Holonomic Techniques, Periods, and Decision Problems

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Abstract

Holonomic techniques have deep roots going back to Wallis, Euler, and Gauss, and have evolved in modern times as an important subfield of computer algebra, thanks in large part to the work of Zeilberger and others over the past three decades. In this talk, I will give an overview of the area, and in particular will present a select survey of known and original results on decision problems for holonomic sequences and functions. (*Holonomic sequences* satisfy linear recurrence relations with polynomial coefficients, and *holonomic functions* satisfy linear differential equations with polynomial coefficients.) I will also discuss some surprising connections to the theory of periods and exponential periods, which are classical objects of study in algebraic geometry and number theory; in particular, I will relate the decidability of certain decision problems for holonomic sequences to deep conjectures about periods and exponential periods, notably those due to Kontsevich and Zagier.

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1 Summary

Holonomic sequences (also known as *P-recursive* or *P-finite* sequences) are infinite sequences of real (or complex) numbers that satisfy a linear recurrence relation with polynomial coefficients. The earliest and best-known example is the Fibonacci sequence, introduced by Leonardo of Pisa in the 12th century; more recently, Apéry famously made use of certain holonomic sequences $\langle u_n \rangle_n$ satisfying the recurrence relation

$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1} \quad (n \in \mathbb{N})$$

to prove that $\zeta(3) := \sum_{n=1}^{\infty} n^{-3}$ is irrational [2]. Holonomic sequences now form a vast subject in their own right, with numerous applications in mathematics and other sciences; see, for instance, the monographs [20, 5, 6] or the seminal paper [24] of Zeilberger.

Any holonomic sequence $\langle u_n \rangle_{n=0}^{\infty}$ naturally gives rise to a *holonomic function* by considering the associated generating power series $\mathcal{F}(x) = \sum_{n=0}^{\infty} u_n x^n$. The recurrence relation defining the holonomic sequence in turn yields a linear differential equation satisfied by the corresponding power series.

There is a voluminous literature devoted to the study of identities for holonomic sequences and functions, and several computer-algebra packages implementing various identity-checking algorithms are also available. However, as noted by Kauers and Pillwein, *"in contrast, [...] almost no algorithms are available for inequalities"* [11]. For example, the *Positivity Problem*

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(i.e., whether every term of a given sequence is non-negative) for C-finite sequences[1] is only known to be decidable at low orders, and there is strong evidence that the problem is mathematically intractable in general [19, 18]; see also [10, 14, 19, 17]. For holonomic sequences that are not C-finite, virtually no decision procedures currently exist for Positivity, although several partial results and heuristics are known (see, for example [15, 11, 16, 23, 21, 22]).

Another extremely important property of holonomic sequences is *minimality*; a sequence $\langle u_n \rangle_n$ is minimal if, given any other linearly independent sequence $\langle v_n \rangle_n$ satisfying the same recurrence relation, the ratio u_n/v_n converges to 0. Minimal holonomic sequences play a crucial rôle, among others, in numerical calculations and asymptotics, as noted for example in [7, 8, 9, 3, 1, 4] – see also the references therein. Unfortunately, there is also ample evidence that determining algorithmically whether a given holonomic sequence is minimal is a very challenging task, for which no satisfactory solution is at present known to exist.

In this talk, I will present a select survey of known and original results on decision problems for holonomic sequences and functions. Some of this work will involve *periods* and *exponential periods*, which are classical objects of study in algebraic geometry and number theory; in particular, I will relate the decidability of certain decision problems for holonomic sequences to deep conjectures about periods and exponential periods, notably those due to Kontsevich and Zagier [13]. Parts of this presentation will be based on the paper [12].

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[1] C-finite sequences are linear recurrent sequences with *constant* coefficients.

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