

Radiation-reaction force and multipolar waveforms for eccentric, spin-aligned binaries in the effective-one-body formalism

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While most binary inspirals are expected to have circularized before they enter the LIGO/Virgo frequency band, a small fraction of those binaries could have non-negligible orbital eccentricity depending on their formation channel. Hence, it is important to accurately model eccentricity effects in waveform models used to detect those binaries, infer their properties, and shed light on their astrophysical environment. We develop a multipolar effective-one-body (EOB) eccentric waveform model for compact binaries whose components have spins aligned or anti-aligned with the orbital angular momentum. The waveform model contains eccentricity effects in the radiation-reaction force and gravitational modes through second post-Newtonian (PN) order, including tail effects, and spin-orbit and spin-spin couplings. We recast the PN-expanded, eccentric radiation-reaction force and modes in factorized form so that the newly derived terms can be directly included in the state-of-the-art, quasi-circular-orbit EOB model currently used in LIGO/Virgo analyses (i.e., the SEOBNRv4HM model).

I. INTRODUCTION

The observation of gravitational waves (GWs) by the LIGO-Virgo detectors [1, 2] have corroborated the existence of binary black holes (BBHs) in our universe. But how and in which astrophysical environments these binaries form is not yet fully understood. However, the masses, spins (magnitude and orientation), and binary eccentricities inferred from GWs provide invaluable clues to determine BBH formation channels [3, 4]. So far, the observed GWs are consistent with binary coalescences of negligible eccentricity, i.e., on quasi-circular orbits [5–8].

In general, binaries are expected to circularize [9, 10] as they approach merger due to the emission of gravitational radiation. But depending on their astrophysical formation channel, a small fraction of binaries could have non-negligible orbital eccentricity, as they enter the frequency bands of current detectors. This can occur in dense stellar environments, such as globular clusters or galactic nuclei, where dynamic capture [11–16] or the Lidov-Kozai mechanism in hierarchical triples [17–19] can lead to eccentric binary inspirals at close separations.

In particular, Ref. [13] (and Ref. [14]) showed that $\sim 5\%$ (or $\sim 10\%$) of all mergers in globular clusters enter the LIGO band with eccentricity $e > 0.1$. Binaries formed via dynamic capture in galactic nuclei are expected to have high eccentricities [16], with 92% having $e > 0.1$ and 50% – 85% having $e > 0.8$ at 10 Hz. For a BBH around a supermassive BH, the Lidov-Kozai mechanism can secularly drive the BBH to eccentricities near unity for some orientations [19]. Hence, inferring those eccentricities from GWs is important for understanding the origin and environment of BBHs. Interestingly,

Ref. [8] pointed out that GW190521 [20] could be consistent with either an eccentric nonprecessing or a quasi-circular precessing binary, which illustrates both the difficulties and prospects of further observations in the upcoming and future LIGO, Virgo and KAGRA runs [21].

While the expected fraction of eccentric GW observations with current detectors is small, neglecting eccentricity for the parameter inference can cause significant bias [22]. This becomes more relevant for LISA where a large fraction of stellar-mass binaries are expected to be eccentric [23–27]. Hence, it is important to develop accurate waveform models for eccentric binaries to detect them, infer their properties, and shed light on their astrophysical environment and formation channels. Several studies developed post-Newtonian (PN) waveform models for eccentric orbits, such as Refs. [28–38], or hybrid models that use PN results for the inspiral and quasi-circular numerical-relativity (NR) simulations near merger [39–41]. Recently, NR simulations for eccentric binaries were reported in Refs. [41–43], and the first NR surrogate model for eccentric BBHs has been developed in Ref. [44].

The effective-one-body (EOB) formalism [45–47] improves inspiral-merger-ringdown waveforms by combining information from PN theory, NR simulations, and the strong-field test-body limit. EOB Hamiltonians have been constructed to include spin [48–56], tidal effects [57–60], information from the small mass-ratio [61–65] and the post-Minkowskian approximations [66–68], and have been refined and calibrated to NR simulations [69–77]. While the EOB Hamiltonian is valid for generic orbits, most EOB waveform models use quasi-circular orbit results for the radiation-reaction (RR) force, gravitational waveform modes, and the calibration with NR simulations.

Recent approaches to extend the EOB formalism to eccentric orbits include Ref. [78], which derived the RR force with eccentricity up to 2PN order, but without

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tail effects and for nonspinning BHs. More recently, Ref. [79] incorporated eccentricity effects in the RR force and in the (2, 2) waveform mode through 1.5PN order, including tail effects, using the Keplerian parametrization and phase variables that evolve only due to RR. References [80, 81] extended the quasi-circular SEOBNRv1 [72] model to eccentric orbits, while Ref. [82] added eccentric corrections in the SEOBNRv4 [74, 83] waveform model, notably in the (2, 2), (2, 1), (3, 3), (4, 4) modes through 2PN order, including spin-orbit (SO) and spin-spin (SS) couplings¹, but not tail effects. They employed these eccentric modes to construct a RR force for eccentric orbits, which however does not include the Schott terms. As argued in Ref. [78] and Sec. II below, these Schott terms are necessary for generic orbits to satisfy the flux-balance equations. Furthermore, Refs. [84, 85] incorporated eccentricity effects in the TEOBResumS_SM [76, 86] model but only at leading PN order in the azimuthal component of the RR force, while using a quasi-circular 2PN-expanded radial RR force without spin or tail effects. They included eccentric corrections to all modes up to $\ell = |m| = 5$, but not to the $m = 0$ mode, and only at leading PN order.

In this paper, we develop a multipolar EOB waveform model for eccentric binaries with the compact-objects' spins aligned or antialigned (henceforth, for short aligned) with the orbital angular momentum. We derive the eccentric PN expressions for the RR force (including the Schott terms) and the gravitational modes up to $\ell = |m| = 6$, including the $m = 0$ mode, through 2PN order, including tail effects, and SO and SS couplings. We recast our results for the RR force and modes in a form that can be directly incorporated in the state-of-the-art, quasi-circular-orbit EOB model currently used in LIGO/Virgo analyses (SEOBNRv4HM [74, 83]).

The paper is structured as follows. In Sec. II, we derive the RR force from the energy and angular momentum fluxes using the balance relations. We use the gauge freedom in the RR force to impose that it reduces to the relation used in SEOBNRv4HM in the quasi-circular-orbit limit. In Sec. III, we obtain initial conditions for eccentric orbits. In Sec. IV, we calculate all the gravitational waveform modes that contribute up to 2PN order relative to the leading order of the (2, 2) mode, i.e., up to the $\ell = |m| = 6$ mode. These higher-order modes are even more important for eccentric orbits than for quasi-circular ones [87]. We conclude in Sec. V with a discussion of results and potential future work. Finally, Appendix A provides the coordinate transformation from harmonic to EOB coordinates, Appendix B includes a derivation of the LO spin-squared contribution to the angular momentum flux, Appendix C lists the spin contributions to the waveform modes in harmonic coordinates,

Appendix D provides some relations for dynamic quantities in the Keplerian parametrization, and Appendix E includes the transformation to tortoise coordinates. We provide our results for the RR force and waveform modes as *Mathematica* files in the Supplemental Material [88].

Notation

We use the metric signature $(-, +, +, +)$, and use units in which $c = G = 1$, but write c explicitly in PN expansions.

We consider an aligned-spin binary with masses m_1 and m_2 , with $m_1 \geq m_2$, we define the following constants:

$$\begin{aligned} M &= m_1 + m_2, & \mu &= \frac{m_1 m_2}{M}, & \nu &= \frac{\mu}{M}, \\ \delta &= \frac{m_1 - m_2}{M}, & X_1 &= \frac{m_1}{M}, & X_2 &= \frac{m_2}{M}. \end{aligned} \quad (1)$$

In the binary's center of mass, we introduce the canonical phase-space variables (R, ϕ, P_R, P_ϕ) , where R is the separation, ϕ the azimuthal angle, P_R the radial momentum, and P_ϕ the angular momentum. The total relative momentum P is given by $P^2 = P_R^2 + P_\phi^2/R^2$. We use the rescaled dimensionless variables

$$\begin{aligned} r &= \frac{R}{M}, & t &= \frac{T}{M}, & p_r &= \frac{P_r}{\mu}, & p_\phi &= \frac{P_\phi}{M\mu}, \\ \hat{H} &= \frac{H}{\mu}, & \hat{S}_i &= \frac{S_i}{M\mu}, & \chi_i &= \frac{S_i}{m_i^2}, \end{aligned} \quad (2)$$

where the dimensionless quantities are denoted with either a hat or a lowercase letter.

The energy and angular momentum fluxes far away from the binary are denoted by Φ_E and Φ_J respectively, and scale as follows:

$$\Phi_E = Gc^5 \tilde{\Phi}_E, \quad \Phi_J = c^5 \frac{\tilde{\Phi}_J}{M}, \quad (3)$$

where quantities with a tilde are the physical dimensionful fluxes. The components of the RR force are denoted by \mathcal{F}_r and \mathcal{F}_ϕ , and are scaled similarly to Φ_E and Φ_J , respectively.

II. RADIATION REACTION FORCE

The RR force accounts for the energy and angular momentum losses by the system, and is added to the right-hand side of the Hamilton equations of motion (EOMs) such that

$$\begin{aligned} \dot{r} &= \frac{\partial \hat{H}}{\partial p_r}, & \dot{p}_r &= -\frac{\partial \hat{H}}{\partial r} + \mathcal{F}_r, \\ \dot{\phi} &= \frac{\partial \hat{H}}{\partial p_\phi}, & \dot{p}_\phi &= -\frac{\partial \hat{H}}{\partial \phi} + \mathcal{F}_\phi, \end{aligned} \quad (4)$$

¹ Our results for those modes are mostly in agreement with Ref. [82] except for the SO part, where we disagree with their findings (their expressions contain two extra SO terms).

where the leading order of $\mathcal{F}_{r,\phi}$ is of order $1/c^5$ (2.5 PN). From the EOMs, with $\partial H/\partial \phi = 0$, the time derivatives of energy and angular momentum are given by

$$\begin{aligned}\dot{E}_{\text{system}} &= \frac{d\hat{H}}{dt} = \dot{r}\mathcal{F}_r + \dot{\phi}\mathcal{F}_\phi, \\ \dot{J}_{\text{system}} &= \frac{dp_\phi}{dt} = \mathcal{F}_\phi.\end{aligned}\quad (5)$$

The energy and angular momentum lost by the system are not equal to the energy and angular momentum fluxes, Φ_E and Φ_J , because of additional contributions to E and J due to interactions with the radiation field. The balance equations are modified by *Schott* terms, as in electrodynamics, that appear as total time derivatives in the balance equations [78]

$$\begin{aligned}\dot{E}_{\text{system}} + \dot{E}_{\text{Schott}} + \Phi_E &= 0, \\ \dot{J}_{\text{system}} + \dot{J}_{\text{Schott}} + \Phi_J &= 0.\end{aligned}\quad (6)$$

Substituting the expressions for the energy and angular momentum losses, we obtain

$$\begin{aligned}\dot{r}\mathcal{F}_r + \dot{\phi}\mathcal{F}_\phi + \dot{E}_{\text{Schott}} + \Phi_E &= 0, \\ \mathcal{F}_\phi + \dot{J}_{\text{Schott}} + \Phi_J &= 0.\end{aligned}\quad (7)$$

The energy and angular momentum fluxes are gauge-independent, but the RR force and Schott terms are gauge-dependent. This coordinate gauge freedom in the RR force was discussed by Iyer and Will in Refs. [89, 90], and by Gopakumar et. al. in Ref. [91]. Bini and Damour showed in Ref. [78] how the gauge freedom in \mathcal{F} is related to the freedom in defining the Schott terms.

A. Summary of the approach used in this paper for the RR force

The aim of this paper is to extend the quasi-circular RR force and gravitational modes employed in the SEOBNRv4HM waveform model to eccentric orbits. The Hamilton equations that describe the dynamics of the SEOBNRv4HM model use the following relations between the RR force and the energy flux for quasi-circular orbits, which are based on results from Refs. [46, 92],

$$\begin{aligned}\mathcal{F}_\phi^{\text{qc}} &= -\frac{\Phi_E^{\text{qc}}}{\Omega}, \\ \mathcal{F}_r^{\text{qc}} &= \mathcal{F}_\phi^{\text{qc}} \frac{p_r}{p_\phi} = -\frac{\Phi_E^{\text{qc}} p_r}{\Omega p_\phi},\end{aligned}\quad (8)$$

with Ω being the (angular) orbital frequency. However, these two relations are only valid for quasi-circular orbits and are not consistent for generic orbits, since they use the circular-orbit relation $\Phi_E^{\text{qc}} = \Omega \Phi_J^{\text{qc}}$ and do not include the Schott terms.

Hence, the approach we use to obtain the RR force is to write a generic ansatz with unknown coefficients for

the Schott terms, and calculate the RR force from the fluxes using the balance equations

$$\begin{aligned}\mathcal{F}_\phi &= -\Phi_J - \dot{J}_{\text{Schott}}, \\ \dot{r}\mathcal{F}_r &= -\Phi_E + \dot{\phi}\Phi_J - \dot{E}_{\text{Schott}} + \dot{\phi}\dot{J}_{\text{Schott}}.\end{aligned}\quad (9)$$

Then, we specify the free unknown coefficients in the Schott terms such that the force reduces to the conditions in Eq. (8) in the limit of quasi-circular orbits, i.e.,

$$\begin{aligned}\mathcal{F}_\phi &= -\Phi_J + \mathcal{O}(\dot{p}_r) + \mathcal{O}(p_r^2), \\ \frac{\mathcal{F}_r p_\phi}{\mathcal{F}_\phi p_r} &= 1 + \mathcal{O}(p_r^2),\end{aligned}\quad (10)$$

since both p_r and \dot{p}_r are zero for circular orbits. Finally, we factorize the RR force into the quasi-circular part used in SEOBNRv4HM times eccentric corrections

$$\mathcal{F}_r = \mathcal{F}_r^{\text{qc}} \mathcal{F}_r^{\text{ecc}}, \quad \mathcal{F}_\phi = \mathcal{F}_\phi^{\text{qc}} \mathcal{F}_\phi^{\text{ecc}}, \quad (11)$$

where the quasi-circular parts are given by Eq. (8), and the eccentric corrections scale as $\mathcal{F}_i^{\text{ecc}} \sim 1 + \dot{p}_r + p_r^2 + \dots$. In the following subsections, we provide the details of these steps.

B. EOB Hamiltonian and angular momentum

The EOB Hamiltonian is calculated from an effective Hamiltonian H_{eff} via the energy map

$$H_{\text{EOB}} = M \sqrt{1 + 2\nu \left(\frac{H_{\text{eff}}}{\mu} - 1 \right)}, \quad (12)$$

with H_{eff} given in Refs. [50, 51, 56]. When calculating the RR force to 2PN, we only need to work with the PN expansion of the EOB Hamiltonian. The nonspinning part to 2PN order is given by

$$\begin{aligned}\hat{H}_{\text{EOB}}^0 &= \frac{c^2}{\nu} + \frac{p^2}{2} - \frac{1}{r} + \frac{1}{c^2} \left[\frac{(\nu-1)p^2}{2r} - \frac{1+\nu}{8} p^4 - \frac{p_r^2}{r} \right. \\ &\quad \left. - \frac{1+\nu}{2r^2} \right] + \frac{1}{c^4} \left[\frac{1+\nu+\nu^2}{16} p^6 + \frac{(1+2\nu)p_r^2}{r^2} \right. \\ &\quad \left. - \frac{(1+\nu-3\nu^2)p^2}{4r^2} + \frac{(1+\nu-3\nu^2)p^4}{8r} \right. \\ &\quad \left. + \frac{(1+\nu)p^2 p_r^2}{2r} - \frac{1-\nu+\nu^2}{2r^3} \right],\end{aligned}\quad (13)$$

the LO (1.5PN) spin-orbit part

$$\hat{H}_{\text{EOB}}^{\text{SO}} = \frac{p_\phi}{2c^3 r^3} [\chi_1 (2 + 2\delta - \nu) + \chi_2 (2 - 2\delta - \nu)], \quad (14)$$

and the LO (2PN) spin-spin part

$$\hat{H}_{\text{EOB}}^{\text{SS}} = \frac{1}{2c^4 r^3} \left\{ \chi_1^2 \left[X_1^4 \left(1 - \frac{p_\phi^2}{r} + r p_r^2 \right) - C_{\text{IES}^2} X_1^2 \right] \right\}$$

$$\begin{aligned}
& + \chi_2^2 \left[X_2^4 \left(1 - \frac{p_\phi^2}{r} + r p_r^2 \right) - C_{2\text{ES}^2} X_2^2 \right] \\
& + 2\chi_1\chi_2 \left[(\nu - 1)\nu - \frac{\nu^2 p_\phi^2}{r} + \nu^2 r p_r^2 \right] \Big\}, \quad (15)
\end{aligned}$$

where $C_{i\text{ES}^2}$ are the spin quadrupole constants, which equal one for BHs.

The orbital frequency expanded to 2PN is given by

$$\begin{aligned}
\Omega \equiv \dot{\phi} &= \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_\phi} \\
&= \frac{p_\phi}{r^2} + \frac{p_\phi}{c^2} \left[\frac{\nu - 1}{r^3} - \frac{(\nu + 1)p^2}{2r^2} \right] \\
&+ \frac{1}{2c^3 r^3} [\chi_1(2 + 2\delta - \nu) + \chi_2(2 - 2\delta - \nu)] \\
&+ \frac{p_\phi}{c^4} \left[\frac{3(\nu^2 + \nu + 1)p^4}{8r^2} + \frac{(-3\nu^2 + \nu + 1)p^2}{2r^3} \right. \\
&\quad \left. + \frac{(\nu + 1)p_r^2}{r^3} - \frac{-3\nu^2 + \nu + 1}{2r^4} \right] \\
&- \frac{p_\phi}{c^4 r^4} (2\nu^2 \chi_1 \chi_2 + \chi_1^2 X_1^4 + \chi_2^2 X_2^4) \quad (16)
\end{aligned}$$

From the EOM $\dot{p}_r = -\partial \hat{H} / \partial r$, we can obtain an expression for $p_\phi(r, \dot{p}_r, p_r)$

$$\begin{aligned}
\frac{p_\phi^2}{r} &= 1 + r^2 \dot{p}_r + \frac{1}{2c^2 r} \left[6 + (\nu + 1)r^4 \dot{p}_r^2 - (\nu - 5)r^2 \dot{p}_r \right. \\
&\quad \left. + r p_r^2 ((\nu + 1)r^2 \dot{p}_r + 4) \right] \\
&- \frac{3\sqrt{r^3 \dot{p}_r + r}}{2c^3 r^2} [\chi_1(2 + 2\delta - \nu) + \chi_2(2 - 2\delta - \nu)] \\
&+ \frac{1}{8c^4 r^2} \left[(\nu^2 + 5\nu + 1)r^6 \dot{p}_r^3 - (\nu^2 - \nu + 1)r^4 \dot{p}_r^4 \dot{p}_r \right. \\
&\quad + 2(5\nu + 8)r^4 \dot{p}_r^2 - (\nu^2 + 7\nu - 63)r^2 \dot{p}_r \\
&\quad - 24(\nu - 3) + 2(8 - 24\nu + 3\nu r^4 \dot{p}_r^2) r p_r^2 \\
&\quad \left. + 2(\nu^2 + \nu + 3)p_r^2 r^3 \dot{p}_r \right] \\
&+ \frac{1}{2c^4 r^2} \left\{ \chi_1^2 [3C_{1\text{ES}^2} X_1^2 + X_1^4 (1 + 4r^2 \dot{p}_r - 2r p_r^2)] \right. \\
&\quad + \chi_2^2 [3C_{2\text{ES}^2} X_2^2 + X_2^4 (1 + 4r^2 \dot{p}_r - 2r p_r^2)] \\
&\quad \left. + \nu \chi_1 \chi_2 [2(\nu + 4\nu r^2 \dot{p}_r + 3) - 4\nu r p_r^2] \right\}, \quad (17)
\end{aligned}$$

which we use to express the noncircular part of the RR force and modes in terms of p_r and \dot{p}_r . It will also be useful below, when taking the circular-orbit limit, to have an expression for p_ϕ as a function of r for circular orbits. Setting $\dot{p}_r = 0 = p_r$ in the previous equation yields

$$\begin{aligned}
\frac{p_\phi^2}{r} \Big|_{\text{circ}} &\equiv 1 + \frac{3}{c^2 r} + \frac{3(3 - \nu)}{c^4 r^2} \\
&- \frac{3}{2c^3 r^{3/2}} [\chi_1(-2\delta + \nu - 2) + \chi_2(2\delta + \nu - 2)]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2c^4 r^2} \left[\chi_1^2 (3C_{1\text{ES}^2} X_1^2 + X_1^4) + 2\nu \chi_1 \chi_2 (3 + \nu) \right. \\
&\quad \left. + \chi_2^2 (3C_{2\text{ES}^2} X_2^2 + X_2^4) \right]. \quad (18)
\end{aligned}$$

C. Energy and angular momentum fluxes

The energy and angular momentum fluxes for nonspinning binaries were derived to 3PN order in harmonic and ADM coordinates in Refs. [93–95]. The 2PN instantaneous part of the fluxes for nonspinning bodies is given in EOB coordinates in Appendix A of Ref. [78]. The leading order reads

$$\begin{aligned}
\Phi_E^{\text{inst}} &= \frac{8\nu^2}{15r^4} (12p^2 - 11p_r^2) + \mathcal{O}\left(\frac{1}{c^2}\right), \\
\Phi_J^{\text{inst}} &= \frac{8\nu^2}{5r^3} p_\phi \left(2p^2 - 3p_r^2 + \frac{2}{r} \right) + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (19)
\end{aligned}$$

The hereditary contributions to the fluxes can be expressed as an infinite sum over Bessel functions [79, 94] that can be evaluated numerically, or resummed analytically [31, 96]. Here, we follow the method from Ref. [79] to obtain the LO tail part (1.5 PN) of the orbit-averaged fluxes in an eccentricity expansion and we extend their derivation to $\mathcal{O}(e^6)$, which yields²

$$\begin{aligned}
\langle \Phi_E^{\text{tail}} \rangle &= \frac{128\pi\nu^2}{5c^3} x^{13/2} \left[1 + \frac{2335}{192} e^2 + \frac{42955}{768} e^4 \right. \\
&\quad \left. + \frac{6204647}{36864} e^6 + \mathcal{O}(e^8) \right], \\
\langle \Phi_J^{\text{tail}} \rangle &= \frac{128\pi\nu^2}{5c^3} x^5 \left[1 + \frac{209}{32} e^2 + \frac{2415}{128} e^4 + \frac{730751}{18432} e^6 \right. \\
&\quad \left. + \mathcal{O}(e^8) \right], \quad (21)
\end{aligned}$$

where $x \equiv \Omega^{2/3}$. The eccentricity e in these equations is defined using the Keplerian parametrization, which is given by

$$r = \frac{1}{u_p(1 + e \cos \chi)}, \quad (22)$$

² Calculating the tail contribution to the fluxes is similar to that for the waveform modes (see Sec. IV B) except for using the integrals [97]

$$\begin{aligned}
\Phi_E^{\text{tail}} &= \frac{4}{5} I_{ij}^{(3)} \int_0^\infty d\tau I_{ij}^{(5)}(t - \tau) \ln\left(\frac{\tau}{b}\right), \\
\Phi_J^{\text{tail}} &= \frac{4}{5} \epsilon_{zij} \left[I_{il}^{(2)} \int_0^\infty d\tau I_{jl}^{(5)}(t - \tau) \ln\left(\frac{\tau}{b}\right) \right. \\
&\quad \left. + I_{jl}^{(3)} \int_0^\infty d\tau I_{il}^{(4)}(t - \tau) \ln\left(\frac{\tau}{b}\right) \right], \quad (20)
\end{aligned}$$

where I_{ij} is the mass quadrupole moment, and $b = 2r_0 e^{-11/12}$ with r_0 a gauge parameter.

where u_p is the inverse semilatus rectum and χ is the relativistic anomaly.

Since we are not using the adiabatic approximation and are not working with orbit-averaged fluxes, we can obtain an approximate expression for the tail contribution to the fluxes by writing an ansatz in terms of (r, p_r, p_ϕ) in a p_r expansion of the form

$$\begin{aligned}\Phi_E^{\text{tail}} &= \frac{128\pi\nu^2 p_\phi}{5c^3 r^4} \left[\frac{1}{r^3} + c_1 \frac{p_r^2}{r^2} + c_2 \frac{p_r^4}{r} + c_3 p_r^6 + \mathcal{O}(p_r^8) \right], \\ \Phi_J^{\text{tail}} &= \frac{128\pi\nu^2}{5c^3 r^2} \left[\frac{1}{r^3} + c_4 \frac{p_r^2}{r^2} + c_5 \frac{p_r^4}{r} + c_6 p_r^6 + \mathcal{O}(p_r^8) \right],\end{aligned}\quad (23)$$

calculate the average of that ansatz in terms of (e, x) (see Appendix D), and then match it to the average flux in Eq. (21) to determine the unknowns c_n . This yields

$$\begin{aligned}\Phi_E^{\text{tail}} &= \frac{128\pi\nu^2 p_\phi}{5c^3 r^4} \left[\frac{1}{r^3} + \frac{415p_r^2}{96r^2} + \frac{5p_r^4}{288r} - \frac{73p_r^6}{11520} + \mathcal{O}(p_r^8) \right], \\ \Phi_J^{\text{tail}} &= \frac{128}{5c^3 r^2} \pi\nu^2 \left[\frac{1}{r^3} + \frac{49p_r^2}{16r^2} - \frac{49p_r^6}{5760} + \mathcal{O}(p_r^8) \right].\end{aligned}\quad (24)$$

The LO (1.5PN) SO fluxes for generic orbits and generic spins were derived in Refs. [98, 99]. (The NLO SO energy flux was derived in Ref. [100].) It should be noted that Ref. [98] used the Tulczyjew-Dixon (covariant) spin supplementary condition (SSC) [101–104], while Ref. [99] used the Newton-Wigner (NW), or canonical, SSC [105, 106]. In this paper, we use the NW SSC since we are working in a canonical Hamiltonian formulation of the spinning two-body dynamics [107, 108]. Changing the velocities in Eq. (17) of Ref. [99] to momenta, which involves spin-orbit terms, the aligned-spin fluxes reduce to

$$\begin{aligned}\Phi_E^{\text{SO}} &= \frac{4\nu^2 p_\phi}{15c^3 r^6} \left\{ \chi_1 \left[p^2(36\nu - 37 - 37\delta) + \frac{4(9 + 9\delta - 4\nu)}{r} \right. \right. \\ &\quad \left. \left. + 9p_r^2(3 + 3\delta - 2\nu) \right] + 1 \leftrightarrow 2 \right\}, \\ \Phi_J^{\text{SO}} &= \frac{4\nu^2}{15c^3 r^3} \left\{ \chi_1 \left[p^4(8\nu - 9\delta - 9) + \frac{9 + 9\delta - 4\nu}{r^2} \right. \right. \\ &\quad \left. \left. + (9 + 9\delta - 24\nu)p^2 p_r^2 - (17 + 17\delta + 6\nu) \frac{p_r^2}{r} \right. \right. \\ &\quad \left. \left. + 15\nu p_r^4 + (11 + 11\delta + 10\nu) \frac{p_r^2}{r} \right] + 1 \leftrightarrow 2 \right\}.\end{aligned}\quad (25)$$

For the LO (2PN) SS contributions, the LO spin₁-spin₂ energy and angular momentum fluxes in harmonic coordinates were derived in Refs. [98, 109], while the spin-squared energy flux was derived in Refs. [110, 111], and we obtain in Appendix B the spin-squared angular momentum flux. Transforming from harmonic to EOB coordinates, using the transformations in Appendix A, we get the following SS contributions to the fluxes for aligned-spins:

$$\Phi_E^{\text{SS}} = \frac{\nu^2}{15c^4 r^6} \left\{ \chi_1^2 \left[C_{1\text{ES}^2}(\delta - 2\nu + 1) (144p^2 - 156p_r^2) \right. \right.$$

$$\begin{aligned}&\left. + 3(96\nu\delta - 47\delta - 96\nu^2 + 190\nu - 47)p^2 \right. \\ &\left. + (149\delta - 280\nu\delta + 280\nu^2 - 578\nu + 149)p_r^2 \right] \\ &+ \nu\chi_1\chi_2 \left[10(28\nu - 33)p_r^2 - 6(48\nu - 47)p^2 \right. \\ &\left. + 1 \leftrightarrow 2 \right\}. \\ \Phi_J^{\text{SS}} &= \frac{\nu^2 p_\phi}{5c^4 r^5} \left\{ \chi_1^2 \left[(32\nu\delta - 15\delta - 32\nu^2 + 62\nu - 15) \frac{1}{r} \right. \right. \\ &\quad \left. \left. + (\delta - 2\delta\nu + 2\nu^2 - 4\nu + 1) (22p_r^2 - 20p^2) \right. \right. \\ &\quad \left. \left. + C_{1\text{ES}^2}(\delta - 2\nu + 1) \left(12p^2 - 30p_r^2 + \frac{24}{r} \right) \right] \right. \\ &\quad \left. + \nu\chi_1\chi_2 \left[\frac{2(23 - 16\nu)}{r} - 8(5\nu - 3)p^2 \right. \right. \\ &\quad \left. \left. + 4(11\nu - 15)p_r^2 \right] + 1 \leftrightarrow 2 \right\}.\end{aligned}\quad (26)$$

The total 2PN energy and angular momentum fluxes are the sum of all the above contributions, i.e.,

$$\begin{aligned}\Phi_E &= \Phi_E^{\text{inst}} + \Phi_E^{\text{tail}} + \Phi_E^{\text{SO}} + \Phi_E^{\text{SS}}, \\ \Phi_J &= \Phi_J^{\text{inst}} + \Phi_J^{\text{tail}} + \Phi_J^{\text{SO}} + \Phi_J^{\text{SS}}.\end{aligned}\quad (27)$$

D. Ansatz for the Schott terms

As an ansatz for the Schott terms E_{Schott} and J_{Schott} , we consider

$$\begin{aligned}J_{\text{Schott}}^{\text{inst}} &= \frac{\nu^2 p_r p_\phi}{r^2} \left[\alpha_1 + \frac{1}{c^2} \left(\alpha_2 p_r^2 + \alpha_3 p^2 + \frac{\alpha_4}{r} \right) \right. \\ &\quad \left. + \frac{1}{c^4} \left(\alpha_5 p_r^4 + \alpha_6 p^2 p_r^2 + \alpha_7 \frac{p_r^2}{r} + \alpha_8 p^4 + \alpha_9 \frac{p^2}{r} \right. \right. \\ &\quad \left. \left. + \frac{\alpha_{10}}{r^2} \right) \right], \\ E_{\text{Schott}}^{\text{inst}} &= \frac{\nu^2 p_r}{r^2} \left[\beta_1 p_r^2 + \beta_2 p^2 + \frac{\beta_3}{r} + \frac{1}{c^2} \left(\beta_4 p_r^4 + \beta_5 p^2 p_r^2 \right. \right. \\ &\quad \left. \left. + \beta_6 \frac{p_r^2}{r} + \beta_7 p^4 + \beta_8 \frac{p^2}{r} + \frac{\beta_9}{r^2} \right) \right. \\ &\quad \left. + \frac{1}{c^4} \left(\beta_{10} p_r^6 + \beta_{11} p^2 p_r^4 + \beta_{12} \frac{p_r^4}{r} + \beta_{13} p^4 p_r^2 \right. \right. \\ &\quad \left. \left. + \beta_{14} \frac{p^2 p_r^2}{r} + \beta_{15} \frac{p_r^2}{r^2} + \beta_{16} p^6 + \beta_{17} \frac{p^4}{r} \right. \right. \\ &\quad \left. \left. + \beta_{18} \frac{p^2}{r^2} + \frac{\beta_{19}}{r^3} \right) \right].\end{aligned}\quad (28)$$

Note that this ansatz for E_{Schott} is more general than the one used in Eq. (4.4) of Ref. [78], since we found that such an ansatz is needed for the RR force to satisfy the conditions in Eq. (8).

For the LO tail, we use the ansatz

$$J_{\text{Schott}}^{\text{tail}} = \frac{\pi\nu^2 p_r}{c^3 r^2} \left(\lambda_1 p_r^2 + \lambda_2 p^2 + \frac{\lambda_3}{r} \right),$$

$$E_{\text{Schott}}^{\text{tail}} = \frac{\pi\nu^2 p_\phi p_r}{c^3 r^4} \left(\lambda_4 p_r^2 + \lambda_5 p^2 + \frac{\lambda_6}{r} \right), \quad (29)$$

while for the LO SO part,

$$\begin{aligned} J_{\text{Schott}}^{\text{SO}} &= \frac{\nu^2 p_r}{c^3 r^2} \left[\chi_1 \left(\sigma_1 p_r^2 + \sigma_2 p^2 + \frac{\sigma_3}{r} \right) \right. \\ &\quad \left. + \chi_2 \left(\sigma_4 p_r^2 + \sigma_5 p^2 + \frac{\sigma_6}{r} \right) \right], \\ E_{\text{Schott}}^{\text{SO}} &= \frac{\nu^2 p_r p_\phi}{c^3 r^4} \left[\chi_1 \left(\sigma_7 p_r^2 + \sigma_8 p^2 + \frac{\sigma_9}{r} \right) \right. \\ &\quad \left. + \chi_2 \left(\sigma_{10} p_r^2 + \sigma_{11} p^2 + \frac{\sigma_{12}}{r} \right) \right], \quad (30) \end{aligned}$$

and for the SS part,

$$\begin{aligned} J_{\text{Schott}}^{\text{SS}} &= \frac{\nu^2 p_\phi p_r}{c^4 r^4} \left[\chi_1^2 (\zeta_1 + C_{1\text{ES}^2} \zeta_2) \right. \\ &\quad \left. + \chi_1 \chi_2 \zeta_3 + \chi_2^2 (\zeta_4 + C_{2\text{ES}^2} \zeta_5) \right], \\ E_{\text{Schott}}^{\text{SS}} &= \frac{\nu^2 p_r}{c^4 r^4} \left[\chi_1^2 \left(\zeta_6 p_r^2 + \zeta_7 p^2 + \frac{\zeta_8}{r} \right) \right. \\ &\quad + \chi_1^2 C_{1\text{ES}^2} \left(\zeta_9 p_r^2 + \zeta_{10} p^2 + \frac{\zeta_{11}}{r} \right) \\ &\quad + \chi_1 \chi_2 \left(\zeta_{12} p_r^2 + \zeta_{13} p^2 + \frac{\zeta_{14}}{r} \right) \\ &\quad + \chi_2^2 \left(\zeta_{15} p_r^2 + \zeta_{16} p^2 + \frac{\zeta_{17}}{r} \right) \\ &\quad \left. + \chi_2^2 C_{2\text{ES}^2} \left(\zeta_{18} p_r^2 + \zeta_{19} p^2 + \frac{\zeta_{20}}{r} \right) \right]. \quad (31) \end{aligned}$$

The total energy and angular momentum Schott terms are the sum of the above contributions, i.e.

$$\begin{aligned} J_{\text{Schott}} &= J_{\text{Schott}}^{\text{inst}} + J_{\text{Schott}}^{\text{tail}} + J_{\text{Schott}}^{\text{SO}} + J_{\text{Schott}}^{\text{SS}}, \\ E_{\text{Schott}} &= E_{\text{Schott}}^{\text{inst}} + E_{\text{Schott}}^{\text{tail}} + E_{\text{Schott}}^{\text{SO}} + E_{\text{Schott}}^{\text{SS}}. \quad (32) \end{aligned}$$

Note that when taking the time derivative of these Schott terms using the EOMs, the LO nonspinning part contributes to the LO SO and SS parts of the RR force.

E. Solving for the eccentric-orbits RR force

Using the fluxes and the Schott terms, the RR force can be calculated from the balance equations (9), which fix some of the unknowns in the ansatz for the Schott terms. The remaining unknowns can be determined by requiring that the RR force satisfies the conditions (10) in the circular-orbits limit.

1. Leading order

At leading order, calculating the RR force with the ansatz in Eqs. (28) and expanding in p_r gives

$$\mathcal{F}_r = \frac{\nu^2}{5p_r r^3} \left(p^2 - \frac{1}{r} \right) \left[p^2 (5\alpha_1 - 5\beta_2 + 16) - 5\frac{\beta_3}{r} \right] + \mathcal{O}(p_r). \quad (33)$$

Requiring that the $1/p_r$ term is zero, leads to the solution

$$\beta_3 = 0, \quad \alpha_1 = \frac{1}{5}(5\beta_2 - 16). \quad (34)$$

Expanding $\mathcal{F}_r p_\phi / (\mathcal{F}_\phi p_r) - 1$ in p_r yields

$$\frac{\mathcal{F}_r p_\phi}{\mathcal{F}_\phi p_r} - 1 = \frac{9(5\beta_1 - 8)p^2 - (45\beta_1 + 30\beta_2 + 88)/r}{15\beta_2 p^2 + 3(32 - 5\beta_2)/r} + \mathcal{O}(p_r^2). \quad (35)$$

Requiring that the first term in that series expansion is zero gives the solution

$$\beta_1 = \frac{8}{5}, \quad \beta_2 = -\frac{16}{3}, \quad \alpha_1 = -\frac{128}{15}. \quad (36)$$

With that solution, we obtain the LO RR force

$$\begin{aligned} \mathcal{F}_\phi^{\text{LO}} &= \frac{8\nu^2}{15r^3} p_\phi \left(10p^2 - 39p_r^2 - \frac{22}{r} \right), \\ \mathcal{F}_r^{\text{LO}} &= -\frac{16\nu^2}{15r^3} p_r \left(-5p^2 + 12p_r^2 + \frac{11}{r} \right). \quad (37) \end{aligned}$$

This force satisfies the conditions in Eq. (10) for circular orbits since

$$\begin{aligned} \mathcal{F}_\phi^{\text{LO}} &= -\Phi_J^{\text{LO}} - \frac{128}{15r^3} p_\phi \nu^2 (2p_r^2 - \dot{p}_r r), \\ \frac{\mathcal{F}_r^{\text{LO}} p_\phi}{\mathcal{F}_\phi^{\text{LO}} p_r} &= 1 + \frac{15p_r^2}{10p^2 - 39p_r^2 - 22/r}. \quad (38) \end{aligned}$$

2. 1PN

Following the same steps as above, we obtain the following solution for the unknowns at 1PN:

$$\begin{aligned} \beta_8 &= \frac{1}{15} (371 - 15\alpha_3 + 268\nu), \\ \alpha_2 &= \frac{1}{35} (35\beta_5 + 108\nu - 22), \\ \beta_6 &= \frac{2}{315} (105\alpha_3 - 1760\nu + 1743), \\ \beta_9 &= -\frac{32}{105} (4\nu - 1), \\ \beta_7 &= \alpha_3 + \frac{2}{35} (4\nu + 93), \\ \alpha_4 &= \frac{893}{21} - \alpha_3 + \frac{2924\nu}{105}, \quad (39) \end{aligned}$$

with 3 arbitrary coefficients out of 9 coefficients at that order. To simplify the resulting expressions for the RR

force, we choose to set all arbitrary coefficients to zero, i.e. $\alpha_3 = \beta_4 = \beta_5 = 0$, which yields the following 1PN contribution to the RR force:

$$\begin{aligned}\mathcal{F}_\phi^{1\text{PN}} &= \frac{\nu^2 p_\phi}{105c^2 r^3} \left[18(4\nu + 93)p_r^2 p^2 - 6(4\nu + 93)p^4 \right. \\ &\quad + 180(7\nu - 5)p_r^4 + (9780\nu + 19198)\frac{p_r^2}{r} \\ &\quad \left. - (484\nu + 3833)\frac{p^2}{r} + \frac{1684\nu + 6213}{r^2} \right], \\ \mathcal{F}_r^{1\text{PN}} &= \frac{\nu^2 p_r}{105c^2 r^3} \left[180(7\nu - 5)p_r^4 - 6(4\nu + 93)p^4 \right. \\ &\quad + 4(691\nu + 3958)\frac{p_r^2}{r} - 198(6\nu - 13)p_r^2 p^2 \\ &\quad \left. - (484\nu + 3833)\frac{p^2}{r} + \frac{1684\nu + 6213}{r^2} \right]. \quad (40)\end{aligned}$$

3. LO tail

Solving for the unknowns at the LO tail contribution leads to the solution

$$\begin{aligned}\lambda_2 &= -\frac{334}{15}, \quad \lambda_3 = \frac{128}{5}, \quad \lambda_4 = -\frac{334}{15} + \lambda_1, \\ \lambda_5 &= -\frac{334}{15}, \quad \lambda_6 = \frac{128}{5}, \quad (41)\end{aligned}$$

with either λ_1 or λ_4 arbitrary. Choosing $\lambda_1 = 0$, the tail contribution to the RR forces becomes

$$\begin{aligned}\mathcal{F}_\phi^{\text{tail}} &= \frac{\pi\nu^2}{c^3 r^2} \left(\frac{334p^4}{15r} - \frac{718p^2}{15r^2} - \frac{334p^2 p_r^2}{5r} - \frac{308p_r^2}{15r^2} \right. \\ &\quad \left. + \frac{49p_r^6}{225} \right), \\ \mathcal{F}_r^{\text{tail}} &= \frac{\pi\nu^2 p_\phi p_r}{c^3 r^4} \left(\frac{334p^2}{15r} - \frac{p_r^4}{18} - \frac{7034p_r^2}{45r} - \frac{718}{15r^2} \right). \quad (42)\end{aligned}$$

4. LO spin-orbit

At LO SO, we obtain the solution

$$\begin{aligned}\sigma_5 &= \frac{1}{15}(-36\delta - 32\nu + 15\sigma_{11} + 36), \\ \sigma_6 &= \frac{1}{15}(-68\delta - 40\nu + 15\sigma_{12} + 68), \\ \sigma_2 &= \frac{1}{15}(36\delta - 32\nu + 15\sigma_8 + 36), \\ \sigma_3 &= \frac{1}{15}(68\delta - 40\nu + 15\sigma_9 + 68), \\ \sigma_{12} &= \frac{8}{15}(4\delta + 3\nu - 4),\end{aligned}$$

$$\begin{aligned}\sigma_9 &= -\frac{8}{15}(4\delta - 3\nu + 4), \\ \sigma_4 &= \frac{1}{15}(-6\delta + 44\nu + 15\sigma_{10} + 6), \\ \sigma_1 &= \frac{1}{15}(6\delta + 44\nu + 15\sigma_7 + 6), \\ \sigma_{11} &= -\frac{2}{15}(9\delta + 22\nu - 9), \\ \sigma_8 &= \frac{2}{15}(9\delta - 22\nu + 9) \quad (43)\end{aligned}$$

where either σ_1 or σ_7 is arbitrary, and either σ_4 or σ_{10} is arbitrary. Choosing $\sigma_7 = \sigma_{10} = 0$, we obtain

$$\begin{aligned}\mathcal{F}_\phi^{\text{SO}} &= \frac{2\nu^2}{15c^3 r^3} \chi_1 \left[p^4(22\nu - 9\delta - 9) + 5(3 + 3\delta + 16\nu)p_r^4 \right. \\ &\quad + (66\nu - 23\delta - 23)\frac{p_r^2}{r} + (179 + 179\delta - 146\nu)\frac{p^2}{r} \\ &\quad \left. + 6(9 + 9\delta - 22\nu)p^2 p_r^2 \right] + 1 \leftrightarrow 2. \quad (44)\end{aligned}$$

$$\begin{aligned}\mathcal{F}_r^{\text{SO}} &= \frac{2\nu^2 p_r p_\phi}{15c^3 r^5} \chi_1 \left[\frac{179\delta - 146\nu + 179}{r} + (22\nu - 9\delta - 9)p^2 \right. \\ &\quad \left. - 5(3 + 3\delta + 16\nu)p_r^2 \right] + 1 \leftrightarrow 2. \quad (45)\end{aligned}$$

5. 2PN no spin

At 2PN, we obtain

$$\begin{aligned}\alpha_6 &= \beta_{13} - \frac{463\nu^2}{63} - \frac{1909\nu}{315} + \frac{922}{315}, \\ \beta_{15} &= -\frac{2\beta_{18}}{3} + \frac{1060\nu^2}{189} - \frac{12116\nu}{189} - \frac{347236}{2835}, \\ \beta_{16} &= -\beta_{17} - \beta_{18} - \frac{5276\nu^2}{315} - \frac{109609\nu}{630} + \frac{9175}{378}, \\ \beta_{14} &= \alpha_7 - \frac{2\beta_{17}}{3} - \frac{2\beta_{18}}{3} + \frac{244\nu^2}{45} - \frac{12466\nu}{135} - \frac{28204}{2835}, \\ \beta_{19} &= -\frac{640\nu^2}{189} + \frac{416\nu}{35} - \frac{2608}{945}, \\ \alpha_8 &= -\beta_{17} - \beta_{18} - \frac{5018\nu^2}{315} - \frac{105491\nu}{630} + \frac{52223}{1890}, \\ \alpha_{10} &= \beta_{18} - \frac{92\nu^2}{9} - \frac{309\nu}{5} - \frac{10019}{315}, \\ \alpha_9 &= \beta_{17} - \frac{78\nu^2}{7} + \frac{1201\nu}{90} + \frac{5711}{126}, \quad (46)\end{aligned}$$

with 8 arbitrary coefficients out of 16. Choosing $\alpha_5 = \alpha_7 = \beta_{10} = \beta_{11} = \beta_{12} = \beta_{13} = \beta_{17} = \beta_{18} = 0$ yields

$$\begin{aligned}
\mathcal{F}_\phi^{2\text{PN}} = & \frac{\nu^2 p_\phi}{c^4 r^3} \left\{ \left(\frac{5276\nu^2}{315} + \frac{109609\nu}{630} - \frac{9175}{378} \right) p^6 + \left(\frac{1519}{54} - \frac{9964\nu^2}{315} - \frac{100847\nu}{630} \right) \frac{p^4}{r} + \left(\frac{3512\nu^2}{315} - \frac{4234\nu}{45} + \frac{3355}{21} \right) \frac{p^2}{r^2} \right. \\
& + \left(\frac{9175}{126} - \frac{5276\nu^2}{105} - \frac{109609\nu}{210} \right) p^4 p_r^2 + \left(\frac{104296}{945} - \frac{15121\nu^2}{105} - \frac{254732\nu}{315} \right) \frac{p_r^2 p^2}{r} + \left(\frac{52}{3} - \frac{152\nu^2}{9} - \frac{310\nu}{9} \right) p_r^6 \\
& + \left(-\frac{185\nu^2}{63} + \frac{2171\nu}{63} - \frac{269}{63} \right) p_r^4 p^2 + \left(\frac{4112}{35} - \frac{278\nu^2}{45} + \frac{344\nu}{35} \right) \frac{p_r^4}{r} + \left(-\frac{1604\nu^2}{315} - \frac{2054\nu}{7} - \frac{8803}{21} \right) \frac{p_r^2}{r^2} \\
& \left. + \left(\frac{8\nu^2}{15} + \frac{9728\nu}{315} - \frac{190244}{2835} \right) \frac{1}{r^3} \right\}, \tag{47}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_r^{2\text{PN}} = & \frac{\nu^2 p_r}{c^4 r^3} \left\{ \left(\frac{5276\nu^2}{315} + \frac{109609\nu}{630} - \frac{9175}{378} \right) p^6 + \left(\frac{1519}{54} - \frac{9964\nu^2}{315} - \frac{100847\nu}{630} \right) \frac{p^4}{r} + \left(\frac{3512\nu^2}{315} - \frac{4234\nu}{45} + \frac{3355}{21} \right) \frac{p^2}{r^2} \right. \\
& + \left(\frac{9713}{126} - \frac{2129\nu^2}{45} - \frac{350537\nu}{630} \right) p^4 p_r^2 + \left(\frac{26459}{945} - \frac{1745\nu^2}{63} - \frac{449227\nu}{315} \right) \frac{p_r^2 p^2}{r} + \left(\frac{52}{3} - \frac{152\nu^2}{9} - \frac{310\nu}{9} \right) p_r^6 \\
& + \left(\frac{293\nu^2}{21} + \frac{1447\nu}{21} - \frac{1361}{63} \right) p_r^4 p^2 + \left(\frac{41612}{315} - \frac{1894\nu^2}{45} + \frac{28648\nu}{315} \right) \frac{p_r^4}{r} + \left(\frac{34528\nu^2}{945} - \frac{481624\nu}{945} - \frac{18761}{45} \right) \frac{p_r^2}{r^2} \\
& \left. + \left(\frac{8\nu^2}{15} + \frac{9728\nu}{315} - \frac{190244}{2835} \right) \frac{1}{r^3} \right\}. \tag{48}
\end{aligned}$$

6. LO spin-spin

At LO SS, we obtain the unique solution

$$\begin{aligned}
\zeta_1 &= \frac{1}{30} [\delta(73 - 128\nu) + 128\nu^2 - 274\nu + 73], \\
\zeta_2 &= -\frac{42}{5}(\delta - 2\nu + 1), \quad \zeta_3 = \frac{2}{15}\nu(64\nu - 261), \\
\zeta_4 &= \frac{1}{30} [\delta(128\nu - 73) + 128\nu^2 - 274\nu + 73], \\
\zeta_5 &= \frac{42}{5}(\delta + 2\nu - 1), \\
\zeta_6 &= -\frac{74}{15}(-2\delta\nu + \delta + 2\nu^2 - 4\nu + 1), \\
\zeta_7 &= \frac{1}{10} [\delta(43 - 80\nu) + 80\nu^2 - 166\nu + 43], \\
\zeta_8 &= 0, \quad \zeta_9 = 2(\delta - 2\nu + 1), \\
\zeta_{10} &= -6(\delta - 2\nu + 1), \quad \zeta_{11} = 0, \\
\zeta_{12} &= \frac{8}{15}(15 - 37\nu)\nu, \quad \zeta_{13} = \frac{2}{5}\nu(40\nu - 63), \\
\zeta_{14} &= 0, \quad \zeta_{15} = -\frac{74}{15}[\delta(2\nu - 1) + 2\nu^2 - 4\nu + 1], \\
\zeta_{16} &= \frac{1}{10} [\delta(80\nu - 43) + 80\nu^2 - 166\nu + 43], \\
\zeta_{17} &= 0, \quad \zeta_{18} = -2(\delta + 2\nu - 1), \\
\zeta_{19} &= 6(\delta + 2\nu - 1), \quad \zeta_{20} = 0. \tag{49}
\end{aligned}$$

With that solution, we get

$$\begin{aligned}
\mathcal{F}_\phi^{\text{SS}} = & \frac{\nu^2 p_\phi}{30c^4 r^5} \left\{ \chi_1^2 \left[24X_1^2 C_{1\text{ES}^2} \left(15p^2 - 90p_r^2 - \frac{49}{r} \right) \right. \right. \\
& + (361\delta - 632\nu\delta + 632\nu^2 - 1354\nu + 361) p_r^2 \\
& \left. \left. + (400\nu\delta - 209\delta - 400\nu^2 + 818\nu - 209) p^2 \right] \right. \\
& \left. + (355\delta - 704\nu\delta + 704\nu^2 - 1414\nu + 355) \frac{1}{r} \right] \\
& + \nu\chi_1\chi_2 \left[(378 - 400\nu)p^2 + (632\nu - 2250)p_r^2 \right. \\
& \left. + \frac{704\nu - 1182}{r} \right] + 1 \leftrightarrow 2 \Big\}, \tag{50}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_r^{\text{SS}} = & \frac{\nu^2 p_r}{30c^4 r^5} \left\{ \chi_1^2 \left[24X_1^2 C_{1\text{ES}^2} \left(15p^2 - 55p_r^2 - \frac{49}{r} \right) \right. \right. \\
& + (301\delta - 512\delta\nu + 512\nu^2 - 1114\nu + 301) p_r^2 \\
& + (400\nu\delta - 209\delta - 400\nu^2 + 818\nu - 209) p^2 \\
& \left. \left. + (355\delta - 704\delta\nu + 704\nu^2 - 1414\nu + 355) \frac{1}{r} \right] \right. \\
& \left. + \nu\chi_1\chi_2 \left[(378 - 400\nu)p^2 + (512\nu - 1410)p_r^2 \right. \right. \\
& \left. \left. + \frac{704\nu - 1182}{r} \right] + 1 \leftrightarrow 2 \Big\}. \tag{51}
\end{aligned}$$

F. Factorizing the RR force into circular and noncircular parts

The total RR force is the sum of the contributions calculated in the previous section, i.e.,

$$\begin{aligned}
\mathcal{F}_\phi &= \mathcal{F}_\phi^{\text{LO}} + \mathcal{F}_\phi^{\text{1PN}} + \mathcal{F}_\phi^{\text{2PN}} + \mathcal{F}_\phi^{\text{tail}} + \mathcal{F}_\phi^{\text{SO}} + \mathcal{F}_\phi^{\text{SS}}, \\
\mathcal{F}_r &= \mathcal{F}_r^{\text{LO}} + \mathcal{F}_r^{\text{1PN}} + \mathcal{F}_r^{\text{2PN}} + \mathcal{F}_r^{\text{tail}} + \mathcal{F}_r^{\text{SO}} + \mathcal{F}_r^{\text{SS}}. \tag{52}
\end{aligned}$$

We have checked that our gauge-dependent RR force agrees with that in Refs. [78, 99, 109] by using the balance equations. Denoting the RR force from those references by $(\bar{\mathcal{F}}_r, \bar{\mathcal{F}}_\phi)$ with corresponding Schott terms

($\bar{E}_{\text{Schott}}, \bar{J}_{\text{Schott}}$), Eqs. (7) lead to

$$\begin{aligned} \dot{r}\bar{\mathcal{F}}_r + \dot{\phi}\bar{\mathcal{F}}_\phi + \dot{\bar{E}}_{\text{Schott}} &= \dot{r}\mathcal{F}_r + \dot{\phi}\mathcal{F}_\phi + \dot{E}_{\text{Schott}}, \\ \bar{\mathcal{F}}_\phi + \dot{\bar{J}}_{\text{Schott}} &= \mathcal{F}_\phi + \dot{J}_{\text{Schott}}. \end{aligned} \quad (53)$$

Then, by writing an ansatz for ($\bar{E}_{\text{Schott}}, \bar{J}_{\text{Schott}}$) with unknown coefficients, we checked that a solution exists, implying that ($\mathcal{F}_r, \mathcal{F}_\phi$) and ($\bar{\mathcal{F}}_r, \bar{\mathcal{F}}_\phi$) are related via a coordinate transformation.

To implement our results in the SEOBNRv4HM model, we factorize the RR force into a quasi-circular part times eccentric corrections as in Eqs. (11) and (8), which read

$$\begin{aligned} \mathcal{F}_\phi &= \mathcal{F}_\phi^{\text{qc}} \mathcal{F}_\phi^{\text{ecc}}, & \mathcal{F}_r &= \mathcal{F}_r^{\text{qc}} \mathcal{F}_r^{\text{ecc}}, \\ \mathcal{F}_\phi^{\text{qc}} &= -\frac{\Phi_E^{\text{qc}}}{\Omega}, & \mathcal{F}_r^{\text{qc}} &= -\frac{\Phi_E^{\text{qc}} p_r}{\Omega p_\phi}, \end{aligned} \quad (54)$$

and for the quasi-circular part we use the unexpanded force used in SEOBNRv4HM, in which the energy flux has the following PN expansion in terms of the orbital velocity $v_\Omega \equiv \Omega^{1/3}$:

$$\begin{aligned} \frac{\Phi_E^{\text{qc}}}{\nu^2} &= \frac{32}{5} v_\Omega^{10} - \frac{2}{105c^2} (980\nu + 1247) v_\Omega^{12} + \frac{128\pi}{5c^3} v_\Omega^{13} \\ &+ \frac{4v_\Omega^{13}}{5c^3} [\chi_1(12\nu - 11\delta - 11) + \chi_2(12\nu + 11\delta - 11)] \\ &+ \frac{2}{2835c^4} v_\Omega^{14} (32760\nu^2 + 166878\nu - 44711) \\ &+ \frac{2}{5c^4} v_\Omega^{14} [(32C_{1\text{ES}^2} + 1)\chi_1^2 X_1^2 + 62\nu\chi_1\chi_2 \\ &+ (32C_{2\text{ES}^2} + 1)\chi_2^2 X_2^2]. \end{aligned} \quad (55)$$

This leads to the eccentric part

$$\begin{aligned} \mathcal{F}_\phi^{\text{ecc}} &= \frac{29p_r^2 r - 10\dot{p}_r r^2 + 12}{12(\dot{p}_r r^2 + 1)^{2/3}} + \frac{[\dots]}{c^2 r (\dot{p}_r r^2 + 1)^{5/3}} + \dots \\ \mathcal{F}_r^{\text{ecc}} &= \frac{7p_r^2 r - 5\dot{p}_r r^2 + 6}{6(\dot{p}_r r^2 + 1)^{2/3}} + \frac{[\dots]}{c^2 r (\dot{p}_r r^2 + 1)^{5/3}} + \dots \end{aligned} \quad (56)$$

The full 2PN expressions are provided in the Supplemental Material [88].

In these eccentric corrections to the RR force, we used \dot{p}_r instead of p_ϕ because it improves the agreement of our model with SEOBNRv4HM in the quasi-circular orbit limit, in which $p_r = 0 = \dot{p}_r$ leading to $\mathcal{F}_{r,\phi}^{\text{ecc}} = 1$. However, having \dot{p}_r on the right-hand side of the EOM for p_r would complicate solving the system of differential equations (4). Therefore, when evolving the EOMs, we simply replace \dot{p}_r in the RR force with the derivative of the Hamiltonian with respect to r calculated numerically, i.e. $\dot{p}_r \rightarrow -\partial\dot{H}_{\text{EOB}}/\partial r$.

SEOBNR waveform models use p_{r^*} (the conjugate momentum to the tortoise radial coordinate r^*) instead of p_r since it improves stability of the EOMs near the EOB event horizon [69, 112]. The two momenta are related by Eq. (E3). In Appendix E, we also obtain Eq. (E8) for the transformation between \dot{p}_r and \dot{p}_{r^*} .

III. INITIAL CONDITIONS

Having determined the RR force that enters the EOMs, we need to specify the initial conditions to be used in evolving the system of equations. In this section, we first review how the initial conditions are implemented in SEOBNRv4HM for quasi-circular orbits [46, 92], and then discuss a simple extension for eccentric orbits.

A. Initial conditions for quasi-circular orbits

Let us recapitulate the initial conditions for quasi-circular/spherical orbits in the SEOBNRv4HM model as derived in Refs. [46, 92]. We start by specifying an initial orbital frequency Ω_0 , with initial orbital phase $\phi_0 = 0$, and solve

$$\left[\frac{\partial H}{\partial r} \right]_0 = 0, \quad \left[\frac{\partial H}{\partial p_\phi} \right]_0 = \Omega_0 \quad (57)$$

for the initial values of r and p_ϕ , while neglecting RR, $p_r \approx 0$. The initial condition for p_r is then obtained by solving

$$[\dot{r}]_0 = \left[\frac{\partial H}{\partial p_r} \right]_0 \quad (58)$$

for p_r , after calculating $[\dot{r}]_0$ using the result from adiabatic evolution [46]

$$[\dot{r}]_0 = \left[\frac{dp_\phi/dt}{dp_\phi/dr} \right]_0 = \left[\frac{\dot{E}}{dE/dr} \right]_0, \quad (59)$$

where \dot{E} is the circular-orbits energy flux, and the derivative $dE/dr = dH/dr$ can be determined using the following equations for circular orbits:

$$dH = \frac{\partial H}{\partial r} dr + \frac{\partial H}{\partial p_r} dp_r + \frac{\partial H}{\partial p_\phi} dp_\phi, \quad (60)$$

$$p_r = 0, \quad dp_r = 0, \quad d\left(\frac{\partial H}{\partial r}\right) = 0. \quad (61)$$

This leads to

$$\frac{d}{dr} \frac{\partial H}{\partial r} = 0 = \frac{\partial^2 H}{\partial r^2} + \frac{\partial^2 H}{\partial r \partial p_\phi} \frac{dp_\phi}{dr}, \quad (62)$$

which can be solved for dp_ϕ/dr to obtain

$$\frac{dp_\phi}{dr} = -\frac{\partial^2 H / \partial r^2}{\partial^2 H / \partial r \partial p_\phi}. \quad (63)$$

Plugging that solution into $dH/dr = (\partial H / \partial p_\phi) dp_\phi / dr$ yields the result in Eq. (4.14) of Ref. [92], which reads

$$\frac{dH}{dr} = -\frac{(\partial H / \partial p_\phi)(\partial^2 H / \partial r^2)}{\partial^2 H / \partial r \partial p_\phi}, \quad (64)$$

and hence

$$\left[\frac{\partial H}{\partial p_r}\right]_0 = [\dot{r}]_0 = \left[-\frac{dE}{dt} \frac{\partial^2 H / \partial r \partial p_\phi}{(\partial H / \partial p_\phi)(\partial^2 H / \partial r^2)}\right]_0. \quad (65)$$

The complete procedure to obtain the initial conditions for the orbital phase-space is now as follows. Given Ω_0 , masses, and spins, we numerically solve the relations in Eq. (57) for the initial values r_0 and p_{ϕ_0} , choosing $\phi_0 = 0$ and assuming $p_r \approx 0$. Using these values, we numerically solve Eq. (65) for the initial value p_{r0} .

B. Initial conditions for eccentric orbits

Since eccentricity is a gauge-dependent concept, we do not need to calculate accurate initial conditions for eccentric orbits in a specific gauge. Instead, we can choose a measure for eccentricity that can be adjusted to be as convenient as possible for numerical implementation. The only strict requirement is that for zero eccentricity $e = 0$ one recovers the quasi-circular case. Hence, we can start with very accurate initial conditions for quasi-circular orbits and perturb them for eccentric orbits.

We choose to specify an initial orbital frequency Ω_0 and an initial eccentricity e_0 using the Keplerian parametrization $1/r = u_p(1 + e \cos \chi)$. We also assume that the orbit starts with $\phi_0 = 0$ at *periastron* ($\chi = 0$), where $p_r = 0$ in absence of RR, which simplifies calculating the initial conditions for r and p_ϕ . An advantage of starting at periastron instead of apastron is that the specified initial frequency is then the maximum orbital frequency (over the first orbit), and can be used to estimate the frequency at which the binary enters a GW detector's frequency band.

To obtain r_0 and p_{ϕ_0} , we solve Eq. (57) with a nonzero \dot{p}_r , i.e.,

$$\left[\frac{\partial H}{\partial r}\right]_0 = -[\dot{p}_r(p_\phi, e)]_0, \quad \left[\frac{\partial H}{\partial p_\phi}\right]_0 = \Omega_0, \quad (66)$$

with $p_r \approx 0$, and $[\dot{p}_r]_0$ given as a 2PN expansion in terms of p_ϕ and e . For quasi-circular orbits, these equations reduce exactly to Eqs. (57) since $\dot{p}_r \propto e$.

To obtain the PN expansion for \dot{p}_r at periastron, we first invert the Hamiltonian at the turning points $r_\pm = 1/(u_p(1 \pm e))$ with $p_r = 0$ and solve for the energy and angular momentum as functions of e and u_p , which are given by Eqs. (D2). Then, we invert $p_\phi(e, u_p)$ to obtain Eq. (D3) for $u_p(p_\phi, e)$ and insert it into the PN expansion for $\dot{p}_r = -\partial H / \partial r$ at periastron ($r = 1/[u_p(1 + e)]$). This yields

$$\begin{aligned} [\dot{p}_r]_0 = & \frac{e_0(e_0 + 1)^2}{p_{\phi_0}^4} + \frac{e_0(e_0 + 1)^2}{2c^2 p_{\phi_0}^6} [e_0^2(5 - \nu) - 4e_0 + \nu + 7] \\ & + \frac{e_0(e_0 + 1)^2 (3e_0^2 - 2e_0 + 3)}{2c^3 p_{\phi_0}^7} \left[(\nu - 2\delta - 2)\chi_1 \right. \\ & \left. + (\nu + 2\delta - 2)\chi_2 \right] + \frac{e_0(e_0 + 1)^2}{8c^4 p_{\phi_0}^8} [3\nu^2 - \nu + 95 \end{aligned}$$

$$\begin{aligned} & - 6e_0^2(\nu^2 + 5\nu - 39) + 8e_0(\nu - 25) \\ & + e_0^4(3\nu^2 - 17\nu + 55) + 8e_0^3(\nu - 7) \\ & + \frac{e_0(e_0 + 1)^2}{2c^4 p_{\phi_0}^8} \left\{ \chi_1^2 [C_{1\text{ES}^2} (3e_0^2 - 2e_0 + 3) X_1^2 \right. \\ & \left. + (5e_0^2 - 6e_0 - 3) X_1^4] + \nu \chi_1 \chi_2 [e_0^2(5\nu + 3) \right. \\ & \left. - 2e_0(3\nu + 1) - 3\nu + 3] + 1 \leftrightarrow 2 \right\}. \quad (67) \end{aligned}$$

The initial condition can now be obtained in analogy to the quasi-circular case: given Ω_0 , e_0 , masses, and spins, we obtain r_0 and p_{ϕ_0} from Eqs. (66) and (67) (assuming $p_r \approx 0$), p_{r0} then follows from Eq. (65) as before, and $\phi_0 = 0$ by convention. We can keep using the circular-orbits energy flux in Eq. (65), instead of replacing \dot{E} with $\Omega \mathcal{F}_\phi$ for eccentric orbits, because the difference on the orbital dynamics is negligible since it involves p_r (which at periastron is numerically much smaller than r or p_ϕ).

To assess the accuracy of the initial conditions for eccentric orbits, we compare the specified value for the eccentricity in the Keplerian parametrization with the value calculated from the orbital frequency (or the frequency of the (2, 2) mode) at periastron Ω_p and apastron Ω_a , which is given by [113]

$$e_\Omega = \frac{\sqrt{\Omega_p} - \sqrt{\Omega_a}}{\sqrt{\Omega_p} + \sqrt{\Omega_a}}, \quad (68)$$

and to calculate it, we follow the steps explained after Eq. (2.8) of Ref. [41]. Since we evaluate e_Ω by evolving the binary over one orbit (including RR), it only holds approximately that $\Omega_p \approx \Omega_a$ in the quasi-circular case. That is, e_Ω does not vanish exactly for quasi-circular orbits, in contrast to our e_0 . Table I shows the value of the eccentricity e_Ω calculated from the orbital frequency compared to the specified eccentricity e_0 , and we see good agreement between the two measures of eccentricity.

TABLE I. Nonspinning initial conditions given the parameters (e_0, Ω_0, ν) , and the eccentricity e_Ω measured from the orbital frequency using Eq. (68). The initial frequency Ω_0 was chosen to give ~ 30 GW cycles between r_0 and $r = 5$.

e_0	Ω_0	ν	r_0	p_{ϕ_0}	p_{r0}	e_Ω
0.01	0.03	0.25	10.4	3.8	-0.0012	0.0087
0.2	0.045	0.25	8.1	3.7	-0.0024	0.19
0.7	0.065	0.25	6.7	4.0	-0.0033	0.69
0.2	0.058	0.1	6.8	3.6	-0.0011	0.20

IV. GRAVITATIONAL WAVEFORM MODES

In this section, we obtain the 2PN waveform modes to 2PN order including LO tail effects, and SO and SS couplings for aligned spins. The instantaneous nonspinning part of the modes was derived for eccentric orbits

in Refs. [114, 115] in harmonic coordinates and we convert their results to EOB coordinates. For the LO tail part, we extend the results of Ref. [79] to $\mathcal{O}(e^6)$ and to higher modes using the Keplerian parametrization, and then convert those results to an expansion in p_r and \dot{p}_r . The spin contributions to the modes were derived to 2PN order for circular orbits in Ref. [116], and here we derive them for eccentric orbits.

The GW spherical harmonic modes $h^{\ell m}$ are the expansion of the complex polarization waveform $h = h_+ - ih_\times$ into spin-weighted $s = -2$ spherical harmonics $Y_{-2}^{\ell m}(\Theta, \Phi)$ such that

$$h_+ - ih_\times = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} Y_{-2}^{\ell m}(\Theta, \Phi). \quad (69)$$

The modes $h^{\ell m}$ can be calculated directly from the radiative multipole moments via [117–119]

$$h^{\ell m} = \frac{1}{\sqrt{2}D_L c^{\ell+2}} \left[U^{\ell m} - \frac{i}{c} V^{\ell m} \right], \quad (70)$$

where D_L is the luminosity distance of the source, and the radiative multipole moments are related to the symmetric trace-free (STF) moments U_L and V_L by

$$\begin{aligned} U^{\ell m} &= \frac{16\pi}{(2\ell+1)!!} \sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \bar{\mathcal{Y}}_L^{\ell m} U_L, \\ V^{\ell m} &= \frac{-32\pi}{(2\ell+1)!!} \sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}} \bar{\mathcal{Y}}_L^{\ell m} V_L, \end{aligned} \quad (71)$$

where $\bar{\mathcal{Y}}_L^{\ell m}$ is the complex conjugate of the STF tensors relating the unit vectors $N_{\langle L \rangle}$ (which point from the source to the detector) to the spherical harmonics basis $Y^{\ell m}(\Theta, \Phi)$ such that

$$Y^{\ell m}(\Theta, \Phi) = \mathcal{Y}_L^{\ell m} N_{\langle L \rangle}(\Theta, \Phi), \quad (72)$$

$$N_{\langle L \rangle}(\Theta, \Phi) = \frac{4\pi\ell!}{(2\ell+1)!!} \sum_{m=-\ell}^{\ell} \bar{\mathcal{Y}}_L^{\ell m} Y^{\ell m}(\Theta, \Phi), \quad (73)$$

$$\bar{\mathcal{Y}}_L^{\ell m} = \frac{(2\ell+1)!!}{4\pi\ell!} \int d\Omega N_{\langle L \rangle} \bar{Y}^{\ell m}, \quad (74)$$

$$\bar{\mathcal{Y}}_L^{\ell m} \mathcal{Y}_L^{\ell m'} = \frac{(2\ell+1)!!}{4\pi\ell!} \delta_{mm'}, \quad (75)$$

and we can express the unit vector \mathbf{N} in terms of the angles Θ and Φ as

$$\mathbf{N} = \sin \Theta \cos \Phi \hat{e}_x + \sin \Theta \sin \Phi \hat{e}_y + \cos \Theta \hat{e}_z. \quad (76)$$

For planar binaries, nonspinning or with aligned spins, it was shown in Ref. [119] that the modes can be determined

using the mass-type multipole moments for even $\ell + m$, or the current-type multipole moments for odd $\ell + m$, i.e.,

$$\begin{aligned} h^{\ell m} &= \frac{1}{\sqrt{2}D_L c^{\ell+2}} U^{\ell m}, & \ell + m \text{ even} \\ h^{\ell m} &= -\frac{1}{\sqrt{2}D_L c^{\ell+3}} V^{\ell m}, & \ell + m \text{ odd.} \end{aligned} \quad (77)$$

We define $H^{\ell m}$ such that

$$h^{\ell m} = -\frac{8\mu}{c^4 D_L} \sqrt{\frac{\pi}{5}} e^{-im\phi} H^{\ell m}, \quad (78)$$

which makes the LO part of $H^{22} = x$ for circular orbits. Note that different conventions for the phase origin contribute a factor of $(-i)^m$ to the modes [120].

In this paper, we compute the modes to 2PN order beyond the leading order of the (2, 2) mode, which means we consider modes up to the $\ell = 6$, $m = \text{even}$ modes. To 2PN order, the instantaneous contributions to the radiative multipole moments coincide with the source multipole moments. Including the hereditary terms that contribute to 2PN, the radiative multipole moments are given by [115, 117, 118]

$$\begin{aligned} U_{ij} &= I_{ij}^{(2)} + \frac{2M}{c^3} \int_0^\infty d\tau I_{ij}^{(4)}(t-\tau) \ln\left(\frac{\tau}{b_1}\right) + \mathcal{O}\left(\frac{1}{c^5}\right), \\ U_{ijk} &= I_{ijk}^{(3)} + \frac{2M}{c^3} \int_0^\infty d\tau I_{ijk}^{(5)}(t-\tau) \ln\left(\frac{\tau}{b_2}\right) + \mathcal{O}\left(\frac{1}{c^5}\right), \\ U_L &= I_L^{(\ell)} + \mathcal{O}\left(\frac{1}{c^3}\right), \\ V_{ij} &= J_{ij}^{(2)} + \frac{2M}{c^3} \int_0^\infty d\tau J_{ij}^{(4)}(t-\tau) \ln\left(\frac{\tau}{b_3}\right) + \mathcal{O}\left(\frac{1}{c^5}\right), \\ V_L &= J_L^{(\ell)} + \mathcal{O}\left(\frac{1}{c^3}\right), \end{aligned} \quad (79)$$

where the constants b_i are gauge parameters that will be eliminated via a phase shift as was done in Ref. [117]. The source multipole moments for nonspinning binaries are given in, e.g., Refs. [115, 117], while the spin contributions to the source moments are given in Refs. [116, 121].

A. Instantaneous nonspinning contributions

The instantaneous contributions to the modes for nonspinning binaries in eccentric orbits were derived in Ref. [114] to 2PN, and in Ref. [115] to 3PN. The results of Ref. [115] are in harmonic coordinates and in terms of the variables $(r, \phi, \dot{r}, \dot{\phi})$. Hence, we can simply transform their results from harmonic to EOB coordinates using the transformations in Appendix A. For the (2, 2) mode we obtain

$$\begin{aligned}
\hat{H}_{\text{inst}}^{22} = & \frac{1}{2} \left(\frac{1}{r} + p^2 - 2p_r^2 \right) + i \frac{p_\phi p_r}{r} + \frac{1}{c^2} \left\{ \left(\frac{\nu}{28} - \frac{5}{28} \right) p^4 + \left(\frac{31\nu}{28} - \frac{157}{84} \right) \frac{p^2}{r} + \left(\frac{5}{14} - \frac{\nu}{14} \right) p_r^2 p^2 + \left(\frac{13}{3} - \nu \right) \frac{p_r^2}{r} \right. \\
& + \left(\frac{\nu}{2} - 2 \right) \frac{1}{r^2} + i \frac{p_\phi p_r}{r} \left[\left(\frac{\nu}{14} - \frac{5}{14} \right) p^2 + \left(\frac{2\nu}{7} - \frac{185}{42} \right) \frac{1}{r} \right] \left. \right\} + \frac{1}{c^4} \left\{ \left(-\frac{17\nu^2}{336} - \frac{13\nu}{336} + \frac{5}{48} \right) p^6 \right. \\
& + \left(-\frac{671\nu^2}{1008} - \frac{1375\nu}{1008} + \frac{481}{504} \right) \frac{p^4}{r} + \left(\frac{127\nu^2}{54} - \frac{1355\nu}{189} - \frac{5519}{3024} \right) \frac{p^2}{r^2} + \left(\frac{17\nu^2}{168} + \frac{13\nu}{168} - \frac{5}{24} \right) p_r^2 p^4 \\
& + \left(-\frac{67\nu^2}{126} + \frac{20\nu}{9} - \frac{659}{504} \right) \frac{p_r^2 p^2}{r} + \left(-\frac{464\nu^2}{189} + \frac{2249\nu}{756} - \frac{811}{3024} \right) \frac{p_r^2}{r^2} + \left(\frac{17\nu^2}{18} - \frac{25\nu}{36} - \frac{919}{1008} \right) \frac{p_r^4}{r} \\
& + \left(\frac{205\nu^2}{252} - \frac{49\nu}{36} + \frac{95}{63} \right) \frac{1}{r^3} + i \frac{p_r p_\phi}{r} \left[\left(-\frac{17\nu^2}{168} - \frac{13\nu}{168} + \frac{5}{24} \right) \frac{p^4}{r} + \left(-\frac{4\nu^2}{21} + \frac{29\nu}{28} + \frac{67}{56} \right) \frac{p^2}{r^2} \right. \\
& \left. + \left(-\frac{523\nu^2}{378} + \frac{1226\nu}{189} + \frac{193}{54} \right) \frac{1}{r^3} + \left(\frac{43\nu^2}{63} - \frac{125\nu}{126} + \frac{787}{504} \right) \frac{p_r^2}{r^2} \right] \left. \right\}. \tag{80}
\end{aligned}$$

The expressions for the other modes that contribute to 2PN, i.e., up to $\ell = |m| = 6$, are provided as a *Mathematica* file in the Supplemental Material [88]. Note that the $(\ell, 0)$ modes are zero for circular orbits but not for eccentric orbits. For example, the LO part of the $(2, 0)$ mode is given by

$$\hat{H}_{\text{inst}}^{20} = \frac{1}{\sqrt{6}} \left(p^2 - \frac{1}{r} \right) + \mathcal{O} \left(\frac{1}{c^2} \right), \tag{81}$$

which is zero for circular orbits since $p^2 = 1/r + \dots$

B. Hereditary contributions

The hereditary contributions to the modes can be calculated analytically in an eccentricity expansion, as was done in Ref. [79] for the $(2, 2)$ mode to $\mathcal{O}(e^2)$, and in Ref. [122] for all modes to 3PN order and to $\mathcal{O}(e^6)$. The results of Ref. [122] use the quasi-Keplerian parametrization, while here we use the Keplerian parametrization following the method developed in Ref. [79], which is based on results from Refs. [94, 117, 123], to derive the leading order tail effects that contribute to the modes up to 2PN order and to $\mathcal{O}(e^6)$. (See Ref. [79] for a discussion of the advantages of the Keplerian parametrization over the quasi-Keplerian parametrization.) We finally convert the eccentricity-expanded tail contributions to an expansion in p_r and \dot{p}_r .

1. modes with even $\ell + m$

The LO mass-type multipole moments are given by [94]

$$I^L = \mu s_\ell r^\ell n^{(L)}, \tag{82}$$

where $s_\ell \equiv X_2^{\ell-1} + (-1)^\ell X_1^{\ell-1}$, and the unit vectors $n^{(L)}$ are related to spherical harmonics via Eq. (73), leading

to

$$I^L = \sum_{m=-\ell}^{\ell} \mathcal{Y}_{\ell m}^L a_{\ell m} r^\ell e^{-im\phi}, \tag{83}$$

with the coefficients (for equatorial orbits)

$$a_{\ell m} \equiv \frac{4\pi \mu s_\ell \ell!}{(2\ell + 1)!!} \bar{Y}_{\ell m} \left(\frac{\pi}{2}, 0 \right). \tag{84}$$

Decomposing the phase ϕ into an oscillatory part and a linearly growing part $\phi = \phi_0 + \omega_\phi t + \Delta\phi_r$ allows expressing the oscillatory part $\Delta\phi_r$ as a Fourier series expansion. Hence,

$$I^L = \sum_{m=-\ell}^{\ell} \mathcal{Y}_{\ell m}^L a_{\ell m} J_{\ell m} e^{-im\psi_\phi}, \tag{85}$$

with the functions $J_{\ell m}$ defined by

$$J_{\ell m} = r^\ell e^{-im\phi_0} e^{-im\Delta\phi_r} = \sum_{k=-\infty}^{\infty} J_{\ell m k} e^{-ik\psi_r}, \tag{86}$$

where ψ_r and ψ_ϕ are the radial and azimuthal angle variables associated with the frequencies $\omega_r = d\psi_r/dt$ and $\omega_\phi = d\psi_\phi/dt$. The coefficients $J_{\ell m k}$ are given by

$$\begin{aligned}
J_{\ell m k} &= \frac{1}{2\pi} \int_0^{2\pi} d\psi_r e^{ik\psi_r} J_{\ell m} \\
&= \frac{\omega_r}{2\pi} \int_0^{2\pi} \frac{d\chi}{\mathcal{P}} r^\ell e^{-im\phi_0} e^{-im\Delta\phi_r} e^{ik\psi_r} \\
&= \frac{\omega_r}{2\pi} u_p^{-\ell-3/2} \int_0^{2\pi} \frac{d\chi}{(1 + e \cos \chi)^{\ell+2}} e^{-im\Delta\phi_r} e^{ik\psi_r}, \tag{87}
\end{aligned}$$

where, in the last line, we assume $\phi_0 = 0$. The function \mathcal{P} denotes the conservative part of $\dot{\chi}$ and is related to

the radial angle ψ_r via $d\psi_r/d\chi = \omega_r/\mathcal{P}$, with $\mathcal{P} = (1 + e \cos \chi)^2 u_p^{3/2}$ at LO (see Ref. [79] for more details).

Thus, the Newtonian mass multipole moments can be expressed as

$$I^L = \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \mathcal{Y}_{\ell m}^L a_{\ell m} J_{\ell m k} e^{-i(k\psi_r + m\psi_\phi)}, \quad (88)$$

where the azimuthal angle is related to the radial angle by $\psi_\phi = \phi - \Delta\phi_r$ with $\Delta\phi_r = \chi - \psi_r$ at LO. This allows us to write the LO tail contribution to the mass-type radiative moments as

$$\begin{aligned} U_L^{\text{tail}} &= \frac{2M}{c^3} \int_0^\infty d\tau I_L^{(\ell+2)}(t-\tau) \ln\left(\frac{\tau}{b}\right), \\ &= (-i)^{\ell+2} \frac{2M}{c^3} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \mathcal{Y}_{\ell m}^{\ell+2} a_{\ell m} J_{\ell m k} \Omega_{mk}^{\ell+2} \\ &\quad \times e^{-i(k\psi_r + m\psi_\phi)} \mathcal{I}(\Omega_{mk}), \end{aligned} \quad (89)$$

where $\Omega_{mk} \equiv m\omega_\phi + k\omega_r$ and

$$\begin{aligned} \mathcal{I}(x) &\equiv \int_0^\infty d\tau e^{ix\tau} \ln\left(\frac{\tau}{b}\right) \\ &= -\frac{1}{x} \left[\frac{\pi}{2} \text{sgn}(x) + i \ln(|x|b) + i\gamma_E \right]. \end{aligned} \quad (90)$$

The exponential $e^{-ik\psi_r}$ can be expressed in terms of χ and e by integrating $d\psi_r = \omega_r d\chi/\mathcal{P}$, with $\omega_r = (u_p - e^2 u_p)^{3/2}$, leading to Eq. (3.41) of Ref. [79], which reads

$$e^{-ik\psi_r} = e^{ik \frac{e\sqrt{1-e^2} \sin \chi}{1+e \cos \chi}} \left(\frac{1 + \sqrt{1-e^2} + e e^{i\chi}}{e + (1 + \sqrt{1-e^2})e^{i\chi}} \right)^k. \quad (91)$$

Hence, the modes with $\ell = 2$ and m even are given by

$$\begin{aligned} h_{\text{tail}}^{2m} &= \frac{\sqrt{24} M a_{2m}}{c^7 R} e^{-im\phi} \sum_{k=-\infty}^{\infty} J_{2mk} \Omega_{mk}^4 \\ &\quad \times e^{im\chi} e^{-i(k+m)\psi_r} \mathcal{I}(\Omega_{mk}), \end{aligned} \quad (92)$$

while the modes with $\ell = 3$ and m odd are given by

$$\begin{aligned} h_{\text{tail}}^{3m} &= -i \frac{4\sqrt{5} M a_{3m}}{3\sqrt{6} c^8 R} e^{-im\phi} \sum_{k=-\infty}^{\infty} J_{3mk} \Omega_{mk}^5 \\ &\quad \times e^{im\chi} e^{-i(k+m)\psi_r} \mathcal{I}(\Omega_{mk}). \end{aligned} \quad (93)$$

To obtain analytical expressions for the modes, we expand the above equations in eccentricity, where the infinite sum over k can be stopped at the order of the expansion in e . The result of that expansion is complicated, but we can perform a phase redefinition in the leading order instantaneous part³ of the form $\phi \rightarrow \phi + x^{3/2} \delta_\phi$, and absorb in δ_ϕ all terms that are not proportional to $\pi^{3/2}$. This modifies the phase at 4PN relative order, which we can ignore when working to 2PN order. (See Ref. [117] for more details.) The result for the (2, 2) mode to $\mathcal{O}(e^6)$ is given by

$$\begin{aligned} \hat{H}_{\text{tail}}^{22} &= \frac{2\pi}{c^3} x^{5/2} \left[1 + e \left(\frac{11e^{-i\chi}}{8} + \frac{13e^{i\chi}}{8} \right) + e^2 \left(\frac{5}{8} e^{-2i\chi} + \frac{7}{8} e^{2i\chi} + 4 \right) + e^3 \left(\frac{121e^{-i\chi}}{32} + \frac{143e^{i\chi}}{32} + \frac{3}{32} e^{-3i\chi} + \frac{1}{12} e^{3i\chi} \right) \right. \\ &\quad + e^4 \left(\frac{25}{16} e^{-2i\chi} + \frac{203}{96} e^{2i\chi} - \frac{5}{96} e^{4i\chi} + \frac{65}{8} \right) + e^5 \left(\frac{55e^{-i\chi}}{8} + \frac{6233e^{i\chi}}{768} + \frac{15}{64} e^{-3i\chi} + \frac{281e^{3i\chi}}{1536} + \frac{53e^{5i\chi}}{7680} \right) \\ &\quad \left. + e^6 \left(\frac{175}{64} e^{-2i\chi} + \frac{1869}{512} e^{2i\chi} - \frac{449e^{4i\chi}}{3840} + \frac{31e^{6i\chi}}{23040} + \frac{30247}{2304} \right) \right], \end{aligned} \quad (94)$$

while for the (2, 0) mode

$$\begin{aligned} \hat{H}_{\text{tail}}^{20} &= \frac{\pi x^{5/2}}{2\sqrt{6} c^3} \left[e (e^{-i\chi} + e^{i\chi}) + e^2 (e^{-2i\chi} + e^{2i\chi} + 2) + e^3 \left(3e^{-i\chi} + 3e^{i\chi} + \frac{1}{4} e^{-3i\chi} + \frac{1}{4} e^{3i\chi} \right) \right. \\ &\quad \left. + e^4 \left(\frac{29}{12} e^{-2i\chi} + \frac{29}{12} e^{2i\chi} + \frac{9}{2} \right) + e^5 \left(\frac{179e^{-i\chi}}{32} + \frac{179e^{i\chi}}{32} + \frac{125}{192} e^{-3i\chi} + \frac{125}{192} e^{3i\chi} \right) \right] \end{aligned}$$

³ One first needs to express the leading order part in terms of the variables (e, x, χ) instead of (r, p_r, p_ϕ) using the relations from Appendix D. For example, for the (2, 2) mode, we obtain

$$h_{\text{LO}}^{22} = \frac{-8\mu}{c^4 D_L} \sqrt{\frac{\pi}{5}} e^{-2i\phi} \frac{x}{1-e^2} \left[1 + \frac{e}{4} (e^{-i\chi} + 5e^{i\chi}) + \frac{e^2}{2} e^{2i\chi} \right].$$

$$+ e^6 \left(\frac{805}{192} e^{-2i\chi} + \frac{805}{192} e^{2i\chi} - \frac{7}{960} e^{-4i\chi} - \frac{7}{960} e^{4i\chi} + \frac{121}{16} \right). \quad (95)$$

The (3, 3) mode is given by

$$\begin{aligned} \hat{H}_{\text{tail}}^{33} = & -\frac{9i\pi\delta}{4c^4} \sqrt{\frac{15}{14}} x^3 \left[1 + e \left(\frac{47e^{-i\chi}}{27} + \frac{19e^{i\chi}}{9} \right) + e^2 \left(\frac{61}{54} e^{-2i\chi} + \frac{91}{54} e^{2i\chi} + \frac{155}{27} \right) + e^3 \left(\frac{691e^{-i\chi}}{108} + \frac{841e^{i\chi}}{108} + \frac{35e^{-3i\chi}}{108} \right. \right. \\ & + \left. \frac{65e^{3i\chi}}{108} \right) + e^4 \left(\frac{32}{9} e^{-2i\chi} + \frac{287}{54} e^{2i\chi} + \frac{5e^{-4i\chi}}{144} + \frac{115e^{4i\chi}}{1728} + \frac{3139}{216} \right) + e^5 \left(\frac{503e^{-i\chi}}{36} + \frac{613e^{i\chi}}{36} + \frac{35e^{-3i\chi}}{36} \right. \\ & \left. \left. + \frac{3095e^{3i\chi}}{1728} - \frac{457e^{5i\chi}}{25920} \right) + e^6 \left(\frac{131}{18} e^{-2i\chi} + \frac{150503e^{2i\chi}}{13824} + \frac{5}{48} e^{-4i\chi} + \frac{151}{810} e^{4i\chi} - \frac{41e^{6i\chi}}{20736} + \frac{219}{8} \right) \right], \quad (96) \end{aligned}$$

and the (3, 1) mode

$$\begin{aligned} \hat{H}_{\text{tail}}^{31} = & \frac{i\pi\delta x^3}{12\sqrt{14}c^4} \left[1 + e (e^{i\chi} - 9e^{-i\chi}) + e^2 \left(-\frac{27}{2} e^{-2i\chi} - \frac{5}{4} e^{2i\chi} - 15 \right) + e^3 \left(-\frac{177}{4} e^{-i\chi} - \frac{19e^{i\chi}}{2} - \frac{25}{4} e^{-3i\chi} - \frac{4}{3} e^{3i\chi} \right) \right. \\ & + e^4 \left(-\frac{89}{2} e^{-2i\chi} - \frac{125}{24} e^{2i\chi} - \frac{15}{16} e^{-4i\chi} - \frac{55e^{4i\chi}}{192} - \frac{1703}{32} \right) + e^5 \left(-\frac{10141}{96} e^{-i\chi} - \frac{2867e^{i\chi}}{96} - \frac{75}{4} e^{-3i\chi} - \frac{629}{192} e^{3i\chi} \right. \\ & \left. \left. - \frac{7}{960} e^{5i\chi} \right) + e^6 \left(-\frac{142903e^{-2i\chi}}{1536} - \frac{2965}{256} e^{2i\chi} - \frac{45}{16} e^{-4i\chi} - \frac{239}{240} e^{4i\chi} + \frac{37e^{6i\chi}}{23040} - \frac{16343}{144} \right) \right]. \quad (97) \end{aligned}$$

We checked that our results agree with those of Ref. [122] after converting between the quasi-Keplerian and Keplerian parametrization, and performing a phase shift.

To express the modes in terms of (r, p_r, p_ϕ) instead of (x, e, χ) , we use the following leading order relations:

$$p_\phi = \frac{1}{\sqrt{u_p}}, \quad p_r = e\sqrt{u_p} \sin \chi, \quad \frac{1}{r} = u_p(1 + e \cos \chi), \quad x = u_p(1 - e^2). \quad (98)$$

As explained above, it is advantageous to replace p_ϕ^2 with \dot{p}_r using $\dot{p}_r = (p_\phi^2 - r)/r^3$ and expand in both p_r and \dot{p}_r (since p_r and \dot{p}_r are both of order e) to obtain

$$\begin{aligned} \hat{H}_{\text{tail}}^{22} = & \frac{2\pi}{c^3} \left\{ \frac{p_\phi}{r^3} + \frac{i\dot{p}_r}{4r^2} + \left[\frac{7}{32} p_\phi p_r^2 \dot{p}_r - \frac{7}{96} p_\phi r^3 \dot{p}_r^3 + i \left(\frac{7p_r^3}{96r} - \frac{7}{32} r^2 p_r \dot{p}_r^2 \right) \right] + \left[\frac{3p_\phi r^5 \dot{p}_r^4}{32} - \frac{p_\phi r^2 p_r^2 \dot{p}_r^2}{8} + \frac{p_\phi p_r^4}{48r} \right. \right. \\ & + \left. i \left(\frac{r^4}{12} p_r \dot{p}_r^3 - \frac{r}{96} p_r^3 \dot{p}_r \right) \right] + \left[\frac{31}{384} p_\phi r p_r^4 \dot{p}_r - \frac{173p_\phi r^7 \dot{p}_r^5}{1920} - \frac{1}{192} p_\phi r^4 p_r^2 \dot{p}_r^3 + i \left(\frac{r^6 p_r}{768} \dot{p}_r^4 - \frac{49}{384} r^3 p_r^3 \dot{p}_r^2 + \frac{89p_r^5}{3840} \right) \right] \\ & \left. + \left[\frac{97p_\phi r^9 \dot{p}_r^6}{1152} + \frac{1}{16} p_\phi r^6 p_r^2 \dot{p}_r^4 - \frac{47}{384} p_\phi r^3 p_r^4 \dot{p}_r^2 + \frac{p_\phi p_r^6}{96} + i \left(-\frac{1}{64} r^8 p_r \dot{p}_r^5 + \frac{137r^5 p_r^3 \dot{p}_r^3}{1152} - \frac{23}{640} r^2 p_r^5 \dot{p}_r \right) \right] \right\}, \\ \hat{H}_{\text{tail}}^{20} = & \frac{\pi}{\sqrt{6}c^3} \left\{ \frac{p_\phi \dot{p}_r}{r} - p_\phi r \dot{p}_r^2 + \left[\frac{3}{4} p_\phi r^3 \dot{p}_r^3 - \frac{1}{4} p_\phi p_r^2 \dot{p}_r \right] + \left[-\frac{7}{12} p_\phi r^5 \dot{p}_r^4 - \frac{p_\phi p_r^4}{6r} \right] + \left[\frac{95}{192} p_\phi r^7 \dot{p}_r^5 + \frac{13}{96} p_\phi r^4 p_r^2 \dot{p}_r^3 \right. \right. \\ & \left. \left. + \frac{11}{192} p_\phi r p_r^4 \dot{p}_r \right] + \left[-\frac{139}{320} p_\phi r^9 \dot{p}_r^6 - \frac{11}{96} p_\phi r^6 p_r^2 \dot{p}_r^4 + \frac{3}{64} p_\phi r^3 p_r^4 \dot{p}_r^2 - \frac{p_\phi p_r^6}{480} \right] \right\}, \\ \hat{H}_{\text{tail}}^{33} = & -\frac{9i\pi\delta}{4c^4} \sqrt{\frac{15}{14}} \left\{ \frac{1}{r^3} + \left[\frac{23\dot{p}_r}{27r} + \frac{10ip_\phi p_r}{27r^3} \right] - \frac{2p_r^2}{27r^2} + \left[i \left(\frac{25}{432} p_\phi p_r^3 \dot{p}_r - \frac{25}{432} p_\phi r^3 p_r \dot{p}_r^3 \right) - \frac{25r^5 \dot{p}_r^4}{1728} + \frac{25}{288} r^2 p_r^2 \dot{p}_r^2 \right. \right. \\ & \left. \left. - \frac{25p_r^4}{1728r} \right] + \left[i \left(\frac{109p_\phi r^5 p_r \dot{p}_r^4}{2592} + \frac{41p_\phi r^2 p_r^3 \dot{p}_r^2}{1296} - \frac{41p_\phi p_r^5}{12960r} \right) + \frac{293r^7 \dot{p}_r^5}{25920} + \frac{41r^4 p_r^2 \dot{p}_r^3}{1296} - \frac{157r p_r^4 \dot{p}_r}{5184} \right] + \left[\frac{2845r^3 p_r^4 \dot{p}_r^2}{41472} \right. \\ & \left. \left. - \frac{1561r^9 \dot{p}_r^6}{207360} - \frac{775r^6 p_r^2 \dot{p}_r^4}{13824} - \frac{653p_r^6}{69120} + i \left(\frac{4211p_\phi r p_r^5 \dot{p}_r}{103680} - \frac{2173p_\phi r^7 p_r \dot{p}_r^5}{103680} - \frac{307p_\phi r^4 p_r^3 \dot{p}_r^3}{3456} \right) \right] \right\}, \\ \hat{H}_{\text{tail}}^{31} = & \frac{i\pi\delta}{12\sqrt{14}c^4} \left\{ \frac{1}{r^3} + \left[\frac{10ip_\phi p_r}{r^3} - \frac{11\dot{p}_r}{r} \right] - \left[\frac{11ip_\phi \dot{p}_r p_r}{2r} + \frac{13p_r^2}{4r^2} + \frac{11r\dot{p}_r^2}{4} \right] + \left[i \left(6p_\phi r p_r \dot{p}_r^2 - \frac{p_\phi p_r^3}{6r^2} \right) + \frac{35}{12} r^3 \dot{p}_r^3 \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{p_r^2 \dot{p}_r}{2} \left] + \left[i \left(\frac{45}{16} p_\phi p_r^3 \dot{p}_r - \frac{61}{16} p_\phi r^3 p_r \dot{p}_r \right) - \frac{269}{192} r^5 \dot{p}_r^4 + \frac{37}{32} r^2 p_r^2 \dot{p}_r^2 - \frac{63 p_r^4}{64 r} \right] + \left[\frac{643}{960} r^7 \dot{p}_r^5 + \frac{5}{16} r^4 p_r^2 \dot{p}_r^3 \right. \\
& + \frac{101}{192} r p_r^4 \dot{p}_r + i \left(\frac{83}{32} p_\phi r^5 p_r \dot{p}_r^4 - \frac{43}{48} p_\phi r^2 p_r^3 \dot{p}_r^2 + \frac{379 p_\phi p_r^5}{480 r} \right) \left. \right] + \left[\frac{317}{768} r^3 p_r^4 \dot{p}_r^2 - \frac{5039 r^9 \dot{p}_r^6}{11520} - \frac{289}{768} r^6 p_r^2 \dot{p}_r^4 - \frac{297 p_r^6}{1280} \right. \\
& \left. + i \left(\frac{151}{576} p_\phi r^4 p_r^3 \dot{p}_r^3 - \frac{4267 p_\phi r^7 p_r \dot{p}_r^5}{1920} + \frac{271}{640} p_\phi r p_r^5 \dot{p}_r \right) \right] \left. \right\}. \quad (99)
\end{aligned}$$

2. modes with odd $\ell + m$

The Newtonian order current quadrupole moment is given by [94]

$$J^{ij} = -\delta \mu r^2 n_k v_l \epsilon^{kl(i} n^{j)} = -\delta r p_\phi \hat{e}_z^{(i} n^{j)}, \quad (100)$$

where \hat{e}_z^i is the unit vector in the z -direction. The term $\hat{e}_z^{(i} n^{j)}$ can be expressed in terms of \mathcal{Y}_{21}^{ij} , as was done in Ref. [124], by defining the complex vector

$$\zeta^i = e_x^i + i e_y^i, \quad (101)$$

which leads to

$$\mathcal{Y}_{21}^{ij} = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \zeta^{(i} e_z^{j)}. \quad (102)$$

Since, for equatorial orbits, $n^i = \cos \phi \hat{e}_x^i + \sin \phi \hat{e}_y^i$ and $\lambda^i = -\sin \phi \hat{e}_x^i + \cos \phi \hat{e}_y^i$, we obtain

$$n^i + i \lambda^i = e^{-i\phi} \zeta^i. \quad (103)$$

Hence,

$$\begin{aligned}
e_z^{(i} n^{j)} &= \text{Re} \left[e^{-i\phi} e_z^{(i} \zeta^{j)} \right] = -2 \sqrt{\frac{2\pi}{15}} \text{Re} \left[e^{-i\phi} \mathcal{Y}_{21}^{ij} \right] \\
&= -\sqrt{\frac{2\pi}{15}} \left(e^{-i\phi} \mathcal{Y}_{21}^{ij} + e^{i\phi} \bar{\mathcal{Y}}_{21}^{ij} \right). \quad (104)
\end{aligned}$$

Since V_{ij} is contracted with $\bar{\mathcal{Y}}_{ij}^{\ell m}$ in Eq. (71), and $\bar{\mathcal{Y}}_{ij}^{\ell m} \mathcal{Y}_{\ell m}^{ij} = 0$, only the term with \mathcal{Y}_{21}^{ij} in the above equation contributes to the modes. Thus, we only need to consider the following part of the current quadrupole

$$J^{ij} = \mu \delta \sqrt{\frac{2\pi}{15}} \mathcal{Y}_{21}^{ij} p_\phi r e^{-i\phi} + \dots \quad (105)$$

Then, we follow the same steps as in the previous subsection. Decomposing the phase into $\phi = \omega_\phi t + \Delta\phi$ leads

to

$$J^{ij} = \mu \delta \sqrt{\frac{2\pi}{15}} \mathcal{Y}_{21}^{ij} p_\phi e^{-i\psi_\phi} J_{11} + \dots, \quad (106)$$

with

$$J_{11} = r e^{-i\Delta\phi} = \sum_{k=-\infty}^{\infty} J_{11k} e^{-ik\psi_r}, \quad (107)$$

and

$$\begin{aligned}
J_{11k} &= \frac{1}{2\pi} \int_0^{2\pi} d\psi_r J_{11} e^{ik\psi_r} \\
&= \frac{\omega_r}{2\pi u_p^{5/2}} \int_0^{2\pi} \frac{d\chi}{(1 + e \cos \chi)^3} e^{-i\Delta\phi} e^{ik\psi_r}. \quad (108)
\end{aligned}$$

Thus, the current quadrupole source moment can be expressed as

$$J^{ij} = \mu \delta \sqrt{\frac{2\pi}{15}} \mathcal{Y}_{21}^{ij} p_\phi \sum_{k=-\infty}^{\infty} J_{11k} e^{-i(k\psi_r + \psi_\phi)} + \dots, \quad (109)$$

and the current quadrupole radiative moment

$$\begin{aligned}
V_{ij}^{\text{tail}} &= \frac{2M\mu\delta}{c^3} \sqrt{\frac{2\pi}{15}} \mathcal{Y}_{21}^{ij} p_\phi \sum_{k=-\infty}^{\infty} J_{11k} \Omega_{1k}^4 \\
&\quad \times e^{-i(k\psi_r + \psi_\phi)} \mathcal{I}(\Omega_{1k}), \quad (110)
\end{aligned}$$

leading to the (2, 1) mode

$$\begin{aligned}
h_{\text{tail}}^{21} &= \frac{16i}{3} \sqrt{\frac{\pi}{5}} \frac{\delta M}{Rc^8} e^{-i\phi} p_\phi \sum_{k=-\infty}^{\infty} J_{11k} \Omega_{1k}^4 e^{i\chi} \\
&\quad \times e^{-i(k+1)\psi_r} \mathcal{I}(\Omega_{1k}), \quad (111)
\end{aligned}$$

with $p_\phi = 1/\sqrt{u_p} = \sqrt{(1-e^2)/x}$. Expanding in eccentricity yields

$$\hat{H}_{\text{tail}}^{21} = \frac{i\pi\delta}{3c^4} x^3 \left[1 + e(3e^{-i\chi} + e^{i\chi}) + e^2 \left(3e^{-2i\chi} + \frac{1}{4}e^{2i\chi} + 6 \right) + e^3 \left(\frac{45e^{-i\chi}}{4} + 4e^{i\chi} + \frac{5}{4}e^{-3i\chi} + \frac{1}{6}e^{3i\chi} \right) \right]$$

$$\begin{aligned}
& + e^4 \left(\frac{19}{2} e^{-2i\chi} + \frac{25}{24} e^{2i\chi} + \frac{3}{16} e^{-4i\chi} + \frac{17}{192} e^{4i\chi} + \frac{493}{32} \right) + e^5 \left(\frac{2375 e^{-i\chi}}{96} + \frac{865 e^{i\chi}}{96} + \frac{15}{4} e^{-3i\chi} + \frac{91}{192} e^{3i\chi} \right. \\
& \left. - \frac{7}{960} e^{5i\chi} \right) + e^6 \left(\frac{29957 e^{-2i\chi}}{1536} + \frac{593}{256} e^{2i\chi} + \frac{9}{16} e^{-4i\chi} + \frac{241}{960} e^{4i\chi} + \frac{37 e^{6i\chi}}{23040} + \frac{8417}{288} \right) \Big], \quad (112)
\end{aligned}$$

which is in agreement with the results of Ref. [122]. In terms of $(r, p_r, p_\phi, \dot{p}_r)$, we obtain

$$\begin{aligned}
\hat{H}_{\text{tail}}^{21} = & \frac{i\pi\delta}{3c^4} \left\{ \frac{1}{r^3} + \left[\frac{\dot{p}_r}{r} - \frac{2ip_\phi p_r}{r^3} \right] + \left[\frac{ip_\phi \dot{p}_r p_r}{2r} - \frac{p_r^2}{4r^2} + \frac{1}{4} r \dot{p}_r^2 \right] + \left[-\frac{ip_\phi p_r^3}{6r^2} - \frac{1}{12} r^3 \dot{p}_r^3 - \frac{1}{2} \dot{p}_r p_r^2 \right] \right. \\
& + \left[i \left(\frac{5}{16} p_\phi p_r^3 \dot{p}_r - \frac{5}{16} p_\phi r^3 p_r \dot{p}_r^3 \right) - \frac{1}{192} 5r^5 \dot{p}_r^4 + \frac{13}{32} r^2 p_r^2 \dot{p}_r^2 - \frac{7p_r^4}{64r} \right] + \left[i \left(\frac{11}{32} p_\phi r^5 p_r \dot{p}_r^4 - \frac{19}{48} p_\phi r^2 p_r^3 \dot{p}_r^2 + \frac{19p_\phi p_r^5}{480r} \right) \right. \\
& + \frac{43}{960} r^7 \dot{p}_r^5 - \frac{3}{16} r^4 p_r^2 \dot{p}_r^3 + \frac{29}{192} r p_r^4 \dot{p}_r \Big] + \left[i \left(-\frac{511 p_\phi r^7 p_r \dot{p}_r^5}{1920} + \frac{115}{576} p_\phi r^4 p_r^3 \dot{p}_r^3 - \frac{61}{640} p_\phi r p_r^5 \dot{p}_r \right) - \frac{79r^9 \dot{p}_r^6}{2304} \right. \\
& \left. \left. - \frac{13}{768} r^6 p_r^2 \dot{p}_r^4 + \frac{17}{768} r^3 p_r^4 \dot{p}_r^2 + \frac{59p_r^6}{1280} \right] \right\}. \quad (113)
\end{aligned}$$

C. Aligned-spin contributions

The spin contributions to the modes were derived for circular orbits in Refs. [116, 120, 125]. To derive the spin part of the modes to 2PN for eccentric orbits, we use the source moments from Refs. [116, 121], which are in harmonic coordinates and in terms of the covariant SSC. Dif-

ferentiating the source moments to obtain the radiative moments (79), and plugging them into Eq. (70), we obtain the modes listed in Appendix C. Transforming from harmonic to EOB coordinates, and from the covariant to the NW SSC using the transformations in Appendix A, we obtain the following spin contributions to the modes:

$$\begin{aligned}
\hat{H}_{\text{spin}}^{22} = & \frac{1}{c^3} \left[\frac{\chi_1}{12r^3} [(6\delta + \nu + 6)p_\phi + 2i(3\delta - \nu + 3)rp_r] + \frac{\chi_2}{12r^3} [(-6\delta + \nu + 6)p_\phi - 2i(3\delta + \nu - 3)rp_r] \right] \\
& + \frac{1}{4c^4 r^3} \left\{ \chi_1^2 [3C_{1\text{ES}^2} X_1^2 - X_1^4 (2p^2 r + 1)] + \chi_2^2 [3C_{2\text{ES}^2} X_2^2 - X_2^4 (2p^2 r + 1)] - 2\nu\chi_1\chi_2 (\nu - 3 + 2\nu p^2 r) \right\}, \\
\hat{H}_{\text{spin}}^{21} = & \frac{i}{4c^2 r^2} [-(1 + \delta)\chi_1 + (1 - \delta)\chi_2] + \frac{\chi_1}{84c^4 r^2} \left[\frac{ip_\phi^2}{r^2} (43\nu\delta - 42\delta + 153\nu - 42) + \frac{prp_\phi}{r} (3\delta(2\nu + 49) - 104\nu + 147) \right. \\
& + \frac{i}{2} p_r^2 (\delta(38\nu + 105) + 74\nu + 105) + \frac{i}{r} (63\delta - 38\nu\delta - 74\nu + 63) \Big] + \frac{\chi_2}{84c^4 r^2} \left[\frac{ip_\phi^2}{r^2} (43\nu\delta - 42\delta - 153\nu + 42) \right. \\
& + \frac{prp_\phi}{r} (3\delta(2\nu + 49) + 104\nu - 147) + \frac{i}{2} p_r^2 (38\nu\delta + 105\delta - 74\nu - 105) + \frac{i}{r} (63\delta - 38\nu\delta + 74\nu - 63) \Big], \\
\hat{H}_{\text{spin}}^{20} = & \frac{\sqrt{3}p_\phi}{2\sqrt{2}c^3 r^3} [(2\delta - \nu + 2)\chi_1 + (-2\delta - \nu + 2)\chi_2] + \frac{\sqrt{3}}{2\sqrt{2}c^4 r^3} \left\{ \chi_1^2 \left[\frac{X_1^4}{3r} (2r^2 p_r^2 - 2p_\phi^2 + r) - C_{1\text{ES}^2} X_1^2 \right] \right. \\
& \left. + \chi_2^2 \left[\frac{X_2^4}{3r} (2r^2 p_r^2 - 2p_\phi^2 + r) - C_{2\text{ES}^2} X_2^2 \right] + \frac{2\nu\chi_1\chi_2}{3} \left[\nu - 3 - 2\nu \frac{p_\phi^2}{r} + 2\nu r p_r^2 \right] \right\}, \\
\hat{H}_{\text{spin}}^{30} = & \frac{-i\nu p_r}{\sqrt{42}c^3 r^2} (\chi_1 + \chi_2), \\
\hat{H}_{\text{spin}}^{31} = & \frac{1}{24\sqrt{14}c^4 r^2} \left\{ \chi_1 \left[\frac{ip_\phi^2}{2r^2} (55\nu\delta - 96\delta + 375\nu - 96) + i \left(p_r^2 - \frac{2}{r} \right) (2\nu\delta - 6\delta + 23\nu - 6) \right. \right. \\
& + \frac{prp_\phi}{r} (-6\nu\delta + 30\delta - 127\nu + 30) \Big] + \chi_2 \left[\frac{ip_\phi^2}{2r^2} (55\nu\delta - 96\delta - 375\nu + 96) + \frac{prp_\phi}{r} (-6\nu\delta + 30\delta + 127\nu - 30) \right. \\
& \left. \left. + i \left(p_r^2 - \frac{2}{r} \right) (2\nu\delta - 6\delta - 23\nu + 6) \right] \right\},
\end{aligned}$$

$$\begin{aligned}
\hat{H}_{\text{spin}}^{32} &= \frac{\nu}{6c^3r^3} \sqrt{\frac{5}{7}} (4p_\phi + irp_r)(\chi_1 + \chi_2), \\
\hat{H}_{\text{spin}}^{33} &= \frac{\sqrt{5}}{8\sqrt{42}c^4r^2} \left\{ \left[-\frac{23i(\delta+1)\nu p_\phi^2}{2r^2} + \frac{(2\nu\delta + 6\delta - 19\nu + 6)p_r p_\phi}{r} + i\left(\frac{2}{r} - p_r^2\right)(2\nu\delta - 6\delta + 23\nu - 6) \right] \chi_1 \right. \\
&\quad \left. + \left[-\frac{23i(\delta-1)\nu p_\phi^2}{2r^2} + \frac{(2\delta\nu + 6\delta + 19\nu - 6)p_r p_\phi}{r} + i\left(\frac{2}{r} - p_r^2\right)(2\delta\nu - 6\delta - 23\nu + 6) \right] \chi_2 \right\}, \\
\hat{H}_{\text{spin}}^{41} &= -i\sqrt{\frac{5}{2}} \frac{\nu}{336c^4r^4} (-10irp_r p_\phi + 6r^2 p_r^2 - 12r + 11p_\phi^2) [(\delta-1)\chi_1 + (\delta+1)\chi_2], \\
\hat{H}_{\text{spin}}^{43} &= \sqrt{\frac{5}{14}} \frac{\nu}{48c^4r^4} (10rp_r p_\phi + 2ir^2 p_r^2 - 4ir - 23ip_\phi^2) [(\delta-1)\chi_1 + (\delta+1)\chi_2]. \tag{114}
\end{aligned}$$

The circular-orbit limit of these modes, when expressed in terms of the orbital frequency, agrees with the results of Refs. [116, 125]. The spin contributions to the (2, 2), (2, 1), and (3, 3) modes for eccentric orbits were calculated in Ref. [82], however we find a small disagreement with their results for the SO part.⁴

D. Factorized modes

The quasi-circular waveform modes used in SEOBNRv4HM are factorized as follows [70, 83, 126, 127]:

$$h_{\ell m}^{\text{F, qc}} = h_{\ell m}^{\text{N, qc}} \hat{S}_{\text{eff}}^{\text{qc}} T_{\ell m}^{\text{qc}} e^{i\delta_{\ell m}} f_{\ell m}^{\text{qc}}, \tag{115}$$

where $h_{\ell m}^{\text{N, qc}}$ is the Newtonian part of the mode, $\hat{S}_{\text{eff}}^{\text{qc}}$ is an effective source term given by

$$\hat{S}_{\text{eff}}^{\text{qc}} = \begin{cases} \hat{H}_{\text{eff}}(v_\Omega) & \ell + m \text{ even} \\ \hat{L}_{\text{eff}} \equiv v_\Omega p_\phi(v_\Omega) & \ell + m \text{ odd} \end{cases}, \tag{116}$$

$T_{\ell m}^{\text{qc}}$ resums the infinite number of “leading logarithms” entering the tail effects, $\delta_{\ell m}$ contains the part of the tail not included in $T_{\ell m}^{\text{qc}}$, and $f_{\ell m}^{\text{qc}}$ contains PN corrections such that the expansion of $h_{\ell m}^{\text{F, qc}}$ agrees with the known PN expansion of the modes. See Refs. [70, 83] for more details and for expressions of these terms.

⁴ The difference between the modes in Ref. [82] (denoted with a bar) and the modes in Eq. (114) (with $C_{1\text{ES}^2} = C_{2\text{ES}^2} = 1$) is given by

$$\begin{aligned}
\hat{H}_{\text{spin}}^{22} - \hat{H}_{\text{spin}}^{22} &= \frac{i\nu p_r}{2c^3r^2} (\chi_1 + \chi_2), \\
\hat{H}_{\text{spin}}^{21} - \hat{H}_{\text{spin}}^{21} &= \frac{i\delta\nu p_r}{6c^4r^3} (rp_r + ip_\phi) (\chi_1 + \chi_2), \\
\hat{H}_{\text{spin}}^{33} - \hat{H}_{\text{spin}}^{33} &= \frac{\sqrt{5}\delta\nu p_r}{8\sqrt{42}c^4r^3} (17p_\phi + 5irp_r) (\chi_1 + \chi_2),
\end{aligned}$$

which is likely due to the coordinate/SSC transformations detailed in Appendix A.

We include the eccentric corrections in the factorized modes as follows:

$$\begin{aligned}
h_{\ell 0}^{\text{F}} &= \hat{S}_{\text{eff}} (1 + T_{\ell 0}^{\text{ecc}}) f_{\ell 0}^{\text{ecc}}, \\
h_{\ell m}^{\text{F}} &= h_{\ell m}^{\text{N, qc}} \hat{S}_{\text{eff}} (T_{\ell m}^{\text{qc}} + T_{\ell m}^{\text{ecc}}) e^{i\delta_{\ell m}} (f_{\ell m}^{\text{qc}} + f_{\ell m}^{\text{ecc}}), \tag{117}
\end{aligned}$$

where the effective source term is given by

$$\hat{S}_{\text{eff}} = \begin{cases} \hat{H}_{\text{eff}}(r, p_r, p_\phi) & \ell + m \text{ even} \\ \hat{L}_{\text{eff}} \equiv v_\Omega p_\phi & \ell + m \text{ odd} \end{cases}, \tag{118}$$

$T_{\ell m}^{\text{ecc}}$ contains the eccentric corrections to the hereditary contributions, $\delta_{\ell m}$ is the same as in the quasi-circular case, and $f_{\ell m}^{\text{ecc}}$ contains the eccentric corrections to the instantaneous contributions (both spinning and nonspinning, including the Newtonian part). For example, for the leading order of the (2, 2) mode, we obtain

$$\begin{aligned}
f_{22}^{\text{ecc}} &= \frac{1}{2(r^2\dot{p}_r + 1)^{1/3}} \left[2 + r^2\dot{p}_r - rp_r^2 - 2(r^2\dot{p}_r + 1)^{1/3} \right. \\
&\quad \left. + 2ip_r\sqrt{r^3\dot{p}_r + r} \right] + \dots \tag{119}
\end{aligned}$$

For the tail part, we simplified the results of Sec. IV B and eliminated the gauge parameter by using a phase shift, which led to the circular part of the tail contribution to the (2, 2) mode simply being $2\pi v_\Omega^5$, however this phase redefinition is not done in SEOBNRv4HM, and the corresponding expression reads $v_\Omega^5 (2\pi + 12i \log(2\epsilon v_\Omega) - 17i/3 + 12i\gamma_E/3)$. Therefore, when including the eccentric corrections in $T_{\ell m}^{\text{ecc}}$, we assume that the phase redefinition was done only for the eccentric part and keep using the same circular part as in SEOBNRv4HM. In addition, since we expanded the tail part in eccentricity to $\mathcal{O}(e^6)$, when factorizing the modes as in Eq. (117) and writing the quasi-circular part in terms of frequency, we reexpand $T_{\ell m}^{\text{ecc}}$ in eccentricity (or p_r and \dot{p}_r). For example, for the (2, 2) mode, we obtain

$$T_{22}^{\text{ecc}} = -\frac{\pi}{4r} \left[4r^{3/2}\dot{p}_r + ip_r (r^2\dot{p}_r + 6) + 2\sqrt{r}p_r^2 + \mathcal{O}(p_r^3) \right]. \tag{120}$$

The full expressions for $T_{\ell m}^{\text{ecc}}$ and $f_{\ell m}^{\text{ecc}}$ are provided in the Supplemental Material [88].

V. CONCLUSIONS

Extending the waveform models used today in GW astronomy from quasi-circular to eccentric orbits is important for future observations with LIGO, Virgo and KAGRA detectors [21], and with new facilities on the ground (Cosmic Explorer and Einstein Telescope), and in space (LISA). In fact, sources with non-negligible eccentricity might come into reach of observations soon and should be routinely included in searches and parameter inference. While this presents a challenge for waveform modeling and data analysis, it also offers the unique opportunity to unveil the formation channels of compact binaries and probe their environment (through eccentricity measurements). In this paper, we constructed an EOB waveform model for eccentric binaries. For this purpose, we obtained analytical results for the RR force and waveform modes to 2PN order, including the leading-order tail effects, and SO and SS couplings for aligned spins.

In particular, we first derived the RR force for eccentric orbits in PN expanded form, and then we recast it in a form that it can be directly incorporated in the quasi-circular RR force employed in the SEOBNRv4HM [74, 83] model, currently used in LIGO/Virgo analyses [1]. We then obtained initial conditions for the binary evolution which generalize those from Ref. [92] to eccentric orbits, and which allow starting the binary's evolution from a specified initial frequency at periastron and an initial eccentricity (in the Keplerian parametrization). We also calculated all the waveform modes that contribute up to 2PN order relative to the leading order of the (2, 2) mode. It should be noted that the $(\ell, 0)$ modes are proportional to the eccentricity and are hence important for eccentric orbits, especially the (2, 0) mode since it starts at the same PN order as the (2, 2) mode. Also the gravitational modes were rewritten in a factorized form to be straightforwardly implemented in the SEOBNRv4HM model.

Our results for the RR force and modes are valid for moderate to high eccentricities during the inspiral phase, since we do not use an eccentricity expansion except for the tail part, which is known analytically as an infinite series expansion. We provided expressions for the tail part in an expansion to $\mathcal{O}(e^6)$, but we checked that expanding to $\mathcal{O}(e^{10})$ produces negligible difference on the waveform even for high eccentricities ($\lesssim 0.9$). If results

for e close to 1 are needed, one could calculate the series expansion for the tail part numerically, or use analytical resummation methods as was done in Refs. [31, 96].

We are currently incorporating the eccentric RR force and gravitational modes of this paper in the inspiral-merger-ringdown quasi-circular-orbit SEOBNRv4HM waveform model (SEOBNRv4EHM [128]) and validating it against NR simulations with eccentricity. We leave to future work the extension of the model to higher PN orders and the inclusion of spin precession.

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Appendix A: Coordinate transformation from harmonic to EOB coordinates

The coordinate transformation from harmonic to EOB coordinates with no spin is given in Appendix A of Ref. [78]. In this appendix, we include LO SO and SS contributions to the transformation. We label harmonic, ADM, and EOB coordinates by $(\mathbf{x}_h, \mathbf{v}_h)$, $(\mathbf{x}_a, \mathbf{p}_a)$, and (\mathbf{x}, \mathbf{p}) , respectively.

1. ADM to EOB transformation

To find the canonical transformation from the ADM Hamiltonian with LO SO and SS using the NW SSC (see e.g. Refs. [129–131]) and the 2PN expansion of the EOB Hamiltonian of Ref. [50], we write an ansatz with unknown coefficients for the generating function $G(\mathbf{x}, \mathbf{p})$, perform the following transformation on the ADM Hamiltonian [45]:

$$\begin{aligned} x_a^i &= x^i + \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial x^j} \frac{\partial^2 G}{\partial p_j \partial p_i} + \mathcal{O}\left(\frac{1}{c^6}\right), \\ p_a^i &= p^i - \frac{\partial G}{\partial x_i} + \frac{\partial G}{\partial x^j} \frac{\partial^2 G}{\partial p_j \partial x^i} + \mathcal{O}\left(\frac{1}{c^6}\right), \end{aligned} \quad (\text{A1})$$

and match it to the EOB Hamiltonian to solve for the unknowns.

The result for the generating function is given by

$$\begin{aligned} G(x, p) &= \frac{p_r}{c^2} \left[-1 - \frac{\nu}{2} + \frac{1}{2} \nu p^2 r \right] + \frac{p_r}{c^4} \left[\frac{1}{8} \nu (3\nu - 1) p^4 r - \frac{1}{8} \nu (\nu + 14) p^2 - \frac{\nu^2 - 7\nu + 1}{4r} + \frac{1}{8} \nu^2 p_r^2 \right] \\ &+ \frac{\nu^2}{2c^4 r} \left[p_r (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 - (\mathbf{n} \cdot \hat{\mathbf{S}}_1 + \mathbf{n} \cdot \hat{\mathbf{S}}_2) (\mathbf{p} \cdot \hat{\mathbf{S}}_1 + \mathbf{p} \cdot \hat{\mathbf{S}}_2) \right], \end{aligned} \quad (\text{A2})$$

which has no LO SO terms since the ADM and EOB Hamiltonian are the same at that order. This generating function

yields

$$\begin{aligned}
\mathbf{x}_a &= \mathbf{x} + \frac{1}{c^2} \left[\mathbf{x} \left(\frac{\nu p^2}{2} - \frac{\nu+2}{2r} \right) + \nu r p_r \mathbf{p} \right] + \frac{1}{c^4} \left\{ \mathbf{x} \left[\frac{3(\nu-2)\nu p^2}{8r} - \frac{1}{8}\nu(\nu+1)p^4 - \frac{\nu(5\nu+16)r^2 p_r^2}{8r^3} - \frac{\nu^2-7\nu+1}{4r^2} \right] \right. \\
&\quad \left. + \mathbf{p} p_r \left[\frac{1}{2}(\nu-1)\nu p^2 r + \frac{3(\nu-2)\nu r r^2 - 2\nu(\nu+2)r^3}{4r^3} \right] + \frac{\nu^2}{2r} \left[(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 \frac{\mathbf{x}}{r} - (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)(\mathbf{n} \cdot \hat{\mathbf{S}}_1 + \mathbf{n} \cdot \hat{\mathbf{S}}_2) \right] \right\}, \\
\mathbf{p}_a &= \mathbf{p} + \frac{1}{c^2} \left[\mathbf{p} \left(\frac{\nu+2}{2r} - \frac{\nu p^2}{2} \right) - \mathbf{x} \frac{(\nu+2)r p_r}{2r^3} \right] + \frac{1}{c^4} \left\{ \mathbf{p} \left[\frac{1}{8}\nu(3\nu+1)p^4 - \frac{\nu(7\nu+2)p^2}{8r} + \frac{\nu(\nu+8)p_r^2}{8r} + \frac{2\nu^2-3\nu+5}{4r^2} \right] \right. \\
&\quad \left. + \mathbf{x} p_r \left[\frac{3(\nu-2)\nu p^2}{8r^2} - \frac{3\nu^2-10\nu+6}{4r^3} + \frac{3\nu^2 p_r^2}{8r^2} \right] + \frac{\nu^2 \mathbf{x}}{r^3} \left[(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 p_r - (\mathbf{n} \cdot \hat{\mathbf{S}}_1 + \mathbf{n} \cdot \hat{\mathbf{S}}_2)(\mathbf{p} \cdot \hat{\mathbf{S}}_1 + \mathbf{p} \cdot \hat{\mathbf{S}}_2) \right] \right. \\
&\quad \left. + \frac{\nu^2}{2r^2} \left[-(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 \mathbf{p} + (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)(\mathbf{p} \cdot \hat{\mathbf{S}}_1 + \mathbf{p} \cdot \hat{\mathbf{S}}_2) \right] \right\}. \tag{A3}
\end{aligned}$$

2. Harmonic to EOB transformation

The transformation from harmonic to ADM coordinates is given by Eq. (E1) of Ref. [78], which is independent of spin since the ADM and harmonic coordinates agree at LO SO and SS. Using that equation together with Eq. (A3), we obtain the following transformation from harmonic to EOB coordinates:

$$\begin{aligned}
\mathbf{x}_h &= \mathbf{x} + \frac{1}{c^2} \left[\mathbf{x} \left(\frac{\nu p^2}{2} - \frac{\nu+2}{2r} \right) + \nu r p_r \mathbf{p} \right] + \frac{1}{c^4} \left\{ \mathbf{x} \left[-\frac{1}{8}\nu(\nu+1)p^4 + \frac{3(\nu-2)\nu p^2}{8r} - \frac{\nu(5\nu+16)p_r^2}{8r} - \frac{\nu^2-7\nu+1}{4r^2} \right] \right. \\
&\quad \left. + \mathbf{p} p_r \left[\frac{1}{4}(\nu-10)\nu + \frac{1}{2}(\nu-1)\nu p^2 r \right] + \frac{\nu^2}{2r} \left[(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 \frac{\mathbf{x}}{r} - (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)(\mathbf{n} \cdot \hat{\mathbf{S}}_1 + \mathbf{n} \cdot \hat{\mathbf{S}}_2) \right] \right\}, \\
\mathbf{v}_h &= \mathbf{p} + \frac{1}{c^2} \left[\mathbf{p} \left(\left(\nu - \frac{1}{2} \right) p^2 - \frac{\nu+4}{2r} \right) - \mathbf{x} \frac{3(\nu+2)p_r}{2r^2} \right] - \frac{1}{4c^3 r^2} \left[\mathbf{n} \times \hat{\mathbf{S}}_1 (3-3\delta+2\nu) + \mathbf{n} \times \hat{\mathbf{S}}_2 (3+3\delta+2\nu) \right] \\
&\quad + \frac{1}{c^4} \left\{ \mathbf{p} \left[\left(\frac{3}{8} - \nu \right) p^4 + \frac{(7\nu^2-41\nu+8)p^2}{8r} + \frac{(-15\nu^2+29\nu+8)p_r^2}{8r} + \frac{-\nu^2+15\nu+1}{2r^2} \right] \right. \\
&\quad \left. + \mathbf{x} p_r \left[\frac{(4-7\nu^2-23\nu)p^2}{8r^2} + \frac{4-3\nu^2+9\nu}{4r^3} + \frac{3\nu(5\nu+1)p_r^2}{8r^2} \right] + \frac{\nu^2}{2r^2} \left[(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)(\mathbf{p} \cdot \hat{\mathbf{S}}_1 + \mathbf{p} \cdot \hat{\mathbf{S}}_2) - (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 \mathbf{p} \right] \right. \\
&\quad \left. + \frac{\nu^2 \mathbf{x}}{r^3} \left[(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 p_r - (\mathbf{n} \cdot \hat{\mathbf{S}}_1 + \mathbf{n} \cdot \hat{\mathbf{S}}_2)(\mathbf{p} \cdot \hat{\mathbf{S}}_1 + \mathbf{p} \cdot \hat{\mathbf{S}}_2) \right] \right\}, \tag{A4}
\end{aligned}$$

and for the scalars $(\phi, r, \dot{\phi}, \dot{r})$, we obtain

$$\begin{aligned}
\phi_h &= \phi + \frac{p_\phi \nu p_r}{c^2 r} + \frac{p_r p_\phi}{c^4} \left[\frac{3(\nu-5)\nu}{4r^2} - \frac{\nu p^2}{2r} - \frac{\nu^2 p_r^2}{r} \right], \\
r_h &= r + \frac{1}{c^2} \left(\frac{\nu}{2} p^2 r + \nu r p_r^2 - 1 - \frac{\nu}{2} \right) + \frac{1}{c^4} \left[\frac{\nu}{8} (3\nu-1)p^2 - \frac{\nu}{8} (\nu+1)p^4 r - \frac{\nu}{4r} (\nu-19) + \frac{\nu}{2} (2\nu-1)p^2 p_r^2 \right. \\
&\quad \left. - \frac{\nu}{8} (3\nu+55)p_r^2 - \frac{1}{2} \nu^2 r p_r^4 + \frac{1}{2r} (X_1^4 \chi_1^2 + 2\nu^2 \chi_1 \chi_2 + X_2^4 \chi_2^2) \right], \\
\dot{\phi}_h &= \frac{p_\phi}{r^2} + \frac{p_\phi}{c^2 r^2} \left[\frac{\nu-1}{2} p^2 - 2\nu p_r^2 - \frac{1}{r} \right] + \frac{1}{2r^3 c^3} \left[\chi_1 (2+2\delta-\nu) + \chi_2 (2-2\delta-\nu) \right] + \frac{p_\phi}{c^4 r^2} \left[4\nu^2 p_r^4 + \frac{(3\nu^2-17\nu+2)p^2}{4r} \right. \\
&\quad \left. - 2(\nu-1)\nu p_r^2 p^2 - \frac{(\nu^2+5\nu-3)p^4}{8} + \frac{4-5\nu^2+65\nu}{4r} p_r^2 - \frac{\nu^2-9\nu+2}{4r^2} + \frac{1}{2r^2} (X_1^4 \chi_1^2 + 2\nu^2 \chi_1 \chi_2 + X_2^4 \chi_2^2) \right], \\
\dot{r}_h &= p_r + \frac{p_r}{c^2} \left[\left(2\nu - \frac{1}{2} \right) p^2 - (2\nu+3) \frac{1}{r} - \nu p_r^2 \right] + \frac{p_r}{c^4} \left[\left(\nu^2 - 2\nu + \frac{3}{8} \right) p^4 + \frac{(\nu^2-55\nu+6)p^2}{4r} + \left(\nu - \frac{5\nu^2}{2} \right) p_r^2 p^2 \right. \\
&\quad \left. + \frac{4-\nu^2+39\nu}{4r} p_r^2 + \frac{3}{2} \nu^2 p_r^4 + \frac{6-5\nu^2+39\nu}{4r^2} + \frac{1}{2r^2} (X_1^4 \chi_1^2 + 2\nu^2 \chi_1 \chi_2 + X_2^4 \chi_2^2) \right]. \tag{A5}
\end{aligned}$$

3. Transformation for the SSC

[121] which are in terms of the covariant SSC. To trans-

When calculating the spin contributions to the wave-form modes, we used the source moments from Refs. [116,

form the resulting modes to the NW SSC, we use the center-of-mass shift [98]

$$x_{A(\text{cov})}^i \rightarrow x_A^i + \frac{1}{2c^3 m_A} (\mathbf{v}_A \times \mathbf{S}_A)^i, \quad (\text{A6})$$

and the spin transformation [132]

$$\mathbf{S}_1^{\text{cov}} = \left(1 - \frac{m_2}{c^2 r}\right) \mathbf{S}_1 + \frac{1}{2c^2} \mathbf{v}_1 (\mathbf{v}_1 \cdot \mathbf{S}_1), \quad (\text{A7})$$

where the spin transformation is only required for the NLO SO part of the 2PN (2, 1) mode.

For the scalars ($r, \phi, \dot{r}, \dot{\phi}, \chi_1, \chi_2$), we obtain the transformations

$$\begin{aligned} r_{\text{cov}} &= r - \frac{\nu r \dot{\phi}}{2c^3} (\chi_1 + \chi_2), \\ \phi_{\text{cov}} &= \phi + \frac{\nu \dot{r}}{2c^3 r} (\chi_1 + \chi_2), \\ \dot{r}_{\text{cov}} &= \dot{r} + \frac{\nu \dot{r} \dot{\phi}}{2c^3} (\chi_1 + \chi_2), \\ \dot{\phi}_{\text{cov}} &= \dot{\phi} - \frac{\nu}{2c^3 r^3} (1 + r\dot{r}^2 - r^3 \dot{\phi}^2) (\chi_1 + \chi_2), \\ \chi_1^{\text{cov}} &= \chi_1 - \frac{\chi_1}{2c^2 r} (1 - \delta), \\ \chi_2^{\text{cov}} &= \chi_2 - \frac{\chi_2}{2c^2 r} (1 + \delta). \end{aligned} \quad (\text{A8})$$

Appendix B: Angular momentum flux at leading-order spin-squared

In this appendix, we derive the angular momentum flux at leading spin-squared (S_i^2) order. Here, we use unscaled variables in *harmonic* coordinates, but we drop the subscript ‘*h*’ for simplify the notation. We denote the orbital angular momentum $\mathbf{L} = \mu \mathbf{r} \times \mathbf{v}$, the relative position $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, and relative velocity $\mathbf{v} = d\mathbf{r}/dt$.

The relative acceleration $\mathbf{a} \equiv \mathbf{a}_1 - \mathbf{a}_2$ with LO SO and SS contributions, in harmonic coordinates and the NW SSC, is given by [98]

$$\begin{aligned} \mathbf{a} &= -M \frac{\mathbf{n}}{r^2} \\ &+ \frac{1}{c^3} \left[3 \left(2 + \frac{3m_2}{2m_1} \right) \mathbf{n} \cdot (\mathbf{v} \times \mathbf{S}_1) \frac{\mathbf{n}}{r^3} - \left(4 + \frac{3m_2}{m_1} \right) \right. \\ &\quad \left. \frac{\mathbf{v} \times \mathbf{S}_1}{r^3} + 3 \left(2 + \frac{3m_2}{2m_1} \right) \frac{\dot{r}}{r^3} \mathbf{n} \times \mathbf{S}_1 + 1 \leftrightarrow 2 \right] \\ &- \frac{3}{c^4 \mu r^4} \left[\mathbf{n} (\mathbf{S}_1 \cdot \mathbf{S}_2) + \mathbf{S}_1 (\mathbf{n} \cdot \mathbf{S}_2) + \mathbf{S}_2 (\mathbf{n} \cdot \mathbf{S}_1) \right. \\ &\quad \left. - 5\mathbf{n} (\mathbf{n} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) \right] \\ &+ \frac{3}{2c^4 r^4} \left[\frac{m_2 C_{1\text{ES}^2}}{m_1 \mu} \left(-\mathbf{n} \mathbf{S}_1^2 + 5\mathbf{n} (\mathbf{n} \cdot \mathbf{S}_1)^2 \right. \right. \\ &\quad \left. \left. - 2\mathbf{S}_1 (\mathbf{n} \cdot \mathbf{S}_1) \right) + 1 \leftrightarrow 2 \right]. \end{aligned} \quad (\text{B1})$$

Since the spin evolution equations start at 1PN order, we can assume $\dot{\mathbf{S}}_1 = 0 = \dot{\mathbf{S}}_2$ for the calculation of the LO fluxes.

The source multipole moments needed are the spin quadrupole I^{ij} and the current quadrupole J^{ij} , which are given by [98, 110, 121]

$$\begin{aligned} I^{ij} &= m_1 x_1^i x_1^j + \frac{3}{c^3} x_1^i (\mathbf{v}_1 \times \mathbf{S}_1)^j - \frac{4}{3c^3} \frac{d}{dt} x_1^i (\mathbf{x}_1 \times \mathbf{S}_1)^j \\ &\quad - \frac{C_{1\text{ES}^2}}{c^4 m_1} S_1^i S_1^j + 1 \leftrightarrow 2 \end{aligned} \quad (\text{B2})$$

$$J^{ij} = m_1 x_1^i (\mathbf{x}_1 \times \mathbf{v}_1)^j + \frac{3}{2c} x_1^i S_1^j + 1 \leftrightarrow 2, \quad (\text{B3})$$

where the indices in angle brackets denote a symmetric trace free part.

To transform from the coordinates of the two bodies x_1^i and x_2^i to the center-of-mass relative coordinates $x^i = x_1^i - x_2^i$, we use [133]

$$x_1^i = \frac{m_2}{M} x^i + \delta x^i, \quad x_2^i = -\frac{m_1}{M} x^i + \delta x^i, \quad (\text{B4})$$

where

$$\delta x^i = -\frac{\nu}{2c^3} \left[\frac{(\mathbf{v} \times \mathbf{S}_1)^i}{m_1} - \frac{(\mathbf{v} \times \mathbf{S}_2)^i}{m_2} \right]. \quad (\text{B5})$$

The energy and angular momentum fluxes in terms of the multipole moments, to the order needed for the LO fluxes, are then calculated from [98, 134]

$$\Phi_E = \frac{1}{5} I_{ij}^{(3)} I_{ij}^{(3)} + \frac{16}{45c^2} J_{ij}^{(3)} J_{ij}^{(3)}, \quad (\text{B6})$$

$$\Phi_J^i = \frac{2}{5} \epsilon_{ijk} I_{jl}^{(2)} I_{kl}^{(3)} + \frac{32}{45c^2} \epsilon_{ijk} J_{jl}^{(2)} J_{kl}^{(3)}. \quad (\text{B7})$$

This yields the LO SO and $S_1 S_2$ fluxes derived in Refs. [98, 99, 109], in addition to the S_i^2 energy flux from Ref. [110]. For the S_i^2 angular momentum flux, we obtain

$$\begin{aligned} \Phi_J^{S_i^2} &= \frac{2m_2^2}{5c^4 r^5} \left[\frac{\mathbf{L}}{\mu r} \mathbf{S}_1^2 - \mathbf{n} \cdot \mathbf{S}_1 (\mathbf{v} \times \mathbf{S}_1) + \mathbf{v} \cdot \mathbf{S}_1 (\mathbf{n} \times \mathbf{S}_1) \right] \\ &+ \frac{2m_2^2 C_{1\text{ES}^2}}{5c^4 M r^4} \left[\frac{\mathbf{L}}{\mu r} \mathbf{S}_1^2 \left(-30\dot{r}^2 + 12v^2 + 24\frac{M}{r} \right) \right. \\ &\quad + \frac{\mathbf{L}}{\mu r} (\mathbf{n} \cdot \mathbf{S}_1)^2 \left(210\dot{r}^2 - 60v^2 - 90\frac{M}{r} \right) \\ &\quad + \mathbf{v} \times \mathbf{S}_1 (\mathbf{n} \cdot \mathbf{S}_1) \left(30\dot{r}^2 - 18v^2 - 12\frac{M}{r} \right) \\ &\quad + 6\mathbf{n} \times \mathbf{S}_1 (\mathbf{v} \cdot \mathbf{S}_1 - \dot{r} \mathbf{n} \cdot \mathbf{S}_1) \frac{M}{r} \\ &\quad - 90 \frac{\mathbf{L}}{\mu r} \dot{r} (\mathbf{n} \cdot \mathbf{S}_1) (\mathbf{v} \cdot \mathbf{S}_1) + 6 \frac{\mathbf{L}}{\mu r} (\mathbf{v} \cdot \mathbf{S}_1)^2 \\ &\quad \left. - 6\dot{r} (\mathbf{v} \cdot \mathbf{S}_1) \mathbf{v} \times \mathbf{S}_1 \right] + 1 \leftrightarrow 2. \end{aligned} \quad (\text{B8})$$

This is in agreement with the recent results of Ref. [135], although our expression appears simpler because of using the individual spins S_i and masses m_i , instead of different combinations of them.

Appendix C: Aligned-spin contributions to the modes in harmonic coordinates

The modes calculated from the source moments of Refs. [116, 121] in *harmonic* coordinates and using the

covariant SSC have the following spin contributions:

$$\begin{aligned}
\hat{H}_{\text{spin}}^{20} &= \frac{\dot{\phi}}{\sqrt{6}c^3r} [\chi_1(1 + \delta + \nu) + \chi_2(1 - \delta + \nu)] - \frac{\sqrt{3}}{2\sqrt{2}c^4r^3} [C_{1\text{ES}^2}X_1^2\chi_1^2 + 2\nu\chi_1\chi_2 + C_{2\text{ES}^2}X_2^2\chi_2^2], \\
\hat{H}_{\text{spin}}^{21} &= -\frac{i}{4r^2} [(1 + \delta)\chi_1 + (\delta - 1)\chi_2] + \frac{i}{168c^4r^3} \left\{ \chi_1 \left[154 + 22\delta(\nu + 7) + 34\nu + 4r^3\dot{\phi}^2(4\nu\delta - 21\delta + 66\nu - 21) \right. \right. \\
&\quad \left. \left. - 2i\dot{r}r^2\dot{\phi}(13\nu\delta + 147\delta - 83\nu + 147) + \dot{r}^2r(-60\delta\nu + 105\delta - 52\nu + 105) \right] + \chi_2 \left[-154 + 22\delta(\nu + 7) - 34\nu \right. \right. \\
&\quad \left. \left. + 4r^3\dot{\phi}^2(4\nu\delta - 21\delta - 66\nu + 21) - 2i\dot{r}r^2\dot{\phi}(13\nu\delta + 147\delta + 83\nu - 147) + \dot{r}^2r(-60\delta\nu + 105\delta + 52\nu - 105) \right] \right\}, \\
\hat{H}_{\text{spin}}^{22} &= -\frac{1}{6c^3r^2} \left\{ \chi_1 \left[r\dot{\phi}(3\delta - 5\nu + 3) + i\dot{r}(3\delta - 8\nu + 3) \right] + \chi_2 \left[r\dot{\phi}(-3\delta - 5\nu + 3) - i\dot{r}(3\delta + 8\nu - 3) \right] \right\} \\
&\quad + \frac{3}{4c^4r^3} [C_{1\text{ES}^2}\chi_1^2X_1^2 + C_{2\text{ES}^2}\chi_2^2X_2^2 + 2\nu\chi_2\chi_1], \\
\hat{H}_{\text{spin}}^{30} &= -\frac{i\nu\dot{r}}{\sqrt{42}c^3r^2} (\chi_1 + \chi_2), \\
\hat{H}_{\text{spin}}^{31} &= \frac{i}{48\sqrt{14}c^4r^3} \left\{ \chi_1 \left[-4 + 20\delta\nu - 4\delta + 20\nu + r^3\dot{\phi}^2(-31\delta\nu - 24\delta + 87\nu - 24) + i\dot{r}r^2\dot{\phi}(-70\delta\nu - 12\delta + 62\nu - 12) \right. \right. \\
&\quad \left. \left. + \dot{r}^2r(-30\delta\nu + 12\delta - 50\nu + 12) \right] + \chi_2 \left[4 + 20\delta\nu - 4\delta - 20\nu + r^3\dot{\phi}^2(-31\delta\nu - 24\delta - 87\nu + 24) \right. \right. \\
&\quad \left. \left. + i\dot{r}r^2\dot{\phi}(-70\delta\nu - 12\delta - 62\nu + 12) + \dot{r}^2r(-30\delta\nu + 12\delta + 50\nu - 12) \right] \right\}, \\
\hat{H}_{\text{spin}}^{32} &= \sqrt{\frac{5}{7}} \frac{\nu}{6c^3r^2} (4r\dot{\phi} + i\dot{r}) (\chi_1 + \chi_2), \\
\hat{H}_{\text{spin}}^{33} &= \sqrt{\frac{5}{42}} \frac{i}{16c^4r^3} \left\{ \chi_1 \left[4 - 20\delta\nu + 4\delta - 20\nu + r^3\dot{\phi}^2(-33\delta\nu + 24\delta - 119\nu + 24) + i\dot{r}r^2\dot{\phi}(-78\delta\nu + 36\delta - 154\nu + 36) \right. \right. \\
&\quad \left. \left. + \dot{r}^2r(30\delta\nu - 12\delta + 50\nu - 12) \right] + \chi_2 \left[-4 - 20\delta\nu + 4\delta + 20\nu + r^3\dot{\phi}^2(-33\delta\nu + 24\delta + 119\nu - 24) \right. \right. \\
&\quad \left. \left. + i\dot{r}r^2\dot{\phi}(-78\delta\nu + 36\delta + 154\nu - 36) + \dot{r}^2r(30\delta\nu - 12\delta - 50\nu + 12) \right] \right\}, \\
\hat{H}_{\text{spin}}^{41} &= -i\sqrt{\frac{5}{2}} \frac{\nu}{336c^4r^3} (11r^3\dot{\phi}^2 - 10i\dot{r}r^2\dot{\phi} + 6\dot{r}^2r - 12) [(\delta - 1)\chi_1 + (\delta + 1)\chi_2], \\
\hat{H}_{\text{spin}}^{43} &= \sqrt{\frac{5}{14}} \frac{\nu}{48c^4r^3} (-23i\dot{r}r^3\dot{\phi}^2 + 10i\dot{r}r^2\dot{\phi} + 2i\dot{r}^2r - 4i) [(\delta - 1)\chi_1 + (\delta + 1)\chi_2]. \tag{C1}
\end{aligned}$$

Appendix D: Keplerian parametrization

This appendix provides expressions for some orbital quantities in the Keplerian parametrization that are needed for calculating the initial conditions, and the tail part of the RR force and waveform modes.

In the Keplerian parametrization,

$$r = \frac{1}{u_p(1 + e \cos \chi)}, \tag{D1}$$

where u_p is the inverse semilatus rectum and χ is the relativistic anomaly. Inverting the Hamiltonian at the turning points $r_{\pm} = 1/(u_p(1 \pm e))$ and solving for the

energy and angular momentum to 2PN order yields

$$\begin{aligned}
E &= \frac{1}{2} (e^2 - 1) u_p - \frac{u_p^2}{8c^2} (e^2 - 1)^2 (\nu - 3) \\
&+ \frac{u_p^3}{16c^4} (e^2 - 1)^2 [e^2 (\nu^2 - 3\nu + 5) - \nu^2 - 5\nu + 27] \\
&+ \frac{(1 - e^2)^2 u_p^{5/2}}{4c^4} [\chi_1 (\nu - 2\delta - 2) + \chi_2 (\nu + 2\delta - 2)] \\
&+ \frac{(1 - e^2)^2 u_p^3}{4c^4} \left[\chi_1^2 (C_{1\text{ES}^2} X_1^2 + X_1^4) + \right. \\
&\quad \left. + 2\nu(1 + \nu)\chi_2\chi_1 + \chi_2^2 (C_{2\text{ES}^2} X_2^2 + X_2^4) \right], \\
p_\phi &= \frac{1}{\sqrt{u_p}} + \frac{\sqrt{u_p}}{2c^2} (e^2 + 3) + \frac{u_p^{3/2}}{8c^4} (e^2 + 3) (3e^2 - 4\nu + 9) \\
&+ \frac{1}{4} (e^2 + 3) u_p [\chi_1 (-2\delta + \nu - 2) + \chi_2 (2\delta + \nu - 2)] \\
&+ \frac{(e^2 + 3) u_p^{3/2}}{4c^4} \left[\chi_1^2 (C_{1\text{ES}^2} X_1^2 + X_1^4) \right. \\
&\quad \left. + \chi_2^2 (C_{2\text{ES}^2} X_2^2 + X_2^4) \right] \\
&+ \frac{\nu u_p^{3/2} \chi_1 \chi_2}{2c^4} [e^2 (3\nu + 1) + \nu + 3]. \tag{D2}
\end{aligned}$$

Inverting $p_\phi(u_p, e)$, we obtain $u_p(p_\phi, e)$

$$\begin{aligned}
u_p(p_\phi, e) &= \frac{1}{p_\phi^2} + \frac{(e^2 + 3)}{c^2 p_\phi^4} + \frac{(e^2 + 3)(2e^2 - \nu + 6)}{c^4 p_\phi^6} \\
&+ \frac{3 + e^2}{2p_\phi^5 c^3} [(\nu - 2\delta - 2)\chi_1 + (\nu + 2\delta - 2)\chi_2] \\
&+ \frac{1}{2p_\phi^6 c^4} \left\{ \nu\chi_1\chi_2 [e^2(3\nu + 1) + \nu + 3] \right. \\
&\quad \left. + \chi_1^2 [C_{1\text{ES}^2} (e^2 + 3) X_1^2 + (3e^2 + 1) X_1^4] \right. \\
&\quad \left. + 1 \leftrightarrow 2 \right\}. \tag{D3}
\end{aligned}$$

Inverting the Hamiltonian to obtain $p_r(E, p_\phi)$, and plugging $E(e, u_p)$ and $p_\phi(e, u_p)$, yields

$$p_r = e\sqrt{u_p} \sin \chi + \frac{e u_p^{3/2}}{2c^2} \sin \chi (e^2 + 2e \cos \chi + 1) + \dots \tag{D4}$$

The radial and azimuthal periods are given respectively by

$$\begin{aligned}
T_r &= \oint dt = \oint \left(\frac{\partial H}{\partial p_r} \right)^{-1} dr = 2 \int_0^\pi \left(\frac{\partial H}{\partial p_r} \right)^{-1} \frac{dr}{d\chi} d\chi \\
&= \frac{2\pi}{(u_p - e^2 u_p)^{3/2}} - \frac{\pi(\nu - 6)}{c^2 \sqrt{u_p - e^2 u_p}} + \dots, \\
T_\phi &= \oint \dot{\phi} dt = \oint \frac{\partial H}{\partial p_\phi} dt = 2\pi + \frac{6\pi u_p}{c^2} + \dots \tag{D5}
\end{aligned}$$

The associated frequencies are

$$\omega_r = \frac{2\pi}{T_r}, \quad \omega_\phi = \frac{2\pi}{T_\phi}. \tag{D6}$$

The dimensionless frequency variable $x \equiv \omega_\phi^{2/3}$ to 2PN order is given by

$$\begin{aligned}
x &= u_p - e^2 u_p + \frac{u_p^2}{3c^2} (e^2 - 1) (e^2(\nu - 6) - \nu) \\
&- \frac{u_p^3}{36c^4} (e^2 - 1) \left\{ e^4 (8\nu^2 - 33\nu + 180) + 8\nu^2 \right. \\
&- 2e^2 \left[3 \left(12\sqrt{1 - e^2} + 5 \right) \nu - 90\sqrt{1 - e^2} + 8\nu^2 + 27 \right] \\
&+ 9 \left(8\sqrt{1 - e^2} - 13 \right) \nu - 180 \left(\sqrt{1 - e^2} - 1 \right) \left. \right\} \\
&+ \frac{u_p^{5/2}}{6c^4} (e^2 - 1) (3e^2 + 1) [\chi_1 (2 + 2\delta - \nu) + 1 \leftrightarrow 2] \\
&- \frac{u_p^3}{2c^4} \left\{ \chi_1^2 [C_{1\text{ES}^2} (e^4 - 1) X_1^2 + (e^2 - 1)^2 X_1^4] \right. \\
&\quad \left. + \nu\chi_1\chi_2 [e^4(\nu + 1) - 2e^2\nu + \nu - 1] + 1 \leftrightarrow 2 \right\}, \tag{D7}
\end{aligned}$$

which can be inverted to obtain $u_p(x, e)$, the 1PN part of which reads

$$u_p = -\frac{x}{e^2 - 1} + \frac{x^2 [e^2(\nu - 6) - \nu]}{3c^2 (e^2 - 1)^2} + \dots \tag{D8}$$

Appendix E: \dot{p}_r in tortoise coordinates

The tortoise-coordinate r_* is defined by [69, 112]

$$\frac{dr_*}{dr} = \frac{\sqrt{D(r)}}{A(r)} \equiv \frac{1}{\xi(r)}, \tag{E1}$$

where $A(r)$ and $D(r)$ are the metric potentials

$$ds_{\text{eff}}^2 = -A(r)dt^2 + \frac{D(r)}{A(r)}dr^2 + r^2 d\Omega^2. \tag{E2}$$

The conjugate momentum to r_* is denoted p_{r_*} , and invariance of the action gives the relation

$$p_{r_*} = p_r \xi(r). \tag{E3}$$

The Hamiltonian and EOMs used in SEOBNRv4 (see Eqs. (10) of Ref. [136]) are expressed in terms of the variables $(r, p_{r_*}, \phi, p_\phi)$. However, the RR force we derived in Sec. II is expressed in terms of (r, p_r, \dot{p}_r) . We use Eq. (E3) to replace p_r with p_{r_*} , and to obtain a relation between \dot{p}_r and the derivatives of $H_{\text{EOB}}(r, p_{r_*}, p_\phi)$. We use the following relations:

$$\begin{aligned}
dH &= \left(\frac{\partial H}{\partial r} \right)_{p_{r_*}} dr + \left(\frac{\partial H}{\partial p_{r_*}} \right)_r dp_{r_*} + \frac{\partial H}{\partial p_\phi} dp_\phi \\
&= \left(\frac{\partial H}{\partial r} \right)_{p_r} dr + \left(\frac{\partial H}{\partial p_r} \right)_r dp_r + \frac{\partial H}{\partial p_\phi} dp_\phi, \tag{E4}
\end{aligned}$$

$$dp_{r_*} = \left(\frac{\partial p_{r_*}}{\partial r} \right)_{p_r} dr + \left(\frac{\partial p_{r_*}}{\partial r} \right)_r dp_r, \tag{E5}$$

leading to

$$\left(\frac{\partial H}{\partial r}\right)_{p_r} = \left(\frac{\partial H}{\partial r}\right)_{p_{r_*}} + \left(\frac{\partial H}{\partial p_{r_*}}\right)_r \left(\frac{\partial p_{r_*}}{\partial r}\right)_{p_r}, \quad (\text{E6})$$

where

$$\left(\frac{\partial p_{r_*}}{\partial r}\right)_{p_r} = p_r \frac{d\xi(r)}{dr}. \quad (\text{E7})$$

Hence,

$$\begin{aligned} \dot{p}_r &= - \left(\frac{\partial H}{\partial r}\right)_{p_r} \\ &= - \left[\left(\frac{\partial H}{\partial r}\right)_{p_{r_*}} + \left(\frac{\partial H}{\partial p_{r_*}}\right)_r \frac{p_{r_*}}{\xi(r)} \frac{d\xi(r)}{dr} \right]. \quad (\text{E8}) \end{aligned}$$

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- [1] R. Abbott *et al.* (LIGO Scientific, Virgo), GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run, (2020), [arXiv:2010.14527 \[gr-qc\]](#).
- [2] B. P. Abbott *et al.* (LIGO Scientific, Virgo), GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs, *Phys. Rev.* **X9**, 031040 (2019), [arXiv:1811.12907 \[astro-ph.HE\]](#).
- [3] B. P. Abbott *et al.* (LIGO Scientific, Virgo), Binary Black Hole Population Properties Inferred from the First and Second Observing Runs of Advanced LIGO and Advanced Virgo, *Astrophys. J. Lett.* **882**, L24 (2019), [arXiv:1811.12940 \[astro-ph.HE\]](#).
- [4] R. Abbott *et al.* (LIGO Scientific, Virgo), Population Properties of Compact Objects from the Second LIGO-Virgo Gravitational-Wave Transient Catalog, (2020), [arXiv:2010.14533 \[astro-ph.HE\]](#).
- [5] B. Abbott *et al.* (LIGO Scientific, Virgo), Search for Eccentric Binary Black Hole Mergers with Advanced LIGO and Advanced Virgo during their First and Second Observing Runs, *Astrophys. J.* **883**, 149 (2019), [arXiv:1907.09384 \[astro-ph.HE\]](#).
- [6] I. M. Romero-Shaw, P. D. Lasky, and E. Thrane, Searching for Eccentricity: Signatures of Dynamical Formation in the First Gravitational-Wave Transient Catalogue of LIGO and Virgo, *Mon. Not. Roy. Astron. Soc.* **490**, 5210 (2019), [arXiv:1909.05466 \[astro-ph.HE\]](#).
- [7] A. H. Nitz, A. Lenon, and D. A. Brown, Search for Eccentric Binary Neutron Star Mergers in the first and second observing runs of Advanced LIGO, *Astrophys. J.* **890**, 1 (2019), [arXiv:1912.05464 \[astro-ph.HE\]](#).
- [8] I. M. Romero-Shaw, P. D. Lasky, E. Thrane, and J. C. Bustillo, GW190521: orbital eccentricity and signatures of dynamical formation in a binary black hole merger signal, *Astrophys. J. Lett.* **903**, L5 (2020), [arXiv:2009.04771 \[astro-ph.HE\]](#).
- [9] P. Peters, Gravitational Radiation and the Motion of Two Point Masses, *Phys. Rev.* **136**, B1224 (1964).
- [10] I. Hinder, B. Vaishnav, F. Herrmann, D. Shoemaker, and P. Laguna, Universality and final spin in eccentric binary black hole inspirals, *Phys. Rev. D* **77**, 081502 (2008), [arXiv:0710.5167 \[gr-qc\]](#).
- [11] J. Samsing and E. Ramirez-Ruiz, On the Assembly Rate of Highly Eccentric Binary Black Hole Mergers, *Astrophys. J. Lett.* **840**, L14 (2017), [arXiv:1703.09703 \[astro-ph.HE\]](#).
- [12] J. Samsing, M. MacLeod, and E. Ramirez-Ruiz, The Formation of Eccentric Compact Binary Inspirals and the Role of Gravitational Wave Emission in Binary-Single Stellar Encounters, *Astrophys. J.* **784**, 71 (2014), [arXiv:1308.2964 \[astro-ph.HE\]](#).
- [13] J. Samsing, Eccentric Black Hole Mergers Forming in Globular Clusters, *Phys. Rev. D* **97**, 103014 (2018), [arXiv:1711.07452 \[astro-ph.HE\]](#).
- [14] C. L. Rodriguez, P. Amaro-Seoane, S. Chatterjee, and F. A. Rasio, Post-Newtonian Dynamics in Dense Star Clusters: Highly-Eccentric, Highly-Spinning, and Repeated Binary Black Hole Mergers, *Phys. Rev. Lett.* **120**, 151101 (2018), [arXiv:1712.04937 \[astro-ph.HE\]](#).
- [15] M. Zevin, J. Samsing, C. Rodriguez, C.-J. Haster, and E. Ramirez-Ruiz, Eccentric Black Hole Mergers in Dense Star Clusters: The Role of Binary-Encounters, *Astrophys. J.* **871**, 91 (2019), [arXiv:1810.00901 \[astro-ph.HE\]](#).
- [16] L. Gondán and B. Kocsis, High Eccentricities and High Masses Characterize Gravitational-wave Captures in Galactic Nuclei as Seen by Earth-based Detectors, (2020), [arXiv:2011.02507 \[astro-ph.HE\]](#).
- [17] F. Antonini and H. B. Perets, Secular evolution of compact binaries near massive black holes: Gravitational wave sources and other exotica, *Astrophys. J.* **757**, 27 (2012), [arXiv:1203.2938 \[astro-ph.GA\]](#).
- [18] F. Antonini, S. Chatterjee, C. L. Rodriguez, M. Morscher, B. Pattabiraman, V. Kalogera, and F. A. Rasio, Black hole mergers and blue stragglers from hierarchical triples formed in globular clusters, *Astrophys. J.* **816**, 65 (2016), [arXiv:1509.05080 \[astro-ph.GA\]](#).
- [19] J. H. VanLandingham, M. Miller, D. P. Hamilton, and D. C. Richardson, The Role of the Kozai-lidov Mechanism in Black Hole Binary Mergers in Galactic Centers, *Astrophys. J.* **828**, 77 (2016), [arXiv:1604.04948 \[astro-ph.HE\]](#).
- [20] R. Abbott *et al.* (LIGO Scientific, Virgo), GW190521: A Binary Black Hole Merger with a Total Mass of $150M_{\odot}$, *Phys. Rev. Lett.* **125**, 101102 (2020), [arXiv:2009.01075 \[gr-qc\]](#).
- [21] B. P. Abbott *et al.* (KAGRA, LIGO Scientific, Virgo), Prospects for observing and localizing gravitational-wave transients with Advanced LIGO, Advanced Virgo and KAGRA, *Living Rev. Rel.* **23**, 3 (2020).
- [22] M. Favata, Systematic parameter errors in inspiraling neutron star binaries, *Phys. Rev. Lett.* **112**, 101101 (2014), [arXiv:1310.8288 \[gr-qc\]](#).
- [23] A. Sesana, Self consistent model for the evolution of eccentric massive black hole binaries in stellar environments: implications for gravitational wave observations, *Astrophys. J.* **719**, 851 (2010), [arXiv:1006.0730 \[astro-ph.CO\]](#).
- [24] K. Breivik, C. L. Rodriguez, S. L. Larson, V. Kalogera, and F. A. Rasio, Distinguishing Between Formation Channels for Binary Black Holes with LISA, *Astrophys. J. Lett.* **830**, L18 (2016), [arXiv:1606.09558 \[as-](#)

- tro-ph.GA].
- [25] B. Willems, V. Kalogera, A. Vecchio, N. Ivanova, F. Rasio, J. Fregeau, and K. Belczynski, Eccentric double white dwarfs as LISA sources in globular clusters, *Astrophys. J. Lett.* **665**, L59 (2007), arXiv:0705.4287 [astro-ph].
- [26] J. Samsing and D. J. D’Orazio, Black Hole Mergers From Globular Clusters Observable by LISA I: Eccentric Sources Originating From Relativistic N -body Dynamics, *Mon. Not. Roy. Astron. Soc.* **481**, 5445 (2018), arXiv:1804.06519 [astro-ph.HE].
- [27] V. Cardoso, C. F. Macedo, and R. Vicente, Eccentricity evolution of compact binaries and applications to gravitational-wave physics, (2020), arXiv:2010.15151 [gr-qc].
- [28] R.-M. Memmesheimer, A. Gopakumar, and G. Schaefer, Third post-Newtonian accurate generalized quasi-Keplerian parametrization for compact binaries in eccentric orbits, *Phys. Rev. D* **70**, 104011 (2004), arXiv:gr-qc/0407049.
- [29] Y. Boetzel, A. Susobhanan, A. Gopakumar, A. Klein, and P. Jetzer, Solving post-Newtonian accurate Kepler Equation, *Phys. Rev. D* **96**, 044011 (2017), arXiv:1707.02088 [gr-qc].
- [30] N. Loutrel and N. Yunes, Eccentric Gravitational Wave Bursts in the Post-Newtonian Formalism, *Class. Quant. Grav.* **34**, 135011 (2017), arXiv:1702.01818 [gr-qc].
- [31] S. Tanay, M. Haney, and A. Gopakumar, Frequency and time domain inspiral templates for comparable mass compact binaries in eccentric orbits, *Phys. Rev. D* **93**, 064031 (2016), arXiv:1602.03081 [gr-qc].
- [32] N. Yunes, K. Arun, E. Berti, and C. M. Will, Post-Circular Expansion of Eccentric Binary Inspirals: Fourier-Domain Waveforms in the Stationary Phase Approximation, *Phys. Rev. D* **80**, 084001 (2009), [Erratum: *Phys.Rev.D* 89, 109901 (2014)], arXiv:0906.0313 [gr-qc].
- [33] E. A. Huerta, P. Kumar, S. T. McWilliams, R. O’Shaughnessy, and N. Yunes, Accurate and efficient waveforms for compact binaries on eccentric orbits, *Phys. Rev. D* **90**, 084016 (2014), arXiv:1408.3406 [gr-qc].
- [34] S. Tiwari, G. Achamveedu, M. Haney, and P. Hemantakumar, Ready-to-use Fourier domain templates for compact binaries inspiralling along moderately eccentric orbits, *Phys. Rev. D* **99**, 124008 (2019), arXiv:1905.07956 [gr-qc].
- [35] A. Klein, Y. Boetzel, A. Gopakumar, P. Jetzer, and L. de Vittori, Fourier domain gravitational waveforms for precessing eccentric binaries, *Phys. Rev. D* **98**, 104043 (2018), arXiv:1801.08542 [gr-qc].
- [36] T. Damour, A. Gopakumar, and B. R. Iyer, Phasing of gravitational waves from inspiralling eccentric binaries, *Phys. Rev. D* **70**, 064028 (2004), arXiv:gr-qc/0404128.
- [37] C. Konigsdorffer and A. Gopakumar, Phasing of gravitational waves from inspiralling eccentric binaries at the third-and-a-half post-Newtonian order, *Phys. Rev. D* **73**, 124012 (2006), arXiv:gr-qc/0603056.
- [38] B. Moore, M. Favata, K. Arun, and C. K. Mishra, Gravitational-wave phasing for low-eccentricity inspiralling compact binaries to 3PN order, *Phys. Rev. D* **93**, 124061 (2016), arXiv:1605.00304 [gr-qc].
- [39] I. Hinder, L. E. Kidder, and H. P. Pfeiffer, Eccentric binary black hole inspiral-merger-ringdown gravitational waveform model from numerical relativity and post-Newtonian theory, *Phys. Rev. D* **98**, 044015 (2018), arXiv:1709.02007 [gr-qc].
- [40] E. Huerta *et al.*, Complete waveform model for compact binaries on eccentric orbits, *Phys. Rev. D* **95**, 024038 (2017), arXiv:1609.05933 [gr-qc].
- [41] A. Ramos-Buades, S. Husa, G. Pratten, H. Estellés, C. García-Quirós, M. Mateu-Lucena, M. Colleoni, and R. Jaume, First survey of spinning eccentric black hole mergers: Numerical relativity simulations, hybrid waveforms, and parameter estimation, *Phys. Rev. D* **101**, 083015 (2020), arXiv:1909.11011 [gr-qc].
- [42] E. A. Huerta *et al.*, Physics of eccentric binary black hole mergers: A numerical relativity perspective, *Phys. Rev. D* **100**, 064003 (2019), arXiv:1901.07038 [gr-qc].
- [43] M. Boyle *et al.*, The SXS Collaboration catalog of binary black hole simulations, *Class. Quant. Grav.* **36**, 195006 (2019), arXiv:1904.04831 [gr-qc].
- [44] T. Islam, V. Varma, J. Lodman, S. E. Field, G. Khanna, M. A. Scheel, H. P. Pfeiffer, D. Gerosa, and L. E. Kidder, Eccentric binary black hole surrogate models for the gravitational waveform and remnant properties: comparable mass, nonspinning case, (2021), arXiv:2101.11798 [gr-qc].
- [45] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, *Phys.Rev.* **D59**, 084006 (1999), arXiv:gr-qc/9811091 [gr-qc].
- [46] A. Buonanno and T. Damour, Transition from inspiral to plunge in binary black hole coalescences, *Phys. Rev. D* **62**, 064015 (2000), arXiv:gr-qc/0001013 [gr-qc].
- [47] T. Damour, P. Jaranowski, and G. Schäfer, Fourth post-Newtonian effective one-body dynamics, *Phys. Rev. D* **91**, 084024 (2015), arXiv:1502.07245 [gr-qc].
- [48] T. Damour, Coalescence of two spinning black holes: an effective one-body approach, *Phys. Rev. D* **64**, 124013 (2001), arXiv:gr-qc/0103018 [gr-qc].
- [49] T. Damour, P. Jaranowski, and G. Schaefer, Effective one body approach to the dynamics of two spinning black holes with next-to-leading order spin-orbit coupling, *Phys. Rev. D* **78**, 024009 (2008), arXiv:0803.0915 [gr-qc].
- [50] E. Barausse and A. Buonanno, An Improved effective-one-body Hamiltonian for spinning black-hole binaries, *Phys. Rev. D* **81**, 084024 (2010), arXiv:0912.3517 [gr-qc].
- [51] E. Barausse and A. Buonanno, Extending the effective-one-body Hamiltonian of black-hole binaries to include next-to-next-to-leading spin-orbit couplings, *Phys. Rev. D* **84**, 104027 (2011), arXiv:1107.2904 [gr-qc].
- [52] A. Nagar, Effective one body Hamiltonian of two spinning black-holes with next-to-next-to-leading order spin-orbit coupling, *Phys. Rev. D* **84**, 084028 (2011), [Erratum: *Phys.Rev.D* 88, 089901 (2013)], arXiv:1106.4349 [gr-qc].
- [53] S. Balmelli and P. Jetzer, Effective-one-body Hamiltonian with next-to-leading order spin-spin coupling for two nonprecessing black holes with aligned spins, *Phys. Rev. D* **87**, 124036 (2013), [Erratum: *Phys.Rev.D* 90, 089905 (2014)], arXiv:1305.5674 [gr-qc].
- [54] T. Damour and A. Nagar, New effective-one-body description of coalescing nonprecessing spinning black-hole binaries, *Phys. Rev. D* **90**, 044018 (2014), arXiv:1406.6913 [gr-qc].
- [55] S. Balmelli and T. Damour, New effective-one-

- body Hamiltonian with next-to-leading order spin-spin coupling, *Phys. Rev. D* **92**, 124022 (2015), arXiv:1509.08135 [gr-qc].
- [56] M. Khalil, J. Steinhoff, J. Vines, and A. Buonanno, Fourth post-Newtonian effective-one-body Hamiltonians with generic spins, (2020), arXiv:2003.04469 [gr-qc].
- [57] T. Damour and A. Nagar, Effective One Body description of tidal effects in inspiralling compact binaries, *Phys. Rev. D* **81**, 084016 (2010), arXiv:0911.5041 [gr-qc].
- [58] D. Bini, T. Damour, and G. Faye, Effective action approach to higher-order relativistic tidal interactions in binary systems and their effective one body description, *Phys. Rev. D* **85**, 124034 (2012), arXiv:1202.3565 [gr-qc].
- [59] J. Steinhoff, T. Hinderer, A. Buonanno, and A. Taracchini, Dynamical Tides in General Relativity: Effective Action and Effective-One-Body Hamiltonian, *Phys. Rev. D* **94**, 104028 (2016), arXiv:1608.01907 [gr-qc].
- [60] T. Hinderer *et al.*, Effects of neutron-star dynamic tides on gravitational waveforms within the effective-one-body approach, *Phys. Rev. Lett.* **116**, 181101 (2016), arXiv:1602.00599 [gr-qc].
- [61] N. Yunes, A. Buonanno, S. A. Hughes, Y. Pan, E. Barausse, M. Miller, and W. Thrope, Extreme Mass-Ratio Inspirals in the Effective-One-Body Approach: Quasi-Circular, Equatorial Orbits around a Spinning Black Hole, *Phys. Rev. D* **83**, 044044 (2011), [Erratum: *Phys. Rev. D* **88**, 109904 (2013)], arXiv:1009.6013 [gr-qc].
- [62] T. Damour, Gravitational Self Force in a Schwarzschild Background and the Effective One Body Formalism, *Phys. Rev. D* **81**, 024017 (2010), arXiv:0910.5533 [gr-qc].
- [63] E. Barausse, A. Buonanno, and A. Le Tiec, The complete non-spinning effective-one-body metric at linear order in the mass ratio, *Phys. Rev. D* **85**, 064010 (2012), arXiv:1111.5610 [gr-qc].
- [64] S. Akcay, L. Barack, T. Damour, and N. Sago, Gravitational self-force and the effective-one-body formalism between the innermost stable circular orbit and the light ring, *Phys. Rev. D* **86**, 104041 (2012), arXiv:1209.0964 [gr-qc].
- [65] A. Antonelli, M. van de Meent, A. Buonanno, J. Steinhoff, and J. Vines, Quasicircular inspirals and plunges from nonspinning effective-one-body Hamiltonians with gravitational self-force information, *Phys. Rev. D* **101**, 024024 (2020), arXiv:1907.11597 [gr-qc].
- [66] T. Damour, Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory, *Phys. Rev. D* **94**, 104015 (2016), arXiv:1609.00354 [gr-qc].
- [67] T. Damour, High-energy gravitational scattering and the general relativistic two-body problem, *Phys. Rev. D* **97**, 044038 (2018), arXiv:1710.10599 [gr-qc].
- [68] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent, and J. Vines, Energetics of two-body Hamiltonians in post-Minkowskian gravity, *Phys. Rev. D* **99**, 104004 (2019), arXiv:1901.07102 [gr-qc].
- [69] Y. Pan, A. Buonanno, L. T. Buchman, T. Chu, L. E. Kidder, H. P. Pfeiffer, and M. A. Scheel, Effective-one-body waveforms calibrated to numerical relativity simulations: coalescence of non-precessing, spinning, equal-mass black holes, *Phys. Rev. D* **81**, 084041 (2010), arXiv:0912.3466 [gr-qc].
- [70] Y. Pan, A. Buonanno, R. Fujita, E. Racine, and H. Tagoshi, Post-Newtonian factorized multipolar waveforms for spinning, non-precessing black-hole binaries, *Phys. Rev. D* **83**, 064003 (2011), [Erratum: *Phys. Rev. D* **87**, no.10, 109901 (2013)], arXiv:1006.0431 [gr-qc].
- [71] Y. Pan, A. Buonanno, A. Taracchini, L. E. Kidder, A. H. Mroué, H. P. Pfeiffer, M. A. Scheel, and B. Szilágyi, Inspiral-merger-ringdown waveforms of spinning, precessing black-hole binaries in the effective-one-body formalism, *Phys. Rev. D* **89**, 084006 (2014), arXiv:1307.6232 [gr-qc].
- [72] A. Taracchini, Y. Pan, A. Buonanno, E. Barausse, M. Boyle, T. Chu, G. Lovelace, H. P. Pfeiffer, and M. A. Scheel, Prototype effective-one-body model for nonprecessing spinning inspiral-merger-ringdown waveforms, *Phys. Rev. D* **86**, 024011 (2012), arXiv:1202.0790 [gr-qc].
- [73] A. Taracchini *et al.*, Effective-one-body model for black-hole binaries with generic mass ratios and spins, *Phys. Rev. D* **89**, 061502 (2014), arXiv:1311.2544 [gr-qc].
- [74] A. Bohé *et al.*, Improved effective-one-body model of spinning, nonprecessing binary black holes for the era of gravitational-wave astrophysics with advanced detectors, *Phys. Rev. D* **95**, 044028 (2017), arXiv:1611.03703 [gr-qc].
- [75] S. Babak, A. Taracchini, and A. Buonanno, Validating the effective-one-body model of spinning, precessing binary black holes against numerical relativity, *Phys. Rev. D* **95**, 024010 (2017), arXiv:1607.05661 [gr-qc].
- [76] A. Nagar *et al.*, Time-domain effective-one-body gravitational waveforms for coalescing compact binaries with nonprecessing spins, tides and self-spin effects, *Phys. Rev. D* **98**, 104052 (2018), arXiv:1806.01772 [gr-qc].
- [77] S. Ossokine *et al.*, Multipolar Effective-One-Body Waveforms for Precessing Binary Black Holes: Construction and Validation, *Phys. Rev. D* **102**, 044055 (2020), arXiv:2004.09442 [gr-qc].
- [78] D. Bini and T. Damour, Gravitational radiation reaction along general orbits in the effective one-body formalism, *Phys. Rev. D* **86**, 124012 (2012), arXiv:1210.2834 [gr-qc].
- [79] T. Hinderer and S. Babak, Foundations of an effective-one-body model for coalescing binaries on eccentric orbits, *Phys. Rev. D* **96**, 104048 (2017), arXiv:1707.08426 [gr-qc].
- [80] Z. Cao and W.-B. Han, Waveform model for an eccentric binary black hole based on the effective-one-body-numerical-relativity formalism, *Phys. Rev. D* **96**, 044028 (2017), arXiv:1708.00166 [gr-qc].
- [81] X. Liu, Z. Cao, and L. Shao, Validating the Effective-One-Body Numerical-Relativity Waveform Models for Spin-aligned Binary Black Holes along Eccentric Orbits, *Phys. Rev. D* **101**, 044049 (2020), arXiv:1910.00784 [gr-qc].
- [82] X. Liu, Z. Cao, and Z.-H. Zhu, A higher-multipole gravitational waveform model for an eccentric binary black holes based on the effective-one-body-numerical-relativity formalism, (2021), arXiv:2102.08614 [gr-qc].
- [83] R. Cotesta, A. Buonanno, A. Bohé, A. Taracchini, I. Hinder, and S. Ossokine, Enriching the Symphony of Gravitational Waves from Binary Black Holes by Tuning Higher Harmonics, *Phys. Rev. D* **98**, 084028 (2018), arXiv:1803.10701 [gr-qc].
- [84] D. Chiamello and A. Nagar, Faithful analytical effective-one-body waveform model for spin-aligned,

- moderately eccentric, coalescing black hole binaries, *Phys. Rev. D* **101**, 101501 (2020), arXiv:2001.11736 [gr-qc].
- [85] A. Nagar, A. Bonino, and P. Retzegno, All in one: effective one body multipolar waveform model for spin-aligned, quasi-circular, eccentric, hyperbolic black hole binaries, (2021), arXiv:2101.08624 [gr-qc].
- [86] A. Nagar, G. Riemenschneider, G. Pratten, P. Retzegno, and F. Messina, Multipolar effective one body waveform model for spin-aligned black hole binaries, *Phys. Rev. D* **102**, 024077 (2020), arXiv:2001.09082 [gr-qc].
- [87] S. Habib and E. Huerta, Characterization of numerical relativity waveforms of eccentric binary black hole mergers, *Phys. Rev. D* **100**, 044016 (2019), arXiv:1904.09295 [gr-qc].
- [88] For the arXiv version, download the ancillary files `RRforce.dat.m` and `modes.dat.m`, which contain the RR force and waveform modes for eccentric orbits.
- [89] B. R. Iyer and C. Will, PostNewtonian gravitational radiation reaction for two-body systems, *Phys. Rev. Lett.* **70**, 113 (1993).
- [90] B. R. Iyer and C. Will, PostNewtonian gravitational radiation reaction for two-body systems: Nonspinning bodies, *Phys. Rev. D* **52**, 6882 (1995).
- [91] A. Gopakumar, B. R. Iyer, and S. Iyer, Second post-Newtonian gravitational radiation reaction for two-body systems: Nonspinning bodies, *Phys. Rev. D* **55**, 6030 (1997), [Erratum: *Phys.Rev.D* 57, 6562 (1998)], arXiv:gr-qc/9703075.
- [92] A. Buonanno, Y. Chen, and T. Damour, Transition from inspiral to plunge in precessing binaries of spinning black holes, *Phys. Rev. D* **74**, 104005 (2006), arXiv:gr-qc/0508067 [gr-qc].
- [93] K. Arun, L. Blanchet, B. R. Iyer, and M. S. Qusailah, Inspiralling compact binaries in quasi-elliptical orbits: The Complete 3PN energy flux, *Phys. Rev. D* **77**, 064035 (2008), arXiv:0711.0302 [gr-qc].
- [94] K. Arun, L. Blanchet, B. R. Iyer, and M. S. Qusailah, Tail effects in the 3PN gravitational wave energy flux of compact binaries in quasi-elliptical orbits, *Phys. Rev. D* **77**, 064034 (2008), arXiv:0711.0250 [gr-qc].
- [95] K. Arun, L. Blanchet, B. R. Iyer, and S. Sinha, Third post-Newtonian angular momentum flux and the secular evolution of orbital elements for inspiralling compact binaries in quasi-elliptical orbits, *Phys. Rev. D* **80**, 124018 (2009), arXiv:0908.3854 [gr-qc].
- [96] N. Loutrel and N. Yunes, Hereditary Effects in Eccentric Compact Binary Inspirals to Third Post-Newtonian Order, *Class. Quant. Grav.* **34**, 044003 (2017), arXiv:1607.05409 [gr-qc].
- [97] L. Blanchet, Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries, *Living Rev. Rel.* **17**, 2 (2014), arXiv:1310.1528 [gr-qc].
- [98] L. E. Kidder, Coalescing binary systems of compact objects to postNewtonian 5/2 order. 5. Spin effects, *Phys. Rev. D* **52**, 821 (1995), arXiv:gr-qc/9506022 [gr-qc].
- [99] J. Zeng and C. M. Will, Application of energy and angular momentum balance to gravitational radiation reaction for binary systems with spin-orbit coupling, *Gen. Rel. Grav.* **39**, 1661 (2007), arXiv:0704.2720 [gr-qc].
- [100] A. Bohé, S. Marsat, and L. Blanchet, Next-to-next-to-leading order spin-orbit effects in the gravitational wave flux and orbital phasing of compact binaries, *Class. Quant. Grav.* **30**, 135009 (2013), arXiv:1303.7412 [gr-qc].
- [101] W. G. Dixon, Extended bodies in general relativity: their description and motion, in *Isolated Gravitating Systems in General Relativity*, edited by J. Ehlers (North Holland, Amsterdam, 1979) pp. 156–219.
- [102] J. Steinhoff, Spin and quadrupole contributions to the motion of astrophysical binaries, *Proceedings, 524th WE-Heraeus-Seminar: Equations of Motion in Relativistic Gravity (EOM 2013)*, *Fund. Theor. Phys.* **179**, 615 (2015), arXiv:1412.3251 [gr-qc].
- [103] W. Tulczyjew, Equations of motion of rotating bodies in general relativity theory, *Acta Phys. Polon.* **18**, 37 (1959), [Erratum: *Acta Phys. Polon.* **18**, 534 (1959)].
- [104] A. D. Fokker, *Relativiteitstheorie* (P. Noordhoff, 1929).
- [105] M. H. L. Pryce, The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles, *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **195**, 62 (1948).
- [106] T. D. Newton and E. P. Wigner, Localized states for elementary systems, *Reviews of Modern Physics* **21**, 400 (1949).
- [107] J. Vines, D. Kunst, J. Steinhoff, and T. Hinderer, Canonical Hamiltonian for an extended test body in curved spacetime: To quadratic order in spin, *Phys. Rev. D* **93**, 103008 (2016), arXiv:1601.07529 [gr-qc].
- [108] E. Barausse, E. Racine, and A. Buonanno, Hamiltonian of a spinning test-particle in curved spacetime, *Phys. Rev. D* **80**, 104025 (2009), [Erratum: *Phys. Rev. D* 85, 069904 (2012)], arXiv:0907.4745 [gr-qc].
- [109] H. Wang and C. M. Will, Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. IV. Radiation reaction for binary systems with spin-spin coupling, *Phys. Rev. D* **75**, 064017 (2007), arXiv:gr-qc/0701047.
- [110] N. T. Maia, C. R. Galley, A. K. Leibovich, and R. A. Porto, Radiation reaction for spinning bodies in effective field theory II: Spin-spin effects, *Phys. Rev. D* **96**, 084065 (2017), arXiv:1705.07938 [gr-qc].
- [111] A. Bohé, G. Faye, S. Marsat, and E. K. Porter, Quadratic-in-spin effects in the orbital dynamics and gravitational-wave energy flux of compact binaries at the 3PN order, *Class. Quant. Grav.* **32**, 195010 (2015), arXiv:1501.01529 [gr-qc].
- [112] T. Damour and A. Nagar, Faithful effective-one-body waveforms of small-mass-ratio coalescing black-hole binaries, *Phys. Rev. D* **76**, 064028 (2007), arXiv:0705.2519 [gr-qc].
- [113] T. Mora and C. M. Will, Numerically generated quasiequilibrium orbits of black holes: Circular or eccentric?, *Phys. Rev. D* **66**, 101501 (2002), arXiv:gr-qc/0208089.
- [114] A. Gopakumar and B. R. Iyer, Second postNewtonian gravitational wave polarizations for compact binaries in elliptical orbits, *Phys. Rev. D* **65**, 084011 (2002), arXiv:gr-qc/0111010.
- [115] C. K. Mishra, K. Arun, and B. R. Iyer, Third post-Newtonian gravitational waveforms for compact binary systems in general orbits: Instantaneous terms, *Phys. Rev. D* **91**, 084040 (2015), arXiv:1501.07096 [gr-qc].
- [116] A. Buonanno, G. Faye, and T. Hinderer, Spin effects on gravitational waves from inspiraling compact binaries at second post-Newtonian order, *Phys. Rev. D* **87**, 044009 (2013), arXiv:1209.6349 [gr-qc].

- [117] L. E. Kidder, Using full information when computing modes of post-Newtonian waveforms from inspiralling compact binaries in circular orbit, *Phys. Rev. D* **77**, 044016 (2008), arXiv:0710.0614 [gr-qc].
- [118] L. Blanchet, G. Faye, B. R. Iyer, and S. Sinha, The Third post-Newtonian gravitational wave polarisations and associated spherical harmonic modes for inspiralling compact binaries in quasi-circular orbits, *Class. Quant. Grav.* **25**, 165003 (2008), [Erratum: *Class. Quant. Grav.* **29**, 239501 (2012)], arXiv:0802.1249 [gr-qc].
- [119] G. Faye, S. Marsat, L. Blanchet, and B. R. Iyer, The third and a half post-Newtonian gravitational wave quadrupole mode for quasi-circular inspiralling compact binaries, *Class. Quant. Grav.* **29**, 175004 (2012), arXiv:1204.1043 [gr-qc].
- [120] K. G. Arun, A. Buonanno, G. Faye, and E. Ochsner, Higher-order spin effects in the amplitude and phase of gravitational waveforms emitted by inspiraling compact binaries: Ready-to-use gravitational waveforms, *Phys. Rev. D* **79**, 104023 (2009), [Erratum: *Phys. Rev. D* **84**, 049901 (2011)], arXiv:0810.5336 [gr-qc].
- [121] L. Blanchet, A. Buonanno, and G. Faye, Higher-order spin effects in the dynamics of compact binaries. II. Radiation field, *Phys. Rev. D* **74**, 104034 (2006), [Erratum: *Phys. Rev. D* **75**, 049903 (2007), Erratum: *Phys. Rev. D* **81**, 089901 (2010)], arXiv:gr-qc/0605140.
- [122] Y. Boetzel, C. K. Mishra, G. Faye, A. Gopakumar, and B. R. Iyer, Gravitational-wave amplitudes for compact binaries in eccentric orbits at the third post-Newtonian order: Tail contributions and postadiabatic corrections, *Phys. Rev. D* **100**, 044018 (2019), arXiv:1904.11814 [gr-qc].
- [123] S. Drasco, E. E. Flanagan, and S. A. Hughes, Computing inspirals in Kerr in the adiabatic regime. I. The Scalar case, *Class. Quant. Grav.* **22**, S801 (2005), arXiv:gr-qc/0505075.
- [124] B. Banihashemi and J. Vines, Gravitomagnetic tidal effects in gravitational waves from neutron star binaries, *Phys. Rev. D* **101**, 064003 (2020), arXiv:1805.07266 [gr-qc].
- [125] N. Siemonsen, J. Steinhoff, and J. Vines, Gravitational waves from spinning binary black holes at the leading post-Newtonian orders at all orders in spin, *Phys. Rev. D* **97**, 124046 (2018), arXiv:1712.08603 [gr-qc].
- [126] T. Damour and A. Nagar, Comparing Effective-One-Body gravitational waveforms to accurate numerical data, *Phys. Rev. D* **77**, 024043 (2008), arXiv:0711.2628 [gr-qc].
- [127] T. Damour, B. R. Iyer, and A. Nagar, Improved resummation of post-Newtonian multipolar waveforms from circularized compact binaries, *Phys. Rev. D* **79**, 064004 (2009), arXiv:0811.2069 [gr-qc].
- [128] A. Ramos-Buades *et al.*, in preparation (2021).
- [129] T. Damour and G. Schäfer, Higher-order relativistic periastron advances and binary pulsars, *Il Nuovo Cimento B (1971-1996)* **101**, 127 (1988).
- [130] T. Damour, P. Jaranowski, and G. Schafer, On the determination of the last stable orbit for circular general relativistic binaries at the third postNewtonian approximation, *Phys. Rev. D* **62**, 084011 (2000), arXiv:gr-qc/0005034 [gr-qc].
- [131] J. Steinhoff, G. Schafer, and S. Hergt, ADM canonical formalism for gravitating spinning objects, *Phys. Rev. D* **77**, 104018 (2008), arXiv:0805.3136 [gr-qc].
- [132] H. Tagoshi, A. Ohashi, and B. J. Owen, Gravitational field and equations of motion of spinning compact binaries to 2.5 postNewtonian order, *Phys. Rev. D* **63**, 044006 (2001), arXiv:gr-qc/0010014.
- [133] C. M. Will, Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. III. Radiation reaction for binary systems with spinning bodies, *Phys. Rev. D* **71**, 084027 (2005), arXiv:gr-qc/0502039.
- [134] K. Thorne, Multipole Expansions of Gravitational Radiation, *Rev. Mod. Phys.* **52**, 299 (1980).
- [135] G. Cho, B. Pardo, and R. A. Porto, Gravitational radiation from inspiralling compact objects: Spin-spin effects completed at the next-to-leading post-Newtonian order, (2021), arXiv:2103.14612 [gr-qc].
- [136] Y. Pan, A. Buonanno, M. Boyle, L. T. Buchman, L. E. Kidder, H. P. Pfeiffer, and M. A. Scheel, Inspiral-merger-ringdown multipolar waveforms of nonspinning black-hole binaries using the effective-one-body formalism, *Phys. Rev. D* **84**, 124052 (2011), arXiv:1106.1021 [gr-qc].