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On members of Lucas sequences which are products of Catalan numbers

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Abstract

We show that if $\{U_n\}_{n\ge 0}$ is a Lucas sequence, then the largest n such that $|U_n| = C_{m_1}C_{m_2}\cdots C_{m_k}$ with $1 \le m_1 \le m_2 \le \cdots \le m_k$, where C_m is the mth Catalan number satisfies $n < 6500$. In case the roots of the Lucas sequence are real, we have $n \in \{1,2,3,4,6,8,12\}$. As a consequence, we show that if $\{X_n\}_{n\ge 1}$ is the sequence of the X coordinates of a Pell equation $X^2-dY^2 = \pm 1$ with a nonsquare integer $d > 1$, then $X_n = C_m$ implies $n = 1$.

1 Introduction

Let $r,\ s$ be coprime nonzero integers with $r^2 + 4s \neq 0$. Let $\alpha,\ \beta$ be the roots of the quadratic equation $\lambda^2 - r\lambda - s = 0$ and assume without loss of generality that $|\alpha| \geq \beta|$. We assume further that α/β is not a root of 1. The Lucas sequences $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ of parameters (r, s) are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad V_n = \alpha^n + \beta^n \qquad \text{for all} \qquad n \geq 0.$$

Alternatively, they can be defined recursively as

$$U_{n+2} = rU_{n+1} + sU_n \quad \text{and} \quad V_{n+2} = rV_{n+1} + sV_n \qquad \text{for all} \quad n \geq 0$$

with initial conditions $U_0 = 0,\ U_1 = 1,\ V_0 = 2,\ V_1 = r$. In case when $r = s = 1$, we get $U_n = F_n$, the nth Fibonacci number. Let

$$B_m := \binom{2m}{m} \quad \text{and} \quad C_m := \frac{1}{m+1}\binom{2m}{m} \quad \text{for} \quad m \geq 0,$$

be the middle binomial coefficient and Catalan number, respectively. For each m, we write D_m for one of the numbers B_m, C_m. Let

$$\mathcal{P}BC := \{\pm \prod_{j=1}^{k} D_{m_j} : D_m \in \{B_m, C_m\},\ k \geq 1,\ 1 \leq m_1 \leq m_2 \leq \cdots \leq m_k\}$$

be the set of integers which are products of middle binomial coefficients and Catalan numbers. Diophantine equations with members of $\mathcal{P}BC$ have been studied before. For example, in [6], the authors characterised all nontrivial solutions of the system of two equations

$$\sum_{i=1}^{n} ip_i = \sum_{j=1}^{r} jq_j \quad \text{and} \quad \prod_{i=1}^{n} B_i^{p_i} = \prod_{j=1}^{r} B_j^{q_j}.$$

This system of equations arose naturally from a question in topology concerning n-dimensional complexes which do not embed in $\mathbb{R}^{2n}$ and characterising non-homotopic pairs of such with the same homology. In [7], it was shown that the largest positive integer solution (n, m) of the Diophantine equation

$$F_n = C_m$$

is $(n, m) = (5, 3)$. In [8], it is shown that if $\{u_n\}_{n\geq 0}$ is any nondegenerate binary recurrence of integers, then the equation $u_n = B_m$ has only finitely many positive integer solutions (n, m). Inspired by these problems, we study here the Diophantine equation obtained by imposing that a member of the Lucas sequences U_n or V_n is a product of middle binomial coefficients of Catalan numbers.

Our theorem is the following.

Theorem 1. *For each m, let $D_m \in \{B_m, C_m\}$. The equation*

$$(1)\quad U_n = \pm D_{m_1} D_{m_2} \cdots D_{m_k}, \quad \textit{where} \quad k \geq 1 \quad \textit{and} \quad 1 \leq m_1 \leq \cdots \leq m_k,$$

implies $n < 6500$ if n is odd and $n \leq 720$ if n is even. Further when α, β are real, then $n \in \{1, 2, 3, 4, 6, 8, 12\}$.

The equation

$$(2)\quad V_n = \pm D_{m_1} D_{m_2} \cdots D_{m_k}, \quad \textit{where} \quad k \geq 1 \quad \textit{and} \quad 1 \leq m_1 \leq \cdots \leq m_k,$$

implies $n < 6500$ and $4 \nmid n$. Further, when α, β are real, then $n \in \{1, 2, 3, 6\}$.

Note that $U_1 = 1 \in \mathcal{P}BC$. For this reason, whenever we look at equation (1), we omit $n = 1$ and assume $n \geq 2$.

We present a corollary regarding X-coordinates of Pell equations which are in $\{C_m, D_m\}$. For a positive integer d which is square-free, let (X_n, Y_n) be the n-th solution of the Pell equation $X^2 - dY^2 = \pm 1$ in positive integers (X, Y)(solution of either $X^2 - dY^2 = 1$ or $X^2 - dY^2 = -1$, not separately). Arithmetic properties of the coordinates X or Y of Pell equations have been studied before. For example, values of n such that X_n is a square have been studied by Ljunggren [5]. He proved that there are at most two such values of n. This was improved later in [11] where it was shown that in fact there is at most one such n except for $d = 1785$, for which both X_1 and X_2 are squares. in [3], a similar result was proved for X_n being a product of factorials. We supplement this with the following result on values of X_n which are in $\{C_m, B_m\}$.

Theorem 2. *Let (X_n, Y_n) be the nth solution in positive integers of the equation $X^2 - dY^2 = \pm 1$ for some squarefree integer d. Then $X_n \in \{C_m, B_m\}$ implies $n = 1$. Similarly, let (W_n, Z_n) be the nth solution in positive integers of the equation $W^2 - dZ^2 = \pm 4$ for some squarefree integer d. Then $W_n \in \{C_m, B_m\}$ implies $n \in \{1, 3\}$ or $n = 2$ with*

$$\begin{aligned}
&d = 2, W_2 = B_2 = 6: \quad 6^2 - 2 \cdot 4^2 = 4, \text{ where } (W_1, Z_1) = (2, 2);\\
&d = 2, W_2 = C_4 = 14: \; 14^2 - 2 \cdot 10^2 = -4, \text{ where } (W_1, Z_1) = (2, 2);\\
&d = 3, W_2 = C_4 = 14: \; 14^2 - 3 \cdot 8^2 = 4, \text{ where } (W_1, Z_1) = (4, 2).
\end{aligned}$$

We believe that there are only finitely many solutions of (1) such that $n \in \{6, 8, 12\}$ regardless of whether α, β are real or complex conjugates, which we are not able to prove. Also we conjecture that there are only finitely many solutions of (2) with $n \in \{3, 6\}$. Recently, the three of us proved similar theorems for members of Lucas sequences $U_n,\ V_n$ which are products of factorials in [3]. The current paper is much inspired by the method of the paper [3].

We give the proof of Theorem 1 in Section 4 and the proof of Theorem 2 in Section 5. Throughout the paper, we use $P(n), \mu(n)$ and $\varphi(n)$ with the regular meaning as being the largest prime factor of n, the Möbius function of n and the Euler phi function of n, respectively. All the computations in this manuscript were carried out in SageMath.

2 Preliminaries

Let n_0 be a positive integer. For an integer ℓ, define

$$M_{n_0}(\ell) := \log\left(\prod_{\substack{p^{\nu_p}\|\ell \\ p\equiv\pm1 \pmod{n_0}}} p^{\nu_p}\right) = \sum_{\substack{p^{\nu_p}\|\ell \\ p\equiv\pm1 \pmod{n_0}}} \nu_p \log p. \tag{3}$$

We prove a number of results to estimate lower and upper bounds for $M_{n_0}(U_n)$ and $M_{n_0}(V_n)$ for some divisors n_0 of n.

To recall the terminology, we take coprime nonzero integers r, s with $r^2+4s \neq 0$ and let α and β be the roots of the equation $\lambda^2 - r\lambda - s = 0$. For $n \geq 0$, we have

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$$

We suppose that α/β is not a root of unity. We assume without loss of generality that $|\alpha| \geq |\beta|$. Further, we may replace (α, β) by $(-\alpha, -\beta)$ if needed. This replacement changes the pair (r, s) to $(-r, s)$, while $|U_n|$ and $|V_n|$ are not affected and hence the values of $M_{n_0}(|U_n|)$ and $M_{n_0}(|V_n|)$ for any divisor n_0 of n. Thus, we may assume that $r > 0$. When α, β are real, these conventions imply that α is positive so $\alpha > |\beta|$. Further, in this case $U_n > 0$ and $V_n > 0$ for all $n \geq 1$.

We begin by proving a lower bound for $M_{n_0}(U_n)$ and $M_{n_0}(V_n)$ for some divisors n_0 of n. Throughout the paper, we use $x := \beta/\alpha$.

Lemma 1. *Let n be a positive integer and $p < p_1$ be distinct primes and $t \geq 0, h > 0, h_1 > 0$ be integers. Let $n_0 \in \{p^h, p^h p_1^{h_1}\}$, $n_0 > 4, n_0 \notin \{6, 12\}$*

be such that $n_0p^t \mid n$. *Then*

(4)
$M_{n_0}(U_n) \geq$

$$\begin{cases} n\left(1-\frac{1}{p^{t+1}}\right)\log|\alpha| + \log\left(\frac{1-x^n}{1-x^{n/p^{t+1}}}\right) - \log(p^{t+1}), & n_0 = p^h; \\ n\left(1-\frac{1}{p^{t+1}}\right)\left(1-\frac{1}{p_1}\right)\log|\alpha| + \log\left(\frac{1-x^n}{1-x^{\frac{n}{p^{t+1}}}}\right) - \log(pp_1)^{t+1}, & n_0 = p^h p_1^{h_1}, \end{cases}$$

and for $n_0 = p^h, p > 2$,

$$M_{n_0}(V_n) \geq n\left(1-\frac{1}{p^{t+1}}\right)\log|\alpha| + \log\left(\frac{1+x^n}{1+x^{n/p^{t+1}}}\right) - \log(p^{t+1}). \tag{5}$$

Proof. Let n_0 be the divisor of n given in the statement of the lemma. Let $m = n_0p^t$. Write

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \left(\frac{(\alpha^{n/m})^m - (\beta^{n/m})^m}{\alpha^{n/m} - \beta^{n/m}}\right)\left(\frac{\alpha^{n/m} - \beta^{n/m}}{\alpha - \beta}\right).$$

Let $\alpha_1 := \alpha^{n/m}$ and $\beta_1 := \beta^{n/m}$ and put

$$U_\ell^1 = \frac{\alpha_1^\ell - \beta_1^\ell}{\alpha_1 - \beta_1} \quad \text{and} \quad V_\ell^1 = \alpha_1^\ell + \beta_1^\ell \qquad \text{for} \quad \ell \geq 1.$$

Then $\{U_\ell^1\}_{\ell\geq 0}$ and$\{V_\ell^1\}_{\ell\geq 0}$ are the Lucas sequences with parameters (r_1, s_1), where $(r_1, s_1) = (\alpha_1 + \beta_1, -\alpha_1\beta_1) = (V_{n/m}, (-1)^{n/m-1}s^{n/m})$. Further, we have $U_n = U_m^1 U_{n/m}$ and $V_n = V_m^1$ implying

$$M_{n_0}(U_n) \geq M_{n_0}(U_m^1) \qquad \text{and} \qquad M_{n_0}(V_n) \geq M_{n_0}(V_m^1).$$

Observe that $U_m^1 = U_{n_0p^t}^1$ is divisible by each $U_{n_0p^i}^1, 0 \leq i \leq t$. Recall that a prime $q \mid U_\ell^1$ is a primitive divisor of U_ℓ^1 if $q \nmid U_{\ell'}^1$ for $\ell' < \ell$ and $q \nmid r_1^2 + 4s_1$. Also the primitive divisors of U_ℓ^1 are all congruent to one of ± 1 modulo ℓ. Hence, the primitive divisors of $U_{n_0p^i}^1$ for $0 \leq i \leq t$ are all congruent to one of ± 1 modulo n_0. We now look at the primitive part of U_ℓ^1. This is the part of U_ℓ^1 built up only with powers of primitive prime divisors of U_ℓ^1. Thus, the primitive parts of $U_{n_0p^i}^1$ for $0 \leq i \leq t$ divide U_m^1. Hence,

$$M_{n_0}(U_n) \geq M_{n_0}(U_m^1) \geq M_{n_0}\left(\prod_{i=0}^{t} U_{n_0p^i}^1\right).$$

For a positive integer ℓ, let

$$\Phi_\ell(\alpha_1,\beta_1) := \prod_{\substack{1\le k\le \ell\\(k,\ell)=1}} (\alpha_1 - e^{2\pi i k/\ell}\beta_1)$$

be the specialisation of the homogenization $\Phi_\ell(X,Y)$ of the ℓ-th cyclotomic polynomial $\Phi_\ell(X)$ in the pair (α_1,β_1). Further, it is well-known (see, for example, [2, Theorem 2.4]), that for $\ell > 4, \ell \notin \{6,12\}$,

$$\prod_{\substack{p^{\nu_p}\| U_\ell^1\\ p \text{ primitive}}} p^{\nu_p} = \frac{\Phi_\ell(\alpha_1,\beta_1)}{\delta_\ell},$$

where $\delta_\ell \in \{1,2,P(\ell)\}$. Since primitive divisors of U_ℓ^1 are congruent to one of ± 1 modulo ℓ, we obtain by taking $\ell = n_0 p^i$ for $0 \le i \le t$ that

$$M_{n_0}(U_n) \ge M_{n_0}\left(\prod_{i=0}^{t} U^1_{n_0 p^i}\right) \ge \left(\prod_{i=0}^{t} |\Phi_{n_0 p^i}(\alpha_1,\beta_1)|\right)(P(n_0))^{-t-1}. \tag{6}$$

Also from the fact that $V_n = V^1_{n_0 p^t}$ is divisible by each $V_{n_0 p^i}, 0 \le i \le t$ (here n_0, p are both odd) and the primitive part of $V_{n_0p^i}$ is exactly the primitive part of $U^1_{2n_0p^i}$, we obtain similarly

$$M_{n_0}(V_n) \ge M_{2n_0}\left(\prod_{i=0}^{t} U^1_{2n_0 p^i}\right) \ge \left(\prod_{i=0}^{t} |\Phi_{2n_0 p^i}(\alpha_1,\beta_1)|\right)(P(n_0))^{-t-1}. \tag{7}$$

Therefore, it remains to estimate the right–hand sides of inequalities (6) and (7).

It is well-known that for a positive integer ℓ,

$$\Phi_\ell(\alpha_1,\beta_1) = \prod_{d|\ell} (\alpha_1^{\frac{\ell}{d}} - \beta_1^{\frac{\ell}{d}})^{\mu(d)}.$$

Hence, we have, by using $\alpha_1^{n_0p^t} = \alpha^n$,

$$\begin{aligned}\prod_{i=0}^{t} \Phi_{n_0p^i}(\alpha_1,\beta_1) &= \prod_{i=0}^{t} \frac{\alpha_1^{p^{h+i}} - \beta_1^{p^{h+i}}}{\alpha_1^{p^{h+i-1}} - \beta_1^{p^{h+i-1}}} = \frac{\alpha_1^{p^{h+t}} - \beta_1^{p^{h+t}}}{\alpha_1^{p^{h-1}} - \beta_1^{p^{h-1}}}\\ &= \frac{\alpha^n - \beta^n}{\alpha^{n/p^{t+1}} - \beta^{n/p^{t+1}}}, \quad n_0 = p^h;\end{aligned} \tag{8}$$

and

(9)

$$\prod_{i=0}^{t}\Phi_{n_0p^i}(\alpha_1,\beta_1)=\prod_{i=0}^{t}\frac{(\alpha_1^{p^{h+i}p_1^{h_1}}-\beta_1^{p^{h+i}p_1^{h_1}})(\alpha_1^{p^{h+i-1}p_1^{h_1-1}}-\beta_1^{p^{h+i-1}p_1^{h_1-1}})}{(\alpha_1^{p^{h+i-1}p_1^{h_1}}-\beta_1^{p^{h+i-1}p_1^{h_1}})(\alpha_1^{p^{h+i}p_1^{h_1-1}}-\beta_1^{p^{h+i}p_1^{h_1-1}})}$$
$$=\frac{(\alpha_1^{p^{h+t}p_1^{h_1}}-\beta_1^{p^{h+t}p_1^{h_1}})(\alpha_1^{p^{h-1}p_1^{h_1-1}}-\beta_1^{p^{h-1}p_1^{h_1-1}})}{(\alpha_1^{p^{h-1}p_1^{h_1}}-\beta_1^{p^{h-1}p_1^{h_1}})(\alpha_1^{p^{h+t}p_1^{h_1-1}}-\beta_1^{p^{h+t}p_1^{h_1-1}})}$$
$$=\left(\frac{\alpha^n-\beta^n}{\alpha^{\frac{n}{p^{t+1}}}-\beta^{\frac{n}{p^{t+1}}}}\right)\left(\frac{\alpha^{\frac{n}{p_1p^{t+1}}}-\beta^{\frac{n}{p_1p^{t+1}}}}{\alpha^{n/p_1}-\beta^{n/p_1}}\right),\quad n_0=p^hp_1^{h_1}.$$

Also,

$$\prod_{i=0}^{t}\Phi_{2n_0p^i}(\alpha_1,\beta_1)=\frac{\alpha_1^{p^{h+t}}+\beta_1^{p^{h+t}}}{\alpha_1^{p^{h-1}}+\beta_1^{p^{h-1}}}=\frac{\alpha^n+\beta^n}{\alpha^{\frac{n}{p^{t+1}}}+\beta^{\frac{n}{p^{t+1}}}},\quad n_0=p^h. \tag{10}$$

From $|\alpha|\ge|\beta|$, we have $|x|\le 1$. Taking out the powers of α in (8)–(10) and further using in (9) the inequality

$$\left|\frac{1-y}{1-y^{p^{t+1}}}\right|\ge\frac{1}{p^{t+1}}\quad\text{valid for all}\quad p,\quad\text{where}\quad y:=x^{\frac{n}{p_1p^{t+1}}}\quad\text{has}\quad |y|\le 1,$$

we get the assertions (4) and (5) from (6) and (7), respectively. □

From the inequality

$$44.72(\log t+2.36)^2+0.16\log^2 t\le 44.88\log^2 t+211.08\log t+249.08,$$

we obtain the following result which is [3, Lemma 4] and which is a consequence of Voutier [12, Lemma 5].

Lemma 2. *Let α and β be complex conjugates with $\log|\alpha|>4$. Let*

$$f(\ell):=44.88\log^2\ell+211.08\log\ell+249.08\quad\text{for}\quad\ell>1. \tag{11}$$

Then for integer $\ell\ge 3$, we have

$$\log|\alpha^\ell-\beta^\ell|\ge\log|\alpha|\left(\ell-f\left(\frac{\ell}{\gcd(\ell,2)}\right)\right) \tag{12}$$

and

$$\log|\alpha^\ell+\beta^\ell|\ge\log|\alpha|\left(\ell-f(\ell)\right). \tag{13}$$

The following lemma gives us range for the parameters (r, s) in case when α is real, positive and lies in an interval $[c_1, c_2]$.

Lemma 3. *Let α, β be real. Assume $\alpha > 0$. Let $c_1 \leq \alpha \leq c_2$ where c_1, c_2 are positive reals and $r^2 + 4s > 0$. For $s > 0$, we have $r < c_2$ and*

$$\max\left\{c_1(c_1 - r), \frac{c_1^2 - r^2}{4}\right\} \leq s \leq c_2(r - c_2).$$

For $s < 0$, we have $c_1 \leq r \leq 2c_2$ and

$$c_2(r - c_2) \leq |s| < \frac{r^2}{4}, \quad \textit{and further} \quad |s| < c_1(r - c_1) \quad \textit{if} \quad r < 2c_1.$$

Proof. We have $2c_1 \leq 2\alpha = r + \sqrt{r^2 + 4s} \leq 2c_2$. This gives the inequality $r^2 + 4s \leq (2c_2 - r)^2$ implying $s \leq c_2(c_2 - r)$. If $2c_1 > r$, we then have $r^2 + 4s \geq (2c_1 - r)^2$ giving $s \geq c_1(c_1 - r)$.

Let $s > 0$. Then $r < \alpha \leq c_2$ giving $r < c_2$ and $s \leq c_2(c_2 - r)$. If $c_1 > r$, then $2c_1 > r$ and therefore $s \geq c_1(c_1 - r)$. Also

$$2c_1 \leq r + \sqrt{r^2 + 4s} \leq 2\sqrt{r^2 + 4s}$$

gives $s \geq \dfrac{c_1^2 - r^2}{4}$ implying

$$s \geq \max\left\{c_1(c_1 - r), \frac{c_1^2 - r^2}{4}\right\}.$$

Let $s < 0$. Then $c_1 \leq \alpha < r < r + \sqrt{r^2 + 4s} \leq 2c_2$ giving $c_1 < r < 2c_2$. Also $r^2 + 4s > 0$ gives $|s| = -s < r^2/4$. From $s \leq c_2(c_2 - r)$, we get

$$|s| = -s \geq c_2(r - c_2).$$

If $r < 2c_1$, then $s \geq c_1(c_1 - r)$ implying $|s| = -s \leq c_1(r - c_1)$. □

The following lemma is proved using Stirling's formula.

Lemma 4. *The function $m \mapsto \log(C_m/2)/m$ is increasing for $m \geq 7$. Hence,*

$$\log\left(\frac{B_m}{2}\right) > \log\left(\frac{C_m}{2}\right) > \begin{cases} m & \text{for} \quad m \geq 14; \\ 1.36m & \text{for} \quad m \geq 400; \\ 1.38m & \text{for} \quad m \geq 2100. \end{cases} \tag{14}$$

Further, given $M \geq 7$ and $m \leq M$, we have

$$\frac{m \log 2m}{\log(C_m/2)} \frac{\log(C_M/2)}{M} \leq 1.0001 \log 2M. \tag{15}$$

Proof. We recall Stirling's formula. For a positive integer ν, we have

$$\sqrt{2\pi\nu}\ e^{-\nu}\nu^{\nu}e^{\frac{1}{12\nu+1}} < \nu! < \sqrt{2\pi\nu}\ e^{-\nu}\nu^{\nu}e^{\frac{1}{12\nu}}.$$

From $C_m = \frac{(2m)!}{(m+1)(m!)^2}$, we have

$$m\log 4 - \sigma_m < \log(C_m/2) < m\log 4 - \tau_m, \tag{16}$$

where

$$\sigma_m := \log 2 + \log(m+1) + \log\sqrt{\pi m} + \frac{1}{6m} - \frac{1}{24m+1}$$
$$\text{and}\quad \tau_m := \log 2 + \log(m+1) + \log\sqrt{\pi m} + \frac{2}{12m+1} - \frac{1}{24m}.$$

We have $C_m < 4^m/\sqrt{\pi m}$ and

$$\frac{4^m}{\sqrt{\pi m}}\left(\frac{m+2}{4m+2}\right)^m = \frac{(1+3/(2m+1))^m}{\sqrt{\pi m}} \le \frac{e^{\frac{3m}{2m+1}}}{\sqrt{\pi m}} < \frac{e^{3/2}}{\sqrt{\pi m}} < 1 \quad \text{for} \quad m \ge 7.$$

Hence, from $C_{m+1}/C_m = (4m+2)/(m+2)$, we get

$$m\log\left(\frac{C_{m+1}}{2}\right) - (m+1)\log\left(\frac{C_m}{2}\right) \ge m\log\left(\frac{C_{m+1}}{C_m}\right) - \log C_m > 0$$

for $m \ge 7$. This shows that $\log(C_m/2)/m$ is an increasing function for $m \ge 7$. Hence, the assertion (14) follows by calculating $\log(C_m/2)/m$ at $m = 14, 400, 2100$, respectively.

From (16), we have

$$\frac{m\log 2m}{\log(C_m/2)} \le \frac{\log 2m}{\log 4 - \sigma_m/m}$$

and the right–hand side is an increasing function of m. Therefore, from $m \le M$ and inequality (16) again, we get

$$\begin{aligned}
\left(\frac{m\log 2m}{\log(\frac{C_m}{2})}\right)\left(\frac{\log(C_M/2)}{M}\right) &\le \frac{\log 2M}{\log 4 - \sigma_M/M}(\log 4 - \tau_M/M) \\
&= (\log 2M)\left(1 + \frac{\sigma_M - \tau_M}{M\log 4 - \sigma_M}\right) \\
&\le (\log 2M)\left(1 + \frac{\frac{1}{24M+1} + \frac{4}{12M+1}}{24M(M\log 4 - \sigma_M)}\right) \\
&\le 1.0001\log 2M,
\end{aligned}$$

since $M \ge 7$, implying the assertion (15). ∎

The next lemma follows easily from the Brun-Titchmarsh inequality given by Montgomery and Vaughan [9, Theorem 2] since $\pi(1; q, l) = 0$ and $\pi(y; q, l) \le \pi(y+1; q, l) - \pi(1; q, l)$. Recall that $\pi(y; q, l)$ stands for the number of primes $p \le y$ and $p \equiv l \pmod q$.

Lemma 5. *Let q be a positive integer, l be coprime to q and $y > q$. Then*

$$\pi(y; q, l) \le \frac{2y}{\varphi(q)\log(y/q)} \quad \text{and} \quad \pi(2y; q, l) - \pi(y; q, l) \le \frac{2y}{\varphi(q)\log(y/q)}.$$

As usual, let

$$\psi(y; q, l) := \sum_{\substack{p^t \le y \\ p \equiv l \pmod m}} \log p \quad \text{and} \quad \theta(y; q, l) := \sum_{\substack{p \le y \\ p \equiv l \pmod m}} \log p.$$

The following estimates are from [10, Table 2]. We have taken into account the estimates for $\theta\#$ defined in [10, Table 2] for $q \in \{8, 16, 24\}$.

Lemma 6. *Let $q \in \{8, 9, 12, 16, 24\}$ or $5 \le q \le 23$ be a prime and ℓ_0 be an integer coprime to q with $\ell_0 \not\equiv 1 \pmod q$. Then for $y \ge q$, we have*

$$\psi(y; q, 1) + \psi(y; q, \ell_0) \le \frac{2y}{\varphi(q)}\left(1 + \frac{\varepsilon_\psi \varphi(q)}{\sqrt{y}}\right) \tag{17}$$

and

$$\theta(y; q, 1) + \theta(y; q, \ell_0) \ge \frac{2y}{\varphi(q)}\left(1 - \frac{\varepsilon_\theta \varphi(q)}{\sqrt{y}}\right), \tag{18}$$

where ε_ψ and ε_θ are given by

q	5	7	8	9	12	16	24	$11 \le q \le 23$
ε_ψ	.807	.78	.927	.789	.863	.774	.745	.912
ε_θ	1.413	1.106	1.5	1.11	1.5	1.03	1.5	1.1

Further,

$$\frac{y}{\varphi(24)}\left(1 - \frac{\varphi(24)}{\sqrt{y}}\right) \le \theta(y; 24, 5) \le \psi(y; 24, 5) \le \frac{y}{\varphi(q)}\left(1 + \frac{0.745\varphi(24)}{\sqrt{y}}\right).$$

As a consequence, we have the following result.

Lemma 7. *Let $q \in \{8, 9, 12, 16, 24\}$ or $5 \le q \le 23$ be a prime and ℓ_0 be an integer coprime to q with $\ell_0 \not\equiv 1 \pmod q$. Then for $y \ge 1500$, we have*

(19)
$$\sum_{l=1,\ell_0} \left(\psi(2y;q,l) - \theta(y;q,l) + \theta\left(\frac{2y}{3};q,l\right) - \theta\left(\frac{y}{2};q,l\right) + \theta\left(\frac{2y}{5};q,l\right)\right)$$
$$\le \frac{y}{\varphi(q)}\left\{\frac{47}{15} + \frac{2\sqrt{2}\varepsilon_\psi}{\sqrt{y}}\left(1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}\right) + \frac{2\varepsilon_\theta}{\sqrt{y}}\left(1 + \frac{1}{\sqrt{2}}\right)\right\},$$

where ε_ψ and ε_θ are given in Lemma 6. Also for each $y \ge 15$, there is a prime $p \equiv 5 \pmod{24}$ with $y + 1 < p \le 2y$. Further, for $y \ge 6$, there is a prime $p \equiv \pm 5 \pmod 8$ with $y+1 < p \le 2y$. And for $y \ge 9$, there is a prime $p \equiv 5 \pmod 1 2$ with $y + 1 < p \le 2y$.

Proof. The assertion (19) is immediate from Lemma 6 and using the inequality $\theta(y;q,l) \le \psi(y;q,l)$ valid for all y. For primes $p \equiv 5 \pmod{24}$, again from Lemma 6, we have

$$\begin{aligned}\theta(2y;24,5) - \theta(y+1;24,5) &\ge \frac{2y}{\varphi(24)}\left(1 - \frac{\varphi(24)}{\sqrt{2y}}\right) \\ &\quad - \frac{y+1}{\varphi(24)}\left(1 + \frac{0.745\varphi(24)}{\sqrt{y+1}}\right) \\ &= \frac{y}{\varphi(24)}\left\{1 - \frac{2\varphi(24)}{\sqrt{2y}} - \frac{1}{y} - \frac{0.745\varphi(q)}{\sqrt{y+1}} - \frac{0.745\varphi(q)}{y\sqrt{y+1}}\right\} > 0,\end{aligned}$$

for $y \ge 400$. Thus, there is a prime $p \equiv 5 \pmod{24}$ with $y + 1 < p \le 2y$ for $y \ge 400$. This is also true for $15 \le y < 400$ by checking at integer values of y. Since a prime congruent to $5 \pmod{24}$ is also congruent to $5 \pmod 8$ and $5 \pmod{12}$, the last two assertions can be obtained by checking it in the range $6 \le y < 15$. □

In the next section, we use use Lemmas 4, 5 and 7 to obtain upper bound for prime powers dividing a product of Catalan numbers and middle binomial coefficients.

3 Upper bound for prime powers dividing a product of Catalan numbers and middle binomial coefficients

For positive integers $1 < m_1 \le m_2 \le \cdots \le m_k$, let

$$\mathcal{D} := \mathcal{D}(m_1, m_2, \ldots, m_k) := \prod_{i=1}^{k} D_{m_i}, \quad D_{m_i} \in \{C_{m_i}, B_{m_i}\}.$$

Let n_0 be a positive integer. Recall the definition of $M_{n_0}(\ell)$ given in (3). We use analytic methods to find an upper bound for

$$M_{n_0}(\mathcal{D}) := \log \left(\prod_{\substack{p^{\nu_p} \| \mathcal{D} \\ p \equiv \pm 1 \pmod{n_0}}} p^{\nu_p} \right) = \sum_{\substack{p^{\alpha_p} \| \mathcal{D} \\ p \equiv \pm 1 \pmod{n_0}}} \nu_p \log p.$$

This is the content of the following lemma.

Lemma 8. *For $n_0 \ge 25$, we have*

$$(20) \qquad M_{n_0}(\mathcal{D}) \le \begin{cases} \left(\frac{3.9}{\varphi(n_0)} + 2.92 \frac{\log 3n_0}{n_0} \right) (\log \mathcal{D} - \log 2), & \text{if } n_0 \text{ is even;} \\ \left(\frac{3.9}{\varphi(n_0)} + 1.46 \frac{\log 3n_0}{n_0} \right) (\log \mathcal{D} - \log 2), & \text{if } n_0 \text{ is odd.} \end{cases}$$

Let $n_0 \in \{9, 16, 24\}$ or $5 \le n_0 \le 23$ be a prime. We have

$$(21) \qquad M_{n_0}(\mathcal{D}) \le \begin{cases} \frac{\delta_0}{\varphi(n_0)} \log \mathcal{D}, & \text{if } \quad m_k < 1500; \\ \frac{\delta_0}{\varphi(n_0)} \left(\log \mathcal{D} - \log 2 \right), & \text{if } \quad m_k \ge 1500; \end{cases}$$

where δ_0 is given by

n_0	5	7	9	16	24	$11 \le n_0 \le 23$
δ_0	2.61	3.19	3.57	2.89	2.746	3.3

Proof. Let t_{1j} and t_{2j} be the number of i's such that $D_{m_i} = C_j$ and $D_{m_i} = B_j$, respectively. Put $t_j = t_{1j} + t_{2j}$. Then

$$\log \mathcal{D} = \sum_{1 \le i \le k} \log D_{m_i} = \sum_{1 < j \le m_k} (t_{1j} \log C_j + t_{2j} \log B_j) \ge \sum_{1 < j \le m_k} t_j \log C_j$$

since $B_m > C_m$. Let $7 < M \le m_k$ be an integer which we will choose later on. Using Lemma 4, we get

$$\begin{aligned}\log \mathcal{D} &\ge \sum_{j\le M} t_j \log C_j + \sum_{j>M} t_j \log C_j \\ &\ge \log 2 + \sum_{j\le M} t_j \log(C_j/2) + \frac{\log(C_M/2)}{M} \sum_{j>M} t_j j,\end{aligned}$$

so that

$$\sum_{j>M} t_j j \le \frac{M}{\log(C_M/2)} \left(\log \mathcal{D} - \log 2 - \sum_{j\le M} t_j \log(C_j/2) \right). \tag{22}$$

Here, as usual, the empty sum is taken to be 0. For a prime number p and a positive integer t, we write $\nu_p(t)$ for the exact exponent of p in the prime factorization of t. Given a positive integer j, let

$$\xi_1(j) := \sum_{p\equiv \pm 1 \pmod{n_0}} \nu_p(C_j) \log p \quad \text{and} \quad \xi_2(j) := \sum_{p\equiv \pm 1 \pmod{n_0}} \nu_p(B_j) \log p.$$

Then $\xi_1(j) \le \xi_2(j)$ and hence $M_{n_0}(\mathcal{D}) \le \sum_j t_j \xi_2(j)$. For a prime p, we have

$$\nu_p(B_j) = \sum_{\ell\ge 1} \left(\left\lfloor \frac{2j}{p^\ell} \right\rfloor - 2 \left\lfloor \frac{j}{p^\ell} \right\rfloor \right) \le \begin{cases} 1, & \text{if } \frac{2j}{2i} < p \le \frac{2j}{2i-1},\ i \in \{1,2\}; \\ 0, & \text{if } \dfrac{2j}{2i+1} < p \le \frac{2j}{2i},\ i \in \{1,2\}; \\ \left\lfloor \frac{\log(2j)}{\log p} \right\rfloor, & \text{if } p \le \dfrac{2j}{5}. \end{cases}$$

Therefore,

$$\begin{aligned}\xi_2(j) &\le \sum_{\substack{(2j)^{1/2} < p \le 2j \\ p\equiv \pm 1 \pmod{n_0}}} \left(\left\lfloor \frac{2j}{p} \right\rfloor - 2 \left\lfloor \frac{j}{p} \right\rfloor \right) \log p + \sum_{\substack{p \le (2j)^{1/2} \\ p\equiv \pm 1 \pmod{n_0}}} \left\lfloor \frac{\log(2j)}{\log p} \right\rfloor \log p \\ &\le \sum_{\substack{p\le 2j \\ p\equiv \pm 1 \pmod{n_0}}} \left\lfloor \frac{\log(2j)}{\log p} \right\rfloor \log p - \sum_{\substack{1\le i\le 2 \\ \frac{2j}{2i-1} < p \le \frac{2j}{2i} \\ p\equiv \pm 1 \pmod{n_0}}} \log p \\ &\le \sum_{\ell\in\{1,-1\}} \left\{ \psi(2j; n_0, \ell) + \sum_{t=2}^{3} \theta(2j/t; n_0, \ell) - \sum_{t=1}^{2} \theta(j/t; n_0, \ell) \right\}.\end{aligned} \tag{23}$$

Recall that $\pi(x; n_0, \ell)$ stands for the number of primes $p \le x$ satisfying the congruence $p \equiv \ell \pmod{n_0}$. We put $\pi_{\pm 1}(x) := \pi(x; n_0, 1) + \pi(x; n_0, -1)$. Then

$$\xi_2(j) \le (\pi_{\pm 1}(2j) - \pi_{\pm 1}(j) + \pi_{\pm 1}(2j/3)) \log(2j), \tag{24}$$

by (23). Let us assume that $n_0 \ge 25$. Let $t > 0$. We split the analysis in two cases according to whether $2j \le (3n_0)^{1+1/t}$ or $2j > (3n_0)^{1+1/t}$.

Assume first that $2j \ge (3n_0)^{1+1/t}$. Then $2j/3n_0 \ge (2j)^{1/(1+t)}$ and therefore

$$\log(j/n_0) \ge \log(2j/3n_0) \ge (\log(2j))/(1+t).$$

From (24) and Lemma 5, we get

$$\xi_2(j) \le \frac{4j \log 2j}{\varphi(n_0) \log(j/n_0)} + \frac{(4j/3) \log 2j}{\varphi(n_0) \log(2j/3n_0)} \le \frac{16(1+t)j}{3\varphi(n_0)}.$$

In the smaller range $2j \le (3n_0)^{1+1/t}$, using the trivial estimates and the fact that primes congruent to one of ± 1 modulo n_0 are of the form $2ln_0 \pm 1$ when n_0 is odd, we get

$$\pi_{1,-1}(2j) - \pi_{1,-1}(j) + \pi_{1,-1}(2j/3) \le \pi_{1,-1}(2j)$$
$$\le \begin{cases} \frac{2j-1}{n_0} + \frac{2j+1}{n_0} = \frac{4j}{n_0}, & \text{if } n_0 \text{ is even;} \\ \frac{2j-1}{2n_0} + \frac{2j+1}{2n_0} = \frac{2j}{n_0}, & \text{if } n_0 \text{ is odd.} \end{cases}$$

Let $\eta := 1, 2$ according to whether n_0 is even or odd, respectively. From (24), we get

$$\xi_2(j) \le \left(\frac{4}{\eta}\right) \frac{j \log 2j}{n_0}.$$

We choose

$$t := \frac{3.0003}{4\eta} \frac{\varphi(n_0) \log 3n_0}{n_0} \qquad \text{and} \qquad M := \left\lfloor \frac{1}{2}(3n_0)^{1+\frac{1}{t}} \right\rfloor.$$

Since $n_0/\varphi(n_0) \ge 2/\eta$, we observe that

$$\frac{1}{2}(3n_0)^{1+\frac{1}{t}} \ge 1.5 n_0 \exp\left(\frac{8}{3.0003}\right) > 686 \quad \text{for} \quad n_0 \ge 32,$$

which together with $M \geq 686$ for each $25 \leq n_0 < 32$ implies $M \geq 686$ for all $n_0 \geq 25$. From (22), we have

$$\begin{aligned}
&M_{n_0}(\mathcal{D}) \leq \sum_j t_j \xi_2(j) \\
\leq &\frac{M}{\log(\frac{C_M}{2})} \frac{16(1+t)}{3\varphi(n_0)} \left(\log \frac{\mathcal{D}}{2} - \sum_{j \leq M} t_j \log\left(\frac{C_j}{2}\right)\right) + \sum_{j \leq M} \frac{4t_j j \log 2j}{\eta n_0} \\
\leq &\frac{M}{\log(\frac{C_M}{2})} \frac{16(1+t)}{3\varphi(n_0)} \log \frac{\mathcal{D}}{2} - \sum_{j \leq M} t_j \left(\frac{M \log(C_j/2)}{\log(C_M/2)} \frac{16(1+t)}{3\varphi(n_0)} - \frac{4j \log 2j}{\eta n_0}\right).
\end{aligned}$$

Since $M > 7$, and we get from (15) and $\log 2M \leq (1 + 1/t) \log 3n_0$ that

$$\begin{aligned}
&\frac{M \log(C_j/2)}{\log(C_M/2)} \frac{16(1+t)}{3\varphi(n_0)} - \frac{4j \log 2j}{\eta n_0} \\
= &\frac{4M \log(C_j/2)}{n_0 \log(C_M/2)} \left(\frac{4(1+t)n_0}{3\varphi(n_0)} - \frac{j \log 2j}{\eta \log(C_j/2)} \frac{\log(C_M/2)}{M} \right) \\
\geq &\frac{16M \log(C_j/2)}{3n_0 \log(C_M/2)} \frac{n_0}{\varphi(n_0)} \left(\frac{(1+t)n_0}{\varphi(n_0)} - \frac{3\varphi(n_0)}{4\eta n_0} \frac{1.0001(1+t)\log 3n_0}{t} \right) \geq 0,
\end{aligned}$$

since

$$t = \frac{3.0003}{4\eta} \frac{\varphi(n_0) \log 3n_0}{n_0}.$$

Therefore, we have from $M \geq 686$ and Lemma 4, that

$$\begin{aligned}
M_{n_0}(\mathcal{D}) &\leq \frac{M}{\log(C_M/2)} \frac{16(1+t)}{3\varphi(n_0)} \log \frac{\mathcal{D}}{2} \\
&\leq \frac{16}{3} \frac{686}{\log(C_{686}/2)} \left(\frac{1}{\varphi(n_0)} + 3.0003 \left(\frac{\log 3n_0}{4\eta n_0} \right) \right) (\log \mathcal{D} - \log 2),
\end{aligned}$$

which gives the assertion (20).

We now consider $n_0 \leq 24$ as given in the statement of the lemma. Then either $n_0 \in \{9, 16, 24\}$, or n_0 is a prime with $5 \leq n_0 \leq 23$. We check with exact computations that for $j \leq 1500$,

$$\xi_1(j) \leq \frac{\delta_0 \log C_j}{\varphi(n_0)} \quad \text{and} \quad \xi_2(j) \leq \frac{\delta_0 \log B_j}{\varphi(n_0)},$$

where δ_0 are given in the statement of the lemma. Hence, we have

$$M_{n_0}(\mathcal{D}) = \sum_{1<j \leq m_k} (t_{1j}\xi_1(j) + t_{2j}\xi_2(j)) \leq \frac{\delta_0}{\varphi(n_0)} \sum_{1<j \leq m_k} (t_{1j} \log C_j + t_{2j} \log B_j),$$

which gives the assertion (21) for $m_k < 1500$.

We now take $m_k \geq 1500$. From (23) and Lemma 7, we get

$$\xi_1(j) \leq \xi_2(j) \leq \frac{\delta_1 j}{\varphi(n_0)} \leq \frac{\delta_1 j}{\log(C_j/2)} \frac{\log(D_j/2)}{\varphi(n_0)} \quad \text{for} \quad j \geq 1500,$$

where

$$\delta_1 = \frac{47}{15} + \frac{2\sqrt{2}\varepsilon_\psi}{\sqrt{1500}}\left(1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}\right) + \frac{2\varepsilon_\theta}{\sqrt{1500}}\left(1 + \frac{1}{\sqrt{2}}\right),$$

and ε_ψ and ε_θ are given in Lemma 6. By Lemma 4, we have

$$\frac{\delta_1 j}{\log(C_j/2)} \leq \frac{\delta_1}{1.37} \quad \text{for each} \quad j \geq 1500,$$

and we find that $\delta_1/1.37 \leq \delta_0$. Thus,

$$\xi_1(j) \leq \xi_2(j) \leq \frac{\delta_0 \log(D_j/2)}{\varphi(n_0)} \quad \text{for each} \quad j \geq 1500,$$

and therefore

$$\begin{aligned} M_{n_0}(\mathcal{D}) &= \sum_{j<1500} (t_{1j}\xi_1(j) + t_{2j}\xi_2(j)) + \sum_{j\geq 1500} (t_{1j}\xi_1(j) + t_{2j}\xi_2(j)) \\ &\leq \frac{\delta_0}{\varphi(n_0)} \sum_{j<1500} (t_{1j}\log C_j + t_{2j}\log B_j) + \frac{\delta_0}{\varphi(n_0)} \sum_{j\geq 1500} \left(t_{1j}\log\frac{C_j}{2} + t_{2j}\log\frac{B_j}{2}\right) \\ &\leq \frac{\delta_0}{\varphi(n_0)} (\log\mathcal{D} - \log 2). \end{aligned}$$

Hence, the assertion (21) follows and the proof is complete. □

4 Proof of Theorem 1

We recall that for $n \geq 0$

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$$

where α and β are the roots of the quadratic equation $\lambda^2 - r\lambda - s = 0$ and r, s are coprime nonzero integers with $r^2 + 4s \neq 0$. We suppose that α/β is not a root of unity. We also recall that we assume that $r > 0$. When α, β are real, these conventions imply that α is positive so $\alpha > |\beta|$ and in this case $U_n > 0$ and $V_n > 0$ for all $n \geq 1$. Further, we put $x = \beta/\alpha$. Thus, $|x| \leq 1$.

Note that $U_1 = 1 \in \mathcal{P}BC$. In fact, if $U_n = \pm 1$ (or $V_n = \pm 1$) then U_n (or V_n) are also in $\mathcal{P}BC$. The equations $U_n = \pm 1$ and $V_n = \pm 1$ are important from the Diophantine point of view. However, such equations have been solved completely and we refer to [2] for more details. For this reason, whenever we study the equations (1) and (2), we omit the cases $n = 1$, $U_n = \pm 1$ and $V_n = \pm 1$. Thus, we also assume that $m_1 > 1$.

We first treat the case of the sequence $\{U_n\}_{n\geq 0}$. Assume that the equation (1) has a solution. Then

$$|U_n| = \mathcal{D} = D_{m_1} \cdots D_{m_k}, \quad D_{m_i} \in \{C_{m_i}, B_{m_i}\}..$$

For a divisor n_0 of n, we will compare the upper bound of $M_{n_0}(\mathcal{D})$ given by Lemma 8 with a lower bound on it obtained by using Lemma 1. We will choose a suitable divisor n_0 of n such that these bounds contradict each other and hence for n with such divisors n_0, $|U_n|$ cannot be a product of Catalan numbers and middle binomial coefficients.

Recall that a prime $p \mid U_n$ is a primitive divisor of U_n if $p \nmid U_t$ for $t < n$ and $p \nmid r^2 + 4s$. Further, the primitive prime divisors of U_n are congruent to one of ± 1 modulo n. From the well known result from [2], we know that a primitive divisor for U_n exist for all $n > 30$. Further, for $5 \leq n \leq 30$, $n \neq 6$, the pairs (r, s) for which a primitive divisor for U_n does not exist are given by

n	(r, s)
5	$(1, 1), (1, -2), (1, -3), (1, -4), (2, -11), (12, -55), (12, -377)$
7	$(1, -2), (1, -5)$
8	$(1, -2), (2, -7)$
10	$(2, -3), (5, -7), (5, -18)$
12	$(1, 1), (1, -2), (1, -3), (1, -4), (1, -5), (2, -15)$
$13, 18, 30$	$(1, -2)$

We checked that for (r, s) given above with $n \geq 5, n \neq 6$, the equation (1) holds in several instances. The roots (α, β) are real only when $(r, s) = (1, 1)$ and then

$$(r, s, n) = (1, 1, 5), \ (1, 1, 12), \quad U_5 = C_3, \ U_{12} = B_1^6 C_2^2 = B_1^2 B_2^2.$$

Hence, we assume now that U_n has a primitive prime divisor p and so $p \equiv \pm 1 \pmod{n}$. Let $P_n := P(U_n)$ be the largest primitive divisor of U_n. From (1), we have that $P_n \mid B_{m_k}$ and so $2m_k \geq P_n + 1$ since P_n is odd. Let Q_n be the least prime congruent to one of ± 1 modulo n. Then $2m_k \geq P_n + 1 \geq Q_n + 1$ and therefore

$$2|\alpha|^n \geq \left| \frac{\alpha^n - \beta^n}{\alpha - \beta} \right| = |U_n| \geq C_{m_k}. \tag{25}$$

From Lemma 4, we have

(26)

$$n \log |\alpha| \geq \log(C_{m_k}/2) \geq \begin{cases} 1.36 m_k \geq 0.68(Q_n + 1) \geq 0.68n, & n \geq 400; \\ 1.38 m_k \geq 0.69(Q_n + 1) \geq 0.69n, & n \geq 4200, \end{cases}$$

since $Q_n \geq n - 1$.

We have

$$\log \mathcal{D} \leq \log |U_n| \leq n \log |\alpha| + \log |1 - x^n|.$$

Now we complete the proof by choosing suitable n_0 and comparing upper and lower bounds of $M_{n_0}(U_n) = \mathcal{M}_{n_0}(\mathcal{D})$. For $n_0 \in \{9, 16, 24\}$, or n_0 an odd prime power, we define

$$g(n_0) := \begin{cases} \frac{\delta_0}{\varphi(n_0)}, & \text{if } n_0 \in \{9, 16, 24\}, \text{ or } n = p \leq 23; \\ \frac{3.9}{\varphi(n_0)} + \frac{1.46 \log 3n_0}{n_0}, & \text{if } n_0 \geq 25 \text{ is odd}, \end{cases} \tag{27}$$

where δ_0 is stated in Lemma 8. By Lemma 8, we have

$$M_{n_0}(\mathcal{D}) \leq g(n_0) \log |U_n| \leq g(n_0) \left(n \log |\alpha| + \log |1 - x^n| \right). \tag{28}$$

Let $p^{h+t} \mid n$, where p is a prime and $h > 0, t \geq 0$ are integers such that $p^h > 4$. Taking $n_0 = p^h$ and using (4) in Lemma 1, we get a lower bound for $M_{n_0}(U_n) = M_{n_0}(\mathcal{D})$ which we compare with (28). We obtain

$$\begin{aligned} & g(n_0) \left(n \log |\alpha| + \log |1 - x^n| \right) \\ \geq & \left(1 - \frac{1}{p^{t+1}} \right) n \log |\alpha| + \log |1 - x^n| - \log |1 - x^{n/p^{t+1}}| - \log(p^{t+1}), \end{aligned}$$

implying

(29)

$$\left(1 - \frac{1}{p^{t+1}} - g(n_0) \right) \leq \frac{(g(n_0) - 1) \log |1 - x^n| + \log |1 - x^{n/p^{t+1}}| + \log(p^{t+1})}{n \log |\alpha|}.$$

We consider different cases.

4.1 The case when n is even

We assume that $n > 720$. We choose $n_0 = p^h$ and t as follows:

$$(n_0, t) \in \{(2^4, 1), (3^2, 1), (5, 1)\} \cup \{(p, 0) : p > 5\}. \tag{30}$$

Since $2^4 \cdot 3^2 \cdot 5 = 720$, we find that for each even $n > 720$, there is some (n_0, t) in (30) with $n_0 p^t \mid n$. From the triangle inequality

$$2|\alpha^{n/2}| \leq |\alpha^{n/2} - \beta^{n/2}| + |\alpha^{n/2} + \beta^{n/2}|,$$

we have either $|\alpha^{n/2} - \beta^{n/2}| \geq |\alpha|^{\frac{n}{2}}$, or $|\alpha^{n/2} + \beta^{n/2}| \geq |\alpha|^{\frac{n}{2}}$. Therefore,

$$|\alpha^n - \beta^n| = |\alpha^{n/2} - \beta^{n/2}||\alpha^{n/2} + \beta^{n/2}|$$
$$\geq \begin{cases} |\alpha|^{\frac{n}{2}}|\alpha^{\frac{n}{2}} + \beta^{\frac{n}{2}}| = |\alpha|^{\frac{n}{2}}|V_{\frac{n}{2}}| \geq |\alpha|^{\frac{n}{2}}, & |\alpha^{\frac{n}{2}} - \beta^{\frac{n}{2}}| \geq |\alpha|^{\frac{n}{2}}; \\ |\alpha - \beta||\frac{\alpha^{\frac{n}{2}} - \beta^{\frac{n}{2}}}{\alpha - \beta}||\alpha|^{\frac{n}{2}} \geq |U_{\frac{n}{2}}||\alpha|^{\frac{n}{2}}, \geq |\alpha|^{\frac{n}{2}} & |\alpha^{\frac{n}{2}} + \beta^{\frac{n}{2}}| \geq |\alpha|^{\frac{n}{2}}, \end{cases}$$

since $V_{n/2}, U_{n/2}$ are integers and $|\alpha - \beta| \geq 1$. Hence,

$$|1 - x^n| = |\alpha|^{-n}|\alpha^n - \beta^n| \geq |\alpha|^{-\frac{n}{2}}.$$

Using the above inequality together with the inequality $|1 - x^{n/p^{t+1}}| \leq 2$ (since $|x| \leq 1$) in (29), we get

$$\frac{\log(2p^{t+1})}{n \log|\alpha|} \geq 1 - \frac{1}{p^{t+1}} - g(n_0) + \frac{g(n_0) - 1}{2} = \frac{1}{2} - \frac{1}{p^{t+1}} - \frac{g(n_0)}{2}.$$

From (26), we have

$$0 \geq \frac{1}{2} - \frac{1}{p^{t+1}} - \frac{g(n_0)}{2} - \frac{\log(2p^{t+1})}{0.68n}.$$

For a fixed choice of $n_0 = p^h$ and t, the right–hand side of the above inequality is an increasing function of n. We check that for (n_0, t) in (30) with $n_0 < 29$, the above inequality is not valid at $n = 720$ and hence it is not valid for any $n \geq 720$. Further, for $n_0 \geq 29$, we have $n_0 = p$ is prime, which together with the observation that $g(p)$ is a decreasing function of p, we obtain

$$0 \geq \frac{1}{2} - \frac{1}{p^{t+1}} - \frac{g(n_0)}{2} - \frac{\log(2p^{t+1})}{0.65n} \geq \frac{1}{2} - \frac{1}{29} - \frac{g(29)}{2} - \frac{\log(2 \times 29)}{0.68n}.$$

We check that the right–most side is positive for $n \geq 720$ and hence we get a contradiction for all $n \geq 720$. Thus, equation (1) has no even solution $n > 720$.

4.2 The case when α, β are complex conjugates

From the previous section, we may assume that either $n > 720$ is odd or n is an even number ≤ 720. Since we are shooting for the inequality $n < 6500$, we may assume that $n \geq 6500$ is odd. Also, we have $Q_n \geq 2n - 1$ which together with $2m_k \geq Q_n + 1$, inequality (25) and Lemma 4 gives

$$\log |U_n| \geq 1.38n \qquad \text{and} \qquad n \log |\alpha| \geq 1.38n.$$

We choose n_0 of the form p^h and t given by

$$(n_0, t) \in \{(3^2, 2), (5, 1), (7, 1)\} \cup \{(p, 0) : p \geq 11\}. \tag{31}$$

Since $3^3 \cdot 5^2 \cdot 7 < 6500$, we find that for each odd $n \geq 6500$, there is some (n_0, t) in (31) with $n_0 p^t \mid n$.

First we consider the case when $\log |\alpha| \leq 4$. We use

$$|1 - x^n| = \frac{|\alpha - \beta||U_n|}{|\alpha|^n} \geq \frac{|U_n|}{|\alpha|^n}, \quad |1 - x^{n/p^{t+1}}| \leq 2 \quad \text{and} \quad \log |\alpha| \leq 4$$

in (4) and compare it with (28) to obtain

$$g(n_0) \log |U_n| \geq M_{n_0}(U_n) \geq \log |U_n| - \frac{4n}{p^{t+1}} - \log(2p^{t+1}).$$

Since $\log |U_n| \geq 1.38n$, we obtain

$$0 \geq 1.38(1 - g(n_0)) - \frac{4}{p^{t+1}} - \frac{\log(2p^{t+1})}{n}. \tag{32}$$

For a fixed choice of $n_0 = p^h$ and t, the right–hand side of the above inequality is an increasing function of n. We check that for (n_0, t) in (31) with $n_0 < 29$, the above inequality is not valid at $n = 6500$ and hence it is not valid for any $n \geq 6500$. Further, for $n_0 \geq 29$, we have $n_0 = p$ is prime and $t = 0$, which together with the observation that $g(p)$ is a decreasing function of p, we obtain

$$\begin{aligned} 0 &\geq 1.38(1 - g(n_0)) - \frac{4}{p^{t+1}} - \frac{\log(2p^{t+1})}{n} \\ &\geq 1.38(1 - g(29)) - \frac{4}{29} - \frac{\log(2 \cdot 29)}{n}. \end{aligned}$$

We check that the right–most side is positive for $n \geq 6500$ and hence we get a contradiction for any $n \geq 6500$. Thus, the equation (1) does not have an odd solution $n \geq 6500$ in case $\log |\alpha| < 4$.

Assume now that $\log|\alpha| > 4$. By Lemma 2, we get

$$\log|1-x^n| \geq -f(n)\log|\alpha|,$$

where $f(n)$ is given by formula (11). Using this inequality along with

$$|1-x^n| \leq 2 \quad \text{and} \quad n\log|\alpha| \geq 4n$$

(since $\log|\alpha| > 4$) in (29), we obtain

$$0 \geq 1 - \frac{1}{p^{t+1}} - g(n_0) + \frac{(1-g(n_0))f(n)}{n} - \frac{\log(2p^{t+1})}{4n}. \tag{33}$$

For a fixed $n_0 = p^h$ and t, the right–hand side of the above inequality is an increasing function of n. We check that for (n_0, t) in (31) with $n_0 < 29$, the above inequality is not valid at $n = 6500$ and hence it is not valid for any $n \geq 6500$. Further, for $n_0 \geq 29$, we have $n_0 = p$ is prime and $t = 0$ and hence the right–hand side of the above inequality is at least

$$1 - \frac{1}{29} - g(29) + \frac{(1-g(29))f(n)}{n} - \frac{\log(2\cdot 29)}{4n}.$$

We check that the above quantity is positive for $n \geq 6500$ and hence we get a contradiction for any $n \geq 6500$. Thus, the equation (1) has no odd solution $n \geq 6500$ in case $\log|\alpha| > 4$.

4.3 The case when α, β are real and $n \geq 5, n \notin \{6, 8, 12, 24\}$

We now consider the case when α and β are real. Recall that in this case $\alpha > 0$ and $U_n > 0$. For the proof of Theorem 1, we may assume that $n \geq 5$, $n \notin \{6, 8, 12, 24\}$. We will consider the case $n = 24$ separately in the next section. We choose $n_0 = p^h$ with $t = 0$ as

$$n_0 \in \{2^4, 3^2\} \cup \{p : p \geq 5\}. \tag{34}$$

Note that each $n \geq 5, n \notin \{6, 8, 12, 24\}$ is divisible by some n_0 in (34).

Let $n_0 = 2^4 = 16$. Then $p = 2$, $4 \mid n$ and hence

$$g(16)\log|1-x^n| < 0 \quad \text{and} \quad \frac{1-x^n}{1-x^{n/p}} = 1 + \sum_{i=1}^{p-1} x^{in/p} > 1.$$

Using this in (29) together with $n \geq 16$ and $\alpha \geq \dfrac{1+\sqrt{5}}{2}$, we get

$$0 \geq 1 - \frac{1}{2} - g(16) - \frac{\log 2}{n\log\alpha} \geq \frac{1}{2} - g(16) - \frac{\log 2}{16\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

We find that the right–most quantity is positive, which is a contradiction. Thus, equation (1) has no solution when α, β are real with $16 \mid n$.

Let $n_0 \neq 2^4$. Then $p > 2$. Writing

$$1 - x^n = (1 - x^{n/p}) \left(\frac{1 - x^n}{1 - x^{n/p}} \right),$$

we have

$$|1 - x^{n/p}| \leq 2 \quad \text{and} \quad \frac{1 - x^n}{1 - x^{n/p}} = \begin{cases} 1 + \sum_{i=1}^{p-1} y^i > 1, & y = x^{\frac{n}{p}} > 0; \\ \frac{1 - y(y^{(p-1)/2})^2}{1-y} \geq \frac{1}{1-y} > \frac{1}{2}, & y = x^{\frac{n}{p}} < 0. \end{cases}$$

Using this in (29), we obtain

$$\begin{aligned} \log \alpha &\leq \frac{(g(n_0) - 1) \log \left(\frac{1-x^n}{1-x^{n/p}} \right) + g(n_0) \log(1 - x^{n/p}) + \log p}{n(1 - 1/p - g(n_0)))} \\ &\leq \frac{(1 - g(n_0)) \log 2 + g(n_0) \log 2 + \log p}{n(1 - 1/p - g(n_0))} \\ &= \frac{\log(2p)}{n(1 - 1/p - g(n_0))}. \end{aligned}$$

This together with $n \geq n_0$ and $\alpha \geq \dfrac{1 + \sqrt{5}}{2}$ gives

$$\begin{aligned} \log \left(\frac{1 + \sqrt{5}}{2} \right) &\leq \log \alpha \leq \frac{\log(2p)}{n(1 - 1/p - g(n_0))} \\ &\leq \begin{cases} \frac{\log 2p}{n_0(1-1/p-g(n_0))}, & n_0 < 29; \\ \frac{\log(2 \cdot 29)}{29(1-1/29-g(29))}, & n_0 = p \geq 29. \end{cases} \end{aligned} \tag{35}$$

We check that the right–most quantity exceeds $\log \left(\dfrac{1 + \sqrt{5}}{2} \right)$ except when $n_0 = p \in \{5, 7\}$. Further, for $n_0 = p \in \{5, 7\}$, putting $n = p\ell$, we obtain by using (26),

$$\log(C_{m_k}/2) \leq n \log \alpha = p\ell \log \alpha \leq \frac{\log 2p}{1 - 1/p - g(p)} \leq \begin{cases} 15.62, & \text{if } p = 5; \\ 8.11, & \text{if } p = 7. \end{cases}$$

This gives $m_k \leq 15, 9$ according to whether $n_0 = p = 5, 7$, respectively. Further $1 \leq \ell \leq 6, 2$, according as $p = 5, 7$, respectively since $\alpha \geq \dfrac{1 + \sqrt{5}}{2}$.

This together with $P(U_n) \le P(B_{m_k})$ yields

$$\log\left(\frac{1+\sqrt{5}}{2}\right) \le \log\alpha \le \begin{cases} \frac{15.62}{5\ell}, & \text{if } p=5;\\ \frac{8.11}{7\ell}, & \text{if } p=7 \end{cases} \quad \text{and} \quad P(U_{p\ell}) \le \begin{cases} 29, & \text{if } p=5;\\ 19, & \text{if } p=7. \end{cases}$$

For the pairs (r,s) given by Lemma 3 with the conditions above, we check that the equation (1) has no solution with $n = p\ell$. Therefore, equation (1) has no solution for α, β real and $n \ge 5, n \notin \{6, 8, 12, 24\}$.

4.4 The case when α, β are real and $n = 24$

Let α, β be real and $n = 24$. Then $|x| < 1$. We have

(36)

$$\log\mathcal{D} = \log U_{24} = \log\left(\frac{\alpha^{24}-\beta^{24}}{\alpha-\beta}\right) = 23\log\alpha + \log\left(\frac{1-x^{12}}{1-x}\right) + \log(1+x^{12}).$$

We take $n_0 = n = 24$. Let $g > 0$ and $\lambda \ge 0$ be such that

(37)

$$M_{24}(\mathcal{D}) \le g\left(\log\mathcal{D} - \lambda\right) \le g(23\log\alpha + \log\left|\frac{1-x^{12}}{1-x}\right| + \log|1+x^{12}| - \lambda\Big).$$

In particular,

$$g \le g_0(24) = \frac{2.746}{8} \quad \text{and} \quad \lambda = 0,$$

by (28). We now take $n_0 = 24, t = 0$ in (4) to get a lower bound for $M_{24}(U_{24}) = M_{24}(\mathcal{D})$ and compare it with (37) to obtain

$$8\log|\alpha| + \log|1+x^{12}| - \log 6 \le g\left(23\log\alpha + \log\left|\frac{1-x^{12}}{1-x}\right| + \log|1+x^{12}| - \lambda\right).$$

This gives

$$(8-23g)\log\alpha \le g\left(\log\left|\frac{1-x^{12}}{1-x}\right| + \left(1-\frac{1}{g}\right)\log|1+x^{12}| + \frac{\log 6}{g} - \lambda\right).$$

Recall that $|x| < 1$. Assume that $x < 0$. Then

$$\frac{1-x^{12}}{1-x} = 1 + x\left(\frac{1-x^{11}}{1-x}\right) < 1 \quad \text{and} \quad 1 + x^{12} > 1,$$

which together with $g < 1$ implies the right hand side of the above inequality is strictly less than $\log 6$.

Assume next that $x > 0$. Then

$$\frac{1-x^{12}}{1-x} = 1 + x + x^2 + \cdots x^{11} < 12,$$

since $x < 1$. For any x_0 with $0 < x_0 < 1$, we have

$$\log\left|\frac{1-x^{12}}{1-x}\right| + \left(1-\frac{1}{g}\right)\log|1+x^{12}|$$
$$\leq \begin{cases} \log 12 + \left(1-\frac{1}{g}\right)\log(1+x_0), & x > x_0^{\frac{1}{12}}, \\ \log\left|\frac{1-x_0}{1-x_0^{\frac{1}{12}}}\right|, & x \leq x_0^{\frac{1}{12}}. \end{cases}$$

Putting

$$y_0 := y_0(g,x_0) = \frac{\log 6}{g} - \lambda + \max\left(\log 12 + \left(1-\frac{1}{g}\right)\log(1+x_0), \log\left|\frac{1-x_0}{1-x_0^{\frac{1}{12}}}\right|\right),$$

we get

$$\log\alpha < \frac{y_0 g}{8-23g} \qquad \text{or} \qquad \alpha < \exp\left(\frac{y_0 g}{8-23g}\right). \tag{38}$$

As stated before, we have

$$g \leq g_0(24) = \frac{2.746}{8} < 0.3433 \quad \text{and} \quad \lambda = 0,$$

by (28). Taking $g = 0.3433, \lambda = 0$ and $x_0 = 0.298$, we get $y_0 \leq 7.21$ and hence $\log\alpha < 23.78$ by (38). However, for $m_k \geq 420$, we have

$$\log\alpha \geq \frac{\log(C_{m_k}/2)}{24} \geq \frac{\log(C_{240}/2)}{24} > 24.82,$$

by (26). Thus, $m_k < 420$.

For each $j < 420$, let

$$\varepsilon_{1j} := \frac{M_{24}(C_j)}{\log C_j} \qquad \text{and} \qquad \varepsilon_{2j} := \frac{M_{24}(B_j)}{\log B_j}.$$

Then $\varepsilon_{1j} = \varepsilon_{2j} = 0$ for $j < 12$ since 23 is the least prime congruent to one of ± 1 modulo 24. We check that $\max(\varepsilon_{1j}, \varepsilon_{2j}) \leq 0.3433$ for $j < 420$. Write $U_{24} = \mathcal{D} = \prod_{i=1}^{k} D_{m_i}$ as

$$\log\mathcal{D} = \sum_j t_{1j}\log C_j + \sum_j t_{2j}\log D_j,$$

where

$$t_{1j} := \#\{i : D_{m_i} = C_{m_i}\} \qquad \text{and} \qquad t_{2j} := \#\{i : D_{m_i} = B_{m_i}\}.$$

Let $\varepsilon_0 \ge \max_j\{\varepsilon_{1j}, \varepsilon_{2j}\}$ for j such that $t_{1j} + t_{2j} > 0$. Then

$$\begin{aligned} M_{24}(U_{24}) &= \sum_j (\varepsilon_{1j} t_{1j} \log C_j + \varepsilon_{2j} t_{2j} \log B_j) \\ &= \varepsilon_0 \sum_j \left(\left(1 - \left(1 - \frac{\varepsilon_{1j}}{\varepsilon_0}\right)\right) t_{1j} \log C_j \right. \\ &\left. + \left(1 - \left(1 - \frac{\varepsilon_{2j}}{\varepsilon_0}\right)\right) t_{2j} \log B_j \right) \\ &\le \varepsilon_0 \left(\log \mathcal{D} - \sum_j t_{1j}\lambda_{1j} - \sum_j t_{2j}\lambda_{2j} \right), \end{aligned}$$

where

$$\lambda_{1j} := \left(1 - \frac{\varepsilon_{1j}}{\varepsilon_0}\right) \log C_j \quad \text{and} \quad \lambda_{2j} := \left(1 - \frac{\varepsilon_{2j}}{\varepsilon_0}\right) \log B_j.$$

It is clear that $\lambda_{1j} \ge 0$ and $\lambda_{2j} \ge 0$.

Suppose that $\alpha \le 100$. Then $t_{1j} + t_{2j} > 0$ implies $j \le m_k \le 85$ by (26) since $\log(C_{86}/2) > 24 \log 100$. For $j \le 85$ and $j \notin \{37, 38, 39, 40, 41, 42, 43\}$, we find that $\varepsilon_{1j}, \varepsilon_{2j} \le 0.29$. Taking $g = 0.29$ and $\lambda = 0$ in (37) and taking $x_0 = 0.25$, we get $y_0 \le 8.12$ and $\alpha < 5.88$ so $\log \alpha \le 1.771$. By (26) again, we have $j \le m_k \le 32$ since $\log(C_{33}/2) > 24 \log 5.88$ and we furthermore have $P(U_{24}) \le P(B_{32}) \le 61$. We check that the equation (1) with $P(U_{24}) \le 61$ and $\alpha \le 5.88$ is not possible. Here, we use Lemma 3 to find all possible pairs (r, s) with $\alpha \le 5.88$. Thus, we assume $t_{1j} + t_{2j} > 0$ for some $j \in \{37, 38, 39, 40, 41, 42, 43\}$. Also

$$\log \alpha \ge \frac{\log(C_{37}/2)}{24} > 2.0379,$$

by (26). Again $t_{1j} + t_{2j} > 0$ implies $j \le m_k \le 46$ since

$$\log C_{37} + \log C_{47} > 24 \log 100 + \log 2 \ge \log U_{24}.$$

Hence, $P(U_{24}) \le P(B_{46}) \le 89$. Further $47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \mid U_{24}$ also since $47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \mid C_j \mid B_j$ for $j \in \{37, 38, 39, 40, 41, 42, 43\}$. For the pairs (r, s) with $2.0379 < \log \alpha \le \log 100$ given by Lemma 3, we check that $P(U_{24}) \leq 89$ and $47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \mid U_{24}$ is not possible. Therefore, equation (1) has no solution when $\alpha \leq 100$.

From now on, we assume that $\alpha > 100$. Suppose that $t_{1j} = 0$ for $j \in \{37, 38\}$. Then we find that $\max(\varepsilon_{1j}, \varepsilon_{2j}) \leq \varepsilon_0 = 0.324$ for $j < 420$ with $j \neq 37, 38$, and also $\varepsilon_{2j} < 0.324$ for $j = 37, 38$. By taking $g = 0.324$ and $\lambda = 0$ in (37) and further $x_0 = 0.28$, we get $y_0 \leq 7.5$ and $\alpha < 84.3$. This is not possible. Therefore, we have $t_{1j} > 0$ for $j = 37$ or $j = 38$. Then $\max(\varepsilon_{1j}, \varepsilon_{1j}) \leq \varepsilon_0 = 0.3433$ for $j < 420$. Taking $g = \varepsilon_0 = 0.3433$ and $\lambda = \sum_j t_{1j}\lambda_{1j} + \sum_j t_{2j}\lambda_{2j}$ in (37) and further taking $x_0 = 0.298$, we obtain $y_0 \leq 7.21 - \lambda$ and

$$\log \alpha < \frac{0.3433\left(7.21 - \sum_j t_{1j}\lambda_{1j} - \sum_j t_{2j}\lambda_{2j}j\right)}{8 - 23 \times 0.3433}.$$

Together with $\alpha > 100$, this gives

$$\sum_j t_{1j}\lambda_{1j} + \sum_j t_{2j}\lambda_{2j} \leq 7.21 - \left(\frac{8}{0.3433} - 23\right)\log 100 \leq 5.8136. \tag{39}$$

We compute the values of λ_{1j} and λ_{2j} for $j < 420$ and find that

$$\lambda_{1j} \leq 5.8136 \quad \text{for} \quad j \in T_1 := \{j : j \leq 6\} \cup \{12, 13, 14, 37, 38, 39, 40, 41\},$$

and

$$\lambda_{2j} \leq 5.8136 \quad \text{for} \quad j \in T_2 := \{j : j \leq 5\} \cup \{12, 37, 38\}.$$

Thus, by (39), we may suppose that $t_{1j} > 0$ implies $j \in T_1$ and $t_{2j} > 0$ implies $j \in T_2$. Recall that we have $t_{1j} > 0$ for $j = 37$ or $j = 38$. Write t_4, t_5, t_{12} for t_{1j} according to whether $j = 4, 5, 12$, respectively. We find that $\lambda_{1j} \geq 2.639, 3.737, 3.111$ according to whether $j = 4, 5, 12$, respectively. Hence, from (39), we have $t_4 \leq 2, t_5 \leq 1$ and $t_{12} \leq 1$. Put

$$\log \mathcal{D}_1 := \sum_{j \in T_1, j \neq 4,5,12} t_{1j} \log C_j + \sum_{j \in T_2} t_{2j} \log B_j,$$

so that

$$\log \mathcal{D} = \log \mathcal{D}_1 + t_4 \log C_4 + t_5 \log C_5 + t_{12} \log C_{12}. \tag{40}$$

We now consider $M_8(\mathcal{D})$ given by (3). We find that $M_8(C_j) < 0.46 \log C_j$ for all $j \in T_1$ except when $j \in \{4, 5, 12\}$ and $M_8(B_j) < 0.46 \log B_j$ for $j \in T_2$ and further

$$M_8(C_4) \leq 0.74, \quad M_8(C_5) \leq 0.53 \quad \text{and} \quad M_8(C_{12}) \leq 0.65.$$

Hence, from (40), (36) and the fact that $\frac{1-x^{12}}{1-x} < 12$, we get

$$\begin{aligned} M_8(\mathcal{D}) <&0.46 \log \mathcal{D}_1 + 0.74 t_4 \log C_4 + 0.53 t_5 \log C_5 + 0.65 t_{12} \log C_{12} \\ <&0.46 \log \mathcal{D} + 0.28 t_4 \log C_4 + 0.07 t_5 \log C_5 + 0.19 t_{12} \log C_{12} \\ <&0.46(23 \log \alpha + \log 12 + \log(1 + x^{12})) \\ &+ 0.56 \log C_4 + 0.07 \log C_5 + 0.19 \log C_{12}, \end{aligned}$$

since $t_4 \leq 2, t_5 \leq 1$ and $t_{12} \leq 1$. Comparing the above inequality with the lower bound of $M_8(\mathcal{D}) = M_8(U_{24})$ given by (4) with $n_0 = 2^3$ and $t = 0$, we obtain

$$\begin{aligned} 4.07 >&0.56 \log C_4 + 0.07 \log C_5 + 0.19 \log C_{12} \\ >&(12 - 0.46 \times 23) \log \alpha + (1 - 0.46) \log(1 + x^{12}) - 0.46 \log 12 - \log 2 \\ >&(12 - 0.46 \times 23) \log 100 - 0.46 \log 12 - \log 2 > 4.7 \end{aligned}$$

since $1 + x^{12} > 0$ and $\alpha > 100$. This is a contradiction. Therefore, equation (1) has no solution with $n = 24$ when α and β are real.

4.5 The case of equation (2)

We now consider the equation (2). Since $V_n = U_{2n}/U_n$, we see that primitive divisors of V_n are the primitive divisors of U_{2n}. From the table listed in the beginning of Section 2, we find that the values of $n \geq 4$ for which V_n does not have a primitive divisor which are given by the instances for which U_{2n} has no primitive divisors belongs to the set $\{4, 5, 6, 9\}$. For $n \in \{4, 5, 6, 9\}$ and corresponding pairs (r, s) (which are given by pairs (r, s) corresponding to $2n$ in the table), we check that the equation (2) has no solution. Hence, for the proof of Theorem 1, we now assume that $n \geq 4$ and further V_n has a primitive divisor which is congruent to one of ± 1 modulo $2n$.

Let $n = 4t$ be even. Then

$$V_{4t} = \alpha^{4t} + \beta^{4t} = (\alpha^{2t} + \beta^{2t})^2 - 2(\alpha\beta)^{2t} = V_{2t}^2 - 2(-s)^{2t}.$$

For an odd prime $p \mid V_{4t}$, we see that 2 is a quadratic residue modulo p and hence $p \equiv \pm 1 \pmod 8$. We observe that both C_m and B_m are divisible by each prime $m+1 < p \leq 2m$. By Lemma 7, there is a prime $p \equiv \pm 5 \pmod 8$ with $m + 1 < p \leq 2m$ for each $m \geq 6$. Thus, equation (2) implies $m_k \leq 5$ which together with the fact that V_n has a primitive prime divisor gives $n = 4t = 4$. Further, $\gcd(V_t, s) = 1$ for all $t \geq 1$ gives $\nu_2(V_{2t}^2 - 2(s)^{2t}) \leq 1$ implying $\nu_2(V_4) \leq 1$. Considering $\nu_2(B_m), \nu_2(C_m)$ for $2 \leq m \leq 5$ and using the fact that V_4 has a primitive prime divisor which is congruent to $\pm 1 \pmod 8$, we get $V_4 = C_4 = 14$. Now $14 = V_4 = \alpha^4 + \beta^4 = r^2(r^2+4s) + 2s^2$ and $\gcd(V_4, s) = 1$ gives s odd and r even. Reducing the above relation modulo 8, we get $14 \equiv 2 \pmod 8$ which is a contradiction. Thus, equation (2) does not have a solution for n even with $4 \mid n$.

From now on, we take n odd with $n \geq 5$ or $2||n$ with $n \geq 6$. We have

$$\log \mathcal{D} = \sum_{i=1}^{k} \log D_{m_i} = \log |V_n| = \log |\alpha^n + \beta^n| = n \log |\alpha| + \log |1 + x^n|.$$

Since V_n has a primitive divisor which is congruent to one of ± 1 modulo $2n$, we have $2m_k - 1 \geq 2n - 1$ or $m_k \geq n$. By Lemma 4 and since $|1 + x^n| \leq 2$, we have

(41)
$$\log |\alpha| + \log 2 \geq \log |V_n| \geq \log 2 + \log C_{m_k/2} \geq \log 2 + 1.38n \text{ for } n \geq 2100.$$

Let p be an odd prime and $h > 0, t \geq 0$ be such that $p^{h+t} \mid n$. Taking $n_0 = p^h$, we use estimate (5) of Lemma 1 to get a lower bound for the quantity $M_{n_0}(\mathcal{D}) = M_{n_0}(V_n)$ and compare it with the upper bound given by Lemma 8 to obtain

$$g(n_0)(n \log |\alpha| + \log |1 + x^n|) \geq \left(1 - \frac{1}{p^{t+1}}\right) n \log |\alpha| + \log \left| \frac{1 + x^n}{1 + x^{n/p}} \right| - \log p^{t+1},$$

implying

(42)
$$\left(1 - \frac{1}{p^{t+1}} - g(n_0)\right) \leq \frac{(g(n_0) - 1) \log |1 + x^n| + \log |1 + x^{n/p^{t+1}}| + \log(p^{t+1})}{n \log |\alpha|},$$

where $g(n_0)$ is given by (27). We consider different cases as in the analysis for U_n.

Let α and β be complex conjugates. We may assume that $n \geq 6500$. We choose $n_0 = p^h$ and t given by (31). Assume that $\log |\alpha| \leq 4$. Then using

$$|1 + x^n| = \frac{|V_n|}{|\alpha|^n}, \quad |1 + x^{n/p^{t+1}}| \leq 2 \quad \text{and} \quad \log |\alpha| \leq 4,$$

along with (41) in (42), we obtain

$$\begin{aligned} 0 \geq & \frac{(1 - g(n_0)) \log |V_n| + \log(2p^{t+1})}{n \log |\alpha|} - \frac{1}{p^{t+1}} \\ \geq & \frac{1.38(1 - g(n_0)) + \log(2p^{t+1})}{4n} - \frac{1}{p^{t+1}}, \end{aligned}$$

which is the inequality (32). As in the case of U_n in Section 4.2, we a get a contradiction. Assume now that $\log|\alpha| > 4$. By Lemma 2, we get

$$\log|1+x^n| \geq -f(n)\log|\alpha|,$$

where $f(n)$ be given by (11). Using this along with $|1+x^{n/p^{t+1}}| \leq 2$ and $n\log|\alpha| > 4n$ (since $\log|\alpha| > 4$) in (42), we obtain the inequality (33). As in the case of U_n in Section 4.2, we a get a contradiction. Therefore, equation (2) has no solution $n \geq 6500$.

Let α and β be real. Then $\alpha > 0$. We take $n \geq 5, n \neq 6$. We choose $n_0 = p^h$ and t given by (34), $n_0 \neq 2^4$. Since p is odd, writing

$$1 + x^n = (1+x^{n/p})\left(\frac{1+x^n}{1+x^{n/p}}\right),$$

we have

$$|1+x^{n/p}| \leq 2 \quad \text{and} \quad \frac{1+x^n}{1+x^{n/p}} = \begin{cases} 1+\sum_{i=1}^{p-1}(-y)^i > 1, & y = x^{\frac{n}{p}} < 0; \\ \frac{1+y^p}{1+y} \geq \frac{1}{1+y} > \frac{1}{2}, & y = x^{\frac{n}{p}} > 0. \end{cases}$$

Using this in (42), we obtain

$$\begin{aligned} \log\alpha &\leq \frac{(g(n_0)-1)\log\left(\frac{1+x^n}{1+x^{n/p}}\right) + g(n_0)\log(1+x^{n/p}) + \log p}{n(1-1/p-g(n_0)))} \\ &\leq \frac{(1-g(n_0))\log 2 + g(n_0)\log 2 + \log p}{n(1-\frac{1}{p}-g(n_0))} = \frac{\log(2p)}{n(1-1/p-g(n_0))}. \end{aligned}$$

This together with $n \geq n_0$ and $\alpha \geq \dfrac{1+\sqrt{5}}{2}$ gives (35). As in the case of U_n in Section 4.3, we a get a contradiction except for $n_0 = p \in \{5,7\}$. Further, for $n_0 = p \in \{5,7\}$, putting $n = p\ell$, we obtain similarly

$$\log\left(\frac{1+\sqrt{5}}{2}\right) \leq \log\alpha \leq \begin{cases} \frac{15.62}{5\ell}, & \text{if } p=5; \\ \frac{8.11}{7\ell}, & \text{if } p=7, \end{cases} \quad \text{and} \quad P(U_{p\ell}) \leq \begin{cases} 29, & \text{if } p=5; \\ 19, & \text{if } p=7. \end{cases}$$

For the pairs (r,s) given by Lemma 3 with the above conditions, we check that the equation (2) has no solution at $n = p\ell$. Therefore, equation (2) has no solution for α, β real and $n \geq 5,\ n \neq 6$.

Writing $n = p\ell$, we have from $P(V_n) \leq P(B_{m_k})$ that

$$\log\left(\frac{1+\sqrt{5}}{2}\right) \leq \log\alpha \leq \begin{cases} \frac{15.62}{5\ell}, & \text{if } p=5; \\ \frac{8.11}{7\ell}, & \text{if } p=7, \end{cases} \quad \text{and} \quad P(V_{p\ell}) \leq \begin{cases} 29, & \text{if } p=5; \\ 19, & \text{if } p=7. \end{cases}$$

For the pairs (r,s) given by Lemma 3 with the conditions above, we check that the equation (1) has no solutions.

$\square$

5 The Proof of Theorem 2

First we prove the following result for $s = \pm 1$.

Lemma 9. *Let $s \in \{\pm 1\}$ and $r \geq 1$. Then $V_n \in \{C_m, B_m, 2C_m, 2B_m\}$ with $n > 1$ and $m > 1$ implies $n = 3$ or*

$$
\begin{aligned}
&n = 2 : (r, s; V_2) = (2, 1; B_2);\\
&n = 2 : (r, s; V_2) = (4, -1; C_4);\\
&n = 3 : (r, s; V_3) = (1, 1; 2C_2), (2, 1; C_4), (5, 1; 2B_4).
\end{aligned}
\tag{43}
$$

Proof. Let $m \geq 2$ and

$$\mathcal{D}_m := \{C_m, B_m, 2C_m, 2B_m\}.$$

By Theorem 1 and $C_2 = 2$, we have $V_n \in \mathcal{D}_m$ implies $n \in \{1, 2, 3, 6\}$. Let $n \in \{2, 6\}$ and $V_n \in \mathcal{D}_m$. Now $V_2 = r^2 + 2s$ and $V_6 = (r^2+2s)((r^2+2s)^2-3)$. Let $p \equiv 5 \pmod{12}$ be a prime such that $p \mid V_2V_6$. Then we either have $r^2 \equiv -2s \pmod{p}$ or $(r^2+2s)^2 \equiv 3 \pmod{p}$. This is not possible since both $\left(\frac{\pm 2}{p}\right) = \left(\frac{3}{p}\right) = -1$, where $(\frac{\cdot}{\cdot})$ is the Legendre symbol. Thus, $p \nmid V_2V_6$ for any prime $p \equiv 5 \pmod{12}$. By Lemma 7, we get $m \leq 8$. Further from $s = \pm 1$, we have $\nu_2(r^2 + 2s) \leq 1$ giving $\nu_2(V_2) = \nu_2(V_6) \leq 1$. Using both $5 \nmid V_2V_6$ and $\nu_2(V_2) = \nu_2(V_6) \leq 1$, we find that if $V_2, V_6 \in \mathcal{D}_m$ with $2 \leq m \leq 14$, then $V_2, V_6 \in \{C_m\}$ implies $m \in \{2, 4, 5, 7\}$; $V_2, V_6 \in \{2C_m\}$ implies $m = 7$; $V_2, V_6 \in \{B_m\}$ implies $m = 2$ and $V_2, V_6 \notin \{2B_m\}$. Now $V_2 = r^2 + 2s = E_m \in \mathcal{D}_m$ gives $r^2 = E_m - 2s$. We check that for the values $m \in \{2, 4, 5, 7\}$, $C_m - 2s$ is a square only when $r = 2, s = -1, C_2 = 2$ and $r = 4, s = -1, C_2 = 2$; $B_2 - 2s = 6 - 2s$ is a square only for $r = 2, s = 1$; and $2C_7 - 2s$ is not a square. These solutions are listed in (43) except that we omit $(r, s) = (2, -1)$ since it gives a degenerate characteristic equation. Let $V_6 = (r^2 + 2s)((r^2 + 2s)^2 - 3) = E_m \in \mathcal{D}_m$. We check that for $r \leq 3$, $V_6 = E_m \in \mathcal{D}_m$ only for $r = 2, s = -1, V_6 = C_2 = 2$ and we omit $(r, s) = (2, -1)$. For $r + 2s \geq 4$, we have $r_1 = r^2 + 2s \geq 14$ and hence $(r_1 - 2)^3 < r_1(r_1^2 - 3) = V_6 \leq C_7 = 429$. This gives $r_1 = r^2 + 2s \leq 9$, which is not possible.

Let $n = 3$ and $V_3 = r(r^2 + 3s) = E_m \in \mathcal{D}_m$. For $r \le 3$, we check that indeed $V_3 = E_m$ only for the pair $(r, s) = (2, 1)$. We now take $r \ge 4$. Using the inequality $r^3 < r(r^2 + 3) = E_m < (r + 1)^3$ when $s = 1$ and $(r - 2)^3 < r(r^2 - 3) = E_m < (r - 1)^3$ when $s = -1$, we get $r = \lfloor E_m^{1/3} \rfloor$ when $s = 1$ and $r = \lfloor E_m^{1/3} + 2 \rfloor$ when $s = -1$. For $m \le 15$, we find that putting $r = \lfloor E_m^{1/3} \rfloor$ gives $r(r^2 + 3) = E_m$ only when $r = 2$, $E_m = C_4 = 7$ and $r = 5$, $E_m = 2B_4 = 140$ which are already listed in (43) (except that we again omit $(r, s) = (2, -1)$) and putting $r = \lfloor E_m^{1/3} + 2 \rfloor$ gives $r(r^2 - 3) \ne E_m$. Thus, we assume that $m \ge 16$.

We observe that $\nu_2(r^2+3) \le 2, \nu_2(r^2-3) \le 1$ and $\nu_3(r^2 \pm 3) \le 1$. Further we observe that primes $p \mid r^2 + 3s$ with $p > 3$ satisfy $\left(\frac{-3s}{p}\right) = 1$. We get $p \equiv 1, 7 \pmod{12}$ if $p > 3$ when $s = 1$ and $p \equiv \pm 1 \pmod{12}$ if $p > 3$ when $s = -1$. We take $n_0 = 12$ and $\ell_0 = 7, -1$, according to whether $s = 1, -1$, respectively. From the equation

$$V_3 = r(r^2 + 3s) = E_m, \quad E_m \in \mathcal{D}_m,$$

we obtain

$$\log(r^2 + 3s) = 2\log r + \log\left(1 + \frac{3s}{r^2}\right) \le \xi(m) + \log 3 + \begin{cases} \log 4, & \text{if} \quad s = 1; \\ \log 2, & \text{if} \quad s = -1, \end{cases}$$

where

$$\xi(m) = \sum_{p \equiv 1, \ell_0 \pmod{12}} \nu_p(2B_m) \log p = \sum_{p \equiv 1, \ell_0 \pmod{12}} \nu_p(B_m) \log p.$$

From

$$\log E_m = \log r(r^2 + 3s) = \frac{3}{2}\left(2\log r + \log\left(1 + \frac{3s}{r^2}\right)\right) - \frac{\log(1 + 3s/r^2)}{2},$$

and

$$\frac{3}{2}\log 4 - \log(1 + 3s/r^2) < \begin{cases} \frac{3}{2}\log 4, & \text{if} \quad s = 1; \\ \frac{3}{2}\log 2 - \log(1 - 3/16) < \frac{3}{2}\log 4, & \text{if} \quad s = -1, \end{cases}$$

since $r \ge 4$, together with $E_m \ge C_m$, we get

$$\log C_m \le \log E_m < \frac{3}{2}\left(\xi(m) + \log 12\right) \quad \text{implying} \quad \frac{2}{3} < \frac{\xi(m)}{\log C_m} + \frac{\log 12}{\log C_m}.$$

Hence,

$$\log C_m < \frac{\log 12}{\frac{2}{3} - \frac{\xi(m)}{\log C_m}}. \tag{44}$$

For $16 \le m \le 35$, we find that $\frac{\xi(m)}{\log C_m} < 0.52$ and therefore

$$\log C_m < \frac{\log 12}{\frac{2}{3} - 0.52} < \log C_{16},$$

which is a contradiction. Thus, we have $m \ge 36$. For $36 \le m < 1500$, we check that $\frac{\xi(m)}{\log C_m} < 0.59$ and hence $\log C_m < \frac{\log 12}{\frac{2}{3} - 0.59} < \log C_{36}$, which is a contradiction again. Thus, $m \ge 1500$. As in the proof of Lemma 8, we get

$$\begin{aligned}\xi(m) \le & \sum_{\substack{(2m)^{1/2} < p \le 2m \\ p \equiv 1, \ell_0 \pmod{12}}} \left(\left\lfloor \frac{2m}{p} \right\rfloor - 2 \left\lfloor \frac{m}{p} \right\rfloor \right) \log p \\ & + \sum_{\substack{p \le (2m)^{1/2} \\ p \equiv 1, \ell_0 \pmod{12}}} \left\lfloor \frac{\log(2m)}{\log p} \right\rfloor \log p \\ & \le \sum_{\ell \in \{1, \ell_0\}} \left\{ \psi(2m; 12, \ell) + \sum_{t=2}^{3} \theta(2m/t; 12, \ell) - \sum_{t=1}^{2} \theta(m/t; 12, \ell) \right\}.\end{aligned}$$

The above inequality with Lemma 7, yields

$$\xi(m) \le \frac{\delta_1 m}{4} = \frac{\delta_1 m}{4 \log C_m} \log C_m \quad \text{for} \quad m \ge 1500,$$

where

$$\delta_1 = \frac{47}{15} + \frac{2\sqrt{2} \times 0.863}{\sqrt{1500}} \left(1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}\right) + \frac{2 \times 1.5}{\sqrt{1500}} \left(1 + \frac{1}{\sqrt{2}}\right).$$

Hence,

$$\frac{\xi(m)}{\log C_m} \le \frac{\delta_1}{4} \frac{m}{\log C_m} \le \frac{\delta_1}{4} \frac{1500}{\log C_{1500}} < 0.62,$$

by Lemma 4. Inserting this last estimate into (44), we get

$$\log C_m < \frac{\log 12}{\frac{2}{3} - 0.62} < 54 < \log C_{50}.$$

This is a contradiction and the proof of Lemma 9 is complete. □

Proof of Theorem 2: Let d be a squarefree positive integer and assume $\varepsilon \in \{\pm 1\}$. Let (X_n, Y_n) be the nth solution of the equation $X^2 - dY^2 = \varepsilon$. Then $X_n = (\alpha^n + \beta^n)/2$ where (α, β) are the two roots of the quadratic $x^2 - (2X_1)x + \varepsilon = 0$. Observe that $C_2 = 2$. Thus, $X_n \in \{C_m, B_m\}$ gives $V_n \in \{2C_m, 2B_m\}$ where $V_n = \alpha^n + \beta^n$ and $s = -\alpha\beta = \pm 1$. Then for $n > 1$, V_n is given by Lemma 9, namely $n = 3$, $V_n \in \{2C_2, 2B_4\}$. Then $X_n = V_n/2 \in \{C_2 = 2, B_4 = 70\}$. The solutions given by

$$\begin{aligned} & 2^2 - 3 \cdot 1^2 = 1, && 2^2 - 5 \cdot 1^2 = -1 \\ \text{and} \quad & 70^2 - 3 \cdot 23 \cdot 71 \cdot 1^2 = 1, && 70^2 - 29 \cdot 13^2 = -1, \end{aligned}$$

are exactly (X_1, Y_1) of the corresponding Pell equations and the assertion of Theorem 2 for X_n follows.

We now consider solutions (W_n, Z_n) of $W^2 - dZ^2 = 4\varepsilon$ with $\varepsilon \in \{\pm 1\}$. Assume that $W_n \in \{C_m, B_m\}$. Note that by putting

$$\alpha := (W_1 + \sqrt{d}Z_1)/2 \quad \text{and} \quad \beta := (W_1 - \sqrt{d}Z_1)/2,$$

we have that $W_n = \alpha^n + \beta^n = V_n$ and $s = -\alpha\beta = \pm 1$. By Lemma 9, we get that either $n = 1$ or $n = 2$ with $V_2 \in \{C_2 = 2, B_2 = 6, C_4 = 14\}$ or $n = 3$ with $V_3 \in \{C_2 = 2, C_4 = 14\}$ or $n = 6$ with $V_6 = C_2 = 2$. For $n \neq 1$, we have solutions

$$d = 2, (W_2, Z_2) = (B_2, 4) \text{ with } 6^2 - 2 \cdot 4^2 = 4 \text{ (and } (W_1, Z_1) = (2, 2));$$
$$d = 2, (W_2, Z_2) = (C_4, 10) \text{ with } 14^2 - 2 \cdot 10^2 = -4 \text{ (and } (W_1, Z_1) = (2, 2));$$
$$d = 3, (W_2, Z_2) = (C_4, 8) \text{ with } 14^2 - 3 \cdot 8^2 = 4 \text{ (and } (W_1, Z_1) = (4, 2)).$$

The solution given by $B_2^2 - 10 \cdot 2^2 = 6^2 - 40 = -4$ is exactly $(W_1, Z_1) = (6, 2)$ for $d = 10$. This finishes the proof of Theorem 2. $\square$

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