Weinberg-Witten theorem is that a conventional theory living on a Minkowski background cannot mimic the properties of gravity.

4.3 Evading the Weinberg-Witten theorem

We have seen that the attributes of massless particles, which are used in the proof of the Weinberg-Witten theorem, are based on the symmetry group of Minkowski spacetime. It should not be surprising that the theorem can be bypassed if we try to modify the fundamental structure of the background geometry.

4.3.1 Noncommutative geometry

There are several examples for approaches which violate Lorentz covariance in order to evade the Weinberg-Witten theorem. The ideas range from introducing a fundamental lattice in the basic theory [29] to considering the emergence of massless particles by spontaneous breaking of Lorentz invariance [30].

Recently it was suggested by H.S. Yang [31] that gravity could emerge from electromagnetism in the context of noncommutative geometry. The concept of noncommutative space can be traced back to the 1930s when Heisenberg first suggested it to give rise to a coordinate uncertainty. This reason for dealing with noncommutative geometry is still attractive, as it introduces a minimal length and a natural cutoff to treat divergences in quantum field theory. Even though the idea of postulating commutation relations of the form

$$[x^i, x^j] = i\theta^{ij} \tag{4.20}$$

is quite self-evident in the context of our world of quantum theories, Connes gives another interesting motivation for why we should think about this approach [32]:

If we consider the action terms representing current physics, i.e. the Einstein-Hilbert action S_{EH} and the action of the Standard Model S_{SM}, we find two different fundamental symmetry groups. The first corresponds to coordinate transformations given by the diffeomorphism group $\mathrm{Diff}(M)$ of an underlying manifold M. The second is the group of local gauge transformations $\mathcal{G}$ represented by maps from the manifold to the small gauge group G

$$\mathcal{G} = \{\text{maps}: \ M \to G\}. \tag{4.21}$$

At the current state of our experiments we believe that

$$G = U(1)_{\mathrm{Y}} \times SU(2) \times SU(3)_{\mathrm{color}}. \tag{4.22}$$

The full group of symmetry transformations of the action

$$S = S_{\mathrm{EH}} + S_{\mathrm{SM}}, \tag{4.23}$$

is then given by the semi-direct product

$$\mathcal{U} = \mathcal{G} \rtimes \mathrm{Diff}(M). \tag{4.24}$$

Nevertheless, the classical theories we know are defined on the manifold M with the invariance group of only S_{EH} as its diffeomorphism group. A much more natural idea is to ask for an underlying space X with diffeomorphism group $\mathcal{U}$. Searching for such a space, we are confronted with the general mathematical result that the connected component of the identitiy in the diffeomorphism group $\mathrm{Diff}(X)$ of an ordinary manifold is always a simple group. Thus, if we want to include the identity in the group of symmetry transformations, this implies that $\mathcal{G}$ cannot be a normal group of $\mathcal{U}$. Therefore an ordinary manifold cannot have a diffeomorphism group with the structure of a semi-direct product.

Following Connes we want to show that noncommutative spaces naturally imply this structure. For an ordinary manifold the algebraic object that corresponds to a diffeomorphism is an automorphism $\alpha \in \mathrm{Aut}(\mathcal{A})$ of the algebra $\mathcal{A}$ of coordinates. In contrast to an ordinary algebra, the noncommutative algebra has inner automorphisms $\mathrm{Int}(\mathcal{A})$ of the form

$$\alpha(x) = uxu^{-1}, \forall x \in \mathcal{A} \quad \text{and} \quad u \in \mathcal{A} \text{ invertible.} \tag{4.25}$$

If we now define the outer automorphisms as the quotient

$$\mathrm{Out}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})/\mathrm{Int}(\mathcal{A}), \tag{4.26}$$

it is generally true that we have an exact sequence of the form

$$1 \to \mathrm{Int}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1. \tag{4.27}$$

Let us consider an example where

$$\mathcal{A} = C^\infty(M, M_n(\mathbb{C})), \tag{4.28}$$

i.e. $\mathcal{A}$ equals the algebra of smooth maps from a manifold M to the algebra of $n \times n$ matrices. It can be shown that the group $\mathrm{Int}(\mathcal{A})$ is locally isomorphic to the group $\mathcal{G}$ of smooth maps from M to the small gauge group G, where $G = PSU(n)$ represents the quotient of $SU(n)$ by its center. In this example Eq. (4.27) becomes

$$1 \to \mathcal{G} \to \mathcal{U} \to \mathrm{Diff}(M) \to 1, \tag{4.29}$$

which implies the statement that $\mathcal{U}$ is the semi-direct product of $\mathcal{G}$ and $\mathrm{Diff}(M)$

$$\mathcal{U} = \mathcal{G} \rtimes \mathrm{Diff}(M).$$

Hence, we find that an underlying noncommutative space naturally includes the idea of a diffeomorphism group with a semi-direct product structure, whereas this structure is not possible for an ordinary manifold. As this is the symmetry group structure of the most fundamental action we have today, it suggests itself to think of building a theory on a space which admits this structure for its diffeomorphism group.

As always, the most interesting question is how this alternative approach can deal with the problem of gravity. As we have already indicated above, there are suggestions that in noncommutative spacetime gravity could emerge from electromagnetism. Of course, this is an idea that first has to pass the Weinberg-Witten theorem. As we have also indicated, the key point in this case is Lorentz invariance. Postulating noncommutative coordinates like in (4.20) singles out special spacetime directions due to the presence of θ^{ij}. It follows that the energy-momentum tensor in noncommutative geometry is not Lorentz covariant and according to the Weinberg-Witten theorem a graviton emerging from electromagnetism as proposed in [31] would be allowed.

4.3.2 AdS/CFT duality

The most popular representative of an emergent theory that evades the Weinberg-Witten theorem is certainly the idea of the AdS/CFT duality incorporating the holographic principle. In 1993 't Hooft [33] and two years later also Susskind [34] suggested that a theory of quantum gravity can be approached by the assumption that our 3+1 dimensional world could be a projection from a 2+1 dimensional theory, that is to say that all degrees of freedom are described by the lower dimensional theory. The idea was that the boundary of a very large region could be regarded as a flat plane at infinity. What we see in three dimensional space could be projected onto this distant screen which contains all necessary information to describe the bulk inside. The name of this idea can be based on a statement by 't Hooft who wrote that "the situation can be compared with a hologram of a three dimensional image on a two dimensional surface" [33].

To see how this can be combined with the concept of AdS/CFT duality it is insightful to go back to the roots of string theory. The idea to describe particles by strings came up in 1970 as an approach to deal with the large number of mesons and hadrons. It was not really successful in this context and was finally superseded by QCD. But the strong interaction remained a source for interesting ideas. As the gauge group of QCD is $SU(3)$, quarks have three color charges. However, it becomes strongly coupled at low energies and therefore cannot be treated in perturbation theory. To solve these difficulties with QCD calculations, 't Hooft suggested in 1974 to treat the strong interaction in terms of a large number N of colors [35]. In this so called 't Hooft limit

$$N \to \infty, \quad \lambda = g^2 N = \text{fixed}, \tag{4.30}$$

where λ defines the 't Hooft coupling and g the coupling constant of the underlying matrix gauge theory, only planar Feynman diagrams survive. In this limit the structure of the considered gauge theory bears resemblance to string theory, which gave rise to a striking idea by Maldacena in 1997 relating gauge and string theories [36].

Naively we could try to think about a relation of a four dimensional gauge theory to a four dimensional string theory. To make life as easy as possible, we want to start with a very symmetric gauge theory. Hence, we consider $SU(N)$ maximally supersymmetric, conformally invariant $\mathcal{N} = 4$ Yang-Mills theory in four spacetime dimensions. In addition to the gauge fields (gluons), this theory contains four fermions and six scalar fields. We have a global R-symmetry that rotates these scalar and fermion fields. The conformal group in four dimensions is $SO(4, 2)$ which includes the usual Poincaré group as well as scale transformations and special conformal transformations.

However, it is well known that string theory in four flat spacetime dimensions is not consistent. Trying to quantize it, one has to introduce an extra field, called *Liouville field* [37], in order to deal with the anomaly of the classical Weyl symmetry of the Polyakov action

$$S_{\mathrm{P}}[X, \gamma] = -\frac{1}{4\pi\alpha} \int_M d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu. \tag{4.31}$$

Here X denotes the embedding coordinates of the string, γ the world sheet metric and α the string tension. The Liouville field can be interpreted as an extra dimension giving an example of how consistency can require the introduction of additional spacetime dimensions.

Hence we have to think of at least a five dimensional background for the corresponding string theory. The space should have four dimensional Poincaré invariance which yields the ansatz

$$ds^2 = f(z)^2 (dx^\mu dx_\mu + dz^2). \tag{4.32}$$

Since we have chosen our gauge theory to be conformally invariant, this should be reflected on the string side of the correspondence implying that the theory should be invariant under rescaling

$$x \to \lambda x. \tag{4.33}$$

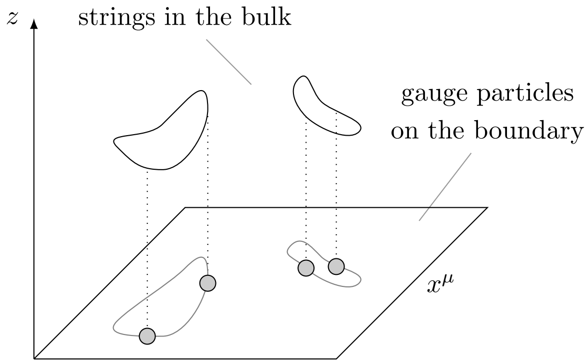

Fig. 2 Schematic idea of the AdS/CFT duality: A string theory on $AdS_5 \times S^5$ could have a dual description in terms of a gauge theory defined on the boundary of $AdS_5 \times S^5$. Here the Weinberg-Witten theorem would allow the graviton, described by strings in the bulk, to emerge as the holographic projection of two gluons living on the boundary. The figure sketches this setup with z and x^μ denoting the coordinates of AdS_5. In this context Minkowski spacetime can be regarded as the projection screen of a ten dimensional string theory.

However, string theory has a scale in form of the string tension. Consequently the only way out is that the scale invariance is an isometry of the background. Thus, under (4.33) we should have $z \to \lambda z$ and choosing $f(z) = R/z$ yields the metric of five dimensional Anti-de Sitter space in Poincaré coordinates

$$ds^2 = R^2 \frac{dx^\mu dx_\mu + dz^2}{z^2} . \tag{4.34}$$

As we consider a supersymmetric gauge theory, the strings should be supersymmetric too. It is well known that consistency then requires spacetime to be of dimension ten. Similarly to the argument in four dimensions, we add five more dimensions to get a ten dimensional space. Thinking about the geometry of these additional five dimensions, we remember that our gauge theory has a $SU(4) \simeq SO(6)$ global symmetry, which suggests a five sphere S^5 for the additional dimensions. This argumentation sketches what Maldacena showed in his famous paper [36]: In the large N limit, $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory can have a dual superstring theory in ten dimensions. Furthermore it is known that every string theory contains a massless excitation of spin 2, which, recalling the arguments in Sect. 4.1.2, can be regarded as a description of gravity. Hence, we find that gauge theories could imply a gravitational theory.

If we now combine the idea of the holographic principle and gauge-string duality, the impressive picture of AdS/CFT duality arises. The graviton could emerge as a hologram of a gauge theory. That is to say that from a gauge theory defined on a screen, a bulk geometry based on string theory could emerge in the limit of a large number of colors. This bulk theory would contain a spin 2 field and thus a theory of gravity. Unfortunately, no boundary theory is known as yet which gives rise to an interior theory describing exactly the four interactions we experience. Overviews over these ideas are given in [38–41].

The concept described above shows a possibility for the graviton to emerge from or to be composed of ordinary particles in the boundary theory, i.e. the hologram of the boundary particles could look like the graviton in our world. But how is this consistent with the Weinberg-Witten theorem? The answer is that for the bulk, containing a gravitational theory, we cannot construct a Lorentz covariant energy-momentum tensor, as described in 4.1.2. This argument is actually the same as that to justify a graviton in Minkowski spacetime. However, now the boundary theory has an energy-momentum tensor which lives in lower dimensions than the graviton and does not contain the graviton itself. In this holographic sense, the spin 2 graviton now could be composed of two spin 1 gluons, as the energy-momentum tensor of the latter lives in a different spactime, and the Weinberg-Witten theorem would be bypassed.

5 Summary

We have seen that, based on the isometry group of Minkowski spacetime, the Poincaré group, one may classify massless particles by their helicity and four-momentum. Basically by using the properties of Poincaré covariance, which is motivated by the form of the background geometry, we have derived the Weinberg-Witten theorem, especially dwelling on the continuity discussion in the proof. The theorem restricts the classification parameter ‘spin’ of massless particles if we impose that these particles are charged under conserved Poincaré covariant currents. The most important consequence of these limitations on massless particles are constraints on an emerging graviton in Minkowski spacetime. We have described why, in the context of gauge theories and gravity, the theorem is not applicable and how emergent theories evade the Weinberg-Witten theorem by modifications of the crucial background geometry.

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