Supersymmetric Boundaries of One-Dimensional Phases of Fermions beyond Symmetry-Protected Topological States

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It has recently been demonstrated that protected supersymmetry emerges on the boundaries of one-dimensional intrinsically fermionic symmetry protected trivial (SPT) phases. Here we investigate the boundary supersymmetry of one-dimensional fermionic phases beyond SPT phases. Using the connection between Majorana edge modes and real supercharges, we compute, in terms of the bulk phase invariants, the number of protected boundary supercharges.

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Much recent interest has been paid to topological phases of matter and their classification. Simplest among them are invertible phases, which are essentially trivial in their bulks but host topologically protected phenomena on their boundaries [1]. The notion of topological phases may be enriched by considering systems invariant under a global symmetry and restricting deformations to symmetric ones. A large class of symmetry enriched invertible phases is given by symmetry protected trivial (SPT) phases—those which would belong to the trivial phase in the absence of the protecting symmetry [2,3]. Beyond SPT phases are invertible phases that remain topologically distinct even without symmetry.

A particularly interesting class of invertible phases is those of fermions in one dimension, which includes, for example, the topological superconductor, whose boundary Majorana modes distinguish it from the trivial superconductor [4]. The absence of noninvertible topological order in one dimension means that every one-dimensional indecomposable phase without symmetry breaking is invertible. The problem of classifying and characterizing these phases has been solved by Fidkowski and Kitaev [5]. Roughly speaking, for the group G_b of symmetries modulo fermion parity, phases are described by three topological invariants: group cochains $\alpha \in C^2(G_b; U(1))$ and $\beta \in C^1(G_b; \mathbb{Z}_2)$, subject to constraints and equivalences, and a value $\gamma \in \mathbb{Z}_2$. The invariant γ measures whether a phase supports an even or odd number of modes on each boundary. Since even numbers of modes may be gapped out by interactions that disrespect the symmetry, a value of $\gamma = 0$ indicates that the phase is SPT. The invariants α and β may also be understood in terms of protected features of the boundary physics.

One such feature is the projectivity of the symmetry action on the boundary. The 2-cochain α always measures the projectivity of the G_b action on the boundary, while the meaning of the 1-cochain β depends on the value of γ. For an SPT phase, $\beta(\bar{g})$ encodes whether the boundary action of a symmetry $\bar{g} \in G_b$ commutes or anticommutes with fermion parity. For a phase that is not SPT, $\beta(\bar{g})$ instead encodes commutation of the $\bar{g}$ action with the central fermionic boundary mode Γ.

While the bulk phase invariants constrain the projective action of the symmetry on the boundary, they do not fix the number and statistics of the boundary degrees of freedom on which the symmetry acts. For example, a system in the trivial phase may have zero energy degrees of freedom on its boundaries, yet it belongs to the same phase as the trivial system obtained by gapping out these degrees of freedom. This is true of $k = 2$ copies of the nontrivial class D Majorana chain or $k = 8$ copies of the class BDI Majorana chain, for example. For an example of a system with nontrivial order, consider a stack of $k = 4$ class BDI Majorana chains. This system has four Majorana zero modes on each of its boundaries, yet it belongs to the same phase as the system obtained by partially gapping out the boundaries in a way that leaves a Kramer's doublet of bosonic zero modes. Despite the collection of boundary modes of a system not being an invariant of the system's phase, it is still possible to make statements about *protected* modes. If the constraint imposed by the phase invariants on the projective action of the symmetry is such that a minimal collection of modes is necessary to realize the constraint, these modes will be present in every system belonging to the phase.

The perspective of characterizing a phase by its boundary degrees of freedom is suited to studying the supersymmetry that emerges on the boundary. Supersymmetry is the existence of parity-odd operators called supercharges that satisfy the relations of a supersymmetry algebra [6]. Supercharges on a zero-dimensional space are closely related to Majorana modes, and the supersymmetry algebra to their Clifford algebra [7]. Supersymmetry is especially interesting when it emerges on the boundaries of topological phases because its supercharges may be protected by bulk topological invariants, meaning the supersymmetry requires no fine-tuning. This occurs, for example, in a recent Letter of Prakash and Wang [8], which constructs two real supercharges on the boundary of some one-dimensional SPT phases and argues that they are protected by the SPT invariant β.

The purpose of the present Letter is to investigate the possibility of protected supersymmetry on the boundaries of one-dimensional fermionic phases beyond SPT phases. Ultimately, we are able to determine the number of protected supercharges as a function of the bulk phase invariants. These findings are stated as Result 1 and Result 2 in the text. We find examples of phases that protect arbitrarily many boundary supercharges. The case of class BDI superconductors is also discussed in detail.

Symmetries and invariants.—We begin by reviewing the symmetry groups and topological invariants of one-dimensional phases of fermions.

A fermionic symmetry group G_f has a central involution p called fermion parity. Centrality of p means there are no parity-odd symmetries in the bulk. In general, the extension $\mathbb{Z}_2^f \to G_f \xrightarrow{b} G_b$ of $G_b = G_f/\mathbb{Z}_2^f$ by $\mathbb{Z}_2^f = \{1, p\}$ does not split as a product group $G_f = G_b \times \mathbb{Z}_2^f$. However, in order for G_f to be realized as the symmetries of a fermionic system that is not SPT, it must split [5,9,10]. Whether a symmetry of G_b is represented unitarily or anti-unitarily is encoded by a map $x: G_b \to \mathbb{Z}_2^T$. The triplet (G_f, p, x) specifies the symmetry class.

Fermionic phases of symmetry class (G_f, p, x) have a classification, due to Ref. [5] (see also Refs. [9–11]) in terms of three invariants

$$\alpha \in C^2(G_b; U(1)), \qquad \beta \in C^1(G_b; \mathbb{Z}_2), \qquad \gamma \in \mathbb{Z}_2, \quad (1)$$

subject to certain constraints and equivalences. In the case of a product group, the invariants α and β represent classes in group cohomology twisted by the action where $\bar{g} \in G_b$ with $x(\bar{g})$ inverts the coefficient.

One way of understanding the invariants is in terms of the action of the symmetries on the algebra of operators on the boundary. Here we briefly review how this works, leaving detailed discussion to Refs. [9,10]. The invariant γ measures whether this algebra is of the form

$$A = \mathrm{End}(U_f) \qquad (\gamma = 0, \text{even}, \text{SPT}) \quad \text{or}$$
$$A = \mathrm{End}(U_b) \otimes C\ell(1) \qquad (\gamma = 1, \text{odd}, \text{not SPT}),$$

where U_f, U_b are vector spaces, End denotes the algebra of matrices on a space, and $C\ell(1)$ is the complex Clifford algebra with one generator. An algebra (and its corresponding system) is referred to as "even" when $\gamma = 0$ or "odd" when $\gamma = 1$. For example, the algebra generated by N Majorana modes is the Clifford algebra $C\ell(N)$, which is isomorphic to an even or odd algebra depending on whether N is even or odd. Systems belong to the same phase in the absence of symmetry if their algebras are related by $\mathrm{End}(U)$ factors [9]; this means that γ encodes whether a system is SPT. The symmetries act on the algebra as follows, according to the invariants:

$$\gamma = 0: g \cdot M = Q_f(g) M Q_f(g)^{-1} \quad (2)$$

$$\gamma = 1: \bar{g} \cdot M \otimes \Gamma^m = (-1)^{\beta(\bar{g})m} Q_b(\bar{g}) M Q_b(\bar{g})^{-1} \otimes \Gamma^m$$
$$p \cdot M \otimes \Gamma^m = (-1)^m M \otimes \Gamma^m, \quad (3)$$

where Q_f, Q_b are projective representations of G_f, G_b, and $g \in G_f$, $\bar{g} = b(g) \in G_b$. The cocycle measuring the projectivity of Q_b is simply α, while that of Q_f is a certain function of α and β [9,10], such that β may be extracted from Q_f as the phase in the commutator $PQ_f(g)P = (-1)^{i\pi\beta(\bar{g})} Q_f(g)$ of the actions of g and fermion parity $P = Q_f(p)$.

Supercharges from zero modes.— We now establish a connection between Majorana zero modes and real supercharges that has been noted, for example, in Ref. [12].

Consider a system with Majorana zero modes γ_i, not necessarily protected. Any set of N modes forms the Clifford algebra $C\ell(N)$. Diagonalize the Hamiltonian of the system as $H = \sum_\mu E_\mu \mathbb{1}_\mu$, where $\mathbb{1}_\mu$ projects onto the eigenspace labeled by μ. Since the modes have zero energy, they must commute with H and so also with the $\mathbb{1}_\mu$. We may define N real supercharges as

$$Q_i = \sum_\mu \sqrt{E_\mu}\, \mathbb{1}_\mu \gamma_i. \quad (4)$$

They satisfy the supersymmetry algebra

$$\{Q_i, Q_j\} = \sum_\mu E_\mu \mathbb{1}_\mu \{\gamma_i, \gamma_j\} = 2\delta_{ij} H. \quad (5)$$

Conversely, given an algebra of real supercharges Q_i, the Clifford algebra is recovered by the inverse to Eq. (4).

Now suppose the N modes are protected by virtue of living on the boundary of a nontrivial phase. This implies that the supersymmetry is protected as well.

When N is odd, fermion parity P can be used to construct an additional supercharge, so that the total number is even [13]; however, the boundary of an odd one-dimensional phase has no local fermionic parity, so the total number of *local* supercharges is still odd.

Counting protected supercharges.—We now investigate when a phase protects $\mathcal{N}$ Majorana zero modes on its boundary. As we just saw, this amounts to studying protected boundary supersymmetry of $\mathcal{N}$ real supercharges. The terms Majorana zero mode and real supercharge are henceforth used interchangeably.

Recall from a previous section that a fermionic system is characterized by an algebra A with a compatible action of the symmetries G_f. This data may be interpreted as the algebra of zero energy degrees of freedom on the boundary and how they transform under symmetry. The form of the algebra is constrained by the phase invariants: γ determines whether the algebra is of the form $\mathrm{End}(U)$ or $\mathrm{End}(U) \otimes C\ell(1)$, while α (and also β in the even case) detects the projectivity of the group action on U.

Given an algebra A, we ask for the largest N, denoted $N(A)$, such that A has a tensor factor decomposition $A = B \hat{\otimes} C\ell(N)$, where the hat over $\otimes$ reminds us to use the tensor product graded by fermionic parity, if neither factor is purely even. The number $N(A)$ represents the number of Majorana modes, or real supercharges, present on the boundary of the system. Since we are interested in fermionic rather than bosonic modes, we require that the $C\ell(N)$ factor is not purely even. If $C\ell(N)$ has at least one odd generator γ_j, any even generator may be replaced by an odd one by the graded isomorphism $\gamma_i' \sim \gamma_i \gamma_j$.

Two systems belong to the same phase if they have the same topological invariants, which is to say that their algebras satisfy the same constraints. We are interested in features protected by the phase; that is, in characteristics of the class $[\alpha, \beta, \gamma]$ of algebras compatible with given values the invariants. The absence or presence of the $C\ell(1)$ factor is a characteristic of the class because it is associated with γ. The invariants α and β provide more flexibility: there may be multiple distinct irreducible projective representations U with the same projectivity class, and so their algebras are associated with systems in the same phase. This means the particular $\mathrm{End}(U)$ factor is *not* a characteristic. In physical terms, a typical zero energy boundary degree of freedom of a system is not a protected feature of the phase, as only some of these are present in every system in the phase. Here we ask, given a phase, what the *protected* zero modes on its boundary are. This amounts to looking at all of the algebras A in the class and asking for the smallest value of $N(A)$:

$$\mathcal{N}(\alpha, \beta, \gamma) = \min_{A \in [\alpha, \beta, \gamma]} N(A). \tag{6}$$

In the following, we will compute the numbers $\mathcal{N}(\alpha, \beta, \gamma)$ of protected boundary Majorana zero modes. We begin with odd phases before turning to SPT phases.

Odd case.—Consider the algebra associated with a system in an odd phase. It has the form $A = M(d) \otimes C\ell(1)$, where $M(d)$ denotes the algebra of complex $d \times d$ matrices. The $C\ell(1)$ factor is generated by an odd zero mode Γ. Our question is whether there exist additional modes.

We claim that the number of odd zero modes is $N = 2k + 1$ where k is the largest whole number such that 2^k divides the degree d of the representation U.

To see this, let $l = d/2^k$ and note that

$$\begin{aligned} M(d) \otimes C\ell(1) &\simeq M(l2^k) \otimes C\ell(1) \\ &\simeq M(l) \otimes C\ell(2k+1). \end{aligned} \tag{7}$$

So far we have counted the modes on the boundary of the system associated with the algebra A. But we are interested only in the modes that are protected by the phase to which this systems belongs. Since the value d is the degree of an irreducible representation with projectivity class α, the number of protected modes, given abstractly by Eq. (6), is the following:

Result 1: The number of real supercharges $\mathcal{N}(\alpha, \beta, 1)$ protected by an odd phase with invariant α is exactly $2k + 1$, where k is the largest number for which 2^k divides d_α, the greatest common divisor of the degrees of the irreducible representations with projectivity class α.

We cannot present a general, explicit formula for $\mathcal{N}$ because no such expression for d_α is known; however, the mathematics literature contains some limited results that hold at least when G_b contains only unitary symmetries. Upper and lower bounds on $\mathcal{N}$ follow from the facts that d_α divides the order of G_b and is divided by the order of α in cohomology (cf. Ref. [14], corollary VI.3.10 and lemma VI.4.1). More can be said when G_b is a finite Abelian group. In this case, every irreducible representation with class α has the same degree $d_\alpha = \sqrt{|G|/|K_\alpha|}$, where K_α is the subgroup on which α is symmetric (cf. Ref. [14], theorem VI.6.6, and Ref. [15]).

For an example that is common in studies of symmetry-enriched phases, consider the group $G_b = \mathbb{Z}_n \times \mathbb{Z}_n$. Its irreducible projective representations are described by clock and shift matrices on spaces of dimension the order of α in $H^2(G_b; U(1)) = \mathbb{Z}_n$ [16]. Let $n = 2^k$ and take α to generate the cohomology group. Then $\mathcal{N}(\alpha, \beta, 1)$ is $2k + 1$. This class of examples can be used to obtain an arbitrarily large number of protected supercharges.

Another important case is odd phases with trivial α, for which the number $\mathcal{N}(0, \beta, 1)$ is exactly 1. To see this, note that if α is trivial, it represents the projectivity class of the trivial representation, which has degree $d = 1$ and so $k = 0$, corresponding to one real supercharge. Since this mode is protected (as in any odd phase) and is the only mode in one system (the one with trivial representation), it is the only protected mode for the phase.

Let us also illustrate how our formula applies to phases of symmetry class BDI—that is, $G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^f$. The construction of supercharges for each $\nu = 0, \ldots, 7$, has been carried out in Ref. [12], and we recover their counting. Using the dictionary, given in Refs. [5,10], between the number of layers ν and the invariants α, β, γ, our formalism recovers this count of supercharges: (i) In the phases with $\gamma = \nu \bmod 2 = 1$, there is at least one protected supercharge. (ii) When $\nu = 1, 7$, α is trivial and so there are no more protected supercharges. We have a total of $\mathcal{N} = 1$. (iii) When $\nu = 3, 5$, α is nontrivial and the smallest irreducible projective representation (the Kramer's doublet) has degree $d = 2$, for a total of $\mathcal{N} = 3$. We also could have recovered this counting from looking at the "minimal" algebras given by the real superdivision algebras: see the table in Ref. [10], and note that $C\ell_{1,0}\mathbb{R}$ and $C\ell_{0,1}\mathbb{R}$ each have a single generator while $\mathbb{H} \otimes C\ell_{0,1}\mathbb{R}$ and $\mathbb{H} \otimes C\ell_{1,0}\mathbb{R}$ each have three.

Even case.—Consider the algebra associated with a system in an SPT phase. It has the form $A = M(a|b)$, where $M(a|b)$ denotes the graded algebra of matrices on the graded vector space $\mathbb{C}^{a|b}$.

We begin by arguing that, since the projective action of G_f on $U = \mathbb{C}^{a|b}$ is irreducible (by assumption), the grading is either purely even $b = 0$ (when β is trivial) or equal $a = b$ (when β is nontrivial). Recall the interpretation of the invariant β in an SPT phase as measuring whether a symmetry acts on $U = \mathbb{C}^{a|b}$ as an odd (invertible) operator. Note that the grading is equal precisely when such an operator exists. Therefore, a nontrivial β implies that the grading is equal. On the other hand, consider the case where β is trivial, meaning the symmetry acts as even operators, i.e., within the even-even and odd-odd blocks. By irreducibility, one of these (up to isomorphism, the odd-odd block) must vanish; therefore, the grading is purely even. This proves our claim.

Next observe that $M(a|a) \simeq M(a) \otimes C\ell(2)$. This means that, if $M(a|b)$ has equal grading $a = b$, it has at least two odd modes. Also note that, for $k > 0$,

$$\begin{aligned} M(c|d)\hat{\otimes}C\ell(2k) &\simeq M(c|d)\hat{\otimes}M(2^{k-1}|2^{k-1}) \\ &\simeq M\big((c+d)2^{k-1}|(c+d)2^{k-1}\big). \end{aligned} \quad (8)$$

This implies the converse: that, if $M(a|b)$ has unequal grading $a \neq b$, it contains no odd modes.

From this we may conclude the following, which was first proved by Prakash and Wang:

Lemma: (cf. Ref. [8]).—If β of an SPT phase is trivial, the number of protected real supercharges $\mathcal{N}(\alpha, 0, 0)$ is 0, while, if β is nontrivial, $\mathcal{N}(\alpha, \beta, 0)$ is at least 2.

Next, we state an SPT analog of Result 1. As discussed briefly in a previous section and more extensively in Refs. [9,10], the projectivity class of the group action on U is a certain lift ω of α to from G_b to G_f that combines the data of α and β. In terms of this class ω, we have

Result 2: The number of real supercharges $\mathcal{N}(\alpha, \beta, 0)$ protected by an SPT phase with invariants α, β is 0 if β is trivial; if β is nontrivial, it is exactly $2k$, where k is the largest number for which 2^k divides d_ω, the greatest common divisor of the degrees of the irreducible representations with projectivity class ω.

The proof uses the reasoning from the odd case. If β is nontrivial, a $C\ell(2)$ factor splits off of the algebra. Then

$$M(a|a) \simeq M(l) \otimes C\ell(2m+2), \quad (9)$$

where $a = l2^m$. The degree d of the representation is $2a$, so $d = l2^k$ for $k = m - 1$. This means that the number of supercharges is $2k$, where k is the largest number for which 2^k divides the degree of the representation. To complete the argument, recall Eq. (6), which says that the number of protected supercharges is the minimum of the number of supercharges over all systems in the phase.

As an example, consider $G_f = \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$. By the argument presented in the odd case, we see that the number of protected supercharges $\mathcal{N}(\alpha, \beta, 0)$ is $2k$ for any phase with α, β such that ω generates the cohomology group. As before, this class of examples can be used to obtain an arbitrarily large number of protected supercharges.

For SPT phases with trivial α, the number $\mathcal{N}(0, \beta, 0)$ is 0 if β is trivial and exactly 2 if β is nontrivial. To see this, note that if α is trivial, the only projectivity in the action of G_f comes from commutators of $Q_f(g)$ with P. Therefore, P may be represented as σ_z and each $Q_f(g)$ with $\beta(\bar{g}) = 0, 1$ as $\mathbb{1}, \sigma_x$, respectively. This representation has degree $d = 2$, so the number of protected supercharges is at most 2. Applying the lemma completes the proof.

Let us look again at the case of class BDI, this time for the even phases. We recover the counting of Ref. [12]: (i) When $\nu = 0, 4$, β is trivial and so there are no protected supercharges. (ii) When $\nu = 2$, β is nontrivial while α is trivial, so there are $\mathcal{N} = 2$ protected supercharges. (iii) When $\nu = 6$, both β and α are nontrivial. These are compatible with a representation of degree $d = 2$ (given by the real graded algebra $C\ell_{0,2}\mathbb{R}$), so there are still only $\mathcal{N} = 2$ protected supercharges. Again, we could have looked at real superdivision algebras, where $\mathbb{R}$ and $\mathbb{H}$ are purely even while $C\ell_{2,0}\mathbb{R}$ and $C\ell_{0,2}\mathbb{R}$ each have two odd generators.

Summary and outlook.—We have found that supersymmetry emerges without fine-tuning on the boundaries of a broad class of one-dimensional symmetry-enriched phases of fermions—both SPT and beyond. The lack of fine-tuning is by virtue of its protection by the topological invariants (α, β, γ) that characterize the bulk phase. For each phase, we computed the number $\mathcal{N}(\alpha, \beta, \gamma)$ of protected real supercharges. Our results extend the recent discovery that intrinsically fermionic SPT phases support at least $\mathcal{N} = 2$ protected boundary supersymmetry [8].

This work opens a line of investigation into the consequences of the emergent supersymmetry for the Sachdev-Ye-Kitaev (SYK) models that arise on the boundaries of one-dimensional fermionic phases that have been many-body localized. The quantum chaotic eigenspectra of these models have been shown to encode information about the bulk topological invariants [17]. In the setting of bulk phases of symmetry class BDI, which support the eightfold way of SYK models on their boundaries, this feature of the spectra and related properties of dynamical correlation functions were shown to be constrained by supersymmetry [12]. Our results raise the possibility of understanding the connection between supersymmetry and these phenomena in a much broader class of phases. It would also be interesting to study how our work generalizes to phases of quantum matter in higher dimensions.

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