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# Super-resolution of tides

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## ABSTRACT

The precision of tide prediction is ultimately limited by the underlying noise spectrum, $S(f)$. For two neighbouring spectral lines at frequencies $f$ and $f+\Delta f$, the variance in the estimate of either amplitude is $3\pi^{-2}S(f)(\Delta f)^{-2}T^{-3}$ (equation 15) where $T$ is the record length. For the case $M_2$ and $N_2$ typical amplitudes are 50 cm and 10 cm respectively, $\Delta f$ is 0.03 cycles per day (cpd); it is usually thought a record length $T=(\Delta f)^{-1}\approx 1$ month is required for adequate resolution. The observed $S(f)$ is typically 1 cm²/cpd. For $T=3$ days the variance is then 1 cm², and the rms amplitude errors equal 2 and 10%, respectively.


## 1. Tidal line spectrum

The motion of the Sun and Moon and the rotation of the Earth are associated with a tide-producing potential

$$\phi=\sum_n (a_n \cos 2\pi f_n t+b_n \sin 2\pi f_n t)$$

summed over an infinite set of denumerable frequencies $f_n$. The latter arise from the non-linear interaction between the various Kepplerian orbital parameters. They can be written as a sum of six basic frequencies each multiplied by some integer:

$$f_n=s_a f_a+s_b f_b\cdots+s_f f_f \quad s=0, \pm 1, \pm 2, \cdots$$

where

$f_a^{-1}=1$ day is the period of Earth's rotation,
$f_b^{-1}=1$ month is the period of Moon's orbital motion,
$f_c^{-1}=1$ year is the period of Sun's orbital motion,
$f_d^{-1}\approx 9$ years is the period of lunar perigee,
$f_e^{-1}\approx 18.6$ years is the period of regression of lunar nodes,
$f_f^{-1}\approx 21{,}000$ years is the period of solar perigee (precession).

We can ignore lower frequencies arising from planetary perturbations.

## 2. Resolution and "beam-splitting"

Consider a record

$$\left.\begin{aligned} x(t)=a_1 \cos 2\pi f_1 t+b_1 \sin 2\pi f_1 t \\ +a_2 \cos 2\pi f_2 t+b_2 \sin 2\pi f_2 t \end{aligned}\right\} \quad (1)$$

consisting of two neighboring "lines"

$$f_1=f_a=1 \text{ cycle per day (cpd)}$$
$$f_2=f_a+f_b=1+.0366=1.0366 \text{ cpd}$$

separated by 1 cycle per month. It has been customary to presume that one month of record is required to evaluate the coefficients; similarly, that a year's record is required to determine the coefficients for frequencies split by 1 cycle per year, 18.6 years of record for regressional splitting, and that the longest existing records ($\approx$200 years) cannot shed any light on precessional splitting. But this presumption that

*a record of length $T$ is required to evaluate coefficients pertaining to frequencies separated by $T^{-1}$* (2)

is clearly false (or at least grossly incomplete). For any 4 known values of $x(t)$ (excepting degeneracies) will provide 4 equations to solve for the 4 unknowns $a_1$, $b_1$, $a_2$, $b_2$, regardless of the frequency separation. If the frequencies in equation (1) are not known, then we require 6 readings of $x(t)$ to determine the 6 unknowns $a_1$, $b_1$, $a_2$, $b_2$, $f_1$, $f_2$. The problem now requires the solution of transcendental (rather than algebraic) equations, but in principle is not so very different from the previous one. In general,

> *if $x(t)$ consists of a sum of $n$ frequencies, then $2n$ readings are required to determine the coefficients if the frequencies are known, and $3n$ readings if they are not known, regardless of frequency separations.* (3)

What is wrong here? Common sense supports the traditional statement (2) and rebels at the assertion (3) that a few hourly readings could tell us anything about regressional splitting. The trouble is that we have presumed that oscillations in sea level $x(t)$ can be described by two (or at most a denumerable set of) *discrete* frequencies, without a superposed *continuous* noise spectrum arising from geophysical sources and from random errors in reading $x(t)$. Equation (1) must be amended to read

$$\left.\begin{aligned} x(t) &= a_1\cos 2\pi f_1 t + b_1 \sin 2\pi f_1 t + a_2 \cos 2\pi f_2 t \\ &\quad + b_2 \sin 2\pi f_2 t + x'(t) \end{aligned}\right\} \quad (4)$$

Noise $x'(t)$ is inevitably present (except in the literature on tide analysis) and an essential factor in the present context. Thus we will demonstrate that some meaningful statements about the frequencies $f_1$ and $f_2$ can be made, provided

$$|f_2 - f_1| > \frac{T^{-1}}{(\text{signal/noise level})^{\frac{1}{2}}} \qquad (5)$$

For very low relative noise levels we may indeed improve upon the classical resolution $T^{-1}$ (statement 2), but we shall never learn anything about regressional splitting from 4 hourly observations because this requires a noise level so low that it simply cannot be achieved.[1] Unlike statement (3), the inequality (5) is no longer at odds with experience. In the almost analogous problem of forming narrow RADAR beams it is well known that the theoretical resolution $(L/\lambda)^{-1}$ associated with the finite aperture $L$ can be exceeded in just this ratio $\sqrt{\text{signal/noise level}}$. This is called "beam splitting.". We shall refer to any improvement over and above the theoretical limit $T^{-1}$ afforded by the "time aperture" of the record as "super-resolution".

Any record $x(t)$ in the interval $0<t<T$ can be completely represented by a line spectrum at frequencies $sT^{-1}$, where $s$ is an integer. In the absence of any further information, this is the simplest spectral presentation consistent with the facts, and nothing can be learned about frequencies differing by less than $T^{-1}$ (in accordance with statement (2)). But in tidal analysis the situation is different: here we may assume that the tides can be universally represented by known Kepplerian frequencies. For any noise-free record $0<t<T$ the Kepplerian representation is no better and no worse than a representation in terms of equally spaced Fourier frequencies. But outside the interval $0<t<T$ the Kepplerian representation is still valid, whereas the Fourier representation is not. In principle, the Kepplerian representation is completely determined by any noise-free record of length $T$. In practice, the determination is limited by the noise.

### 3. Fourier-Stieltjes representation

We confine ourselves to the simplest possible example that exhibits super-resolution: two neighboring lines and a noise. Then

$$\begin{aligned} x(t) &= A_1 e^{i\omega_1 \tau} + A_1^* e^{-i\omega_1 \tau} + A_2 e^{i\omega_2 \tau} \\ &\quad + A_2^* e^{-i\omega_2 \tau} + x'(t) \end{aligned}$$

is an infinite stationary time series. We represent $x(t)$ as a Fourier-Stieltjes integral

$$x(t) = \int_{-\infty}^{\infty} dX(\omega) e^{i\omega t} \qquad (6)$$

where

$$\begin{aligned} dX(\omega) &= d\omega[A_1\delta(\omega-\omega_1) + A_1^*\delta(\omega+\omega_1) \\ &\quad + A_2\delta(\omega-\omega_2) + A_2^*\delta(\omega+\omega_2)] \\ &\quad + dX'(\omega) \end{aligned}$$

[1] If for no other reason than that $x$ would have to be measured to a small fraction of the wave length of light.



and $dX'(\omega)$ is the component of a stationary noise (not necessarily Gaussian), with the property

$$< dX'(\omega) dX'(\hat{\omega}) > = S(\omega)\delta(\omega+\hat{\omega}) d\omega d\hat{\omega} \qquad (7)$$

so that $<(x'(t))^2> = \int_{-\infty}^{\infty} S(\omega) d\omega$ is the total noise "power", and $S(\omega)$ the power spectrum of the noise.

### 4. Finite record length

$x(t)$ is measured in the interval $t=-\frac{1}{2}T$ to $t=+\frac{1}{2}T$. Let

$$\begin{aligned} h(t) &= 1 \quad \text{for} \quad |t| < \frac{1}{2}T \\ &= 0 \quad \text{otherwise} \end{aligned}$$

designate the box function, and

$$H(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt = \frac{\sin \frac{1}{2}\omega T}{\pi\omega}$$

its Fourier transform. It follows from (6) and the convolution theorem that

$$\left.\begin{aligned} X(\omega_1) &= A_1 H(0) + A_2 H(w_1 - w_2) + X'(\omega_1) \\ X(\omega_2) &= A_1 H(\omega_2-\omega_1) + A_2 H(0) + X'(\omega_2) \end{aligned}\right\} \qquad (8)$$

where

$$X'(\omega_j) = \int_{-\infty}^{\infty} dX'(\omega) H(\omega_j - \omega). \qquad (9)$$

Equation (8) states that the spectrum at $\omega_1$ is due to three terms: (i) the line at $\omega_1$; (ii) some sideband effect from the line at $\omega_2$, with the degree of interaction depending on record length through $H(\omega_1-\omega_2)$; and (iii) the noise density at $\omega_1$.

### 5. Least-square solutions

Since the noise density is not known, a precise determination of $A_j$ is not feasible. Rather, equations (8) can give only estimates of $A_j$. In particular, we desire the expected value and variance of these estimates. For this purpose we generate a noise-free time series

$$y(t) = B_1 e^{i\omega_1 t} + B_1^* e^{-i\omega_1 t} + B_2 e^{i\omega_2 t} + B_2^* e^{-i\omega_2 t}$$

and determine the coefficients $B$ by minimizing

$$\int_{-\infty}^{\infty} < x(t) - y(t) >^2 h(t) dt$$

Terms in $\omega_1+\omega_2$ can be neglected, since $H(\omega_1+\omega_2) \ll H(\omega_1-\omega_2)$; also $H(\omega_2-\omega_1) = H(\omega_1-\omega_2) = H(\Delta\omega)$. The result then is

$$\left.\begin{aligned} X(\omega_1) &= B_1 H(0) + B_2 H(\Delta\omega) \\ X(\omega_2) &= B_1 H(\Delta\omega) + B_2 H(0) \end{aligned}\right\} \qquad (10)$$

and comparison with (8) yields

$$\left.\begin{aligned} B_1 &= A_1 + \delta A_1, \\ \delta A_1 &= \frac{X'(\omega_1) H(0) - X'(\omega_2) H(\Delta\omega)}{H^2(0) - H^2(\Delta\omega)} \end{aligned}\right\} \qquad (11)$$

and similarly for $B_2$. The expected values of $X'(\omega_j)$ are zero. Thus $E(\delta A_j)=0$ and $E(B_j)=A_j$.

We need the expected value of $|\delta A_j|^2$. First we introduce the definitions (9) into (11):

$$\delta A_j = \int_{-\infty}^{\infty} dX'(\omega) \Phi_j(\omega) d\omega$$

where

$$\Phi_1(\omega) = \frac{H(\omega-\omega_1) H(0) - H(\omega-\omega_2) H(\Delta\omega)}{H^2(0) - H^2(\Delta\omega)} \qquad (12)$$

and similarly for $\Phi_2(\omega)$. Then we make use of the orthogonality property (7) to derive

$$\left.\begin{aligned} &< \delta A_j \delta A_j^* > \\ &= \iint dX'(\omega)\Phi_j(\omega) dX'^*(\hat{\omega}) \Phi_j^*(\hat{\omega}) d\omega d\hat{\omega} \\ &= \int S(\omega) |\Phi_j(\omega)|^2 d\omega . \end{aligned}\right\} \qquad (13)$$

To summarize: we measure the spectra $X(\omega_j)$ at the known frequencies $\omega_1$ and $\omega_2$ and solve for $B_j$ according to (10). These are the expected values of the harmonic coefficients $A_j$. Their variance is found from the convolution (13) upon the noise spectrum $S(\omega)$. This is the formal solution of our problem.

### 6. The doublet kernel

The kernel $\Phi_j(\omega)$ can be greatly simplified by allowing for the fact that $(\omega_2-\omega_1) \ll \frac{1}{2}(\omega_2+\omega_1)$. It is convenient to replace $\omega_j$ by $2\pi f_j$. Then write

$$f_0 = \frac{1}{2}(f_1+f_2), \qquad \Delta f = f_2 - f_1$$

$$r = \Delta f \cdot T,$$

$$f - f_1 = f - f_0 + \frac{1}{2}\Delta f, \qquad f - f_2 = f - f_0 - \frac{1}{2}\Delta f,$$

$$H(z)=\frac{\sin \pi zT}{2\pi z}=\frac{T}{2}I(z)\,, \qquad I(z)=\frac{\sin \pi zT}{\pi zT}$$

Then from (12)

$$\left.\begin{aligned}\Phi_1(f)&=\frac{I(f-f_1)I(0)-I(f-f_2)I(\Delta f)}{I^2(0)-I^2(\Delta f)}\\&=\frac{3}{\pi r}\left[I'(f-f_0)+\frac{\pi r}{6}I(f-f_0)+\cdots\right]\end{aligned}\right\} \quad (14a)$$

and similarly

$$\Phi_2(f)=\frac{3}{\pi r}\left[-I'(f-f_0)+\frac{\pi r}{6}I(f-f_0)-\cdots\right] \quad (14b)$$

where $I'$ is the derivative of $I$ with respect to its argument ($\pi zT$).

## 7. White noise

The simplest case is that of a noise spectrum which does not vary appreciably in the vicinity of the spectral doublet ($\omega_1$, $\omega_2$), so that we can replace $S(\omega)$ by $S(\omega_0)$ in (13). Also[2] $S(\omega)d\omega=S(f)df$. To the first order in $r$, we then have

$$\left.\begin{aligned}<|\delta A_j|^2>&=S(f_0)\left(\frac{3}{\pi r}\right)^2\int_{-\infty}^{\infty}I'^2(f-f_0)df\\&=\frac{9}{\pi^3}\frac{S(f_0)}{r^2T}\int_{-\infty}^{\infty}\left[\frac{d}{dx}\left(\frac{\sin x}{x}\right)\right]^2dx\\&=\frac{3}{\pi^2}\frac{S(f_0)}{r^2T}=\frac{3}{\pi^2}\frac{S(f_0)}{(\Delta f)^2T^3}\end{aligned}\right\} \quad (15)$$

## 8. A numerical experiment

We have generated artificial time series consisting of a series of 8 doublets superimposed on a white noise. The doublets had the following amplitudes and frequencies:

$a_1=20,\quad f_1=0.2,\qquad a_2=10,\quad f_2=0.2002$

$a_1=30,\quad f_1=0.3,\qquad a_2=15,\quad f_2=0.3002$

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$a_1=90,\quad f_1=0.9,\qquad a_2=45,\quad f_2=0.9002.$

The length of the series is $N=1000$ values. Let $\Delta t$ designate some arbitrary interval between successive readings. Then $T=N\ \Delta t$ is the record length. The above frequencies are in Nyquist units, $1/2\ \Delta t$. Thus the resolution coefficient equals

$$r=\Delta f\cdot T=.0002\ (1/2\ \Delta t)\cdot N\Delta t=0.1$$

for each of the 8 doublets, and it is assumed that the interaction between one doublet and any other doublet is negligible. Six independent sets of random numbers were generated, each having a mean square $v^2=500$. This mean square value is equally distributed among all frequencies between $-1/2\Delta t$ and $1/2\Delta t$, so that the spectral density equals

$$S(f)=\frac{v^2}{1/\Delta t}=v^2\Delta t=500\Delta t\,.$$

We then have according to (15)

$$\begin{aligned}<|\delta A_j|^2>&=\frac{3}{\pi^2}\frac{v^2\Delta t}{(\frac{1}{2}\cdot .0002N)^2N\Delta t}\\&=\frac{12}{\pi^2}\frac{v^2}{(.0002)^2N^3}=15\end{aligned}$$

Fig. 1 shows the computed values of $a_1$ and $a_2$ for each of the six independent noise sets. For zero noise the computed values should be along the heavy lines $a_1$ and $a_2$ in accordance with the assumed values. The presence of noise introduces a scatter. The computed rms departures $\pm\sqrt{15}$ from the assumed values are shown by the thin lines.

## 9. Tidal noise spectrum

Suppose the noise spectrum is due entirely to round-off error, tide gauges being read to the nearest "least count" ($lc$) of 0.1 feet. The mean-square error is $(1/12)\cdot(lc)^2$. This is distributed equally in the frequency range $\pm 1/2\Delta t$, with $\Delta t=1/24$ day designating the interval between readings. The round-off spectrum is accordingly

$$S(f)=\frac{(1/12)\cdot(lc)^2}{1/\Delta t}=.032\ \mathrm{cm^2/cpd}$$

where "cpd" is "cycles per day". Suppose we are attempting to observe monthly splitting from a 3-day record. Then

$$T=3\ \text{days},\qquad r=\Delta f\cdot T=\frac{1}{30}\cdot 3=0.1$$

$$<|\delta A|^2>=0.33\ \mathrm{cm^2}$$

[2] In engineering practice, the definition $<x^2(t)>=\int_0^{\infty}S_E(f)df$ is customary, whereas here $<x^2(t)>=\int_{-\infty}^{\infty}S(f)\ df$, so that $S(f)=\frac{1}{2}S_E(f)$.



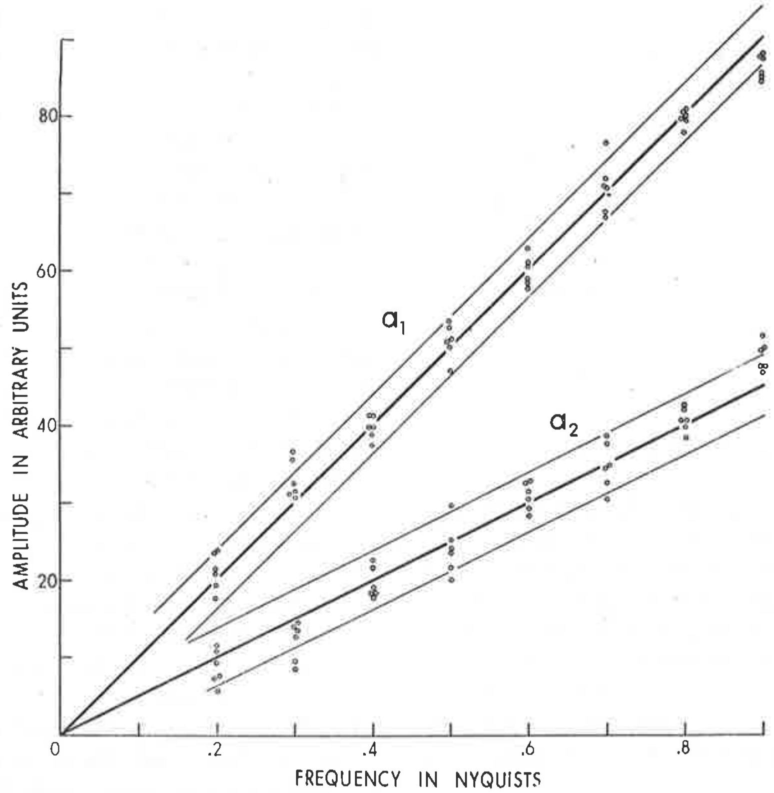


Fig. 1. The dots designate computed amplitudes of doublets $a_1\cos 2\pi f_1t+a_2\cos 2\pi f_2t$ at frequencies .2, .3, ..., .9 Nyquists. In each case $f_2-f_1=.0002$ Nyquists. We arbitrarily chose $a_1=100\,f$ and $a_2=50\,f$.

or a rms error of 0.6 cm. The amplitude is of the order of 20 cm, and some meaningful measure of monthly splitting can then be attained from a 3-day record.

But in fact the geophysical noise level (due to atmospheric excitation) far exceeds the instrumental noise level. Munk and Bullard[3] (1963) estimate $S(f)=1\ \mathrm{cm^2/cpd}$ at tidal frequencies. With this higher value of noise level, $<|\delta A|^2>=10\ \mathrm{cm^2}$, and the signal-to-noise power reduces to $400\ \mathrm{cm^2}/10\ \mathrm{cm^2}=40:1$.

## 10. Tidal cusps

The previous estimate of $1\ \mathrm{cm^2/cpd}$ is based on measurements of the noise spectrum well to one side or the other of the tidal line clusters. Within the line cluster the spectrum rises sharply, apparently as a result of non-linear interaction between the lines themselves and the rising noise spectrum near "zero" frequency. This leads to "cusps" in the noise spectrum at the strong lines. Let

$$C_j^2=\int_{f_j-\delta}^{f_j+\delta}S(f)df$$

designate the energy in the cusp centered on $f_j$. The approximation is now the reverse of that leading to (15). There we assumed that $S(f)$ was nearly constant over the range of integration and could be taken outside the integral sign. Now we shall assume that $\Phi_j(f)$ can be taken outside:

$$\begin{aligned}<|\delta A_1|^2>&=|\Phi_1(f_1)|^2\int_{f_1-\delta}^{f_1+\delta}S(f_1)df\\&\quad+|\Phi_1(f_2)|^2\int_{f_2-\delta}^{f_2+\delta}S(f)df\\&=|\Phi_1(f_1)|^2C_1^2+|\Phi_1(f_2)|^2C_2^2\end{aligned}$$

[3] Due to a numerical error the published value is 0.1 cm²/cpd.

But

$$\frac{d}{dx}\frac{\sin x}{x}=-\frac{1}{3}x+\cdots$$

$$I'(f_1-f_0)=-\frac{1}{3}\pi(f_1-f_0)T+\cdots=+\frac{1}{6}\pi r+\cdots$$

$$I'(f_2-f_0)=-\frac{1}{3}\pi(f_2-f_0)T+\cdots=-\frac{1}{6}\pi r+\cdots$$

so that

$$\Phi_1(f_1)=1\ , \qquad \Phi_1(f_2)=0$$

$$\Phi_2(f_1)=0\ , \qquad \Phi_2(f_2)=1$$

plus smaller terms. The result is

$$<|\delta A_j|^2>=C_j^2 \tag{16}$$

so that the variance of the line estimate is limited by the noise energy in its own cusp.

Some estimates (unpublished) show that the noise spectrum rises to 300 $cm^2$/cpd within a band $\pm$ .01 cpd of the $M_2$ line. Very roughly the energy in the cusp is (300 $cm^2$/cpd) (.02 cpd)=6 $cm^2$, and $<|\delta A_j|^2>=6\,cm^2$ as compared to 10 $cm^2$ for a white noise $S(f_0)=1\,cm^2$. The uncertainties introduced by the cusps are of comparable magnitude to those introduced by the underlying white noise.

## 11. On tide prediction

The discovery of the tidal cusps separates tide prediction into two classes: (i) the short-range problem, and (ii) the long-range problem.

In the short-range problem the uncertainties associated with the tidal cusps can be largely removed. These spectral cusps have a simple interpretation in the time domain: they arise from the non-linear interaction of the tides with the fluctuating "mean **sea** level". Mean sea level may vary by 10 cm in a decade, and the tidal constants are altered by this variation. Suppose the problem is to predict the 1964 tides at some station, for which the tidal constants were determined in 1950. We can improve the prediction by allowing for the modification of the tidal constants due to the change in sea level between 1950 and 1964. The easiest way is to measure the 1963 sea level and assume that the 1964 sea level will be the same. A better method is to perform a Wiener-type prediction. With this additional effort (keeping track of the changing sea level) the prediction error can then be improved to the extent to which it is due to cusp energy (equation 16). But clearly this improvement is limited to such short ranges for which meaningful predictions can be made. This is of the order of $\delta^{-1}$ (about one year), where $\delta$ is the cusp width in the frequency domain.

Predictions beyond $\delta^{-1}$ constitute the long-range problem, and these incorporate the uncertainties associated with the cusps as well as the underlying flat noise spectrum. Such long-range predictions can be made many centuries in advance. Ultimately they are limited by the variable rotation of the Earth and the anomalies in the Moon's orbit.

### References

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