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NOTES

A NOTE ON THE PARALLEL DIFFUSION COEFFICIENT

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ABSTRACT

The gradient-induced anisotropy and the parallel diffusion coefficient are related to the pitch-angle diffusion coefficient. The treatment is more complete than that given previously by Jokipii and by Hasselmann and Wibberenz in order to show the limits of applicability and the physical meaning of divergences resulting for certain types of magnetic-field spectra.

Random fluctuations in the magnetic field lead to pitch-angle scattering of particles traveling along the mean field lines. For weak scattering (pitch-angle relaxation time τ large compared with the gyration period), the process can be described by a Fokker-Planck equation for the particle distribution. In the quasi-homogeneous limit (horizontal scales of mean properties large compared with the mean free path $\lambda = \tau \times$ particle velocity V), the Fokker-Planck equation can be integrated with respect to pitch angle, yielding a diffusion equation for the mean particle density.

The relation between the parallel diffusion coefficient $D_\|$ and the pitch-angle diffusion coefficient D_μ has been given by Jokipii (1966, hereinafter referred to as J1), and Hasselmann and Wibberenz (1968, hereinafter referred to as HW). Except for a relativistic generalization in HW, and a factor $\frac{1}{2}$ misprint in J1, the expressions given in both papers are identical:

$$D_\| = \tfrac{1}{4} V^2 \int_{-1}^{+1} \left[\int_0^{\mu'} \frac{1-\mu^2}{D_\mu} d\mu \right] \mu' d\mu' \tag{1}$$

(J1, eq. [28]; HW, eq. [7.5], with $\gamma = 1$ [notation changed to that of J1]), where $\mu = V_\|/V$, $D_\mu = \frac{1}{2}\langle(\Delta\mu)^2\rangle/\Delta t$.

In a Note in this Journal, Jokipii (1968, hereinafter called J2) corrected the factor $\frac{1}{2}$ and at the same time presented another derivation of the parallel diffusion coefficient, which led to a different expression:

$$D_\| = \tfrac{1}{9} V^2 \left[\int_0^1 D_\mu d\mu \right]^{-1} . \tag{2}$$

Equations (1) and (2) coincide for the special case of an axisymmetric, transverse, unpolarized magnetic field and a k^{-1} power-law spectrum. In this case, $D_\mu \sim 1 - \mu^2$. In general, the two expressions for $D_\|$ differ appreciably. It has been shown in HW and in Wibberenz, Hasselmann, and Hasselmann (1969), that the tensor structure of the magnetic-field spectrum and the exponent of the power law have a strong effect on particle diffusion. Thus it is important to determine the correct expression for the parallel diffusion coefficient.

All three derivations (J1, HW, and J2) are based on the assumption that the pitch-angle distribution is close to the isotropic equilibrium distribution n_{eq} = const.,

$$n(\mu; x, t) = \tfrac{1}{2}\rho(x, t) + n'(\mu; x, t) \tag{3}$$

where $n' \ll n$, and $\rho(x, t)$ is the particle density, $\rho = \int n d\mu$, where the integration is from -1 to $+1$. By definition,

$$\int_{-1}^{+1} n' d\mu = 0 . \tag{4}$$

In J2, the additional assumption is made that n' is proportional to μ. In general, this is not the case (cf. HW, Figs. 7–9). Of the examples considered in HW, a linear dependence was found only for an axisymmetric transverse field with zero circular polarization and a k^{-1} power-law spectrum (Fig. 7, upper panel). This is the case mentioned above, for which equations (1) and (2) give the same result.

Since the correct expression (1) was derived in J1 under rather restrictive assumptions, and since details were omitted in HW, a more complete presentation may serve to clarify the approximations involved. The diffusion limit of the Fokker-Planck equation applies if the particle density is slowly varying,

$$\partial\rho/\partial t = O(\rho/T) , \quad \partial\rho/\partial x_{\|} = O(\rho/L) , \tag{5}$$

where $L \gg \lambda_{\|}$ and $T \gg \tau$ ($\lambda_{\|} = V\tau$ = mean free path, $\tau = O(1/D_\mu)$ = pitch-angle relaxation time). In this case, the particles have time to adjust locally to a near-isotropic equilibrium, so that $n' \ll \rho$. Since we are concerned here with parallel diffusion only, we assume there is no variation of ρ perpendicular to the field lines.

Substituting equation (3) into the Fokker-Planck equation

$$\frac{\partial n}{\partial t} + V\mu \frac{\partial n}{\partial x_{\|}} - \frac{\partial}{\partial \mu}\left(D_\mu \frac{\partial n}{\partial \mu}\right) = 0 \tag{6}$$

and subtracting the mean with respect to μ, we obtain

$$\frac{\partial n'}{\partial t} + V\mu \frac{\partial n'}{\partial x_{\|}} - \tfrac{1}{2}V \int_{-1}^{+1} \mu \frac{\partial n'}{\partial x_{\|}} d\mu - \frac{\partial}{\partial \mu}\left(D_\mu \frac{\partial n'}{\partial \mu}\right) = -\tfrac{1}{2}V\mu \frac{\partial \rho}{\partial x_{\|}} . \tag{7}$$

The differential operators determining the response of n' to the source term on the right-hand side of equation (7) can be characterized by different time scales. The first three terms on the left-hand side are of order T^{-1}, VL^{-1}, and VL^{-1}, respectively. The fourth term, the pitch-angle diffusion, is of the order τ^{-1}. Under condition (5) this is large compared with the first three terms, so that the anisotropy is determined to lowest order by the local equilibrium between pitch-angle diffusion and the source term proportional to the gradient on the right-hand side.

The solution is

$$n' = -V \frac{\partial \rho}{\partial x_{\|}} \left[\tfrac{1}{4} \int_0^\mu \frac{1 - (\mu')^2}{D_\mu} d\mu' + \alpha \right] , \tag{8}$$

where the constant α is determined by equation (4).

Substituting equation (8) into equation (6) and integrating over μ, we obtain the diffusion equation

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x_{\|}}\left(D_{\|} \frac{\partial \rho}{\partial x_{\|}}\right) = 0 , \tag{9}$$

with $D_{\|}$ as given by equation (1).

The numerical results derived from equation (1) or (2), respectively, do not differ appreciably as long as D_μ is finite for all μ and does not approach zero sufficiently rapidly. For pure power-law spectra $k^{-1} \ldots k^{-1.5}$ this is the case under restrictive assumptions about the magnetic-field fluctuations, e.g., in the above-mentioned case of axisymmetric, transverse fluctuations.

However, in general, expressions (1) and (2) differ in their singular behavior. If the pitch-angle scattering is very small for particular pitch angles—in the examples considered in HW, at 0°, or at 90°—then particles with these pitch angles propagate virtually undisturbed along the field lines. If these "escape holes" are sufficiently large, the net particle propagation is *convective* rather than diffusive, and the formal expression for the diffusion coefficient diverges. Mathematically, the assumption that the pitch-angle diffusion dominates over the convective terms on the left-hand side of equation (7) is no longer valid.

The divergences in equation (1) are therefore physically meaningful. One specific example is the case of pure power-law magnetic-field spectra, $f(k) \sim k^q$. The mean free path $\lambda_\parallel = 3D_\parallel/V$, which is a function of particle rigidity P only, depends on P according to $\lambda_\parallel \sim P^{2+q}$, provided that $q > -2$. The limiting case $q = -2$ would yield $\lambda_\parallel =$ const. if equation (2) were applied. However, equation (1) leads to escape-hole divergences for $q = -2$, in particular $\lambda_\parallel \rightarrow \infty$ for axisymmetric transverse fluctuations.

The details of the divergences depend strongly on the type of fluctuations assumed. In some cases, an escape-hole divergence ($D_\mu \rightarrow 0$) at certain pitch angles occurs simultaneously with infinities of D_μ at other pitch angles. The behavior of $\lambda_\parallel$ for various field models is discussed in Wibberenz *et al.* (1969). It is shown that if the spectra flatten at small wavenumbers, isotropic fluctuations yield finite $\lambda_\parallel$ even if the spectra fall off very steeply at high wavenumbers. Thus care is necessary in deriving values of the mean free path from steep magnetic-field spectra. This holds already in the framework of the weak-interaction formalism, quite independent of the complications arising if a large portion of the spectral density is contained in discontinuities (see Sari and Ness 1969).

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