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SUSY in the Sky with Gravitons

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Abstract

Picture yourself in the wave zone of a gravitational scattering event of two massive, spinning compact bodies (black holes, neutron stars or stars). We show that this system of genuine astrophysical interest enjoys a hidden $\mathcal{N}=2$ supersymmetry, at least to the order of spin-squared (quadrupole) interactions in arbitrary D spacetime dimensions. Using the $\mathcal{N}=2$ supersymmetric worldline action, augmented by finite-size corrections for the non-Kerr black hole case, we build a quadratic-in-spin extension to the worldline quantum field theory (WQFT) formalism introduced in our previous work, and calculate the two bodies' deflection and spin kick to sub-leading order in the post-Minkowskian expansion in Newton's constant G. For spins aligned to the normal vector of the scattering plane we also obtain the scattering angle. All D-dimensional observables are derived from an eikonal phase given as the free energy of the WQFT that is invariant under the $\mathcal{N}=2$ supersymmetry transformations.

Contents

1 Introduction

Our Universe is populated by massive astrophysical bodies (black holes, neutron stars and stars) that rotate intrinsically — they carry spin. Along their trajectories through space and time scattering events may occur that are mediated by the gravitational force. The two scattered bodies will be deflected, their spins will be altered and the scattering process will emit gravitational Bremsstrahlung, which could be detected in future-generation gravitational wave observatories on or near Earth. In this work we show that this astrophysical system enjoys a hidden, extended ($\mathcal{N} = 2$) supersymmetry that constrains the dynamics of the spinning scattering process — at least to the order of spin-squared (or quadrupole) interactions. The supersymmetry is manifested by anti-commuting worldline vectors ψ^a attributed to the spin tensor of the body ($S^{ab} \sim \bar{\psi}^{[a}\psi^{b]}$). These fermionic degrees of freedom are a natural ingredient in the recently developed worldline quantum field theory (WQFT) description of massive spinning bodies [1–3]. The appearance of supersymmetry in such a problem of genuine astrophysical interest was first realized in ref. [4] by studying hidden symmetries of the spinning black hole (Kerr-Newman) solution.

The investigation of classical gravitational scattering has a long history in general relativity [5–10], being naturally performed in a post-Minkowskian (PM) perturbative expansion in Newton's constant G about a flat spacetime background ($g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}$) with the graviton field $h_{\mu\nu}$. This *unbound* setup is different to the intensively studied post-Newtonian (PN) expansion in both G and relative velocity ($\frac{Gm}{r} \sim v^2$) often used for binary inspirals, i.e. massive bodies on *bound* orbits with separation r. The gravitational radiation emitted in the inspiral finally leads to a merger and is now routinely detected in gravitational wave observatories [11–14]. Even though Bremsstrahlung-emitting gravitational scattering events appear to be out of reach for current gravitational wave observatories due to their non-periodic signal and lower intensity [15–17] they represent interesting targets for future searches, calling for precision calculations.

It has also been emphasized recently that gravitational scattering is relevant for the study of binary inspirals. There exist various options for mapping between the bound and unbound problems, including reconstructing a gravitational two-body potential [18–22], or mapping directly between physical observables such as the scattering angle (unbound) and periastron advance (bound) [23, 24]. The scattering problem per se is the natural habitat for quantum field theory (QFT) that was largely designed to describe the scattering of elementary particles in collider experiments. Applying the classical limit of such a perturbative Feynman-diagrammatic expansion of the path integral of matter-coupled gravity in a PM expansion has proven to be highly efficient, and there are two popular QFT-based approaches.

The worldline effective field theory (EFT) formalism [25–27] models massive bodies as point-like particles moving along their worldlines coupled to the gravitational field. Spin effects are naturally incorporated by introducing a spin tensor and co-moving frame along the worldline [28–30]. Similarly, finite-size and tidal effects may be systematically included by coupling the worldline degrees of freedom to higher-dimensional operators dressed with Wilson coefficients (or Love numbers) [31]. Integrating out the graviton fluctuations in the path integral yields an effective action for the interaction of two spinning bodies — see Refs. [28, 29] for reviews of the PN framework. Impressively high orders in the PN [32–37] and PM [38–41] expansions have also been established.

The second rapidly developing approach is the amplitudes-based formalism [19, 42–47]. Massive elementary particles (scalars, spin-1/2 fermions, etc.) minimally coupled to the gravitational field are the avatars of black holes, neutron stars or stars; their $2 \to 2$ quantum scattering amplitudes are constructed by employing modern innovations such as generalized unitarity and the double copy, see [48–51] for reviews. Yet, the $2 \to 2$ amplitude is only the starting point for a subtle classical limit [52–54]. This procedure then yields the gravitational potential between two massive spinning bodies and observables such as the deflection and spin kick have been derived from it. A direct path from amplitudes to classical observables was introduced in ref. [52]. Nevertheless, the inclusion of spin poses certain challenges, in particular going beyond spin-squared interactions [55] due to the known hurdle of constructing an interacting higher-spin quantum field theory. Tidal and finite-size effects have also been included [56–58]. Amplitudes-based approaches can employ the powerful tools developed in collider physics and have led to impressively high-order calculations without spin [44–46, 54, 59–61] and first results with spin [55, 62] including radiation effects [63–71]. A related variant of the amplitude approach is the heavy-particle EFT [72–75], which enables a more straightforward classical limit of the amplitude from the outset.

The WQFT is a new formalism that unites these two approaches and clarifies their interrelations [1–3]. The worldline path integral representation of a massive particle's graviton-dressed propagator corresponds to the EFT worldline theory's path integral. Inserting this into the QFT S-matrix (represented as a time-ordered correlation function) a precise connection between the EFT and amplitudes-based approaches was given for the spinless case [1], which involves the *classical* $\hbar \to 0$ limit. A key feature of the WQFT approach — distinguishing it from the EFT approach — is that *both* the graviton field $h_{\mu\nu}$ *and* the worldline fluctuations about straight-line backgrounds (in the scattering case) are integrated out in the path integral. In short, the worldline degrees of freedom are also quantized. This leads to an economic Feynman-diagram-based perturbative solution to the equations of motion of the matter-graviton system.

In the WQFT, direct access to classical observables, such as the spinless particle deflection [1] and the time-domain asymptotic gravitational waveform [2, 3] emerging from the scattering encounter, has been made. The approach is economic in the sense that (a) one only computes the classically relevant contributions, circumventing the "super-classical" subtleties of the amplitudes-based approach, and (b) there is no need to determine a non-observable and gauge-dependent effective potential, thereafter solving the resulting equations of motion perturbatively as is done in the EFT approach. In fact, the WQFT formalism is to date the only approach used to successfully construct the asymptotic gravitational waveform of the QFT-based approaches with or without spin [2, 3].[1]

In this paper we report on the inclusion of spin degrees of freedom in the WQFT. As mentioned above spin is introduced through anti-commuting worldline vectors,

[1] See refs. [47, 76, 77] for work on the gravitational Bremsstrahlung *integrand* or the related problem in electrodynamics in the amplitudes and EFT based approaches.

building upon refs. [4, 78–80]. In those works it was shown that the higher-spin $\mathcal{N}/2$ field equations (in flat space) follow from quantizing an $\mathcal{N}$-extended supersymmetric particle augmenting the coordinate vector $x^\mu(\tau)$ by $\mathcal{N}$ anti-commuting vectors ψ^μ. In (generic) curved spacetimes this is only possible up to $\mathcal{N} = 2$ supersymmetry (or spin-1) as we will discuss in section 2. This limits our approach to the gravitational scattering of spinning objects up to interactions of quadratic order in spin (quadrupoles) at present. In our recent letter [3] we have already used this formalism to establish the asymptotic waveform of a spinning gravitational Bremsstrahlung event, extending the seminal work of Crowley, Kovacs and Thorne [5–7] to the spinning case.

Our spinning WQFT formalism innovates over existing approaches to classical spin based on corotating-frame variables [28, 29] in the effective field theory (EFT) of compact objects [25–27, 81, 82] — see refs. [83–85] for the construction of PM integrands and refs. [30, 38] for the computation of worldline and spin deflections (shown to be in agreement with the amplitude based results [55, 62]). The EFT approach was applied to radiation also in the PN approximation [86–90]— see refs. [91, 92] for more traditional methods. Other approaches to PM spin effects can be found in refs. [72, 73, 93–102].

This paper is organized as follows. In section 2 we review the supersymmetric worldline formalism and give a detailed analysis of the $\mathcal{N} = 2$ supersymmetric particle in a curved background. In section 3 we show that this theory is equivalent to the traditional description of spinning particles using spin and body-fixed frame fields, and explain how finite-size effects may be incorporated while maintaining the supersymmetry up to the desired quadratic-in-spin order. In section 4 we lay out the spinning WQFT and establish its Feynman rules, the relationship of the eikonal phase of a scattering event to the free energy of the spinning WQFT and show the hidden supersymmetry properties of the eikonal phase. Finally, in section 5 we compute a larger class of observables: the eikonal phase up to 2PM (next-to-leading) order from which we derive the deflection and spin kick. We close with concluding remarks and in the appendices give details of the $\mathcal{N} = 2$ supersymmetry of the gauge-fixed spinning WQFT action and on the computation of the necessary integrals arising in section 5.

2 Supersymmetric worldline actions

Extending the WQFT to include spin calls for a worldline theory of a relativistic spinning particle. In this section we review the first-order formulation of spinning particle actions where spin is represented by anti-commuting vector fields. Our main focus is the $\mathcal{N} = 2$ supersymmetric theory in a generic curved background, which represents massive spinning bodies up to quadratic order in spin.

2.1 Putting spin on the worldline

To begin with recall the first-order formulation of the massless non-spinning particle in a gravitational background:

$$\tilde{S}_0 = -\int \mathrm{d}\tau \left[p_M \dot{x}^M - e\, H\right] , \qquad \text{with} \quad H = \tfrac{1}{2} p_M p_N g^{MN} . \tag{2.1}$$

The transition to the massive case is easily done through a dimensional reduction setting $p^M = (p^\mu, m)$, which yields $H = \frac{1}{2}(p^2 - m^2)$. We work with mostly minus signature and take the spacetime manifold to be $M^{1,D-1} \times S^1$. Eliminating the momentum through its algebraic equations of motion, $p^\mu = e^{-1}\dot{x}^\mu$, yields the second-order action

$$S_0 = -\int \mathrm{d}\tau \left[\frac{e^{-1}}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{e}{2} m^2\right] , \tag{2.2}$$

which is the "Polyakov" formulation of the non-spinning worldline theory enjoying a local reparametrization invariance $\tau \to \tau'(\tau)$ under which e transforms as $e \to e\,\dot{\tau}'$. One may also eliminate the einbein e using its algebraic equations of motion to arrive at the usual action for a massive particle $m \int \mathrm{d}s$. However, it is more convenient to choose the proper-time gauge $e = 1/m$ as this linearizes the graviton worldline interaction [1, 38].

In order to include spin for the particle one adds real anti-commuting *vector* fields $\psi^a_\alpha(\tau)$ to the worldline degrees of freedom [78, 103, 104]. Here $\alpha = 1, \ldots \mathcal{N}$ is a flavor index while $a = 0, \ldots, D-1$ is the flat tangent space index related to the curved spacetime index $\mu = 0, \ldots, D-1$ via the vielbein $e^a_\mu(x)$, with $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ as usual. Of particular interest are the supersymmetric worldline theories which have been shown to describe the propagation of relativistic quantum fields of spin $1/2$ for $\mathcal{N} = 1$ supersymmetry [103], spin 1 for $\mathcal{N} = 2$ supersymmetry and generally spin $\mathcal{N}/2$ for $\mathcal{N}$-extended supersymmetry [78, 104] in a *flat* spacetime background, respectively. The $\mathcal{N}$-extended supersymmetric spinning particle generalization of eq. (2.1) in a *flat* $(D+1)$-dimensional spacetime background is[2]

$$\tilde{S}_N = -\int \mathrm{d}\tau \left[p_M \dot{x}^M + \tfrac{i}{2}\psi^A_\alpha \dot{\psi}^B_\alpha \eta_{AB} - e\, H - i\chi^\alpha Q_\alpha - \frac{1}{2} f_{\alpha\beta} M^{\alpha\beta}\right] \tag{2.3}$$

with the conserved charges

$$H = \tfrac{1}{2}p^2 , \qquad Q_\alpha = p \cdot \psi_\alpha , \qquad M_{\alpha\beta} = i\psi_\alpha \cdot \psi_\beta . \tag{2.4}$$

The Lagrange multipliers of the einbein $e(\tau)$, the $\mathcal{N}$ gravitinos $\chi^a(\tau)$ and the $\mathrm{O}(\mathcal{N})$ gauge field $f_{\alpha\beta}(\tau)$ enforce the conservation of these charges. This gauged theory may be thought of as a *locally* supersymmetric worldline theory, i.e. $\mathcal{N}$-extended

[2] We take $M = (\mu, D)$ and $A = (a, D)$ with an S^1 in the $(D+1)$th dimension.

1D supergravity, supersymmetrizing the reparametrization invariance of the spinless case eq. (2.1). Upon using the Poisson (resp. Dirac) brackets for the worldline fields

$$\{x^M, p_N\}_{\text{P.B.}} = \delta^M_N\,, \qquad \{\psi^A_\alpha, \psi^B_\beta\}_{\text{P.B.}} = -i\delta_{\alpha\beta}\,\eta^{AB}\,, \tag{2.5}$$

one derives the $\mathcal{N}$-extended supersymmetry algebra

$$\begin{aligned} \{Q_\alpha, Q_\beta\}_{\text{P.B.}} &= -2i\delta_{\alpha\beta}H\,, \qquad \{H, Q_\beta\}_{\text{P.B.}} = \{H, M_{\alpha\beta}\}_{\text{P.B.}} = 0\,, \\ \{M_{\alpha\beta}, Q_\gamma\}_{\text{P.B.}} &= -2\delta_{\gamma[\alpha}Q_{\beta]}\,, \qquad \{M_{\alpha\beta}, M^{\gamma\delta}\}_{\text{P.B.}} = -4\delta_{[\alpha}{}^{[\gamma}\,M_{\beta]}{}^{\delta]}\,. \end{aligned} \tag{2.6}$$

The charges $M_{\alpha\beta}$ generate an $\mathrm{O}(\mathcal{N})$ R-symmetry algebra. Again, a dimensional reduction of the theory from $D+1$ to D dimensions, thereby setting $p^M = (p^\mu, m = \text{const})$ and $(\psi^A = \psi^a_\alpha, \theta_\alpha)$, yields the massive theory in D dimensions — as in the spinless case of eq. (2.2).

The relevant question for our application is whether this theory may be embedded in an *arbitrary curved* spacetime background whilst preserving supersymmetry. This is only possible for $\mathcal{N} \leq 2$ describing a spin 0, 1/2 or spin 1 particle in a *generic* curved background [78–80, 104, 105]. For the $\mathcal{N} = 4$ spinning particle a consistent quantization requires the backgound spacetime to be an Einstein manifold, i.e. $R_{\mu\nu} = \lambda g_{\mu\nu}$ [106]. As stated earlier, our focus is now the spinning $\mathcal{N} = 2$ superparticle in a generic curved background, which allows us to describe spinning massive bodies up to *quadratic* order in spin.

2.2 $\mathcal{N} = 2$ spinning superparticle in a curved background

Following ref. [80] in order to prepare for the massive theory via dimensional reduction we consider the $\mathcal{N} = 2$ spinning superparticle in the curved background $M^{1,D-1} \times S^1$. It is convenient to combine the two real Grassmann fields into one complex Grassmann worldline field via $\psi^A = \frac{1}{\sqrt{2}}(\psi^A_1 + i\psi^A_2)$ and $\bar{\psi}^A = \frac{1}{\sqrt{2}}(\psi^A_1 - i\psi^A_2)$. We note the Poisson (resp. Dirac) brackets

$$\{x^M, p_N\}_{\text{P.B.}} = \delta^M_N\,, \qquad \{\psi^A, \bar{\psi}^B\}_{\text{P.B.}} = -i\eta^{AB}\,. \tag{2.7}$$

The covariantization of the super-charges (2.4) takes the form

$$Q = \psi^a\, e^\mu_a(x)\, \pi_\mu + \theta\, m\,, \qquad \bar{Q} = \bar{\psi}^a\, e^\mu_a(x)\, \pi_\mu + \bar{\theta}\, m\,, \tag{2.8}$$

where we have split off the fifth dimension explicitly $\psi^5 := \theta$, $p_5 = m$ and introduced the covariantized four-momentum

$$\pi_\mu := p_\mu - i\omega_{\mu ab}\bar{\psi}^a\psi^b\,, \tag{2.9}$$

with the spin connection $\omega_\mu{}^{ab} = e^a_\nu(\partial_\mu e^{\nu b} + \Gamma^\nu_{\mu\lambda}e^{\lambda b})$. The Hamiltonian may now be derived from the Poisson bracket $\{Q, \bar{Q}\}_{\text{P.B.}}$. Using $\{\pi_\mu, \pi_\nu\}_{\text{P.B.}} = iR_{\mu\nu ab}\bar{\psi}^a\psi^b$ and $\{\psi^\mu, \pi_\nu\}_{\text{P.B.}} = \Gamma^\mu_{\nu\rho}\psi^\rho$ one finds

$$\{Q, \bar{Q}\}_{\text{P.B.}} = -2i\Big[\underbrace{\tfrac{1}{2}(g^{\mu\nu}\pi_\mu\pi_\nu - m^2 - R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d)}_{H}\Big] - 2\,\underbrace{\pi_\mu\bar{\psi}^a\psi^b\,T^\mu{}_{ab}}_{T}\,. \tag{2.10}$$

Note that the last term T couples to the torsion tensor $T^{\nu}{}_{\mu\rho} := \Gamma^{\nu}_{[\mu\rho]}$, i.e. the anti-symmetric part of the affine connection that vanishes in Einstein gravity. In addition $\{Q, Q\}_{\text{P.B.}} = 0$ vanishes due to the cyclic identity for the Riemann tensor. Finally, in the complex basis for the Grassmann variables the internal R-symmetry turns into a U(1) symmetry generated by the charge

$$J = \bar{\psi}^a \psi^b \eta_{ab} - \bar{\theta}\theta \,, \tag{2.11}$$

which obeys $\{J, Q\}_{\text{P.B.}} = -iQ$ and $\{J, \bar{Q}\}_{\text{P.B.}} = i\bar{Q}$.

In summary, the gauged first-order form of the $\mathcal{N} = 2$ superparticle action in a curved spacetime background takes the form

$$\begin{aligned} \tilde{S} &= -\int \mathrm{d}\tau \left[p_\mu \dot{x}^\mu + i\bar{\psi}^a \dot{\psi}^b \eta_{ab} - i\bar{\theta}\dot{\theta} - e\, H - i\bar{\chi}Q - i\chi\bar{Q} + a\, J \right] \\ &= -\int \mathrm{d}\tau \Big[p_\mu \dot{x}^\mu + i\bar{\psi}^a \dot{\psi}^b \eta_{ab} - i\bar{\theta}\dot{\theta} - \frac{e}{2}\left(g^{\mu\nu}\pi_\mu \pi_\nu - m^2 - R_{abcd}\bar{\psi}^a \psi^b \bar{\psi}^c \psi^d \right) \\ &\qquad - i\pi_\mu(\bar{\chi}\psi^\mu - \bar{\psi}^\mu \chi) - im(\bar{\chi}\theta - \bar{\theta}\chi) + a(\bar{\psi}\cdot\psi - \bar{\theta}\theta) \Big] \,, \end{aligned} \tag{2.12}$$

with the einbein e, the complex gravitino χ and the U(1) gauge field a. Eliminating p_μ by its algebraic equations of motion yields the action

$$\begin{aligned} S = -\int \mathrm{d}\tau \Big[& \frac{e^{-1}}{2} g_{\mu\nu}(\dot{x}^\mu - i\bar{\chi}\psi^\mu - i\chi\bar{\psi}^\mu)(\dot{x}^\nu - i\bar{\chi}\psi^\nu - i\chi\bar{\psi}^\nu) \\ & + i\bar{\psi}^a(\dot{\psi}_a + \dot{x}^\mu \omega_{\mu ab}\psi^b) + \frac{e}{2} R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d + \frac{e}{2}m^2 + a(\bar{\psi}^a\psi_a - \bar{\theta}\theta) \\ & - i\bar{\theta}\dot{\theta} - im(\bar{\chi}\theta + \chi\bar{\theta}) \Big] \,. \end{aligned} \tag{2.13}$$

A number of comments are now in order. Firstly, this theory is invariant under *local* reparametrization, supersymmetry and U(1) gauge symmetries under which both the worldline fields x^μ, ψ_a as well as $e, \chi, \bar{\chi}, a$ transform — see refs. [79, 80] for explicit formulae. Secondly, as shown in ref. [79], one may add a Chern-Simons term for the gauge field a to the action

$$S_{CS} = s \int \mathrm{d}\tau\, a \,, \qquad s = \text{const} \,, \tag{2.14}$$

which is invariant under the U(1) transformation $\delta a = \dot{\alpha}$, $\delta\psi_a = i\alpha\, \psi_a$. Finally, one may set the Lagrange multipliers χ and a to zero, yielding the constraints

$$\begin{aligned} &\bar{\chi} = 0 : \quad g_{\mu\nu}\psi^\mu \dot{x}^\nu + em\theta = 0 \,, \qquad \chi = 0 : \quad g_{\mu\nu}\bar{\psi}^\mu \dot{x}^\nu + em\bar{\theta} = 0 \,, \\ &a = 0 : \quad \eta_{ab}\bar{\psi}^a \psi^b - \bar{\theta}\theta = s \,. \end{aligned} \tag{2.15}$$

We shall see in section 3 that these may be related to the spin-supplementary condition (SSC) as well as the conservation of the total spin in the traditional formulation of describing spinning compact objects.

The resulting second-order theory is then

$$S^{\mathcal{N}=2} = -\int \mathrm{d}\tau \Big[\frac{e^{-1}}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + i\bar{\psi}^a(\dot{\psi}_a + \dot{x}^\mu \omega_{\mu ab}\psi^b) + \frac{e}{2} R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d + \frac{e}{2}m^2 - i\bar{\theta}\dot{\theta}\Big]. \tag{2.16}$$

The fermionic extra dimensional contribution $\bar{\theta}\dot{\theta}$ is free and couples only to the other fields via the constraints (2.15) — we drop it from now on. Moreover, it is convenient to further make the gauge choice $e = 1/m$ and rescale the fermions by $\sqrt{m}$:

$$\psi^a \to \sqrt{m}\,\psi^a\,, \qquad \bar{\psi}^a \to \sqrt{m}\,\bar{\psi}^a\,. \tag{2.17}$$

This renders the mass-shell constraint as

$$\dot{x}^\mu \dot{x}_\mu = 1 + R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\,. \tag{2.18}$$

Therefore, strictly speaking τ is not the proper time in this gauge. This gauge fixed spinning worldline action (2.16) (setting $e = 1/m$ and rescaling the fermions) is invariant under the following global $\mathcal{N} = 2$ supersymmetry transformations:

$$\begin{aligned} &\delta x^\mu = i e^\mu_a(\bar{\epsilon}\psi^a + \epsilon\bar{\psi}^a)\,, \\ &\delta\psi^a = -\epsilon e^a_\mu\,\dot{x}^\mu - \delta x^\mu \omega_\mu{}^a{}_b\psi^b\,, \quad \delta\bar{\psi}^a = -\bar{\epsilon} e^a_\mu\,\dot{x}^\mu - \delta x^\mu \omega_\mu{}^a{}_b\bar{\psi}^b\,, \end{aligned} \tag{2.19}$$

the analysis of which is relegated to appendix A. There is also a manifest global U(1) symmetry

$$\delta_a\psi^a = ia\,\psi^a\,, \qquad \delta_a\bar{\psi}^a = -ia\,\bar{\psi}^a\,, \qquad \delta_a x^\mu = 0\,, \tag{2.20}$$

as well as the remnant of the reparametrization symmetry generated by H of eq. (2.10), which is given by the commutator of two supersymmetries (2.19).

3 Comparison with a spinning compact body

In this section we demonstrate that the $\mathcal{N} = 2$ spinning superparticle action (2.16) represents an alternative description of the classical physics of a spinning compact body moving in a gravitational background, up to and including terms quadratic in the spin (quadrupoles). For this we need to augment the spinning superparticle action by an additional term capturing finite-size effects and distinguishing black holes from (neutron) stars. Surprisingly, supersymmetry is preserved in an approximate sense under this extension of the theory and may be linked to the spin-supplementary condition (SSC).

3.1 Traditional worldline action

The traditional worldline action of a spinning compact body describes spin via a body-fixed frame ${\Lambda_A}^\mu(\tau)$ along the worldline using the flat indices $A, B, \ldots$ that are distinct from the vielbein indices $a, b, \ldots$ from above. The first-order action takes the form [81, 82, 107]

$$S^{\rm spin} = -\int {\rm d}\tau \left[\pi_\mu \dot{x}^\mu + \frac{1}{2} S_{\mu\nu} \underbrace{{\Lambda_A}^\mu \frac{D\Lambda^{A\nu}}{D\tau}}_{=:\,\Omega^{\mu\nu}} - \lambda(\pi_\mu \pi^\mu - \mathcal{M}^2) - \chi_\mu S^{\mu\nu}\left(\frac{\pi_\nu}{\sqrt{\pi^2}} + \Lambda_{0\nu}\right)\right]. \tag{3.1}$$

Here $\frac{D}{D\tau} := \dot{x}^\mu \nabla_\mu$ is the covariant derivative along the curve and the ortho-normal Lorentz body-fixed frame ${\Lambda_A}^\mu$ satisfies $g_{\mu\nu}{\Lambda_A}^\mu {\Lambda_B}^\nu = \eta_{AB}$. Using ${\Lambda_A}^\mu$ one may construct the (antisymmetric) angular velocity tensor $\Omega^{\mu\nu}$ as shown above. The Legendre dual of $\Omega^{\mu\nu}$ is the intrinsic angular momentum or spin tensor $S^{\mu\nu}$. The Lagrange multipliers λ and χ_μ enforce the mass-shell constraints $\pi^2 = \mathcal{M}^2$ and the spin-supplementary condition (SSC) respectively; the latter arises from the necessity of constraining the antisymmetric $S^{\mu\nu}$ to carry only the physical rotational degrees of freedom of the compact body (i.e. three angles in 4d). If we gauge fix the time-like component of the body-fixed frame ${\Lambda_0}^\mu = \pi^\mu/\sqrt{\pi^2}$ then the SSC takes the covariant form $S_{\mu\nu}\pi^\nu = 0$. Instead, we find it more convenient to gauge fix via the choice $\chi_\mu = 0$ for the Lagrange multiplier (analogous to fixing e above) so we may drop the last term in the action, which approximately is equivalent to the covariant SSC $S_{\mu\nu}\pi^\nu = 0 + \mathcal{O}(S^3)$.

Starting at quadratic order in spins and linear order in curvature, the parity-invariant mass-shell constraint $\pi^2 = \mathcal{M}^2$ receives curvature couplings [107]

$$\mathcal{M}^2 = m^2 - \tfrac{1}{4} R_{\mu\nu\alpha\beta} S^{\mu\nu} S^{\alpha\beta} + C_E E_{\mu\nu} S^{\mu\rho} P_{\rho\sigma} S^{\nu\sigma} + \mathcal{O}(S^3)\,, \tag{3.2}$$

where C_E is a Wilson coefficient induced by finite-size deformations (for a Kerr black hole $C_E = 0$).[3] We have introduced the "electric" curvature tensor $E_{\mu\nu}$ and projector $P_{\mu\nu}$ as

$$E_{\mu\nu} = R_{\mu\alpha\nu\beta}\pi^\alpha \pi^\beta / m^2\,, \tag{3.3}$$

$$P_{\mu\nu} = g_{\mu\nu} - \pi_\mu \pi_\nu / \pi^2\,. \tag{3.4}$$

In four spacetime dimensions one may introduce the Pauli-Lubanski spin vector $S^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} S_{\alpha\beta}\pi_\nu/m$ and the additional term takes the form $C_E E_{\mu\nu} S^\mu S^\nu$. Finally, $S^{\mu\nu}S_{\mu\nu} = 2s^2$ is a constant along the worldline.

3.2 Identification with the $\mathcal{N} = 2$ spinning particle

Identification of the traditional action for a spinning compact body eq. (3.1) with the $\mathcal{N} = 2$ supersymmetric worldline theory of eq. (2.16) is made by identifying the

[3] In the literature one often finds $C_E \to C_E - 1$.

spin tensor using

$$\boxed{S^{\mu\nu} = -2ie^{\mu}_{a}e^{\nu}_{b}\bar{\psi}^{[a}\psi^{b]}\,.} \tag{3.5}$$

We have $S^{ab} = e^{a}_{\mu}e^{b}_{\nu}S^{\mu\nu}$ in a local Minkowski frame. So in a sense the worldline fermions ψ^a are the quarks of the spin field S^{ab}. This identification is well-motivated by noting that the SO(1,3) Lorentz algebra:

$$\{S^{ab}, S^{cd}\}_{\text{P.B.}} = \eta^{ac}S^{bd} + \eta^{bd}S^{ac} - \eta^{bc}S^{ad} - \eta^{ad}S^{bc}\,, \tag{3.6}$$

follows from $\{\psi^a, \bar{\psi}^b\}_{\text{P.B.}} = -i\eta^{ab}$ and provides the normalization in eq. (3.5). Note that the mass-shell constraint eq. (3.2) for $C_E = 0$ directly maps with (3.5) to the Hamiltonian of the $\mathcal{N} = 2$ theory (2.12), i.e. the Kerr black hole. Upon (seemingly) sacrificing supersymmetry the finite-size C_E term may also be included by adding

$$H_E := m\, C_E E_{ab}\, \bar{\psi}^a\psi^b\, P_{cd}\bar{\psi}^c\psi^d \tag{3.7}$$

to the spinning superparticle Hamiltonian H in eq. (2.12). Using eq. (3.5) one can easily show that this agrees with the corresponding term in the spinning particle action (3.1). We will discuss the implications of this for SUSY at the end of this section.

How do we prove the equivalence of our $\mathcal{N} = 2$ worldline SUSY theory (2.12), augmented by eq. (3.7), with the traditional formulation of eq. (3.1)? Taking inspiration from refs. [30, 81], we compare the *Routhians* $\mathcal{R}$ of the two theories. The Routhian to be considered is a Legendre transform of the Lagrangian with respect only to the spin degrees of freedom: it is a Hamiltonian with respect to the spin, but a Lagrangian with respect to the position of the particle. In the traditional spin formalism (3.1) switching to the Routhian conveniently eliminates all dependence on angular velocity tensor $\Omega^{\mu\nu}$:

$$\begin{aligned} \mathcal{R}^{\text{spin}} &:= -\frac{1}{2}S_{ab}\,\Lambda^{a}_{A}\dot{\Lambda}^{A\,b} - L^{\text{spin}} \\ &= \pi_\mu\dot{x}^\mu - \frac{1}{2}\omega_{\mu,ab}\dot{x}^\mu S^{ab} + \lambda(\pi_\mu\pi^\mu - \mathcal{M}^2)\,, \end{aligned} \tag{3.8}$$

where we used that $\Lambda^{a}_{A}\frac{D\Lambda^{A\,b}}{D\tau} = \Lambda^{a}_{A}\,\dot{\Lambda}^{A\,b} - \omega_{\mu}{}^{ab}\dot{x}^\mu$. Comparing this to the spinning superparticle action in eq. (2.12) augmented by the finite size term (3.7) the corresponding $\mathcal{N} = 2$ SUSY Routhian takes the form

$$\begin{aligned} \mathcal{R}^{\mathcal{N}=2} &:= -i\eta_{ab}\bar{\psi}^a\dot{\psi}^b - L^{\mathcal{N}=2} - L^E = p_\mu\dot{x}^\mu + e\,(H + H_E) \\ &= \pi_\mu\dot{x}^\mu + i\omega_{\mu,ab}\dot{x}^\mu\bar{\psi}^a\psi^b \\ &\quad - \frac{e}{2}\left(\pi_\mu\pi^\mu - m^2 + R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d + m\,C_E E_{ab}\,\bar{\psi}^a\psi^b\,P_{cd}\,\bar{\psi}^c\psi^d\right)\,, \end{aligned} \tag{3.9}$$

having set $\theta = \bar{\theta} = 0$ and used the definition of π^μ of eq. (2.9) in the last step. Upon identifying $\lambda = \frac{e}{2}$ we have a perfect match $\mathcal{R}^{\text{spin}} = \mathcal{R}^{\mathcal{N}=2}$, thereby also reproducing the finite-size term coupling to C_E in $\mathcal{M}$ of $\mathcal{R}^{\rm spin}$. We also note that the evolution of a generic function $F(\psi^a, \bar{\psi}^a)$ of the spinors is given by Hamilton's equation:

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} = \{F, \mathcal{R}\}_{\rm P.B.}\,. \tag{3.10}$$

Choosing F as $S^{\mu\nu}$ we see that $\mathcal{R}^{\rm spin} = \mathcal{R}^{\mathcal{N}=2}$ guarantees consistency of the spin tensor's equation of motion between these two descriptions.

From our perspective, the Routhian has now served its purpose and we will not need it again. Our approach is therefore qualitatively different from other EFT-based methods, e.g. ref. [30], wherein one proceeds by solving Hamilton's equation (3.10) for $S^{\mu\nu}$. In that context, as we have discussed, the main benefit of the Routhian (3.8) over the Lagrangian (3.1) is that it does not depend on the angular velocity tensor $\Omega^{\mu\nu}$. However, by introducing the Grassmann vector ψ^a at an early stage we have already gained this benefit. We will therefore continue to profit from our use of a fully Lagrangian-based formalism, using the SUSY action (2.12) augmented by the SUSY-breaking term (3.7), treating position and spin of the particles on an equal footing. This will allow us to perform calculations using a set of Feynman rules, to be derived in section 4.

3.3 Approximate supersymmetry of the finite-size term

Finally, we discuss the implications of adding the SUSY-breaking finite-size term H_E (3.7) to the Hamiltonian. In fact, this term *does preserve SUSY approximately* in the sense that the SUSY variation of it vanishes up to terms irrelevant for spin-squared interactions. This may be seen from the bracket of H_E with the (undeformed) supercharge Q: as H_E is quartic in fermions the leading term in the bracket is cubic in fermions and reads

$$\{Q, H_E\}_{\rm P.B.} = mC_E\big(\underbrace{E_{\mu\nu}\pi^\mu}_{=0}\, e^\nu_b\psi^b P_{cd}\bar{\psi}^c\psi^d + E_{ab}\psi^a\,\psi^b\,\underbrace{\pi^\gamma P_{\gamma\rho}}_{=0}\, e^\rho_d\psi^d\big) + \mathcal{O}(\psi^5)\,, \tag{3.11}$$

which vanishes by virtue of eqs. (3.3) and (3.4). Hence the full Hamiltonian $H + H_E$ is *approximately* supersymmetric. The SUSY-violating terms $\mathcal{O}(\psi^5)$ above will in turn receive corrections from the SUSY variation of putative spin-cubed additions to the Hamiltonian.

In fact, one may also incorporate H_E into the SUSY algebra (2.6) in this approximate sense. Deforming the SUSY generators Q and $\bar{Q}$ by $\mathcal{O}(\psi^5)$ terms via

$$Q \to Q' = Q + \frac{\psi\cdot\pi}{\pi^2}H_E\,, \qquad \bar{Q} \to \bar{Q}' = \bar{Q} + \frac{\bar{\psi}\cdot\pi}{\pi^2}H_E\,, \tag{3.12}$$

gives rise to the Poisson brackets

$$\begin{aligned}\{Q', \bar{Q}'\}_{\rm P.B.} &= -2iH + \{Q, \frac{\bar{\psi}\cdot\pi}{\pi^2}H_E\}_{\rm P.B.} + \{\bar{Q}, \frac{\psi\cdot\pi}{\pi^2}H_E\}_{\rm P.B.} + \mathcal{O}(\psi^6) \\ &= -2i(H + H_E) + \mathcal{O}(\psi^6)\,,\end{aligned} \tag{3.13}$$

and $\{Q', Q'\}_{\rm PB} = \mathcal{O}(\psi^6)$. Hence, we have an approximate closure of the $\mathcal{N} = 2$ SUSY algebra. As our analysis is valid only up to spin-squared order we may neglect these higher-order fermionic terms and violations. This entails that the SUSY constraints — alias SSC — hold only approximately for $C_E \neq 0$, i.e. we have $\pi \cdot \psi = \pi \cdot \bar{\psi} = 0 + \mathcal{O}(\psi^5)$ and $S_{\mu\nu}\pi^\nu = 0 + \mathcal{O}(S^3)$. In appendix A we discuss the Lagrangian version of this result in the gauge-fixed theory.

Finally, the U(1) R-symmetry is preserved by the finite-size extension. Having set the Lagrange multiplier a to zero we also have the constraint $\bar{\psi} \cdot \psi = s$, which corresponds the conservation of spin length $S_{\mu\nu}S^{\mu\nu} = 2s^2$.

4 Spinning supersymmetric WQFT

In this section we use the $\mathcal{N} = 2$ SUSY worldline action of section 2.2 to build a spinning generalization of the WQFT formalism [1], valid up to quadratic order in the spins. For each massive body i we start from the worldline action (2.16):

$$S^{(i)} = -m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu + i\bar{\psi}_{i,a} \frac{D\psi_i^a}{D\tau} + \frac{1}{2} R_{abcd} \bar{\psi}_i^a \psi_i^b \bar{\psi}_i^c \psi_i^d \right] , \tag{4.1}$$

where $\frac{D\psi_i^a}{D\tau} = \dot{\psi}_i^a + \dot{x}^\mu {\omega_\mu}^{ab} \psi_{i,b}$ and include the finite-size corrections (3.7)

$$S_E^{(i)} := -m_i\, C_{E,i} \int \mathrm{d}\tau\, R_{a\mu b\nu} \dot{x}_i^\mu \dot{x}_i^\nu \bar{\psi}_i^a \psi_i^b\, P_{cd}\, \bar{\psi}_i^c \psi_i^d \,, \tag{4.2}$$

with two distinct Wilson coefficients $C_{E,1}$ and $C_{E,2}$. Note that the projector reads $P_{ab} = \eta_{ab} - e_{a\mu} e_{b\nu} \dot{x}^\mu \dot{x}^\nu / \dot{x}^2$ here. The two bodies interact via ordinary general relativity, appropriately described by the D-dimensional Einstein-Hilbert action and gauge-fixing term:

$$S_{\rm EH} = -\frac{2}{\kappa^2} \int \mathrm{d}^D x \sqrt{-g}\, R \,, \qquad S_{\rm gf} = \int \mathrm{d}^D x \left(\partial_\nu h^{\mu\nu} - \tfrac{1}{2} \partial^\mu {h^\nu}_\nu \right)^2 , \tag{4.3}$$

where $\kappa = \sqrt{32\pi G}$ with Newton's constant G; the harmonic gauge-fixing term enforces $\partial_\nu h^{\mu\nu} = \frac{1}{2}\partial^\mu {h^\nu}_\nu$.

In order to describe a scattering encounter we expand the worldline fields about their undeflected straight-line trajectories:

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau) \,, \qquad \psi_i^a(\tau) = \Psi_i^a + {\psi'}_i^a(\tau) \,, \tag{4.4}$$

with the new fields $z_i^\mu(\tau)$ and ${\psi'}_i^a(\tau)$ as perturbations. We also introduce the constant part of the spin tensor $S_i^{\mu\nu}(\tau)$ in the local frame:

$$\mathcal{S}_i^{ab} = -2i \bar{\Psi}_i^{[a} \Psi_i^{b]} \,. \tag{4.5}$$

The bulk metric is expanded around a flat Minkowski vacuum:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)\,, \tag{4.6}$$

using a mostly minus metric signature $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$. The corresponding expansions of the vielbein e^a_μ and spin connection $\omega_\mu{}^{ab}$ are

$$e^a_\mu = \eta^{a\nu}\left(\eta_{\mu\nu} + \tfrac{\kappa}{2}h_{\mu\nu} - \tfrac{\kappa^2}{8}h_{\mu\rho}h^\rho{}_\nu + \mathcal{O}(\kappa^3)\right)\,, \tag{4.7a}$$
$$\omega_\mu{}^{ab} = -\kappa\partial^{[a}h^{b]}{}_\mu - \tfrac{\kappa^2}{2}h^{\nu[a}(\partial^{b]}h_{\mu\nu} - \partial_\nu h^{b]}{}_\mu + \tfrac{1}{2}\partial_\mu h^{b]}{}_\nu) + \mathcal{O}(\kappa^3)\,. \tag{4.7b}$$

In this perturbative framework we no longer distinguish between curved $\mu,\nu,\ldots$ and tangent $a,b,\ldots$ indices. The background parameters b_i^μ, v_i^μ and $\mathcal{S}_i^{\mu\nu}$, along with the masses of the two bodies m_i, constitute the physical data regarding the scattering in question. We may set $b\cdot v_i = 0$, where $b^\mu = b_2^\mu - b_1^\mu$ is the relative impact parameter, and $v_i\cdot\Psi_i = v_i\cdot\bar{\Psi}_i = 0$, implying $\mathcal{S}_i^{\mu\nu}v_{i\mu} = 0$, without loss of generality — more on this in section 4.4.

In the WQFT framework $h_{\mu\nu}(x)$, $z_i^\mu(\tau_i)$ and $\psi'^\mu_i(\tau_i)$ are promoted to quantum fields. The quantum theory is defined by the partition function:

$$\begin{aligned}\mathcal{Z}_{\mathrm{WQFT}} = e^{i\chi} := \mathrm{const}\times\int D[h_{\mu\nu}]\int\prod_{i=1}^{2}D[z_i,\psi'_i,\mathfrak{a}_i,\mathfrak{b}_i,\mathfrak{c}_i]\\ \times\exp\left[i\Big(S_{\mathrm{EH}} + S_{\mathrm{gf}} + \sum_{i=1}^{2}\big(S^{(i)} + S_E^{(i)} + S^{(i)}_{\mathrm{ghost}}\big)\Big)\right]\,,\end{aligned} \tag{4.8}$$

where χ is the *eikonal phase*; the overall constant ensures that $\mathcal{Z}_{\mathrm{WQFT}} = 1$ in the non-interacting case ($\kappa = 0$), so $\chi = 0$. The additional terms $S^{(i)}_{\mathrm{ghost}}$ arise from the need to write down a covariant path integral measure [108]:

$$\begin{aligned}\mathcal{D}[x] &= D[x]\prod_{0\leq\sigma\leq T}\sqrt{-\mathrm{det}g_{\mu\nu}[x(\tau)]}\\ &= D[x]\int D[\mathfrak{a},\mathfrak{b},\mathfrak{c}]\,\exp\Big[\underbrace{-i\int_0^T\mathrm{d}\tau\,\left(\tfrac{1}{2}g_{\mu\nu}(\mathfrak{a}^\mu\mathfrak{a}^\nu + \mathfrak{b}^\mu\mathfrak{c}^\nu)\right)}_{iS_{\mathrm{ghost}}}\Big]\,,\end{aligned} \tag{4.9}$$

which brings the "Lee-Yang" ghosts $\mathfrak{a}_i^\mu$ (commuting) and $\mathfrak{b}_i^\mu$, $\mathfrak{c}_i^\mu$ (anti-commuting) into the theory. On top, dimensional regularization of the path integral induces a finite counter term in terms of the Ricci scalar evaluated along the worldline trajectory $-\frac{1}{2}R[x(\tau)]$ [108, 109]. However, both of these additions turn out to be irrelevant for the classical considerations that we are interested in in this work. Observables are calculated as expectation values in the WQFT:

$$\begin{aligned}\Big\langle \mathcal{O}(h,\{x_i,\psi_i\})\Big\rangle := \mathcal{Z}_{\rm WQFT}^{-1}\int D[h_{\mu\nu}]\int\prod_{i=1}^{2}D[z_i,\psi'_i,\mathfrak{a}_i,\mathfrak{b}_i,\mathfrak{c}_i]\,\mathcal{O}(h,\{x_i,\psi_i\})\\ \times\exp\Big[i\Big(S_{\rm EH}+S_{\rm gf}+\sum_{i=1}^{2}\big(S^{(i)}+S_E^{(i)}+S_{\rm ghost}^{(i)}\big)\Big)\Big]\,,\end{aligned}\tag{4.10}$$

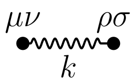
To straightforwardly compute both these observables and the eikonal phase χ in practice we derive a set of momentum-space Feynman rules. These build on the non-spinning rules already derived in ref. [1], and have already been presented in ref. [3].

4.1 Feynman rules

As the Feynman rules originating from the Einstein-Hilbert action (4.3) are fairly conventional we do not dwell on them here; the graviton is simply re-expressed in momentum space as

$$h_{\mu\nu}(x)=\int_k e^{ik\cdot x}h_{\mu\nu}(-k)\,,\tag{4.11}$$

where $\int_k := \int\frac{{\rm d}^Dk}{(2\pi)^D}$ (the negative sign convention gives Feynman vertices with outgoing momenta). Spatial integration $\int{\rm d}^Dx$ in eq. (4.3) gives rise to momentum-conserving delta functions at each n-point momentum-space graviton vertex; also including the gauge-fixing term $S_{\rm gf}$ we read off the graviton propagator from the two-point function:



$$= i\frac{P_{\mu\nu;\rho\sigma}}{k^2+i\epsilon}\,,\qquad\qquad P_{\mu\nu;\rho\sigma}:=\eta_{\mu(\rho}\eta_{\sigma)\nu}-\tfrac{1}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\,.\tag{4.12}$$

The Feynman $i\epsilon$ prescription used here is consistent with purely conservative scattering, and will be sufficient at the 2PM order we shall be working at.

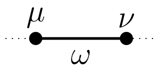
Next we consider the supersymmetric worldline action (4.1). The quadratic terms in z_i^μ and $\psi_i'^\mu$ are

$$S^{(i)}\big|_{\rm quadratic}=-m_i\int{\rm d}\tau\,\eta_{\mu\nu}\left[\frac{1}{2}\dot z_i^\mu\dot z_i^\nu+i\bar\psi_i'^\mu\dot\psi_i'^\nu\right]\,.\tag{4.13}$$

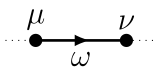
Writing the worldline fields in energy space,

$$z_i^\mu(\tau)=\int_\omega e^{i\omega\tau}z_i^\mu(-\omega)\,,\qquad\qquad \psi_i'^\mu(\tau)=\int_\omega e^{i\omega\tau}\psi_i'^\mu(-\omega)\,,\tag{4.14}$$

where $\int_\omega := \int\frac{{\rm d}\omega}{2\pi}$, we read off the two worldline propagators:



$$= -i\frac{\eta^{\mu\nu}}{2m_i}\left(\frac{1}{(\omega+i\epsilon)^2}+\frac{1}{(\omega-i\epsilon)^2}\right)\,,\tag{4.15}$$



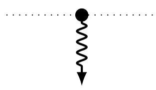
$$= -i\frac{\eta^{\mu\nu}}{2m_i}\left(\frac{1}{\omega+i\epsilon}+\frac{1}{\omega-i\epsilon}\right)\,.\tag{4.16}$$

As explained in ref. [1], the choice of $i\epsilon$ prescription determines the precise interpretation of the background parameters: with advanced or retarded prescriptions $b^\mu_{i,\pm\infty}$, $v^\mu_{i,\pm\infty}$ and $\Psi^\mu_{i,\pm\infty}$ are identified with the undeflected particle trajectories in the limit $\tau \to \pm\infty$. To leading order in G the time-symmetric $i\epsilon$ prescription used here averages over these two possibilities, hence $\Psi^\mu_i = \frac{1}{2}(\Psi^\mu_{i,+\infty} + \Psi^\mu_{i,-\infty}) + \mathcal{O}(G^2)$, $v^\mu_i = \frac{1}{2}(v^\mu_{i,+\infty} + v^\mu_{i,-\infty}) + \mathcal{O}(G^2)$ and $b^\mu_i = \frac{1}{2}(b^\mu_{i,+\infty} + b^\mu_{i,-\infty}) + \mathcal{O}(G^2)$.

Finally, we read off interaction vertices from the supersymmetric worldline action (4.1) and finite-size term (4.2). The worldline fields are written in energy space (4.14); as the graviton coupling to the worldline implicitly depends on z^μ_i we re-write it thus:

$$
\begin{aligned}
h_{\mu\nu}(x_i(\tau)) &= \int_k e^{ik\cdot(b_i+v_i\tau+z_i(\tau))} h_{\mu\nu}(-k) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_k e^{ik\cdot(b_i+v_i\tau)} \left(k\cdot z_i(\tau)\right)^n h_{\mu\nu}(-k) \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{k,\omega_1,\dots,\omega_n} e^{ik\cdot b_i} e^{i(k\cdot v_i + \sum_{j=1}^n \omega_j)\tau} \left(\prod_{j=1}^{n} k\cdot z_i(-\omega_j) \right) h_{\mu\nu}(-k)\,. \qquad (4.17)
\end{aligned}
$$

Integration of the actions over proper time gives rise to energy-conserving delta functions $\bar{\delta}(\omega) := 2\pi\delta(\omega)$, and we read off Feynman vertices from the interaction terms. The simplest vertex is the single-graviton source (suppressing the i subscripts):


$h_{\mu\nu}(k)$

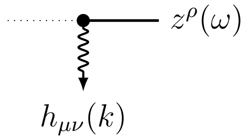
$$
\begin{aligned}
&= -i\frac{m\kappa}{2} e^{ik\cdot b} \bar{\delta}(k\cdot v) \qquad (4.18) \\
&\times \left(v^\mu v^\nu + i(k\cdot\mathcal{S})^{(\mu} v^{\nu)} - \frac{1}{2}(k\cdot\mathcal{S})^\mu (k\cdot\mathcal{S})^\nu + \frac{C_E}{2} v^\mu v^\nu (k\cdot\mathcal{S}\cdot\mathcal{S}\cdot k) \right),
\end{aligned}
$$

where $(k\cdot\mathcal{S})^\mu := k_\nu \mathcal{S}^{\nu\mu}$, representing the linearized (in $h_{\mu\nu}$) stress-energy tensor. In the non-spinning case ($\mathcal{S}^{\mu\nu} = 0$) this precisely reproduces the corresponding vertex of ref. [1]; for a Kerr black hole ($C_E = 0$) the expression is consistent with an exponential representation of the linearized stress-energy tensor [93, 98, 100] apparently valid to all orders in spin:

$$
\begin{aligned}
h_{\mu\nu}(-k) T^{\mu\nu}(k) &= m e^{ik\cdot b} \bar{\delta}(k^2) \bar{\delta}(k\cdot v) (v\cdot\epsilon)^2 \exp\left(i\frac{k\cdot\mathcal{S}\cdot\epsilon}{v\cdot\epsilon} \right) \qquad (4.19) \\
&= m e^{ik\cdot b} \bar{\delta}(k^2) \bar{\delta}(k\cdot v) \epsilon_\mu \epsilon_\nu \left(v^\mu v^\nu + i(k\cdot\mathcal{S})^{(\mu} v^{\nu)} - \frac{1}{2}(k\cdot\mathcal{S})^\mu (k\cdot\mathcal{S})^\nu \right) + \mathcal{O}(\mathcal{S}^3)\,,
\end{aligned}
$$

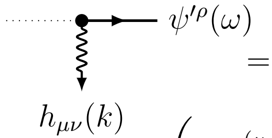
where the on-shell graviton is $h_{\mu\nu}(k) = \bar{\delta}(k^2)\epsilon_\mu\epsilon_\nu$.

As the higher-point vertices become rapidly more complicated we provide only the ones required to compute the 2PM eikonal phase in section 4.3. Firstly, the graviton coupling to a single deflection mode is



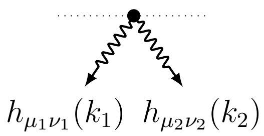
$$
= \frac{m\kappa}{2} e^{ik\cdot b} \hat{\delta}(k\cdot v + \omega) \tag{4.20}
$$
$$
\times \Bigg(2\omega v^{(\mu}\delta^{\nu)}_\rho + v^\mu v^\nu k_\rho + i(k\cdot\mathcal{S})^{(\mu}(k_\rho v^{\nu)} + \omega\delta^{\nu)}_\rho) + \frac{1}{2} k_\rho (k\cdot\mathcal{S})^\mu (\mathcal{S}\cdot k)^\nu
$$
$$
+ \frac{C_E}{2}\Big(\left(2\omega v^{(\mu}\delta^{\nu)}_\rho + v^\mu v^\nu k_\rho\right)(k\cdot\mathcal{S}\cdot\mathcal{S}\cdot k) - \omega^2 k_\rho (\mathcal{S}\cdot\mathcal{S})^{\mu\nu} + 2\omega^2 (k\cdot\mathcal{S}\cdot\mathcal{S})^{(\mu}\delta^{\nu)}_\rho \Big)\Bigg),
$$

which again reproduces the non-spinning case when $\mathcal{S}^{\mu\nu} = 0$. The coupling to a single Grassmann-odd vector is



$$
= -im\kappa e^{ik\cdot b}\hat{\delta}(k\cdot v + \omega) \tag{4.21}
$$
$$
\times \left(k_{[\rho}\delta^{(\mu}_{\sigma]}\left(v^{\nu)} - i(\mathcal{S}\cdot k)^{\nu)}\right) + iC_E\left(v^{(\mu}k_\lambda + \omega\delta^{(\mu}_\lambda\right)\left(v^{\nu)}k_{[\rho} + \omega\delta^{\nu)}_{[\rho}\right)\mathcal{S}^\lambda{}_{\sigma]}\right)\bar{\Psi}^\sigma .
$$

The vertex with an outgoing $\bar{\psi}'^\rho(\omega)$ line is the same, except with $\bar{\Psi}^\sigma \to \Psi^\sigma$. Finally, we require the two-graviton emission vertex from the worldline:

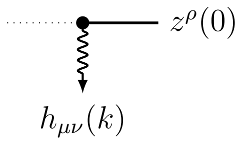


$$
= -\frac{m\kappa^2}{4} e^{i(k_1+k_2)\cdot b}\hat{\delta}((k_1+k_2)\cdot v) \tag{4.22}
$$
$$
\begin{aligned}
\times \Bigg(& (k_1\cdot\mathcal{S})^{\mu_2} v^{\mu_1}\eta^{\nu_1\nu_2} - \mathcal{S}^{\mu_1\mu_2}\left(v^{\nu_1}k_1^{\nu_2} - \tfrac{1}{2}k_1\cdot v\eta^{\nu_1\nu_2}\right) \\
& + i\big((\mathcal{S}\cdot k_1)^{\mu_1}(\mathcal{S}\cdot k_1)^{\mu_2} + \tfrac{1}{2}(\mathcal{S}\cdot k_2)^{\mu_1}(\mathcal{S}\cdot k_1)^{\mu_2} - \tfrac{1}{2}\mathcal{S}^{\mu_1\mu_2}(k_1\cdot\mathcal{S}\cdot k_2)\big)\eta^{\nu_1\nu_2} \\
& + \tfrac{i}{4}k_1\cdot k_2\mathcal{S}^{\mu_1\nu_2}\mathcal{S}^{\mu_2\nu_1} - ik_1^{\nu_2}(\mathcal{S}\cdot(k_1+k_2))^{\mu_1}\mathcal{S}^{\mu_2\nu_1} \\
& + i\,C_E\Big(\big(2k_1\cdot v(\mathcal{S}\cdot\mathcal{S}\cdot(k_1+k_2))^{\mu_2}v^{\mu_1} - \tfrac{1}{2}(k_1\cdot v)^2(\mathcal{S}\cdot\mathcal{S})^{\mu_1\mu_2} \\
& - \tfrac{1}{2}(k_1\cdot\mathcal{S}\cdot\mathcal{S}\cdot k_2)v^{\mu_1}v^{\mu_2}\big)\eta^{\nu_1\nu_2} - \tfrac{1}{2}k_1\cdot k_2(\mathcal{S}\cdot\mathcal{S})^{\nu_1\nu_2}v^{\mu_1}v^{\mu_2} \\
& + k_1^{\nu_2}(\mathcal{S}\cdot\mathcal{S}\cdot k_2)^{\nu_1}v^{\mu_1}v^{\mu_2} - k_1^{\nu_2}(\mathcal{S}\cdot\mathcal{S}\cdot k_1)^{\mu_2}v^{\mu_1}v^{\nu_1} - k_1^{\nu_2}(\mathcal{S}\cdot\mathcal{S}\cdot k_2)^{\mu_2}v^{\mu_1}v^{\nu_1} \\
& - (\mathcal{S}\cdot\mathcal{S})^{\mu_2\nu_2}\left(k_1\cdot v k_2^{\nu_1} - \tfrac{1}{2}k_1\cdot k_2 v^{\nu_1}\right)v^{\mu_1}\Big)\Bigg) + (1\leftrightarrow 2),
\end{aligned}
$$

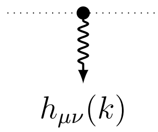
which is implicitly symmetrized on (μ_1,ν_1) and (μ_2,ν_2).

4.2 Recursive properties

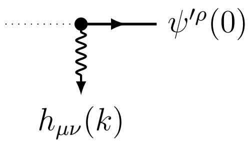
The Feynman rules (4.18), (4.20) and (4.21) satisfy recursive properties:



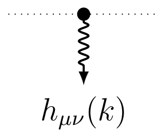
$$= \frac{\partial}{\partial b^\rho}$$



, (4.23a)

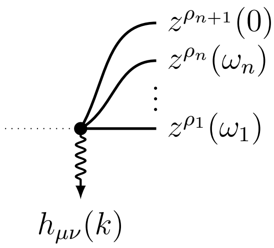


$$= \frac{\partial}{\partial \Psi^\rho}$$

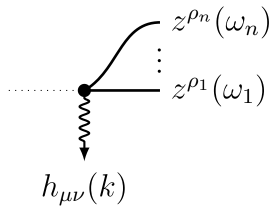


. (4.23b)

In ref. [1] (the non-spinning case) the first relationship was generalized to n points:

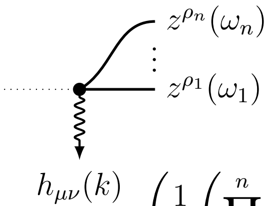


$$= \frac{\partial}{\partial b^{\rho_{n+1}}}$$



. (4.24)

In words: a vertex with $(n+1)$ external z^μ particles, and $\omega_{n+1} = 0$, is given by a derivative with respect to the impact parameter b^μ of the corresponding n-point vertex. We claim this continues to hold when spin is included, and that eq. (4.23b) generalizes similarly, regardless of what other external lines are present on the vertex. In the non-spinning case we confirmed this recursive property using an analytic expression for the worldline vertices:



$$= i^{n-1} m\kappa\, e^{ik\cdot b} \delta\!\!\!{}^{-}\!\left(k\cdot v + \sum_{i=1}^{n}\omega_i\right)\times \tag{4.25}$$

$$\left(\frac{1}{2}\left(\prod_{i=1}^{n} k_{\rho_i}\right) v^\mu v^\nu + \sum_{i=1}^{n}\omega_i \left(\prod_{j\neq i}^{n} k_{\rho_j}\right) v^{(\mu}\delta^{\nu)}_{\rho_i} + \sum_{i<j}^{n}\omega_i\omega_j \left(\prod_{l\neq i,j}^{n} k_{\rho_l}\right)\delta^{(\mu}_{\rho_i}\delta^{\nu)}_{\rho_j}\right).$$

With the inclusion of spin, however, we no longer have such a compact expression and therefore argue differently.

At the Lagrangian level these properties follow straightforwardly from

$$\frac{\partial L^{(i)}(\tau)}{\partial b_i^\mu} = \frac{\partial L^{(i)}(\tau)}{\partial z_i^\mu(\tau)}\,, \qquad\qquad \frac{\partial L^{(i)}(\tau)}{\partial \Psi_i^\mu} = \frac{\partial L^{(i)}(\tau)}{\partial \psi_i'^\mu(\tau)}\,, \tag{4.26}$$

where $S^{(i)} + S_E^{(i)} = \int \mathrm{d}\tau\, L^{(i)}(\tau)$ (we are now ignoring the ghosts). The former is true simply because the Lagrangian $L^{(i)}$ depends on b_i^μ and z_i^μ only implicitly via $x_i^\mu(\tau) =$

$b_i^\mu + \tau v_i^\mu + z_i^\mu(\tau)$; the latter because $L^{(i)}$ depends on spin only via $\psi_i^\mu(\tau) = \Psi_i^\mu + \psi_i'^\mu(\tau)$ (and its complex conjugate). In energy space the action therefore generically depends on $x_i^\mu(\omega_j) = \delta\!\!\!{}^-(\omega_j)b_i^\mu - i\delta\!\!\!{}^-{}'(\omega_j)v_i^\mu + z_i^\mu(\omega_j)$ and $\psi_i^\mu(\omega_j) = \delta\!\!\!{}^-(\omega_j)\Psi_i^\mu + \psi_i'^\mu(\omega_j)$ for the collection of energies $\{\omega_j\}$. So, in this case a derivative with respect to the background parameter b_i^μ or Ψ_i^μ is equivalent to one with respect to the corresponding perturbation $z_i^\mu(\omega_j)$ or $\psi_i'^\mu(\omega_j)$, if we set $\omega_j = 0$ (as implied by the delta function). In the next section we use these properties to obtain observables from the eikonal phase χ.

4.3 Observables from the eikonal phase

The eikonal phase χ is a scalar with a privileged role in the WQFT, containing knowledge of both the classical impulse $\langle\Delta p_{i,\mu}\rangle$ and spin kick $\langle\Delta S_{i,\mu\nu}\rangle$. To recover these observables we use the recursive properties of the worldline vertices (4.26); from the action $S = S_{\rm EH} + S_{\rm gf} + \sum_{i=1}^2 \int \mathrm{d}\tau\, L^{(i)}(\tau)$ these may be re-expressed as

$$\frac{\partial S}{\partial b_i^\mu} = \int_{-\infty}^{\infty} \mathrm{d}\tau \left(\frac{\partial L^{(i)}}{\partial x_i^\mu(\tau)} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L^{(i)}}{\partial \dot{x}_i^\mu(\tau)}\right) - \underbrace{\left[p_{i,\mu}\right]_{\tau=-\infty}^{\tau=\infty}}_{\Delta p_{i,\mu}}, \tag{4.27a}$$

$$\frac{\partial S}{\partial \Psi_i^\mu} = \int_{-\infty}^{\infty} \mathrm{d}\tau \left(\frac{\partial L^{(i)}}{\partial \psi_i^\mu(\tau)} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L^{(i)}}{\partial \dot{\psi}_i^\mu(\tau)}\right) + im_i \underbrace{\left[\bar{\psi}_{i,\mu}\right]_{\tau=-\infty}^{\tau=\infty}}_{\Delta\bar{\psi}_{i,\mu}}, \tag{4.27b}$$

where $p_{i,\mu} = -\frac{\partial L^{(i)}}{\partial \dot{x}_i^\mu}$ and $im_i\bar{\psi}_{i,\mu} = \frac{\partial L^{(i)}}{\partial \dot{\psi}_i^\mu}$. As the first terms define the classical (Euler-Lagrange) equations of motion for x_i^μ and ψ_i^μ, their expectation values in the WQFT vanish. Therefore, by taking derivatives of the free energy $-i\log\mathcal{Z}_{\rm WQFT} = \chi$ of eq. (4.8) with respect to b_i^μ, Ψ_i^μ and $\bar{\Psi}_i^\mu$ and using eq. (4.27) we see that[4]

$$\langle\Delta p_{i,\mu}\rangle = -\frac{\partial\chi}{\partial b_i^\mu}, \tag{4.28a}$$

$$im_i\langle\Delta\psi_{i,\mu}\rangle = \frac{\partial\chi}{\partial\bar{\Psi}_i^\mu} = -2i\Psi_i^\nu\frac{\partial\chi}{\partial\mathcal{S}_i^{\mu\nu}}, \tag{4.28b}$$

$$im_i\langle\Delta\bar{\psi}_{i,\mu}\rangle = \frac{\partial\chi}{\partial\Psi_i^\mu} = -2i\bar{\Psi}_i^\nu\frac{\partial\chi}{\partial\mathcal{S}_i^{\mu\nu}}. \tag{4.28c}$$

We have used the fact that χ depends on Ψ_i^μ and $\bar{\Psi}_i^\mu$ only implicitly via $\mathcal{S}_i^{\mu\nu} = -2i\bar{\Psi}_i^{[\mu}\Psi_i^{\nu]}$. The expectation value of the spin kick $\Delta S_i^{\mu\nu}$ is therefore recovered as

$$\begin{aligned}\langle\Delta S_i^{\mu\nu}\rangle &= -2i\bar{\Psi}_i^{[\mu}\langle\Delta\psi_i^{\nu]}\rangle - 2i\langle\Delta\bar{\psi}_i^{[\mu}\rangle\Psi_i^{\nu]} \\ &= \frac{4}{m_i}\mathcal{S}_i^{\rho[\mu}\frac{\partial\chi}{\partial\mathcal{S}_{i,\nu]}{}^{\rho}}.\end{aligned} \tag{4.29}$$

[4]Fermionic derivatives act to the right.

For any field perturbed around a background expectation value, the expected "kick" of that field is therefore extracted from the eikonal phase by taking a derivative with respect to the corresponding background parameter.

In the special case of aligned spins to the scattering plane we can also determine the scattering angle θ, given in terms of the momentum impulse as

$$\sin\left(\frac{\theta}{2}\right) = \frac{|\Delta p_i|}{2p_\infty}\,, \qquad\qquad p_\infty = \frac{m_1 m_2}{E}\sqrt{\gamma^2-1}\,, \tag{4.30}$$

where $|\Delta p_i| := \sqrt{-\langle\Delta p_i\rangle^2}$, p_∞ is the center-of-mass momentum, the total energy is $E = \sqrt{m_1^2+m_2^2+2\gamma m_1 m_2}$ and $\gamma = v_1\cdot v_2$. From the eikonal phase the scattering angle is directly recovered as

$$\sin\left(\frac{\theta}{2}\right) = -\frac{1}{2p_\infty}\frac{\partial\chi}{\partial|b|}\,. \tag{4.31}$$

Using these relations, in sections 5.2 and 5.3 we will calculate the momentum impulse, spin kick and aligned-spin scattering angle at 1PM and 2PM order respectively.

4.4 Background field symmetries

Invariance of the action under the SUSY transformations (2.19) (see appendix A for details), the U(1) symmetry (2.20) and translation invariance along the worldline has physical consequences for these observables derived from the eikonal phase. After integrating out the worldline fluctuations z_i^μ and $\psi_i'^\mu$, for each transformation there is a flat-space remnant of these symmetries on the background parameters:

$$\delta b_i^\mu = \xi_i v_i^\mu + i\bar\epsilon_i \Psi_i^\mu + i\epsilon_i\bar\Psi_i^\mu\,, \tag{4.32a}$$

$$\delta v_i^\mu = 0\,, \tag{4.32b}$$

$$\delta\Psi_i^\mu = -\epsilon_i v_i^\mu - i\alpha_i\Psi_i^\mu\,, \tag{4.32c}$$

$$\delta\bar\Psi_i^\mu = -\bar\epsilon_i v_i^\mu + i\alpha_i\bar\Psi_i^\mu\,, \tag{4.32d}$$

for constant shift parameters ξ_i, ϵ_i, $\bar\epsilon_i$ and α_i. Hence the eikonal phase χ depending only on the background parameters will be invariant under

$$\delta\chi = \frac{\partial\chi}{\partial b_i^\mu}\delta b_i^\mu + \frac{\partial\chi}{\partial\Psi_i^\mu}\delta\Psi_i^\mu + \frac{\partial\chi}{\partial\bar\Psi_i^\mu}\delta\bar\Psi_i^\mu = 0 \tag{4.33}$$

for both $i = 1,2$. For each parameter we recover a constraint:

$$\xi_i: \quad 0 = v_i^\mu\langle\Delta p_{i,\mu}\rangle\,, \tag{4.34a}$$

$$\epsilon_i: \quad 0 = \langle\Delta p_{i,\mu}\rangle\bar\Psi_i^\mu + m_i v_i^\mu\langle\Delta\bar\psi_{i,\mu}\rangle\,, \tag{4.34b}$$

$$\bar\epsilon_i: \quad 0 = \langle\Delta p_{i,\mu}\rangle\Psi_i^\mu + m_i v_i^\mu\langle\Delta\psi_{i,\mu}\rangle\,, \tag{4.34c}$$

$$\alpha_i: \quad 0 = \bar\Psi_i^\mu\langle\Delta\psi_{i,\mu}\rangle + \langle\Delta\bar\psi_{i,\mu}\rangle\Psi_i^\mu\,. \tag{4.34d}$$

These four constraints respectively imply conservation of p_i^2, $p_i \cdot \bar{\psi}_i$, $p_i \cdot \psi_i$ and $\psi_i \cdot \bar{\psi}_i$ between initial and final asymptotic states, i.e. the energy/mass, conserved super-charges and spin length.[5]

How do we re-interpret the latter three constraints in terms of the classical spin tensors $S_i^{\mu\nu}$? Using $\langle \Delta S_i^{\mu\nu} \rangle = -2i\bar{\Psi}_i^{[\mu} \langle \Delta \psi_i^{\nu]} \rangle - 2i \langle \Delta \bar{\psi}_i^{[\mu} \rangle \Psi_i^{\nu]}$ we have

$$\begin{aligned} m_i v_{i,\mu} \langle \Delta S_i^{\mu\nu} \rangle + \langle \Delta p_{i,\mu} \rangle S_i^{\mu\nu} = &-i(\langle \Delta p_{i,\mu} \rangle \bar{\Psi}_i^{\mu} + m_i v_i^{\mu} \langle \Delta \bar{\psi}_{i,\mu} \rangle) \Psi_i^{\nu} \\ &- i(\langle \Delta p_{i,\mu} \rangle \Psi_i^{\mu} + m_i v_i^{\mu} \langle \Delta \psi_{i,\mu} \rangle) \bar{\Psi}_i^{\nu} \\ &- i m_i (v_i \cdot \bar{\Psi}_i \langle \Delta \psi_i^{\nu} \rangle + v_i \cdot \Psi_i \langle \Delta \bar{\psi}_i^{\nu} \rangle) \,. \end{aligned} \tag{4.35}$$

Therefore, $p_{i,\mu} S_i^{\mu\nu}$ is conserved only if, in addition to eqs. (4.34b) and (4.34c), we choose $v_i \cdot \Psi_i = v_i \cdot \bar{\Psi}_i = 0$. This is consistent with our observation in section 3 that the $\mathcal{N} = 2$ SUSY theory agrees with the spinning particle action only when we use the covariant SSC: $\pi_{i,\mu} S_i^{\mu\nu} = 0$, which is implied by $\pi_i \cdot \psi_i = \pi_i \cdot \bar{\psi}_i = 0$. Meanwhile,

$$\mathcal{S}_{i,\mu\nu} \langle \Delta S_i^{\mu\nu} \rangle = 2\Psi_i \cdot \bar{\Psi}_i (\bar{\Psi}_i^{\mu} \langle \Delta \psi_{i,\mu} \rangle + \langle \Delta \bar{\psi}_{i,\mu} \rangle \Psi_i^{\mu}) \,, \tag{4.36}$$

so preservation of the spin lengths $\mathrm{tr}(S_i \cdot S_i) = -2s_i^2$ is guaranteed by eq. (4.34d).

One should note that the background symmetries (4.32) are gauge fixed by our previous requirements that $b \cdot v_i = 0$ and $v_i \cdot \Psi_i = v_i \cdot \bar{\Psi}_i = 0$, the latter implying $v_{i,\mu} \mathcal{S}_i^{\mu\nu} = 0$. In terms of the shifts (4.32) these constraints are achieved using

$$\begin{aligned} &\epsilon_i = v_i \cdot \Psi_i \,, \qquad \bar{\epsilon}_i = v_i \cdot \bar{\Psi}_i \,, \qquad \alpha_i = 0 \,, \\ &\xi_1 = \frac{b \cdot (\gamma v_2 - v_1) - v_1 \cdot \mathcal{S}_2 \cdot v_2 + \gamma v_1 \cdot \mathcal{S}_1 \cdot v_2}{\gamma^2 - 1} \,, \\ &\xi_2 = \frac{b \cdot (v_2 - \gamma v_1) + v_1 \cdot \mathcal{S}_1 \cdot v_2 - \gamma v_1 \cdot \mathcal{S}_2 \cdot v_1}{\gamma^2 - 1} \,. \end{aligned} \tag{4.37}$$

However, no information is lost: full dependence on terms of the form $b \cdot v_i$ and $v_{i,\mu} \mathcal{S}_i^{\mu\nu}$ is restored to the eikonal phase by shifting

$$\begin{aligned} \mathcal{S}_i^{\mu\nu} &\to \mathcal{S}_i^{\mu\nu} + 2(v_i \cdot \mathcal{S}_i)^{[\mu} v_i^{\nu]} \,, \\ b^{\mu} &\to b^{\mu} + \xi_2 v_2^{\mu} - \xi_1 v_1^{\mu} + \mathcal{S}_2^{\mu\nu} v_{2,\nu} - \mathcal{S}_1^{\mu\nu} v_{1,\nu} \,, \end{aligned} \tag{4.38}$$

with ξ_i as given above.

Given our imposition of these background constraints, in order to truly "check" the background symmetries of the eikonal phase one should calculate it without making these requirements a priori. Up to the 2PM order described in section 5 we have done so, as a separate calculation which involved generalizing the Feynman rules in section 4.1 to the inclusion of such terms. Notice also that the background field symmetries (4.32) continue to apply when $C_{E,i} \neq 0$: although local SUSY is

[5] Although the true supercharges $Q_i = \pi_i \cdot \psi_i$, $\bar{Q}_i = \pi_i \cdot \bar{\psi}_i$ involve the covariantized momentum $\pi_{i,\mu} = p_{i,\mu} - i\omega_{\mu ab} \bar{\psi}_i^a \psi_i^b$, the spacetime is asymptotically flat so $\omega_{\mu ab} = 0$ at the boundary.

spoiled by the presence of additional curvature couplings in the action, approximate SUSY persists up to spin-squared effects as discussed in section 3.3. So we continue to expect conservation of energy, spin length, and the SSC. In ref. [2] the same approximate SUSY was also seen acting on the leading-PM waveform $\langle h_{\mu\nu}(k)\rangle$.

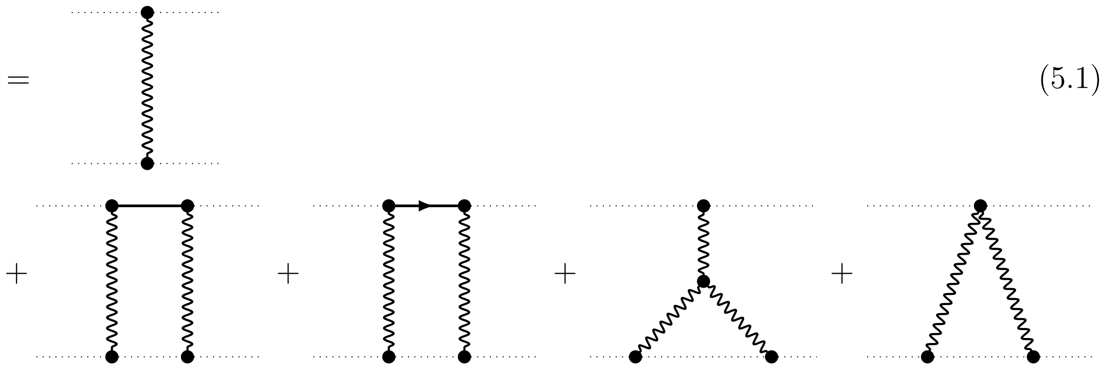
5 The eikonal phase and derived observables

In this section we compute the eikonal phase $\chi = -i\log \mathcal{Z}_{\rm WQFT}$ up to 2PM order, and from it derive the momentum impulse $\langle \Delta p_i^\mu\rangle$, spin kick $\langle \Delta S_i^{\mu\nu}\rangle$ and aligned-spin scattering angle θ using the relationships established in section 4.3.

5.1 2PM eikonal phase

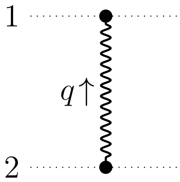
Up to 2PM order the eikonal phase is given as the sum of four vacuum diagrams in the WQFT:

$i\chi =$



$+\,\mathcal{O}(G^3)\,,$ (5.1)

where mirror diagrams $(1 \leftrightarrow 2)$ are left implicit and we sum over both directions of the arrowed line (representing a propagating spin mode $\psi_i'^{a}$). Explicit expressions are obtained using the Feynman rules given in section 4.1. For example, the 1PM contribution only involves the graviton source vertex (4.18) and has the explicit form



$$
\begin{aligned}
&= i\frac{\kappa^2 m_1 m_2}{4}\int_q e^{iq\cdot b}\bar\delta(q\cdot v_1)\bar\delta(q\cdot v_2)\frac{P_{\mu\nu;\rho\sigma}}{q^2+i\epsilon} \\
&\times\left(v_1^\mu v_1^\nu - i(q\cdot\mathcal{S}_1)^\mu v_1^\nu - \tfrac{1}{2}(q\cdot\mathcal{S}_1)^\mu(q\cdot\mathcal{S}_1)^\nu + \tfrac{C_{E,1}}{2}v_1^\mu v_1^\nu(q\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot q)\right) \\
&\times\left(v_2^\rho v_2^\sigma + i(q\cdot\mathcal{S}_2)^\rho v_2^\sigma - \tfrac{1}{2}(q\cdot\mathcal{S}_2)^\rho(q\cdot\mathcal{S}_2)^\sigma + \tfrac{C_{E,2}}{2}v_2^\rho v_2^\sigma(q\cdot\mathcal{S}_2\cdot\mathcal{S}_2\cdot q)\right),
\end{aligned}
\tag{5.2}
$$

where we discard all terms above $\mathcal{O}(\mathcal{S}^2)$; $b^\mu = b_2^\mu - b_1^\mu$ and we integrate over the off-shell momentum q of the exchanged graviton. Similar expressions are easily assembled for the other diagrams; for a worldline propagator of either type (z^μ or ψ'^μ) we perform a one-dimensional integral $\int_\omega$ over the intermediate energy ω.

An important practical consideration is our use of the constant spinors Ψ_i^μ: in general, we prefer final results to be expressed in terms of the physically relevant antisymmetric spin tensors $\mathcal{S}_i^{\mu\nu} = -2i\bar{\Psi}_i^{[\mu}\Psi_i^{\nu]}$. This motivates our writing the interaction vertices (4.18)–(4.21) in terms of $\mathcal{S}^{\mu\nu}$ wherever possible — so that most of the graphs in eq. (5.1) depend on Ψ_i^μ only via $\mathcal{S}_i^{\mu\nu}$. The only exception is the third diagram, which carries an overall factor $\bar{\Psi}_1^\mu\Psi_1^\nu$ due to the spinor vertex (4.21) (appropriately contracted with the rest of the expression). Manifest dependence on $\mathcal{S}_1^{\mu\nu}$ is only recovered once the counterpart diagram with the arrowed line pointing in the opposite direction is included: except for its overall dependence on Ψ_1^μ the expression is identical, and we recover $\bar{\Psi}_1^\mu\Psi_1^\nu + \Psi_1^\mu\bar{\Psi}_1^\nu = i\mathcal{S}_1^{\mu\nu}$ as an overall factor.

As the techniques used to integrate these expressions are now well-established (see e.g. refs. [38, 110]) we relegate those details to appendix B, and here simply present our results. As explained in section 4.3, $\chi = \sum_{n=1}^\infty G^n\chi^{(n)}$ depends only on the orthogonal components of b^μ and $\mathcal{S}_i^{\mu\nu}$ with respect to the velocities v_i^μ, so we set $b\cdot v_i = 0$ and $(v_i\cdot\mathcal{S}_i)^\mu = 0$ (the covariant SSC) without loss of generality. At 1PM order the various D-dimensional contributions are[6]

$$\chi^{(1)}\big|_{\mathcal{S}_1^0\mathcal{S}_2^0} = \frac{2\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-2)((D-2)\gamma^2-1)m_1m_2}{(D-2)|b|^{D-4}\sqrt{\gamma^2-1}}\,, \tag{5.3a}$$

$$\chi^{(1)}\big|_{\mathcal{S}_1\mathcal{S}_2^0} = \frac{4\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-1)\gamma m_1m_2}{|b|^{D-3}\sqrt{\gamma^2-1}}\,\hat{b}\cdot\mathcal{S}_1\cdot v_2\,, \tag{5.3b}$$

$$\begin{aligned}\chi^{(1)}\big|_{\mathcal{S}_1\mathcal{S}_2} = {}& \frac{2\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-1)m_1m_2}{|b|^{D-2}(\gamma^2-1)^{3/2}} \\ &\times\Big((\gamma^2-1)\big(\gamma\,\mathrm{tr}(\mathcal{S}_1\cdot\mathcal{S}_2)-(D-2)(\hat{b}\cdot\mathcal{S}_1\cdot v_2\,\hat{b}\cdot\mathcal{S}_2\cdot v_1-\gamma\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot\hat{b})\big) \\ &\quad -v_2\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot v_1\Big)\,,\end{aligned} \tag{5.3c}$$

$$\begin{aligned}\chi^{(1)}\big|_{\mathcal{S}_1^2\mathcal{S}_2^0} = {}& \frac{2\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-1)m_1m_2}{(D-2)|b|^{D-2}(\gamma^2-1)^{3/2}} \\ &\times\Big((\gamma^2-1)\big((D-2)^2(\hat{b}\cdot\mathcal{S}_1\cdot v_2)^2+(D-2)\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot\hat{b}-2s_1^2\big) \\ &\quad+\big(D-1-\gamma^2(D-2)\big)v_2\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot v_2 \\ &\quad-C_{E,1}\big((D-2)\gamma^2-1\big)\Big((\gamma^2-1)\big((D-2)\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot\hat{b}-2s_1^2\big) \\ &\qquad+v_2\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot v_2\Big)\Big)\,,\end{aligned} \tag{5.3d}$$

where $\hat{b}^\mu := b^\mu/|b|$, $|b| = \sqrt{-b\cdot b}$, $\mathrm{tr}(\mathcal{S}_i\cdot\mathcal{S}_i) = -2s_i^2$ and we recall that $\gamma = v_1\cdot v_2$. As

[6] The zeroth-order-spin contribution (5.3a) is logarithmically divergent in $D = 4$ dimensions — this pole is unphysical and affects neither the impulse nor spin kick.

the 2PM results are more involved we provide them here only in $D = 4$ dimensions:

$$\chi^{(2)}\big|_{\mathcal{S}_1^0\mathcal{S}_2^0} = \frac{3\pi(5\gamma^2-1)(m_1+m_2)m_1m_2}{4|b|\sqrt{\gamma^2-1}}\,, \tag{5.4a}$$

$$\chi^{(2)}\big|_{\mathcal{S}_1\mathcal{S}_2^0} = \frac{\pi\gamma(5\gamma^2-3)(4m_1+3m_2)m_1m_2}{4|b|^2(\gamma^2-1)^{3/2}}\hat{b}\cdot\mathcal{S}_1\cdot v_2\,, \tag{5.4b}$$

$$\begin{aligned}\chi^{(2)}\big|_{\mathcal{S}_1\mathcal{S}_2} = &\frac{\pi(m_1+m_2)m_1m_2}{4|b|^3(\gamma^2-1)^{5/2}}\\ &\times\big((\gamma^2-1)(\gamma(5\gamma^2-3)(2\,\mathrm{tr}(\mathcal{S}_1\cdot\mathcal{S}_2)+3\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot\hat{b})\\ &\quad-9(5\gamma^2-1)\hat{b}\cdot\mathcal{S}_1\cdot v_2\,\hat{b}\cdot\mathcal{S}_2\cdot v_1)-3(3\gamma^2-1)v_2\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot v_1\big)\,,\end{aligned} \tag{5.4c}$$

$$\begin{aligned}\chi^{(2)}\big|_{\mathcal{S}_1^2\mathcal{S}_2^0} = &\frac{\pi\, m_1m_2}{64|b|^3(\gamma^2-1)^{5/2}}\\ \times\big(&8(\gamma^2-1)((13\gamma^4-42\gamma^2+21)m_1-4(3\gamma^2-1)m_2)s_1^2\\ &-6(\gamma^2-1)((29\gamma^4-66\gamma^2+29)m_1-4(3\gamma^2-1)m_2)\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot\hat{b}\\ &+24(\gamma^2-1)((31\gamma^2-11)m_1+3(5\gamma^2-1)m_2)(\hat{b}\cdot\mathcal{S}_1\cdot v_2)^2\\ &-6((49\gamma^4-90\gamma^2+33)m_1+4(5\gamma^4-9\gamma^2+2)m_2)v_2\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot v_2\\ &+4C_{E,1}(\gamma^2-1)((125\gamma^4-138\gamma^2+29)m_1+2(45\gamma^4-42\gamma^2+5)m_2)s_1^2\\ &-3C_{E,1}(\gamma^2-1)((155\gamma^4-174\gamma^2+35)m_1+4(30\gamma^4-29\gamma^2+3)m_2)\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot\hat{b}\\ &-3C_{E,1}((95\gamma^4-102\gamma^2+23)m_1+4(15\gamma^4-13\gamma^2+2)m_2))v_2\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot v_2\big)\,.\end{aligned} \tag{5.4d}$$

For the full D-dimensional expressions we refer the reader to the ancillary file attached to the `arXiv` submission of this paper.

5.2 1PM observables

The impulse, spin kick and aligned-spin scattering angle are derived from the eikonal phase χ using

$$\langle\Delta p_{i,\mu}\rangle = -\frac{\partial\chi}{\partial b_i^\mu}\,,\qquad \langle\Delta S_i^{\mu\nu}\rangle = \frac{4}{m_i}\mathcal{S}_i^{\rho[\mu}\frac{\partial\chi}{\partial\mathcal{S}_{i,\nu]}{}^\rho}\,,\qquad \sin\left(\frac{\theta}{2}\right) = -\frac{1}{2p_\infty}\frac{\partial\chi}{\partial|b|}\,, \tag{5.5}$$

where p_∞ is the centre-of-mass momentum (4.30). Care should be taken with these derivatives as the 1PM eikonal phase given in eq. (5.3) depends only on the $(D-2)$-dimensional orthogonal components of $b^\mu = b_2^\mu - b_1^\mu$ and $\mathcal{S}_i^{\mu\nu}$ with respect to the velocities v_i^μ. As explained in section 4.3, full dependence on terms of the form $b\cdot v_i$ and $(v_i\cdot\mathcal{S}_i)^\mu$ is restored to the eikonal phase using the SUSY shifts in eq. (4.38), after which the derivatives (5.5) may be taken without issue. One may then safely re-impose $b\cdot v_i = 0$ and $(v_i\cdot\mathcal{S}_i)^\mu = 0$ on the resulting physical observables. Notice that eq. (5.5) implies conservation of momentum $\langle\Delta p_2^\mu\rangle = -\langle\Delta p_1^\mu\rangle$ as all dependence on b_i^μ comes via the relative impact parameter $b^\mu = b_2^\mu - b_1^\mu$ (we are free to choose a space-time origin). At higher-PM orders this implies that the scattering is *conservative*, i.e. by this procedure we miss radiation-reaction effects for which $\langle\Delta p_2^\mu\rangle \neq -\langle\Delta p_1^\mu\rangle$.

The $\mathcal{O}(G)$ part of the impulse $\langle\Delta p_1^\mu\rangle=\sum_{n=1}^\infty G^n\langle\Delta p_1^\mu\rangle^{(n)}$ is given by

$$\langle\Delta p_1^\mu\rangle^{(1)}\big|_{\mathcal{S}_1^0\mathcal{S}_2^0}=\frac{4\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-1)((D-2)\gamma^2-1)m_1m_2}{(D-2)|b|^{D-3}\sqrt{\gamma^2-1}}\hat{b}^\mu\,, \tag{5.6a}$$

$$\langle\Delta p_1^\mu\rangle^{(1)}\big|_{\mathcal{S}_1\mathcal{S}_2^0}=-\frac{4\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-1)\gamma m_1m_2}{|b|^{D-2}\sqrt{\gamma^2-1}}\Big((D-2)\hat{b}\cdot\mathcal{S}_1\cdot v_2\,\hat{b}^\mu+(\mathcal{S}_1\cdot v_2)^\mu\Big)\,, \tag{5.6b}$$

$$\begin{aligned}\langle\Delta p_1^\mu\rangle^{(1)}\big|_{\mathcal{S}_1\mathcal{S}_2}&=\frac{4\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2})m_1m_2}{|b|^{D-1}\sqrt{\gamma^2-1}}\Big(\hat{b}\cdot\mathcal{S}_1\cdot v_2(v_1\cdot\mathcal{S}_2)^\mu+\hat{b}\cdot\mathcal{S}_2\cdot v_1(v_2\cdot\mathcal{S}_1)^\mu\\&+\Big(\gamma D\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot\hat{b}-D\hat{b}\cdot\mathcal{S}_1\cdot v_2\hat{b}\cdot\mathcal{S}_2\cdot v_1+\gamma\,\mathrm{tr}(\mathcal{S}_1\cdot\mathcal{S}_2)-\frac{v_2\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot v_1}{\gamma^2-1}\Big)\hat{b}^\mu\\&+\gamma(\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot P_{12})^\mu+\gamma(P_{12}\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot\hat{b})^\mu\Big)\,,\end{aligned} \tag{5.6c}$$

$$\begin{aligned}\langle\Delta p_1^\mu\rangle^{(1)}\big|_{\mathcal{S}_1^2\mathcal{S}_2^0}&=\frac{2\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-1)m_1m_2}{|b|^{D-1}\sqrt{\gamma^2-1}}\Big(-2(D-2)\hat{b}\cdot\mathcal{S}_1\cdot v_2(v_2\cdot\mathcal{S}_1)^\mu\\&+\Big((D-2)D(\hat{b}\cdot\mathcal{S}_1\cdot v_2)^2+D\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot\hat{b}+\frac{D-1-(D-2)\gamma^2}{\gamma^2-1}v_2\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot v_2\\&-2s_1^2\Big)\hat{b}^\mu+2(\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot P_{12})^\mu-C_{E,1}\big((D-2)\gamma^2-1\big)\Big(2(\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot P_{12})^\mu\\&+\big(D\hat{b}\cdot\mathcal{S}_1\cdot\mathcal{S}_1\cdot\hat{b}+\mathrm{tr}(\mathcal{S}_1\cdot P_{12}\cdot\mathcal{S}_1)\big)\hat{b}^\mu\Big)\Big)\,,\end{aligned} \tag{5.6d}$$

and $\langle\Delta p_2^\mu\rangle=-\langle\Delta p_1^\mu\rangle$. These expressions are manifestly orthogonal to v_i^μ, as required by eq. (4.34a), which is apparent given our use of the projector $P_{12}^{\mu\nu}$ to the $(D-2)$-dimensional space orthogonal to these velocities:

$$P_{12}^{\mu\nu}=\eta^{\mu\nu}+\frac{1}{\gamma^2-1}\Big[v_1^\mu v_1^\nu-2\gamma v_1^{(\mu}v_2^{\nu)}+v_2^\mu v_2^\nu\Big]\,. \tag{5.7}$$

The $\mathcal{O}(G)$ part of the spin kick $\langle \Delta S_1^{\mu\nu}\rangle = \sum_{n=1}^{\infty} G^n \langle \Delta S_1^{\mu\nu}\rangle^{(n)}$ is

$$\langle \Delta S_1^{\mu\nu}\rangle^{(1)}\big|_{\mathcal{S}_1^1\mathcal{S}_2^0} = -\frac{8\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-1)m_2}{|b|^{D-3}\sqrt{\gamma^2-1}} \times \left(\frac{(\hat{b}\cdot\mathcal{S}_1)^{[\mu}{v_1}^{\nu]}}{D-2} + \gamma(v_2\cdot\mathcal{S}_1)^{[\mu}\hat{b}^{\nu]} - \gamma(\hat{b}\cdot\mathcal{S}_1)^{[\mu}{v_2}^{\nu]}\right), \tag{5.8a}$$

$$\begin{aligned}\langle \Delta S_1^{\mu\nu}\rangle^{(1)}\big|_{\mathcal{S}_1^2\mathcal{S}_2^0} &= \frac{8\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-1)m_2}{|b|^{D-2}\sqrt{\gamma^2-1}} \\ &\times \Bigg(-(\mathcal{S}_1\cdot\mathcal{S}_1\cdot\hat{b})^{[\mu}\hat{b}^{\nu]} - (D-2)\hat{b}\cdot\mathcal{S}_1\cdot v_2\Big((\mathcal{S}_1\cdot v_2)^{[\mu}\hat{b}^{\nu]} - (\mathcal{S}_1\cdot\hat{b})^{[\mu}{v_2}^{\nu]}\Big) \\ &\quad + \frac{\gamma(\mathcal{S}_1\cdot\mathcal{S}_1\cdot v_2)^{[\mu}{v_1}^{\nu]}}{(\gamma^2-1)(D-2)} + \frac{1+(D-2)\gamma^2-D}{(\gamma^2-1)(D-2)}(\mathcal{S}_1\cdot\mathcal{S}_1\cdot v_2)^{[\mu}{v_2}^{\nu]} \\ &\quad + C_{E,1}\big((D-2)\gamma^2-1\big)\Big((\mathcal{S}_1\cdot\mathcal{S}_1\cdot\hat{b})^{[\mu}\hat{b}^{\nu]} - \frac{(\mathcal{S}_1\cdot\mathcal{S}_1\cdot v_2)^{[\mu}(\gamma v_1 - v_2)^{\nu]}}{(\gamma^2-1)(D-2)}\Big)\Bigg),\end{aligned} \tag{5.8b}$$

$$\begin{aligned}\langle \Delta S_1^{\mu\nu}\rangle^{(1)}\big|_{\mathcal{S}_1^1\mathcal{S}_2^1} &= \frac{4\pi^{2-\frac{D}{2}}\Gamma(\frac{D}{2}-1)m_2}{|b|^{D-2}\sqrt{\gamma^2-1}} \\ &\times \Bigg(\gamma(D-2)\big((\mathcal{S}_1\cdot\hat{b})^{[\mu}(\mathcal{S}_2\cdot\hat{b})^{\nu]} - (\mathcal{S}_1\cdot\mathcal{S}_2\cdot\hat{b})^{[\mu}\hat{b}^{\nu]}\big) - \frac{(\mathcal{S}_1\cdot v_2)^{[\mu}(\mathcal{S}_2\cdot v_1)^{\nu]}}{\gamma^2-1} \\ &\quad - \frac{(\mathcal{S}_1\cdot\mathcal{S}_2\cdot v_1)^{[\mu}(\gamma v_1 - v_2)^{\nu]}}{\gamma^2-1} - 2\gamma(\mathcal{S}_1\cdot\mathcal{S}_2)^{[\mu\nu]} \\ &\quad - (D-2)\big((\mathcal{S}_1\cdot\hat{b})^{[\mu}{v_2}^{\nu]} - (\mathcal{S}_1\cdot v_2)^{[\mu}\hat{b}^{\nu]}\big)\hat{b}\cdot\mathcal{S}_2\cdot v_1\Bigg).\end{aligned} \tag{5.8c}$$

The other components are $\langle \Delta\mathcal{S}_1^{\mu\nu}\rangle^{(1)}|_{\mathcal{S}_1^0\mathcal{S}_2^1} = \langle \Delta\mathcal{S}_1^{\mu\nu}\rangle^{(1)}|_{\mathcal{S}_1^0\mathcal{S}_2^2} = 0$; $\langle \Delta\mathcal{S}_2^{\mu\nu}\rangle^{(1)}$ is recovered by simple relabelling. Finally, the $\mathcal{O}(G)$ part of the scattering angle $\theta = \sum_{n=1}^{\infty} G^n\theta^{(n)}$ emerging in the case of aligned spins is

$$\begin{aligned}\theta^{(1)} = \frac{2\pi^{2-\frac{D}{2}}(D-3)\Gamma(\frac{D}{2}-1)E}{|b|^{D-3}(\gamma^2-1)} &\Bigg(\frac{2(D-2)\gamma^2-2}{(D-3)(D-2)} + 2\gamma\sqrt{\gamma^2-1}\frac{s_1+s_2}{|b|} \\ &+ \big((D-2)\gamma^2 - \tfrac{3}{4}D + 2\big)\frac{(s_1+s_2)^2}{|b|^2} - \frac{D-4}{4}\frac{(s_1-s_2)^2}{|b|^2} \\ &- \big((D-2)\gamma^2-1\big)\left(\frac{C_{E,1}s_1^2 + C_{E,2}s_2^2}{|b|^2}\right)\Bigg).\end{aligned} \tag{5.9}$$

To specify aligned spins we have inserted

$$\mathcal{S}_1^{\mu\nu} = 2s_1\frac{b^{[\mu}(\gamma v_1 - v_2)^{\nu]}}{|b|\sqrt{\gamma^2-1}}, \qquad \mathcal{S}_2^{\mu\nu} = 2s_2\frac{b^{[\mu}(v_1 - \gamma v_2)^{\nu]}}{|b|\sqrt{\gamma^2-1}}, \tag{5.10}$$

with the normalizations ensuring that $\mathrm{tr}(\mathcal{S}_i\cdot\mathcal{S}_i) = -2s_i^2$. The aligned-spin tensors live in the subspace spanned by b^μ, v_1^μ and v_2^μ, which together with the SSC $(v_i\cdot\mathcal{S}_i)^\mu = 0$

defines them uniquely. This definition ensures planar dynamics and includes the conventional definition in four spacetime dimensions.

5.3 2PM observables

The 2PM momentum impulse and spin kick are again derived from the eikonal phase by taking derivatives with respect to the background parameters b_i^μ and $\mathcal{S}_i^{\mu\nu}$ (5.5); however, an additional subtlety is the interpretation of these background parameters (and v_i^μ). In general, we prefer to express observables in terms of $b^\mu_{\pm\infty}$, $v^\mu_{i,\pm\infty}$ and $\mathcal{S}^\mu_{i,\pm\infty}$ taken in the far past or future:

$$x_i^\mu(\tau) \xrightarrow[\tau\to\pm\infty]{} b^\mu_{i,\pm\infty} + \tau v^\mu_{i,\pm\infty}\,, \tag{5.11a}$$

$$S_i^{\mu\nu}(\tau) \xrightarrow[\tau\to\pm\infty]{} \mathcal{S}^{\mu\nu}_{i,\pm\infty}\,. \tag{5.11b}$$

With the time-symmetric worldline propagators (4.15) the currently used background parameters are $b^\mu = \frac{1}{2}(b^\mu_{+\infty}+b^\mu_{-\infty})+\mathcal{O}(G^2)$, $v_i^\mu = \frac{1}{2}(v^\mu_{i,+\infty}+v^\mu_{i,-\infty})+\mathcal{O}(G^2)$ and $\mathcal{S}_i^{\mu\nu} = \frac{1}{2}(\mathcal{S}^{\mu\nu}_{i,+\infty}+\mathcal{S}^{\mu\nu}_{i,-\infty})+\mathcal{O}(G^2)$. To leading order in G the transition is straightforwardly accomplished using $\langle\Delta b^\mu\rangle$, the momentum impulse $\langle\Delta p_i^\mu\rangle = m_i(v^\mu_{i,+\infty}-v^\mu_{i,-\infty})$ and the spin kick $\langle\Delta S_i^{\mu\nu}\rangle = \mathcal{S}^{\mu\nu}_{i,+\infty}-\mathcal{S}^{\mu\nu}_{i,-\infty}$:

$$b^\mu_{\pm\infty} = b^\mu \pm \frac{\langle\Delta b^\mu\rangle}{2} + \mathcal{O}(G^2)\,, \tag{5.12a}$$

$$v^\mu_{i,\pm\infty} = v^\mu_i \pm \frac{\langle\Delta p_i^\mu\rangle}{2m_i} + \mathcal{O}(G^2)\,, \tag{5.12b}$$

$$\mathcal{S}^{\mu\nu}_{i,\pm\infty} = \mathcal{S}^{\mu\nu}_i \pm \frac{\langle\Delta \mathcal{S}_i^{\mu\nu}\rangle}{2} + \mathcal{O}(G^2)\,. \tag{5.12c}$$

Using $0 = v_i^\mu\langle\Delta p_{i,\mu}\rangle$ (4.34a) it also follows that $\gamma_{\pm\infty} = \gamma + \mathcal{O}(G^2)$. Given our 1PM results in section 5.2 we lack only the 1PM expression for $\langle\Delta b^\mu\rangle$; those results are unaffected as the parameters differ by terms $\mathcal{O}(G)$.

Conservation of angular momentum at 1PM order $\langle\Delta J^{\mu\nu}\rangle = \mathcal{J}^{\mu\nu}_{+\infty} - \mathcal{J}^{\mu\nu}_{-\infty} = 0$ gives us an $\mathcal{O}(G)$ expression for $\langle\Delta b^\mu\rangle$. The total angular momentum at future/past infinity is

$$\begin{aligned}\mathcal{J}^{\mu\nu}_{\pm\infty} &= \sum_{i=1}^{2} m_i\left(2b^{[\mu}_{i,\pm\infty}v^{\nu]}_{i,\pm\infty} + \mathcal{S}^{\mu\nu}_{i,\pm\infty}\right)\\ &= 2(b_{1,\pm\infty}+b_{2,\pm\infty})^{[\mu}P^{\nu]} + 2q^{[\mu}_{\pm\infty}b^{\nu]}_{\pm\infty} + \sum_{i=1}^{2} m_i\mathcal{S}^{\mu\nu}_{i,\pm\infty}\,,\end{aligned} \tag{5.13}$$

where $P^\mu := \frac{1}{2}(m_1 v^\mu_{1,\pm\infty}+m_2 v^\mu_{2,\pm\infty})$ and $q^\mu_{\pm\infty} := \frac{1}{2}(m_1 v^\mu_{1,\pm\infty}-m_2 v^\mu_{2,\pm\infty})$. We note that $\langle\Delta p_1^\mu\rangle = -\langle\Delta p_2^\mu\rangle = q^\mu_{+\infty}-q^\mu_{-\infty}$ due to conservation of linear momentum. To isolate the term depending on the relative impact parameter $b^\mu = b_2^\mu - b_1^\mu$ we introduce a projector to the $(D-1)$-dimensional space orthogonal to P^μ:

$$\Lambda^\mu{}_\nu := \delta^\mu{}_\nu - \frac{P^\mu P_\nu}{P^2}\,. \tag{5.14}$$

In effect, contraction with this projector specializes us to the spacelike components of $\mathcal{J}^{\mu\nu}_{\pm\infty}$ in the center-of-mass frame:

$$(\Lambda\cdot\mathcal{J}_{\pm\infty}\cdot\Lambda)^{\mu\nu} = 2(\Lambda\cdot q_{\pm\infty})^{[\mu}(\Lambda\cdot b_{\pm\infty})^{\nu]} + \sum_{i=1}^{2} m_i(\Lambda\cdot\mathcal{S}_{i,\pm\infty}\cdot\Lambda)^{\mu\nu}\,. \tag{5.15}$$

Contracting $\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\langle\Delta J^{\rho\sigma}\rangle = 0$ with $q^\mu = \frac{1}{2}(m_1 v_1^\mu - m_2 v_2^\mu)$ we rearrange to find an $\mathcal{O}(G)$ expression for $\langle\Delta b^\mu\rangle$:

$$\langle\Delta b^\mu\rangle^{(1)} = \frac{\Lambda^{\mu\nu}}{q\cdot\Lambda\cdot q}\left(-q_\nu b^\rho\langle\Delta p_{1,\rho}\rangle^{(1)} + \sum_{i=1}^{2} m_i\langle\Delta S_{i,\nu\rho}\rangle^{(1)}\Lambda^{\rho\sigma}q_\sigma\right)\,, \tag{5.16}$$

having inserted the expressions for $b^\mu_{\pm\infty}$, $v^\mu_{i,\pm\infty}$ and $\mathcal{S}^{\mu\nu}_{i,\pm\infty}$ (5.12). We have also used the requirement that $b_{\pm\infty}\cdot v_{i,\pm\infty} = 0$, which using eq. (5.12) implies $P_\mu\langle\Delta b^\mu\rangle = 0$ and $q_\mu\langle\Delta b^\mu\rangle = -b_\mu\langle\Delta p_1^\mu\rangle$.

Our preference is to re-express observables in terms of background parameters taken in the far past $\tau\to-\infty$; the switch should be performed only *after* taking derivatives of the eikonal (5.5). The 2PM observables then pick up corrections from the 1PM observables in section 5.2. As the results are quite lengthy we provide them here only in $D=4$ dimensions up to linear order in spin; for the full D-dimensional quadratic-in-spin expressions we refer the reader to the accompanying ancillary file. Firstly, the momentum impulse:

$$\langle\Delta p_1^\mu\rangle^{(2)}\big|_{\mathcal{S}_1^0\mathcal{S}_2^0} = \frac{m_1m_2}{|b|^2}\Bigg(\frac{3\pi(5\gamma^2-1)(m_1+m_2)}{4\sqrt{\gamma^2-1}}\hat{b}^\mu \tag{5.17a}$$
$$- 2\frac{(2\gamma^2-1)^2}{(\gamma^2-1)^2}\big((\gamma m_1+m_2)v_1^\mu - (\gamma m_2+m_1)v_2^\mu\big)\Bigg)\,,$$

$$\langle\Delta p_1^\mu\rangle^{(2)}\big|_{\mathcal{S}_1^1\mathcal{S}_2^0} = \frac{m_1m_2}{|b|^2}\Bigg(\frac{\hat{b}\cdot\mathcal{S}_1\cdot v_2}{|b|}\Bigg(-\frac{3\pi\gamma(5\gamma^2-3)}{4(\gamma^2-1)^{3/2}}(4m_1+3m_2)\hat{b}^\mu \tag{5.17b}$$
$$+\frac{16\gamma^2(2\gamma^2-1)m_1+2\gamma(12\gamma^2-5)m_2}{(\gamma^2-1)^2}v_1^\mu - \frac{16\gamma(2\gamma^2-1)m_1+2(8\gamma^4-1)m_2}{(\gamma^2-1)^2}v_2^\mu\Bigg)$$
$$-\frac{2(4\gamma^2-1)m_2+8\gamma(2\gamma^2-1)m_1}{\gamma^2-1}\frac{(\hat{b}\cdot\mathcal{S}_1)^\mu}{|b|} + \frac{\pi\gamma(5\gamma^2-3)(4m_1+3m_2)}{4(\gamma^2-1)^{3/2}}\frac{(v_2\cdot\mathcal{S}_1)^\mu}{|b|}\Bigg)\,.$$

Here, and from this point on, the $-\infty$ subscripts on $\hat{b}^\mu_{-\infty}$, $v^\mu_{i,-\infty}$ and $\mathcal{S}^{\mu\nu}_{i,-\infty}$ should be considered implicit. The $\mathcal{O}(G^2)$ part of the spin kick is

$$\langle \Delta S_1^{\mu\nu} \rangle^{(2)} = \frac{m_2^2}{|b|^2(\gamma^2-1)} \bigg(4(\hat{b}\cdot\mathcal{S}_1)^{[\mu}\hat{b}^{\nu]} - 16\gamma\hat{b}\cdot\mathcal{S}_1\cdot v_2(2\gamma v_2 - v_1)^{[\mu}\hat{b}^{\nu]} - 16\gamma^2(v_2\cdot\mathcal{S}_1)^{[\mu}{v_2}^{\nu]} + \frac{\pi\gamma(5\gamma^2-3)(4m_1+3m_2)}{2\sqrt{\gamma^2-1}m_2}\Big((\hat{b}\cdot\mathcal{S}_1)^{[\mu}{v_2}^{\nu]} - (v_2\cdot\mathcal{S}_1)^{[\mu}\hat{b}^{\nu]}\Big) - \frac{\pi(5\gamma^4+6\gamma^2-3)m_1 + 3\pi(3\gamma^2-1)m_2}{2\sqrt{\gamma^2-1}m_2}(\hat{b}\cdot\mathcal{S}_1)^{[\mu}{v_1}^{\nu]} + \frac{4(2\gamma^2-1)^2 m_1 + 4\gamma(4\gamma^2-3)m_2}{(\gamma^2-1)m_2}(v_2\cdot\mathcal{S}_1)^{[\mu}{v_1}^{\nu]} \bigg) \tag{5.18}$$

When expressed in terms of the new background parameters, different relationships are satisfied by these observables:

$$m_i^2 v_i^2 = (m_i v_i^\mu + \langle \Delta p_i^\mu \rangle)^2 \,, \tag{5.19a}$$
$$(\mathcal{S}_i^{\mu\nu})^2 = (\mathcal{S}_i^{\mu\nu} + \langle \Delta S_i^{\mu\nu} \rangle)^2 \,, \tag{5.19b}$$
$$m_i v_{i,\mu} \mathcal{S}_i^{\mu\nu} = (m_i v_{i,\mu} + \langle \Delta p_{i,\mu} \rangle)(\mathcal{S}_i^{\mu\nu} + \langle \Delta S_i^{\mu\nu} \rangle) \,. \tag{5.19c}$$

However, the interpretation is still the same: supercharges are conserved between initial and final states. Lastly, the 2PM scattering angle in $D = 4$ is

$$\theta^{(2)} = \frac{E(m_1+m_2)}{|b|^2} \bigg(\frac{3\pi(5\gamma^2-1)}{4(\gamma^2-1)} + \frac{\pi\gamma(5\gamma^2-3)}{2(\gamma^2-1)^{3/2}} \Big(\frac{3m_2+4m_1}{m_1+m_2}\frac{s_1}{|b|} + (1\leftrightarrow 2) \Big) + \frac{3\pi}{2(\gamma^2-1)^2|b|^2} \bigg[\frac{(95\gamma^4-102\gamma^2+15)m_1 + 4\big(15\gamma^4-15\gamma^2+2\big)m_2}{8(m_1+m_2)} s_1^2 + (1\leftrightarrow 2) - C_{E,1}\frac{(125\gamma^4-138\gamma^2+29)m_1 + 2(45\gamma^4-42\gamma^2+5)m_2}{16(m_1+m_2)} s_1^2 + (1\leftrightarrow 2) + (20\gamma^4-21\gamma^2+3)s_1 s_2 \bigg] \bigg) \,, \tag{5.20}$$

where the specialization to aligned spins was given earlier (5.10). The spin-free part of our 2PM scattering angle in D dimensions (provided in the ancillary file) agrees with earlier results [110, 111].

6 Conclusions

The $\mathcal{N} = 2$ supersymmetric worldline action provides an alternative description of a compact spinning object up to terms quadratic in spin $\mathcal{O}(\mathcal{S}^2)$ (quadrupoles). Using this equivalence we have shown how quadratic-in-spin effects may be incorporated into the worldline quantum field theory (WQFT) prescription for scattering massive bodies in a curved background [1]. The classical spin tensors $S_i^{ab} = -2i\bar{\psi}_i^{[a}\psi_i^{b]}$ (in a local frame e_a^μ) are considered composite fields, built from the complex Grassmann-valued vectors ψ_i^a living on each worldline i. Conveniently, this provides for a Lagrangian worldline formalism that involves neither a body-fixed frame nor angular velocity tensor. The technology was previously used to obtain the far-field time-domain waveform from a scattering of two massive bodies (black holes, neutron stars or stars) to leading order in G [3]; here we elaborated on it further.

While Kerr black holes are privileged, and represented by the unique $\mathcal{N}=2$ supersymmetric theory, finite-size effects may also be incorporated starting at $\mathcal{O}(\mathcal{S}^2)$ by adding terms that only preserve SUSY approximately (up to $\mathcal{O}(\psi^5)$). The conserved supercharges have a natural physical interpretation: conservation of energy, spin length and the spin-supplementary condition (SSC) $p_\mu S^{\mu\nu}=0$ along each worldline. While these are conserved locally in the supersymmetric theory, with the inclusion of finite-size effects they are only conserved approximately up to $\mathcal{O}(\mathcal{S}^3)$. The analogue can be seen in ref. [3], where the time-domain waveform is approximately supersymmetric when finite-size effects are included and exactly supersymmetric in the Kerr-black hole case.

Our main result is an explicit expression for the D-dimensional eikonal phase $\chi=-i\log\mathcal{Z}_{\text{WQFT}}$ up to $\mathcal{O}(G^2)$ (2PM order), where $\mathcal{Z}_{\text{WQFT}}$ is the partition function of the WQFT. This was obtained as a sum of tree-level vacuum diagrams integrated over the momenta (or energies on the worldlines) of internal lines. From the eikonal phase we showed how one may derive three key observables: the momentum impulse $\langle\Delta p_1^\mu\rangle=-\langle\Delta p_2^\mu\rangle$, spin kicks $\langle\Delta S_i^{\mu\nu}\rangle$ and (for aligned spins) the scattering angle θ. In $D=4$ dimensions these observables agree with previous results [30, 62]. The requirements of energy conservation and preservation of both the spin-supplementary condition (SSC) and spin length follow naturally from the supersymmetry. Another important subtlety is the interpretation of background parameters b_i^μ, v_i^μ and $\mathcal{S}_i^{\mu\nu}$: in the eikonal phase these are defined at an intermediate point of the scattering, and an interpolation is needed to relate them to those in the far past ($\tau\to-\infty$).

Our work offers numerous follow-up opportunities, which we will explore enthusiastically. Naturally, one wonders about the prospects for extending the formalism beyond quadratic order in the spins. As explained in section 2.1, in a flat spacetime background there exist $\mathcal{N}$-supersymmetric worldline theories with real Grassmann-valued vectors $\psi_\alpha^a(\tau)$ carrying flavor indices $\alpha=1,\ldots,\mathcal{N}$ that generically describe the propagation of spin-$\mathcal{N}/2$ particles [78, 103, 104]. The main obstacle is generalizing these theories to an arbitrary curved spacetime background whilst preserving supersymmetry. Yet we have seen that perturbative deformations of the supercharges yielding an approximate supersymmetry are possible; it would be worthwhile to revisit the issue under these premises. Fortunately, the higher spin limitation does not exist in gauge theories: the so-called $\sqrt{\text{Kerr}}$ theory [112], which enjoys a complex worldsheet description [102], is a natural candidate for study. Given ongoing research on the double copy in WQFT [113], this could provide a window on higher spins for the Kerr black hole and is left for future work.

We also see excellent prospects for applying the spinning WQFT formalism to higher-PM order calculations. As explained in ref. [1] we are not limited to using the eikonal phase: we can also compute $\langle \Delta p_i^\mu \rangle$ directly by drawing graphs with an outgoing deflection mode z_i^μ; similarly we can obtain $\langle \Delta \psi_i^\mu \rangle$, and therefore $\langle \Delta S_i^{\mu\nu} \rangle$ via eq. (4.29), by drawing graphs with an outgoing $\psi_i'^\mu$ line. There already has been excellent progress at 3PM order in the non-spinning case [39, 71, 75, 114] including radiation reaction effects [41, 60].

Finally, in the non-spinning case a link between scalar-graviton S-matrix elements and operator expectation values in the WQFT has been formally provided by a worldline path integral "Feynman-Schwinger" representation of the graviton-dressed scalar propagator [1]. We would like to extend this link to include spin effects — again, gauge theory theory will provide a useful starting point given that the n-dressed electron propagator is already known [115, 116] (see refs. [117–119] for comprehensive reviews). In gravity, such a dressed propagator is not currently known, and when obtained will provide for a complete theoretical map between the different PM-based approaches to spinning black hole scattering.

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A Supersymmetry

The relevant part of the $e = 1/m$ gauge-fixed $\mathcal{N} = 2$ worldline action in curved space (2.16) reads

$$S = -m \int \mathrm{d}\tau \left[\tfrac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + i \bar{\psi}^a \dot{\psi}_a + i \dot{x}^\mu \omega_{\mu ab} \bar{\psi}^a \psi^b + \tfrac{1}{2} R_{abcd} \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d \right] . \tag{A.1}$$

and we now want to prove its SUSY invariance. The SUSY transformations of x^μ, ψ^a and $\bar{\psi}^a$ were quoted in eq. (2.19):

$$\delta x^\mu = i e_a^\mu (\bar{\epsilon} \psi^a + \epsilon \bar{\psi}^a) , \quad \delta \psi^a = -\epsilon e_\mu^a \, \dot{x}^\mu - \delta x^\mu \omega_\mu{}^a{}_b \psi^b , \tag{A.2}$$

and are augmented by

$$\delta e_\mu^a = \partial_\nu e_\mu^a \, \delta x^\nu , \quad \delta \omega_{\mu ab} = \partial_\nu \omega_{\mu ab} \, \delta x^\nu , \quad \delta R_{abcd} = \partial_\nu R_{abcd} \, \delta x^\nu . \tag{A.3}$$

In order to show the invariance of the action S we analyze the variation δS order-by-order in the fermions ψ^a. At linear order only the variations of the first three terms in eq. (A.1) contribute and one finds

$$\delta S|_{\rm lin} = -m\int d\tau\Big(g_{\mu\nu}\dot{x}^\mu i\bar{\epsilon}\dot{\psi}^\nu - \dot{x}^\mu e^a_\mu\, i\bar{\epsilon}\dot{\psi}^a + \left[\tfrac{1}{2}e^\rho_a\partial_\rho g_{\mu\nu} - \omega_{(\mu\nu)a}\right] i\bar{\epsilon}\psi^a\dot{x}^\mu\dot{x}^\nu + \text{c.c}\Big)\,. \tag{A.4}$$

Using $\frac{1}{2}e^\rho_a\partial_\rho g_{\mu\nu} - \omega_{(\mu\nu)a} = -g_{\rho(\mu}\partial_{\nu)}e^\rho_a$ the last term is rewritten as $-g_{\rho\mu}\dot{x}^\mu(\frac{d}{d\tau}e^\rho_a)i\bar{\epsilon}\psi^a$. Noting that the first two terms combine to the same expression, but an opposite sign, we see the vanishing of the linear in ψ^a variation $\delta S|_{\rm lin}$. At cubic order one picks up contributions only from the last three terms in eq. (A.1) and finds

$$\delta S|_{\rm cubic} = -m\int d\tau\Big([\partial_\rho\omega_{\mu ab} - \partial_\mu\omega_{\rho ab} - \omega_{\mu ac}\,\omega_\rho{}^c{}_b + \omega_{\rho ac}\,\omega_\mu{}^c{}_b + R_{\mu\rho ab}]i\dot{x}^\mu\delta x^\rho\bar{\psi}^a\psi^b\,, \tag{A.5}$$

which vanishes identically using the spin-connection based definition of the Riemann-tensor. At quintic order one merely considers the variation of the last term in eq. (A.1). Here the three types of contributions conspire to yield

$$\delta S|_{\rm quintic} = \tag{A.6}$$
$$-m\int d\tau[\partial_\mu R_{abcd} + \omega_{\mu a}{}^e R_{ebcd} + \omega_{\mu b}{}^e R_{aecd} + \omega_{\mu c}{}^e R_{abed} + \omega_{\mu d}{}^e R_{abce}]\,\delta x^\mu\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\,,$$

which constitutes the covariant derivative of the Riemann-tensor $\nabla_\mu R_{abcd}$. By virtue of $\nabla_\mu e^a_\nu = 0$ we need to show the vanishing of the term $\nabla_\mu R_{\alpha\nu\beta\rho}\,\psi^\mu\psi^\nu\psi^\rho$ (and analogously for $\psi\to\bar{\psi}$). Using the cyclicity of the Riemann tensor in its last three indices, $R_{\alpha\nu\beta\rho} = -R_{\alpha\beta\rho\nu} - R_{\alpha\rho\nu\beta}$, and the anti-commuativity of the ψ's we have

$$\nabla_\mu R_{\alpha\nu\beta\rho}\,\psi^\mu\psi^\nu\psi^\rho = \frac{1}{2}\nabla_\mu R_{\alpha\beta\nu\rho}\,\psi^\mu\psi^\nu\psi^\rho\,. \tag{A.7}$$

This expression vanishes by virtue of the Biancchi identity $\nabla_\mu R_{\alpha\beta\nu\rho} + \nabla_\nu R_{\alpha\beta\rho\mu} + \nabla_\rho R_{\alpha\beta\mu\nu} = 0$.

For completeness, let us now also look at the supersymmetry variation of the finite-size term (4.2) relevant for (neutron) stars:

$$S_E = -m\,C_E\int \mathrm{d}\tau\, R_{a\mu b\nu}\dot{x}^\mu\dot{x}^\nu\bar{\psi}^a\psi^b\; P_{cd}\,\bar{\psi}^c\,\psi^d\,. \tag{A.8}$$

Varying this under (A.2) we produce terms of order three and five in the worldline fermions ψ^a. The order-five terms would also receive contributions from putative order-six terms (yielding spin^3 effects) in the effective worldline theory that we have not considered. Hence, of relevance here are only the order-three terms in the supersymmetry variation of (A.8), which in fact come close to vanishing: varying the first two fermions in S_E yields zero due to $R_{\rho\mu b\nu}\dot{x}^\mu\dot{x}^\nu\dot{x}^\rho = 0$. One is left with

$$\delta S_E\Big|_{\psi^3} = m\, C_E \int \mathrm{d}\tau\, R_{a\mu b\nu}\dot{x}^\mu\dot{x}^\nu\bar{\psi}^a\psi^b\, g_{\rho\sigma}(\bar{\epsilon}\, P_{\rho\sigma}\psi^\rho\dot{x}^\sigma + \epsilon\, P_{\rho\sigma}\bar{\psi}^\rho\dot{x}^\sigma)\,. \tag{A.9}$$

The terms in the bracket vanish by virtue of the projector property: $P_{\rho\sigma}\dot{x}^\sigma = 0$. Hence the finite-size term S_E is supersymmetric approximately, i.e.

$$\delta S_E = \mathcal{O}(\psi^3)\,. \tag{A.10}$$

As we would need to include a new layer of ψ^6 terms in order to describe spinning massive objects at the spin-cubed order, the SUSY variation of these not-considered terms would induce $\mathcal{O}(\psi^3)$ terms which would talk to the above.

B Integrals

To integrate the 1PM contribution to the eikonal phase χ given in eq. (5.2) we require expressions for the following class of D-dimensional Fourier transforms:

$$I_\nu^{\mu_1\mu_2\cdots\mu_n}(D) := \int_q e^{iq\cdot b}\,\bar{\delta}(q\cdot v_1)\bar{\delta}(q\cdot v_2)\left(-q^2\right)^\nu q^{\mu_1}q^{\mu_2}\cdots q^{\mu_n}\,, \tag{B.1}$$

with $n \leq 2$;[7] the generalization to $\nu \neq -1$ becomes relevant at 2PM order. The scalar integral is straightforwardly evaluated in a $(D-2)$-dimensional space orthogonal to v_1^μ and v_2^μ (see e.g. ref. [1]) with the well-known result:

$$I_\nu(D) = \frac{4^\nu}{\pi^{(D-2)/2}\sqrt{\gamma^2-1}}\frac{\Gamma(\frac{D-2}{2}+\nu)}{\Gamma(-\nu)}\left(-b\cdot P_{12}\cdot b\right)^{-\frac{D-2}{2}-\nu}\,, \tag{B.2}$$

where the projector $P_{12}^{\mu\nu}$ to the $(D-2)$-dimensional space orthogonal to v_i^μ was given in eq. (5.7). The generalization to higher-rank integrals follows easily by taking derivatives with respect to b^μ:

$$I_\nu^{\mu_1\mu_2\cdots\mu_n}(D) = (-i)^n\frac{\partial^n}{\partial b_{(\mu_1}\partial b_{\mu_2}\cdots\partial b_{\mu_n)}}I_\nu(D)\,. \tag{B.3}$$

It is important not to impose $b\cdot v_i = 0$ until after these derivatives have been taken — hence our use of the projector $P_{12}^{\mu\nu}$.

To integrate the various contributions appearing at 2PM order we additionally require full knowledge of the following family of integrals:

$$J_{\nu_1,\nu_2,\nu_3}^{\mu_1\mu_2\cdots\mu_n}(D) := \int_\ell \frac{\bar{\delta}(\ell\cdot v_1)}{(\ell^2+i\epsilon)^{\nu_1}((\ell-q)^2+i\epsilon)^{\nu_2}(\ell\cdot v_2+i\epsilon)^{\nu_3}}\ell^{\mu_1}\ell^{\mu_2}\cdots\ell^{\mu_n}\,, \tag{B.4}$$

[7] An $\mathcal{O}(\mathcal{S}^\alpha)$ contribution generically requires integrals with rank $n=\alpha$.

with $v_1^\mu \leftrightarrow v_2^\mu$ related by symmetry and $n \leq 3$. The scalar integral is straightforwardly evaluated by choosing the rest frame of the first body, $v_1^\mu = (1, \mathbf{0})$, and performing the resulting $(D-1)$-dimensional one-loop integral:

$$J_{\nu_1,\nu_2,\nu_3}(D) = (4\pi)^{\frac{1-D}{2}} (-1)^{\nu_1+\nu_2+\nu_3} \left(-q^2\right)^{\frac{D-1}{2}-\nu_1-\nu_2-\frac{\nu_3}{2}} \left(\frac{4}{1-\gamma^2}\right)^{\frac{\nu_3}{2}} \Gamma_{\nu_1,\nu_2,\nu_3}(D-1)\,, \tag{B.5}$$

where

$$\Gamma_{\nu_1,\nu_2,\nu_3}(D) := \frac{\Gamma(\nu_1+\nu_2+\frac{\nu_3}{2}-\frac{D}{2})\Gamma(\frac{\nu_3}{2})}{2\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \frac{\Gamma(\frac{D}{2}-\nu_1-\frac{\nu_3}{2})\Gamma(\frac{D}{2}-\nu_2-\frac{\nu_3}{2})}{\Gamma(D-\nu_1-\nu_2-\nu_3)}\,. \tag{B.6}$$

When $\nu_3 = 0$ we use $\Gamma(\frac{\nu_3}{2}) = 2\Gamma(\nu_3)$.[8] Higher-rank integrals are expanded on a basis of tensors living in the $(D-1)$-dimensional space orthogonal to v_1^μ: namely $P_1^{\mu\nu}$, $P_1^{\mu\nu} v_{2,\nu}$, and q^μ, where $P_1^{\mu\nu} := \eta^{\mu\nu} - v_1^\mu v_1^\nu$ is the projector orthogonal to v_1^μ. The coefficients are found by contracting an ansatz with these tensors, and for example with $n = 1$ one can easily show that

$$J^\mu_{\nu_1,\nu_2,\nu_3} = \frac{q^\mu}{2q^2}\left(q^2 J_{\nu_1,\nu_2,\nu_3} + J_{\nu_1-1,\nu_2,\nu_3} - J_{\nu_1,\nu_2-1,\nu_3}\right) + \frac{P_1^{\mu\nu} v_{2,\nu}}{\gamma^2-1} J_{\nu_1,\nu_2,\nu_3-1}\,, \tag{B.7}$$

which holds in any dimension D. Similar relations hold for $n = 2$ and $n = 3$.

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