

Homological Stability, Characteristic Classes and the Minimal Genus Problem

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Thorben Gerhard Kastenholz

aus

Bonn, Deutschland

Bonn, September 2020

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelm-Universität Bonn

1 Gutachter: Prof. Dr. Ursula Hamenstädt
2 Gutachter: Prof. Dr. Wolfgang Lück

Datum der mündlichen Prüfung: 14.12.2020
Erscheinungsjahr: 2021

Summary:

The purpose of this thesis is to study the (co-)homological properties of the classifying space of subsurface bundles in a trivial background bundle with fiber a manifold M , in particular we will investigate

- homological stability phenomena of this moduli space if M is simply-connected and at least 5-dimensional and the subsurfaces are equipped with tangential structures
- the representability of second homology classes by surfaces in general topological spaces. In the case of manifolds this yields a measure for the failure of homological stability if M is not simply-connected.

In the introduction we will also briefly touch on how to proceed from these homological stability results to determining the stable characteristic classes of subsurface bundles.

Contents

Introduction	5
1. Homological Stability for Spaces of Embedded Subsurfaces with Tangential Structure	5
2. Minimal Genus of Second Homology Classes	8
3. Acknowledgements	10
Bibliography	11
Chapter 1. Homological Stability for Spaces of Subsurfaces with Tangential Structure	13
1. Introduction	13
2. Spaces of Subsurfaces and Tangential Structures	18
3. Stabilization Maps	24
4. Resolutions via Semi-Simplicial Spaces	29
5. Resolutions of the Space of Subsurfaces	31
6. k -Triviality and π_0 -stabilization	37
7. Proving Homological Stability	40
8. Homological Stability for Capping off Boundary Components	43
9. Tangential Structures that fulfil Homological Stability	46
10. Spaces of Pointed Subsurfaces and their Stabilization	52
11. Homological Stability for Symplectic Subsurfaces	59
Bibliography	63
Chapter 2. The Minimal Genus of Homology Classes in a Finite 2-Complex	65
1. Introduction	65
2. Attainable Sets	69
3. The Cokernel of the Hurewicz Map	73
4. Generalized 2-Complexes	75
5. Reduction to Cellwise Coverings	76
6. Effective Computability	86
7. Undecidability	89
Bibliography	91

Introduction

One of the most useful tools in the study of fiber bundles have been characteristic classes. They allow a translation of geometric questions about the fiber bundle to algebraic questions about certain cohomology classes. An example, which greatly captures this idea, is the Euler class of an oriented vector bundle, which is the primary obstruction for a non-vanishing section of the vector bundle.

By general abstract arguments the study of characteristic classes of specific fiber bundles is equivalent to determining the cohomology groups of the classifying space of these bundles. A classical example is given by complex vector bundles, where the integral cohomology ring of the classifying space, the infinite complex Grassmannians, is generated by the so called Chern classes (See [MS74]).

Characteristic classes of surface bundles have been the focus of mathematical research for decades: One of the first steps towards determining these characteristic classes has been Harers homological stability result (see [Har85]), which says that the stabilization maps induced from inclusions between groups of orientation-preserving diffeomorphisms of surfaces that fix the boundary

$$\mathrm{Diff}_{\partial}^{+}(\Sigma_{g,b}) \rightarrow \mathrm{Diff}_{\partial}^{+}(\Sigma_{g',b'})$$

induce isomorphisms in integral homology in a range of degrees, increasing with g , that tends to infinity as g tends to infinity. A characteristic class $\sigma \in H^{*}(B\mathrm{Diff}_{\partial}^{+}(\Sigma_{g,b}))$ is called stable if it lies in the image of $H^{*}(B\mathrm{Diff}_{\partial}^{+}(\Sigma_{g',b'}))$ with respect to the aforementioned stabilization maps and provided that g' is large. Determining the stable (rational) characteristic classes has been an important open problem, until it was proven by Madsen and Weiss in [MW07] that all (rational) stable characteristic classes are polynomials in the so called Mumford-Morita-Miller classes (See [Mor01] for an introduction to these classes). Harers stability result played a prominent role in their proof.

The purpose of this thesis is to study the classifying space of subsurface bundles in a trivial background bundle with fiber a manifold M , in particular we will study homological stability of this moduli space if M is simply-connected and the failure thereof if M is not simply-connected. In this introduction we will also briefly touch on how to proceed from these homological stability results to determining the stable characteristic classes of subsurface bundles. The actual determination will be carried out in an upcoming paper by the author.

1. Homological Stability for Spaces of Embedded Subsurfaces with Tangential Structure

The goal of this introduction is to put the results of Chapter 1 into a broader mathematical context and to explain the authors motivation for his investigation of these phenomena. For an in detail introduction to the results of Chapter 1 the reader is advised to consider the introduction of that chapter.

Harers homological stability result plays an important role, because the Madsen-Weiss theorem actually determines the rational (co)-homology of the stable diffeomorphism group i.e.

$$H^*\left(\operatorname{colim}_g \operatorname{Diff}_\partial^+(\Sigma_{g,1}); \mathbb{Q}\right)$$

and Harers result allows to relate this to the rational cohomology of the diffeomorphism groups of $\Sigma_{g,1}$ and Σ_g .

A model for the classifying space of the diffeomorphism group of surfaces is given by

$$(1) \quad \operatorname{Emb}(\Sigma_{g,b}, \mathbb{R}^\infty; d) / \operatorname{Diff}_\partial^+(\Sigma_{g,b})$$

where $\operatorname{Emb}(\Sigma_{g,b}, [0, \infty) \times \mathbb{R}^{\infty-1}; d)$ denotes the space of embeddings of $\Sigma_{g,b}$ into $[0, \infty) \times \mathbb{R}^{\infty-1}$ which meet $\{0\} \times \mathbb{R}^{\infty-1}$ in a fixed embedding $d: \partial\Sigma_{g,b} \rightarrow \{0\} \times \mathbb{R}^{\infty-1}$ and the group of orientation-preserving diffeomorphisms of the surface that fix the boundary pointwise acts on this space via precomposition. The stabilization maps are given in this case by adding a fixed embedding of a surface into $[-1, 0] \times \mathbb{R}^{\infty-1}$ and then shifting the resulting surfaces to lie in $[0, \infty) \times \mathbb{R}^{\infty-1}$.

In [CMRW17] it was proven that homological stability still holds if one replaces $[0, \infty) \times \mathbb{R}^{\infty-1}$ in (1) by a simply-connected and at least 5-dimensional manifold M with non-empty boundary. The goal of Chapter 1 is to expand their result to include a certain notion of tangential structure explained below. Similarly, there has been an extension of Harer result to include tangential structures in [RW16] and the paper in Chapter 1 used some of the ideas in that paper and adapted them to the setting in [CMRW17] to achieve a similar extension.

The common definition of tangential structure is not well suited to spaces of subsurface, since it neither captures tangential structures of the normal bundle nor potential relative structures stemming from specific structures on M . Consequently we define a **space of θ -structures of subplanes of TM** to be a Hurewicz-fibration $\theta: B_2(M) \rightarrow \operatorname{Gr}_2(TM)$, where $\operatorname{Gr}_2(TM)$ denotes the Grassmannian of oriented 2-planes. Every embedding $e: \Sigma_{g,b} \rightarrow M$ comes equipped with a lift $\operatorname{Gr}(De)$

$$\begin{array}{ccc} & & \operatorname{Gr}_2(TM) \\ & \nearrow \operatorname{Gr}(De) & \downarrow \pi_{\operatorname{Gr}} \\ \Sigma_{g,b} & \xrightarrow{e} & M \end{array}$$

which sends a point to its oriented tangent plane in TM . A **θ -structure** for an embedding e is a map $T_e: \Sigma_{g,b} \rightarrow B_2(M)$ such that the following diagram commutes:

$$(2) \quad \begin{array}{ccc} & & B_2(M) \\ & \nearrow T_e & \downarrow \theta \\ & & \operatorname{Gr}_2(TM) \\ & \nearrow \operatorname{Gr}(De) & \downarrow \pi_{\operatorname{Gr}} \\ \Sigma_{g,b} & \xrightarrow{e} & M \end{array}$$

Examples of such tangential structures are given in the introduction of Chapter 1 and they include for example framings of the normal bundle. We define the **space of embeddings with θ -structure** as follows:

$$\operatorname{Emb}^\theta(\Sigma_{g,b}, M; \delta^T) := \left\{ (e, T_e) \left| \begin{array}{l} e: \Sigma_{g,b} \rightarrow M \text{ is an embedding and} \\ T_e: \Sigma_{g,b} \rightarrow B_2(M) \text{ makes Diagram (2) commute} \\ T_e|_{\partial\Sigma_{g,b}} = \delta^T \end{array} \right. \right\}$$

where δ^T denotes the germ of an embedding with tangential structure $\delta: \partial\Sigma_{g,b} \times [0, \epsilon) \rightarrow M$ such that $\partial\Sigma_{g,b}$ maps to ∂M (the introduction of Chapter 1 contains a detailed explanation of what is meant by a germ here). The group of orientation preserving diffeomorphisms of $\Sigma_{g,b}$ that fix the boundary pointwise operates on $\operatorname{Emb}^\theta(\Sigma_{g,b}, M; \delta^T)$ via precomposition and we denote the quotient

by $\mathcal{E}_{g,b}^\theta(M; \delta^T)$. Elements of this space will be called **subsurfaces with θ -structure**. By taking a fixed subsurface with θ -structure P in $\partial M \times [0, 1]$ which agrees with δ^T at $\partial M \times \{0\}$ one obtains stabilization maps

$$(3) \quad \begin{aligned} - \cup P : \mathcal{E}_{g,b}^\theta(M; \delta^T) &\rightarrow \mathcal{E}_{g',b'}^\theta(M \cup_{\partial M \times \{0\}} \partial M \times [0, 1]; \delta'^T) \\ W &\mapsto W \cup P \end{aligned}$$

Again, we are sweeping some details under the rug and Chapter 1 explains the stabilization process more thoroughly.

We say a space of tangential structures $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ satisfies **homological stability** if all stabilization maps as in (3) induce isomorphisms in homology with integral coefficients in a range of degrees monotonously increasing and tending to ∞ with g . If M is simply-connected and at least 5-dimension, then the main theorem of Chapter 1 provides criteria for when a space of tangential structures satisfies homological stability.

THEOREM 1.1. *Suppose M is an at least 5-dimensional, simply-connected manifold with non-empty boundary and $\partial^0 M$ is a codimension 0 simply-connected submanifold of ∂M together with a space of θ -structures on subplanes of TM . Suppose further that θ stabilizes on connected components, i.e.*

$$(- \cup P)_* : \pi_0(\mathcal{E}_{g,b}^\theta(M; \delta^T)) \rightarrow \pi_0(\mathcal{E}_{g',b'}^\theta(M \cup_{\partial^0 M \times \{0\}} \partial^0 M \times [0, 1]; \delta'^T))$$

and

$$(- \cup P)_* : \pi_0(\mathcal{E}_{g,b}^\theta(\partial^0 M \times [-1, 0]; \delta^T)) \rightarrow \pi_0(\mathcal{E}_{g',b'}^\theta(\partial^0 M \times [-1, 1]; \delta'^T))$$

are isomorphisms for any P if g is high enough, then θ satisfies homological stability.

This theorem is an adaptation of Theorem 1.2 in [RW16] to spaces of subsurfaces. Using this theorem one can prove that many examples satisfy homological stability. Again the introduction of Chapter 1 goes into more depth. One example worth mentioning is that a space of tangential structures whose fiber $\text{Fib}_B(\theta)$ is simply-connected satisfies homological stability.

The space $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ is the moduli space of subsurfaces with θ -structure and it classifies $\Sigma_{g,b}$ -bundles with θ -structure that are embedded into a trivial M -bundle. By twisting this space with $B\text{Diff}_\partial^+(M)$ one obtains the classifying space for $\Sigma_{g,b}$ -bundles embedded in M -bundles. More precisely for spaces of subsurfaces without tangential structure, the space

$$(4) \quad B(\Sigma_{g,b}, \delta; M, \delta_M) := \mathcal{E}_{g,b}^+(M; \delta) \times \text{Emb}(M, [0, \infty) \times \mathbb{R}^{\infty-1}; \delta_M) / \text{Diff}^+(M)$$

where the diffeomorphism group acts by postcomposition with the inverse on the first component and by precomposition on the second component, is a model for the classifying space of subsurface bundles of M -bundles. A simple relative Serre spectral sequence argument shows that homological stability holds for these classifying spaces as well if M is simply-connected and at least 5-dimensional. By replacing $\mathcal{E}_{g,b}^+(M; \delta)$ by $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ and $\text{Diff}_\partial^+(M)$ by the group of homeomorphisms of $B_2(M)$ that cover diffeomorphisms of $\text{Gr}_2(TM)$ induced from diffeomorphisms of M in (4) yields the classifying space for subsurface bundles with θ embedded in M -bundles. Again if θ satisfies homological stability, then these classifying spaces satisfy homological stability as well by the same relative Serre spectral sequence argument.

As with Harers stability theorem, a homological stability result is worth the most, if it is possible to compute the stable (co)homology. One can adapt the proof of the Madsen-Weiss Theorem to obtain a similar statement about the stable rational cohomology of $B(\Sigma_{g,b}, \delta; M, \delta_M)$ (one can also produce a similar result for spaces of subsurfaces with θ -structure). The easiest version of the result is as follows:

THEOREM 1.2. *Let $W_h := \#_h(S^n \times S^n) \# D^n$, let $2n > 4$ and let $\mathcal{B} \subset H^*(BSO(2n); \mathbb{Q})$ be the set of monomials in the classes $e, p_{n-1}, \dots, p_{\lfloor (n+1)/4 \rfloor}$ of total degree greater than $2n$. Then there is an isomorphism*

$$\mathbb{Q}[\{e_2^2, \dots, e_{2n}^{2n}, e_\perp\} \cup \mathcal{B}] \rightarrow \text{colim}_{g,h \rightarrow \infty} H^*(B(\Sigma_{g,b}, \delta; W_h, \delta_M); \mathbb{Q})$$

In this theorem e_2^i corresponds to the Mumford-Morita-Miller classes associated to powers of the Euler class of the vertical tangent bundle of the surface bundle (i.e. just the push-forward of powers of the Euler class of that bundle), e_\perp corresponds to the pushforward of the Euler class of the "vertical normal bundle", and the classes in \mathcal{B} correspond to the generalized Mumford-Morita-Miller classes associated to the vertical tangent bundle of the W_h -bundle.

One can prove this theorem by following the general strategy of the proof of the Madsen-Weiss Theorem. One starts with nested cobordism categories developed by Hoekzema in [Hoe18], in particular $\mathcal{C}_{2 < 2n}$. Objects of this category are $2n - 1$ dimensional manifolds with 1-dimensional submanifolds and morphisms are given by $2n$ -dimensional bordisms that contain a 2-dimensional bordism of the 1-dimensional submanifolds. In this setting Hoekzema defined a scanning map to a double Thom space over the manifold of $2, 2n$ flags. This is a key piece to carrying out an adapted version of the techniques presented in [GRW10]. Afterwards one can adapt the proof in [GRW14] to simplify the occurring categories in order to carry out a group completion argument, which uses the fact that there is a homological stability result for $B(\Sigma_{g,b}, \delta; M, \delta_M)$. This proof strategy will be carried out in a paper that is currently in preparation. Similar but more intricate arguments also work for spaces of subsurfaces with tangential structures. Again the homological stability result plays a crucial role in relating the computed stable cohomology to the characteristic classes of non-stable bundles.

2. Minimal Genus of Second Homology Classes

Again the goal of this introduction is to give a broader mathematical context for the findings of Chapter 1 and to tie them into the theme of characteristic classes. For a more detailed discussion the reader is advised to consider the introduction of that particular chapter.

The results of Chapter 1 rely heavily on M being simply-connected. If M has non-trivial fundamental group, then in general the maps induced by the stabilization maps

$$((-\cup P)_*) : \pi_0(\mathcal{E}_{g,b}^\theta(M; \delta^T)) \rightarrow \pi_0(\mathcal{E}_{g',b'}^\theta(M \cup_{\partial M \times \{0\}} \partial M \times [0, 1]; \delta'^T))$$

are never surjective. This is in part due to the following phenomenon: There is a map

$$(5) \quad \begin{array}{c} \mathcal{E}_{g,b}^\theta(M; \delta^T) \rightarrow H_2(M, \partial M) \\ W \mapsto [W, \partial W] \end{array}$$

and while every class lies in the image of some space of subsurfaces (provided that the dimension of M is at least 3), there is in general no overall bound on this genus. More precisely, the aforementioned map factors through bordism homology

$$\mathcal{E}_{g,b}^\theta(M; \delta^T) \rightarrow \Omega_2(M, \partial M) \rightarrow H_2(M, \partial M)$$

and while the second map is an isomorphism, the first map is not surjective in general. The following definition is meant to capture this difference:

DEFINITION 2.1. Given a topological pair (X, A) (where A is allowed to be empty) and a homology class $\alpha \in H_2(X, A)$, we say that a map $f: (\Sigma, \partial\Sigma) \rightarrow (X, A)$ **represents** α if $f_*([\Sigma, \partial\Sigma]) = \alpha$. Here Σ is a not necessarily connected, compact, oriented surface.

We define the **minimal genus of** α , denoted by $g(\alpha)$ as the minimal genus among all domains of maps representing α , where the genus of a non-connected surface is the sum of the genera of the connected components.

Furthermore we define the following invariant of surfaces, where the sum is taking over the connected components,

$$\chi^-(\Sigma) = \sum_{\Sigma'} \max\{0, -\chi(\Sigma')\}$$

and again we define $\chi^-(\alpha)$ to be the minimal χ^- among all domains of map representing α .

If X is connected, then using connected sums, there is always a connected surface representing α and having minimal genus. Hence the minimal genus is exactly the first g , where an α lies in the image of the map in (5). Of course if X is simply-connected (as in the first part of this introduction) the minimal genus is always 0 by the Hurewicz Theorem.

The minimal genus and χ^- have many striking properties, some of which have been studied by me and Mark Pedron and I present our results in 2. We focused on the case, where A is empty, i.e. we are considering $\alpha \in H_2(X)$. The most striking phenomenon for the minimal genus and χ^- , in this case, is that they only depend on the fundamental group of X . More precisely if X is path-connected: There is a map $X \rightarrow K(\pi_1(X), 1)$ inducing the identity on fundamental groups. Furthermore by a classical result by Hopf (see [Hop42]) this map induces an isomorphism, where H denotes the Hurewicz map,

$$H_2(X)/H(\pi_2(X)) \rightarrow H_2(K(\pi_1(X), 1)) \cong H_2(\pi_1(X))$$

Clearly the minimal genus and χ^- descend to $H_2(X)/H(\pi_2(X))$ and Mark Pedron and I prove that the minimal genus and χ^- of α both agree with the minimal genus and χ^- of its image in $H_2(K(\pi_1(X), 1)) \cong H_2(\pi_1(X))$. Additionally if a map between topological spaces induces an isomorphism on fundamental groups, then the minimal genus and χ^- of a homology class agree with the minimal genus and χ^- of their image.

Furthermore, utilizing Hopfs Formula for second group homology, there is an algebraic description of the minimal genus. Given a presentation $\langle S \mid R \rangle$ of a group G or in other words given a presentation

$$1 \rightarrow R \rightarrow F_S \rightarrow G \rightarrow 1$$

where F_S is the free group generated by S and R is the subgroup generated by the relations in the presentation, then $H_G(2)$ is isomorphic to

$$(6) \quad H_2(G) \cong R \cap [F_S, F_S]/[R, F_S]$$

The minimal genus of a homology class is now the minimal commutator length of the representatives in $R \cap [F_S, F_S]$ of the corresponding $[R, F_S]$ -coset.

It was by proven by Gordon in [Gor95] that given a presentation of a group, it is impossible to algorithmically determine its second group homology. Using a similar approach Mark Pedron and I were able to show the following:

THEOREM 2.2 (Minimal Genus is Undecidable). *Let B be a positive integer, and i be either χ^- or g . Then there is no algorithm, taking as input an (encoding of a) finite 2-complex X and a homology class $\alpha \in H_2(|X|)$, which outputs whether $i(\alpha) \leq B$.*

Notice that the restriction to 2-complexes is inessential, since every group occurs as the fundamental group of a 2-complex. Even though the general problem is undecidable, Mark Pedron and I succeeded in showing that the minimal genus and χ^- is computable for 2-complexes, that possess so called weight functions (See Section 6 in Chapter 2 for definition of these weight functions). Unfortunately we were not able to identify the groups that allowed a weight function on a presentation complex.

To come back to characteristic classes, a very important group, whose minimal genus problem is of interest, is the mapping class group of a surface $\text{Mod}(\Sigma_{g,b}) := \pi_0(\text{Diff}_\partial^+(\Sigma_{g,b}))$, the group of isotopy classes of orientation preserving diffeomorphisms that fix the boundary. By the Earle-Eells Theorem in [EE69] the mapping class group is homotopy equivalent to the whole diffeomorphism group (provided that the genus is at least 2), hence there is no real distinction between their classifying spaces. The second homology of $\text{Mod}(\Sigma_{g,b})$ has been computed by Harer in [Har83] (with a mistake that was fixed in [Mor87]) to be \mathbb{Z} if g is at least 5. Determining the minimal genus for $H_{\text{Mod}(\Sigma_g)}(2)$ is intimately tied to the following question on Kirbys list of problems in low-dimensional topology [EK95]:

QUESTION 1. *What is the minimal genus of a surface Σ such that there exists a fiber bundle*

$$\Sigma_g \rightarrow X \rightarrow \Sigma$$

where the total space has non-zero signature?

The connection is due to the fact, that $H^2(\text{Mod}(\Sigma_g)) \cong \text{Hom}(H_{\text{Mod}(\Sigma_g)}(2), \mathbb{Z})$ is generated (up to a multiple) by the signature cocycle, i.e. the cocycle that associates to a map $\Sigma \rightarrow B\text{Mod}(\Sigma_g)$ the signature of the total space of the corresponding bundle. Now, the question above translates to determining the minimum of the minimal genus of all non-zero homology classes in the second homology of the mapping class group. The minimal genus problem for specific homology classes corresponds to the aforementioned question where the signature is required to be a fixed number (which has to be a multiple by 4 by results in [HKR07]). Unfortunately the approach by Mark Pedron and myself is not applicable to the mapping class group, but an algorithmic approach to the question might still be feasible.

The stabilization maps between mapping class groups of surfaces induce isomorphisms on second homology provided that the genus of the surfaces is high enough by Harers homological stability theorem. Unfortunately an isomorphism on second homology does not have to preserve the minimal genus as is illustrated by degree one maps between surfaces. Hence relating the minimal genus problem for mapping class groups of different surfaces can be understood as a more thorough study of the isomorphism provided by Harers homological stability result and the work by Mark Pedron and myself serves as a first step to such an understanding.

3. Acknowledgements

First and foremost I would like to thank my advisor Ursula Hamenstädt for supporting me mathematically and financially during my PhD. Obviously this thesis and much of my mathematical development would not have been possible without her.

Secondly I would like to thank my father, who unfortunately passed away before I could finish my PhD. He always inspired me to improve myself, never give up and never settle for mediocrity. I hope that he would have been proud of me and I'm saddened that he is no longer here to witness this next step of my journey.

I would also like to thank my girlfriend Anna for always supporting me and keeping me grounded in many kinds of ways. Mark Pedron was not only a coauthor of the paper presented in Chapter 2, but also very helpful throughout the writing of the paper in Chapter 1 and we also had countless fruitful discussion about mathematics and much more.

During my PhD I was financially supported by the SPP2026 "Geometry at Infinity" of the DFG and by the European Research Council Advanced Grant "Moduli" [Grant Number 290399].

Bibliography

- [CMRW17] Federico Cantero Morán and Oscar Randal-Williams, *Homological stability for spaces of embedded surfaces*, *Geom. Topol.* **21** (2017), no. 3, 1387–1467. MR 3650077
- [EE69] Clifford J. Earle and James Eells, *A fibre bundle description of Teichmüller theory*, *J. Differential Geometry* **3** (1969), 19–43. MR 276999
- [EK95] Rob Kirby (Ed.) and Edited Rob Kirby, *Problems in low-dimensional topology*, *Proceedings of Georgia Topology Conference, Part 2*, Press, 1995, pp. 35–473.
- [Gor95] C. McA. Gordon, *Some embedding theorems and undecidability questions for groups*, *Combinatorial and geometric group theory (Edinburgh, 1993)*, *London Math. Soc. Lecture Note Ser.*, vol. 204, Cambridge Univ. Press, Cambridge, 1995, pp. 105–110. MR 1320278

- [GRW10] Søren Galatius and Oscar Randal-Williams, *Monoids of moduli spaces of manifolds*, *Geom. Topol.* **14** (2010), no. 3, 1243–1302. MR 2653727
- [GRW14] ———, *Stable moduli spaces of high-dimensional manifolds*, *Acta Math.* **212** (2014), no. 2, 257–377. MR 3207759
- [Har83] John Harer, *The second homology group of the mapping class group of an orientable surface*, *Invent. Math.* **72** (1983), no. 2, 221–239. MR 700769
- [Har85] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, *Ann. of Math. (2)* **121** (1985), no. 2, 215–249. MR 786348
- [HKR07] Ian Hambleton, Andrew Korzeniewski, and Andrew Ranicki, *The signature of a fibre bundle is multiplicative mod 4*, *Geom. Topol.* **11** (2007), 251–314. MR 2302493
- [Hoe18] Renee Hoekzema, *Higher orientability and spaces of nested manifolds*, Unpublished Thesis at the University of Oxford, 2018.
- [Hop42] Heinz Hopf, *Fundamentalgruppe und zweite Bettische Gruppe*, *Comment. Math. Helv.* **14** (1942), 257–309. MR 0006510
- [Mor87] Shigeyuki Morita, *Characteristic classes of surface bundles*, *Invent. Math.* **90** (1987), no. 3, 551–577. MR 914849
- [Mor01] ———, *Geometry of characteristic classes*, *Translations of Mathematical Monographs*, vol. 199, American Mathematical Society, Providence, RI, 2001, Translated from the 1999 Japanese original, Iwanami Series in Modern Mathematics. MR 1826571
- [MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, *Annals of Mathematics Studies*, No. 76. MR 0440554
- [MW07] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford’s conjecture*, *Ann. of Math. (2)* **165** (2007), no. 3, 843–941. MR 2335797
- [RW16] Oscar Randal-Williams, *Resolutions of moduli spaces and homological stability*, *J. Eur. Math. Soc. (JEMS)* **18** (2016), no. 1, 1–81. MR 3438379

Homological Stability for Spaces of Subsurfaces with Tangential Structure

Given a manifold with boundary, one can consider the space of subsurfaces of this manifold meeting the boundary in a prescribed fashion. It is known that these spaces of subsurfaces satisfy homological stability if the manifold has at least dimension five and is simply-connected. We introduce a notion of tangential structure for subsurfaces and give a general criterion for when the space of subsurfaces with tangential structure satisfies homological stability provided that the manifold is simply-connected and has dimension $n \geq 5$. Examples of tangential structures such that the spaces of subsurface with that tangential structure satisfy homological stability are framings or spin structures of their tangent bundle, or k -frames of the normal bundle provided that $k \leq n - 2$.

Furthermore we introduce spaces of pointedly embedded subsurfaces and construct stabilization maps, as well as prove homological stability for these. This is used to prove homological stability for spaces of symplectic subsurfaces.

1. Introduction

The term homological stability refers to the following phenomenon in algebraic topology: Consider a sequence of spaces X_n indexed by the natural numbers together with continuous maps $g_n: X_n \rightarrow X_{n+1}$, called **stabilization maps**. Such a sequence satisfies *homological stability* if $(g_n)_*: H_*(X_n) \rightarrow H_*(X_{n+1})$ is an isomorphism for $* \leq \varphi(n)$, where $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ is a non-decreasing function such that $\lim_{n \rightarrow \infty} \varphi(n) = \infty$. This phenomenon has been studied intensely in the context of group homology. For example it was proven in [Nak61] that the classifying spaces of the symmetric groups BS_n together with the maps induced from the inclusion $\{0, \dots, n\} \rightarrow \{0, \dots, n, n+1\}$ satisfy homological stability.

Harer showed in [Har85] that homological stability holds true for the mapping class group of closed oriented surfaces of genus g denoted by Mod_g . Even though in this situation we do not have stabilization maps $\text{Mod}_g \rightarrow \text{Mod}_{g+1}$, there are geometrically motivated maps $H_*(\text{Mod}_g) \rightarrow H_*(\text{Mod}_{g+1})$ that are only defined in a range of degrees, where they turn out to be isomorphisms. Actually, homological stability for the mapping class group of surfaces is more naturally formulated for surfaces with boundary, since their mapping class groups have stabilization maps stemming from the inclusions of the surfaces. Then the homology of the mapping class group of closed surfaces can be related to the homology of the mapping class group of surfaces with boundary in a range of degrees increasing with the genus. A similar situation will repeat itself in the present text as well.

Similar results were obtained for the mapping class groups of non-orientable surfaces (see [Wah08]), the spin mapping class groups (see [Har90]) and the mapping class groups of 3-manifolds (see [HW10]). In [RW14] it was shown, among other things, that homological stability holds for the framed mapping class group and generalized spin mapping class groups of surfaces.

Since models for the classifying space of the mapping class group resemble the spaces studied in this paper, the example of homological stability of the mapping class groups of a surface is of particular interest for this paper. Let Σ_g denote a closed, orientable and connected surface of

genus g , and $\text{Diff}^+(\Sigma_g)$ its orientation preserving diffeomorphism group. We will denote the space of smooth embeddings of Σ_g into a possibly infinite-dimensional manifold M by $\text{Emb}(\Sigma_g, M)$. A model for the classifying space $B\text{Mod}_g$ is given as follows:

$$B\text{Mod}_g := \text{Emb}(\Sigma_g, \mathbb{R}^\infty) / \text{Diff}^+(\Sigma_g)$$

Here the diffeomorphism group of the surface acts via precomposition. Replacing \mathbb{R}^∞ in this construction by a finite dimensional manifold M yields the moduli space of submanifolds of M diffeomorphic to Σ_g . Hence points in this quotient will be referred to as **subsurfaces**. This space classifies Σ_g -bundles together with an embedding in a trivial M -bundle. These spaces and their relatives will be the main focus of this paper.

Analogously to homological stability of the mapping class group of surfaces, it will be more natural to define stabilization maps for subsurfaces with boundary: Let $\Sigma_{g,b}$ denote an orientable, compact, connected surface of genus g with b boundary components. Suppose M has non-empty boundary, then we can consider subsurfaces diffeomorphic to $\Sigma_{g,b}$ that intersect ∂M in a fixed set of embedded circles δ . One can define stabilization maps:

$$\text{Emb}(\Sigma_{g,b}, M; \delta) / \text{Diff}^+(\Sigma_{g,b}) \rightarrow \text{Emb}(\Sigma_{g',b'}, M; \delta) / \text{Diff}^+(\Sigma_{g',b'})$$

These change the number of boundary components and the genus. It was shown in [CMRW17] that if M is at least 5-dimensional and simply-connected, then these stabilization maps induce isomorphism in integral homology in a range of degrees that depends on the genus g .

In this paper we will combine the techniques of [CMRW17] with the methods introduced in [RW16], a paper where a unified approach to homological stability for mapping class groups of surfaces with tangential structures was given, to prove homological stability for spaces of subsurfaces that are equipped with certain relative tangential structures. Furthermore we will extend these results to yield homological stability results for certain spaces of closed subsurfaces of closed manifolds. Let us proceed by introducing these relative tangential structures.

Subsurfaces with Tangential Structure: Let M denote a smooth manifold possibly with boundary, let $\text{Gr}_2(TM)$ denote the Grassmannian of oriented 2-bundles in TM and note that an embedding $e: \Sigma_{g,b} \rightarrow M$ lifts to the Grassmannian as a map

$$\begin{aligned} \text{Gr}(De) : \Sigma_{g,b} &\rightarrow \text{Gr}_2(TM) \\ x &\mapsto De(T_x \Sigma_{g,b}) \end{aligned}$$

Let $B_2(M)$ denote a topological space together with a continuous Hurewicz-fibration $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$. We will call a lift T_e of $\text{Gr}(De)$ to $B_2(M)$ a **θ -structure for e** and refer to the pair (e, T_e) as an **embedding with tangential structure**. This is encapsulated in the following diagram:

$$(7) \quad \begin{array}{ccc} & B_2(M) & \\ & \nearrow T_e & \downarrow \theta \\ & \text{Gr}_2(TM) & \\ \Sigma_{g,b} & \nearrow & \downarrow \\ \Sigma_{g,b} & \xrightarrow{e} & M \end{array}$$

We will refer to $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ as a **space of θ -structures of subplanes of TM** .

EXAMPLE 1.1. Fix a metric and an orientation on M , then there is an isomorphism $\text{Gr}_2(TM) \cong \text{Gr}_{n-2}(TM)$. This isomorphism corresponds to taking the normal bundle of the subsurfaces and a tangential structure corresponds to a tangential structure on the normal bundle.

Particular examples of these tangential structures are provided by the various Stiefel-manifolds associated to the tautological bundle over $\mathrm{Gr}_{n-2}(TM)$. In this case an embedding with tangential structure is an embedding together with a k -frame of its normal bundle.

Other examples are given by $B_2(M) = \mathrm{Gr}_2(TM) \times X$ where X is an arbitrary topological space.

A different example will be introduced in Section 11.

EXAMPLE 1.2. This definition of tangential structures should be thought of as a relative tangential structure in contrast to the usual definition of a tangential structure in terms of a fibration $\theta': B \rightarrow \mathrm{Gr}_2(\mathbb{R}^\infty)$. Nevertheless one can recover the usual definition as a special case of the present definition by pulling back $\theta': B \rightarrow \mathrm{Gr}_2(\mathbb{R}^\infty)$ via a classifying map of the tautological bundle over $\mathrm{Gr}_2(TM)$ to get a tangential structure θ in our sense.

We define the **space of embeddings with θ -structure** as follows:

$$\mathrm{Emb}^\theta(\Sigma_{g,b}, M) := \left\{ (e, T_e) \left| \begin{array}{l} e: \Sigma_{g,b} \rightarrow M \text{ is an embedding and} \\ T_e: \Sigma_{g,b} \rightarrow B_2(M) \text{ makes Diagram (7) commute} \end{array} \right. \right\}$$

The definition of the topology of this space will be given in Section 2. Let φ denote an orientation preserving diffeomorphism of $\Sigma_{g,b}$, note $\mathrm{Gr}(De \circ \varphi) = \mathrm{Gr}(De) \circ \varphi$. Therefore $\mathrm{Diff}^+(\Sigma_{g,b})$ has a natural and free action on $\mathrm{Emb}^\theta(\Sigma_{g,b}, M)$ by precomposition.

In order to have a reasonable notion of stabilization, we will need to restrict ourselves to embeddings that intersect the boundary of M in a fixed fashion:

Suppose that ∂M is non-empty and that we have specified a connected component $\partial^0 M$ of ∂M .

We say that two embeddings with tangential structure $e_1, e_2 \in \mathrm{Emb}^\theta(\Sigma_{g,b}, M)$ **have the same jet along ∂M** if the ∞ -jet at $\partial\Sigma_{g,b}$ of e_1 and e_2 agree, and $T_{e_1}|_{\partial\Sigma_{g,b}} = T_{e_2}|_{\partial\Sigma_{g,b}}$.

Fix an ∞ -jet at $\partial\Sigma_{g,b}$ of embeddings with tangential structure of $\Sigma_{g,b}$ into M , denoted by δ^T . Consider the space of embeddings, whose ∞ -jet at $\partial\Sigma_{g,b}$ agrees with δ^T . Note that the group of diffeomorphisms that fixes the ∞ -jet of the identity at $\partial\Sigma_{g,b}$, denoted by $\mathrm{Diff}_\partial^+(\Sigma_{g,b})$, acts freely on this subspace of the space of embeddings. We denote the quotient by $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ and call it the **space of subsurfaces with θ -structure**, elements of this space are called **subsurfaces with θ -structure**. The fixed boundary condition enables us to stabilize these spaces in the following way:

We can embed $\partial^0 M \times [0, 1]$ into M via a collar that identifies $\partial^0 M \times \{1\}$ with $\partial^0 M$ via the identity and we pullback $\theta: B_2(M) \rightarrow \mathrm{Gr}_2(TM)$ to $\mathrm{Gr}_2(\partial^0 M \times [0, 1])$. This yields a space of θ -structures on subplanes in $\partial^0 M \times [0, 1]$. Let P denote a subsurface with tangential structure of $\partial^0 M \times [0, 1]$ such that every connected component of P has a non-empty intersection with $\partial^0 M \times \{0\}$. If we assume that $P \cap \partial^0 M \times \{0\}$ agrees with the image of the jet δ^T and that the higher jets are compatible as well, then the following map is well-defined:

$$\begin{aligned} - \cup P: \mathcal{E}_{g,b}^\theta(M; \delta^T) &\rightarrow \mathcal{E}_{g',b'}^\theta(M \cup_{\partial^0 M \times \{0\}} \partial^0 M \times [0, 1]; \delta'^T) \\ W \subset M &\mapsto W \cup P \subset M \cup_{\partial^0 M \times \{0\}} \partial^0 M \times [0, 1] \end{aligned}$$

Here g', b' and δ'^T depend on the topology of P . By identifying $M \cup_{\partial^0 M \times \{0\}} \partial^0 M \times [0, 1]$ with M , we get stabilization maps of spaces of subsurfaces with tangential structure of M . We say that a tangential structure θ **satisfies homological stability** if $- \cup P$ induces isomorphisms in integral homology in a range of degrees that increases with g and an epimorphism in the next degree i.e. the reduced homology of the mapping cone of $- \cup P$ vanishes in a range of degrees increasing with g .

With these notions we can paraphrase our main result, which can be understood as an adaptation of Theorem 1.2 in [RW16] to spaces of subsurfaces:

THEOREM 1.3. *Suppose M is an at least 5-dimensional, simply-connected manifold with non-empty boundary and $\partial^0 M$ is a codimension 0 simply-connected submanifold of ∂M together with a space of θ -structures on subplanes of TM . Suppose further that θ stabilizes on connected components, i.e.*

$$(- \cup P)_* : \pi_0(\mathcal{E}_{g,b}^\theta(M; \delta^T)) \rightarrow \pi_0(\mathcal{E}_{g',b'}^\theta(M \cup_{\partial^0 M \times \{0\}} \partial^0 M \times [0, 1]; \delta'^T))$$

and

$$(- \cup P)_* : \pi_0(\mathcal{E}_{g,b}^\theta(\partial^0 M \times [-1, 0]; \delta^T)) \rightarrow \pi_0(\mathcal{E}_{g',b'}^\theta(\partial^0 M \times [-1, 1]; \delta'^T))$$

are isomorphisms for any P if g is high enough, then θ satisfies homological stability.

See Theorem 7.3 for the most general statement including the bounds and a wider class of tangential structures. Furthermore Theorem 7.3 does not need that $\partial^0 M$ is simply-connected. Theorem 9.1 is the more precise version of 1.3.

The following theorem (See Theorem 9.2 for a more precise statement, that also includes the bounds) yields a large number of tangential structures which satisfy the assumptions in Theorem 1.3.

THEOREM 1.4. *In the setting of Theorem 1.3, a space of θ -structures of subplanes $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ fulfils the assumptions in Theorem 1.3 i.e.*

$$(- \cup P)_* : \pi_0(\mathcal{E}_{g,b}^\theta(M; \delta^T)) \rightarrow \pi_0(\mathcal{E}_{g',b'}^\theta(M \cup_{\partial^0 M \times \{0\}} \partial^0 M \times [0, 1]; \delta'^T))$$

and

$$(- \cup P)_* : \pi_0(\mathcal{E}_{g,b}^\theta(\partial^0 M \times [-1, 0]; \delta^T)) \rightarrow \pi_0(\mathcal{E}_{g',b'}^\theta(\partial^0 M \times [-1, 1]; \delta'^T))$$

is an isomorphism if g is high enough, if the homotopy fiber of $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ is simply-connected.

In particular such a tangential structure satisfies homological stability.

Examples of such spaces of tangential structures are given by k -framings of the normal bundle, where $k \leq n - 2$. It would be interesting to see whether one can use Theorem 9.1 to prove that homological stability also holds for framings of the normal bundle, similar to the situation in [RW14]. Another example, related to symplectic structures, will be introduced in Section 11.

Furthermore we prove in Theorem 9.6 that a tangential structure θ as in Example 1.2 satisfies the assumptions of Theorem 1.3, if the corresponding moduli spaces of abstract surfaces $\mathcal{M}^{\theta'}(\Sigma_{g,b})$ introduced in Definition 1.1 in [RW16] fulfil homological stability (see Theorem 7.1 in [RW16] for assumptions implying homological stability). In particular it is proven in [RW16], that homological stability for the abstract moduli space holds, if the stabilization maps in [RW16] yield bijections

$$\pi_0(\mathcal{M}^{\theta'}(\Sigma_{g,b})) \rightarrow \pi_0(\mathcal{M}^{\theta'}(\Sigma_{g,b}))$$

provided that g is large enough.

Examples of such "non-relative" tangential structures are given by framings of the tangent bundle, or spin structures of the subsurfaces. As was mentioned before, homological stability for these was proven in [RW14].

Since some tangential structures are more naturally defined in the context of closed subsurfaces of closed manifolds, we will prove a version of Theorem 1.3 for closed subsurfaces:

Homological Stability for Pointed Subsurfaces: Let M denote a closed manifold with space of θ -structures of subplanes of TM , fix $x \in \Sigma_g$, $x_M \in M$ and $B \in \theta^{-1}(\text{Gr}_2(T_{x_M} M))$, define

$$\text{Emb}_B^\theta(\Sigma_g, M) := \left\{ (e, T_e) \in \text{Emb}^\theta(\Sigma_g, M) \mid e(x) = x_M, T_e(x) = B \right\}$$

As before, the group of orientation-preserving diffeomorphisms of Σ_g that fix x , denoted by $\text{Diff}_x(\Sigma_g)$ admits a natural and free action on this space of embeddings by precomposition. We

will denote the quotient of this action by $\mathcal{E}_B^\theta(\Sigma_g, M)$. This space is called the **space of pointedly embedded θ -subsurfaces of M** . In Section 10 we will construct a stabilization map

$$\sigma_g: \mathcal{E}_B^\theta(\Sigma_g, M) \rightarrow \mathcal{E}_B^\theta(\Sigma_{g+1}, M)$$

which is heuristically given by flattening the subsurfaces in a neighborhood of x_M and then taking a connected sum with a torus in this neighborhood. We also prove that this stabilization map yields homological stability for many kinds of tangential structures (Theorem 10.17). Parts of this theorem can be summarized as follows:

THEOREM 1.5. *Suppose (M, x_M) is an at least 5-dimensional, simply-connected, pointed manifold and suppose further that θ is a space of tangential structures such that the homotopy fiber of θ is simply-connected, then σ_g induces an isomorphism in integral homology in degrees $* \leq \frac{2}{3}g - 1$ and an epimorphism in degrees less than $\frac{2}{3}g$.*

Symplectic Subsurfaces: As an application of this, we prove in Section 11 a homological stability result for spaces of symplectic subsurfaces: Let (M, ω) denote a simply connected symplectic manifold of dimension at least 6 with a base point $x_M \in M$. We will call an embedding $e: \Sigma_g \rightarrow M$ an oriented symplectic embedding if $e^*\omega$ is a symplectic form on Σ_g and $\int_{\Sigma_g} e^*\omega > 0$. Fixing a symplectic 2-plane $B_\omega \in \text{Gr}_2(T_{x_M}M)$, i.e. $\omega|_{B_\omega}$ is non-degenerate, we can consider $\text{SEmb}_{B_\omega}(\Sigma_g, M)$, the space of oriented symplectic embeddings that map the tangent space of the base point of the surface to B_ω . Being an oriented symplectic embedding is invariant under the action of $\text{Diff}_x(\Sigma_g)$ by precomposition. We denote the quotient by $\mathcal{S}_{B_\omega}(\Sigma_g, M)$ and call it the **space of symplectic subsurfaces which are tangential to B** .

A proposition in the context of the h-principle, namely a variation of Theorem 12.1.1 in [EM02], allows us to relate the space of symplectic subsurfaces to spaces of subsurfaces with a certain tangential structure (See Proposition 11.8). Using Theorem 1.5, one obtains:

THEOREM 1.6. *Let (M, ω) denote a simply-connected symplectic manifold of dimension at least 6. For every $x_M \in M$ and every symplectic 2-plane B_ω in $T_{x_M}M$. There is a homomorphism of integral homology*

$$\sigma_g: H_*(\mathcal{S}_{B_\omega}(\Sigma_g, M)) \rightarrow H_*(\mathcal{S}_{B_\omega}(\Sigma_{g+1}, M))$$

And this homomorphism induces an isomorphism for $ \leq \frac{2}{3}g - 1$ and an epimorphism for $* \leq \frac{2}{3}g$.*

Even though the present stabilization map exists only on the level of homology, the author is hopeful that with the presented methods, it is possible to construct an actual map $\mathcal{S}_{B_\omega}(\Sigma_g, M) \rightarrow \mathcal{S}_{B_\omega}(\Sigma_{g+1}, M)$ that realizes the map on homology (see the end of Section 11 for a more detailed discussion of this).

It was shown in [Li05] that many second homology classes can be represented by symplectic subsurfaces. Furthermore essential tools in the theory of symplectic manifolds are J -holomorphic curves and their moduli spaces (See [MS12] for an introduction), embedded J -holomorphic curves are essentially the same as symplectic subsurfaces and their moduli spaces are connected to the space of symplectic subsurfaces. We hope that the homological stability result might give some new insight into these moduli spaces.

Overview: The first two sections describe spaces of subsurfaces with boundary, tangential structures and related concepts used throughout the paper (Section 2) and their stabilization maps (Section 3). Furthermore we show that every stabilization map can be decomposed in to simpler maps, given by stabilizing with a pair of pants or a disk) for which we will prove homological stability.

Section 4 introduces the most important tool in the proof of Theorem 1.3 and Theorem 7.3: Semi-simplicial resolutions. In Section 5 a semi-simplicial resolution of the space of subsurfaces is built and its behaviour with respect to stabilization maps is studied.

In Section 6 we introduce the notion of k -triviality, a concept developed in [RW16], and adapt it to spaces of subsurfaces instead of abstract surfaces. Using this notion we introduce in Section 7 the actual stability bounds and prove most of Theorem 7.3 the stronger version of Theorem 1.3. Section 8 finishes the proof by explaining how to remove boundary components.

In Section 9 we establish criteria for spaces of tangential structures to be suitable for application of Theorem 7.3 and in doing so prove among other results the strengthened version of Theorem 1.4. In Section 10, we extend the notions of spaces of subsurfaces to spaces of pointedly embedded subsurfaces and construct a stabilization map for these.

Lastly in Section 11, we apply the results of Section 9 and Section 10 to prove Theorem 1.6, the homological stability result for symplectic subsurfaces.

Most of the setting and the general proof strategy in this paper were inspired by [CMRW17], and represent an adaptation of their ideas to the newly introduced setting of tangential structures for subsurfaces. However the introduction of the concept of k -triviality (adapted from [RW16]) made the proof more intricate.

The results of Sections 9 to 11 are new and allow a much broader application as well as an extension of the results of [CMRW17].

Acknowledgements: I would like to thank my PhD advisor Ursula Hamenstädt for many helpful discussions and suggestions and furthermore for introducing me to the topic of homological stability. I would also like to thank Mark Pedron for plenty of discussions about this subject and for helping me making the present text much more accessible. Frederico Cantero helped me understand parts of [CMRW17] and he helped me to put things into the correct context, for which I am very grateful. Furthermore I would like to express my gratitude for numerous corrections and hints at possible extensions of the results of this paper by an anonymous referee. Lastly I would like to thank Peter Michor for an answer on MathOverflow ([hm]) without which I would not have been able to finish the proof of Proposition 10.4.

2. Spaces of Subsurfaces and Tangential Structures

The following section will introduce spaces of subsurfaces, tangential structures and will also lay most of the technical foundation for the rest of this paper.

2.1. Spaces of Embeddings and Embedded Submanifolds

Let $(M, \partial M)$ denote a compact smooth manifold with boundary and let $\partial^0 M$ denote a codimension 0 submanifold (possibly with boundary) of ∂ . We call an embedding $c: \partial M \times [0, 1) \rightarrow M$ such that $c((x, 0)) = x$ a **collar of M** .

We say that M is **collared** if there is a fixed collar c of M . Note that if M is collared, then

$$M^k := M \cup_{\partial^0 M} \partial^0 M \times [0, k]$$

has a canonical smooth structure.

DEFINITION 2.1. Let M_1 and M_2 denote smooth collared manifolds. A **smooth map $f: M_1 \rightarrow M_2$ between collared manifolds** is a smooth map that maps ∂M_1 to ∂M_2 and there exist neighborhoods of ∂M_1 and ∂M_2 such that in these neighborhoods f is a product of a smooth map and the identity with respect to the collars of M_1 and M_2 .

The definition of **collared embedding** and **collared diffeomorphism** is completely analogous.

Since every occurring manifold and map will be assumed to be a collared manifold or a collared map we will drop the adjective.

A collared embedding from M_1 to M_2 maps the boundary of M_1 to the boundary of M_2 and the map is transverse to the boundary of M_2 . If $A_1 \subset M_1$ and $A_2 \subset M_2$ are submanifolds, we call an embedding $e: (M_1, A_1) \rightarrow (M_2, A_2)$ an **embedding of pairs** if $e^{-1}(A_2) = A_1$. We will be interested in spaces associated to the **space of (collared) embeddings** $\text{Emb}(M_1, M_2)$. This space is equipped with the C^∞ -topology.

DEFINITION 2.2. Let $e_1, e_2: M_1 \rightarrow M_2$ denote embeddings. We say that e_1 and e_2 **have the same jet along** ∂M_2 if there exists an open neighbourhood $U_{\partial M_1}$ of ∂M_1 such that $e_1|_{U_{\partial M_1}} = e_2|_{U_{\partial M_1}}$. This defines an equivalence relation on $\text{Emb}(M_1, M_2)$ and we denote the quotient map by

$$\mathcal{J}: \text{Emb}(M_1, M_2) \rightarrow J_\partial(M_1, M_2)$$

We write $\text{Emb}(M_1, M_2; d) := \mathcal{J}^{-1}(d)$.

From here on forth let us specialize to the case where the domain of the embedding is a compact connected oriented surface of genus g with b boundary components denoted by $\Sigma_{g,b}$. We denote by $\text{Diff}^+(\Sigma_{g,b})$ the **group of orientation-preserving diffeomorphisms of $\Sigma_{g,b}$** . This group acts freely on the space of embeddings and on J_∂ via precomposition. We define

$$\begin{aligned} \mathcal{E}_{g,b}^+(M) &:= \text{Emb}(\Sigma_{g,b}, M) / \text{Diff}^+(\Sigma_{g,b}) \\ \mathcal{J}_\partial(\Sigma_{g,b}, M) &:= J_\partial(\Sigma_{g,b}, M) / \text{Diff}^+(\Sigma_{g,b}) \end{aligned}$$

The first space will be called the **space of subsurfaces of genus g and b boundary components**. We also define

$$\mathcal{E}^+(M; b) := \bigsqcup_g \mathcal{E}_{g,b}^+(M; \delta)$$

An element $W \in \mathcal{E}_{g,b}^+(M)$ is an unparametrized embedding or in other words a submanifold $W \subset M$ diffeomorphic to $\Sigma_{g,b}$. We will call such an unparametrized embedding a **subsurface of M** .

Since the map \mathcal{J} is equivariant with respect to the $\text{Diff}^+(\Sigma_{g,b})$ -actions, we get an induced map between the quotient spaces also denoted by \mathcal{J} . We define $\mathcal{E}_{g,b}^+(M; \delta)$ as $\mathcal{J}^{-1}(\delta)$ for some $\delta \in \mathcal{J}_\partial(\Sigma_{g,b}, M)$. Note that if $\delta = [d]$ then

$$\mathcal{E}_{g,b}^+(M; \delta) \cong \text{Emb}(\Sigma_{g,b}, M; d) / \text{Diff}_\partial^+(\Sigma_{g,b})$$

Here $\text{Diff}_\partial^+(\Sigma_{g,b})$ denotes the group of diffeomorphisms of $\Sigma_{g,b}$ that agree with the identity on a neighbourhood of the boundary.

As a slight abuse of notation we will also write δ for the image of some representative of $\delta \in \mathcal{J}_\partial(\Sigma_{g,b}, M)$ in ∂M . We will call δ a **boundary condition**.

2.2. Tangential structures

Let M denote a manifold possibly with boundary. There is the fiber bundle $\text{Gr}_2(TM) \rightarrow M$, where $\text{Gr}_2(TM)$ denotes the space of oriented 2-planes in TM . For the purpose of this paper a **space of θ -structures of subplanes of TM** will denote a continuous Hurewicz-fibration $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$. For a submanifold A of M we have an induced map $\text{Gr}_2(TA) \rightarrow \text{Gr}_2(TM)$ and we will denote the pullback of $B_2(M)$ along this inclusion by $B_2(A)$.

If W is an oriented subsurface, then $\text{Gr}_2(W) \rightarrow W$ is a two-sheeted covering with a distinguished section. In this particular case we will identify the correctly oriented connected component of $\text{Gr}_2(W)$ with W and consider $B_2(W)$ as a fibration over W .

If we have an embedding $e: \Sigma_{g,b} \rightarrow M$, then we get an induced map

$$\text{Gr}(De): \Sigma_{g,b} \rightarrow \text{Gr}_2(TM)$$

by mapping $x \in \Sigma_{g,b}$ to $De(T_x \Sigma_{g,b})$ considered as an oriented 2-plane. We will call this map the **Grassmannian differential of e** .

DEFINITION 2.3. Given an embedding $e: \Sigma_{g,b} \rightarrow M$, we call a map $T_e: \Sigma_{g,b} \rightarrow B_2(M)$ a **θ -structure of e** if the following diagram commutes:

$$\begin{array}{ccc} & & B_2(M) \\ & \nearrow T_e & \downarrow \theta \\ \Sigma_{g,b} & \xrightarrow{\text{Gr}(De)} & \text{Gr}_2(TM) \end{array}$$

We will denote by $\text{Emb}^\theta(\Sigma_{g,b}, M)$ the **space of embeddings with θ -structure**. This space is topologized as a subspace of $\text{Map}(\Sigma_{g,b}, B_2(M)) \times \text{Emb}(\Sigma_{g,b}, M)$, where the first factor is equipped with the compact-open topology and the second factor with the C^∞ -topology. Even though elements of $\text{Emb}^\theta(\Sigma_{g,b}, M)$ consist of pairs of maps, we will usually say that e is an embedding with θ -structure. In this case we will denote the underlying θ -structure by T_e . We will say that two embeddings with θ -structure e_1, e_2 have **the same θ -jet along ∂M** if they have the same jet along ∂M and $T_{e_1}|_{\partial\Sigma_{g,b}} = T_{e_2}|_{\partial\Sigma_{g,b}}$.

For a diffeomorphism $\phi \in \text{Diff}^+(\Sigma_{g,b})$, we have $\text{Gr}(De)(x) = \text{Gr}(De \circ \phi^{-1})(\phi(x))$. Hence the group $\text{Diff}^+(\Sigma_{g,b})$ acts on $\text{Emb}^\theta(\Sigma_{g,b}, M)$ via precomposition, this action preserves the equivalence relation of having the same θ -jet. If we restrict the embeddings to have some fixed θ -jet, then we denote the corresponding space of embeddings with θ -structure by δ^T to obtain $\text{Emb}^\theta(\Sigma_{g,b}, M; \delta^T)$ and we denote the quotient of this subspace by the action of $\text{Diff}^+(\Sigma_{g,b})$ by $\mathcal{E}_{g,b}^\theta(M; \delta^T)$. Similarly, we obtain $\mathcal{J}_\partial^\theta(\Sigma_{g,b}, M)$ and $\mathcal{E}^\theta(M; \delta^T)$. We will call the second space the **space of subsurfaces with θ -structure** and an element in it will be called a **subsurface with θ -structure of M** . If some θ -jet δ^T of an embedding with θ -structure is given, we will sometimes write δ for the image of the underlying boundary condition without θ -structure.

If θ is the identity map, we will write $+$ instead of Id . Let us recall parts of an example of the introduction:

EXAMPLE 2.4. Let γ^\perp denote the complement of the tautological 2-plane bundle on $\text{Gr}_2(TM)$ i.e. $\pi_{\text{Gr}_2(TM)}^* TM / \gamma$, where $\pi_{\text{Gr}_2(TM)}: \text{Gr}_2(TM) \rightarrow M$ denotes the projection and γ denotes the tautological bundle which includes into $\pi_{\text{Gr}_2(TM)}^* TM$. Let $S(\gamma^\perp) := \gamma^\perp \setminus 0$ denote the complement of the image of the zero section of this bundle. Then $\theta: S(\gamma^\perp) \rightarrow \text{Gr}_2(TM)$ is a Hurewicz-fibration. For an embedding $e: \Sigma_{g,b} \rightarrow M$ the space $B_2(e(\Sigma_{g,b}))$ is the complement of the zero-section of the normal bundle and a θ -structure corresponds to a continuous section of this bundle. In this case $\text{Emb}^\theta(\Sigma_{g,b}, M)$ is the space of embeddings together with a continuous section of the normal bundle.

REMARK 2.5. In some situations it makes sense to replace $\text{Map}(\Sigma_{g,b}, B_2(M))$ by $C^\infty(\Sigma_{g,b}, B_2(M))$ in Definition 2.3. Of course this requires some kind of smooth structure on $B_2(M)$. In Example 2.4 this would make the sections of the normal bundle smooth sections. One also has to alter the definition of having the same θ -jet to include the jet of the θ -structures. Everything proven in this paper works in that case as well.

REMARK 2.6. If we have a subsurface with a tangential structure W and some isotopy $([0, 1], 0) \rightarrow (\text{Diff}_\partial^+(M), \text{Id})$, we can move W along M using this isotopy to obtain W_t . Since θ is a Hurewicz-fibration we can also equip W_t with a tangential structure and if we say nothing about tangential structures in this context, we will assume implicitly that a lift of the aforementioned path in $\mathcal{E}_{g,b}^+(M; \delta)$ is chosen. Note that the tangential structure of W_1 is not unique i.e. there might be many different tangential structures for W_t .

Lastly, using a collar M^k and M are diffeomorphic and since collars are unique up to isotopy, the associated diffeomorphism is also unique up to isotopy. This enables us to equip M^k with a unique space of θ -structures and we will always assume that M^k has this space of θ -structures. A space of θ -structures on a manifold M induces a space of θ -structures on every submanifold and

we will always assume that submanifolds are equipped with the induced structure. In particular $\partial M \times [0, k]$ is equipped with a space of θ -structures.

2.3. Retractable Spaces and Fibrations

The following arguments will be used frequently throughout this paper to prove that certain maps are locally trivial fibrations and the ideas date back to [Cer61].

DEFINITION 2.7. Let G denote a topological group and X a G -space. We say that X is **G -locally retractile** if every point $x \in X$ possesses an open neighborhood U_x and a continuous map $\xi: (U_x, x) \rightarrow (G, \text{Id})$, called the **G -local retraction around x** , such that $\xi(x') \cdot x = x'$ for all $x \in U_x$. Note that this implies that ξ is a homeomorphism onto its image.

The following lemma will be used several times throughout this paper.

LEMMA 2.8. *If X is G -locally retractile and $g: X' \rightarrow X$ denotes a G -equivariant map, then g is a locally trivial fibration.*

PROOF. For $x \in X$ choose an U_x and a G -local retraction $\xi: U_x \rightarrow G$ around x . Then

$$\begin{aligned} g^{-1}(\{x\}) \times U_x &\rightarrow g^{-1}(U_x) \\ (x_{X'}, x_X) &\mapsto \xi(x_X) \cdot x_{X'} \end{aligned}$$

gives the desired local trivialisation as is easily checked. \square

As before let $\text{Diff}_\partial^+(M)$ denote the group of diffeomorphisms of M that agree with the identity on a neighbourhood of ∂M . The following proposition can be found in [Cer61] (Chapter 2.2, Theorem 5).

PROPOSITION 2.9. *Let d denote the jet of an embedding from A into M , then $\text{Emb}(A, M; d)$ is $\text{Diff}_\partial^+(M)$ -locally retractile.*

The following proposition can be found in [CMRW17] with a proof that relies on a reference to [BF81] or [Mic80]. We will give a sketch of a proof of the proposition, because similar ideas will occur later on.

PROPOSITION 2.10. *The space $\mathcal{E}_{g,b}^+(M; \delta)$ is $\text{Diff}_\partial^+(M)$ -locally retractile.*

PROOF. Let $W \subset M$ denote an element of $\mathcal{E}_{g,b}^+(M; \delta)$ and $N_{W,M}$ a tubular neighborhood of W with corresponding projection $\pi_{N_{W,M}}: N_{W,M} \rightarrow W$. We define

$$U_W = \left\{ W' \in \mathcal{E}_{g,b}^+(M; \delta) \mid W' \subset N_{W,M} \text{ and } \pi_{N_{W,M}}|_{W'} \text{ is a diffeomorphism from } W' \text{ to } W \right\}$$

which is an open subset of $\mathcal{E}_{g,b}^+(M; \delta)$, because the diffeomorphisms form an open subset of the maps from W' to W (See Proposition 1.4.2 in Chapter 2 in [Cer61]). For W' in U_W we can define an isotopy of the embedded subsets by pushing W' along the fiber of $N_{W,M}$ onto W . Furthermore we can use the isotopy extension theorem (See Theorem 6.1.1 in [Wal16]), to extend these isotopies continuously to isotopies of M . The resulting diffeomorphisms at time 1 of these isotopies give us the desired $\text{Diff}_\partial^+(M)$ -local retraction. \square

The last Proposition implies the following useful observation:

COROLLARY 2.11. *The two maps*

$$\begin{aligned} \text{Emb}(\Sigma_{g,b}, M; d) &\rightarrow \mathcal{E}_{g,b}^+(M; \delta) \\ \text{Emb}^\theta(\Sigma_{g,b}, M; d^T) &\rightarrow \mathcal{E}_{g,b}^\theta(M; \delta^T) \end{aligned}$$

are locally trivial fibrations.

PROOF. The only thing that might need clarification is the second assertion, but to prove this one only has to note that

$$\begin{array}{ccc} \text{Emb}^\theta(\Sigma_{g,b}, M; d^T) & \longrightarrow & \text{Emb}(\Sigma_{g,b}, M; d) \\ \downarrow & & \downarrow \\ \mathcal{E}_{g,b}^\theta(M; \delta^T) & \longrightarrow & \mathcal{E}_{g,b}^+(M; \delta) \end{array}$$

is a pullback diagram. \square

Moreover we have the following central lemma:

LEMMA 2.12. *The forgetful map $\pi_\theta: \mathcal{E}_{g,b}^\theta(M; \delta^T) \rightarrow \mathcal{E}_{g,b}^+(M; \delta)$ is a Hurewicz fibration. The fiber over a surface W is given by $\Gamma(B_2(W))$, the space of sections of $B_2(W)$. The same holds true for the forgetful map $\text{Emb}^\theta(\Sigma_{g,b}, M; d^T) \rightarrow \text{Emb}^+(\Sigma_{g,b}, M; d^T)$*

PROOF. By Theorem 42.3 and the corollary in 27.4 in [KM97] (together with the fact that the quotient of a nuclear spaces by a closed subspaces is a nuclear space) we conclude that $\mathcal{E}_{g,b}^+(M; \delta)$ is paracompact and Hausdorff, hence we conclude that a locally trivial fiber bundle over this space is actually a Hurewicz fibration by Theorem 13 in Section 2.7 of [Spa95]. Hence it suffices to prove that π_θ is a locally trivial fiber bundle. In order to do this we want to apply Lemma 2.8.

Let $\text{Homeo}_M(B_2(M))$ denote the group of homeomorphisms of $B_2(M)$ which cover a diffeomorphism of $\text{Gr}_2(TM)$ induced from a diffeomorphism of M that fixes a neighbourhood of the boundary i.e. the following diagram commutes and the lower map is a diffeomorphism

$$\begin{array}{ccc} B_2(M) & \longrightarrow & B_2(M) \\ \downarrow & & \downarrow \\ \text{Gr}_2(TM) & \xrightarrow{\text{Gr}(Df)} & \text{Gr}_2(TM) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array}$$

This group acts on $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ and $\mathcal{E}_{g,b}^+(M; \delta)$ by post-composition. Proposition 2.10 states that $\mathcal{E}_{g,b}^+(M; \delta)$ is $\text{Diff}_\partial^+(M)$ -locally retractile. We will show that it is also $\text{Homeo}_M(B_2(M))$ -locally retractile. For a $W \in \mathcal{E}_{g,b}^+(M; \delta)$, let $\xi: U_W \rightarrow \text{Diff}_\partial^+(M)$ denote a $\text{Diff}_\partial^+(M)$ -local retraction around W . Note that $\text{Diff}_\partial^+(M)$ is locally contractible and it is paracompact (For paracompactness see Lemma 41.11 of [KM97] to conclude that $\text{Diff}_\partial^+(M)$ with the C^∞ topology is metrizable). Hence there is a contractible, closed neighbourhood of the identity in $\text{Diff}_\partial^+(M)$. By shrinking U_W we can assume that $\xi(U_W)$ maps to such a contractible neighbourhood of the identity.

We get a map $\text{Gr}(Dev): \text{Gr}_2(TM) \times U_W \rightarrow \text{Gr}_2(TM)$ which maps (B, W) to $\text{Gr}(D\xi(W))(B)$ and we can use the contractibility of U_W together with its paracompactness and the compactness of M to conclude that the pullback $\text{Gr}(Dev)^* B_2(M)$ is isomorphic to the pullback $\pi_{U_W}^* B_2(M)$, where $\pi_{U_W}: \text{Gr}_2(TM) \times U_W \rightarrow \text{Gr}_2(TM)$ denotes the projection. Using this isomorphism, one can construct a lift of ξ to conclude that $\mathcal{E}_{g,b}^+(M; \delta)$ is $\text{Homeo}_M(B_2(M))$ -locally retractile. By Lemma 2.8 we conclude that the forgetful map π_θ is a locally trivial fiber bundle.

The specification of the fibre is evident. The proof for the second map is completely analogous. \square

We will also need the following standard lemma from algebraic topology, whose proof is omitted:

LEMMA 2.13. *Suppose we have the following diagram of topological spaces*

$$X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3$$

where g_1 is surjective. If g_1 and $g_2 \circ g_1$ are fibrations, then g_2 is a fibration as well.

2.4. Tubular neighborhoods and Thickened Embeddings

If $A \subset M$ denotes a submanifold, we will write $N(A, M)$ for the normal bundle of A i.e. $TM|_A/TA$. A tubular neighbourhood of A is an embedding $e_{A,M}^T: N(A, M) \rightarrow M$ such that $e_{A,M}^T|_A$ is the identity and the composition

$$TA \oplus N(A, M) \xrightarrow{\cong} T(N(A, M))|_A \xrightarrow{De_{A,M}^T} TM|_A \xrightarrow{\pi_{N(A, M)}} N(A, M)$$

agrees with the projection onto the second factor. If $(A, A') \subset (M, A)$ denotes an embedded pair, we define a tubular neighborhood of (A, A') in (M, A) to be a tubular neighborhood $e_{A,M}^T: N(A, M) \rightarrow M$ of A such that $e_{A,M}^T|_{N(A', A)}$ is a tubular neighbourhood of A' in A . If A has a boundary we assume that every tubular neighbourhood of A is actually a tubular neighbourhood of $(A, \partial A)$ in $(M, \partial M)$.

From the vector bundle $N(A, M)$, one can obtain a disk bundle by compactifying this fiberwise using a sphere at infinity to obtain $\overline{N(A, M)}$. An embedding of $\overline{N(A, M)} \rightarrow M$ is called a closed tubular neighborhood if its restriction to $N(A, M)$ is a tubular neighborhood. We denote by

$$\overline{\text{Tub}}(A, M) \subset \text{Emb}(\overline{N(A, M)}, M)$$

the subspace of closed tubular neighborhoods. Similar notation will occur if we consider tubular neighborhoods of pairs.

A proof of the following lemma is sketched in Section 2.5 of [CMRW17].

LEMMA 2.14. *If (A, A') is an embedded pair of compact submanifolds in (M, A) , then $\overline{\text{Tub}}(A, M)$ and $\overline{\text{Tub}}((A, A'), (M, A))$ are contractible.*

DEFINITION 2.15. We call an embedding, possibly equipped with a θ -structure, $e: \Sigma_{g,b} \rightarrow M$ together with a closed tubular neighbourhood of $e(\Sigma_{g,b})$ a **thickened embedding of $\Sigma_{g,b}$** . We will write $\overline{\text{TEmb}}(\Sigma_{g,b}, M; d)$ for the **space of thickened embeddings** and $\overline{\text{TEmb}}^\theta(\Sigma_{g,b}, M; d^T)$ for the **space of thickened embeddings with θ -structure**. We could furthermore replace $\Sigma_{g,b}$ by a pair $(\Sigma_{g,b}, A')$ and M by (M, A) to obtain **spaces of thickened embeddings of pairs**.

The topology of the space of thickened embeddings is quite tricky and we refer the reader to Section 2.5 in [CMRW17] for details. The important thing to note about this space is the following proposition:

PROPOSITION 2.16. *$\text{Diff}_\partial^+(M)$ acts via post composition on $\overline{\text{TEmb}}(A, M; d)$ and this space is $\text{Diff}_\partial^+(M)$ -locally retractile.*

COROLLARY 2.17. *The forgetful map $\overline{\text{TEmb}}(\Sigma_{g,b}, M; d) \rightarrow \text{Emb}(\Sigma_{g,b}, M; d)$ is a locally trivial fibration with fiber over a basepoint $e \in \text{Emb}(\Sigma_{g,b}, M; d)$ given by the space of tubular neighborhoods of $e(\Sigma_{g,b})$.*

Furthermore the forgetful map $\overline{\text{TEmb}}^\theta(\Sigma_{g,b}, M; d) \rightarrow \text{Emb}^\theta(\Sigma_{g,b}, M; d)$ is also a locally trivial fibration with fiber the space of tubular neighborhoods of some $e(\Sigma_{g,b})$.

PROOF. The first assertion follows as $\text{Emb}(\Sigma_{g,b}, M; d)$ is $\text{Diff}_\partial^+(M)$ -locally retractile. The second assertion follows since the following diagram is a pullback diagram:

$$\begin{array}{ccc} \overline{\text{TEmb}}^\theta(\Sigma_{g,\delta}, M; d^T) & \longrightarrow & \overline{\text{TEmb}}(\Sigma_{g,b}, M; d) \\ \downarrow & & \downarrow \\ \text{Emb}^\theta(\Sigma_{g,b}, M; d^T) & \longrightarrow & \text{Emb}(\Sigma_{g,b}, M; d) \end{array}$$

□

We will also need the following lemma:

LEMMA 2.18. *For every submanifold A of $\Sigma_{g,b}$, the restriction maps*

$$\begin{aligned} \text{Emb}^\theta(\Sigma_{g,b}, M; d^T) &\rightarrow \text{Emb}^\theta(A, M; d^T) \\ \overline{\text{TEmb}}^\theta(\Sigma_{g,b}, M; d^T) &\rightarrow \overline{\text{TEmb}}^\theta(A, M; d^T) \end{aligned}$$

are fibrations

The proof will use the following lemma:

LEMMA 2.19. *Let X denote a metric space and $A \subset X$ a closed subset, with a closed neighborhood U_A such that A is a strong deformation retract of U_A via a strong deformation $\Phi(x, t)$. Then $X \times \{0\} \cup A \times [0, 1]$ is a strong deformation retract of $X \times [0, 1]$.*

The assumptions about X are quite restricting and the lemma should be true in a more general setting, but for the present context it certainly suffices.

PROOF. Define $d: U_A \rightarrow \mathbb{R}$ as

$$d(x) = \frac{d(x, A)}{d(x, A) + d(x, \partial U_A)}$$

As assumed above let $\Phi(-, t)$ denote a homotopy between the identity on U_A and a retraction of the neighborhood. Then the deformation retract in the product space and the corresponding homotopy is given by the following:

$$\Psi((x, s), t) = \begin{cases} (\Phi(x, 0), st) & \text{for } d(x) - s \geq 0 \\ \left(\Phi\left(x, (1-t)\left(\frac{s-d(x)}{s}\right)\right), s - (1-t)d(x) \right) & \text{for } d(x) - s \leq 0 \\ (x, st) & \text{for } x \notin U_A \end{cases}$$

□

PROOF OF LEMMA 2.18. Note that both statements follow from Lemma 2.8 if $\theta = \text{Id}$. So consider the following lifting problem:

$$\begin{array}{ccccc} D^N \times \{0\} & \xrightarrow{\bar{g}} & \text{Emb}^\theta(\Sigma_{g,b}, M; d^T) & \longrightarrow & \text{Emb}^+(\Sigma_{g,b}, M; d^T) \\ \downarrow & \nearrow \text{---} & \downarrow & & \downarrow \text{fib} \\ D^N \times [0, 1] & \xrightarrow{g} & \text{Emb}^\theta(A, M; d^T) & \longrightarrow & \text{Emb}^+(A, M; d^T) \end{array}$$

Since the right vertical map is a fibration, we can lift g to $\text{Emb}^+(\Sigma_{g,b}, M; d^T)$. Such a map corresponds to an embedding $e: \Sigma_{g,b} \times D^N \times [0, 1] \rightarrow M \times D^N \times [0, 1]$ which is the identity on the $D^N \times [0, 1]$ component. Furthermore this embedding is equipped with a tangential structure on $\Sigma_{g,b} \times D^N \times \{0\} \cup A \times [0, 1]$ which stems from g and \bar{g} . Lifting g corresponds to an extension of this tangential structure. By Lemma 2.19, $\Sigma_{g,b} \times D^N \times \{0\} \cup A \times [0, 1]$ is a deformation retract of $\Sigma_{g,b} \times D^N \times [0, 1]$, hence $e^*B_2(M)$ is isomorphic to $e^*|_{\Sigma_{g,b} \times D^N \times \{0\} \cup A \times [0, 1]} B_2(M)$. Composition with such an isomorphism provides the required extension of the tangential structure. The proof for the spaces of thickened embeddings is completely analogous. □

3. Stabilization Maps

We will be interested in two kinds of stabilization maps. Up to homotopy equivalences, they are the same, but they will play different roles in the proof and therefore will be distinguished. They are motivated by the definition of inner and outer cobordisms in [RW16].

3.1. Boundary Bordisms and Stabilization Maps

Let M denote an at least 5-dimensional manifold and $\partial^0 M$ a codimension 0 submanifold of ∂M . By definition, a boundary condition $\delta^T \in \mathcal{J}_\partial^\theta(\Sigma_{g,b}, M)$ that intersects the boundary of $\partial^0 M$ transversely yields a submanifold with θ -structure in $\partial^0 M \times [0, 1]$ by taking a product with $[0, 1]$. This specifies a θ -jet along $\partial^0 M \times \{0\}$. Together with a θ -jet $\bar{\delta}^T$ along $\partial^0 M \times \{1\}$ and using collars one obtains boundary conditions in $\mathcal{J}_\partial^\theta(\Sigma_{g',b+b'}, \partial^0 M \times [0, 1])$ for any g' and b' , which we will denote by $\delta^T \cup \bar{\delta}^T$. Let us denote $\bigsqcup_{g'} \mathcal{E}_{g',b+b'}^\theta(\partial^0 M \times [0, 1]; \delta^T \cup \bar{\delta}^T)$ by $\mathcal{C}^\theta(\partial^0 M; \delta^T, \bar{\delta}^T)$ and call it the space of **boundary bordisms**.

Given an element $P \in \mathcal{C}^\theta(\partial^0 M; \delta^T, \bar{\delta}^T)$ one obtains a continuous

$$\begin{aligned} - \cup P: \mathcal{E}_{g,b}^\theta(M; \delta^T) &\rightarrow \mathcal{E}_{g',b'}^\theta(M^1; \bar{\delta}^T) \\ W &\mapsto W \cup P \cup \delta'^T \times [0, 1] \end{aligned}$$

where δ' consists of those boundary conditions that are not met by the boundary bordism. These maps are called **stabilization maps**.

We will call $P \cap \partial^0 M \times \{0\}$ the **incoming boundary of P** and $P \cap \partial^0 M \times \{1\}$ the **outgoing boundary of P** . We have the following lemma, which says that P decomposes into elementary pieces, for which it will be easier to prove homological stability.

PROPOSITION 3.1. *Let M be a manifold with space of θ -structures of subplanes of TM . If P is a boundary bordism as above then there exists P_1, \dots, P_k such that:*

- (i) P_i is a boundary bordism sitting inside of $\partial^0 M \times [i-1, i]$ and all of its connected components but one are homeomorphic to a cylinder.
- (ii) $P_1 \cup \dots \cup P_k \simeq k \cdot P \subset \partial^0 M \times [0, k]$ (The product means that $(x, t) \in \partial^0 M \times [0, 1]$ is mapped to $(x, kt) \in \partial^0 M \times [0, k]$) and \simeq means in this case that the underlying submanifolds are isotopic relative to their boundary and their tangential structures are homotopic
- (iii) the projection onto the second coordinate of $\partial^0 M \times [i-1, i]$ restricted to P_i is a Morse function with at most one critical point.
- (iv) If any such critical point happens to be a minimum or a maximum it has to be a global maximum or minimum of $P_1 \cup \dots \cup P_k$.

REMARK 3.2. Since we implicitly required every connected component of P to meet $\partial^0 M \times \{0\}$ the last part of the lemma says among other things that there is no minimum.

To prove this proposition we will need to use surgery along a half-disk, a process we will explain now. We define

- (i) $D_+^2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0, \|(x_1, x_2)\| \leq 1\}$
- (ii) $\partial^0 D_+^2 := \{(x_1, x_2) \in \partial D_+^2 \mid x_2 = 0\}$
- (iii) $\partial^1 D_+^2 := \{(x_1, x_2) \in \partial D_+^2 \mid \|(x_1, x_2)\| = 1\}$

Let $W \subset M$ denote a subsurface of M . We fix an embedded circle $c \subset \partial^1 D_+^2 \times [0, 1]$ which contains $(\partial^1 D_+^2 \setminus (B_\epsilon(1) \cup B_\epsilon(-1))) \times \{0, 1\}$ and connects these two connected arcs via two small circle segments. Fix once and for all an embedding $a: D^2 \rightarrow D_+^2 \times [0, 1]$ such that $a(\partial D^2) = c$ and $a(D^2 \setminus \frac{1}{2}D^2) \subset \partial^1 D_+^2 \times [0, 1]$ and such that $a(\mathbb{R} \cdot (1, 0) \cap \frac{1}{2}D^2)$ is very close to $\partial^0 D_+^2 \times \{\frac{1}{2}\}$ (see Figure 1 to clarify the definition of c and a). Note that $a(D^2)$ cuts $\partial^1 D_+^2 \times [0, 1]$ into two connected components and let X denote the "inner one" i.e. the one that does not contain $\partial^1 D_+^2 \times \{0, 1\}$.

DEFINITION 3.3. Let M denote a manifold as above and W a subsurface of M . Let $e_{D_+^2}: (D_+^2 \times [0, 1], \partial^1 D_+^2 \times [0, 1]) \rightarrow (M, W)$ denote an embedding, then we will denote $(W \setminus e_{D_+^2}(X)) \cup e_{D_+^2}(a(D^2))$ by $W \natural e_{D_+^2}$. The above requirements for a ensure that this is a smooth manifold.

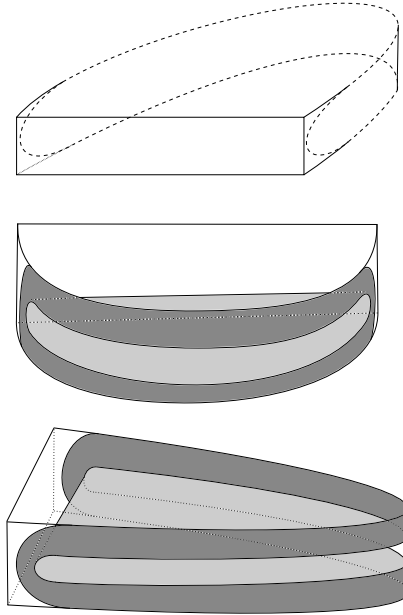


FIGURE 1. The dashed circle in the first picture represents c and the union of the gray areas in the other two pictures represents the image of a (of course the boundary between the two areas has to be smoothed, but this exceeds the drawing skills of the author)

Suppose that the dimension of M is at least 5 and that W is a subsurface with tangential structure. In this case W and $W \natural_{e_{D_+^2}}$ are isotopic and we can use this to give $W \natural_{e_{D_+^2}}$ some tangential structure. Even though this is not well-defined we will denote any such subsurface with tangential structure by $W \natural_{e_{D_+^2}}$.

Assume that the dimension of M is at least 5 and W is a subsurface, given a map of a half-disk $e_{D_+^2} : (D_+^2, \partial^1 D_+^2) \rightarrow (M, W)$, which is an embedding at every point except at $(\pm 1, 0)$, where $De_{D_+^2}(T_{(\pm 1, 0)} \partial^1 D_+^2)$ agrees with $De_{D_+^2}(T_{(\pm 1, 0)} \partial^0 D_+^2)$. We can use the sufficiently high codimension to ensure that $e_{D_+^2}$ extends to an embedding $e'_{D_+^2} : (D_+^2 \times [0, 1], \partial^1 D_+^2) \rightarrow (M, W)$ such that $e_{D_+^2}(\partial^0 D_+^2)$ agrees with $e'_{D_+^2}(a(\mathbb{R} \cdot (1, 0) \cap \frac{1}{2} D^2))$. In this case we will still write $W \natural_{e_{D_+^2}}$ for the surgery along this extension of $e_{D_+^2}$. Armed with this definition we can start proving the previous lemma.

PROOF OF PROPOSITION 3.1. We start by showing that there is an isotopy of P such that $\pi_{[0, 1]} : \partial^0 M \times [0, 1] \rightarrow [0, 1]$ restricted to P is a Morse function with the property that every critical value corresponds to a unique critical point, which proves the first three parts of the lemma. Then we proceed by explaining how to remove local minima. Using the symmetry between minima and maxima given by the flip map $(x, t) \mapsto (x, 1 - t)$ this also explains how to eliminate local maxima.

We have an open inclusion

$$\text{Emb}^\theta(P, \partial^0 M \times [0, 1]) \rightarrow C^\infty(P, \partial^0 M \times [0, 1]) \cong C^\infty(P, [0, 1]) \times C^\infty(P, \partial^0 M)$$

The isomorphism stems from the identification of maps into products as products of maps and the described map has the following form $e \mapsto (\pi_{\partial^0 M} \circ e, \pi_{[0, 1]} \circ e)$. That this map is indeed

open is proven in [Cer61] Chapter 2, Section 1.2. Since Morse functions form an open and dense subset of all real functions we get that in $C^\infty(P, [0, 1]) \times C^\infty(P, \partial^0 M)$ the subset $\text{Morse}(P, [0, 1]) \times C^\infty(P, \partial^0 M)$ is open and dense as well. Since the space of embeddings is also open there exists an embedding, such that the projection onto the second component is a Morse function, such that the embedding is arbitrarily close to the inclusion from P into $\partial^0 M$. Since the space of embeddings is locally path-connected (see Theorem 44.1 of [KM97]), we conclude that P is isotopic to a subsurface such that the projection restricted to this subsurface is a Morse function. It is easy to see that we can arrange the critical points to have different values by moving them up or down a little.

For the fourth part of the lemma chose a riemannian metric g on $\partial^0 M \times [0, 1]$ such that ∂_t (the directional derivative of the interval coordinate) is orthogonal to $T_{(x,t)}\partial^0 M \times \{t\}$ for every (x, t) and assume that $\pi_{[0,1]}|_P$ is a Morse function with distinct critical values.

Before we can continue we have to fix some notation: Consider a critical point x of $\pi_{[0,1]}|_P$, which is not a maximum. Consider the negative gradient flow of $\pi_{[0,1]}|_P$ in a neighborhood of x . Then there have to be some flow lines γ which converge to x meaning that $\lim_{T \rightarrow -\infty} \gamma(T) = x$ but since P is compact this gradient flow line has to either meet a boundary component or it is an embedding of \mathbb{R} into P in which case the compactness of P ensures that $\lim_{T \rightarrow \infty} \gamma(T)$ exists and has to be a critical value as well. In this case we call x the starting point of γ and the other limit point the endpoint of γ . Furthermore note that if γ ends at a point x , there also has to be flow line of the gradient flow, which starts at x and goes in the opposite direction as γ , by which we mean that in a Morse chart centered around x γ is given by $e^{1/2T}v$, then the other curve is given by $-e^{1/2T}v$.

Now let us proceed with the proof of the theorem. Let p denote a minimum for $\pi_{[0,1]}|_P$ or in other words a 0-handle with respect to the handle-decomposition imposed by $\pi_{[0,1]}|_P$. Since every connected component of P meets $\partial^0 M \times \{0\}$ there has to be a 1-handle, which cancels the 0-handle given by p . In other words there has to exist a gradient flow line γ_1 for $\pi_{[0,1]}|_P$ starting at p , which ends at an index 1 critical point q . Following the negative gradient flow along the opposite direction defined by γ_1 at q gives us a gradient flow line γ_2 that ends at some critical point p' which is lower with respect to its t -component than q . By reparametrizing and including their endpoint and starting point we can consider γ_1 and γ_2 as arcs starting and ending at critical points. We denote by γ the concatenation of γ_1 and γ_2 . By changing P a little bit but fixing $\pi_{[0,1]}|_P$ we can arrange that there is no open segment of γ , where $\pi_{\partial^0 M} S^1 \gamma$ is constant. Fix a strictly increasing or strictly decreasing function $f: [0, 1] \rightarrow [0, 1]$ such that $f(0) = \pi_{[0,1]}(p')$ and $f(1) = \pi_{[0,1]}(p)$. Then define

$$\begin{aligned} H'' : [0, 1] \times [0, 1] &\rightarrow \partial^0 M \times [0, 1] \\ (t', s) &\mapsto (\pi_{\partial^0 M}(\gamma(s)), (1-t')\pi_{[0,1]}(\gamma(s)) + t'f(s)) \end{aligned}$$

Using the main theorem of [Whi36], which says that embeddings are dense in the space of mappings, we can replace H'' by a C^∞ close map H' such that H' factors through an embedding of a half disc H' (this is possible since H'' is constant on $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$) such that $D\pi_{[0,1]} \circ H'(TD_+^2) = \langle \partial_t \rangle$, $D\pi_{[0,1]} \circ H'(T\partial^0 D_+^2) = \lambda \partial_t$ for λ a negative function or λ a positive function depending on the height of p and p' , and that $H'|_{\partial^1 D_+^2} = \gamma$. We can further replace H' by H , using the denseness of transversal maps, such that H still fulfils the conditions above and that $H(D_+^2) \cap P = \gamma$.

Now the idea is to do surgery along H to cancel pairs of critical points to get rid of minima (In fact we have to change H a little bit for technical reasons, but this doesn't change the idea). This strategy works since P and $P \natural H$ are isotopic by using a deformation of H to γ .

We have to consider three and a half cases: First consider the case, where p' is also a minimum. Without loss of generality we assume that p' is lower than p . In this case we replace γ by the curve that ends a little bit before p' with respect to the flow time and alter $H|_{\partial^0 D_+^2}$ a little bit such that

it is tangential to γ at $H(\pm 1, 0)$. We alter γ a bit further such that it starts a little bit before p in the sense that we go a short time in the opposite direction of γ at p and proceed in the same fashion as before to produce an \bar{H} with the aforementioned properties. Then $P\natural\bar{H}$ is isotopic to P , $\pi_{[0,1]}|_{P\natural\bar{H}}$ is still a Morse function with the same critical points as $\pi_{[0,1]}|_P$ except for p and q (this is possible by choosing a very small extension of \bar{H} in the definition of surgery). In this case we have eliminated a minimum and start this process anew with another minimum.

The next case is the case where p' is an index 1 critical point which is lower than p . In this case we alter γ exactly as in the previous case and then $\pi_{[0,1]}|_{P\natural H}$ is still a Morse function with the same critical points as $\pi_{[0,1]}|_P$ except for p and q . In this case we have again eliminated a minimum without producing any new critical points.

The last full case, where p' is an index 1 critical point, which is higher than p , is a little bit different compared to the previous two quite similar cases and we will have to alter H and γ a little bit. In this case we let γ start not at p but at $\gamma(\epsilon)$ for some small epsilon and choose $H(\partial^0 D_+^2)$ to be tangential to γ at its starting point. Furthermore we alter γ at its endpoint by stopping a short time before arriving at p' and then going down with respect to $\pi_{[0,1]}$ avoiding p' and points that lie on the gradient flow line which ends at p' and stopping at a point which is lower than p' . As before $\pi_{[0,1]}|_{P\natural H}$ is a Morse function but the only critical point that could have vanished on $P\natural H$ is q as can be seen by considering the derivative of H . Since deleting only one critical point changes the Euler characteristic of P , we know that this procedure has to create an index 1 critical point somewhere, but the only place where this can occur is in a small neighborhood of $H(-1, 0)$ because at the other boundary point of γ we can arrange $H|_{\partial^0 D_+^2}$ to be tangential to γ . Another way to see this, is to note that $H|_{\partial^0 D_+^2}$ becomes part of a new gradient flow line of $\pi_{[0,1]}|_{P\natural H}$, which has to flow down to p after passing a neighborhood of $H(-1, 0)$, where it has to flow up. Because every point close to $H(-1, 0)$ is a point that can flow down along $\text{grad}(\pi_{[0,1]}|_{P\natural H})$ to a critical point that is not p , we conclude that the gradient flow line $H|_{\partial^0 D_+^2}$ left the open unstable manifold of p . This procedure moves the peak of γ below p' so that p' does not play a role anymore in the canceling of p .

All in all these considerations imply that we can proceed again as before but we will not run into p' again essentially reducing the number of critical points that come into question for this construction.

The last "half" case is the case, where γ does not run into a critical point after passing the index 1-critical point, but rather runs into the boundary $P \cap \partial^0 M \times \{0\}$. This case is handled completely analogous to the first two cases. \square

This lemma allows us to split every stabilization map into a composition of maps which are given by taking the union with some P_i . The third and fourth condition limit the topology of P_i to be either a pair of pants with incoming boundary consisting of one or two circles (a single index 1 critical point) and corresponding outgoing boundary or a disk with incoming boundary a circle (a single index 2 critical point). Accordingly we will write

$$\begin{aligned} \alpha_{g,\delta}: \mathcal{E}_{g,b}^\theta(M; \delta^T) &\rightarrow \mathcal{E}_{g+1,b-1}^\theta(M^1; \bar{\delta}^T) \\ \beta_{g,\delta}: \mathcal{E}_{g,b}^\theta(M; \delta^T) &\rightarrow \mathcal{E}_{g,b+1}^\theta(M^1; \bar{\delta}^T) \\ \gamma_{g,\delta}: \mathcal{E}_{g,b}^\theta(M; \delta^T) &\rightarrow \mathcal{E}_{g,b-1}^\theta(M^1; \bar{\delta}^T) \end{aligned}$$

for the corresponding stabilization maps i.e. $\alpha_{g,b}$ corresponds to a map which is given by stabilization with a pair of pants with incoming boundary two circles, $\beta_{g,b}$ is a map which is given by stabilization with a pair of pants with incoming boundary a single circle and $\gamma_{g,b}$ is given by taking the union with a disk. We will say that such a map is a **stabilization map of type α, β or γ** respectively. Even though the homotopy class of these stabilization maps depend on the

isotopy type of the boundary bordism, we will suppress P from the notation. We will call P the **corresponding boundary bordism** and whenever we need it, we will clarify that it corresponds to some fixed stabilization map. In summary, one has:

COROLLARY 3.4. *Let M be a manifold with a space of θ -structures of subplanes of TM . Then every stabilization map can be written as a composition of maps of type α , β and γ .*

3.2. Subset Bordisms

Suppose M'' and M' are closed (as a subset) codimension-0-submanifolds, possibly with corners, of M such that $M' \subset M''$. Here we do not require that the boundaries of these submanifolds lie in the boundary of M . We deliberately do not go into detail about manifolds with corners as in the application later on the situation will be more transparent. Suppose we have a 2-dimensional submanifold with θ -structure $U \subset \overline{M'' \setminus M'}$, then we have the following map:

$$\begin{aligned} - \cup U : \mathcal{E}_{g,b}^\theta(M'; \delta^T) &\rightarrow \mathcal{E}_{g',b'}^\theta(M''; \delta'^T) \\ W &\mapsto W \cup U \end{aligned}$$

Here δ^T is a boundary condition that contains (but does not necessarily equal!) $U \cap \partial M'$ and g' and b' depend on the topology of U . We call U a **subset bordism**.

REMARK 3.5. Note that if $\overline{M'' \setminus M'}$ is homeomorphic to a ball and either M'' or M' has no corners, then one can find homeomorphisms that "smooth the corners" $\Phi' : M' \rightarrow \tilde{M}'$ and $\Phi'' : M'' \rightarrow \tilde{M}''$, where \tilde{M}' does not have corners such that

$$\begin{array}{ccc} \mathcal{E}_{g,b}^\theta(M'; \delta^T) & \xrightarrow{-\cup U} & \mathcal{E}_{g',b'}^\theta(M''; \delta'^T) \\ \downarrow \Phi'_* & & \downarrow \Phi''_* \\ \mathcal{E}_{g,b}^{\tilde{\theta}}(\tilde{M}'; \tilde{\delta}^T) & \xrightarrow{\varphi} & \mathcal{E}_{g',b'}^{\tilde{\theta}}(\tilde{M}''; \tilde{\delta}'^T) \end{array}$$

where Φ'_* and Φ''_* are induced by the aforementioned homeomorphisms and such that φ is a stabilization map and the diagram commutes up to homotopy. In other words, in this case $-\cup U$ behaves up to homotopy like a stabilization map. In particular their homological properties (being an isomorphism or epimorphism etc.) agree.

Examples of subset bordisms will be the approximate augmentations, which will be introduced in Section 5.

4. Resolutions via Semi-Simplicial Spaces

The following section is based entirely on the fourth section of [CMRW17] and all its statements and proofs or references to proofs can be found there. Let Δ_{inj}^{op} denote the category, whose objects are non-empty finite ordinals and whose morphisms are injective order preserving maps. A **semi-simplicial space** is a contravariant functor $X_\bullet : \Delta_{inj}^{op} \rightarrow \mathbf{Top}$, and X_i will denote the image of $[i] := \{0, \dots, i\}$. We denote by $\partial_{j,i}$ the face maps that stem from the inclusion $\{0, \dots, i-1\} \rightarrow \{0, \dots, i\}$ that misses $j \in \{0, \dots, i\}$.

A semi-simplicial space X_\bullet together with a continuous map $\epsilon_{X_0} : X_0 \rightarrow X$ is called a **semi-simplicial space augmented over X** if

$$(8) \quad \epsilon_{X_0} \circ \partial_{0,1} = \epsilon_{X_0} \circ \partial_{1,1}$$

Using the face maps and (8) one obtains one map for every level of the semi-simplicial space, the **ith augmentation map**, $X_i \rightarrow X$, that we will denote by ϵ_{X_i} . We will usually write $\epsilon_{X_\bullet} : X_\bullet \rightarrow X$ for the collection of all augmentation maps and call this the **augmentation**. If X_\bullet and X'_\bullet denote two possibly augmented semi-simplicial spaces, we call a natural transformation between the functors a **semi-simplicial map**, if the semi-simplicial spaces are augmented we furthermore

require the maps given by the natural transformation to commute with the augmentations. In that case we will denote the map by g_\bullet , the maps on the levels of the simplicial set by g_i and the map between the topological spaces by g .

There is a geometric realization functor (compare [ERW19])

$$|\cdot| : \mathbf{Semi\text{-}simplicial\ spaces} \rightarrow \mathbf{Top}$$

and we call a semi-simplicial space augmented over X an n -**resolution** if the induced map between the geometric realization of the semi-simplicial space and X is n -connected. If the induced map is a weak equivalence, we will call the augmented semi-simplicial space a **resolution of X** .

Furthermore we call an augmented semi-simplicial space $\epsilon_{X_\bullet} : X_\bullet \rightarrow X$ an **augmented topological flag complex** if

- (i) the product map $X_i \rightarrow X_0 \times_X \dots \times_X X_0$, given by the product of the face maps, is an open embedding
- (ii) a tuple $(x_0, \dots, x_i) \in X_0 \times_X \dots \times_X X_0$ is in X_i if and only if for each $0 \leq j < j' \leq i$ we have $(x_j, x_{j'}) \in X_1$.

The following three lemmas will be crucial for the proofs of the following sections and can be found in Section 4 of [CMRW17].

LEMMA 4.1. *Let $\epsilon_{X_\bullet} : X_\bullet \rightarrow X$ be an augmented topological flag complex. Suppose that*

- (i) $\epsilon_{X_0} : X_0 \rightarrow X$ has local sections that is ϵ_{X_0} is surjective and for each $x_0 \in X_0$ such that $\epsilon_{X_0}(x_0) = x \in X$ there is a neighbourhood U_x of x and a map $s : U_x \rightarrow X_0$ such that $\epsilon_{X_0} \circ s(x') = x'$ for all $x' \in U_x$ and $s(x) = x_0$.
- (ii) given any finite collection $\{x_0^1, \dots, x_0^n\} \subset X_0$ in a single fiber of ϵ_{X_0} over some $x \in X$, there is an $x_0^\infty \in \text{Fib}_x(\epsilon_{X_0})$ such that each (x_0^1, x_0^∞) is a 1-simplex

then $|\epsilon_{X_\bullet}| : |X_\bullet| \rightarrow X$ is a weak equivalence.

LEMMA 4.2. *Let $\epsilon_{X_\bullet} : X_\bullet \rightarrow X$ denote an augmented semi-simplicial space. If each $\epsilon_{X_i} : X_i \rightarrow X$ is a fibration and $\text{Fib}_x(\epsilon_{X_i})$ denotes its fiber at $x \in X$, then the realization of the semi-simplicial space $\text{Fib}_x(\epsilon_{X_\bullet})$ is weakly homotopy equivalent to the homotopy fiber of $|\epsilon_{X_\bullet}|$ at x .*

DEFINITION 4.3. If $g : X_1 \rightarrow X_2$ denotes a continuous map, we will write $C(g)$ for the pair (M_g, X_1) , where M_g denotes the mapping cylinder of g .

The following lemma hides an occurring spectral sequence argument in the proof. The proof of the lemma can be found in Criterion 4.4. in [CMRW17] and it is an abstraction of the first part of the proof of Theorem 9.3 in [RW16].

LEMMA 4.4. *Let $g_\bullet : X_\bullet \rightarrow X'_\bullet$ be a map of augmented semi-simplicial spaces such that $X_\bullet \rightarrow X$ is an $(n-1)$ -resolution and $X'_\bullet \rightarrow X'$ is an n -resolution.*

Suppose there is a sequence of path connected based spaces (B_i, b_i) and maps $p_i : X'_i \rightarrow B_i$, and form the map

$$g_i : \text{HoFib}_{b_i}(p_i \circ g_i) \rightarrow \text{HoFib}_{b_i}(p_i)$$

induced by the composition with g_i .

$$\begin{array}{ccc} \text{HoFib}_{b_i}(p_i \circ g_i) & \xrightarrow{g_i} & \text{HoFib}_{b_i}(p_i) \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{g_i} & X'_i \longrightarrow B_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X' \end{array}$$

Suppose that there is a $c \leq n + 1$ such that

$$H_q(C(g_i)) = 0 \text{ when } q + i \leq c, \text{ except if } (q, i) = (c, 0)$$

Then the map induced in homology by the composition of the inclusion of the fiber and the augmentation map

$$H_q(C(g_0)) \longrightarrow H_q(C(f_0)) \xrightarrow{\epsilon_\bullet} H_q(C(g))$$

is an epimorphism in degrees $q \leq c$.

If in addition $H_c(C(g_0)) = 0$, then $H_q(C(g)) = 0$ in degrees $q \leq c$.

5. Resolutions of the Space of Subsurfaces

The goal of this section is to establish a resolution of $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ and to understand how this resolution behaves with respect to stabilization maps. This will enable us in Section 7 to relate spaces of subsurfaces and homological properties of maps between them to homological properties of maps between spaces of subsurfaces with smaller genera.

5.1. Constructing a Resolution for the Space of Subsurfaces

Recall the following notation introduced in Section 3:

- (i) $D_+^2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0, \|(x_1, x_2)\| \leq 1\}$
- (ii) $\partial^0 D_+^2 := \{(x_1, x_2) \in \partial D_+^2 \mid x_2 = 0\}$
- (iii) $\partial^1 D_+^2 := \{(x_1, x_2) \in \partial D_+^2 \mid \|(x_1, x_2)\| = 1\}$

From here on forth let us fix an at least 5-dimensional simply-connected manifold M together with a non-empty codimension 0 submanifold of the boundary denoted by $\partial^0 M$ and a space of tangential structures $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$. Furthermore let δ^T denote some fixed boundary condition for subsurfaces with tangential structure in M . Lastly, let ℓ denote a codimension 0 ball in $\partial^0 M$ that intersects δ in two intervals ℓ_0 and ℓ_1 , which we label and orientate according to the orientation of δ once and for all.

To define a resolution of the space of subsurfaces, we will need the following definition:

DEFINITION 5.1. Given $W \in \mathcal{E}^\theta(M; \delta^T)$, we call an embedding

$$a: (D_+^2, \partial^1 D_+^2) \rightarrow (M, W)$$

which maps $\partial^0 D_+^2$ to ℓ and $(1, 0)$ to ℓ_0 and $(-1, 0)$ to ℓ_1 an **arc in W with embedded boundary isotopy**. We will call a thickening of an arc in W with embedded boundary isotopy (a, \mathbf{a}) a **thickened arc in W with embedded boundary isotopy** if the image of \mathbf{a} restricted to the normal bundle of $\partial^0 D_+^2$ lies in ℓ . In this case we will call the image of $a|_{\partial^1 D_+^2}$ the **underlying arc of a** and the image of \mathbf{a} restricted to the normal bundle of $a(\partial^1 D_+^2)$ the **thickened underlying arc of a** and we will write \mathbf{a}_W for it. For notational reasons we will usually just write a for the thickened arc in W with embedded boundary isotopy even though the correct notation would include \mathbf{a} .

An embedding (without the presence of a subsurface) $a: D_+^2 \rightarrow M$ such that $a(\partial^1 D_+^2) \subset M \setminus \partial M$, $a(\partial^0 D_+^2) \subset \ell$ and $a((-1)^k, 0) \in \ell_k$, will be called an **arc with embedded boundary isotopy** i.e. we drop W from the previous notation. Similarly for a **thickened arc with embedded boundary isotopy**.

DEFINITION 5.2. Let $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_\bullet$ denote the following semi-simplicial space: The space of i -simplices consists of tuples $(W, (a^0, \mathbf{a}^0), \dots, (a^i, \mathbf{a}^i))$ such that:

- (i) $W \in \mathcal{E}_{g,b}^\theta(M; \delta^T)$ is a surface with tangential structure in M .

- (ii) All the (a^j, \mathbf{a}^j) are thickened arcs in W with embedded boundary isotopy.
- (iii) The images of all \mathbf{a}^j are disjoint.
- (iv) W without all underlying thickened arcs is connected, i.e. the arc system consisting of the underlying arcs is coconnected.
- (v) The starting and endpoints of the underlying arcs are ordered from 0 to i in ℓ_0 and ordered from i to 0 in ℓ_1 (Note that this makes sense as ℓ_k is oriented). In this case, we will say that the arc system consisting of the underlying arcs is **ordered**.

The j -th face map forgets the j -th embedding and we topologize the set of i -simplices as a subspace of

$$\mathcal{E}_{g,b}^\theta(M; \delta^T) \times \overline{\text{TEmb}}((D_+^2)^{i+1}, M; \ell)$$

where in $\overline{\text{TEmb}}((D_+^2)^{i+1}, M; \ell)$ the boundary condition means just that $\partial^0 D_+^2$ maps to ℓ . This semi-simplicial space possesses an augmentation map ϵ_\bullet to $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ which forgets the thickened arcs with embedded boundary isotopy.

NOTATION. Sometimes we want to distinguish between the cases, where the intersection of δ and ℓ meets a single connected component of δ or two different components. To emphasize this we will sometimes write $\mathcal{O}_{g,\delta}^\theta(M, \delta^T; \ell)_\bullet$ for the single component case and $\mathcal{O}_{g,\delta}^{\prime\prime\theta}(M, \delta^T; \ell)_\bullet$ for the different components case.

The general proof strategy of the following proposition was communicated to me by Frederico Cantero as a proposed fix to some errors in the proof of Proposition 5.3 in [CMRW17].

PROPOSITION 5.3. *Suppose that M is an at least 5-dimensional and simply-connected manifold, then $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_\bullet$ is a $(g-1)$ -resolution of $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ i.e.*

$$\epsilon_\bullet: |\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_\bullet| \rightarrow \mathcal{E}_{g,b}^\theta(M; \delta^T)$$

is $(g-1)$ -connected or in other words the homotopy fiber of this map is $(g-2)$ -connected.

PROOF. Since $\mathcal{E}_{g,b}^+(M; \delta^T)$ is $\text{Diff}_\partial^+(M)$ -locally retractile and the augmentation maps

$$\mathcal{O}_{g,b}^+(M, \delta^T; \ell)_\bullet \rightarrow \mathcal{E}_{g,b}^+(M; \delta^T)$$

are equivariant with respect to the natural action of $\text{Diff}_\partial^+(M)$ via post composition, Lemma 2.8 implies that the augmentation maps are locally trivial fibrations. Using Lemma 4.2 we get that the homotopy fiber $\text{Fib}_W(|\epsilon_\bullet|)$ is weakly homotopy equivalent to $|\text{Fib}_W(\epsilon_\bullet)|$. The i -simplices of $\text{Fib}_W(\epsilon_\bullet)$ are disjoint, ordered thickened arcs in W with embedded boundary isotopy.

Fix $x_k \in \ell_k$ and charts U_{x_k} centered at x_k . We say that an embedding $e: [0, 1] \rightarrow W$ meets x_k in a radial fashion if its image in U_{x_k} is a straight ray meeting 0.

Let $X(W; x_0, x_1)_\bullet$ denote the following semi-simplicial set: Its set of zero simplices consists of embeddings of an arc into W that meets $x_0 \in \ell_0$ and $x_1 \in \ell_1$ in a radial fashion together with a tubular neighborhood of said arc and we furthermore require these arcs to be non-isotopic to a part of the boundary. The set of i -simplices is given by collections of 0-simplices such that the complement of the arcs is connected, and such that the tubular neighborhoods are disjoint except for their intersection with U_{x_k} , lastly we require that the ordering of the arcs at x_k with respect to the angle of the corresponding ray to be order-preserving at x_0 and order-reversing at x_1 .

There is a map $g: \text{Fib}_W(\epsilon_\bullet) \rightarrow X(W, x_0, x_1)_\bullet$ given by sending an element in the fiber to the underlying thickened arcs and then adding a collar of W to W and then joining the arcs to x_k in a controlled way to produce an element in $X(W, x_0, x_1)_\bullet$.

Let $\text{Fib}_W(\epsilon_\bullet)^\delta$ denote the semi-simplicial set (not space) $\text{Fib}_W(\epsilon_\bullet)$ i.e. the set of i -simplices is given by the underlying set of $\text{Fib}_W(\epsilon_i)$. Now we want to apply Theorem A.7 of [Kup13], which says that, if

- (i) $X(W, x_0, x_1)_\bullet$ is weakly Cohen-Macaulay of dimension $(g-1)$ i.e. it is $(g-2)$ -connected and the link of every p -simplex is $(g-2-p-2)$ -connected.
- (ii) $\text{Fib}_W(\epsilon_\bullet)$ is Hausdorff and an ordered flag space (See [Kup13] for the definition of ordered flag space)
- (iii) $|g| : |\text{Fib}_W(\epsilon_\bullet)^\delta| \rightarrow X(W, x_0, x_1)_\bullet$ is simplexwise injective i.e. for every p -simplex $\{a_0, \dots, a_p\}$ with $p \geq 1$ we have $g(a_i) \neq g(a_j)$ for all $i \neq j$.
- (iv) For all finite collections $\{a_1, \dots, a_j\} \subset \text{Fib}_W(\epsilon_0)$ and $e_0 \in X(W, x_0, x_1)_\bullet$ such that $(e_0, g(a_i))$ is a 1-simplex in $X(W, x_0, x_1)_\bullet$ then there exists an element $a_0 \in \text{Fib}_W(\epsilon_0)$ such that $g(a_0) = e_0$ and such that (a_0, a_i) is a 1-simplex in $\text{Fib}_W(\epsilon_\bullet)$.

then $|\text{Fib}_W(\epsilon_\bullet)|$ is $(g-2)$ -connected.

It is an easy observation that the second and third condition hold. For the first condition, note that Theorem 2.9 in [Nar15] proves that $|X(W, x_0, x_1)_\bullet|$ is $(g-2)$ -connected, so we only have to prove that the link of a p -simplex is $(g-2-p-2)$ -connected. Given a p -simplex σ_p , the link of this simplex consists of all arc systems whose union with the arc system of σ_p is still a coconnected arc system. Consider the closed complement of the thickened arcs in σ_p denoted by $W(\sigma_p)$ (where this means, that we take the complement of the tubular neighborhoods, then take the closure in W and then resolve the singularities at x_k). Evidently every simplex in the link of σ_p consists of arcs that lie in $W(\sigma_p)$. Furthermore the points x_k correspond to $p+2$ points in $W(\sigma_p)$ and each of these points corresponds to an interval between two consecutive arcs in σ_p in the ordering of arcs connecting the points x_k . We will denote the subspace of the link of σ_p consisting of those arcs that are bigger than all arcs in σ_p in the ordering of the arcs connecting x_k by Y_\bullet . Unbending corners, Y_\bullet is isomorphic to $X(W(\sigma_p), x'_0, x'_1)_\bullet$ which is at least $(g-p-3)$ -connected by Theorem 2.9 in [Nar15] and a simple Euler characteristic computation.

We will show, using the techniques of Section 2.1 (i.e. bad and good simplices) in [HV17], that the link has the same connectivity as its subspace Y_\bullet . We call a simplex in the link of σ_p bad if all its arcs are smaller than some arc in σ_p . These certainly satisfy the conditions for a set of bad simplices in [HV17]. Furthermore given a bad simplex σ'_k , then $G_{\sigma'_k}$ is given by $X(W(\sigma_p \cup \sigma'_k), x'_0, x'_1)_\bullet$, which is $(g-p-3-k-1)$ -connected. Thus by (b) of Corollary 2.2 in [HV17] the link is at least $(g-2-p-1)$ -connected.

For the fourth condition, note that since $(e_0, g(a_i))$ is a 1-simplex, in particular e_0 and the underlying arc of $g(a_i)$ are coconnected and ordered correctly, we can find a thickened embedded arc \bar{a}_0 in W such that $g(\bar{a}_0) = e_0$ (this expression makes sense, since g only takes the underlying thickened arcs of the elements in $\text{Fib}_W(\epsilon_\bullet)$) and such that the pairs consisting of \bar{a}_0 and the thickened underlying arcs of a_i form an ordered and coconnected arc system. Using that M is simply-connected and the main result of [Hae61] we can find an extension of \bar{a}_0 to an arc in M with embedded boundary isotopy. Since the dimension of M is at least 5 we conclude that a small perturbation of this embedding yields an embedding disjoint from all the other a_i by transversality. Adding a sufficiently small tubular neighborhood finishes the proof of the fourth assumption and therefore implies the desired claim about the connectivity of $\text{Fib}_W(\epsilon_\bullet)$.

To finish the proof, just observe that the following diagram is a pullback diagram, where the lower map denotes the map forgetting the tangential structure.

$$\begin{array}{ccc}
 |\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_\bullet| & \longrightarrow & |\mathcal{O}_{g,b}^+(M, \delta^T; \ell)_\bullet| \\
 \downarrow & & \downarrow \\
 \mathcal{E}_{g,b}^\theta(M; \delta^T) & \longrightarrow & \mathcal{E}_{g,b}^+(M; \delta)
 \end{array}$$

□

We will need to establish some more notation.

DEFINITION 5.4. A **thickened strip with embedded boundary isotopy** consists of the following data:

- (i) A thickened arc with embedded boundary isotopy $a: D_+^2 \rightarrow M$
- (ii) A subbundle L of $N(a(\partial^1 D_+^2), M)$

such that the restriction $\mathbf{a}|_{L|_{(1,0)}}$ lies in ℓ_0 and $\mathbf{a}|_{L|_{(-1,0)}}$ lies in ℓ_1 . We will denote the restriction of \mathbf{a} to L by \mathbf{a}_L . If we add a tangential structure for the image of \mathbf{a}_L , which agrees with the tangential structure specified by δ^T , wherever this makes sense, we will call this a **thickened strip with tangential structure and embedded boundary isotopy**. We will call the image of \mathbf{a}_L the **strip of** (a, L) .

NOTATION. Similar as before we will usually suppress \mathbf{a} and \mathbf{a}_L from this notation and only write a for the tuple (a, \mathbf{a}, L) .

DEFINITION 5.5. Let $A^\theta(M; \delta^T, \ell)_i$ denote the set of tuples (a^0, \dots, a^i) such that all the a^k are thickened strips with tangential structures and embedded boundary isotopies such that all the images of \mathbf{a}^k are disjoint and the starting and endpoints of the underlying arcs lie in ℓ_0 and ℓ_1 and they are ordered from 0 to i in ℓ_0 and ordered from i to 0 in ℓ_1 .

We topologize this as a subset of $\overline{\text{TEmb}}(D_+^2 \times [i], M; \ell) \times \text{Emb}^\theta(I \times I \times [i], M)$, where the thickened arcs with embedded boundary isotopies correspond to elements in the first factor and the strips and their tangential structure to the second factor.

There is a continuous map $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i \rightarrow A^\theta(M; \delta^T, \ell)_i$ which forgets the surface but keeps the tubular neighbourhood in the surface and the tangential structure on it.

LEMMA 5.6. *The restriction map $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i \rightarrow A^\theta(M; \delta^T, \ell)_i$ is a Serre fibration and the fiber over a point $a = (a_0, \dots, a_i)$ can be identified with $\mathcal{E}_{g-i-1, b+i+1}^\theta(M(a); \delta(a)^T)$ if we are considering $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i$. If we are considering $\mathcal{O}_{g,b}^{\prime\theta}(M, \delta^T; \ell)_i$ instead, then the fiber is given by $\mathcal{E}_{g-i, b+i-1}^\theta(M(a); \delta(a)^T)$.*

Here we define $M(a) = \overline{M \setminus \cup_j \mathbf{a}_j}$ and the boundary condition $\delta(a)^T$ is given by the boundary of the image of the strips together with the intersection of the boundary condition δ and the submanifold $M(a)$.

REMARK 5.7. Note that $M(a)$ is a manifold with corners and $\delta(a)^T$ is a boundary condition for a manifold with corners, but we can circumvent this by fixing a homeomorphism from $M(a)$ to M that is a diffeomorphism at all points except the corner points. Heuristically such a homeomorphism pushes the dent which came from removing a to the outside.

Note that the boundary condition $\delta(a)^T$ maps the corners of $\Sigma_{g,\delta}(a)$, i.e. the surface with the corresponding arcs removed, to the corners of $M(a)$. Therefore postcomposing a subsurface with the aforementioned homeomorphism yields an embedded subsurface without corners in a manifold without corners. Using this we can treat the occurring spaces of subsurfaces with corners of manifolds with corners up to homeomorphism like ordinary spaces of subsurfaces.

PROOF. We will construct the following diagram and prove that the maps labelled fib are fibrations to prove the claim:

$$(9) \quad \begin{array}{ccc} \text{Emb}^\theta(\Sigma_{g,b}, M; \delta^T) & \xrightarrow{\text{fib}} & \text{Emb}^\theta(\sigma \times [-1, 1], M; \delta') \\ \text{fib} \uparrow & & \text{fib} \uparrow \\ \mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i & \xrightarrow{\text{fib}} & A^\theta(M; \delta^T, \ell)_i \\ \downarrow & \nearrow & \\ \mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i & & \end{array}$$

Fix a set σ of $i + 1$ disjoint arcs in $\Sigma_{g,b}$ starting at $d^{-1}(\ell_0)$ and ending at $d^{-1}(\ell_1)$, and fulfilling the ordering condition of Definition 5.2, and such that their complement is connected. Here d is a parametrization of δ . Then, the top map is the restriction map, which was proven to be a fibration in Lemma 2.18. The right vertical map forgets the boundary isotopy. Note that this is a fibration as the following diagram is a pullback and the right hand map is a fibration by Lemma 2.8:

$$\begin{array}{ccc} A^\theta(M; \delta^T, \ell)_i & \longrightarrow & A^\theta(M; \delta^T, \ell)_i \\ \downarrow & & \downarrow \\ \text{Emb}^\theta([0, 1] \times [0, 1] \times X_i, M; \delta') & \longrightarrow & \text{Emb}^+([0, 1] \times [0, 1] \times X_i, M; \delta') \end{array}$$

The definition of the space $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i$ is as follows: We define $\overline{\mathcal{O}}_{g,b}^\theta(M, \delta^T; \ell)_i$ just like the arc resolution, but without the boundary isotopies (i.e. subsurfaces with arc systems). We define $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i$ via the following diagram in which the space in the top left corner of both squares are defined as pullbacks:

$$\begin{array}{ccccc} \mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i & \longrightarrow & \mathcal{O}_{g,b}^+(M, \delta^T; \ell)_i & \longrightarrow & \text{Emb}(\Sigma_{g,b}, M; \delta^T) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i & \longrightarrow & \mathcal{O}_{g,b}^+(M, \delta^T; \ell)_i & \longrightarrow & \overline{\mathcal{O}}_{g,b}^+(M, \delta^T; \ell)_i \end{array}$$

Note that $\text{Diff}_\sigma^+(\Sigma_{g,b})$ the group of diffeomorphisms that fix the boundary and σ acts on the space of embeddings in the diagram and the left most map is the quotient map associated to this action. It is a fibration by Lemma 2.8 and the fact that $\overline{\mathcal{O}}_{g,b}^+(M, \delta^T; \ell)_i$ is $\text{Diff}_\partial^+(M)$ -locally retractile. Since the left vertical map is a fibration, all vertical maps in the diagram are fibrations. Furthermore the top square in (9) is also easily seen to be a pullback diagram. Hence all maps in that square are fibrations. Since all maps in (9), except for the diagonal one, are fibrations Lemma 2.13 implies that the diagonal map is a fibration as well.

Lastly we have to determine the fiber over a . Note that removing $i + 1$ strips from a surface increases its Euler characteristic by $i + 1$, since it corresponds to taking out $i + 1$ one-cells. To calculate the genus and the number of boundary components of the fiber it is enough to specify its number of boundary components. We will take out the $i + 1$ strips consecutively and there are two cases we have to distinguish: Either all boundary points lie in the same connected component of the boundary of the surface (Case 1) or they all lie in different connected components (Case 2). But note that removing the first arc in Case 2, reduces the calculation to Case 1 for the rest of the arcs. But in the first case we see that taking out an arc increases the number of boundary components by one and the requirement for the ordering of the arcs ensures that the consecutive arcs all connect the same connected component of the boundary.

All in all we conclude that in Case 2 the number of boundary components changes to $b + i - 1$ and in the first case it changes to $b + i + 1$. Using the Euler-characteristic formula for surfaces, we can compute the corresponding genus to get the above specifications of the fiber. \square

DEFINITION 5.8. For some $(a^0, \dots, a^i) \in A^\theta(M; \delta^T, \ell)_i$ we call the composition of the inclusion of the fiber in Lemma 5.6 with the augmentation of $\epsilon_i: \mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i \rightarrow \mathcal{E}_{g,b}^\theta(M; \delta^T)$ the **i -approximate augmentation** of the resolution $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i$ over the i -simplex (a^0, \dots, a^i) .

5.2. Stabilizing the arc resolution

We want to extend the maps $\alpha_{g,b}$ and $\beta_{g,b}$ to the aforementioned resolution as shown below:

$$\begin{array}{ccc} \mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i & \dashrightarrow & \mathcal{O}_{g+1,b-1}^\theta(M^1, \bar{\delta}^T; \bar{\ell})_i \\ \downarrow \epsilon_i & & \downarrow \epsilon_i \\ \mathcal{E}_{g,b}^\theta(M; \delta^T) & \xrightarrow{\alpha_{g,b}} & \mathcal{E}_{g+1,b-1}^\theta(M^1; \bar{\delta}^T) \end{array}$$

and similarly for $\beta_{g,b}$. Let P denote a subsurface in $\partial^0 M \times I$ used in the definition of a stabilization map. We define $\bar{\ell} = \ell \times \{1\}$ and $\bar{\ell}_i = \ell_i \times \{1\}$ for $i \in \{0, 1\}$ and we assume without loss of generality that $P \cap (\ell \times I) = (\ell \cap \delta) \times I$, in particular $\bar{\ell} \cap \bar{\delta} = (\ell \cap \delta) \times \{1\}$, where $\bar{\delta}^T$ denotes the boundary condition of the image of the stabilization map. (Here we isotope P not relative the boundary to get that $\ell \times I$ is contained in P).

Define \tilde{a} for $a \in A^\theta(M; \delta^T, \ell)_0$ as $a(\partial^0 D_+^2) \times I$. This allows us to extend the stabilization maps to $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i$ as follows

$$(W, (a^0, \dots, a^i)) \mapsto (W \cup P (a^0 \cup \tilde{a}^0, \dots, a^i \cup \tilde{a}^i))$$

and we write \bar{a} for $a \cup \tilde{a}$. This yields the dashed lifts. Since they commute with the face maps and with the augmentation maps we get a map of augmented semi-simplicial spaces

$$\alpha_{g,b}^\bullet: \mathcal{O}_{g,b}^{\prime\theta}(M, \delta^T; \ell)_\bullet \rightarrow \mathcal{O}_{g+1,b-1}^{\prime\theta}(M^1, \bar{\delta}^T; \bar{\ell})_\bullet$$

which is augmented over $\alpha_{g,b}$. If $\bar{\ell}$ intersects two different components of the new boundary condition, then we obtain analogously

$$\beta_{g,b}^\bullet: \mathcal{O}_{g,b}^{\prime\theta}(M, \delta^T; \ell)_\bullet \rightarrow \mathcal{O}_{g,b+1}^{\prime\theta}(M^1, \bar{\delta}^T; \bar{\ell})_\bullet$$

augmented over $\beta_{g,b}$.

All of these considerations imply the following corollary:

COROLLARY 5.9. *The semi-simplicial pair $C(\alpha_{g,b}^\bullet)$ together with the natural augmentation map to $C(\alpha_{g,b})$ is a g -resolution i.e. the map between pairs is g -connected.*

The semi-simplicial pair $C(\beta_{g,b}^\bullet)$ together with the natural augmentation map to $C(\beta_{g,b})$ is a $(g-1)$ -resolution i.e. the map between pairs is $(g-1)$ -connected.

There is a commutative square

$$\begin{array}{ccc} \mathcal{O}_{g,b}^{\prime\theta}(M, \delta^T; \ell)_i & \xrightarrow{\alpha_{g,b}^\bullet} & \mathcal{O}_{g+1,b-1}^{\prime\theta}(M^1, \bar{\delta}^T; \bar{\ell})_i \\ \downarrow & & \downarrow \\ A^\theta(M; \delta^T, \ell)_i & \xrightarrow{a \mapsto \bar{a}} & A^\theta(M^1; \bar{\delta}^T, \bar{\ell})_i \end{array}$$

using collars one sees that the lower map is a homotopy equivalence. By commutativity of the above square we get a map between the fibers over the points a and \bar{a}

$$\mathcal{E}_{g-i,b+i+1}^\theta(M(a); \delta(a)^T) \rightarrow \mathcal{E}_{g-i,b+i}^\theta(M^1(\bar{a}); \bar{\delta}^T(\bar{a}))$$

If P denotes the bordism defining the map $\alpha_{g,b}$ in question, then this map is given by taking the union with $P(a) := \overline{P} \setminus \cup_i a_i$. These are again manifolds with corners, but this is quite easily fixed using similar techniques as in Remark 3.5 and Remark 5.7.

This map between the fibers is quite easily seen to be of the form $\beta_{g-i,b+i-1}$. As the map $A^\theta(M; \delta^T, \ell)_i \rightarrow A^\theta(M^1; \bar{\delta}^T, \bar{\ell})_i$ is a homotopy equivalence, we conclude that the space of subsurfaces $\mathcal{E}_{g-i,b+i-1}^\theta(M(a); \delta^T(a))$ is homotopy equivalent to the homotopy fiber of the composition of the map $\mathcal{O}_{g,b}^\theta(M, \delta^T; \ell)_i \rightarrow A^\theta(M; \delta^T, \ell)_i$ with the aforementioned homotopy equivalence. Moreover we have shown that the map between the fibers is a stabilization map of the form $\beta_{g-i,b+i-1}$. This is expressed in the following diagram:

$$\begin{array}{ccc}
\mathcal{E}_{g-i,b+i-1}^\theta(M(a); \delta^T(a)) & \xrightarrow{\beta_{g-i,b+i-1}} & \mathcal{E}_{g-i,b+i}^\theta(M^1(\bar{a}); \bar{\delta}^T(\bar{a})) \\
\downarrow & & \downarrow \\
\mathcal{O}_{g,b}^{\prime\theta}(M, \delta^T; \ell)_i & \xrightarrow{\alpha_{g,b}^i} & \mathcal{O}_{g+1,b-1}^\theta(M^1, \bar{\delta}^T; \bar{\ell})_i \\
\swarrow & & \searrow \\
& A^\theta(M^1; \bar{\delta}^T, \bar{\ell})_i &
\end{array}$$

Repeating the same procedure for maps of the form $\beta_{g,b}$, we obtain the following corollary:

COROLLARY 5.10. *The induced map between the homotopy fibers of*

$$(\alpha_{g,b}^i) \rightarrow A^\theta(M^1; \bar{\delta}^T, \bar{\ell})_i$$

is given by $\beta_{g-i,b+i-1}$ and analogously the induced map between the homotopy fibers of

$$(\beta_{g,b}^i) \rightarrow A^\theta(M^1; \bar{\delta}^T, \bar{\ell})_i$$

is given by $\alpha_{g-i-1,b+i+1}$.

DEFINITION 5.11. We call the map $C(\beta_{g-i,b+i-1}) \rightarrow C(\alpha_{g,b})$ given by the composition of the inclusion of the fiber in the previous corollary into $C(\alpha_{g,b}^i)$ with the projection onto $C(\alpha_{g,b})$ the **relative i -approximate augmentation** of the resolution $(\alpha_{g,b}^\bullet)$. Analogously we call $C(\alpha_{g-i-1,b+i+1}) \rightarrow C(\beta_{g,b})$ defined as in the previous case the **relative i -approximate augmentation** of the resolution $(\beta_{g,b}^\bullet)$. If we want to emphasize the point $a \in A^\theta(M; \delta^T, \ell)_i$ over which the fiber lives, then we call it the **relative i -approximate augmentation over a** .

6. k -Triviality and π_0 -stabilization

The following notions are inspired by the corresponding definitions in Section 6 of [RW16] and will serve as the main requirements of a tangential structure in order to fulfil homological stability. The following definition is an adaptation of Definition 6.1 in [RW16].

DEFINITION 6.1. We say a subset bordism $U \subset \overline{M \setminus M'}$ absorbs a boundary bordism $P \subset \partial^0 M$ if $U \cap \partial^0 M \times [0, 1] \subset P$ and there exists a $Z \subset (M \setminus M')^1 \cup \partial^0 M \times [1, 2]$, where $\partial^0 M' = \partial^0 M \cap M'$ (note that M^1 can be embedded into M^2 and Z is a subset of the difference of these two) such that

$$\begin{array}{ccc}
\mathcal{E}^\theta(M'; \delta^T(U)) & \xrightarrow{P(U)} & \mathcal{E}^\theta(M^1; \delta^T(U)) \\
\downarrow U & \nearrow Z & \downarrow U^1 \\
\mathcal{E}^\theta(M; \delta^T) & \xrightarrow{P} & \mathcal{E}^\theta(M^1; \delta^T) \\
\downarrow \simeq & \nearrow & \downarrow \simeq \\
\mathcal{E}^\theta(M^2; \delta^T) & \xrightarrow{P+1} & \mathcal{E}^\theta(M^3; \delta^T)
\end{array}$$

the following diagram commutes up to homotopy, where the lower maps labeled by \simeq are given gluing on a cylinder.

REMARK 6.2. While this seems hard to check it is actually just a question about the path components of the occurring bordism spaces that define the spaces. This will be further emphasized in the proof of Proposition 6.9.

The following definition is the analogue of the first part of Definition 6.2 in [RW16] of test pairs of height k :

DEFINITION 6.3. Given a boundary condition δ , a sequence of pairs of disjoint embedded intervals $\ell_0^i \subset \delta$ and $\ell_1^i \subset \delta$ for $1 \leq i \leq k$, one can glue 0-handles along ℓ_0^i and ℓ_1^i to δ . Doing this for the first $i - 1$ pairs of intervals yields an (abstract) 1-manifold with $k - i$ embedded pairs of intervals which we denote by δ^i .

We call such a sequence together with two more intervals $\bar{\ell}_0$ and $\bar{\ell}_1$ **genus maximizing of length k for maps of type α and boundary condition δ** if:

- (i) All occurring embedded intervals are disjoint and $\ell_0^i \subset \bar{\ell}_0$ and $\ell_1^i \subset \bar{\ell}_1$
- (ii) ℓ_0^1 and ℓ_1^1 lie in different connected components of δ
- (iii) ℓ_0^i and ℓ_1^i lie in the same connected component of δ^i if i is even and in different components if i is odd.

Such a sequence is **genus maximizing of length k for maps of type β and boundary condition δ** if:

- (i) All occurring embedded intervals are disjoint and $\ell_0^i \subset \bar{\ell}_0$ and $\ell_1^i \subset \bar{\ell}_1$
- (ii) ℓ_0^1 and ℓ_1^1 lie in the same connected components of δ
- (iii) ℓ_0^i and ℓ_1^i lie in the same connected component of δ^i if i is odd and in different components if i is even.

Furthermore given a genus maximizing sequence of length k , we call a sequence of codimension 0 balls ℓ^i all contained in a ball $\bar{\ell}$ such that $\ell^i \cap \delta = \ell_0^i \cup \ell_1^i$ and $\bar{\ell} \cap \delta = \bar{\ell}_0 \cup \bar{\ell}_1$ together with a sequence of $a^i \in A^\theta(M(a^1, \dots, a^{i-1}); (\delta^{i-1})^T, \ell^i)_0$ for $1 \leq i \leq k$ **compatible with (ℓ_0^i, ℓ_1^i)** . We will usually suppress ℓ from this notation. In this case δ^i is naturally a boundary condition and not just an abstract manifold.

REMARK 6.4. Note that it is always possible to chose $\bar{\ell}$ and ℓ^i which intersect δ as required. If ∂M is simply-connected, then this ball is unique up to isotopy.

We will be interested in tangential structures for which certain sequences of approximate augmentations of the resolutions introduced in Section 5 absorb stabilization maps of type α and β .

DEFINITION 6.5. Given a genus maximizing sequence of length k with compatible a^i , we call the composition of all the 0-th approximate augmentations corresponding to the strips $a^i \in A^\theta(M(\bigcup_{j \in \{1, \dots, i\}} a^j); (\delta^i)^T, \ell^i)_0$ the associated **augmentation composition of length k** and denote it by

$$\sigma_{g,b}^k: \mathcal{E}_{g',b'}^\theta \left(M \left(\bigcup_{j \in \{1, \dots, k\}} a^j \right); (\delta^k)^T \right) \rightarrow \mathcal{E}_{g,b}^\theta(M; \delta^T)$$

where

$$g' = \begin{cases} g - l & \text{if the sequence is genus maximizing for maps of type } \alpha \text{ and } k \text{ is } 2l+1 \\ g - l & \text{if the sequence is genus maximizing for maps of type } \alpha \text{ and } k \text{ is } 2l \\ g - (l + 1) & \text{if the sequence is genus maximizing for maps of type } \beta \text{ and } k \text{ is } 2l+1 \\ g - l & \text{if the sequence is genus maximizing for maps of type } \beta \text{ and } k \text{ is } 2l \end{cases}$$

$$b' = \begin{cases} b - 1 & \text{if the sequence is genus maximizing for maps of type } \alpha \text{ and } k \text{ is } 2l+1 \\ b & \text{if the sequence is genus maximizing for maps of type } \alpha \text{ and } k \text{ is } 2l \\ b + l & \text{if the sequence is genus maximizing for maps of type } \beta \text{ and } k \text{ is } 2l+1 \\ b & \text{if the sequence is genus maximizing for maps of type } \beta \text{ and } k \text{ is } 2l \end{cases}$$

Note that the augmentation compositions are subset bordisms.

If we have a stabilization map defined via P , where the defining boundary bordism P fulfils $P \cap ((\bar{\ell}_0 \cup \bar{\ell}_1) \times [0, 1]) = (\bar{\ell}_0 \cup \bar{\ell}_1) \times [0, 1]$, in this case we call $-\cup P$ **compatible with the genus maximizing sequence**. Then $-\cup P$ lifts to a map

$$-\cup P \left(\bigcup_{j \in \{1, \dots, k\}} a^j \right) : \mathcal{E}_{g', b'}^\theta \left(M \left(\bigcup_{j \in \{1, \dots, k\}} a^j \right); (\delta^k)^T \right) \rightarrow \mathcal{E}_{g'', b''}^\theta \left(M \left(\bigcup_{j \in \{1, \dots, k\}} \bar{a}^j \right)^1; (\delta^k)^T \right)$$

where g' and b' are as before, and if P is of type α or β and the sequence is genus maximizing for maps of same type, then $-\cup P \left(\bigcup_{j \in \{1, \dots, k\}} a^j \right)$ is a map of the same type as before if k is even, and of the other type if k is odd. We call this map a **relative augmentation compositions of length k** and denote it by $\mathfrak{a}_{g, b}^k$ if $-\cup P$ is of type alpha, and $\mathfrak{b}_{g, b}^k$ in the other case.

REMARK 6.6. Note that by the dimension restriction on M , and ∂M being simply-connected, we can find an isotopy that changes a P into one that is compatible with the genus maximizing sequence.

Again this definition is inspired by the second part of Definition 6.2 in [RW16].

DEFINITION 6.7. We call a space of θ -structures k -**trivial** if for every genus maximizing sequence of length k and a compatible sequence a^i , and every compatible stabilization map of type α or β , the boundary bordism P is absorbed by the subset bordism defining the relative augmentation compositions of length k associated to the genus maximizing sequence.

In Section 9 we will prove that it suffices to have stabilisation of connected components in order to obtain k -triviality for some k . To state this we need the following definition, which is the analogue of Definition 6.3 in [RW16]:

DEFINITION 6.8. We say a tangential structure $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ π_0 -**stabilizes at genus g** if all stabilization maps

$$\alpha_{g', b}: \mathcal{E}_{g', b}^\theta(M; \delta^T) \rightarrow \mathcal{E}_{g'+1, b-1}^\theta(M^1; \bar{\delta}^T)$$

$$\beta_{g', b}: \mathcal{E}_{g', b}^\theta(M; \delta^T) \rightarrow \mathcal{E}_{g', b+1}^\theta(M^1; \bar{\delta}^T)$$

induce bijections on π_0 for all $g' \geq g$ and they induce a surjection on π_0 for all $g' \geq g - 1$.

We say a tangential structure $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ π_0 -**stabilizes at the boundary at genus g** if all stabilization maps:

$$\alpha_{g', b}: \mathcal{E}_{g', b}^\theta(\partial^0 M \times [0, 1]; \delta^T) \rightarrow \mathcal{E}_{g'+1, b-1}^\theta(\partial^1 M \times [0, 2]; \bar{\delta}^T)$$

$$\beta_{g', b}: \mathcal{E}_{g', b}^\theta(\partial^0 M \times [0, 1]; \delta^T) \rightarrow \mathcal{E}_{g', b+1}^\theta(\partial^0 M \times [0, 2]; \bar{\delta}^T)$$

induce bijections on π_0 for all $g' \geq g$ and they induce a surjection on π_0 for all $g' \geq g - 1$.

The rather abstract property of k -triviality is already implied by π_0 -stabilization. This is encapsulated in the following proposition, which will be proven in Section 9 and is an adaptation of Proposition 6.5 in [RW16].

PROPOSITION 6.9. *Suppose M is an at least 5-dimensional manifold which is simply connected and $\partial^0 M$ is a simply-connected codimension 0 submanifold of ∂M . Suppose further that $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ π_0 -stabilizes at the boundary at g , then it is $2g + 1$ -trivial.*

7. Proving Homological Stability

In order to prove homological stability we will need the stability bounds. The following definition stems from Definition 6.9 in [RW16].

DEFINITION 7.1. Given two natural numbers $k \geq 1$ and $m \geq 0$, we define the following recursive functions $A, B, SA, SB: \mathbb{Z} \rightarrow \mathbb{Z}$ as follows:

(i)

$$A(g) = B(g) = SA(g) = SB(g) = -1$$

for $g \leq m - 2$ and

$$A(m - 1) = B(m) = SA(m) = SB(m) = 0$$

(ii) We define

$$SA(g) = \min \begin{cases} A(g - 1) + 1 \\ B(g - 1) + 1 \\ SA(g - 1) + 1 \\ SB(g - 1) + 1 \\ 0 \text{ if } g \leq 0 \end{cases} \quad SB(g) = \begin{cases} A(g - 2) + 1 \\ B(g - 1) + 1 \\ SA(g - 2) + 1 \\ SB(g - 1) + 1 \\ 0 \text{ if } g \leq 0 \end{cases}$$

(iii) If $k = 1$, we define:

$$A(g) = \min \begin{cases} A(g - 1) + 1 \\ SA(g) \end{cases} \quad B(g) = \begin{cases} B(g - 1) + 1 \\ SA(g) \end{cases}$$

(iv) If $k = 2l$ with $l > 0$, we define:

$$A(g) = \min \begin{cases} A(g - 1) + 1 \\ B(g - l) + 1 \\ \max\{SA(g + 1 - l), 0\} \\ \max\{SB(g + 1 - l), 0\} \end{cases} \quad B(g) = \begin{cases} A(g - l - 1) + 1 \\ B(g - 1) + 1 \\ \max\{SA(g - l), 0\} \\ \max\{SB(g + 1 - l), 0\} \end{cases}$$

(v) If $k = 2l + 1$ with $l > 0$, we define:

$$A(g) = \min \begin{cases} A(g - l - 1) + 1 \\ \max\{SA(g - l), 0\} \\ \max\{SB(g + 1 - l), 0\} \end{cases} \quad B(g) = \begin{cases} B(g - l - 1) + 1 \\ \max\{SA(g - l), 0\} \\ \max\{SB(g - l), 0\} \end{cases}$$

REMARK 7.2. If $k = 1$ and $m = 0$, then $A(g) = SA(g) = \lfloor \frac{1}{3}(2g + 1) \rfloor$ and $B(g) = SB(g) = \lfloor \frac{2}{3}g \rfloor$ gives the aforementioned functions.

The goal of the rest of this section is to prove the most of the first two parts of the following theorem. The proof will be an adaptation of the proof of the main theorem of [CMRW17] to include k -triviality:

THEOREM 7.3. *Suppose M is an at least 5-dimensional, simply-connected manifold with non-empty boundary and $\partial^0 M$ is a codimension 0 submanifold of ∂M . Suppose further that $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ is a k -trivial space of θ -structures on subplanes of TM which π_0 -stabilizes at m , then*

- the homology of $C(\alpha_{g,b})$ vanishes in degrees $* \leq A(g)$
- the homology of $C(\beta_{g,b})$ vanishes in degrees $* \leq B(g)$. If one of the newly created boundary components is contractible in $\partial^0 M$ and the tangential structure on that boundary component vanishes in $\pi_1(B_2(M))$, then $\beta_{g,b}$ is a monomorphism in all degrees
- the homology of $C(\gamma_{g,b})$ vanishes in degrees $* \leq B(g)$. If $b > 1$, then $\gamma_{g,b}$ induces an epimorphism in all degrees

where A and B are defined as in Definition 7.1 and $C(-)$ denotes the mapping pair as in Definition 4.3.

The proof will proceed via induction. More precisely we will use induction for the following two statements:

- A_h : For all $g \leq h$ and for all $\alpha_{g,b}$, the reduced homology of $C(\alpha_{g,b})$ vanishes in degrees $* \leq A(g)$
- B_h : For all $g \leq h$ and for all $\beta_{g,b}$, the reduced homology of $C(\beta_{g,b})$ vanishes in degrees $* \leq B(g)$

Furthermore we will also need the following two auxiliary statements:

- SA_h : For all $g \leq h$ and all strips with embedded boundary isotopy, the relative 0-approximate augmentation induces a surjection $H_*(C(\beta_{g,b-1})) \rightarrow H_*(C(\alpha_{g,b}))$ in homology in degrees $* \leq SA(g)$.
- SB_h : For all $g \leq h$ and all strip with embedded boundary isotopy, the relative 0-approximate augmentation induces a surjection $H_*(C(\alpha_{g-1,b+1})) \rightarrow H_*(C(\beta_{g,b}))$ in degrees $* \leq SB(g)$.

Note that the start of the induction is provided by the assumption that the tangential structure π_0 -stabilizes at genus m .

We will also denote the **mapping cone** of a map g by $\bar{C}(g)$, if g is an inclusion $A \rightarrow X$ we will write $\bar{C}(X, A)$ instead. The following technical lemma will be needed in the proof. It is Lemma 7.5 in [CMRW17].

LEMMA 7.4. *Suppose we have a map of pairs*

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow g & & \downarrow f \\ A' & \xrightarrow{j} & X' \end{array}$$

such that there exists a map $t: X \rightarrow A'$ making the bottom triangle commute up to a homotopy H . Then the induced maps between mapping cones $(f, g): \bar{C}(X, A) \rightarrow \bar{C}(X', A')$ factors as $\bar{C}(X, A) \xrightarrow{p} \bar{C}A \cup_i \bar{C}X \xrightarrow{h} \bar{C}(X', A')$, where p comes from the Puppe sequence of the pair (X, A) .

Additionaly if there is a homotopy G , which makes the bottom triangle commute, then the composite $\bar{C}A \cup_i \bar{C}X \xrightarrow{h} \bar{C}(X', A') \xrightarrow{p'} \bar{C}A' \cup_j \bar{C}X'$ is nullhomotopic.

We have the following proposition of convoluted implications finishing the proof of Theorem 7.3:

PROPOSITION 7.5. *With the assumptions of Theorem 7.3 we have the following implications:*

- (i) B_h implies SA_h .
 - (ii) A_{h-1} implies SB_h
 - (iii) If $k = 1$, then SA_h and A_{h-1} imply A_h
- If $k > 1$ then SA_h, SB_h and $\begin{cases} B_{h-l} & \text{if } k = 2l \\ A_{h-l-1} & \text{if } k = 2l + 1 \end{cases}$ imply A_h

(iv) If $k = 1$, then SB_h and B_{h-1} imply B_h

$$\text{If } k > 1 \text{ then } SB_h, SA_{h-1} \text{ and } \begin{cases} A_{h-l-1} & \text{if } k = 2l \\ B_{h-l-1} & \text{if } k = 2l + 1 \end{cases} \text{ imply } B_h$$

PROOF OF (I) AND (II). We will prove (i), the other part is completely analogous. By corollary 5.10 we get the following diagram:

$$\begin{array}{ccccc} \mathcal{E}_{h-j, b+j-1}^\theta \left(M(a); \delta(a)^T \right) & \xrightarrow{\beta_{h-j, b+j-1}} & \mathcal{E}_{h-j, b+j}^\theta \left(M(\bar{a})^1; \delta'(\bar{a})^T \right) & & \\ \downarrow & & \downarrow & & \\ \mathcal{O}_{h,b}^{\prime\prime\theta} \left(M, \delta^T; \ell \right)_j & \xrightarrow{(\alpha_{h,b})_\bullet} & \mathcal{O}_{h+1, b-1}^{\prime\theta} \left(M^1, \delta'^T; \bar{\ell} \right)_j & \longrightarrow & A^\theta \left(M; \delta^T, \ell \right)_j \\ \downarrow & & \downarrow & & \\ \mathcal{E}_{h,b}^\theta \left(M; \delta^T \right) & \xrightarrow{\alpha_{h,b}} & \mathcal{E}_{h+1, b-1}^\theta \left(M^1; \delta'^T \right) & & \end{array}$$

Furthermore the augmentation $\mathcal{O}_{h,b}^{\prime\prime\theta} \left(M, \delta^T; \ell \right)_\bullet \rightarrow \mathcal{E}_{h,b}^\theta \left(M; \delta^T \right)$ is $(h-2)$ -connected by Proposition 5.3. Since $M(a)$ is homeomorphic to M , B_h implies that $H_q(\beta_{h-j, b+j-1}) = 0$ for $q \leq B(h-j)$. This gives us all the ingredients to apply Lemma 4.4 with $c = SA(h)$, using that $SA(h) \geq B(g-i) + i$. The lemma implies that the induced map $H_q(C(\beta_{h,b-1})) \rightarrow H_q(C(\alpha_{h,b}))$ is an epimorphism for $q \leq SA(h)$, which is exactly the statement of SA_h . \square

PROOF OF (III) AND (IV). We will prove (iii) for $k = 2l$ here. The proof for odd k is completely analogous and we will indicate the necessary minor changes for the proof of (iv).

Pick a genus maximizing sequence $(\ell_j^i)_{1, \dots, k+1}$ of length $k+1$ for maps of type α with a compatible sequence $(a^i)_{1, \dots, k+1}$ of arcs with embedded boundary isotopy such that $(a^i)_{2, \dots, k, 1}$ is also a genus maximizing sequence of length $k+1$. To get such a sequence, choose for example $k+1$ disjoint intervals in the two components of δ and then label them positively oriented from 1 to $k+1$, this gives the desired genus maximizing sequence (For (iv) choose $2k+2$ disjoint intervals in the single component of δ and label them as follows: $\ell_0^{k+1}, \dots, \ell_0^1, \ell_1^{k+1}, \dots, \ell_1^1$; in this case ℓ_0^1 and ℓ_1^1 switch if one moves a^1 to the end of the sequence).

Now form the following diagram, where we denote $a^1 \cup \dots \cup a^{k+1}$ by d , $\bar{a}^1 \cup \dots \cup \bar{a}^{k+1}$ by \bar{d} , $a^2 \cup \dots \cup a^{k+1}$ by c , and \bar{c} analogously:

$$\begin{array}{ccccccc} \mathcal{E}_{h-l, b-1}^\theta \left(M(d); \delta(d)^T \right) & \longrightarrow & \mathcal{E}_{h-l, b}^\theta \left(M^1(\bar{d}); \delta'(\bar{d})^T \right) & \longrightarrow & \bar{C}(\beta_{h-l, b-1}) & \longrightarrow & \Sigma \mathcal{E}_{h-l, b-1}^\theta \left(M(d); \delta(d)^T \right) \\ \downarrow \sigma_{h-l, b}^1(a^1) & & \downarrow \sigma_{h-l, b}^1(\bar{a}^1) & & \downarrow & & \downarrow \Sigma \sigma_{h-l, b}^1(a^1) \\ \mathcal{E}_{h-l, b}^\theta \left(M(c); \delta(c)^T \right) & \longrightarrow & \mathcal{E}_{h-l+1, b-1}^\theta \left(M^1(\bar{c}); \delta'(\bar{c})^T \right) & \longrightarrow & \bar{C}(\alpha_{h-l, b}) & \xrightarrow{p} & \Sigma \mathcal{E}_{h-l, b}^\theta \left(M(c); \delta(c)^T \right) \\ \downarrow \sigma_{h, b}^k(c) & & \downarrow \sigma_{h+1, b-1}^k(\bar{c}) & & \downarrow & \swarrow h & \\ \mathcal{E}_{h, b}^\theta \left(M; \delta^T \right) & \longrightarrow & \mathcal{E}_{h+1, b-1}^\theta \left(M^1; \delta'^T \right) & \longrightarrow & \bar{C}(\alpha_{h, b}) & & \\ \sigma_{h, b}^1(a^1) \uparrow & & \sigma_{h+1, b-1}^1(\bar{a}^1) \uparrow & & \uparrow & \swarrow h' & \\ \mathcal{E}_{h, b-1}^\theta \left(M(a^1); \delta(\bar{a}^1)^T \right) & \longrightarrow & \mathcal{E}_{h, b}^\theta \left(M^1(\bar{a}^1); \delta'(\bar{a}^1)^T \right) & \longrightarrow & \bar{C}(\beta_{h, b-1}) & \xrightarrow{p'} & \Sigma \mathcal{E}_{h, b-1}^\theta \left(M(a^1); \delta(\bar{a}^1)^T \right) \\ \sigma_{h, b-1}^k(d) \uparrow & & \sigma_{h, b}^k(\bar{d}) \uparrow & & \uparrow & \swarrow h'' & \\ \mathcal{E}_{h-l, b-1}^\theta \left(M(d); \delta(d)^T \right) & \longrightarrow & \mathcal{E}_{h-l, b}^\theta \left(M^1(\bar{d}); \delta'(\bar{d})^T \right) & \longrightarrow & \bar{C}(\beta_{h-l, b-1}) & \longrightarrow & \Sigma \mathcal{E}_{h-l, b-1}^\theta \left(M(d); \delta(d)^T \right) \end{array}$$

Here the rows are given by the corresponding Puppe sequences, note that the first and last line agree. The vertical maps are augmentation compositions and we have indicated in brackets which arcs with boundary isotopy are used there.

Since θ is k -trivial we can find a $Z \subset c \cup \partial^0 M(a^1) \times [1, 2]$ providing a diagonal map for the lower left square. Note that $Z \cup (\bar{a}^1 \setminus a^1) \subset c \cup \partial^0 M \times [1, 2]$ gives a bordism that provides a diagonal map for the second square. This implies that the diagonal map of the lower square postcomposed with $\sigma_{h,b}^1(a^1)$ agrees up to homotopy with the composition of $\sigma_{h-l,b}^1(\bar{a}^1)$ and the diagonal map of the second square. Using this observation and Lemma 7.4, one obtains the following:

$$h \circ \Sigma \sigma_{h-l,b}^1(a^1) \simeq (\sigma_{h,b}^1(a^1), \sigma_{h+1,b-1}^1(\bar{a}^1)) \circ h'' \simeq h' \circ p' \circ h'' \simeq *$$

This implies that h induces the zero morphism in homology in those degrees, where $\Sigma \sigma_{h-l,b}^1(a^1)$ induces a surjective map in homology. Note that by replacing the occurring manifolds with corners by manifolds with boundary, $\sigma_{h-l,b}^1(a^1)$ becomes equivalent to a map of type $\beta_{h-l,b}$, hence is surjective in degrees $B(h-l)$. Hence $h \circ p$, which is homotopic to $\bar{C}(\alpha_{h-l,b}) \rightarrow \bar{C}(\alpha_{h,b})$, the relative augmentation composition on mapping cones instead of mapping pairs, is the null homomorphism in those degrees as well. Since this map is defined as a composition of relative 0-approximate augmentations (on cones not pairs), which are epimorphisms in a range of degrees (depending on the genus occurring in the approximate augmentations) by SA_h and SB_h , one has $H_*(C(\alpha_{h,b}))$ vanishes in all degrees $* \leq \min\{SA(h-l+1), SB(h-l+1), B(h-l)\}$. This finishes the proof as $A(h)$ is by definition smaller or equal than the left hand side. \square

8. Homological Stability for Capping off Boundary Components

All that is left to do to finish the proof of Theorem 7.3 is to prove the homological stability statement for maps of type γ and prove that $\beta_{g,b}$ induces a monomorphism in integral homology provided that one of the newly created boundary components is contractible and the tangential structure on it extends to a disk. To do this we will first observe that homological stability for maps of type $\gamma_{g,b}$ where $b > 1$ is an easy corollary of homological stability of maps of type $\beta_{g,b}$ and then we will establish everything needed to use Lemma 4.4 again to relate the homological stability for maps of type $\gamma_{g,1}$ to maps of type $\gamma_{g,b}$ for $b > 1$.

So consider a subsurface $P \subset \partial^0 M \times [0, 1]$, which defines a map of type $\beta_{g,b}$. Suppose that one of the boundary components of the outgoing boundary of the pair of pants component of P is contractible in $\partial^0 M \times [0, 1]$. Fix a contraction and, using the main result of [Hae61], realize it up to homotopy as an embedding $(D^2, \partial D^2) \rightarrow (\partial^0 M \times [0, 1], \partial^0 M \times \{1\})$.

By flipping this embedded contraction (i.e. flipping the interval direction) and equipping it with a tangential structure extending the one on the boundary given by P , yields a bordism defining a map of type $\gamma_{g,b}$ and by construction $\gamma_{g,b+1} \circ \beta_{g,b}$ is a union with a cylinder and hence the composition is a homotopy equivalence and induces an isomorphism in homology. This implies in particular that $\beta_{g,b}$ induces a monomorphism in integral homology in all degrees.

Similarly given a disk defining a map of type $\gamma_{g,b}$ such that there exists another component of δ in the same connected component of $\partial^0 M$, we can find a pair of pants defining a map $\beta_{g,b-1}$ such that $\gamma_{g,b} \circ \beta_{g,b-1}$ is a homotopy equivalence. We have already shown that $\beta_{g,b}$ induces an isomorphism in homology in degrees $\leq B(g) - 1$ and an epimorphism in the next degree. Moreover in this case $\beta_{g,b-1}$ induces a monomorphism in all degrees from which we conclude that it actually induces an isomorphism in all degrees $\leq B(g)$. This implies that $\gamma_{g,b}$, as a left inverse for this map, induces an isomorphism in these degrees as well. All in all this gives:

LEMMA 8.1. *Suppose M fulfils the assumptions of Theorem 7.3: If one of the new boundary components of the bordism that defines a map of type $\beta_{g,b}$ is contractible in $\partial^0 M$, then $\beta_{g,b}$ induces a monomorphism in homology in all degrees .*

Similarly if there exists another boundary component of δ in the same connected component of $\partial^0 M$, where the disk component of the bordism defining $\gamma_{g,b}$ lies, then $\gamma_{g,b}$ induces an epimorphism in homology in all degrees and furthermore an isomorphism in degrees $\leq B(g) + 1$.

Therefore we only have to concern ourselves with the case where there is no more connected component of δ in the same connected component as the disk component of the bordism defining $\gamma_{g,b}$. We will tackle this case using a certain resolution, which lets us relate $C(\gamma_{g,b})$ to $C(\gamma_{g,b+j})$. This section is very similar to Section 5 albeit it is much easier and does not require any inductive arguments.

Let ℓ denote a codimension 0 ball that is in the same connected component of $\partial^0 M$ as the disk component of the bordism defining some fixed map of type $\gamma_{g,b}$ and that is disjoint from δ .

DEFINITION 8.2. Fix an embedded subsurface $W \in \mathcal{E}_{g,b}^\theta(M; \delta^T)$. We call an embedding

$$y: ([0, 1], \frac{1}{2}) \rightarrow (M, W)$$

such that $y(0) \in \ell$ and $y(1) \in M \setminus \partial M$ a **boundary path of W** . If we add a tubular neighborhood of y in the pair (M, W) denoted by $\mathbf{y} = (\mathbf{y}_M, \mathbf{y}_W)$ such that $\mathbf{y} \cap \partial M \subset \ell$, then we will call the pair (y, \mathbf{y}) a **thickened boundary path of W** .

If we only consider embeddings as above $y: ([0, 1], 0) \rightarrow (M, \partial M)$ without a particular subsurface, we will drop the W from the notation i.e. we will call these just **boundary paths**.

We call a boundary path together with a 2-plane B in the normal bundle at $y(\frac{1}{2})$ a **thickened boundary path**. A **thickened boundary path with tangential structure** is a thickened boundary path with a tangential structure on $\mathbf{y}(B)$.

With this notation at hand we can proceed to define another resolution of $\mathcal{E}_{g,b}^\theta(M; \delta^T)$.

DEFINITION 8.3. Let $\mathcal{Q}_{g,b}^\theta(M; \delta^T, \ell)_\bullet$ denote the semi-simplicial space, whose i -simplices are tuples $(W, (y^0, \mathbf{y}^0), \dots, (y^i, \mathbf{y}^i))$, such that:

- (i) $W \in \mathcal{E}_{g,b}^\theta(M; \delta^T)$
- (ii) (y^j, \mathbf{y}^j) is a thickened boundary path of W .
- (iii) The neighbourhoods $\mathbf{y}^0, \dots, \mathbf{y}^i$ are pairwise disjoint.

The j -th face map forgets the j -th boundary path and we topologize this as a subspace of $\mathcal{E}_{g,b}^\theta(M; \delta^T) \times \overline{\text{TEmb}}([0, 1] \times X_i, M; q)$. There is also an augmentation map ϵ_\bullet to $\mathcal{E}_{g,b}^\theta(M; \delta^T)$, which forgets the boundary paths.

LEMMA 8.4. *If M is connected and has dimension at least 3, then the semi-simplicial space $\mathcal{Q}_{g,b}^\theta(M; \delta^T, \ell)_\bullet$ is a resolution of $\mathcal{E}_{g,b}^\theta(M; \delta^T)$.*

PROOF. There is the following commutative square:

$$\begin{array}{ccc} \mathcal{Q}_{g,b}^\theta(M; \delta^T, \ell)_\bullet & \longrightarrow & \mathcal{E}_{g,b}^\theta(M; \delta^T) \\ \downarrow & & \downarrow \\ \mathcal{Q}_{g,b}^+(M; \delta, \ell)_\bullet & \longrightarrow & \mathcal{E}_{g,b}^+(M; \delta) \end{array}$$

By Lemma 2.8 $\mathcal{E}_{g,b}^+(M; \delta)$ is $\text{Diff}_\partial^+(M)$ -locally retractile. The group $\text{Diff}_\partial^+(M)$ acts on $\mathcal{Q}_{g,b}^+(M; \delta, \ell)_\bullet$ and the lower map is equivariant, hence the bottom map is a Hurewicz-fibration. Furthermore it is obvious that the square is a pullback square, from which we conclude that the top map is also a Hurewicz-fibration.

It is easy to see that $\mathcal{Q}_{g,b}^+(M; \delta, \ell)_\bullet$ is a topological flag complex. We want to apply Lemma 4.1 to show that it is a resolution. Since the augmentation is a fiber bundle (and the fiber is non-empty as M is connected) this flag complex has local sections. All that is left to show to apply the lemma is that for $\{y^1, \dots, y^n\} \subset \text{Fib}_W(\epsilon_0)$, there exists an y^∞ such that (y^i, y^∞) is a 1-simplex for every

i. Since M is at least 3-dimensional we conclude that $M \setminus (\mathbf{y}^0 \cup \dots \cup \mathbf{y}^n)$ is still path-connected, hence we can find a boundary path in this complement. \square

DEFINITION 8.5. Let $R^\theta(M, \ell)_i$ denote the set of tuples $(\mathbf{y}^0, \dots, \mathbf{y}^i)$, where \mathbf{y}^j are thickened boundary paths with tangential structure such that all \mathbf{y}^j are disjoint. We topologize this as a subspace of $\overline{\text{TEmb}}([0, 1] \times X_i, M; q) \times \text{Emb}(D^2 \times X_i, M)$.

There is a map from $\mathcal{Q}_{g,b}^\theta(M; \delta^T, \ell)_\bullet$ to $R^\theta(M, \ell)_i$, which forgets the surface, but keeps the tubular neighbourhood in the surface and the corresponding tangential structure on it.

LEMMA 8.6. *For a point $y = (y_0, \dots, y_i) \in R^\theta(M, \ell)_i$ there is a homotopy fiber sequence:*

$$\mathcal{E}_{g,b+i+1}^\theta(M(y); \delta^T(y)) \longrightarrow \mathcal{Q}_{g,b}^\theta(M; \delta^T, \ell)_i \longrightarrow R^\theta(M, \ell)_i$$

where $M(y)$ denotes $\overline{M \setminus \left(\bigcup_j \mathbf{y}_j \right)}$ and $\delta^T(\mathbf{y})$ denotes $\delta^T \cup \left(\bigcup_j \partial \mathbf{y}_j^{Disk} \right)$ where \mathbf{y}_j^{Disk} denotes the embedding with θ -structure of D^2 into M associated to the two-plane in the definition of thickened boundary path with tangential structure.

The proof of this lemma is completely analogous to the proof of Lemma 5.6 and is therefore omitted.

The issues here that arise from the occurring manifolds with corners are dealt with in the exact same fashion as in the discussion after Lemma 5.6.

Similar to Section 5, we want to understand how this resolution behaves under stabilization maps. Let us take a look at the following diagram, in which we want to construct the dashed arrow:

$$\begin{array}{ccc} \mathcal{Q}_{g,b}^\theta(M; \delta^T, \ell)_\bullet & \dashrightarrow & \mathcal{Q}_{g,b-1}^\theta(M^1; \bar{\delta}^T, \bar{\ell})_\bullet \\ \downarrow & & \downarrow \\ \mathcal{E}_{g,b}^\theta(M; \delta^T) & \xrightarrow{\gamma_{g,b}} & \mathcal{E}_{g,b-1}^\theta(M^1; \bar{\delta}^T) \end{array}$$

Let us denote the surface defining $\gamma_{g,b}$ by P . Replace P by a different but isotopic bordism if necessary to ensure that $(\ell \times [0, 1]) \cap P = \emptyset$. We will write $\bar{\ell}$ for $\ell \times [0, 1]$. For a $y \in R^\theta(M, \ell)_i$ we define $\tilde{y}^j = y^j(0) \times [0, 1]$ and $\bar{y}^j = y^j \cup \tilde{y}^j$. The dashed arrow is now given by mapping W to $W \cup P$ and y^j to \bar{y}^j . These maps commute with the face and augmentation maps and so they define a map of augmented semi-simplicial spaces, which we denote by $\gamma_{g,b}^\bullet$.

There is a map $R^\theta(M, \ell)_i \rightarrow R^\theta(M^1, \bar{\ell})_i$ defined by $y \mapsto \bar{y}$. This map is a homotopy equivalence. We have the following diagram:

$$\begin{array}{ccccc} \mathcal{E}_{g,b+i+1}^\theta(M(y); \delta(y)^T) & \longrightarrow & \mathcal{Q}_{g,b}^\theta(M; \delta^T, \ell)_\bullet & \longrightarrow & R^\theta(M, \ell)_i \\ \downarrow \gamma_{g,b+i+1} & & \downarrow \gamma_{g,b}^i & & \downarrow \simeq \\ \mathcal{E}_{g,b+i}^\theta(M^1(\bar{y}); \delta'(\bar{y})^T) & \longrightarrow & \mathcal{Q}_{g,b-1}^\theta(M^1; \delta'^T, \ell)_\bullet & \longrightarrow & R^\theta(M^1, \ell)_i \end{array}$$

The commutativity of the right square gives us a map between the fibers and it is easy to see that the left vertical arrow is given by $\gamma_{g,b+i+1}$. Putting this together yields:

COROLLARY 8.7. *The induced map between the homotopy fibers of*

$$(\gamma_{g,b}^i) \rightarrow R^\theta(M, \ell)_i$$

is given by $\gamma_{g,b+i+1}$.

Finally with all these tools at hand we are able to finish the proof of Theorem 7.3 by concluding:

PROPOSITION 8.8. *Suppose M fulfils the requirements of Theorem 7.3, then*

$$H_*(C(\gamma_{g,b})) = 0$$

for $* \leq B(g)$.

PROOF. We want to apply Lemma 4.4. The role of B_i will be played by $R^\theta(M, \ell)_i$ and $\mathcal{Q}_{g,b}^\theta(M; \delta^T, \ell)_\bullet$ will be the two resolutions, $\gamma_{g,b}^\bullet$ will be the map between them. Corollary 8.7 specified the occurring homotopy fibers.

We choose $n = \infty$ and $c = B(g) + 1$. By Lemma 8.1 and 8.7 we conclude that the homology of the reduced mapping cone of the map between the fibers equals zero in the desired range i.e.

$$H_*(C(\gamma_{g,b+i+1})) = 0$$

for $* \leq B(g)$, which finishes the proof of the proposition. \square

9. Tangential Structures that fulfil Homological Stability

The goal of this section is to establish criteria for when a space of tangential structures satisfies k -triviality, which was the key requirement for a tangential structure to fulfil homological stability, and prove Proposition 6.9, which will be fundamental in the proofs of these criteria. This will yield more easily applicable versions of Theorem 7.3.

9.1. Stability of Connected Components implies k -triviality

PROOF OF PROPOSITION 6.9. We will prove k -triviality for maps of type α , the case of maps of type β is completely analogous. Let (ℓ_0^i, ℓ_1^i) together with $\bar{\ell}_0$ and $\bar{\ell}_1$ denote a genus maximizing sequence of length $2g+1$ for a map of type α and compatible a^i as in the definition of k -triviality. Let P denote a stabilization bordism such that $P \cap ((\bar{\ell}_0 \cup \bar{\ell}_1) \times [0, 1]) = (\bar{\ell}_0 \cup \bar{\ell}_1) \times [0, 1]$. Furthermore let us denote $\bigcup_i a_L^i$ by a_L and $\bigcup_i \bar{a}_L^i$ by \bar{a}_L and $\bigcup_i a^i$ by a and analogously for \bar{a} . We want to prove that a_L absorbs P or in other words there exists a $Z \subset \bar{a} \cup \partial^0 M \times [1, 2]$ such that

$$\begin{array}{ccc} \mathcal{E}^\theta(M(a); \delta^T(U)) & \xrightarrow{-\cup P(a)} & \mathcal{E}^\theta(M(\bar{a})^1; \delta^T(\bar{a})) \\ \downarrow -\cup a & \swarrow -\cup Z & \downarrow -\cup \bar{a}^1 \\ \mathcal{E}^\theta(M; \delta^T) & \xrightarrow{-\cup P} & \mathcal{E}^\theta(M^1; \delta^T) \\ \downarrow \simeq & \swarrow & \downarrow \simeq \\ \mathcal{E}^\theta(M^2; \delta^T) & \xrightarrow{-\cup(P+2)} & \mathcal{E}^\theta(M^3; \delta^T) \end{array}$$

commutes up to homotopy.

Let us denote $\overline{\partial a_L \setminus \delta}$ by $\partial^0 a_L$ and $\overline{\partial \bar{a}_L \setminus \delta^T}$ by $\partial^0 \bar{a}_L$. We define

$$(10) \quad \delta_a^T := \partial^0 a_L \cup (\delta^T \setminus a)$$

$$(11) \quad \delta_{\bar{a}}^T := \partial^0 \bar{a}_L \cup (\delta^T \setminus \bar{a})$$

We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E}_{g-1,4}^\theta(\bar{a} \cup \partial^0 M \times [1, 2]; \delta_{\bar{a}}^T) & \xrightarrow{-\cup P(\bar{a})} & \mathcal{E}_{g,3}^\theta(a \cup \partial^0 M \times [0, 2]; \delta_a^T) \\ \downarrow -\cup P+2 & & \downarrow -\cup P+2 \\ \mathcal{E}_{g,3}^\theta(\bar{a} \cup \partial^0 M \times [1, 3]; \delta_{\bar{a}}^T) & \xrightarrow{-\cup P(\bar{a})} & \mathcal{E}_{g+1,2}^\theta(a \cup \partial^0 M \times [0, 3]; \delta_a^T) \end{array}$$

Note that the occurring spaces in this diagram are manifolds with corners, this can be fixed by the usual smoothing the angle techniques explained in Remark 5.7 and Remark 5.7, furthermore note that in this diagram $-\cup P(a)$ is a complicated subset bordism and not a stabilization bordism.

Note that the non-cylindrical parts of the occurring bordisms in the first diagram occur as elements of the spaces in this second diagram. For example the bordism for the composition of the two arrows on the right of the first diagram is given by $\bar{a}_L \cup \delta'^T \times [1, 3]$ which is an element of the lower left space. Analogously the bordism of the composition on the left hand side $a_L \cup \delta^T \times [0, 2]$ is an element of the top right space. That the genus and the number of boundary conditions is as stated follows from the requirement that the sequence of intervals is genus maximizing and that P contains $(\bar{\ell}_0 \cup \bar{\ell}_1) \times [0, 1]$.

Z will be an element of the top left space and the compositions with $-\cup Z$ in the first diagram correspond to composition in the second diagram. Furthermore note that the left map is π_0 -surjective and the right hand is π_0 -injective. Hence chose an element Z in $\mathcal{E}_{g-1,4}^\theta(\bar{a} \cup \partial^0 M \times [1, 2]; a)$ such that $Z \cup P + 2$ lies in the same connected component as $\bar{a}_L \cup \delta'^T \times [1, 3]$. Then the lower triangle in the first diagram commutes up to homotopy by construction.

Since the second diagram is commutative the image of $Z \cup P(\bar{a})$ under the right map is isotopic to $\bar{a}_L \cup \delta'^T \times [1, 3] \cup P(a)$ which in turn is isotopic to $a_L \cup \delta^T \times [0, 2] \cup P + 2$, since this is the image of $a_L \cup \delta^T \times [0, 2]$ under the right hand map, which is π_0 -injective, we conclude that the upper triangle in the first diagram commutes as well. \square

As a corollary to Proposition 6.9 and Theorem 7.3 one obtains:

THEOREM 9.1. *Suppose M is an at least 5-dimensional, simply-connected manifold with non-empty boundary and $\partial^0 M$ is a codimension 0 simply-connected submanifold of ∂M with a space of θ -structures of subplanes of TM . Suppose further that θ π_0 -stabilizes at m_1 and π_0 -stabilizes at the boundary at m_2 , then*

- (i) *The homology of $C(\alpha_{g,b})$ vanishes in degrees $* \leq A(g)$*
- (ii) *The homology of $C(\beta_{g,b})$ vanishes in degrees $* \leq B(g)$*
- (iii) *The homology of $C(\gamma_{g,b})$ vanishes in degrees $* \leq B(g) + 1$*

Where k in the definition of A and B (Definition 7.1) equals $2m_2 + 1$.

9.2. Criteria for Stabilization of Connected Components

The goal of this subsection is to prove the following theorem:

THEOREM 9.2. *Suppose that M is a simply connected manifold of dimension at least 5 such that $\partial^0 M$ is simply-connected as well and $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ is a space of θ -structures of subplanes of TM where $\text{Fib}_B(\theta)$ is simply-connected, then*

- (i) *The homology of $C(\alpha_{g,b})$ vanishes in degrees $* \leq \lfloor \frac{1}{3}(2g + 1) \rfloor$*
- (ii) *The homology of $C(\beta_{g,b})$ vanishes in degrees $* \leq \lfloor \frac{2}{3}g \rfloor$*
- (iii) *The homology of $C(\gamma_{g,b})$ vanishes in degrees $* \leq \lfloor \frac{2}{3}g + 1 \rfloor$*

The proof will use Theorem 9.1 and Remark 7.2.

As was explained in the introduction, examples of spaces of θ -structures with simply-connected fiber are given by k framings of the normal bundle of the subsurfaces, where $k \leq n - 2$.

In order to apply Theorem 9.1, we need a better understanding of the space of connected components of $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ if the fiber of the space of tangential structures is simply-connected.

LEMMA 9.3. *Suppose that M is simply-connected and of dimension at least 5, and that the fiber $\text{Fib}_B(\theta)$ is simply connected, then $\pi_0(\text{Emb}^\theta(S^2, M))$ carries a canonical group structure and there*

is an action of this group on $\pi_0(\mathcal{E}_{g,b}^\theta(M; \delta^T))$, given by connected sum, such that it is an affine $\pi_0(\text{Emb}^\theta(S^2, M))$ -set.

PROOF. The proof is divided into three steps. We will first construct the group structure on $\pi_0(\text{Emb}^\theta(S^2, M))$ by relating this space to the second homotopy groups of $B_2(M)$ and $\text{Gr}_2(TM)$, then we will relate this group to $\pi_0(\text{Emb}^\theta(\Sigma_{g,b}, M; \delta^T))$ using the simply-connectedness of M . Lastly, we will see that the identifications we proved along the way behave very well with respect to the action of the diffeomorphism group of the surface to get an understanding of $\pi_0(\mathcal{E}_{g,b}^\theta(M; \delta^T))$.

For the first step, consider the following tower of fibrations:

$$\begin{array}{ccc} & B_2(M) & \\ & \downarrow \theta & \\ & \text{Gr}_2(TM) & \\ \text{Gr}(De) \nearrow & & \downarrow \\ \Sigma_{g,b} \xrightarrow{e} & & M \end{array}$$

Let us calculate $\pi_2(\text{Gr}_2(TM))$. Consider the following diagram, which stems from the fact that the fibration $\text{Gr}_2(TM) \rightarrow M$ is the pullback of $\text{Gr}_2(\gamma)$, where γ denotes the tautological bundle of some $\text{Gr}_n(\mathbb{R}^N)$

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \pi_3(M) & \longrightarrow & \pi_2(\text{Gr}_2(\mathbb{R}^n)) & \longrightarrow & \pi_2(\text{Gr}_2(TM)) & \longrightarrow & \pi_2(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \text{Id} & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 = \pi_3(\text{Gr}_n(\mathbb{R}^N)) & \longrightarrow & \pi_2(\text{Gr}_2(\mathbb{R}^n)) & \longrightarrow & \pi_2(\text{Gr}_2(\gamma)) & \longrightarrow & \pi_2(\text{Gr}_n(\mathbb{R}^N)) & \longrightarrow & 0 \end{array}$$

Hence the upper long exact sequence breaks down to the following:

$$0 \longrightarrow \mathbb{Z} \cong \pi_2(\text{Gr}_2(\mathbb{R}^n)) \longrightarrow \pi_2(\text{Gr}_2(TM)) \xrightarrow{\text{Gr}(D-)} \pi_2(M) \longrightarrow 0$$

Since M is simply-connected, we have an isomorphism $\pi_2(M) \simeq \pi_0(\text{Map}(S^2, M))$ and this group is isomorphic to $\pi_0(\text{Emb}(S^2, M))$ by the main result of [Hae61]. Therefore the split in the aforementioned diagram is given by taking an embedding as a representative and then applying $\text{Gr}(D-)$. From here on forth we consider $\pi_0(\text{Emb}(S^2, M))$ as a subgroup of $\pi_2(\text{Gr}_2(TM))$.

Consider the the following long exact sequence:

$$\dots \longrightarrow \pi_2(\text{Fib}_B(\theta)) \longrightarrow \pi_2(B_2(M)) \xrightarrow{(\theta)_*} \pi_2(\text{Gr}_2(TM)) \longrightarrow 0$$

We claim that $(\theta)_*^{-1}(\pi_0(\text{Emb}(S^2, M)))$ is isomorphic to $\pi_0(\text{Emb}^\theta(S^2, M))$. There is a canonical map $\pi_0(\text{Emb}^\theta(S^2, M)) \rightarrow (\theta)_*^{-1}(\pi_0(\text{Emb}(S^2, M)))$, which is evidently surjective. For injectivity of this map suppose two elements $[e], [e'] \in \pi_0(\text{Emb}^\theta(S^2, M))$ map to the same element in $(\theta)_*^{-1}(\pi_0(\text{Emb}(S^2, M)))$. Hence there is a homotopy H from T_e to $T_{e'}$ as maps into $B_2(M)$. The main theorem of [Hae61] implies that $\theta \circ H$ is actually homotopic via \bar{H} relative to e and e' to an isotopy. We have the following commutative diagram, where the dashed arrows exists since θ is a fibration:

$$\begin{array}{ccc} S^2 \times [0, 1] \times \{0\} \cup S^2 \times \{0, 1\} \times [0, 1] & \xrightarrow{H \cup e \cup e'} & B_2(M) \\ \downarrow & \dashrightarrow & \downarrow \\ S^2 \times [0, 1] \times [0, 1] & \xrightarrow{\bar{H}} & \text{Gr}_2(TM) \end{array}$$

The indicated lift of \bar{H} , when evaluated at $(-, 1)$, gives us the desired isotopy from $[e]$ to $[e']$ as embeddings with θ -structure. All in all this implies that $\pi_0(\text{Emb}^\theta(S^2, M))$ has a group structure, which stems from the isomorphism to $(\theta)_*^{-1}(\pi_2(\text{Gr}_2(TM)))$. For notational reasons, let us denote $\pi_0(\text{Emb}^\theta(S^2, M))$ by π^θ .

For the second step, consider the cofibration $\Sigma_{g,b}^1 \rightarrow \Sigma_{g,b}$, where $\Sigma_{g,b}^1$ denotes a small tubular neighbourhood of the 1-skeleton of $\Sigma_{g,b}$ to which only a single 2-cell is attached via σ . Since the inclusion $\Sigma_{g,b}^1 \rightarrow \Sigma_{g,b}$ is a cofibration, $\text{Map}(\Sigma_{g,b}, M) \rightarrow \text{Map}(\Sigma_{g,b}^1, M)$ is a fibration, with fiber over an embedding e given by $\text{Map}((D^2, \partial D^2), (M, \mathfrak{d}))$, where \mathfrak{d} is the image of the boundary of the single two cell, denoted by D^2 , glued to $\Sigma_{g,b}^1$ under e . Again by [Hae61] and transversality, $\pi_0(\text{Map}((D^2, \partial D^2), (M, \mathfrak{d})))$ is isomorphic to $\pi_0(\text{Emb}(D^2, M \setminus e(\Sigma_{g,b}^1); \mathfrak{d}))$. Similar considerations as before show that the a priori pointed set $\pi_0(\text{Emb}(D^2, M \setminus e(\Sigma_{g,b}^1); \mathfrak{d}))$ can be considered as a subgroup of $\pi_2(\text{Gr}_2(TM), \mathfrak{d})$, where we consider \mathfrak{d} (or its image to be more precise) as a subset of $\text{Gr}_2(TM)$ via the Grassmannian differential.

Since $\text{Fib}_B(\theta)$ is simply-connected, there exists an up to homotopy unique θ -structure on \mathfrak{d} , which we denote by \mathfrak{d}^T .

Let us introduce the shorthand notation π_{rel}^θ for $(\theta)_*^{-1}(\pi_0(\text{Emb}(D^2, M \setminus e(\Sigma_{g,b}^1); \mathfrak{d})))$, here $\theta: (B_2(M), \mathfrak{d}^T) \rightarrow (\text{Gr}_2(TM), \mathfrak{d})$ is considered as a map of pairs. We claim that π_{rel}^θ considered as a subgroup of $\pi_2(B_2(M))$ acts transitively and freely on π_{rel}^θ . In order to prove this claim, consider the long exact sequence of the pair $(B_2(M), \mathfrak{d}^T)$:

$$0 \longrightarrow \pi_2(B_2(M)) \longrightarrow \pi_2(B_2(M), \mathfrak{d}^T) \xrightarrow{\partial} \pi_1(\mathfrak{d}^T)$$

That the action is free follows immediately from this long exact sequence. To prove that this action is transitive consider the following diagram given by considering long exact sequences of pairs and the corresponding maps between them:

$$\begin{array}{ccccccc} \pi^\theta & \longrightarrow & \pi_{\text{rel}}^\theta \subset \partial^{-1}([\mathfrak{d}^T]) & & & & \\ \text{||} \cap & & \text{||} \cap & & & & \\ 0 & \longrightarrow & \pi_2(B_2(M)) & \longrightarrow & \pi_2(B_2(M), \mathfrak{d}^T) & \longrightarrow & \pi_1(\mathfrak{d}^T) \longrightarrow 0 \\ & & \downarrow (\theta)_* & & \downarrow (\theta)_* & & \downarrow \cong \\ 0 & \longrightarrow & \pi_2(\text{Gr}_2(TM)) & \longrightarrow & \pi_2(\text{Gr}_2(TM), \mathfrak{d}) & \longrightarrow & \pi_1(\mathfrak{d}) \longrightarrow 0 \\ & & \text{Gr}(D-) \left(\downarrow (\pi_{\text{Gr}_2(TM)})_* \right) & & \text{Gr}(D-) \left(\downarrow (\pi_{\text{Gr}_2(TM)})_* \right) & & \downarrow \cong \\ 0 & \longrightarrow & \pi_2(M) & \longrightarrow & \pi_2(M, \mathfrak{d}) & \longrightarrow & \pi_1(\mathfrak{d}) \longrightarrow 0 \end{array}$$

The commutativity of the lower left square gives us that $\pi_0(\text{Emb}(S^2, M))$ acts freely and transitively on $\pi_0(\text{Emb}(D^2, M \setminus e(\Sigma_{g,b}^1); \mathfrak{d}))$ and taking preimages with respect to θ does not change the transitivity of the action.

Furthermore the same arguments as for π^θ before show that π_{rel}^θ is isomorphic to the pointed set $\pi_0(\text{Emb}^\theta(D^2, M \setminus e(\Sigma_{g,b}^1); \mathfrak{d}^T))$.

We claim that π_{rel}^θ is isomorphic to $\pi_0(\text{Emb}^\theta(\Sigma_{g,b}, M))$. Consider the long exact sequence in homology of the pair $(B_2(M), \delta^T)$ (As before δ is the boundary condition for the spaces of subsurfaces) :

$$0 \longrightarrow H_2(B_2(M)) \simeq \pi_2(B_2(M)) \longrightarrow H_2(B_2(M), \delta^T) \simeq \pi_2(B_2(M), \delta^T) \xrightarrow{\partial'} H_1(\delta^T) \longrightarrow 0$$

Note that $B_2(M)$ is simply-connected since $\text{Fib}_B(\theta)$ and $\text{Gr}_2(TM)$ are simply-connected. Likewise the pair $(B_2(M), \mathfrak{d}^T)$ is simply-connected. This yields the claimed isomorphisms by the Hurewicz-Theorem. The long exact sequence implies that $\pi^\theta \subset \pi_2(B_2(M)) \cong H_2(B_2(M))$ acts freely on $\partial'^{-1}([\delta^T])$. We have a map from π_{rel}^θ to $\partial'^{-1}([\delta^T])$ by extending an embedding by $e|_{\Sigma_{g,b}^1}$ and considering the image of the fundamental class of $\Sigma_{g,b}$. By definition this map factorizes through $\pi_0(\text{Emb}^\theta(\Sigma_{g,b}, M; \delta^T))$ and it is equivariant with respect to the π^θ -action on $H_2(B_2(M))$ and $\partial'^{-1}([\delta^T])$. This implies that the map from π_{rel}^θ to $\pi_0(\text{Emb}^\theta(\Sigma_{g,b}, M; \delta^T))$ is injective.

To see that this map is surjective consider the fibration (proven to be a fibration in Lemma 2.18) $\text{Emb}^\theta(\Sigma_{g,b}, M; \delta^T) \rightarrow \text{Emb}^\theta(\Sigma_{g,b}^1, M; \delta^T)$ and consider its long exact sequence:

$$\pi_{\text{rel}}^\theta = \pi_0(\text{Emb}^\theta(D^2, M \setminus e(\Sigma_{g,b}^1); \bar{\delta}^T)) \rightarrow \pi_0(\text{Emb}^\theta(\Sigma_{g,b}, M; \delta^T)) \rightarrow \pi_0(\text{Emb}^\theta(\Sigma_{g,b}^1, M; \delta^T))$$

Hence the map is surjective, as $\pi_0(\text{Emb}^+(\Sigma_{g,b}^1, M; \delta))$ has only one element by the main result of [Hae61], and the simply-connectedness of $\text{Fib}_B(\theta)$.

Let us proceed with the last step. We have proven so far that the π^θ -equivariant map from π_{rel}^θ to $\pi_0(\text{Emb}^\theta(\Sigma_{g,b}, M; \delta^T))$ is an isomorphism and that the map from there to $\partial'^{-1}([\delta^T])$ is an injection. But this map factorizes through $\pi_0(\mathcal{E}_{g,b}^\theta(M; \delta^T))$, since an orientation preserving diffeomorphism does not change the image of the fundamental class. Therefore we conclude that $\pi_0(\mathcal{E}_{g,b}^\theta(M; \delta^T))$ is an affine π^θ -set. \square

From this we obtain almost immediately the following corollary.

COROLLARY 9.4. *If M is simply-connected and of dimension at least 5, $\partial^\theta M$ is simply-connected as well and the fiber of θ is simply-connected, then θ π_0 -stabilizes at 0 and π_0 -stabilizes at the boundary at 0 as well.*

PROOF. The stabilization maps are given by gluing in surfaces, which induce $\pi_0(\text{Emb}^\theta(S^2, M))$ -equivariant maps between $\pi_0(\mathcal{E}_{g,b}^\theta(M; \delta^T))$ and $\pi_0(\mathcal{E}_{g+1,b-1}^\theta(M; \delta^T))$ or $\pi_0(\mathcal{E}_{g,b+1}^\theta(M; \delta^T))$. Since both of these spaces are affine $\pi_0(\text{Emb}^\theta(S^2, M))$ -spaces, the map is a bijection. Repeating the same argument for $\partial^\theta M \times [0, 1]$ proves the claim. \square

Applying this corollary to M together with Theorem 9.1 for $m_1 = m_2 = 0$, one obtains Theorem 9.2.

9.3. Non-relative Tangential Structures

As was explained in the introduction, the notion of tangential structure in this paper is a generalization of the usual notion to include relative tangential structures. This subsection will deal with non-relative tangential structures.

There is a (up to homotopy) canonical map $\text{Gr}_2(TM) \rightarrow \text{Gr}_2(\mathbb{R}^\infty) \simeq BSO(2)$ and recall that in the common definition of tangential structure one considers a fibration $\theta: B_2(\text{Gr}_2(\mathbb{R}^\infty)) \rightarrow \text{Gr}_2(\mathbb{R}^\infty)$. In this case one obtains classifying spaces for surface bundles equipped with a θ -structure, denoted by $\mathcal{M}^\theta(\Sigma_{g,b})$ (we are a bit sloppy here, since there are also boundary conditions involved). We will not define these classifying spaces here and the reader is advised to consult [RW16] in particular Definition 1.1 in that paper.

DEFINITION 9.5. We say that a tangential structure $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ (in the sense of this paper) is a **non-relative tangential structure stemming from θ'** if there exists a tangential

structure (in the common sense)

$$\theta' : B_2(\mathrm{Gr}_2(\mathbb{R}^\infty)) \rightarrow \mathrm{Gr}_2(\mathbb{R}^\infty)$$

such that the following diagram is a pullback diagram:

$$\begin{array}{ccc} B_2(M) & \longrightarrow & B_2(\mathrm{Gr}_2(\mathbb{R}^\infty)) \\ \downarrow \theta & & \downarrow \theta' \\ \mathrm{Gr}_2(TM) & \longrightarrow & \mathrm{Gr}_2(\mathbb{R}^\infty) \end{array}$$

The following theorem states that for the question of whether a non-relative tangential structure fulfils homological stability, it does not matter, whether one considers it as tangential structures of abstract surfaces or subsurfaces of a (finite-dimensional) manifold.

THEOREM 9.6. *Suppose M is a simply-connected at least 5-dimensional manifold and that $\partial^0 M$ is simply-connected. Let $\theta : B_2(TM) \rightarrow \mathrm{Gr}_2(TM)$ denote a non-relative tangential structure stemming from θ' such that $\mathcal{M}^{\theta'}(\Sigma_{g,b})$ fulfils homological stability as stated in Theorem 7.1 in [RW16] (Note that the definitions of F and G in the theorem, agree with our definition of A and B).*

Then θ π_0 -stabilizes at genus g , and also π_0 -stabilizes at the boundary at genus g , where g is the first degree in which the first homology of $C(\alpha_{g,b})$, of $C(\beta_{g,b})$, and of $\mathrm{MappingPair}\gamma_{g,b}$ vanishes, here these stabilization maps are stabilization maps of abstract surfaces in the sense of Subsection 1.3 in [RW16]. Hence θ is $2g + 1$ -trivial and $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ fulfils homological stability as stated in Theorem 7.3.

In particular θ fulfils homological stability if θ' stabilizes on connected components of $\mathcal{M}^{\theta'}(\Sigma_{g,b})$

PROOF. By Theorem 9.1 it suffices to prove that $\mathcal{E}_{g,b}^\theta(M; \delta^T)$ stabilizes on connected components. There is the following pullback diagram, where the horizontal maps stems from a fixed embedding of M into \mathbb{R}^∞ ,

$$\begin{array}{ccc} \mathcal{E}_{g,b}^\theta(M; \delta^T) & \longrightarrow & \mathcal{M}^{\theta'}(\Sigma_{g,b}) \\ \downarrow & & \downarrow \\ \mathcal{E}_{g,b}^+(M; \delta) & \longrightarrow & \mathcal{M}^+(\Sigma_{g,b}) \end{array}$$

Here $\mathcal{M}^+(\Sigma_{g,b})$ is the classifying space of oriented surface bundles. Let us denote the fiber of the vertical maps of this pullback diagram by $F_{g,b}$. Let us prove stabilization of connected components for a stabilization map of type $\alpha_{g,b}$, the case of another stabilization map is treated analogously.

One obtains the following diagram, where the rows are exact sequences and the vertical maps are stemming from stabilization maps $\alpha_{g,b}$:

$$\begin{array}{ccccccccc} H_1(\mathcal{M}^{\theta'}(\Sigma_{g,b})) & \longrightarrow & H_1(\mathcal{M}^+(\Sigma_{g,b})) & \longrightarrow & H_0(F_{g,b}) & \longrightarrow & H_0(\mathcal{M}^{\theta'}(\Sigma_{g,b})) & \longrightarrow & H_0(\mathcal{M}^+(\Sigma_{g,b})) \\ \downarrow \cong_{g \text{ large}} & & \downarrow \cong_{g \text{ large}} & & \downarrow \cong \text{ Five Lemma} & & \downarrow \cong_{g \text{ large}} & & \downarrow \cong_{g \text{ large}} \\ H_1(\mathcal{M}^{\theta'}(\Sigma_{g+1,b-1})) & \longrightarrow & H_1(\mathcal{M}^+(\Sigma_{g+1,b-1})) & \longrightarrow & H_0(F_{g+1,b-1}) & \longrightarrow & H_0(\mathcal{M}^{\theta'}(\Sigma_{g+1,b-1})) & \longrightarrow & H_0(\mathcal{M}^+(\Sigma_{g+1,b-1})) \end{array}$$

From this we can conclude that if $(\alpha_{g,b})_* : H_1(\mathcal{M}^{\theta'}(\Sigma_{g,b})) \rightarrow H_1(\mathcal{M}^{\theta'}(\Sigma_{g+1,b-1}))$ and $(\alpha_{g,b})_* : H_0(\mathcal{M}^+(\Sigma_{g,b})) \rightarrow H_0(\mathcal{M}^+(\Sigma_{g+1,b-1}))$ are isomorphisms, then the induced map between the fibers $(\alpha_{g,b})_* : H_0(F_{g,b}) \rightarrow H_0(F_{g+1,b-1})$ is an isomorphism as well.

Hence we obtain:

$$\begin{array}{ccccccc}
H_1(\mathcal{E}_{g,b}^+(M; \delta)) & \longrightarrow & H_0(F_{g,b}) & \longrightarrow & H_0(\mathcal{E}_{g,b}^\theta(M; \delta^T)) & \longrightarrow & H_0(\mathcal{E}_{g,b}^+(M; \delta)) \longrightarrow 0 \\
\downarrow \cong \text{if } g \text{ is large enough} & & \downarrow \cong \text{if } g \text{ is large enough} & & \downarrow \cong \text{Five Lemma} & & \downarrow \cong \text{if } g \text{ is large enough} \\
H_1(\mathcal{E}_{g,b}^+(M; \delta)) & \longrightarrow & H_0(F_{g+1,b-1}) & \longrightarrow & H_0(\mathcal{E}_{g+1,b-1}^{\theta'}(M; \delta^T)) & \longrightarrow & H_0(\mathcal{E}_{g,b}^+(M; \delta)) \longrightarrow 0
\end{array}$$

Repeating the same argument for $\partial^0 M \times [0, 1]$, we conclude that θ π_0 -stabilizes and π_0 -stabilizes at the boundary. The rest of the proof follows from Theorem 9.1. \square

As was explained in the introduction, examples of such tangential structures are given by framings of the tangent bundle of the subsurface or spin structures on the subsurfaces (See [RW16] for the corresponding statement about the moduli spaces of abstract surfaces with these tangential structures).

10. Spaces of Pointed Subsurfaces and their Stabilization

While stabilizing at the boundary, as we have done so far, is the most intuitive way to stabilize subsurfaces, it is not suitable for many kinds of tangential structures. In particular it is not well suited for symplectic subsurfaces, which we will consider in Section 11. In order to deal with this, we will introduce spaces of pointedly embedded subsurfaces and construct stabilization maps between them that are heuristically given by taking the connected sum with a fixed torus and then prove homological stability for these stabilization maps.

10.1. Spaces of Pointedly Embedded Subsurfaces

Let M denote a closed smooth n -manifold, where $n \geq 5$ and let us abbreviate $\Sigma_{g,0}$ by Σ_g . Fix $x \in \Sigma_g$, $x_M \in M$ and an oriented 2-plane with tangential structure $B \in \theta^{-1}(\text{Gr}_2(T_{x_M}M))$, by abuse of notation we will also write B for the underlying 2-plane. We will consider the space

$$\text{Emb}_B^\theta(\Sigma_g, M) := \left\{ e \in \text{Emb}^\theta(\Sigma_g, M) \mid e(x) = x_M \text{ and } T_e(x) = B \right\}$$

equipped with the same topology as in the non-pointed case. Let $\text{Diff}_x(\Sigma_g)$ denote the subgroup of $\text{Diff}^+(\Sigma_g)$ that fixes x . This group acts on $\text{Emb}_B^\theta(\Sigma_g, M)$ freely via precomposition and we denote the quotient by $\mathcal{E}_B^\theta(\Sigma_g, M)$. Elements in this space will be called **pointed subsurfaces of M**

REMARK 10.1. In this situation, one could replace M by a manifold with boundary and Σ_g by $\Sigma_{g,b}$ and include some boundary condition δ^T into this notation. Since all proofs presented here, will work verbatim for the pointed case with boundary, the results hold in that case as well.

As before the following lemma will play a central role.

LEMMA 10.2. *The forgetful map $\pi_\theta: \mathcal{E}_B^\theta(\Sigma_g, M) \rightarrow \mathcal{E}_B^+(\Sigma_g, M)$ is a Hurewicz fibration..*

The proof is verbatim the same as the proof of Lemma 2.12. One only has to replace the occurring homeomorphism and diffeomorphism groups by their pointed analogues and note that $\mathcal{E}_B^+(\Sigma_g, M)$ is $\text{Diff}_{x_M}(M)$ -locally retractile, where this is the group of diffeomorphisms fixing $T_{x_M}M$ pointwise.

10.2. Stabilization Maps for Pointedly Embedded Subsurfaces

The goal of this subsection is to construct stabilization maps

$$\sigma_g: \mathcal{E}_B^\theta(\Sigma_g, M) \rightarrow \mathcal{E}_B^\theta(\Sigma_{g+1}, M)$$

As the first and most complicated step, we will show that there are homotopy equivalent spaces of subsurfaces where the subsurfaces meet a neighbourhood of x_M in a prescribed way. From there it will be quite easy to define the stabilization maps.

Let V denote a coordinate chart centered around x_M diffeomorphic to \mathbb{R}^n such that the fixed tangent plane B corresponds to \mathbb{R}^2 in \mathbb{R}^n , where this inclusion is given by specifying the last $n-2$ coordinates to be zero. Fix once and for all $0 < R \in \mathbb{R}$ and we write $D_R^2 := \mathbb{R}^2 \cap B_R(0)$.

DEFINITION 10.3. $\mathcal{E}_{B,\text{flat}}^\theta(\Sigma_g, M)$ is the subspace of $\mathcal{E}_B^\theta(\Sigma_g, M)$ consisting of those pointed subsurfaces such that their intersection with the open ball of a radius R in V equals D_R^2 , and such that the tangential structure is constantly B on $B_{\frac{R}{2}}(0)$ with respect to some fixed trivialization of $B_2(D_R^2)$.

We will need the following two results to define pointed stabilization maps. The first proposition will deal with the case without tangential structure and its proof will occupy most of this section, the subsequent corollary will extend this to the case with tangential structures.

PROPOSITION 10.4. *There is a coordinate chart centered at x_M such that the inclusion*

$$\mathcal{E}_{B,\text{flat}}^+(\Sigma_g, M) \rightarrow \mathcal{E}_B^+(\Sigma_g, M)$$

is a homotopy equivalence with homotopy inverse denoted by Ψ_{flat} .

REMARK 10.5. It is far easier to prove that this inclusion is a weak homotopy equivalence, but we will need a homotopy inverse to the inclusion (i.e. maps that flatten the subsurfaces) to construct the stabilization maps.

COROLLARY 10.6. *The inclusion $\mathcal{E}_{B,\text{flat}}^\theta(\Sigma_g, M) \rightarrow \mathcal{E}_B^\theta(\Sigma_g, M)$ is a homotopy equivalence with homotopy inverse denoted by Ψ_{flat} as well.*

PROOF. We define $\mathcal{E}_{B,\text{flat}}^\theta(\Sigma_g, M)^*$ as $\pi_\theta^{-1}(\mathcal{E}_{B,\text{flat}}^+(\Sigma_g, M))$, where

$$\pi_\theta: \mathcal{E}_B^\theta(\Sigma_g, M) \rightarrow \mathcal{E}_B^+(\Sigma_g, M)$$

denotes the projection. Note that the following diagram is a pullback diagram:

$$\begin{array}{ccc} \mathcal{E}_{B,\text{flat}}^\theta(\Sigma_g, M)^* & \hookrightarrow & \mathcal{E}_B^\theta(\Sigma_g, M) \\ \downarrow & & \downarrow \\ \mathcal{E}_{B,\text{flat}}^+(\Sigma_g, M) & \hookrightarrow & \mathcal{E}_B^+(\Sigma_g, M) \end{array}$$

It is a standard fact that a pullback of a homotopy equivalence along a Hurewicz-fibration produces an induced map, which is also a homotopy equivalence (see for example Proposition 2.3 in [Fra]). This fact together with Lemma 10.2 (which implies that the right arrow is a fibration) implies that the top map is a homotopy equivalence. Furthermore the homotopy inverse is a map over the homotopy inverse of the base map.

So all that is left to do is trivializing the tangential structure on the disk but this is easily done by using a fixed trivialization of $B_2(D_R^2) \cong D_R^2 \times \theta^{-1}(\text{Gr}_2(T_0 D_R^2))$ and just continuously extend the tangential structure of B to $D_{\frac{R}{2}}^2$. \square

Proof of Proposition 10.4 In order to proof Proposition 10.4, we will define two subspaces of $\mathcal{E}_B^+(\Sigma_g, M)$, which both contain $\mathcal{E}_{B,\text{flat}}^+(\Sigma_g, M)$, and fit in the following diagram

$$\begin{array}{ccccccc} & & \Psi_1 & & \Psi_2 & & \Psi_3 \\ & & \curvearrowright & & \curvearrowright & & \curvearrowright \\ \mathcal{E}_B^+(\Sigma_g, M) & \longleftarrow & \mathcal{E}_{B,R}^+(\Sigma_g, M) & \longleftarrow & \mathcal{E}_{B,\text{almost}}^+(\Sigma_g, M) & \longleftarrow & \mathcal{E}_{B,\text{flat}}^+(\Sigma_g, M) \end{array}$$

The hooked arrows are inclusions and the Ψ_i represent their respective homotopy inverses. We will denote $\Psi_3 \circ \Psi_2 \circ \Psi_1$ by Ψ_{flat} . Let us define these spaces:

DEFINITION 10.7. Let $\pi_2: \mathbb{R}^n \rightarrow \mathbb{R}^2$ denote the projection onto the first two coordinates i.e. the map that sends (x_1, \dots, x_n) to (x_1, x_2) . For $r \in (0, \infty)$, we define

$$\text{Emb}_{B,r}^+(\Sigma_g, M) := \left\{ e \in \text{Emb}_B^\theta(\Sigma_g, M) \left| \begin{array}{l} \exists U_x \text{ a closed neighborhood of } x \text{ s.th.} \\ \pi_2 \circ e|_{U_x} \text{ is a diffeomorphism} \\ \text{with image } \overline{B_r(0)} \subset \mathbb{R}^2 \end{array} \right. \right\}$$

The action of $\text{Diff}_x(\Sigma_g)$ via precomposition on $\text{Emb}_B^\theta(\Sigma_g, M)$ restricts to a free action on the subspace $\text{Emb}_{B,r}^+(\Sigma_g, M)$ and we denote the quotient by $\mathcal{E}_{B,r}^+(\Sigma_g, M)$. For a $W \in \mathcal{E}_{B,r}^+(\Sigma_g, M)$ we will write $W_{r,x}$ for the image of U_x in M .

Let $\mathcal{E}_{B,\text{almost}}^+(\Sigma_g, M)$ denote the subspace of $\mathcal{E}_B^+(\Sigma_g, M)$ of such subsurfaces W such that $D_R^2 \subset W \cap B_R(0)$

REMARK 10.8. Note that the subtle difference between $\mathcal{E}_{B,\text{almost}}^+(\Sigma_g, M)$ and $\mathcal{E}_{B,\text{flat}}^+(\Sigma_g, M)$ is that subsurfaces in the latter intersect $B_R(0)$ exactly in D_R^2 , while surfaces in the former only have to contain D_R^2 , but there can be more parts of these surfaces that intersect $B_R(0)$.

After the following three auxillary lemmas we will prove that all the aforementioned inclusions are indeed homotopy equivalences by constructing their inverse and the necessary homotopies. This will finish the proof of Proposition 10.4.

LEMMA 10.9. *Let $\iota: X' \rightarrow X''$ denote an inclusion and let $\Psi_\iota: X'' \rightarrow X' \subset X''$ be a map such that there exists a homotopy $H: X'' \times [0, 1] \rightarrow X''$ such that $H(-, 0) = \text{Id}_{X''}$ and $H(-, 1) = \Psi_\iota$ and $H(X' \times [0, 1]) \subset X'$, then ι is a homotopy equivalence with homotopy inverse Ψ_ι .*

PROOF. H is already a homotopy between $\iota \circ \Psi_\iota$ and the identity of X'' , therefore we only need to construct a homotopy between $\Psi_\iota \circ \iota$ and the identity on X' . But $\text{Id}_{X''}|_{X'} = \text{Id}_{X'}$, therefore the necessary homotopy is again given by H . \square

This lemma will be used to prove that the Ψ_i are homotopy inverse to the aforementioned inclusions. The next rather technical lemma will be used towards the end of the proof of Proposition 10.4 to show that a certain map is continuous.

LEMMA 10.10. *Suppose we have a diagram of the following form:*

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{g} & \mathbb{R} \cup \{\infty\} \\ \downarrow \pi_1 & & \\ X_1 & \xrightarrow{g_{\text{inf}}} & \mathbb{R} \cup \{\infty\} \end{array}$$

where X_2 is compact and $g_{\text{inf}}(x_1) = \inf_{x_2 \in X_2} g(x_1, x_2)$. Then g_{inf} is continuous.

PROOF. Since intervals of the form $(C, \infty]$ and $(-\infty, C)$ form a subbasis of the topology of $\mathbb{R} \cup \{\infty\}$ it suffices to show that their preimages are open. We will first show that $g_{\text{inf}}^{-1}((C, \infty])$ is open for every C . For this one only has to note that $g_{\text{inf}}^{-1}((C, \infty]) = \pi_1((g^{-1}((-\infty, C]))^c)$ which is open since g is continuous and π_1 is closed since X_2 is compact.

Now we will proof that $g_{\text{inf}}^{-1}((-\infty, C])$ is open, observe that $g_{\text{inf}}^{-1}((-\infty, C]) = \pi_1(g^{-1}((-\infty, C)))$. This is open since π_1 is an open map. \square

Lastly we need the following lemma before we can start the proof of Proposition 10.4.

LEMMA 10.11. $\mathcal{E}_{B,r}^+(\Sigma_g, M)$ is an open subset of $\mathcal{E}_B^+(\Sigma_g, M)$ for every $r \in (0, \infty)$ and if ϵ_k denotes a sequence of positive real numbers that converges to zero, then

$$\mathcal{E}_B^+(\Sigma_g, M) = \bigcup_k \mathcal{E}_{B,\epsilon_k}^+(\Sigma_g, M)$$

PROOF. By the inverse function theorem, for every $W \in \mathcal{E}_B^+(\Sigma_g, M)$ there is an $r > 0$ such that $\pi_2|_{\pi_2^{-1}(\overline{B_r(0)} \cap W)}$ restricted to the connected component of $\pi_2^{-1}(\overline{B_r(0)} \cap W)$ containing x_M is a diffeomorphism. This implies that every $W \in \mathcal{E}_B^+(\Sigma_g, M)$ is contained in some $\mathcal{E}_{B,r}^+(\Sigma_g, M)$ for r small enough.

To finish the proof we have to show that $\mathcal{E}_{B,r}^+(\Sigma_g, M)$ is an open subset of $\mathcal{E}_B^+(\Sigma_g, M)$, which is equivalent to its preimage in $\text{Emb}_B^+(\Sigma_g, M)$ being open. Take some embedding e in this set. There is some closed neighborhood U_x of x in Σ_g such that $\pi_2|_{e(U_x)}$ is a diffeomorphism onto $\overline{B_r(0)}$. Furthermore this implies that the differential of $\pi_2 \circ e$ is an isomorphism on these points. Since the isomorphisms form an open subset of all linear maps we conclude that there is actually an open neighborhood U'_x of U_x such that on the closure of this neighborhood the differential of $\pi_2 \circ e$ is still an isomorphism. Since $\pi_2 \circ e|_{\overline{U'_x}}$ is proper because $\overline{U'_x}$ is compact and a local diffeomorphism onto \mathbb{R}^2 by the requirement that the differential is an isomorphism, we conclude that it is a covering of its image. Since a closed disk can only cover a closed disk via a diffeomorphism we see that this map is actually a diffeomorphism onto some closed disk in \mathbb{R}^2 containing $\overline{B_r(0)}$. Let 2ε denote $\min_{x' \in \partial U'_x} \|\pi_2 \circ e(x')\| - r > 0$. Consider the following neighbourhood of e :

$$U_e := \left\{ e' \in \text{Emb}_B^+(\Sigma_g, M) \left| \begin{array}{l} \sup_{U'_x} \|e - e'\| < \varepsilon \\ D\pi_2 \circ e'|_{TU'_x} \text{ is an isomorphism} \end{array} \right. \right\}$$

With our previous considerations we conclude that U_e lies in the preimage of $\mathcal{E}_{B,r}^+(\Sigma_g, M)$ in the embedding space $\text{Emb}_B^+(\Sigma_g, M)$. As it is by definition an open neighbourhood in the C^1 topology and the C^∞ topology is finer than all the C^k topologies we conclude that it is in fact an open neighborhood of e . \square

LEMMA 10.12. If V sits inside another coordinate chart V' and the coordinate change is given by $(x_1, \dots, x_n) \mapsto (\arctan(x_1), \dots, \arctan(x_n))$, then the inclusion $\mathcal{E}_{B,r}^+(\Sigma_g, M) \rightarrow \mathcal{E}_B^+(\Sigma_g, M)$ is a homotopy equivalence for any $r \in (0, \infty)$ with homotopy inverse Ψ_1 .

PROOF. By Theorem 42.3 and the corollary in 27.4 in [KM97] (together with the fact that the quotient of a nuclear spaces by a closed subspaces is a nuclear space) we conclude that $\mathcal{E}_B^+(\Sigma_g, M)$ is paracompact and Hausdorff and by Lemma 27.8 in the same source, we conclude that it is actually metrizable (This line of argument was communicated to me in [hm]). Therefore we can find a continuous function $\varphi_{\text{radius}}: \mathcal{E}_B^+(\Sigma_g, M) \rightarrow (0, r]$, which is r on $\mathcal{E}_{B,\text{flat}}^+(\Sigma_g, M)$ and for every W it is smaller or equal than the maximal r' such that $\mathcal{E}_{B,r'}^+(\Sigma_g, M)$ contains W (This uses the fact that $\mathcal{E}_{B,\text{flat}}^+(\Sigma_g, M)$ is closed in $\mathcal{E}_B^+(\Sigma_g, M)$).

Let us now construct the homotopy inverse to the inclusion from $\mathcal{E}_{B,R}^+(\Sigma_g, M) \rightarrow \mathcal{E}_B^+(\Sigma_g, M)$. Let $\psi_1: (0, \infty) \rightarrow \text{Diff}_c^+(M)$ denote the following continuous map into the compactly supported diffeomorphisms: Fix a smooth monotone function $\varphi: \mathbb{R} \rightarrow [0, 1]$ such that $\varphi \equiv 0$ on an open neighborhood of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\varphi \equiv 1$ outside of $(-\pi, \pi)$. We define ψ_1 via the following formula:

$$\psi_1(t)((x_1, \dots, x_n)) = \left(\left(\frac{r}{t}\varphi(x_1) + (1 - \varphi(x_1))\right)x_1, \left(\frac{r}{t}\varphi(x_2) + (1 - \varphi(x_2))\right)x_2, x_3, \dots, x_n \right)$$

Since this is the identity outside the cube $(-\pi, \pi)^n$ we can consider this as a compactly supported diffeomorphism of M by extending it via the identity. Let Ψ_1 denote

$$\begin{aligned} \Psi_1: \mathcal{E}_B^+(\Sigma_g, M) &\rightarrow \mathcal{E}_B^+(\Sigma_g, M) \\ W &\mapsto \psi_1(\varphi_{\text{radius}}(W))(W) \end{aligned}$$

This maps $\mathcal{E}_B^+(\Sigma_g, M)$ into $\mathcal{E}_{B,r}^+(\Sigma_g, M)$. Furthermore this gives us a homotopy equivalence between $\mathcal{E}_{B,r}^+(\Sigma_g, M)$ and $\mathcal{E}_B^+(\Sigma_g, M)$ since φ_{radius} can be homotoped to be the map that is constantly r via linear interpolation and such a homotopy induces maps $\mathcal{E}_B^+(\Sigma_g, M) \times [0, 1] \rightarrow \mathcal{E}_B^+(\Sigma_g, M)$ as in Lemma 10.9. Hence Ψ_1 is indeed a homotopy inverse to the inclusion. \square

LEMMA 10.13. *If V sits inside another coordinate chart V' and the coordinate change is given by $(x_1, \dots, x_n) \mapsto (\arctan(x_1), \dots, \arctan(x_n))$, then the inclusion $\mathcal{E}_{B,\text{almost}}^+(\Sigma_g, M) \rightarrow \mathcal{E}_{B,r}^+(\Sigma_g, M)$ is a homotopy equivalence with homotopy inverse denoted by Ψ_2 .*

PROOF. Fix a smooth function $\varphi_1: \mathbb{R}^2 \rightarrow \mathbb{R}$ that is invariant under rotation and 1 on $\overline{B_R(0)}$ and 0 outside of $B_{2R}(0)$. We define a map

$$H: \mathcal{E}_{B,2R}^+(\Sigma_g, M) \times [0, 1] \rightarrow \mathcal{E}_{B,2R}^+(\Sigma_g, M)$$

as follows: Note that π_2 extends to V' . For an $x' \in B_{2R}(0)$ we will denote the corresponding point in W_{2R,x_M} by $W(x')$. Fix a smooth monotone function $\varphi_2: [[0, \infty) \rightarrow \mathbb{R}$ such that $\varphi_2^{-1}(1) = [0, \sqrt{n}\frac{\pi}{4}]$ and $\varphi_2^{-1}(0) = (2\sqrt{n}\pi, \infty)$. Now we define

$$\begin{aligned} \psi_2: \mathcal{E}_{B,2R}^+(\Sigma_g, M) \times [0, 1] &\rightarrow \text{Diff}_c^+(M) \\ \psi_2(W, t)(x') &= x' + \varphi_2(\|\pi_2(x')\|) \varphi_1(\pi_2(x')) (\pi_2 x'(-) W(x')) \end{aligned}$$

Note that $W(x')$ is only defined when $\|\pi_2(x')\| \leq 2R$ but in that case $\varphi_1(\pi_2(x')) = 0$. The map ψ_2 is well-defined i.e. the image is indeed a diffeomorphism, since a map

$$\begin{aligned} \mathbb{R}^{n-2} &\rightarrow \mathbb{R}^{n-2} \\ x' &\mapsto x' + g(\|x'\|) v \end{aligned}$$

for v some vector and a positive, monotonously decreasing and bounded function g , is a diffeomorphism if $\max \|g\| \|v\| < 1$ because this ensure the derivative to be an isomorphism which implies that the map is locally injective. Furthermore such a map extends to a degree 1 map from S^{2n-1} into itself and therefore it also has to be surjective. Hence it is a diffeomorphism, if the derivative of g is small enough, but we can chose g a posteriori with a small enough derivative.

Lastly we have to argue that ψ_2 is a continuous map as can be seen since it can be written in a neighborhood of W in terms of the image of W and the sections of the "normal" bundle of W (where we require the "normal" bundle to have $\{0\} \times \mathbb{R}^{n-2}$ as fiber over W_{2R,x_M}), which is a neighborhood of W in $\mathcal{E}_{B,2R}^+(\Sigma_g, M)$.

We define $H(W, t) = \psi_2(W, t)(W)$. Note that $\mathcal{E}_{B,R}^+(\Sigma_g, M)$ includes into $\mathcal{E}_{B,2R}^+(\Sigma_g, M)$ and with a similar construction as in Lemma 10.12 we can see that this inclusion is actually a homotopy equivalence with homotopy inverse, which is given by postcomposing with a diffeomorphism of M , denoted by $\Psi_{R,2R}$. Note that H fixes $\Psi_{R,2R}(\mathcal{E}_{B,\text{almost}}^+(\Sigma_g, M))$ and hence by Lemma 10.9, we conclude that the inclusion $\mathcal{E}_{B,\text{almost}}^+(\Sigma_g, M) \rightarrow \mathcal{E}_{B,R}^+(\Sigma_g, M)$ followed by $\Psi_{R,2R}$ is a homotopy equivalence with homotopy inverse $H(-, 1)$. We get the following diagram:

$$\begin{array}{ccc} & \mathcal{E}_{B,2R}^+(\Sigma_g, M) & \\ \Psi_{R,2R} \nearrow & & \searrow H(-,1) \\ \mathcal{E}_{B,R}^+(\Sigma_g, M) & \xleftarrow{\cong} & \mathcal{E}_{B,\text{almost}}^+(\Sigma_g, M) \end{array}$$

This diagram shows that $H(-, 1) \circ \Psi_{R, 2R}$ is a homotopy inverse to the inclusion of $\mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M)$ into $\mathcal{E}_{B, R}^+(\Sigma_g, M)$ and we denote it by Ψ_2 . \square

Lastly we have the following lemma which finishes the proof of Proposition 10.4.

LEMMA 10.14. *The inclusion $\mathcal{E}_{B, \text{flat}}^+(\Sigma_g, M) \rightarrow \mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M)$ is a homotopy equivalence with homotopy inverse denoted by Ψ_3 .*

PROOF. Consider the function $\varphi: \mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M) \rightarrow \mathbb{R}$ that sends W to $\inf_{x' \in W \setminus D_R^2} \|x'\|$. Note that φ is bigger than zero for every surface and bounded by R . Assume for now that this function is continuous. We define $\psi_3: [0, R] \times [0, 1] \rightarrow \text{Diff}_c^+(\mathbb{R}^n)$ as follows: $\psi_3(t_1, t_2)$ is the diffeomorphism that is the identity outside the ball of radius $2R$ and the identity for $t_2 = 0$, and $\psi_3(t_1, t_2)$ sends the ball of radius t_1 bijectively onto the ball of radius $Rt_2 + (1 - t_2)t_1$ and finally only acts on the norm component if an element in \mathbb{R}^n is represented in polar coordinates. We then define

$$H: \mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M) \times [0, 1] \rightarrow \mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M)$$

$$(W, t) \mapsto \psi_3\left(\frac{\varphi(W)}{2}, t\right)(W)$$

Then $H(-, 1)$ is a homotopy inverse to the inclusion of $\mathcal{E}_{B, \text{flat}}^+(\Sigma_g, M)$ into $\mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M)$ by Lemma 10.9. We denote it by Ψ_3 .

So the only thing left to prove is that φ is continuous. Let $\text{Emb}_{B, \text{almost}}^+(\Sigma_g, M)$ denote the subspace of those embeddings in $\text{Emb}_B^+(\Sigma_g, M)$ whose $\text{Diff}_x(\Sigma_g)$ -equivalence class lie in $\mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M)$ and we denote by π_{Emb} the projection of this space to $\mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M)$. Let $\text{ev}: \text{Map}(X_1, X_2) \times X_1 \rightarrow X_2$ denote the canonical evaluation map. Let $\bar{A} = \text{ev}^{-1}((D_R^2)^c) \subset \text{Emb}_{B, \text{almost}}^+(\Sigma_g, M) \times \Sigma_g$. $\text{Diff}_x(\Sigma_g)$ acts on $\text{Emb}_{B, \text{almost}}^+(\Sigma_g, M)$ as well as \bar{A} (Here we let $\text{Diff}_x(\Sigma_g)$ from the left on the Σ_g component of \bar{A}). We denote the quotient of this action of \bar{A} by A .

The projection onto the first factor from \bar{A} to $\text{Emb}_{B, \text{almost}}^+(\Sigma_g, M)$ descends to a map from A to $\mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M)$ which we denote by π_A and we claim that this map is a fiber bundle. Let $W \in \mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M)$ denote an embedded subsurface and $N_{W, M}$ a tubular neighborhood of W with corresponding projection π_W . Consider the following set

$$U_W = \left\{ W' \in \mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M) \mid W' \subset N_{W, M} \text{ and } \pi_W|_{W'} \text{ is a diffeomorphism} \right\}$$

which is open because $N_{W, M}$ is open and the space of diffeomorphisms is an open subset of $\text{Map}(W', W)$ (see Section 1.2 of the second chapter of [Cer61]). We will prove that π_A is a locally trivial fibration. Note that

$$\begin{aligned} \pi_A^{-1}(U_W) &\rightarrow \pi_A^{-1}(\{W\}) \times U_W & \text{and} & & \pi_A^{-1}(\{W\}) \times U_W & \rightarrow \pi_A^{-1}(U_W) \\ [e, x'] &\mapsto ([e_W, \pi_W(x')], [e]) & & & ([e_W, x'], [e]) & \mapsto [e, \pi_W|_{W'}^{-1}(x')] \end{aligned}$$

are inverses of another. Here e_W is any representative of W . Therefore we conclude that π_A is a locally trivial fibration, whose fiber is given by Σ_g with an open disc removed.

So we have the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\text{ev}} & M \xrightarrow{\|-\|} \mathbb{R} \cup \{\infty\} \\ \downarrow \pi_A & & \\ \mathcal{E}_{B, \text{almost}}^+(\Sigma_g, M) & \xrightarrow{\varphi} & \mathbb{R} \cup \{\infty\} \end{array}$$

where ev denotes again the evaluation map, which descends from $\text{Emb}_{B, \text{almost}}^+(\Sigma_g, M) \times \Sigma_g$ to A and $\|-\|$ denotes the norm mapping in the chart V extended by ∞ to the complement of V . We give $\mathbb{R} \cup \{\infty\}$ the topology coming from open sets in \mathbb{R} and neighborhoods of ∞ have the form

$(C, \infty]$. Note that $\varphi(W)$ is given by taking the infimum over the composition of the horizontal maps for all elements in $\pi_A(W)$ and this always lands in $(0, R]$. Since π_A is locally a trivial fibration, Lemma 10.10 shows that φ is locally continuous and therefore continuous. \square

This finishes the proof of Proposition 10.4.

REMARK 10.15. At no point in the proof did we use that the submanifolds were two-dimensional. The corollary would also hold true for higher dimensional submanifolds.

To proceed with the construction of the stabilization maps we need the following corollary of our previous considerations:

COROLLARY 10.16. *There is a homotopy equivalence*

$$\Phi: \mathcal{E}_{B, \text{flat}}^\theta(\Sigma_g, M) \rightarrow \mathcal{E}_{g,1}^\theta\left(\overline{M \setminus B_{\frac{R}{2}}(x_M)}; \delta^T\right)$$

where $B_{\frac{R}{2}}(x_M)$ is the preimage of $B_{\frac{R}{2}}(x_M) \subset V$ in M and δ^T is the preimage of the intersection of B with the sphere of radius $\frac{R}{2}$ in V with the tangential structure stemming from the constant one. We denote its homotopy inverse by Ψ_{+B}

PROOF. Removing $B_{\frac{R}{2}}(x_M)$ yields a map $\Phi: \mathcal{E}_{B, \text{flat}}^\theta(\Sigma_g, M) \rightarrow \mathcal{E}_{g,1}^\theta\left(\overline{M \setminus B_{\frac{R}{2}}(x_M)}; \delta^T\right)$ its image is the subspace of $\mathcal{E}_{g,1}^\theta\left(\overline{M \setminus B_{\frac{R}{2}}(x_M)}; \delta^T\right)$ of those subsurfaces that intersect $V \setminus B_{\frac{R}{2}}(0)$ like $B \cap V \setminus B_{\frac{R}{2}}(0)$, and we will denote it by $\mathcal{E}_{g,1}^\theta\left(\overline{M \setminus B_{\frac{R}{2}}(x_M)}; \delta^T\right)^{\text{flat}}$. Evidently Φ is a homeomorphism onto its image. Furthermore the inclusion $\mathcal{E}_{g,1}^\theta\left(\overline{M \setminus B_{\frac{R}{2}}(x_M)}; \delta^T\right)^{\text{flat}} \rightarrow \mathcal{E}_{g,1}^\theta\left(\overline{M \setminus B_{\frac{R}{2}}(x_M)}; \delta^T\right)$ is a homotopy equivalence, as can be seen using collars. \square

We define the **pointed stabilization maps** σ_g via the composition indicated in the following diagram:

$$\begin{array}{ccccc} \mathcal{E}_B^\theta(\Sigma_g, M) & \xrightarrow{\Psi_{\text{flat}}} & \mathcal{E}_{B, \text{flat}}^+(\Sigma_g, M) & \xrightarrow{\Phi} & \mathcal{E}_{g,1}^\theta\left(\overline{M \setminus B_{\frac{R}{2}}(x_M)}; \delta^T\right) \\ \downarrow \sigma_g & & & & \downarrow \beta_{g,1} \\ & & & & \mathcal{E}_{g,2}^\theta\left(\overline{M \setminus B_{\frac{R}{2}}(x_M)}^1; \delta'^T\right) \\ & & & & \downarrow \alpha_{g,2} \\ \mathcal{E}_B^\theta(\Sigma_{g+1}, M) & \longleftarrow & \mathcal{E}_{B, \text{flat}}^+(\Sigma_{g+1}, M) & \xleftarrow{\Psi_{+B}} & \mathcal{E}_{g,1}^\theta\left(\overline{M \setminus B_{\frac{R}{2}}(x_M)}^2; \delta^T\right) \end{array}$$

Note that we required the boundary condition of the lower right space of subsurfaces to agree with the boundary condition of the top right space.

In the pointed case, we say that a space of θ -structures π_0 -stabilizes at g if $\sigma_{g'}$ induces an isomorphism on π_0 provided that $g' \geq g$ and a surjection for all $g' \geq g - 1$. Since σ_g is defined in terms of homotopy equivalences (Corollary 10.6 and Corollary 10.16) and stabilization maps of type α and β , one immediately obtains (using Theorem 9.2 for the second part):

THEOREM 10.17. *Suppose (M, x_M) is an at least 5-dimensional, simply-connected, pointed manifold with a k -trivial space of θ -structures of subplanes of TM $\theta: B_2(M) \rightarrow \text{Gr}_2(TM)$ which π_0 stabilizes at m , then the homology of $C(\sigma_g)$ vanishes in degrees $* \leq \min\{A(g), B(g)\}$ where A and B are defined as in Definition 7.1.*

In particular if $\text{Fib}_B(\theta)$ is simply-connected, then the homology of $C(\sigma_g)$ vanishes in degrees $ \leq \lfloor \frac{2}{3}g \rfloor$.*

11. Homological Stability for Symplectic Subsurfaces

In this last section we will explain how to use Theorem 10.17 to prove homological stability for pointed symplectic subsurfaces of a given closed simply-connected symplectic manifold (M, ω) (i.e. ω is a point-wise non-degenerate 2-form on M) of dimension $2n \geq 6$. The proof will use a classical theorem, that is proven via the h-principle, to relate symplectic subsurfaces to tangential structures as they were defined in the previous parts of this paper.

Let $\pi_{\text{Gr}}: \text{Gr}_2(TM) \rightarrow M$ denote the Grassmannian of oriented two-planes in TM and Σ_g an oriented, connected, closed surface of genus g . Fix $x \in \Sigma_g$, $x_M \in M$ and $B_\omega \in \text{Gr}_2(T_{x_M}M)$ such that $\omega|_{B_\omega}$ is non-degenerate and the orientation of B_ω agrees with the orientation induced by $\omega|_{B_\omega}$.

DEFINITION 11.1. We say an embedding $e: \Sigma_g \rightarrow M$ is an **oriented, pointed symplectic embedding** if

- (i) $e(x) = x_M$ and $De(T_x\Sigma_g) = B_\omega$
- (ii) $\omega|_{De(T_x\Sigma_g)}$ is non-degenerate.
- (iii) $\int_{\Sigma_g} e^*\omega > 0$

We equip the set of symplectic embeddings with the C^∞ -topology and we denote this space by $\text{SEmb}_{B_\omega}(\Sigma_g, M)$.

Similarly we say that a smooth map $F: \Sigma_g \rightarrow \text{Gr}_2(TM)$ is a **formal solution to the oriented symplectic embedding problem** if

- (i) $\pi_{\text{Gr}} \circ F$ is an embedding such that $(\pi_{\text{Gr}} \circ F)(x) = x_M$ and $D(\pi_{\text{Gr}} \circ F)(T_x\Sigma_g) = F(x) = B_\omega$
- (ii) $\omega|_{F(x')}$ is non-degenerate for every point $x' \in \Sigma_g$
- (iii) $\int_{\Sigma_g} (\pi_{\text{Gr}} \circ F)^*\omega > 0$

Note that the last condition ensures that the orientations of Σ_g induced by ω and by the orientation of Σ_g agree.

REMARK 11.2. For an oriented, pointed symplectic embedding $e: \Sigma_g \rightarrow M$, the Grassmannian differential $\text{Gr}(De): \Sigma_g \rightarrow \text{Gr}_2(TM)$ is a formal solution to the oriented symplectic embedding problem.

DEFINITION 11.3. We call a continuous map $H: \Sigma_g \times [0, 1] \rightarrow \text{Gr}_2(TM)$ a **solution homotopy** if

- (i) There exists an embedding $e: \Sigma_g \rightarrow M$ such that $H(-, 0)$ agrees with $\text{Gr}(De)$, $e(x) = x_M$ and $De(T_xM) = B_\omega$
- (ii) The following diagram commutes:

$$\begin{array}{ccc} \Sigma_g \times [0, 1] & \xrightarrow{H} & \text{Gr}_2(TM) \\ \downarrow \pi_{\Sigma_g} & & \downarrow \pi_{\text{Gr}} \\ \Sigma_g & \xrightarrow{e} & M \end{array}$$

- (iii) For all $t \in [0, 1]$, we have $H(x, t) = B_\omega$
- (iv) $H(-, 1)$ is a formal solution to the oriented symplectic embedding problem

The set of solution homotopies is a subspace of $\text{Emb}(\Sigma_g, M) \times \text{Map}(\Sigma_g, \text{Map}([0, 1], \text{Gr}_2(TM)))$ and hence inherits a topology, where the first factor is equipped with the C^∞ topology and the second one with the compact-open topology.

Note that there is an inclusion from the space of symplectic embeddings into the space of solution homotopies by sending an embedding to the constant solution homotopy over that embedding. The proof of the following theorem can be found in Section 12 of [EM02].

THEOREM 11.4. *If (M, ω) is an at least six-dimensional symplectic manifold, then the inclusion of the symplectic embeddings into the space of solution homotopies is a weak equivalence.*

11.1. Interpreting Solution Homotopies as Tangential Structures:

Inspired by Theorem 11.4, we want to construct a space of θ -structures of subplanes of TM such that the space of subsurfaces with that tangential structures is the space of solution homotopies.

We call an element $B_\omega \in \text{Gr}_2(TM)$ a **symplectic 2-plane** if $\omega|_{B_\omega}$ is non-degenerate and we denote by $\text{Gr}_2^\omega(TM)$ the (open) subspace of symplectic two-planes of TM . Analogously we define $\text{Gr}_2^\omega(T_{x'}M)$. For a $B'_\omega \in \text{Gr}_2(T_{x'}M)$ we will denote the homotopy fiber of the inclusion $\text{Gr}_2^\omega(T_{x'}M) \rightarrow \text{Gr}_2(T_{x'}M)$ by $\mathcal{P}_2(M)_{B'_\omega}$. More explicitly this is the space of paths in $\text{Gr}_2(T_{x'}M)$ starting at B'_ω and ending in $\text{Gr}_2^\omega(T_{x'}M)$. If we allow B'_ω to vary, we obtain the mapping path space $\mathcal{P}_2(M)_{x'}$ of the inclusion $\text{Gr}_2^\omega(T_{x'}M) \rightarrow \text{Gr}_2(T_{x'}M)$, i.e.

$$\mathcal{P}_2(M)_{x'} := \{\gamma: [0, 1] \rightarrow \text{Gr}_2(T_{x'}M) \mid \gamma(1) \in \text{Gr}_2^\omega(T_{x'}M)\}$$

This space is a subset of the following space, which is equipped with the compact open topology

$$\mathcal{P}_2(M) := \{\gamma: [0, 1] \rightarrow \text{Gr}_2(TM) \mid \gamma(1) \in \text{Gr}_2^\omega(TM) \text{ and } \pi_{\text{Gr}} \circ \gamma \text{ is constant}\}$$

If we trivialize $\text{Gr}_2(TM) \rightarrow M$ via a Darboux chart defined on $U \subset M$, we conclude that the inclusion $\text{Gr}_2^\omega(TM) \rightarrow \text{Gr}_2(TM)$ is locally equivalent to $\text{Gr}_2^\omega(T_{x'}M) \times U \rightarrow \text{Gr}_2(T_{x'}M) \times U$. This implies that

$$\begin{aligned} \theta_\omega: \mathcal{P}_2(M) &\rightarrow \text{Gr}_2(TM) \\ \gamma &\mapsto \gamma(0) \end{aligned}$$

is locally equivalent to $\mathcal{P}_2(M)_{x'} \times U \rightarrow \text{Gr}_2(T_{x'}M) \times U$, which implies that this map is a Hurewicz-fibration as it is the product of two Hurewicz-fibrations, one being the projection of the mapping path space $\mathcal{P}_2(M)_{x'} \rightarrow \text{Gr}_2(T_{x'}M)$ and the other one being the identity. Using Theorem 13 in Chapter 2.7 of [Spa95], we conclude that it is indeed a Hurewicz-fibration as it is locally a Hurewicz-fibration and the base space $\text{Gr}_2(TM)$ is paracompact.

LEMMA 11.5. *The fiber of*

$$\begin{aligned} \theta_\omega: \mathcal{P}_2(M) &\rightarrow \text{Gr}_2(TM) \\ \gamma &\mapsto \gamma(0) \end{aligned}$$

is simply-connected.

PROOF. The claim that θ_ω is indeed a fibration and hence a space of θ_ω -structures of subplanes was just shown. Using Proposition 6.9 and Corollary 9.4 it suffices to show that the fiber of θ_ω is simply-connected:

The fiber over an oriented 2-plane B'_ω in $T_{x'}M$ is given by the set of paths in $\text{Gr}_2(T_{x'}M)$ from B'_ω to $\text{Gr}_2^\omega(T_{x'}M)$. As we have mentioned before, this is the homotopy fiber of $\text{Gr}_2^\omega(T_{x'}M) \rightarrow \text{Gr}_2(T_{x'}M)$. Choose some almost complex structure J on TM compatible with the symplectic form ω . Let $V_2^\mathbb{R}(T_{x'}M)$ denote the compact 2-Stiefel manifold i.e. the space of real (orthogonal) 2-frames of $T_{x'}M$ and let $V_2^{\text{sympl}}(T_{x'}M)$ denote the space of 2-frames (v_1, v_2) such that $\omega(v_1, v_2) = 1$. Furthermore define $V_1^\mathbb{C}(T_{x'}M)$ as the compact Stiefel manifold of complex 1-frames (i.e. just non-zero vectors). Then we have an inclusion

$$\begin{aligned} V_1^\mathbb{C}(T_{x'}M) &\rightarrow V_2^{\text{sympl}}(T_{x'}M) \\ v &\mapsto (v, J(v)) \end{aligned}$$

and we claim that this is a homotopy equivalence with homotopy inverse given by the projection on the first vector. Indeed the homotopy is given by

$$((v_1, v_2), t) \mapsto (v_1, (1-t)v_2 + tJ(v_1))$$

which is easily seen to be well-defined. Therefore we get the following maps between fiber sequences:

$$\begin{array}{ccc}
S^1 \simeq Sp(2, \mathbb{R}) & \longrightarrow & SO(2) \cong S^1 \\
\downarrow & & \downarrow \\
V_1^{\mathbb{C}}(T_{x'}M) \simeq V_2^{sympl}(T_{x'}M) & \longrightarrow & V_2^{\mathbb{R}}(T_{x'}M) \\
\downarrow & & \downarrow \\
Gr_2^{\omega}(T_{x'}M) & \longrightarrow & Gr_2(T_{x'}M)
\end{array}$$

Note that $V_1^{\mathbb{C}}(T_{x'}M) \cong S^{2n-1}$ and $V_2^{\mathbb{R}}(T_{x'}M) \cong STS^{2n-1}$ (the unit sphere bundle of the tangent bundle of S^{2n-1}). Since both of these spaces are $(2n-3)$ -connected and the top horizontal map is a homotopy equivalence, we conclude that the map from $Gr_2^{\omega}(T_{x'}M) \rightarrow Gr_2(T_{x'}M)$ induces an isomorphism on π_i for $i \leq 3$ or in other words $\pi_i(Gr_2(T_{x'}M), Gr_2^{\omega}(T_{x'}M))$ for $i \leq 2$, which implies that the homotopy fiber is two-connected, since

$$\pi_{i+1}(Gr_2(T_{x'}M), Gr_2^{\omega}(T_{x'}M)) \cong \pi_i(\text{HoFib}_{B_{\omega}}(Gr_2^{\omega}(T_{x'}M) \rightarrow Gr_2(T_{x'}M)))$$

□

11.2. Spaces of Symplectic Subsurfaces and Homological Stability:

The group $\text{Diff}_x(\Sigma_g)$ acts freely on $\text{SEmb}_{B_{\omega}}(\Sigma_g, M)$ via precomposition, since being a symplectic embedding is independent of the parametrization. We denote the quotient by this group action by $\mathcal{S}_{B_{\omega}}(\Sigma_g, M)$.

Elements of this space are referred to as **pointed symplectic subsurfaces**. By a slight abuse of notation, let B_{ω} also denote the path in $\mathcal{P}_2(M)$ that is constantly B_{ω} . The inclusion of the space of symplectic embeddings into the space of solution homotopies descends to an inclusion

$$\mathcal{S}_{B_{\omega}}(\Sigma_g, M) \rightarrow \mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)$$

Since Condition (iii) in Definition 11.1 implies that ω evaluates positively on the fundamental class of pointed symplectic subsurfaces, this inclusion can not be π_0 -surjective. In order to use this inclusion later on we will alter its image. We define:

$$\begin{aligned}
\text{Emb}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)^{\omega} &:= \left\{ e \in \text{Emb}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M) \mid \int_{\Sigma_g} e^* \omega > 0 \right\} \\
\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)^{\omega} &:= \left\{ W \in \mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M) \mid \int_W \omega > 0 \right\}
\end{aligned}$$

With these definition the following corollary follows immediately:

COROLLARY 11.6. $\text{Emb}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)^{\omega}$ is the space of solution homotopies.

Observe that the stabilization map of $\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)$ defined in Section 10 restricts to $\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)^{\omega}$ namely

$$\sigma_g : \mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)^{\omega} \rightarrow \mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_{g+1}, M)^{\omega}$$

since for every $W \in \mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)$ we have $[W] = [\sigma_g(W)] \in H_2(M)$ and if $[\omega]$ denotes the real cohomology class corresponding to ω , then $\int_W \omega > 0$ is equivalent to $[\omega]([W]_{\mathbb{R}}) > 0$, where $[W]_{\mathbb{R}}$ denotes the real fundamental class. Since $\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)^{\omega}$ is a union of connected components of $\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)$ and $(\sigma_g)_* : \pi_0(\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)) \rightarrow \pi_0(\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_{g+1}, M))$ is an isomorphism, Theorem 10.17 implies that homological stability holds for $\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)^{\omega}$ as well i.e.

$$(\sigma_g)_* : H_*(\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_g, M)^{\omega}) \rightarrow H_*(\mathcal{E}_{B_{\omega}}^{\theta_{\omega}}(\Sigma_{g+1}, M)^{\omega})$$

is an isomorphism for $* \leq \frac{2}{3}g - 1$ and an epimorphism in the next degree.

REMARK 11.7. The same arguments show that even in the general case, homological stability still holds for all kinds of homological constraints we could put on $\mathcal{E}_{B_\omega}^\theta(\Sigma_{g+1}, M)$ as long as these constraints depend only on $H_2(M)$.

The following proposition allows us to relate homological stability for the space of solution homotopies to $\mathcal{S}_{B_\omega}(\Sigma_g, M)$.

PROPOSITION 11.8. $\mathcal{S}_{B_\omega}(\Sigma_g, M) \rightarrow \mathcal{E}_{B_\omega}^{\theta_\omega}(\Sigma_g, M)^\omega$ is a weak equivalence.

PROOF. We have the following map between fibrations:

$$\begin{array}{ccc} \text{Diff}_x(\Sigma_g) & \xrightarrow{\text{Id}} & \text{Diff}_x(\Sigma_g) \\ \downarrow & & \downarrow \\ \text{SEmb}_{B_\omega}(\Sigma_g, M) & \longrightarrow & \text{Emb}_{B_\omega}^{\theta_\omega}(\Sigma_g, M)^\omega \\ \downarrow & & \downarrow \\ \mathcal{S}_{B_\omega}(\Sigma_g, M) & \longrightarrow & \mathcal{E}_{B_\omega}^{\theta_\omega}(\Sigma_g, M)^\omega \end{array}$$

Note that the lower right map is a fibration by 10.2, the lower map on the left side is a fibration as a restriction of this fibration and the horizontal arrow in the middle is a weak equivalence by Theorem 11.4. Therefore we conclude from the 5-Lemma and the long exact sequence of homotopy groups for a fibration that the inclusion is in fact a weak equivalence. \square

This proposition together with Theorem 10.17 and the previously discussed fact that $\mathcal{E}_{B_\omega}^\theta(\Sigma_g, M)^\omega$ also fulfils homological stability immediately implies:

THEOREM 11.9. *Let (M, ω) denote a simply-connected symplectic manifold of dimension at least 6. For every $x_M \in M$ and every symplectic 2-plane B_ω in $T_{x_M}M$. There is a homomorphism of integral homology*

$$g_g: H_*(\mathcal{S}_{B_\omega}(\Sigma_g, M)) \rightarrow H_*(\mathcal{S}_{B_\omega}(\Sigma_{g+1}, M))$$

And this homomorphism induces an isomorphism for $ \leq \frac{2}{3}g - 1$ and an epimorphism for $* \leq \frac{2}{3}g$.*

The main problem here is that the present text does not provide an actual stabilization map for symplectic subsurfaces in the spirit of the pointed stabilization maps σ_g , but only an abstract one that uses the inverses of the aforementioned weak equivalence and therefore only exists on the level of homology.

Nevertheless the author is confident that the construction in Section 10 can be adapted to work in the symplectic setting. One would have to be more careful in choosing the deformations so that the embedded subsurfaces stay symplectic.

Furthermore there are nice symplectic embeddings of tori with a hole into \mathbb{R}^{2n} , which should provide the necessary tori in $B_R(0)$ to provide an actual stabilization map in the symplectic case, which would be defined like σ_g i.e. it would be defined by taking the connected sum with one of these symplectic tori and map symplectic subsurfaces to symplectic subsurfaces.

Bibliography

- [BF81] E. Binz and H. R. Fischer, *The manifold of embeddings of a closed manifold*, Differential geometric methods in mathematical physics (Proc. Internat. Conf., Tech. Univ. Clausthal, Clausthal-Zellerfeld, 1978), Lecture Notes in Phys., vol. 139, Springer, Berlin-New York, 1981, With an appendix by P. Michor, pp. 310–329. MR 613004
- [Cer61] Jean Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France **89** (1961), 227–380. MR 140120

- [CMRW17] Federico Cantero Morán and Oscar Randal-Williams, *Homological stability for spaces of embedded surfaces*, *Geom. Topol.* **21** (2017), no. 3, 1387–1467. MR 3650077
- [EM02] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, vol. 48, American Mathematical Society, Providence, RI, 2002. MR 1909245
- [ERW19] Johannes Ebert and Oscar Randal-Williams, *Semisimplicial spaces*, *Algebr. Geom. Topol.* **19** (2019), no. 4, 2099–2150. MR 3995026
- [Fra] Martin Frankland, *Math 527 - homotopy theory homotopy pullbacks*, http://www.home.uni-osnabrueck.de/mfrankland/Math527/Math527_0308_20170508.pdf, Accessed: 27.5.2018.
- [Hae61] André Haefliger, *Plongements différentiables de variétés dans variétés*, *Comment. Math. Helv.* **36** (1961), 47–82. MR 145538
- [Har85] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, *Ann. of Math. (2)* **121** (1985), no. 2, 215–249. MR 786348
- [Har90] ———, *Stability of the homology of the moduli spaces of Riemann surfaces with spin structure*, *Math. Ann.* **287** (1990), no. 2, 323–334. MR 1054572
- [hm] Peter Michor ([https://mathoverflow.net/users/26935/peter michor](https://mathoverflow.net/users/26935/peter%20michor)), *Paracompactness of quotient by group action*, *MathOverflow*, URL:<https://mathoverflow.net/q/299651> (version: 2018-05-07).
- [HV17] Allen Hatcher and Karen Vogtmann, *Tethers and homology stability for surfaces*, *Algebr. Geom. Topol.* **17** (2017), no. 3, 1871–1916. MR 3677942
- [HW10] Allen Hatcher and Nathalie Wahl, *Stabilization for mapping class groups of 3-manifolds*, *Duke Math. J.* **155** (2010), no. 2, 205–269. MR 2736166
- [KM97] Andreas Kriegl and Peter W. Michor, *The convenient setting of global analysis*, *Mathematical Surveys and Monographs*, vol. 53, American Mathematical Society, Providence, RI, 1997. MR 1471480
- [Kup13] A. Kupers, *Homological stability for unlinked circles in a 3-manifold*, *ArXiv:1310.8580v2* (2013).
- [Li05] Tian-Jun Li, *Existence of symplectic surfaces*, *Geometry and topology of manifolds*, *Fields Inst. Commun.*, vol. 47, Amer. Math. Soc., Providence, RI, 2005, pp. 203–217. MR 2189933
- [Mic80] Peter W. Michor, *Manifolds of differentiable mappings*, *Shiva Mathematics Series*, vol. 3, Shiva Publishing Ltd., Nantwich, 1980. MR 583436
- [MS12] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, second ed., *American Mathematical Society Colloquium Publications*, vol. 52, American Mathematical Society, Providence, RI, 2012. MR 2954391
- [Nak61] Minoru Nakaoka, *Homology of the infinite symmetric group*, *Ann. of Math. (2)* **73** (1961), 229–257. MR 131874
- [Nar15] S. Nariman, *Stable homology of surface diffeomorphism groups made discrete*, *ArXiv:1506.00033v1* (2015).
- [RW14] Oscar Randal-Williams, *Homology of the moduli spaces and mapping class groups of framed, r-Spin and Pin surfaces*, *J. Topol.* **7** (2014), no. 1, 155–186. MR 3180616
- [RW16] ———, *Resolutions of moduli spaces and homological stability*, *J. Eur. Math. Soc. (JEMS)* **18** (2016), no. 1, 1–81. MR 3438379
- [Spa95] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1995, Corrected reprint of the 1966 original. MR 1325242
- [Wah08] Nathalie Wahl, *Homological stability for the mapping class groups of non-orientable surfaces*, *Invent. Math.* **171** (2008), no. 2, 389–424. MR 2367024
- [Wal16] C. T. C. Wall, *Differential topology*, *Cambridge Studies in Advanced Mathematics*, vol. 156, Cambridge University Press, Cambridge, 2016. MR 3558600
- [Whi36] Hassler Whitney, *Differentiable manifolds*, *Ann. of Math. (2)* **37** (1936), no. 3, 645–680. MR 1503303

The Minimal Genus of Homology Classes in a Finite 2-Complex

We study surface representatives of homology classes of finite complexes which minimize certain complexity measures, including their genus and Euler characteristic. Our main result is that up to surgery at nullhomotopic curves minimizers are homotopic to cellwise coverings to the 2-skeleton. From this we conclude that the minimizing problem is in general algorithmically undecidable, but can be solved for 2-dimensional CAT(-1)-complexes.

1. Introduction

Fix a singular integral homology class $\alpha \in H_2(X)$ of a CW-complex X . We are interested in maps of closed oriented (possibly disconnected) surfaces $f: \Sigma \rightarrow X$ representing α , i.e., continuous maps f (not necessarily embeddings) such that $f_*([\Sigma]) = \alpha$. Consider the following two invariants of the representing surface (sums are taken over the connected components of Σ):

$$g(\Sigma) = \sum_{\Sigma'} g(\Sigma') \quad \text{and} \quad \chi^-(\Sigma) = \sum_{\Sigma'} \max\{0, -\chi(\Sigma')\}$$

The problem of minimizing g or χ^- among all representatives of a given homology class has been studied under various restrictions on Σ , X , and f . If X is a low-dimensional manifold, the case where one requires f to be an embedding has received a lot of attention (see below).

If Σ is connected (and not a sphere), $\chi^-(\Sigma) = 2g(\Sigma) - 2$, hence the two minimizing problems coincide under the restriction of Σ being connected. But in general, $\chi^-(\Sigma) = 2g(\Sigma) - 2n_0(\Sigma)$, where $n_0(\Sigma)$ is the number of non-spherical components of Σ . Hence the problems can (and will) differ. To capture this difference, we define the **attainable set** $\mathcal{A}(\alpha, X)$ as

$$\mathcal{A}(\alpha, X) = \{(\chi^-(\Sigma), g(\Sigma)) \mid f: \Sigma \rightarrow X, f_*([\Sigma]) = \alpha\}.$$

Let us from now on write $\chi^-(\alpha)$ and $g(\alpha)$ for the minimum of these invariants over all the surface representatives of α .

1.1. Results

In Theorem C, we provide an algorithm which computes the attainable set for some class of finite 2-complexes X , which includes all CAT(-1)-2-complexes. Conversely we show that for general finite 2-complexes one cannot compute g and χ^- .

THEOREM A (Minimal Genus is Undecidable). *Let B be a positive integer, and i be either χ^- or g . Then there is no algorithm, taking as input an (encoding of a) finite 2-complex X and a homology class $\alpha \in H_2(|X|)$, which outputs whether $i(\alpha) \leq B$.*

The restriction to 2-complexes is inessential due to the fact that the attainable set of α depends only on the image of α in the homology of the fundamental group of X (Corollary 3.6).

This paper is joint work with Mark Pedron

A crucial ingredient for our investigation of representatives of a homology class is an extension of a theorem by Edmonds ([**Edm79**]) in Theorem B. This extension can be found at the end of the introduction. It yields the following corollary.

COROLLARY 1.1. *Given a combinatorial generalized 2-complex X and a second homology class $\alpha \in H_2(X)$, for every representative (Σ, f) of α there exists a new representative (Σ', f') of α such that $\chi^-(\Sigma') \leq \chi^-(\Sigma)$, $g(\Sigma') \leq g(\Sigma)$, and f' is a cellwise covering without folds.*

By a combinatorial generalized 2-complex we mean a 2-complex whose 2-cells can be arbitrary compact, connected surfaces with boundary such that the gluing maps are componentwise either constant or a local diffeomorphism. Up to homotopy equivalence this yields the same topological spaces as ordinary 2-complexes. Having no folds means that the map is locally injective outside of a finite set contained in the preimage of the zero cells.

In Section 2 we prove basic results about the structure of attainable sets. In Section 3 we relate the attainable sets of a space to its fundamental group and in particular show that the attainable sets of a space can be retrieved from the attainable sets of its 2-skeleton. In Section 4 we will introduce generalized 2-complexes in order to prove Theorem B in Section 5. Theorem C will be proven in Section 6. Lastly in Section 7 we prove Theorem A.

1.2. Background

For three-dimensional compact oriented manifolds Thurston ([**Thu86**]) introduced a pseudo-norm on their second homology: Let M denote such a 3-manifold, then $\|-\|_{\text{T}} : H_2(M) \rightarrow \mathbb{Z}$ assigns to every homology class the minimum of χ^- over all *embedded* representatives of that class. In particular, he showed that $\|-\|_{\text{T}}$ is multiplicative, i.e. $\|n\alpha\|_{\text{T}} = n \|\alpha\|_{\text{T}}$. For non-primitive classes α these representatives are forced to be disconnected. Gromov ([**Gro82**]) introduced an a priori unrelated pseudo-norm $\|-\|_{\text{G}}$ on homology with real coefficients by considering the infimum of the ℓ^1 -norm of real cycles representing the homology class. He showed that this relates to the stabilization of χ^- via $\|\alpha\|_{\text{G}} = 2 \lim_{n \rightarrow \infty} \frac{\chi^-(n\alpha)}{n}$. Using Thurston's results and the theory of taut foliations, Gabai ([**Gab83**], Corollary 6.18) showed that $\|\alpha\|_{\text{G}} = 2 \|\alpha\|_{\text{T}}$. We can arrange these results in the following chain of inequalities:

$$\|\alpha\|_{\text{T}} \geq \chi^-(\alpha) \geq \frac{\chi^-(n\alpha)}{n} \xrightarrow{\text{Gromov}} \frac{1}{2} \|\alpha\|_{\text{G}} \stackrel{\text{Gabai}}{=} \|\alpha\|_{\text{T}}$$

Hence $\chi^-(\alpha)$, $\|\alpha\|_{\text{T}}$, and $\frac{1}{2} \|\alpha\|_{\text{G}}$ are equal, thus the Thurston norm can be defined using arbitrary (non-embedded) surfaces, *even without stabilization*. This result clearly distinguishes χ^- from g . In general, even though $\lim_{n \rightarrow \infty} \frac{\chi^-(n\alpha)}{n} = 2 \lim_{n \rightarrow \infty} \frac{g(n\alpha)}{n}$ (this is true for arbitrary topological spaces), g is not minimized by an embedded surface. Additionally, while the equality above shows that χ^- is multiplicative on homology classes, g (and even $(g-1)$) does not enjoy such multiplicative properties. Let us illustrate this with an example:

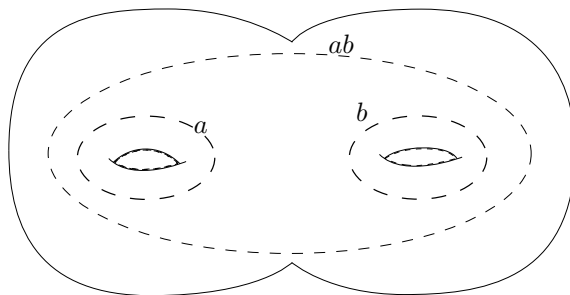


FIGURE 1. The space X in Example 1.2

EXAMPLE 1.2 (Genus 2 Handlebody With Three Tori Removed). Let X be the link complement in a genus 2 handlebody as depicted in Figure 1. If a, b denote the generators of the fundamental group of the handlebody, the 3 link components represent the conjugacy classes of a, b and ab . Each of the components is surrounded by an embedded torus. The fundamental classes of these tori are a basis of $H_2(X)$. The space X is homotopy equivalent to a 2-complex given by gluing 3 squares as depicted in Figure 2 and identifying opposite sides. The classes given by the 3 squares correspond to these fundamental classes. Let α be the sum of the 3 squares with the orientation induced by a fixed orientation of the surrounding plane in Figure 2. There are two

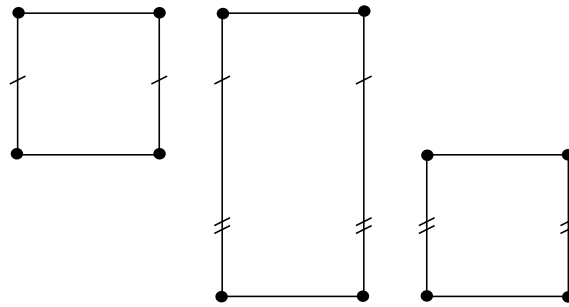


FIGURE 2. The Gluing Pattern of X in Example 1.2

obvious surface representatives of α one is given by 3 tori, the other one is a genus 2 surface bounding the handlebody. Using Corollary 1.1, an elementary but tedious consideration of small combinations of the squares shows that α can not be represented by less than 3 tori. Hence $\mathcal{A}(\alpha, X)$ looks as depicted in Figure 3. In particular, there is no representative simultaneously minimizing

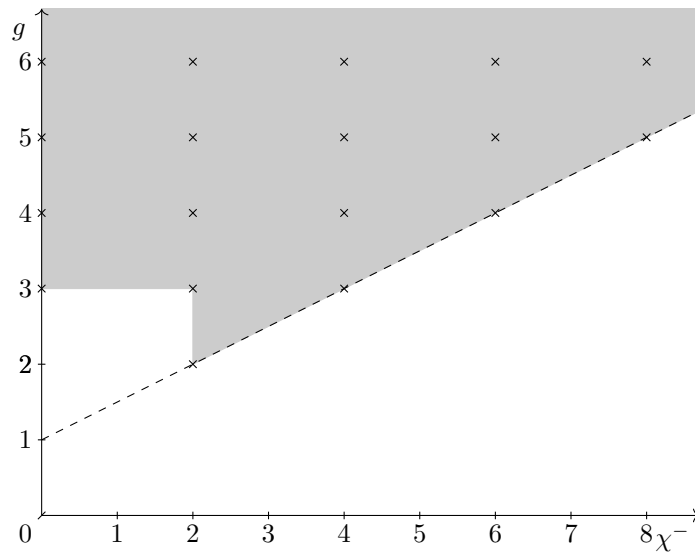


FIGURE 3. The Attainable Set of X in Example 1.2

χ^- and g . Because 2α is represented by 3 tori as well, we see that $g(2\alpha) = 3 \neq 4 = 2g(\alpha)$, hence g is not multiplicative.

For four-dimensional manifolds the situation is different, as in this case the restriction to embedded surfaces can influence the minimal genus drastically. The Thom conjecture states that given an embedded smooth projective curve with genus g in CP^2 (which is automatically connected), every other embedded representative of the same homology class has genus greater or equal than g . Using Seiberg-Witten invariants, this was proven by Kronheimer and Mrowka ([KM94]). Note that

since $\mathbb{C}P^2$ is simply-connected, every second homology class can be represented by a non-embedded sphere. Their result was further strengthened by Ozsváth and Szabó ([OS00]), again using Seiberg-Witten invariants. They showed that every connected symplectic subsurface of a symplectic four-manifold has minimal genus among all embedded representatives of the same homology class.

There is also a relative analogue of the minimal genus of a homology class, defined via extensions of maps $f: S^1 \rightarrow X$ to a connected surface with one boundary component. The minimal genus of such an extension is the commutator length of the conjugacy class of the element in $\pi_1(X, x)$ corresponding to $f: S^1 \rightarrow X$. One can also stabilize the commutator length to obtain the stable commutator length scl. See [Cal09a] for an introduction. Some of our methods are similar to approaches in that context (in particular [Che19], [Cal09b], and [Cal08]).

Similar to how commutator length carries more information than the stable commutator length, the attainable set loses most of its structure when stabilized (Lemma 2.5). Let us illustrate this with an example:

EXAMPLE 1.3 (Three Octagons). The necessary calculations for this example are deferred to Example 6.5. Consider the 2-complex X given by the gluing pattern in Figure 4 and identifying opposite sides. Each of the three cells is a closed surface of genus 2, hence its second homology has rank 3 and is generated by its cells. We consider the homology class α which is given as the sum of the three cells. The attainable set of α is depicted in the left diagram of Figure 5. In this example χ^- and g are never minimized by the same representative. Moreover, all g -minimizers are connected (up to sphere components). In the right diagram $\mathcal{A}(n\alpha, X)$ is rescaled, dividing by n . One can see how the subtle structure of the attainable set vanishes under stabilization.

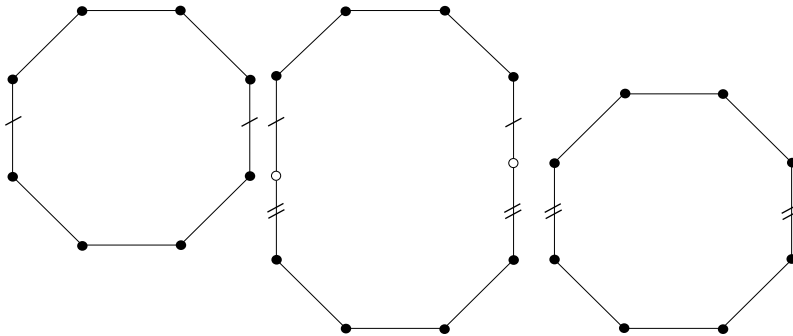


FIGURE 4. Gluing three octagons, Example 1.3

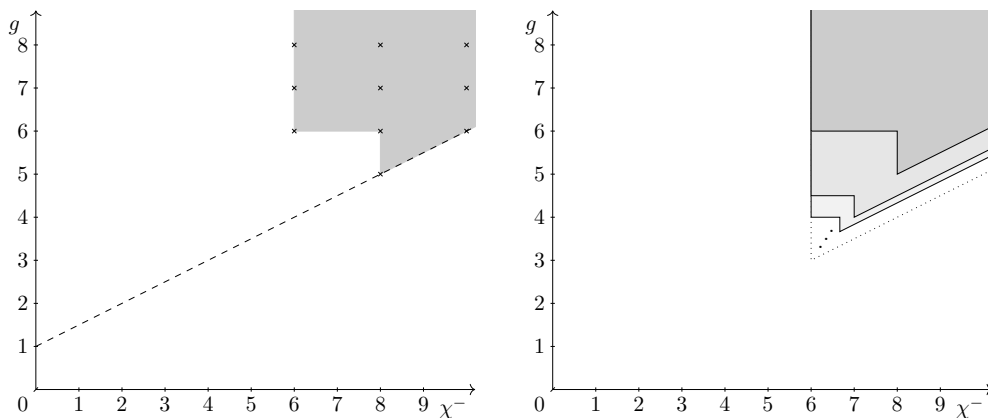


FIGURE 5. Attainable sets in Example 1.3

1.3. Compression Preorder

In order to get a better understanding of the attainable set, we consider a preorder¹ \leq_c on unparameterized homotopy classes of representatives of a homology class, such that g and χ^- are monotonous with respect to this preorder. This is based on the following construction:

DEFINITION 1.4 (Compression). Given two representatives (Σ, f) and (Σ', f') of the same homology class, we will say that (Σ', f') is **obtained from** (Σ, f) **by compression at** e if

- (i) there exists an embedding $e: S^1 \times D^1 \rightarrow \Sigma$ and an extension

$$\begin{array}{ccc} S^1 \times D^1 & \xrightarrow{e} & \Sigma \\ \downarrow & & \downarrow f \\ D^2 \times D^1 & \xrightarrow{e'} & X \end{array}$$

- (ii) there exists a homeomorphism $\phi: \Sigma' \rightarrow \Sigma \setminus e(S^1 \times D^1) \cup_e D^2 \times \partial D^1$ and a homotopy between $\left(f|_{\Sigma \setminus e(S^1 \times D^1)} \cup e'|_{D^2 \times \partial D^1} \right) \circ \phi$ and f .

Consider maps $f_i: \Sigma_i \rightarrow X$, $i \in \{1, 2\}$, representing the fixed homology class α . We define the **compression preorder** by saying that $(\Sigma_1, f_1) \leq_c (\Sigma_2, f_2)$ if (Σ_1, f_1) can be obtained from (Σ_2, f_2) by a finite sequence of the following moves:

- (I) Compression at an essential embedded annulus
- (II) Compression at an inessential embedded annulus
- (III) The inverse of move (II)

It is necessary to allow the inverse of move (II) as otherwise the preorder would not have minima². The compression preorder gives the universal measurement of geometric complexity of surface representatives of a given homology class. The important role of (χ^-, g) comes from the fact that the assignment

$$(\Sigma, f) \mapsto (\chi^-(\Sigma), g(\Sigma))$$

is strictly monotone³ from the compression preorder to the product order on the attainable set. In particular, (χ^-, g) reflects minima⁴.

This allows us to formulate Theorem B which describes minima of the compression preorder up to equivalence in the preorder⁵ of generalized 2-complexes:

THEOREM B (Normal Form). *Let X denote a combinatorial generalized 2-complex and Σ a closed oriented surface. Then every map $f: \Sigma \rightarrow X$ that admits no squeezes is equivalent in the compression preorder to a cellwise covering without folds.*

ACKNOWLEDGEMENTS. The authors would like to thank Ursula Hamenstädt for her support and many helpful discussions. We would like to thank the current and former members of the Bonn Geometry Group for introducing us to new topics and sharing their insights with us.

2. Attainable Sets

We continue to denote by X a CW-complex and by $\alpha \in H_2(X)$ an integral homology class. As described in the introduction, the invariants $\chi^-(\alpha)$ and $g(\alpha)$ are neither independent nor does one

¹i.e., $A \leq_c A$, and $A \leq_c B \wedge B \leq_c C \implies A \leq_c C$, but not necessarily $A \leq_c B \wedge B \leq_c A \implies A = B$

²A minimum A of a preorder is an element such that for all elements B , $B \leq A \implies A \leq B$.

³ $(\Sigma_1, f_1) \leq_c (\Sigma_2, f_2) \implies (\chi^-(\Sigma_1), g(\Sigma_1)) \leq (\chi^-(\Sigma_2), g(\Sigma_2))$ and $(\Sigma_1, f_1) \leq_c (\Sigma_2, f_2) \wedge (\Sigma_2, f_2) \not\leq_c (\Sigma_1, f_1) \implies (\chi^-(\Sigma_2), g(\Sigma_2)) \not\leq (\chi^-(\Sigma_1), g(\Sigma_1))$

⁴Reflects minima: $(\chi^-(\Sigma), g(\Sigma))$ minimum $\implies (\Sigma, f)$ minimum

⁵ A and B are called equivalent if $A \leq_c B \wedge B \leq_c A$

determine the other. To capture their interaction we defined the attainable set $\mathcal{A}(\alpha, X)$ as

$$\mathcal{A}(\alpha, X) = \{(\chi^-(\Sigma), g(\Sigma)) \mid f: \Sigma \rightarrow X, f_*([\Sigma]) = \alpha\}.$$

Its geometry describes the way those two invariants restrain each other. In this section we present results regarding its asymptotic structure. If α is representable by a union of spheres, its attainable set is

$$\mathcal{A}(\alpha, X) = \mathcal{A}(0, X) = \{(\chi^-, g) \in 2\mathbb{Z} \times \mathbb{Z} \mid 0 \leq \chi^- \leq 2g - 2\} \cup \{(0, 0)\}.$$

This attainable set is an exception in many ways, so from now on in this section *we assume that α is not representable by a union of spheres*, i.e., $g(\alpha) > 0$. Be aware, though, that this does not exclude multiples of α to be representable by a union of spheres.

2.1. Saturation

By the inequalities $0 \leq \chi^-(\Sigma) \leq 2g(\Sigma) - 2$ for a surface Σ representing α , the attainable set is contained in a cone whose edges are parallel to the lines defined by $\chi^- = 0$ and $\chi^- = 2g$. We show that the attainable set is invariant under saturation moves parallel to these axes.

LEMMA 2.1 (Invariance under Saturation Moves). *For $\alpha \in H_2(X)$ with $g(\alpha) > 0$, one has*

$$(0, 0) \neq (\chi^-, g) \in \mathcal{A}(\alpha, X) \implies (\chi^-, g + 1), (\chi^- + 2, g + 1) \in \mathcal{A}(\alpha, X).$$

PROOF. Let Σ be a surface representing α with $(\chi^-(\Sigma), g(\Sigma)) = (\chi^-, g)$. Taking the connected or disconnected sum with a torus at a non-spherical component changes the coordinates as required. \square

Hence $\mathcal{A}(\alpha, X)$ will agree, up to a finite set, with the set

$$(12) \quad \begin{aligned} &\{(\chi^-, g) \in 2\mathbb{Z} \times \mathbb{Z} \mid \chi^-(\alpha) \leq \chi^- \leq 2g - 2A(\alpha)\}, \\ &2A(\alpha) = \min \{2g - \chi^- \mid (g, \chi^-) \in \mathcal{A}(\alpha, X)\}. \end{aligned}$$

We will identify the meaning of this number below.

2.2. Connected Representative

Let us call a surface Σ essentially connected if it has exactly one non-spherical component. If Σ is an essentially connected surface representing α , one necessarily has $\chi^-(\Sigma) = 2g(\Sigma) - 2$. Conversely, if $(\chi^-, g) \in \mathcal{A}(\alpha, X)$ satisfies $\chi^- = 2g - 2$, one can find an essentially connected surface Σ representing α such that $(\chi^-(\Sigma), g(\Sigma)) = (\chi^-, g)$. Hence α is represented by an essentially connected surface if and only if $A(\alpha) = 1$. More generally, we have the following lemma:

LEMMA 2.2. *Let $X = \bigsqcup X_i$ be the decomposition of X into its path-components and let $\alpha = (\alpha_i)_i \in \bigoplus H_2(X_i) \cong H_2(X)$ be a homology class of X . Then $A(\alpha)$ equals the number of i such that α_i is not representable by a sphere.*

PROOF. If Σ is a surface representing α having $n_0(\Sigma)$ non-spherical components, then $2g(\Sigma) - \chi^-(\Sigma) = 2n_0(\Sigma)$. Taking the minimum of both sides over all Σ representing α proves the assertion. \square

Thus we can write $n_0(\alpha)$ instead of $A(\alpha)$. The coordinates of all surfaces Σ representing α with $n_0(\Sigma) = n_0(\alpha)$ lie on a positive ray. Hence there is a minimal point on this ray. Let us write $(\chi_c^-(\alpha), g_c(\alpha))$ for its coordinates.

PROPOSITION 2.3 (Attainable Set Bounds). *Consider $\alpha \in H_2(X)$ with $g(\alpha) > 0$.*

(a) *The attainable set is invariant under the following (conditional) saturation move:*

$$(\chi^-, g) \in \mathcal{A}(\alpha, X) \wedge \chi^- < 2g - 2n_0(\alpha) \implies (\chi^- + 2, g) \in \mathcal{A}(\alpha, X)$$

(b) $\mathcal{A}(\alpha, X) \subseteq \{(\chi^-, g) | g_c(\alpha) \leq g \wedge \chi^-(\alpha) \leq \chi^- \leq 2g - 2n_0(\alpha)\}$.

PROOF. If $\chi^-(\Sigma) < 2g(\Sigma) - 2n_0(\alpha)$, then there have to exist two non-spherical components of Σ that get mapped to the same connected component of X . Therefore one can take the connected sum of these components. This proves Part (a).

For Part (b), it is trivial that $\chi^-(\alpha) \leq \chi^-$, and Lemma 2.2 shows that $\chi^- \leq 2g - 2n_0(\alpha)$. By the saturation move of Part (a), for any point $(\chi^-, g) \in \mathcal{A}(\alpha, X)$, the point $(2g - 2n_0(\alpha), g)$ is contained in $\mathcal{A}(\alpha, X)$ as well. This point lies on the ray defining $g_c(\alpha)$. Therefore we have $g_c(\alpha) \leq g(\Sigma)$. \square

The saturation moves from Lemma 2.1 and Proposition 2.3, Part (a), together with the inequalities of Proposition 2.3, Part (b), are depicted in Figure 6.

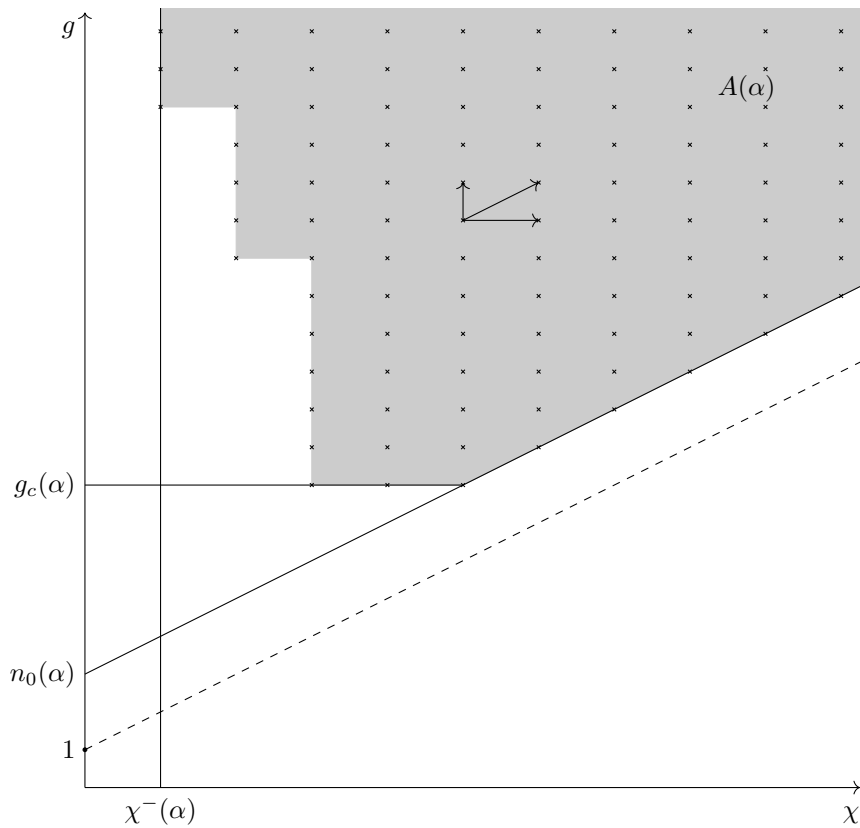


FIGURE 6. An attainable set with saturation moves and lines depicting the bounds from Proposition 2.3.

2.3. Stabilization

The relationship between $g(\alpha)$ and $\chi^-(\alpha)$ strengthens for multiples of α . Note that the limit $\lim_{n \rightarrow \infty} \frac{\chi^-(n\alpha)}{n}$ exists, since taking covers of a representative shows that the sequence is monotonically decreasing. Let us denote the limit by $\chi_{\text{stable}}^-(\alpha)$. Similarly, we define $g_{\text{stable}}(\alpha)$. One has the following well known stabilization phenomenon:

LEMMA 2.4. *For every $\alpha \in H_2(X)$, one has $\chi_{\text{stable}}^-(\alpha) = 2g_{\text{stable}}(\alpha)$.*

This stabilization phenomenon can be derived from a stabilization result for the attainable set. Let us denote the **stable attainable set** (set of limit points) $\lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{A}(n\alpha, X)$ by $\mathcal{A}_{\text{stable}}(\alpha, X)$.

LEMMA 2.5. *For any $\alpha \in H_2(X)$, one has*

$$(13) \quad \mathcal{A}_{\text{stable}}(\alpha, X) = \{(\chi^-, g) \in \mathbb{R}^2 \mid \chi_{\text{stable}}^-(\alpha) \leq \chi^- \leq 2g\}.$$

PROOF. If a multiple of α is representable by spheres, then $\chi_{\text{stable}}^-(\alpha)$ is zero and the claim follows trivially. Let us assume for the rest of the proof that no multiple of α is representable by spheres.

For every representative Σ of $n\alpha$ one has $n\chi_{\text{stable}}^-(\alpha) \leq \chi^-(\Sigma) \leq 2g(\Sigma)$. Hence $\mathcal{A}_{\text{stable}}(\alpha, X)$ is contained in the right hand side of Equation (13). It remains to show the reverse inclusion.

Given a point $(\chi^-, g) \in \mathcal{A}_{\text{stable}}(\alpha, X)$, the saturation moves described in Lemma 2.1 imply that $\mathcal{A}_{\text{stable}}(\alpha, X)$ contains all points between the rays $(\chi^-, g+r)$ and $(\chi^- + 2r, g+r)$ for $r \in \mathbb{R}_{\geq 0}$. Therefore, it suffices to show that $(\chi_{\text{stable}}^-(\alpha), 0.5 \chi_{\text{stable}}^-(\alpha)) \in \mathcal{A}_{\text{stable}}(\alpha, X)$, i.e., that there is a sequence Σ_n representing $n\alpha$ with $\chi_{\text{stable}}^-(\alpha) = \lim_{n \rightarrow \infty} \frac{\chi^-(\Sigma_n)}{n} = \lim_{n \rightarrow \infty} 2 \frac{g(\Sigma_n)}{n}$.

Let Σ_n denote a representative of $n\alpha$ such that $\chi^-(\Sigma_n) = \chi^-(n\alpha)$. Let $\Sigma_{n,m}$ denote an m -fold covering space of Σ_n such that $\Sigma_{n,m}$ has the same number of connected components as Σ_n . This implies that $m\chi^-(\Sigma_n) = \chi^-(\Sigma_{n,m}) = 2g(\Sigma_{n,m}) - 2n_0(\Sigma_n)$. We conclude that, for $m(n) = n_0(\Sigma_n)$, one has

$$\chi_{\text{stable}}^-(\alpha) = \lim_{n \rightarrow \infty} \frac{\chi^-(\Sigma_{n,m(n)})}{n \cdot m(n)} = \lim_{n \rightarrow \infty} \frac{2g(\Sigma_{n,m(n)}) - 2n_0(\Sigma_n)}{n \cdot n_0(\Sigma_n)} = \lim_{n \rightarrow \infty} \frac{2g(\Sigma_{n,m(n)})}{n \cdot m(n)}.$$

□

PROOF OF LEMMA 2.4. By Lemma 2.5 we can find a sequence Σ_n representing $n\alpha$ such that $\lim_{n \rightarrow \infty} \frac{2g(\Sigma_n)}{n} = \chi_{\text{stable}}^-(\alpha)$. Therefore we have the following chain of inequalities:

$$\chi_{\text{stable}}^-(\alpha) = \lim_{n \rightarrow \infty} \frac{2g(\Sigma_n)}{n} \geq \lim_{n \rightarrow \infty} \frac{2g(n\alpha)}{n} \geq \lim_{n \rightarrow \infty} \frac{\chi^-(n\alpha)}{n} = \chi_{\text{stable}}^-(\alpha)$$

Since both outer terms are the same, we conclude that all four terms are equal. □

Lemma 2.5 implies that the delicate structure of $\mathcal{A}(\alpha, X)$, i.e. the finite difference to the cone described in Equation (12), vanishes when passing to the stabilization of the attainable set.

2.4. Simplicial Volume and Further Invariants

From the attainable set of a homology class α one can recover $g(\alpha)$ as $\min \{g \mid (\chi^-, g) \in \mathcal{A}(\alpha, X)\}$ and $\chi^-(\alpha)$ as $\min \{\chi^- \mid (\chi^-, g) \in \mathcal{A}(\alpha, X)\}$. One can generalize this by taking the minimum of a linear combination of g and χ^- :

$$l_{p,q}(\alpha) = \min \{p\chi^- + qg \mid (\chi^-, g) \in \mathcal{A}(\alpha, X)\}$$

For example, the simplicial volume⁶ is $l_{1,2}$. By the description of the attainable set in Equation (12), $l_{p,q}$ is finite if and only if (p, q) lies in the dual cone

$$\{(p, q) \in \mathbb{R}^2 \mid 0 \leq q \wedge 2p + q \geq 0\}$$

These $l_{p,q}$ are convex and positively homogeneous in (p, q) . As a corollary of Proposition 2.3 we have:

LEMMA 2.6. *If $p \leq 0$ and $2p + q \geq 0$, then $l_{p,q}(\alpha) = (p + 2q)g(\alpha) - 2pn_0(\alpha)$.*

PROOF. For a representative Σ of α with $(\chi^-, g) = (\chi^-(\Sigma), g(\Sigma))$, one has

$$p\chi^- + qg \geq p(2g - 2n_0(\alpha)) + qg \geq (2p + q)g(\alpha) - 2pn_0(\alpha),$$

and for a representative of (χ_c^-, g_c) we have equalities. □

⁶here: minimal number of triangles in an *integral* simplicial representation, ignoring sphere components

Apart from these restraints, the $l_{p,q}$ seem to be independent.

3. The Cokernel of the Hurewicz Map

In this section we show that the attainable sets of homology classes of a topological space X are determined by the fundamental group of the space. This fact is essential because it allows us to restrict our attention to 2-complexes. More precisely we show that the attainable set of a homology class is determined by its image in $H_2(\pi_1(X, *))$ under the natural morphism $H_2(X) \rightarrow H_2(\pi_1(X, *))$. Secondly, we show in Lemma 3.4 and Corollary 3.6 how to relate the attainable sets between X and $K(\pi_1(X, x), 1)$.

For every path-connected pointed CW-complex (X, x) we have a Hurewicz-homomorphism

$$h_2: \pi_2(X, x) \rightarrow H_2(X)$$

whose cokernel is independent of the base point x . An important observation in [Hop42] is the following lemma.

LEMMA 3.1. *For every path-connected pointed CW-complex (X, x) , the natural sequence*

$$\pi_2(X, x) \xrightarrow{h_2} H_2(X) \rightarrow H_2(\pi_1(X, x)) \rightarrow 0$$

is exact, hence $H_2(\pi_1(X, x))$ is the cokernel of the Hurewicz homomorphisms.

PROOF. By gluing in cells of dimension 3 and higher we can include X into an Eilenberg-MacLane space $\iota: X \rightarrow K(\pi_1(X, x), 1)$ with ι inducing an isomorphism on fundamental groups. Let us abbreviate $K(\pi_1(X, x), 1)$ by X_1 .

Now we have the following diagram with exact rows and vertical maps given by Hurewicz homomorphisms:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_3(X_1, X, x) & \xrightarrow{\sim} & \pi_2(X, x) & \longrightarrow & 0 \\ & & \downarrow h_3 & & \downarrow h_2 & & \downarrow \\ & & H_3(X_1, X) & \longrightarrow & H_2(X) & \xrightarrow{\iota_*} & H_2(X_1) \end{array}$$

The morphism ι_* in the diagram is surjective because ι is given by adding cells of dimension 3 and higher. The morphism h_3 is surjective by the relative Hurewicz theorem. Hence ι_* is the cokernel of h_2 . \square

LEMMA 3.2. *For every path-connected pointed CW-complex (X, x) and any two classes $\alpha_1, \alpha_2 \in H_2(X)$ which agree in $H_2(\pi_1(X, x))$ one has $\mathcal{A}(\alpha_1, X) = \mathcal{A}(\alpha_2, X)$.*

PROOF. Because α_1 and α_2 agree in $H_2(\pi_1(X, x))$, by Lemma 3.1 there exists a map $f: S^2 \rightarrow X$ such that $\alpha_1 + f_*([S^2]) = \alpha_2$. This implies that $\mathcal{A}(\alpha_1, X) \subseteq \mathcal{A}(\alpha_2, X)$. Similarly one obtains the other inclusion. \square

Together with Lemma 3.2, this shows that the attainable set only depends on the induced class in the group homology of the fundamental group, and X . Therefore we extend the notation of attainable set and write $\mathcal{A}(\alpha, X)$ for a class $\alpha \in H_2(\pi_1(X, x))$ meaning the attainable set of any preimage of α in the homology of X .

Now we want to realize this homological statement on the level of spaces. Therefore we analyze how attainable sets, and more generally, the compression preorder, behave under maps between topological spaces.

DEFINITION 3.3 (Minimizer Surjective). We call a map $f: X_1 \rightarrow X_2$ between CW-complexes **minimizer surjective**, if

- (a) $f_*: H_2(X_1) \rightarrow H_2(X_2)$ is surjective and
- (b) for every $\alpha \in H_2(X_1)$ and every closed, oriented surface $f_2: \Sigma_2 \rightarrow X_2$ representing $f_*(\alpha)$ there exists a closed, oriented surface $f_1: \Sigma_1 \rightarrow X_1$ representing α and $(\Sigma_1, f \circ f_1) \leq_c (\Sigma_2, f_2)$.

Being minimizer surjective is invariant under homotopy. Note that being minimizer surjective implies that one can lift minimizers of \leq_c of representatives of a homology class up to equivalence to representatives of every class in its preimage. The following lemma shows that a minimizer surjective map allows to shift computing attainable sets between its domain and codomain in both directions.

LEMMA 3.4. *Given a minimizer surjective map $f: X_1 \rightarrow X_2$ between path-connected CW-complexes and a $\alpha \in H_2(X_1)$, then one has $\mathcal{A}(\alpha, X_1) = \mathcal{A}(f_*(\alpha), X_2)$.*

PROOF. Since postcomposing a representative of α by f yields a representative of $f_*(\alpha)$, one has $\mathcal{A}(\alpha, X_1) \subset \mathcal{A}(f_*(\alpha), X_2)$.

Now fix a point $(\chi_2^-, g_2) \in \mathcal{A}(f_*(\alpha), X_2)$. If $\chi_2^- = 2g_2 = 0$, then $f_*(\alpha)$ is represented by a sphere and, because f is minimizer surjective, α is also represented by a sphere, hence $(0, 0) \in \mathcal{A}(\alpha, X_1)$. Otherwise, $n_0(f_*(\alpha)) = 1$, which implies $\chi_2^- \leq 2g_2 - 2$. Since f is minimizer surjective, there exists a point $(\chi_1^-, g_1) \in \mathcal{A}(\alpha, X_1)$ such that $\chi_2^- \geq \chi_1^-$ and $g_2 \geq g_1$. Lemma 2.1 implies that $(\chi_1^-, g_2) \in \mathcal{A}(\alpha, X_1)$, and Proposition 2.3 (a) together with $\chi_2^- \leq 2g_2 - 2$ and $n_0(\alpha) \leq 1$ implies that $(\chi_2^-, g_2) \in \mathcal{A}(\alpha, X_1)$. \square

PROPOSITION 3.5 (Canonical Map to $K(\pi_1(X, x), 1)$ is minimizer surjective). *For a path-connected pointed CW-complex (X, x) the canonical inclusion $\iota: X \rightarrow K(\pi_1(X, x), 1)$ is minimizer surjective.*

PROOF. The space X includes into $K(\pi_1(X, x), 1)$ such that the 2-skeleton of X agrees with the 2-skeleton of $K(\pi_1(X, x), 1)$. The cellular approximation theorem implies that we can lift every map from a surface to $K(\pi_1(X, x), 1)$ along ι up to homotopy. The preimages in $H_2(X)$ of a homology class in $H_2(\pi_1(X, x))$ differ by elements in the image of the Hurewicz homomorphism, hence ι is minimizer surjective. \square

COROLLARY 3.6 (Attainable set determined by fundamental group). *Given two path-connected pointed CW-complexes (X_1, x_1) , (X_2, x_2) , and an isomorphism $f: \pi_1(X_1, x_1) \xrightarrow{\sim} \pi_1(X_2, x_2)$, the induced isomorphism*

$$H_2(\pi_1(X_1, x_1)) \xrightarrow{f_*} H_2(\pi_1(X_2, x_2))$$

preserves attainable sets, i.e., for every $\alpha \in H_2(\pi_1(X_1, x_1))$, one has $\mathcal{A}(\alpha, X_1) = \mathcal{A}(f_(\alpha), X_2)$.*

Because every group can be realized as the fundamental group of a 2-complex, this implies that the task of computing attainable sets can be reduced to the case of 2-complexes.

EXAMPLE 3.7 (Hypercube). Consider $K(\mathbb{Z}^n, 1) = (S^1)^n$ and let X be its 2-skeleton (in the product cell structure where S^1 has exactly one 0-cell and one 1-cell). Then the inclusion of X into $K(\mathbb{Z}^n, 1)$ is minimizer surjective. In this case, we can say a bit more about the minimal genus. There are canonical isomorphisms $H_1(X) \cong \mathbb{Z}^n$ and $H_2(X) \cong \bigwedge^2 \mathbb{Z}^n$. Because $\pi_1(X, x)$ is abelian, any surface Σ representing a homology class α can be compressed to a disjoint union of tori with the same genus. Hence $g(\alpha)$ is the minimal number of summands in a decomposition of α into elementary wedges:

$$\alpha = \sum_{i=1}^{g(\alpha)} v_i \wedge w_i, \quad v_i, w_i \in \mathbb{Z}^n$$

The following proposition is an easy corollary from Corollary 1.1

PROPOSITION 3.8 (Free Product is Minimizer Surjective). *The quotient map $K(G_1, 1) \sqcup K(G_2, 1) \rightarrow K(G_1 * G_2, 1)$ is minimizer-surjective.*

4. Generalized 2-Complexes

In this section we generalize the definition of 2-complexes by allowing the cells to have nonzero genus. Technically, this is not necessary for statements about computability, as all subsequent statements hold for usual 2-complexes, and any generalized 2-complex can be subdivided to be a usual 2-complex. However, this subdivision increases the number of cells significantly and complicates the subsequent computations for given spaces one might be interested in. Hence we introduce this notion to allow for concrete computations that could not be handled without the use of a computer otherwise.

DEFINITION 4.1 (Manifold Models). For each dimension 0, 1, 2, we fix models for each diffeomorphism type of compact, connected manifolds, which are either a point or have non-empty boundary:

$$\mathcal{M}(0) = \{*\} \quad \text{a point}$$

$$\mathcal{M}(1) = \{I\} \quad \text{an interval}$$

$$\mathcal{M}(2) = \{\Sigma_{g,b,o} \mid b \geq 1\} \xrightarrow[(g,b,o)]{\sim} \mathbb{N}_0 \times \mathbb{N}_{\geq 1} \times \{+, -\}.$$

where $\Sigma_{g,b,o}$ denotes a compact surface of genus g with b boundary components and o indicating whether it is orientable.

Additionally, we fix

- for each orientable model $M \in \left\{ *, I, (\Sigma_{g,b,+})_{g,b} \right\}$ an orientation,
- for each model M a closed collar $c_M: \partial M \times [0, 1] \rightarrow M$.

DEFINITION 4.2 (Generalized 2-Complex). A **generalized 2-complex** X is given by the following data:

- (i) Three sets $X(0), X(1), X(2)$, whose elements are called **cells** of dimension 0, 1, 2, respectively,
- (ii) maps M indicating the **cell type**

$$M: X(0) \rightarrow \mathcal{M}(0), M: X(1) \rightarrow \mathcal{M}(1), M: X(2) \rightarrow \mathcal{M}(2),$$

- (iii) for each cell $x \in X(d)$, a (continuous) **gluing map** to the previous skeleton (see (iv))

$$r_x: \partial M(x) \rightarrow X^{(d-1)},$$

- (iv) for each dimension $d \in \{-1, 0, 1, 2\}$, a **skeleton** $X^{(d)}$ which is, starting with $X^{(-1)} = \emptyset$, an inductive choice of a pushout (in the category of topological spaces) for the following diagram:

$$\begin{array}{ccc} \coprod_{x \in X(d)} \partial M(x) & \xrightarrow{\coprod_{x \in X(d)} r_x} & X^{(d-1)} \\ \downarrow & & \vdots \\ \coprod_{x \in X(d)} M(x) & \xrightarrow{\coprod_{x \in X(d)} \phi_x} & X^{(d)} \end{array}$$

We call $|X| = X^{(2)}$ the **geometric realization**. The components ϕ_x of the horizontal dotted arrow are called **cell inclusions**. They are injective on the interior and hence define a smooth structure on the image of the interior. One can always choose the pushout in such a way that the vertical dotted arrow is an inclusion on the underlying sets, we will silently assume this whenever it is convenient.

LEMMA 4.3 (Homology of a Generalized 2-Complex). *The second homology group of a generalized 2-complex X can be computed as*

$$H_2(|X|) = \ker \left(\bigoplus_{\substack{x \in X(2), \\ x \text{ orientable}}} H_2(M(x), \partial M(x)) \rightarrow H_1(X^{(1)}) \right).$$

In particular, if $\iota_O: X_O \subseteq X$ denotes the subcomplex of orientable cells, the inclusion $|X_O| \rightarrow |X|$ induces an isomorphism $H_2(|X_O|) \xrightarrow{\sim} H_2(|X|)$.

PROOF. This follows from the long exact sequence for the pair $(|X|, X^{(1)})$, the isomorphism $H_2(|X|, X^{(1)}) \xrightarrow{\sim} \bigoplus_{x \in X(2)} H_2(M(x), \partial M(x))$, and the fact that $H_2(M(x), \partial M(x))$ vanishes for non-orientable cells x . \square

5. Reduction to Cellwise Coverings

In this section we prove Theorem B which extends a theorem of Edmonds (Theorem 1.1 of [Edm79], see also [Sko87] for a version we will use later) from maps between surfaces to maps from surfaces to generalized 2-complexes. The theorem of Edmonds states that every map of non-zero degree between closed, connected surfaces factors up to homotopy as a composition of a pinch map and a branched covering. This theorem completely solves the description of the attainable set of surfaces: Minimizers do not allow pinches and computing g and χ^- for branched coverings shows that minimizers have to be non-branched coverings. An equivalent formulation, which emphasizes its application to the minimizer problem, is that every map of non-zero degree between closed, connected surfaces which admits no pinches is homotopic to a branched covering. Speaking precisely, it is this formulation that we extend in Theorem B.

Let us define what a pinch is. We will also need the related definition of a squeeze (both can be found in [Sko87]):

DEFINITION 5.1 (Squeeze and Pinch). Let X denote a topological space. A continuous map $f: \Sigma \rightarrow X$ **admits a squeeze** if there exists a non-nullhomotopic simple closed curve in Σ such that f maps this curve to a nullhomotopic curve in X or if Σ is S^2 and f is nullhomotopic. It is said to **admit a pinch** if there exists a closed subsurface $\Sigma' \subset \Sigma$ with a single boundary component that is not a 2-disk and such that $f|_{\Sigma'}$ is nullhomotopic.

If a map admits no squeezes, it also admits no pinches. In Theorem B we require the map to not admit squeezes, which is a stronger requirement than the one in the cited theorem. This stems from the fact that maps to generalized 2-complexes may possess complicated squeezes. The reason for considering the equivalence class in the compression preorder instead of homotopy classes as in the main theorem of [Edm79] is more subtle and stems from the presence of complicated second homotopy groups for generalized 2-complexes on the one side and restrictions for branched coverings between surfaces of genus 0 on the other side. The result suffices for studying compression preorder minimizers, because they cannot admit squeezes and are only defined in terms of equivalence classes.

In our case, the role of branched coverings in the theorem of Edmonds is taken by cellwise coverings without folds. If the target generalized 2-complex is a surface, these notions are equivalent for non-zero degree maps without squeezes, and the branch locus is within the 0-cells.

DEFINITION 5.2 (Cellwise Covering). We call a map $f: |X| \rightarrow |Y|$ between generalized 2-complexes a **cellwise covering** if the preimage of every open cell consists of open cells of the same dimension and the restriction to this preimage is a covering. We call a map $f: \Sigma \rightarrow |X|$ a **cellwise covering** if there exists a generalized 2-complex \mathcal{S} and a homeomorphism $\phi: |\mathcal{S}| \rightarrow \Sigma$ such that $f \circ \phi$ is a cellwise covering.

Furthermore we call a map $f: \Sigma \rightarrow |X|$ a **cellwise covering without folds** if it is a cellwise covering and additionally the map $f \circ \phi$ is injective in a neighborhood of the inner of the 1-cells of \mathcal{S} .

For later use we need a version of the main result of [Edm79] for surfaces with boundary (again see [Sko87] for a strengthened version). In order to state this we need some more definitions:

DEFINITION 5.3 (Geometric Degree, Allowable). A map between compact surfaces $f: \Sigma \rightarrow \Sigma'$ is called **proper** if $f^{-1}(\partial\Sigma') = \partial\Sigma$. A homotopy H between proper maps is called **proper** if $H^{-1}(\partial\Sigma') = \partial\Sigma \times I$. The **geometric degree** of a proper map $f: \Sigma \rightarrow \Sigma'$ denoted by $\mathcal{G}(f)$ is defined as the smallest natural number d such that for some 2-disk $D^2 \subset \Sigma'$ there exists a map f' that is properly homotopic to f such that $f'|_{f'^{-1}(D)}$ is a d -fold covering.

We call a proper map $f: \Sigma \rightarrow \Sigma'$ **allowable** if $f|_{\partial\Sigma}$ is a $\mathcal{G}(f)$ -fold covering.

The relative version of Edmond's result in [Sko87] states that every allowable map of non-zero geometric degree between closed surfaces that admits no pinches is homotopic relative to the boundary to a branched covering.

5.1. Block Decomposition

The most complicated behavior of generalized 2-complexes and maps to them occurs in a neighborhood of their 1-skeleton. In this subsection, we give models for neighborhoods around points in the 1-skeleton (which we call blocks), and in the following section we give a normal form result for maps into these neighborhoods. The block structure depends on the following definition:

DEFINITION 5.4 (Plasma, Membrane, Singular Set). For a 2-dimensional model manifold M as in Definition 4.1, we define the **plasma** M_p as the closure of the complement of the image of the closed collar, $M_p = \overline{M} \setminus \text{Im } c_M$, and the **membrane** M_m as the boundary of the plasma, $M_m = \partial M_p$.

For a cell x of a generalized 2-complex X , we denote by c_x the image of the cell inclusion, by p_x the image of the plasma and by m_x the image of the membrane.

We call the closure of the complement of $\bigcup_{x \in X^{(i+1)}} p_x$ in $X^{(i+1)}$ the **i -singular set of X** and denote it by $W_i(X)$. It is a closed neighborhood of $X^{(i)}$ inside $X^{(i+1)}$.

DEFINITION 5.5 (Block, Singular Set Projection). Let x denote a 2-cell, then we define the **2-block** B_x to be p_x . The closure of the complement of all 2-blocks is the 1-singular set $W_1(X)$.

For $i \in \{0, 1\}$, there is a projection $\pi_i: W_i(X) \rightarrow X^{(i)}$, given by the union of the collar projections. The left square in the following diagram is a pushout (π_x is the projection onto the first element):

$$\begin{array}{ccccc} \bigsqcup_{x \in X^{(i+1)}} \partial\mathcal{M}(x) & \xrightarrow{\times\{0\}} & \bigsqcup_{x \in X^{(i+1)}} \partial\mathcal{M}(x) \times I & \xrightarrow{(\pi_x)_x} & \bigsqcup_{x \in X^{(i+1)}} \partial\mathcal{M}(x) \\ \downarrow \bigsqcup r_x & & \downarrow \bigsqcup \phi_x \circ c_{\mathcal{M}(x)} & & \downarrow \bigsqcup r_x \\ X^{(i)} & \longrightarrow & W_i(X) & \xrightarrow{\pi_i} & X^{(i)} \end{array}$$

Therefore we can define the **singular set projection** π_i as the pushout of $\bigsqcup r_x \circ \pi_x$ and the identity $X^{(i)} \rightarrow X^{(i)}$. For every $x \in X(1)$ we define the **1-block** B_x to be $\pi_1^{-1}(p_x)$, and for every $x \in X(0)$ we define the **0-block** B_x to be $(\pi_0 \circ \pi_1)^{-1}(p_x)$. Note that all blocks are closed, X is the union of its blocks, and blocks only intersect in their boundaries. Two different i -blocks, for a fixed i , never intersect. They define a tri-colored decomposition on X .

If the gluing maps of a generalized 2-complex are too wild, they may force maps $\Sigma \rightarrow |X|$ to always have folds. Hence we need the following definition:

DEFINITION 5.6 (Combinatorial Generalized 2-Complex). We call a generalized 2-complex X **combinatorial**, if for every 2-cell x and every connected component C of $\partial M(x)$, one of the following holds:

- The restriction $r_x|_C$ is constant and the image is a 0-cell.
- The preimage of $X^{(0)}$ under $r_x|_C$ is a finite union of points and $r_x|_C$ maps every component of their complement diffeomorphically to an open 1-cell.

Here the smooth structure on the open 1-cells stems from the smooth structure of the underlying cells as explained in Definition 4.2.

REMARK 5.7. Since homotopic gluing maps give homotopy equivalent geometric realizations, every generalized 2-complex is homotopy equivalent to a combinatorial one.

DEFINITION 5.8 (Block Boundary Graph). Let X be a combinatorial generalized 2-complex. The union of all boundaries of blocks can be given a graph structure in the following way:

- The intersection of a 2-block and a 1-block is a disjoint union of closed intervals,
- the intersection of a 2-block and a 0-block is a disjoint union of closed intervals and circles, and
- the intersection of a 1-block and a 0-block is a disjoint union of stars, each of which is a graph with the middle point and the endpoints of the rays as vertices.

In total, this decomposes the union of all boundaries of blocks into closed intervals and circles which meet only in endpoints of intervals. We denote the set of intervals and circles by $E_\partial(X)$ (each element is a subset of X), and the set of endpoints of intervals by $V_\partial(X)$. Note that the interior of each element of $E_\partial(X)$ is smoothly embedded in a 2-cell.

Let us assume from here on forth that X is combinatorial. Given a 1-cell x , the 1-block belonging to x has a standard form (See Figure 7): Let n_x denote the number of incoming 2-cells counted with multiplicity, i.e., the number of connected components of the preimage of p_x under $\bigsqcup r_{x'}$, where x' ranges over all 2-cells. Then B_x is isomorphic to

$$I \times I \times \{1, \dots, n_x\} / (0, s, i) \sim (0, s, j)$$

Here the first coordinate parameterizes the collar of the adjacent 2-cells and the second coordinate parameterizes p_x .

Similarly, given a 0-cell x , we can give a standard form for its 0-blocks (See Figure 7): Let n_x denote the number of essentially incoming boundaries of 2-cells i.e. the number of preimages of $p_x = c_x$ under the restriction of all gluing maps of 2-cells to the components where the gluing map is a local diffeomorphism. Let us silently identify $\{1, \dots, n_x\}$ with this preimage. Let n_x^0 denote the number of boundaries of 2-cells that get glued inessentially to x . Again let us silently identify $\{1, \dots, n_x^0\}$ with the set of those boundaries. Analogously define k_x as the number of incoming 1-cells counted with multiplicity and let us silently identify $\{1, \dots, k_x\}$ with the incoming edges. Note that for every boundary point in an essential component of a 2-cell that maps to x under the gluing map, a small neighborhood of this boundary point maps to the incoming 1-edges. Hence every such point specifies two (possibly agreeing) elements of $\{1, \dots, k_x\}$. By fixing an orientation of the boundary of all 2-cells, we can order these two elements and hence we get two maps (picking the first resp. the second element)

$$s, t: \{1, \dots, n_x\} \rightarrow \{1, \dots, k_x\}$$

Then the 0-block belonging to x is homeomorphic to

$$\begin{array}{l} I \times I \times \{1, \dots, n_x\} \cup I \times \{1, \dots, k_x\} \\ \cup I \times S^1 \times \{1, \dots, n_x^0\} \end{array} / \begin{array}{l} (0, x, i) = (0, y, j) = (0, l) = (0, z, i) \\ (x, 0, i) = (x, s(i)) \\ (x, 1, i) = (x, t(i)) \end{array}$$

Here $x, y, l \in I$ and $z \in S^1$ and the first coordinate parameterizes the collar of the corresponding cells. We will call these coordinate descriptions of the block the **standard form of B_x** .

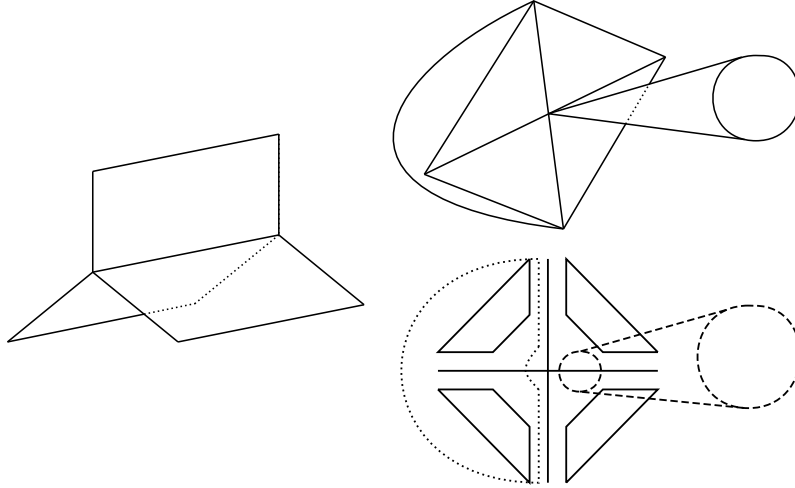


FIGURE 7. A typical 1-block and a typical 0-block. Below the 0-block is its gluing pattern as described in the standard form

5.2. Maps of Surfaces to Generalized 2-Complexes

Now that we have a model for a neighborhood of the 1-skeleton, we want to lift this along a map $\Sigma \rightarrow |X|$. This will require the following definition.

DEFINITION 5.9 (Transverse to the Membrane). Let X denote a combinatorial generalized 2-complex. We call a map $f: \Sigma \rightarrow |X|$ **transverse to the membranes** if

- (a) f is smooth when restricted to the preimage of an open 2-cell x and f is transverse to m_x
- (b) The map $\pi_1 \circ f|_{f^{-1}(W_1(X))}$ is transverse to m_x for any $x \in 1(X)$.

LEMMA 5.10. *Let X denote a combinatorial generalized 2-complex, then every map $f: \Sigma \rightarrow X$ is homotopic to a map that is transverse to the membranes.*

This lemma can be proven by first replacing the restriction of the map to the preimage of a neighborhood of the plasma by a homotopic map that is transverse to the membranes of the 2-cells and the preimage of the membranes of the 1-cells under π_1 intersected with the membrane of the 2-cell. Secondly, using the block structure one can now replace the restriction of the map to the preimage of the singular set by a map that is transverse to the preimage of the membranes of 1-cells under π_1 and that is homotopic relative to the boundary of the singular set to the original restriction. Gluing these two maps together yields a homotopic map that is transverse to the membranes. Because this uses the block structure we need the generalized 2-complex to be combinatorial.

In order to prove Theorem B we will consecutively manipulate maps in order to turn them into a cellwise covering without folds. In the first step we will tighten the map to remove unnecessary folds in the singular set. This is captured by the following definition.

DEFINITION 5.11 (No Backtracking and Allowable). Let $f: \Sigma \rightarrow |X|$ denote a map that is transverse to the membranes. Let Σ_∂ denote the preimage of the union of all boundaries of blocks (which is an embedded graph in Σ). We say that f **has no backtracking** if f is locally injective on Σ_∂ .

Let X denote a generalized 2-complex. We say that a map $\Sigma \rightarrow |X|$ is **allowable** if it is transverse to the membranes and the restriction to all preimages of plasma of 2-cells is allowable.

After replacing a map with an allowable one without backtracking, we will use the main result of [Sko87] for surfaces with boundary to turn the map into a branched covering on the preimage of open 2-cells. By pushing the branching points into the 1-skeleton, we will get an honest covering. Then we deal with 1-cells by moving potential singularities into the 0-cells. This proof strategy is captured in the following lemma:

LEMMA 5.12. *Let X denote a combinatorial generalized 2-complex and $f: \Sigma \rightarrow X$ a continuous map which admits no squeezes. Then f is equivalent in the compression preorder to \tilde{f} such that \tilde{f} satisfies everything below:*

- (a) \tilde{f} is transverse to the membranes
- (b) \tilde{f} admits no backtracking and is allowable
- (c) the restriction of \tilde{f} to the preimage of the plasma of any 2-cell is a branched covering
- (d) the restriction of \tilde{f} to the preimage of the plasma of any 2-cell is a covering
- (e) The preimage of the 1-block belonging to any x consists of rectangles which can be parameterized such that \tilde{f} is the inclusion of

$$I \times I \times \{l, m\} / (0, s, m) \sim (0, s, l) \rightarrow I \times I \times \{1, \dots, n\} / (0, s, i) \sim (0, s, j)$$

- into the standard form of the 1-block belonging to x , where $\{1, \dots, n_x\} \ni l \neq m \in \{1, \dots, n_x\}$
- (f) a connected component of the preimage of any 0-block can either be parameterized as

$$S^1 \times I / (0, z) \sim (0, z')$$

or it can be parameterized as

$$\begin{aligned} I \times I \times \mathbb{Z}/k\mathbb{Z} \cup I \times \mathbb{Z}/k\mathbb{Z} / (0, x, i) = (0, y, j) = (0, l) \\ (x, 0, i) = (x, i) \\ (x, 1, i) = (x, i + 1) \end{aligned}$$

In this parameterization \tilde{f} is componentwise the inclusion with respect to the standard form of the 0-block.

In order to prove (b) in Lemma 5.12 we will need to measure how folded a map is. This is encapsulated in the following definition.

DEFINITION 5.13 (Folding Degree d). Let X be a combinatorial generalized 2-complex, and consider the set of edges $E_\partial(X)$ of the block boundary graph, each element being an interval or circle in X . We can decompose $E_\partial(X)$ into the set of edges $E_\partial(X)_\infty^f$ where there exists a point in the edge which has an infinite preimage and its complement $E_\partial(X)_{fin}^f$. We can define $V_\partial(X)_\infty^f$ and $V_\partial(X)_{fin}^f$ analogously. We define the **folding degree** $d(f)$ of a map $f: \Sigma \rightarrow |X|$:

$$d(f) := \left(\left| E_\partial(X)_\infty^f \right| + \left| V_\partial(X)_\infty^f \right|, \sum_{C \in E_\partial(X)_{fin}^f} \sup \{ |f^{-1}(p)| \mid p \in C \setminus V_\partial(X) \} + \sum_{p \in V_\partial(X)_{fin}^f} |f^{-1}(p)| \right)$$

We denote the first coordinate by d_∞ and the second coordinate by d_{fin} .

LEMMA 5.14. *Let X be a generalized 2-complex and $f: \Sigma \rightarrow |X|$ be a map from a closed oriented surface which is transverse to the membranes, admits no squeezes, and has minimal $d(f)$ in its equivalence class of the compression preorder (Here the minimum is taken in the lexicographic order). Then f has no backtracking and $d_\infty(f) = 0$.*

PROOF. We will first show that if d is minimal, then f has to be locally injective on the preimage of membranes of 1-cells and the adjacent edges in the block boundary graph. Then we will proceed with the membranes of 2-cells. We will do this by showing that if f was not already of a particular form, then $d(f)$ could not have been minimal.

Let S denote a star in the block boundary graph consisting of a point in the membrane of a 1-cell and all its adjacent edges. The preimage of S under f consists of a collection of arcs and circles. Let us first concentrate on the arcs and then on the circles.

By transversality, f is a diffeomorphism in a neighborhood of the boundary of these arcs. Pick an arc in the preimage and denote it by a' , and fix a tubular neighborhood $N(S)$ of S such that its preimage is a tubular neighborhood of the preimage of S . Let us denote the restriction of this tubular neighborhood to a' by $N(a')$. We have to distinguish two cases: Either a' connects different boundary points of the star or it maps both endpoints to the same boundary point.

Relative to two endpoints of the star there exists a deformation of the star, linear on the rays, which deforms the star to the union of the two edges connecting the two boundary points. Using the extra coordinate of the tubular neighborhood, we can extend this homotopy to $N(S)$ such that it is the identity on the part of boundary of the tubular neighborhood that is not a tubular neighborhood of the boundary of the star. Postcomposing $f|_{N(a')}$ with the homotopy corresponding to the image of the endpoints of a' yields a new map such that the image of a' is an interval. Using an extension of the linear homotopy between $f|_{a'}$ and the unique affine map with the same endpoints as $f|_{a'}$ to $N(a')$ yields a new map, which is affine on a' .

Let us now assume that a' maps both its endpoints to the same boundary point. We can again use the deformation of the star relative to the boundary point in the image and an arbitrary second boundary point described before (if there is no second boundary point the argument works without the deformation) so that the image of a' is an interval. Since the image of the arc is contained in two rays (or a single ray) we can treat it as lying in an interval. Postcomposing the map on $N(a')$ with the isotopy depicted in Figure 8 yields a map that maps a' to the plasma of a 2-cell. The resulting map is still transverse to the membrane since it was transverse to the preimage of the membranes under the isotopy.

Doing this for all arcs in the preimage of all stars yields a new map which is locally injective on every arc in the preimage of the stars. Let us denote this map by f' .

So suppose that there exists a circle in the preimage of a star. Since f admits no squeeze this circle necessarily bounds a disk. If the restriction of f' to this disk is homotopic relative its boundary to a constant map, we can use this homotopy to remove all preimages of the block boundary graph in this disk. If the restriction of f does not admit such a homotopy we can perform a compression at its boundary and then assume that it has such a homotopy. Doing this for all circles yields a new map which is locally injective on the preimage of all stars, is transverse to the membranes and equivalent to f . If f was not already of this form, then $d(f') < d(f)$. Therefore we conclude that f was already locally injective on the preimage of stars.

Let us now focus on membranes of 2-cells. The arguments that follow are almost identical to the previous ones. As was noted before, since f is transverse to the membranes, it is a diffeomorphism in a neighborhood of a preimage of a vertex of the block boundary graph in the membranes of 2-cells.

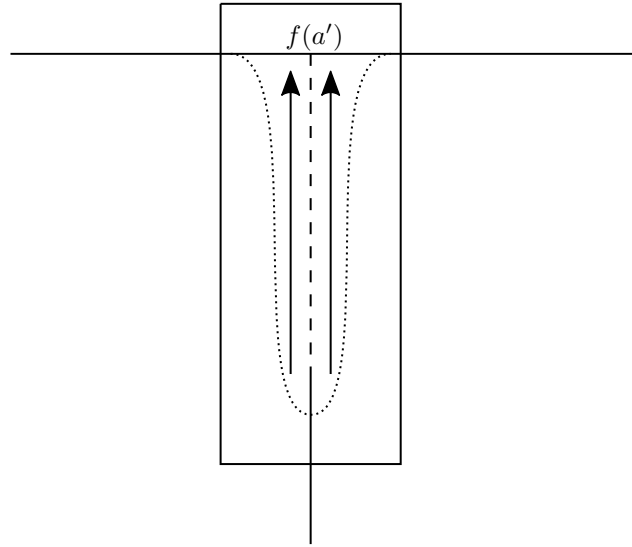


FIGURE 8. The isotopy pushes the dashed line to the part of the membrane between its endpoints. This is only defined on the interval and the indicated neighborhood not on the whole star.

Consider a single edge a in the block boundary graph of X lying in the membrane of a 2-cell. By transversality the preimage of a consists of arcs and circles. Again let us concentrate on arcs first and then deal with circles. Let us pick an arc in the preimage and denote it by a' . Fix a tubular neighborhood $N(a)$ of a such that its preimage is a tubular neighborhood of the preimage. Let us denote the restriction of this tubular neighborhood to a' by $N(a')$.

If a' connects different vertices in the block boundary graph, then using an extension of the linear homotopy between $f|_{a'}$ and the unique affine map with the same endpoints as $f|_{a'}$ to $N(a')$ yields a new map, which is affine on this arcs in the preimage. Otherwise use an isotopy on the membrane similar to the one depicted in Figure 8 to map a' completely into the image of its adjacent edge in the block boundary graph. Note that by transversality the neighboring edges have to map to the same edge in the block boundary graph.

Doing this for all arcs in the preimage of the membranes of 2-cells yields a new map. After finitely many repetitions, this process ends with a map that is locally injective except for circle components that map to the inner of edges, and we can remove those up to equivalence in the same way we did for circles in the preimage of stars. This yields a new map which is locally injective on the preimage of all stars, is transverse to the membranes and equivalent to f . If f was not already of this form, then $d(f') < d(f)$. Therefore we conclude that f was already locally injective on the preimage of membranes of 2-cells.

By compactness of Σ , every map without backtracking f has $d_\infty(f) = 0$. \square

LEMMA 5.15. *Let X be a generalized 2-complex and $f: \Sigma \rightarrow |X|$ be a map from a closed oriented surface which is transverse to the membranes, admits no squeezes, and has minimal $d(f)$ in its equivalence class of the compression preorder (Here the minimum is taken in the lexicographic order). Then f is allowable.*

PROOF. By Lemma 5.14, f has no backtracking. Therefore f is a covering on the preimage of the membranes of 2-cells.

To prove that f is allowable, assume that f is not allowable. Then there is a 2-cell x , such that f is not allowable over p_x . We abbreviate $f|_{f^{-1}(p_x)}$ by f_x . This means that there is a component c of m_x such that the degree of $f_x|_{f^{-1}(c)}: f^{-1}(c) \rightarrow c$ is greater than $\mathcal{G}(f_x)$.

By the definition of geometric degree, there is a map f'_x homotopic to f_x , still mapping to p_x , which is a $\mathcal{G}(f_x) = \mathcal{G}(f'_x)$ -fold covering over an open disk $D^2 \subset p_x$. Let f' be defined by extending f'_x by f . Since the homotopy between f and f' is supported in the inner of the plasma of x , we conclude that $d(f) = d(f')$. Fix an embedded path $y \subset c_x$ starting at a point $* \in m_x \cap V_\partial(X)$ and ending in ∂D^2 , transverse to f'_x . We will use y to reduce d by merging two boundary components of $f'^{-1}(p_x)$ over y .

The preimage of y consists of a collection of arcs with one or two endpoints in $f'^{-1}(m_x)$. Since the degree of $f'_x|_{f'^{-1}(c)} : f'^{-1}(c) \rightarrow c$ is greater than $\mathcal{G}(f'_x)$, there exists an arc y' in the preimage of y that connects two points in $f'^{-1}(c)$. By orientability of Σ , y' connects two different components of $f'^{-1}(m_x)$.

Fix a small tubular neighborhood $N(y)$ of y such that $f_x^{-1}(N(y))$ is a tubular neighborhood of y' as well. Let us denote this tubular neighborhood by $N(y')$. Note that by counting preimages $f(y')$ cannot intersect D^2 . Let f'' be defined as f' outside of $N(y')$ and in $N(y')$ be defined as f' postcomposed with the diffeomorphism depicted in Figure 9. Evidently f'' is homotopic to f' furthermore note that f'' is still transverse to the membranes since the vertical tangent vectors (in Figure 9) lie in the image of the differential of f' and the only points where these do not suffice to have transversality to the preimage of the membrane under the diffeomorphism in Figure 9 lie in D^2 and hence do not have a preimage in $N(y')$.

By construction $f''^{-1}(*)$ has two more points than $f'^{-1}(*)$ and the other summands of $d_{fin}(f'')$ agree with the ones in $d_{fin}(f)$. \square

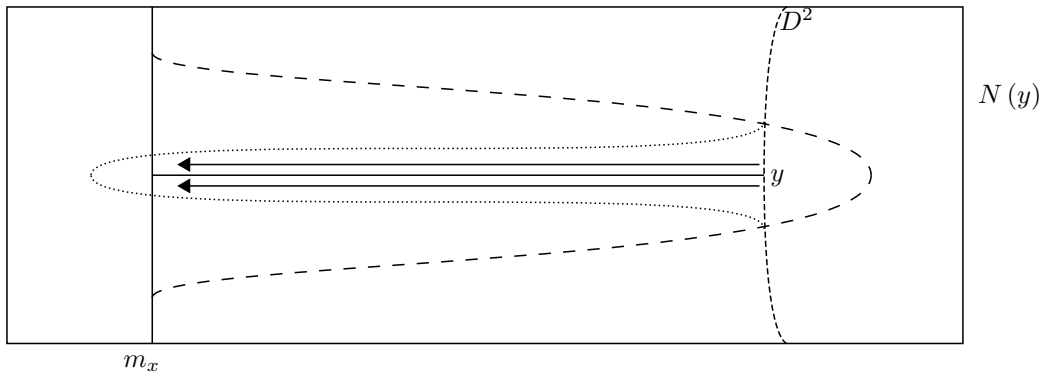


FIGURE 9. The diffeomorphism is supported in $N(y)$ and pushes the dashed portion of D between the endpoints of the dotted line along y to the dotted line. The other dashed line is the preimage of the portion of the plasma between the dotted line under this diffeomorphism.

PROOF OF LEMMA 5.12. Since representatives in the equivalence class of f differ by compression at an inessential curve, there always exists a representative that minimizes d and admits no squeezes if f admitted no squeezes to begin with. Lemma 5.10 implies that such a representative is equivalent to a map that is transverse to the membranes which minimizes d in its equivalence class, hence satisfies Part (a) Lemma 5.14 and Lemma 5.15 imply that this map also satisfies Part (b).

So let us assume that f fulfills (a) and (b), then Theorem 1.1 in [Sko87] implies that there is a homotopy of the restriction of f to the preimage of all plasma of 2-cells relative to the boundary of this preimage such that the resulting map is a branched covering over the plasma of 2-cells. Hence for this map Part (a) to Part (c) hold.

Now let us assume that f fulfills Part (a), (b) and (c). Pick an embedded path from a branching point in the plasma of a cell to the intersection of the 1-singular set with that cell. Postcompose f

with an isotopy, supported in a tubular neighborhood of this path, that pushes along this path. By repeating this for every other branching point, we obtain a homotopic map that satisfies Parts (a)-(d).

Let us assume that f already fulfills (a) to (d). In order to find a map fulfilling (a) to (e), we will first alter f so that the preimage of 1-blocks consists of rectangles and so that the previous parts of the lemma still hold. Then we will alter the resulting map on these rectangles.

Let x denote a 1-cell. Since f is transverse to the membranes, a component of the preimage of the 1-block B_x is bounded by embedded closed polygons. Since f admits no squeezes, the embedded closed polygons bound disks and each such component has genus 0. Then we can perform surgery at each embedded closed polygon using the nullhomotopy collapsing it in the block. Since the curve bounded a disk this does not change the equivalence class in the compression preorder and by construction this did not change d . Then the preimage of the 1-block consists of disks and spheres. Removing these spheres does not alter the equivalence class of f . Hence we can assume without loss of generality that the preimage of the 1-blocks consists of disks. Furthermore since f has no backtracking, the edges of the polygon map alternately to $\pi_1^{-1}(m_x)$ and $\bigcup_{x' \in X(2)} (m_{x'} \cap B_x)$. Furthermore, if we number the edges cyclically then every edge labeled i of the polygon maps to a different edge of ∂B_x than the edges labelled $i - 2$ and $i + 2$.

Let us decompose m_x into the point m_x^0 and the other point m_x^1 . Since f admits no backtracking we also know that the edges of the polygon mapping to $\pi_1^{-1}(m_x)$ map alternately to $\pi_1^{-1}(m_x^0)$ and $\pi_1^{-1}(m_x^1)$.

Connect the middle points of all edges in $f^{-1}(\pi_1^{-1}(m_x^0))$ to the center point of the polygon. This yields an embedded star S in the polygon and we can find a map f' homotopic to f which agrees with f outside the inner of the polygon and such that f' maps the star to m_x^0 . Hence we obtain the following commutative diagram:

$$\begin{array}{ccc} \Sigma & \xrightarrow{f'} & X \\ \downarrow & \nearrow & \\ \Sigma/S & & \end{array}$$

The left map is a homotopy equivalence. Let us denote the diagonal map by \tilde{f}' . The map \tilde{f}' is in general not transverse to the membranes anymore. Let us fix this: Now the preimage of B_x consists of rectangles that possibly intersect in a single point in their boundary, and the restriction of \tilde{f}' to the boundary of such a rectangle is a homeomorphism. Therefore we can replace \tilde{f}' by a homotopic map that has the standard form on these rectangles.

Postcomposing this map with an isotopy as depicted on the left side of Figure 10 yields a new map that we denote \bar{f}' , which is again transverse to the membranes. This procedure does not change d and therefore Part (a) and (b) hold by Lemma 5.14 and Lemma 5.15 for \bar{f}' as well. Since f agrees with \bar{f}' over the plasma of the 2-cells, Part (c) and (d) are still true for \bar{f}' . The preimage of B_x consists of disjoint rectangles as depicted in Figure 10. Furthermore \bar{f}' embeds such a rectangle into B_x . Hence \bar{f}' fulfills Parts (a) to (e).

So let us assume that f fulfills Parts (a)-(e). Similar to the previous situation we can assume without loss of generality that the preimage of a 0-block consists of polyhedra and disks. Since the 0-block is homeomorphic to the cone of its boundary, and the preimage is homeomorphic to the componentwise cone of its boundary, we can replace f by a new map that agrees with f outside a neighborhood of the polyhedra and disks and still has those as the preimage of the 0-block and so that the new map has the standard form on the boundary of these polyhedra. Now a map between cones is homotopic relative the boundary to the cone of the boundary map. Hence we can find a homotopic map satisfying (a)-(f). \square

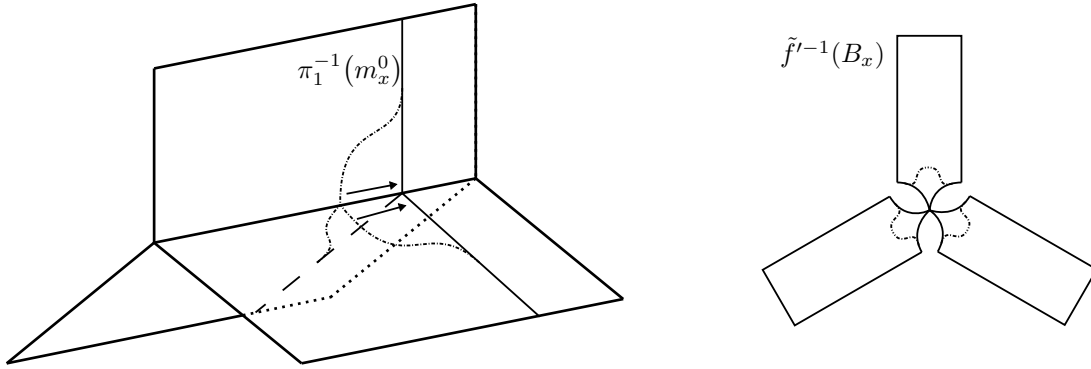


FIGURE 10. The isotopy pushes the indicated union of lines to $\pi_1^{-1}(m_x^0)$. The change to the preimage of the 1-Block is depicted on the right.

PROOF OF THEOREM B. We can assume that f is in the form described in Lemma 5.12. Let us construct a generalized 2-complex \mathcal{S} that is homeomorphic to Σ . Let $\mathcal{S}(i)$ denote the set of connected components of all preimages of open i -cells of X . Let $M: \mathcal{S}(2) \rightarrow \mathcal{M}(2)$ denote the map that sends such a connected component to the homeomorphism type of the preimage of the plasma of the open cell in that connected component. Note that the boundary of a connected component of the preimage of an open i -cell, necessarily lies in the preimage of the $(i-1)$ -skeleton. Since f is of the form described in Lemma 5.12, the inner of the preimage of the plasma is homeomorphic to the preimage of the inner of the corresponding cell. This homeomorphism can be extended so that the boundary of the preimage of the plasma maps to the boundary of the preimage of the corresponding open cell. Define the gluing maps of \mathcal{S} accordingly. Then it is evident that there is a homeomorphism $\Phi|\mathcal{S}| \rightarrow \Sigma$ and that $f \circ \Phi$ is a cellwise covering without folds. \square

As an application of Theorem B we prove the following interesting property of the attainable set:

PROPOSITION 5.16. *Given a generalized 2-complex X , let X_O denote the subcomplex given by the orientable cells. Then the inclusion $\iota_O: X_O \rightarrow X$ induces an isomorphism $H_2(X_O) \rightarrow H_2(X)$ and for all $\alpha \in H_2(X_O)$ we have $\mathcal{A}(\alpha, X) = \mathcal{A}(\iota_{O*}(\alpha), X)$.*

PROOF. By Lemma 4.3, ι_O induces an isomorphism on second homology.

Given a non-orientable $x \in \mathcal{M}(2)$ we write $\pi_x: \hat{x} \rightarrow x$ for its orientation cover. We write \hat{X} for the generalized 2-complex, where the cell type of every non-orientable 2-cell is replaced by its orientation cover and the gluing maps are given by the composition of the covering projection and the corresponding gluing map of X . Note that for every non-orientable $x \in X(2)$, we have $\pi_{x*}([\partial M(\hat{x})]) = 0 \in H_1(\partial M(x))$. By Corollary 1.1 every representative of a homology class has a smaller representative which is a cellwise covering and therefore lifts to $|\hat{X}|$. Let us denote such a representative by $f: \Sigma \rightarrow |X|$ and a lift by \hat{f} .

Consider a non-orientable generalized 2-cell x . There are two cases: Either the preimage of p_x is empty, in which case there is nothing to show, or $\hat{f}|_{\hat{f}^{-1}(p_x)}$ is a degree $d > 0$ cover.

So let us assume that we are in the second case. Since Σ is orientable and $M(\hat{x})$ is connected, we know that the fundamental class of $f^{-1}(p_x)$ gets mapped to $d[p_{\hat{x}}]$.

We know that $[\partial f^{-1}(p_x)]$ maps to $d[\partial p_{\hat{x}}]$ and therefore $[\partial f^{-1}(p_x)]$ maps to 0 in $H_1(\partial p_x)$. Let us denote a connected component of ∂p_x by c , and the intersection of $f^{-1}(c)$ and a connected component of $f^{-1}(p_x)$ by \tilde{c} . Then $f|_{\tilde{c}}$ extends to a sphere with $|\pi_0(\tilde{c})|$ holes such that the image of this sphere is contained in c . Indeed the image of the fundamental class of \tilde{c} in $H_1(\partial p_x)$ being zero, implies the existence of such an extension.

Therefore we can replace $f: \Sigma \rightarrow |X|$ by deleting the preimages of all non-orientable cells and replacing them by the previously constructed spheres with holes, one for every c . Since a disjoint union of spheres with holes has a lower genus and a lower χ^- than all connected surfaces with the same number of boundary components, we conclude that this procedure reduces g and χ^- . The resulting map is evidently homotopic to a map that avoids the non-orientable cells. \square

REMARK 5.17. While all the attainable sets of X_O and X agree, it is not true in general that ι_O is minimizer surjective.

6. Effective Computability

To talk about the computability of invariants of generalized 2-complexes, it has to be explained how a generalized 2-complex can be encoded combinatorially in a machine-understandable way. There are several ways of fixing such an encoding. We will briefly describe one option here:

We require each cell set to be given as an explicit set of labels (natural numbers, or words in the usual alphabet for readability). The gluing maps of 1-cells are given by denoting for each 1-cell the labels of its start- and endpoint. The gluing map of each boundary component of a 2-cell is given by a non-empty sequence of 1-cells and their formal inverses, or a 0-cell. Every combinatorial generalized 2-complex can be encoded this way. For example, the complex from Example 1.2 has the following encoding:

```
( ( [0-cells]
  p [label]
),
( [1-cells]
  ( a [label], (p [start], p [end]) ),
  ( b [label], (p [start], p [end]) ),
  ( c [label], (p [start], p [end]) )
),
( [2-cells]
  ( X [label],
    ( 1 [number of boundary components],
      0 [genus],
      + [orientable: yes]
    ),
    ( a, b, a^{-1}, b^{-1} ) [gluing map]
  ),
  ( Y [label],
    ( 1 [number of boundary components],
      0 [genus],
      + [orientable: yes]
    ),
    ( a, c, a^{-1}, c^{-1} ) [gluing map]
  ),
  ( Z [label],
    ( 1 [number of boundary components],
      0 [genus],
      + [orientable: yes]
    ),
    ( a, b, c, a^{-1}, c^{-1}, b^{-1} ) [gluing map]
  )
)
)
```


In this section we will establish the computability of the attainable set for a large class of spaces. This class of spaces is characterized by the existence of a "weight function".

Let X be a generalized combinatorial 2-complex.

DEFINITION 6.1 (Corners). The connected components of the subset of $\bigsqcup_{x \in X(2)} \partial M(x)$ that are glued to 0-cells are either points or circles. Let the **set of corners of 2-cells** $C_2(X)$ be the set of those connected components which are points.

Let $r_{2,2}$ and $r_{2,0}$ be the canonical maps

$$(r_{2,0}, r_{2,2}) : C_2(X) \rightarrow X(0) \times X(2).$$

Let the **set of corners of 1-cells** $C_1(X)$ denote the set of points in $\bigsqcup_{x \in X(1)} M(x)$ that get mapped to 0-cells by the gluing map. Let $r_{1,1}$ and $r_{1,0}$ be the canonical maps

$$(r_{1,0}, r_{1,1}) : C_1(X) \rightarrow X(0) \times X(1).$$

Let $e^+, e^- : C_2(X) \rightarrow C_1(X)$ be the maps with $r_{1,0} \circ e^+ = r_{2,0}$, $r_{1,0} \circ e^- = r_{2,0}$, which send a corner to the corresponding corner of the image of the boundary segment to its right (left) under the gluing map (right and left are given by an orientation on the boundary of the 2-cell, the choice of the orientation is irrelevant).

DEFINITION 6.2 (Link Sequence). For a zero-cell x , a **link sequence around x** is a sequence $(c_1, o_1), \dots, (c_n, o_n) \in r_{2,0}^{-1}(x) \times \{+, -\}$ such that $e^{o_i}(c_i) = e^{-o_{i+1}}(c_{i+1})$. It is said to be **without folds** if no corner is followed by its inverse.

DEFINITION 6.3 (Weight Function). A **weight function** for X is a map $w : C_2(X) \rightarrow \mathbb{Q}$ such that

- (1) the **weighted Euler characteristic** $\chi_w(x)$ of x is negative, where χ_w is defined as

$$\chi_w(x) = \chi(M(x)) - 0.5 \times \text{Seg}(x) + \text{Circ}(x) + \sum_{c \in r_{2,0}^{-1}(x)} w(c)$$

Here the number of boundary segments $\text{Seg}(x)$ is defined as the number of connected components of the preimage of $X^{(0)}$ under $r_x : \partial M(x) \rightarrow X^{(1)}$ which are points, and the number of boundary circles $\text{Circ}(x)$ as the number of those connected components which are circles.

- (2) For each non-empty link sequence $((c_1, o_1), \dots, (c_n, o_n))$ without folds we have:

$$w(c_1) + \dots + w(c_n) \geq 1$$

REMARK 6.4. A 2-dimensional M_{-1} polyhedral complex (See Section I.7 of [BH99]) is a $CAT(-1)$ -space (See Section II.1 of [BH99]) if and only if the angles of the incoming edges at every vertex define a weight function (Theorem 5.5 in Section II.5 of [BH99]).

Conversely every 2-complex that has a weight function with only positives weights can be turned into a $CAT(-1)$ -space by choosing a metric of constant curvature -1 on each 2-cell.

THEOREM C (Effective Computability). *There is an algorithm, taking as input an (encoding of a) connected generalized 2-complex X , a weight function w on X , and a homology class α in $H_2(X)$, which computes $\mathcal{A}(\alpha, X)$.*

PROOF OF THEOREM C. We prove that all minima in the product order $\mathcal{A}(\alpha, X)$ have representatives $f : \Sigma \rightarrow X$ which are cellwise covers without folds and such that the sum of all degrees over all 2-cells of f is bounded by a computable constant $C(\alpha)$ depending only on X , α and w . Then there are only finitely many of such (Σ, f) , and they are enumerable. Hence one can compute the attainable set by computing g and χ^- of these representatives, and then saturate (Proposition 2.3, Lemma 2.1) to obtain $\mathcal{A}(\alpha, X)$.

Let (Σ, f) be any representative of α . Such a representative can be constructed combinatorially by taking $n(x)$ many copies of every cell x of X and gluing them arbitrarily, where $n(x)$ is defined by $H_2(|X|) \ni \alpha \mapsto \sum_{x \in X(2)} n(x) \phi_{x*}([M(x), \partial M(x)]) \in H_2(|X|, X^{(1)})$. The first step in the computation of $\mathcal{A}(\alpha, X)$ is the construction of such an arbitrary representative. In particular, we obtain the number $B = 2g(\Sigma)$, which we need as an upper bound in the next step. This number may depend on the choice of the gluing.

Let (Σ', f') be a representative of a minimum of $\mathcal{A}(\alpha, X)$. We want to prove that $\chi^-(\Sigma') \leq B$. To prove this, note that the minimality implies $(\chi^-(\Sigma') - 2, 0.5 \chi^-(\Sigma') + n_0(\Sigma')) \notin \mathcal{A}(\alpha, X)$. Hence $0.5 \chi^-(\Sigma') \leq 0.5 \chi^-(\Sigma') + n_0(\Sigma') \leq g_c(\alpha) \leq g(\Sigma)$.

By Corollary 1.1, every minimum of $\mathcal{A}(\alpha, X)$ has a representative which is a cellwise covering without folds. Let (Σ', f') be such a representative, and consider Σ' to be a generalized 2-complex with the induced cell structure i.e. we identify Σ' with the corresponding \mathcal{S} in Definition 5.2. Then, denoting by d_x the covering degree of f' over the cell x , we have:

$$\begin{aligned} -\chi^-(\Sigma') \leq \chi(\Sigma') &= \sum_{x \in \Sigma'(2)} \left(\chi(M(x)) \right) - |\Sigma'(1)| + |\Sigma'(0)| \\ (14) \qquad &= \sum_{x \in X(2)} d_x \left(\chi(M(x)) - 0.5 \text{Seg}(x) \right) + |\Sigma'(0)| \end{aligned}$$

$$\begin{aligned} (15) \qquad &\leq \sum_{x \in X(2)} d_x \left(\chi(M(x)) - 0.5 \text{Seg}(x) + \sum_{c \in r_{2,2}^{-1}(x)} w(c) + \text{Circ}(x) \right) \\ &= \sum_{x \in X(2)} d_x \left(\chi_w(x) \right) \\ &\leq \left(\sum_{x \in X(2)} d_x \right) \max_c \chi_w(c) \end{aligned}$$

Equation (14) holds because every 1-cell of Σ' appears two times as a boundary segment. Inequality (15) holds because of the link condition (2) in Definition 6.3. Because of $\max_c \chi_w(c) < 0$ we obtain:

$$\sum_{x \in X(2)} d_x \leq -\chi^-(\Sigma') / \max_c \chi_w(c) \leq -B / \max_c \chi_w(c)$$

□

EXAMPLE 6.5 (Three Octagons, continued). We show how the attainable sets from Example 1.3 can be computed using the following weight function: Every black corner in Figure 11 has weight $\frac{1}{8}$ and every white corner has weight $\frac{1}{2}$. With this weight function, the weighted Euler characteristic of each cell is -2 .

In a first step, we construct a connected representative Σ_n^c for $n\alpha$ with $(\chi^-(\Sigma_n^c), g(\Sigma_n^c)) = (6n + 2, 3n + 2)$. We do this by gluing n copies of each cell in a chain, and then gluing these chains together, as depicted in Figure 12, again opposite sides are also identified. This complex has $(4n - 2)$ 0-cells, $13n$ 1-cells and $3n$ 2-cells, hence the formulas for $\chi^-(\Sigma_n^c)$ and $g(\Sigma_n^c)$.

Hence, for any (χ^-, g) -minimizer Σ_n representing $n\alpha$, $6n \leq \chi^-(\Sigma_n) \leq \chi^-(\Sigma_n^c) \leq 6n + 2$. Each representative which is a cellwise cover without folds covers each cell exactly n times (by the weighted Euler characteristic bound: a priori one cell could be covered $(n + 1)$ times, but this would yield a homological degree of $(n + 1)$ or $(n - 1)$, not n).

As a second step, we construct a non-connected representative Σ_n^{nc} for the class $n\alpha$ such that $(\chi^-(\Sigma_n^{\text{nc}}), g(\Sigma_n^{\text{nc}})) = (6n, 3n + 3)$ by taking a regular connected n -fold cover of each cell. This representative has three connected components of genus $(n + 1)$, hence the formulas.

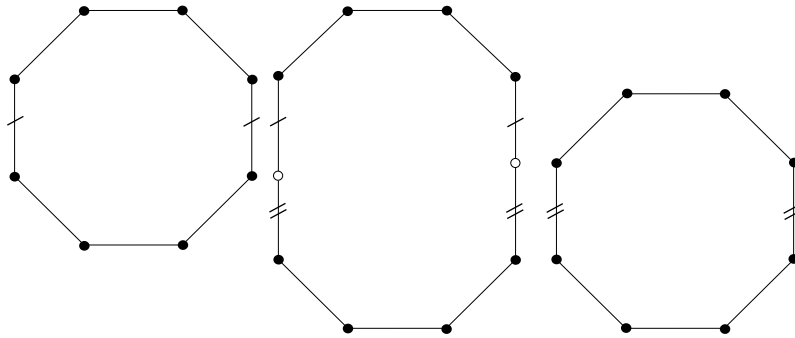


FIGURE 11. Weights on three octagons, Example 6.5

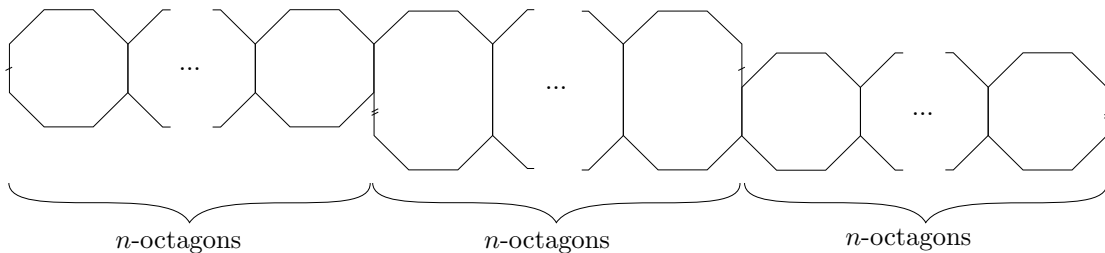


FIGURE 12. Connected representative, Example 6.5

Now all that is left to show is that there is no surface Σ_n representing the class $n\alpha$ with $(\chi^-(\Sigma_n^{\text{nc}}), g(\Sigma_n^{\text{nc}})) = (6n, 3n + 2)$. For such a surface, the estimate for χ^- given by the weighted Euler characteristic would be sharp. Hence, the sum over the weights for each link used in the cover has to be 1. One can see that this is only possible if each corner of weight $\frac{1}{2}$ is glued to another corner of weight $\frac{1}{2}$. But then the representative is a disjoint union of surfaces each mapping to single cells, hence it has at least 3 components, contradicting $n_0(\Sigma_n) = g(\Sigma_n) - \frac{1}{2}\chi^-(\Sigma_n) = 2$

In conclusion, the only minima of the attainable set are $(6n + 2, 3n + 2)$ and $(6n, 3n + 3)$.

7. Undecidability

We need the following construction introduced in [Gor95]: Given a finite presentation

$$\Pi = (q_1, \dots, q_m; r_1, \dots, r_n)$$

of a group G_Π and w a word in q_1, \dots, q_m , let Π_w denote the presentation given by Π together with extra generators a, α, b, β and additional relations:

- (i) $a\alpha a^{-1} = b^2$
- (ii) $\alpha a \alpha^{-1} = b\beta b^{-1}$
- (iii) $a^{2^i} q_i \alpha^{2^i} = \beta^{2^{i+1}} b \beta^{-2^{i+1}}, 1 \leq i \leq m$
- (iv) $[w, a] = \beta^2 b \beta^{-2}$
- (v) $[w, \alpha] = \beta b \beta b^{-1} \beta^{-1}$

The group presented by Π_w will be denoted as $G_{\Pi, w}$. The following properties of $G_{\Pi, w}$ are proven in the proof of Theorem 3 in [Gor95]:

- If w represents the unit element in G_Π , $G_{\Pi, w}$ is trivial.
- Otherwise, a and b have infinite order in $G_{\Pi, w}$.

- In any case, a and b are commutators in $G_{\Pi,w}$, namely $a = [\alpha^{-1}\beta^{-1}w\beta\alpha, \alpha^{-1}\beta^{-1}\alpha\beta\alpha]$, which follows from the additional relations (i) and (v), and $b = [\beta^{-2}w\beta^2, \beta^{-2}a\beta^2]$, which follows from the additional relation (iv).

PROOF OF THEOREM A. First we will prove this for $i = g$. Let $\Pi = (Q; R)$ be a finite presentation of a group G_{Π} . Let w be a word in the generators Q .

We define for every (Π, w) as above a 2-complex $X_{\Pi,w}$ and a class $\alpha_{\Pi,w} \in H_2(|X_{\Pi,w}|)$ in the following way: $X_{\Pi,w}$ is the union of the presentation complex of Π_w and a cell of type $\Sigma_{B,2}$ glued to it via a and b . The class $\alpha_{\Pi,w}$ is the fundamental class of $\Sigma_{B,2}$ plus the sum of the classes of the two commutators above.

Now this class is represented by a surface of genus bounded by B if and only if w was trivial. Hence, if there were an algorithm that computes minimal genus, there would be an algorithm solving the word problem, which is known to be undecidable (See [Nov58]).

Every minimizing representative for the problem above is automatically essentially connected. Because for essentially connected surfaces χ^- and g are related, both problems are undecidable. \square

Bibliography

- [BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, 1 ed., Grundlehren der mathematischen Wissenschaften, vol. 319, Springer, 1999.
- [Cal08] Danny Calegari, *Surface subgroups from homology*, *Geom. Topol.* **12** (2008), no. 4, 1995–2007. MR 2431013
- [Cal09a] ———, *scl*, *MSJ Memoirs*, vol. 20, Mathematical Society of Japan, Tokyo, 2009. MR 2527432
- [Cal09b] ———, *Stable commutator length is rational in free groups*, *J. Amer. Math. Soc.* **22** (2009), no. 4, 941–961. MR 2525776
- [Che19] Lvzhou Chen, *Scl in graphs of groups*, arXiv:1904.08360, 2019.
- [Edm79] Allan L. Edmonds, *Deformation of maps to branched coverings in dimension two*, *Ann. of Math. (2)* **110** (1979), no. 1, 113–125. MR 541331
- [Gab83] David Gabai, *Foliations and the topology of 3-manifolds*, *J. Differential Geom.* **18** (1983), no. 3, 445–503. MR 723813
- [Gor95] C. McA. Gordon, *Some embedding theorems and undecidability questions for groups*, *Combinatorial and geometric group theory (Edinburgh, 1993)*, *London Math. Soc. Lecture Note Ser.*, vol. 204, Cambridge Univ. Press, Cambridge, 1995, pp. 105–110. MR 1320278
- [Gro82] Mikhael Gromov, *Volume and bounded cohomology*, *Publications Mathématiques de l’IHÉS* **56** (1982), 5–99 (en). MR 84h:53053
- [Hop42] Heinz Hopf, *Fundamentalgruppe und zweite Bettische Gruppe*, *Comment. Math. Helv.* **14** (1942), 257–309. MR 0006510
- [KM94] P. B. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, *Math. Res. Lett.* **1** (1994), no. 6, 797–808. MR 1306022
- [Nov58] P. S. Novikov, *On the algorithmic insolvability of the word problem in group theory*, *American Mathematical Society Translations, Ser 2, Vol. 9*, American Mathematical Society, Providence, R. I., 1958, pp. 1–122. MR 0092784
- [OS00] Peter Ozsváth and Zoltán Szabó, *The symplectic Thom conjecture*, *Ann. of Math. (2)* **151** (2000), no. 1, 93–124. MR 1745017
- [Sko87] Richard Skora, *The degree of a map between surfaces*, *Math. Ann.* **276** (1987), no. 3, 415–423. MR 875337
- [Thu86] William P. Thurston, *A norm for the homology of 3-manifolds*, *Mem. Amer. Math. Soc.* **59** (1986), no. 339, i–vi and 99–130. MR 823443