COMPLEX Final Scientific Report, Volume 2
Non-linearities and System-Flips

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With contributions from the COMPLEX Consortium

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C O M P L E X

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7. Lake Systems

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Nature is often expected to respond to gradual changes in a smooth way. However, studies of lakes, coral reefs, oceans, forests and arid lands have shown that smooth change can be interrupted by sharp (or catastrophic) shifts to different regimes (Scheffer *et al.*, 2001; Carpenter, 2003). One of the best-studied catastrophic shifts is the sudden loss of transparency and vegetation observed in shallow lakes, i.e. lakes with a depth less than 3 meters, as a result of human activities. Initially shallow lakes have clear water and a rich submerged vegetation. However, nutrient loading may change this. For instance nutrients arrive in the lake as a result of the use of artificial fertilizers on surrounding land; they are washed into the lake by rainfall.

Due to heavy use of fertilizers, at some point lakes might flip from a clear state to a turbid state that is caused by a dominance of phytoplankton. Lakes are hard to restore to the clear water state in the sense that the nutrient loads have to be reduced far below the level where the flip occurred before the lake returns to a clear state. In this case the lake is said to show *hysteresis*. In some cases the turbidity of the lake is even irreversible. The positive feedback through the effect on the submerged vegetation is one explanation for this *hysteresis effect*. The critical points at which the system flips (shifts) in a way that is not instantly reversible (or irreversible) are called *tipping points*.

The lake model that is used in this study gives a very simplified representation of these complex ecological feedback mechanisms that are active in a shallow lake. Indeed, the lake model in this study should be viewed as a metaphor for general ecological systems with tipping points, thresholds, nonlinearities, and irreversibilities so that the analysis developed here will have a wider applicability (cf. Scheffer, 2009).

Lake Dynamics

The dynamics of a lake, which was described above, can be modeled as a single non-linear difference equation

$$x_t = u_t + (1 - \mathrm{b})x_{t-1} + h(x_{t-1})$$

Here, x_t is the concentration of phosphorus, one of the main nutrients, in the lake. Artificial fertilisers containing phosphorus are used on the fields surrounding the lake. The phosphorus is washed into the lake by rainfall, yields a net inflow, u_t of phosphorus. The parameter, b denotes the sedimentation rate at which phosphorus leaves the water column and enters the sediment at the bottom of the lake. The last term models the internal production of phosphorus in the lake, e.g. through re-suspension of the sediment, and is assumed to be an S-shape function that has its inflection point at the point,

$x = 1$:

$$h(\mathrm{x}) = \frac{x^q}{1 + x^q}$$

The exponent, q, the *responsiveness* of the lake, is proportional to the steepness of $h(x)$ at $x = 1$. Thus, steeper function h (resulting from higher q values) creates stronger *hysteresis.*

For a constant pollution loading, $u_t = u$ for all t, the fixed points of the lake are solutions of the equation

$$u = g(x) = b\,x - \frac{x^q}{1+x^q}$$

which is illustrated for $b = 0.6$, $q = 2$ and $q = 4$ in Fig. 7.1.

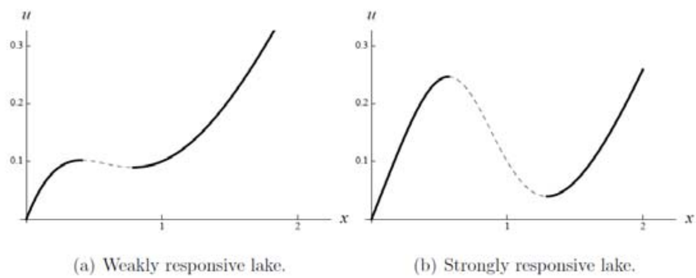


***Figure 7.1**. Location of fixed points for constant pollution streams $u_t = u$ for all t, plotted for b= 0.6 , and for (a) weakly (q = 2) and (b) strongly responsive lakes (q = 4). Indicated are stable (solid) and unstable fixed points (dashed).*

For both values of q there is a range of u-values such that there are multiple steady states. However, the range is bigger for $q = 4$ than for $q = 2$. If the system starts in a low pollution steady state, and if u is then raised very slowly past the first critical value it switches to a high pollution steady state. A small subsequent decrement of u will not move the system back to the clean branch of steady states. For this, the pollution flow has to be lowered significantly, below the second critical value.

There is a value, $b = b^*$, such that for $b < b^*$ the lake can be trapped in the high pollution steady state of phosphorus. This happens if the first flip, which occurs at $u = \bar{u}$, is irreversible. The critical value is $b \approx 0.57$ for $q = 4$ and $b = 0.5$ for

$q = 2$ (see Fig. 7.2 for the case $q = 4$). In that case, only after a change in the value of b the lake can be restored to a clear state.

The sedimentation parameter is set the sequel of this section, it is assumed that the $b = 0.6$ so that the lake displays hysteresis but a flip to a low pollution steady state is reversible.

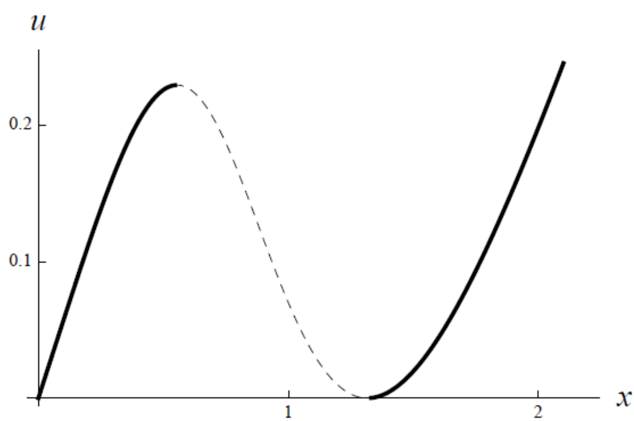


Figure 7.2. *Irreversibility; location of fixed points for constant pollution streams $u_t = u$ for all t, plotted for $b = 0.57$ and $q = 4$. Indicated are stable (solid) and unstable fixed points (dashed).*

Economics of lakes: optimal pollution management

In the ecological literature, management of shallow lakes is mostly interpreted as preventing the lake to flip or, if it flips, as restoring the lake in its original state. However, in this study the economics of the problem is analyzed in the sense of the trade-offs between the utility of the agricultural activities, which are responsible for the release of phosphorus, and the utility of a clear water lake (cf. Maler *et al.*, 2003). When the lake flips to a turbid state, the value of the ecological services of the lake decreases, but there is a high level of agricultural activities. It depends, of course, on the relative weight attached to these welfare components whether it is better to keep the lake clear or to use it as a waste dump.

Note that if it is better to keep the lake clear, it is very costly to let the lake flip first because of the *hysteresis*. The *complexity* of the lake optimal management problem derives from the *non-linear dynamics* of the lake that leads to a non-convex optimal control problem featuring several system parameters. The lake optimal management problems therefore have a rich structure that is the existence of *tipping points*. In such problems, depending on the values of these parameters, there may exist multiple steady states that are the long-run outcome of an optimal management policy. Also, the structure of optimal solutions may change if parameters are varied. In this study the bifurcation analysis developed Moghayer (2012) is used to classify the qualitative characteristics of the set of optimal solutions for different values of the model parameters.

In the lake pollution management problem, a social manager has to weigh the interest of the farmers that derive income from the use of artificial fertilizers against that of the lake users that suffer from pollution damage to the lake. Following Maler *et al.* (2003), the social utility functional is modelled as

$$J = \sum_{1}^{\infty} \left(\log u_t - c x_{t-1}^2 \right) e^{-\rho t}$$

Here, c is the social preference parameter, and $\rho > 0$ the discount rate. The social manager tries to optimally manage the phosphorus pollution stream

$$\mathbf{u} = \{\mathrm{u}_t\}_{t=1}^{\infty}$$

that originates from the use of artificial fertilisers given that the concentration of x_t phosphorus in the lake follows the lake dynamic. The optimization problem is to maximize

$$J = \sum_{1}^{\infty} \left(\log u_t - c x_{t-1}^2 \right) e^{-\rho t}$$

subject to the lake dynamic

$$x_t = u_t + (1 - \mathrm{b}) x_{t-1} + \frac{x^q}{1 + x^q}$$

State space and control space are given as X = U = (0,∞), respectively.

The discrete Pontryagin function is

$$P = \log u - c x^2 + y \left(u + (1 - \mathrm{b})\, \mathrm{x} + \frac{x^q}{1 + x^q} \right),$$

Where y is the co-state. The necessary condition

$$\frac{\partial P}{\partial u}=0$$ takes the form $$0=\frac{\partial P}{\partial u}=\frac{1}{u}+y$$

Solving for u yields that u = -1/y. Substituting out u, the discrete Hamilton function is obtained as

$$\mathrm{H}=-\log(-\mathrm{y})-\mathrm{cx}^2+y\left((1-\mathrm{b})\,\mathrm{x}+\frac{x^q}{1+x^q}\right)$$

The necessary conditions read as

$$x_t=\frac{\partial H}{\partial y}=-\frac{1}{y_t}+(1-\mathrm{b})\,\mathrm{x}_{t-1}+\frac{x_{t-1}^q}{1+x_{t-1}^q}$$

$$e^{\rho}y_{t-1}=\frac{\partial H}{\partial x}=-2cx_{t-1}+y_t\left((1-\mathrm{b})+\mathrm{q}(\frac{x_{t-1}^{q-1}}{1+x_{t-1}^q})^{\wedge 2}\right)$$

Solving the second equation for y_t and substituting into the first yields the phase map, which

determines the state-costate dynamics by:

$$\mathrm{z}_t=(x_t,y_t)=\varphi(x_{t-1},y_{t-1})=\varphi(\mathrm{z}_{t-1})$$

where

$$\varphi(\mathrm{x},\mathrm{y})=\left(-\frac{g'(\mathrm{x})}{e^{\rho}y+2cx}+g(\mathrm{x}),\frac{e^{\rho}y+2cx}{g'(\mathrm{x})}\right)$$

with

$$g(\mathrm{x}):=(1-b)\,\mathrm{x}+\frac{x^q}{1+x^q}\,.$$

Solution structure and qualitative changes

In the rest of the section, the value of ρ is fixed to ρ = 0.03. For $b = 0.6$ and $q = 4$, in Fig. 7.3(f) fixed points and their stable and unstable manifolds are plotted for a range of values of the cost parameter, c; for all these values, the phase map has two saddle fixed points Z_- and Z_+.

Recall that the stable manifold of a fixed point $\bar{z}$, $W^s_{\bar{z}}$, is the set of all points whose forward iterates converge to $\bar{z}$:

$$W^s_{\bar{z}} = \{z \in \mathbb{R}^2 : \lim_{t \to +\infty} \varphi^t(z) = \bar{z}\}$$

Analogously the unstable manifold of $\bar{z}$, $W^u_{\bar{z}}$, consists of the points backward asymptotic to $W^s_{\bar{z}}$:

$$W^u_{\bar{z}} = \{z \in \mathbb{R}^2 : \lim_{t \to -\infty} \varphi^t(z) = \bar{z}\}$$

In Moghayer and Wagener (2008) it is shown that for every $x_0 \in \mathbb{R}$, the problem to optimise

$$J = \sum_1^{\infty} \left(\log u_t - c x_{t-1}^2 \right) e^{-\rho t}$$

subject to the lake dynamic

$$x_t = u_t + (1 - b) x_{t-1} + \frac{x^q}{1 + x^q}$$

has a solution. Moreover, the state-co-state trajectory of such a solution is either on $W^s_{z_+}$ or $W^s_{z_-}$.

Fig. 7.3(d) shows the bifurcation diagram of the lake system in the (b,c)-parameter space. The dashed curve represents saddle-node bifurcations, separating the region of values of the parameters for which the phase map has a fixed point from the region of multiple fixed points. Solid lines indicate indifference-attractor bifurcation curves, separating three parameter regions: *low pollution* region for which the clean steady state is globally optimal, (ii) the *high pollution* region for which the turbid steady state is globally optimal, and (iii) the *dependent on the initial state* region for which both the clean steady state and turbid steady state are locally optimal.

For the values of the physical parameters b and economic parameter c in the *unique equilibrium* region the phase map φ has a unique fixed point. This is a saddle, see Fig. 7.3(a). The long run pollution level depends then on the values of the parameters c and b, changing within the region. If the pair (b, c) corresponds to a point of the *dependent on the initial state* region, the phase map φ has always two saddle fixed points characterized by respectively low pollution and high pollution (see Fig. 7.3(c))***Figure 7.3***. The clear state of the lake corresponds to a high level of water services and a low level of agricultural activities, whereas the turbid state corresponds to a high level of agricultural activities and a low level of water services. Depending on the initial pollution load, the social planner steers the lake to the clear or to the turbid steady state. If the pair (b, c) is in the *low pollution* region the optimal policy steers the lake to the clean steady state independently of the initial state of the lake; the clear state of the lake is globally optimal (see Fig. 7.3(a)***Figure 7.3***. If (b, c) is in the *high pollution* region, see Fig. 7.3(e &f), the optimal orbit lies on the stable manifold of the polluted equilibrium. Regardless of the initial state of the lake, the optimal policy steers the lake to the turbid state, that is the turbid steady state is globally optimal.

For a pair (b, c) in the *dependent on the initial state* region, there exist an indifference threshold, see Fig. 7.3(c)). If the initial state is below the threshold then the clean steady state is optimal, whereas if the initial state is above the threshold then the turbid steady state is optimal. Therefore, for a pair (b, c) in the *dependent on the initial state* region the lake is steered to the clear state only if it is initially not very polluted, otherwise it is steered to the turbid state. Note that at the *indifference threshold*, two different policies are radically different and non-equivalent, one corresponding to high agricultural activity, high pollution and convergence to the polluted steady state, whereas the other is characterized by lower pollution and convergence to clear steady state.

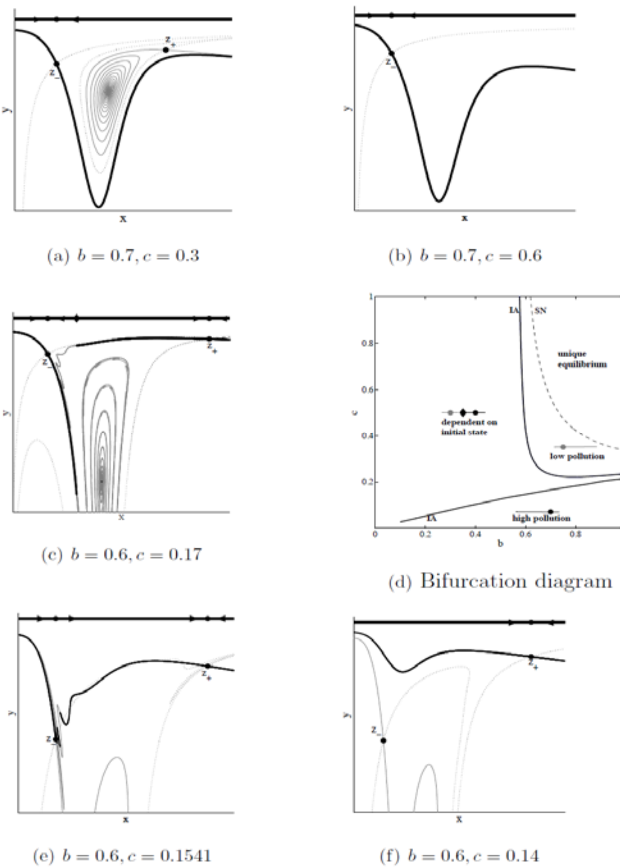
Conclusion

In this section, outcomes of the lake pollution problem and the long-term interest conflicts of the lake users have been presented, in the context of dynamic social planning. A characteristic feature of this problem, and of pollution problems in general, is the qualitative dichotomy in possible outcomes in the presence of tipping points: the lake (or the ecosystem, or the climate) ends up in either a clean or in a polluted state, both of which, if attained, is stablised by some kind of feedback mechanism.

This results to a qualitative aspect in socio-economic outcomes: the decision maker has to decide for or against production, for or against conserving the ecosystem. This qualitative distinction between the possible socially optimal outcomes enables us

to present the outcome of the analyses in the form of a bifurcation diagram presented in the last subsection, which gives a graphical overview of the qualitative characteristics of the solutions, depending on the parameters of the problem. The most critical region in these bifurcation diagrams is the "history dependent" region: in these cases, neglect by an actual decision maker that allows the ecosystem to flip can lead to large irrecoverable welfare losses.

The lake pollution problem is a prototype of a non-linear ecological-economic problem with multiple equilibria, thresholds, and irreversibility. Indeed, lake system, as mentioned in Scheffer (2009) is "a subtle twist on the Greek's view of our mind that is Mikos-Kosmos reflecting the entire world". It is also extensive enough to harbor many scales of complexity therefore served our purpose to present it as an illustration example that covers most of the definition, concepts and notions which were discussed in the previous sections of this report.



(a) $b = 0.7, c = 0.3$ (b) $b = 0.7, c = 0.6$

(c) $b = 0.6, c = 0.17$ (d) Bifurcation diagram

(e) $b = 0.6, c = 0.1541$ (f) $b = 0.6, c = 0.14$

Figure 7.3. *Plot (d) shows the bifurcation diagram of the lake system. The dashed curve represents saddle-node bifurcations, separating the region of values of the parameters for which the phase map has a fixed point from the region of multiple fixed points. Solid lines indicate indifference-attractor bifurcation curves Solid lines indicate stable manifolds, dotted lines unstable manifolds; optimal solutions are marked by thick lines; the vertical line through the indifference threshold is dashed. At the top of the graph, the optimal dynamics are indicated; attractors are marked by a circle, the indifference threshold by a diamond.*