

# A low-rank solution method for Riccati equations with indefinite quadratic terms

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**Abstract:** Algebraic Riccati equations with indefinite quadratic terms play an important role in applications related to robust controller design. While there are many established approaches to solve these in case of small-scale dense coefficients, there is no approach available to compute solutions in the large-scale sparse setting. In this paper, we develop an iterative method to compute low-rank approximations of stabilizing solutions of large-scale sparse continuous-time algebraic Riccati equations with indefinite quadratic terms. We test the developed approach for dense examples in comparison to other established matrix equation solvers, and investigate the applicability and performance in large-scale sparse examples.

**Keywords:** algebraic Riccati equation, large-scale sparse matrices, low-rank approximation, iterative numerical method

**Novelty statement:** We propose an iterative algorithm for computing low-rank approximations of stabilizing solutions of continuous-time algebraic Riccati equations with indefinite quadratic term. This is the first such approach that applies for general setups with large-scale sparse coefficient matrices.

## 1 Introduction

Many concepts in systems and control theory are connected to solutions of algebraic Riccati equations. Prominent examples are the linear-quadratic regulator (LQR) and linear-quadratic Gaussian (LQG) controller design [2, 39, 46, 57] and corresponding model order reduction methods [36] as well as the characterization of passivity and contractivity of input-output systems and properties-preserving model reduction methods for these [3, 28, 50].

They also appear, for example, in applications with differential games [6,27]. In this paper, we consider algebraic Riccati equations with indefinite quadratic terms of the form

$$A^T X E + E^T X A + E^T X \left( B_1 B_1^T - B_2 B_2^T \right) X E + C^T C = 0, \quad (1)$$

with  $A, E \in \mathbb{R}^{n \times n}$ ,  $B_1 \in \mathbb{R}^{n \times m_1}$ ,  $B_2 \in \mathbb{R}^{n \times m_2}$ ,  $C \in \mathbb{R}^{p \times n}$  and  $E$  invertible, and we are interested in symmetric positive semi-definite stabilizing solutions of (1). Equations of the form (1) usually occur in  $\mathcal{H}_\infty$ -control theory and related topics, e.g., robust controller design [29,47,49].

In the case of coefficients of small size  $n$  and classical LQR/LQG Riccati equations, i.e.,  $B_1 = 0$  in (1) such that the quadratic term is negative semi-definite, there is a variety of different numerical approaches to compute the stabilizing solution: Direct approaches that compute eigenvalue decompositions of underlying Hamiltonian or even matrix pencils [1,4,42], iterative methods that work on the underlying spectrum of the Hamiltonian matrix [7,13,51], or iterative approaches, which compute a sequence of matrices converging to the stabilizing solution [37,55]. The problem becomes more complicated in the opposite case with symmetric positive semi-definite quadratic term, i.e., in (1) setting  $B_2 = 0$ , which occurs in bounded-real and positive-real problems [3,28,50]. This already reduces the number of applicable methods. While methods from the classical setting that do not take the definiteness of the quadratic term into account can still be applied here [4,7,42,51], only few, iterative Newton-type methods, which converge to the desired solution, have been developed for this problem class [8,22,61]. The amount of applicable methods narrows down even further when considering the general case of indefinite quadratic terms like in (1). A new type of iterative method for the solution of (1) was developed in [40,41] to overcome accuracy problems of classical solution approaches and the general lack of iterative methods for this problem type. This new method computes a sequence of matrices converging to the stabilizing solution of (1) by solving classical Riccati equations with symmetric negative semi-definite quadratic terms.

However, in this paper, we will mainly focus on the case of large-scale sparse coefficient matrices in (1) arising, for example, from the discretization of partial differential equations, with  $n \gtrsim 10^5$  and low-rank quadratic term and right-hand side such that  $m_1, m_2, p \ll n$ . This leads to another multitude of problems. First of all, the dense computation methods from above cannot be applied anymore since they may transform the original data and compute full solutions, which easily becomes unfeasible in terms of memory requirements. Also, these methods usually use matrix operations that cannot be efficiently used in the large-scale sparse setting, which heavily increases the running time of algorithms. As in the dense case, there exists a variety of methods for the classical LQR/LQG case ( $B_1 = 0$ ), for example, the low-rank Newton method [20,62] with an underlying low-rank alternating direction implicit method [18–20,38,44] for solving the occurring large-scale sparse Lyapunov equations (LR-Newton-ADI), the Riccati alternating direction implicit method (RADI) [11], the incremental low-rank subspace iteration (ILRSI) [45], and projection-based methods that construct approximating subspaces such that internally small-scale dense Riccati equations need to be solved, e.g., [35,56]. See also [12,21,38] for overviews and numerical comparisons of large-scale sparse solvers for this special case of Riccati equations with negative semi-definite quadratic terms. For the opposite case of Riccati equations with positive semi-definite quadratic terms ( $B_2 = 0$ ), only the Newton method from [22] is known to have a low-rank extension to the large-scale sparse matrix case. However, there are no methods known to solve the general case (1) with an indefinite quadratic term in the large-scale sparse setting. The goal of this paper is to develop an extension of the approach described in [40,41] that can be applied in the case of large-scale

sparse coefficient matrices.

In the upcoming [Section 2](#), we first recap the algorithm from [\[40, 41\]](#) and summarize some important results. Afterwards, we reformulate the steps of the algorithm to fit the large-scale sparse system case and extend it further to singular  $E$  matrices arising in certain applications. In [Section 3](#), we test the new algorithm first on Riccati equations with dense coefficients, for a comparison to other established methods, and afterwards on equations with large-scale sparse coefficients. The paper is concluded in [Section 4](#).

## 2 Riccati iteration method

In this section, we describe the idea of the Riccati iteration method from [\[40, 41\]](#) and extend the approach to large-scale sparse systems.

Thereby, we will use the following notation. We will denote symmetric positive semi-definite matrices by  $X = X^\top \geq 0$ , for  $X \in \mathbb{R}^{n \times n}$ , and use the Loewner ordering  $X_1 \geq X_2$  for two symmetric matrices  $X_1, X_2 \in \mathbb{R}^{n \times n}$ , if  $X_1 - X_2 \geq 0$ . We call the matrix triple  $(A, B, E)$  stabilizable, with  $A, E \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $E$  invertible, if there exists a feedback matrix  $K \in \mathbb{R}^{m \times n}$  such that the matrix pencil  $\lambda E - (A - BK)$  has only eigenvalues with negative real parts.

### 2.1 Basic algorithm

In this section, we formulate the fundamental algorithms for the iterative computation of approximate solutions to the following general problem.

**Problem 1** (Stabilizing solutions of indefinite Riccati equations). Given matrices  $A, E \in \mathbb{R}^{n \times n}$ ,  $B_1 \in \mathbb{R}^{n \times m_1}$ ,  $B_2 \in \mathbb{R}^{n \times m_2}$  and  $C \in \mathbb{R}^{p \times n}$ , with  $E$  invertible and  $n, m_1, m_2, p \in \mathbb{N}$ , compute a matrix  $X = X_\infty \in \mathbb{R}^{n \times n}$ , if it exists,

- (a) which solves the Riccati equation [\(1\)](#) with indefinite quadratic term

$$A^\top X E + E^\top X A + E^\top X \left( B_1 B_1^\top - B_2 B_2^\top \right) X E + C^\top C = 0, \quad \text{and}$$

- (b) which is symmetric positive semi-definite and such that the pencil

$$\lambda E - \left( A + (B_1 B_1^\top - B_2 B_2^\top) X_\infty E \right)$$

is stable, i.e., all its eigenvalues lie in the open left half-plane.

For computing the solution to [Problem 1](#) numerically, the *Riccati iteration* method was first described in [\[40, 41\]](#) for the standard equation case ( $E = I_n$ ). Here, we will summarize the method and some theoretical results using directly the generalized case with  $E$  invertible. The underlying idea of the algorithm is to consider the Riccati operator

$$\mathcal{R}(X) := A^\top X E + E^\top X A - E^\top X (B_2 B_2^\top - B_1 B_1^\top) X E + C_1^\top C_1, \quad (2)$$

associated with [\(1\)](#), and a splitting of the final solution into the sum of consecutive solutions of updated algebraic Riccati equations with semi-definite quadratic terms. For two symmetric matrices  $X_1 = X_1^\top$  and  $X_2 = X_2^\top$ , one can show that

$$\mathcal{R}(X_1 + X_2) = \mathcal{R}(X_1) + \tilde{A}^\top X_2 E + E^\top X_2 \tilde{A} - E^\top X_2 (B_2 B_2^\top - B_1 B_1^\top) X_2 E$$

**Algorithm 1:** Riccati iteration (RI).

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**Input:**  $A, B_1, B_2, C, E$  from (1), convergence tolerance  $\tau$ .  
**Output:** Stabilizing approximate solution  $X_k \approx X_\infty$  of (1), with  $X_k = X_k^\top \geq 0$ .

- 1 Initialize  $X_0 = 0, k = 0$ .
- 2 **repeat**
- 3     Update the iteration matrix  $A_k = A + (B_1 B_1^\top - B_2 B_2^\top) X_k E$ .
- 4     Solve the definite Riccati equation
 
$$A_k^\top W_{k+1} E + E^\top W_{k+1} A_k - E^\top W_{k+1} B_2 B_2^\top W_{k+1} E + \mathcal{R}(X_k) = 0$$
 for the stabilizing solution  $W_{k+1} \in \mathbb{R}^{n \times n}$ .
- 5     Update the solution matrix  $X_{k+1} = X_k + W_{k+1}$ .
- 6     **if**  $(A + B_1 B_1^\top X_{k+1} E, B_2, E)$  *is stabilizable* **then**
- 7         Increment  $k \leftarrow k + 1$ .
- 8     **else**
- 9         **error** // There is no stabilizing  $X = X^\top \geq 0$  that solves (1).
- 10    **end**
- 11 **until**  $\|B_1 W_k E\|_2 < \tau$

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holds, where  $\tilde{A} := A - (B_2 B_2^\top - B_1 B_1^\top) X_1 E$ . Accordingly, for  $X_2$  as a solution to the algebraic Riccati equation with negative semi-definite quadratic term

$$0 = \mathcal{R}(X_1) + \tilde{A}^\top X_2 E + E^\top X_2 \tilde{A} - E^\top X_2 B_2 B_2^\top X_2 E, \quad (3)$$

the residual reads

$$\mathcal{R}(X_1 + X_2) = E^\top X_2 B_1 B_1^\top X_2 E.$$

An iterative use of this relation (3), together with the initial solution  $X_0 = 0$ , yields the *Riccati iteration (RI)* method. The resulting algorithm is given in [Algorithm 1](#).

The following proposition lays out the theoretical foundation of the Riccati iteration method.

**Proposition 1** (Properties of the Riccati iteration [40, 41]). *If  $(A, B_2, E)$  is stabilizable,  $(A, C, E)$  has no unobservable purely imaginary modes and there exists a stabilizing solution  $X_\infty = X_\infty^\top \geq 0$  for (1), then the following statements hold for the iteration in [Algorithm 1](#):*

- (a)  $(A + B_1 B_1^\top X_k E, B_2, E)$  is stabilizable for all  $k = 0, 1, \dots$ ,
- (b)  $W_k = W_k^\top \geq 0$  for all  $k = 0, 1, \dots$ ,
- (c) the eigenvalues of the matrix pencil  $\lambda E - (A + B_1 B_1^\top X_k E - B_2 B_2^\top X_{k+1} E)$  lie in the left open half-plane for all  $k = 0, 1, \dots$ ,
- (d)  $\mathcal{R}(X_{k+1}) = E^\top W_{k+1} B_1 B_1^\top W_{k+1} E$  for all  $k = 0, 1, \dots$ ,
- (e)  $X_\infty \geq \dots \geq X_{k+1} \geq \dots \geq X_1 \geq X_0 = 0$ ,
- (f) the iteration converges to the stabilizing solution of (1),  $\lim_{k \rightarrow \infty} X_k = X_\infty$ , and
- (g) the convergence is locally quadratic.

Basically, the conditions of [Proposition 1](#) guarantee convergence of the iterants to the desired solution of [Problem 1](#) as well as the stabilization property for every intermediate iteration step. The following remark gives some insight about the convergence of the stabilizing solutions of [\(1\)](#) and the convergence of the Riccati iteration.

**Remark 1** (Definiteness of stabilizing solutions). In general, if a real stabilizing solution of [\(1\)](#) exists, it does not need to be symmetric positive semi-definite; see examples in [\[40\]](#). Other approaches that rely on the underlying Hamiltonian matrix pencil of [\(1\)](#) are capable of computing also indefinite stabilizing solutions. However, these are undesired in many applications. The Riccati iteration converges only if the stabilizing solution exists and it is symmetric positive semi-definite, which follows from Parts (e) and (f) of [Proposition 1](#) and classical theory about the LQR/LQG Riccati equations that are solved in every iteration step.

## 2.2 Factorized low-rank formulation for large-scale sparse equations

The Riccati iteration method in [Algorithm 1](#) cannot be directly applied to the large-scale case. Most importantly, the iterants  $X_k$  will be dense  $n \times n$ -matrices such that memory will become a limiting factor already for moderate dimensions  $n$ . With the guarantee that all intermediate solutions  $X_k$  as well as all updates  $W_{k+1}$  are symmetric positive semi-definite (cf. Parts (a) and (e) of [Proposition 1](#)), the approximation by low-rank factorizations provides a potential remedy as in the symmetric semi-definite case; see, e.g., [\[12, 21\]](#). The basic idea is to rewrite the intermediate stages of the Riccati iteration via Cholesky-like low-rank factorizations such that  $X_k \approx Z_k Z_k^\top$  and  $W_{k+1} \approx Y_{k+1} Y_{k+1}^\top$ , with  $Z_k \in \mathbb{R}^{n \times r_k}$  and  $Y_{k+1} \in \mathbb{R}^{n \times q_{k+1}}$  for all  $k = 0, 1, \dots$  and the relevant equations in terms of the factors  $Z_k$  and  $Y_k$ . In particular, with the intermediate solution in Step 6 of [Algorithm 1](#) reformulated as

$$Z_{k+1} Z_{k+1}^\top \approx X_{k+1} = X_k + W_{k+1} \approx Z_k Z_k^\top + Y_{k+1} Y_{k+1}^\top = \begin{bmatrix} Z_k & Y_{k+1} \end{bmatrix} \begin{bmatrix} Z_k^\top \\ Y_{k+1}^\top \end{bmatrix},$$

we will develop an iteration for the solution factors without ever forming the full solution explicitly.

The residual Riccati equation in Step 4 of [Algorithm 1](#) that defines the update  $W_{k+1}$  is given via

$$A_k^\top W_{k+1} E + E^\top W_{k+1} A_k - E^\top W_{k+1} B_2 B_2^\top W_{k+1} E + \mathcal{R}(X_k) = 0, \quad (4)$$

and we observe that

1. in the initial step with  $X_0 = 0$ , we have  $\mathcal{R}(X_0) = C^\top C$ , and
2. for all further steps, with Part (d) of [Proposition 1](#), it holds

$$\mathcal{R}(X_{k+1}) = E^\top W_{k+1} B_1 B_1^\top W_{k+1} E = \underbrace{(B_1^\top W_{k+1} E)^\top}_{\in \mathbb{R}^{n \times m_1}} \underbrace{(B_1^\top W_{k+1} E)}_{\in \mathbb{R}^{m_1 \times n}},$$

for all  $k = 0, 1, \dots$ , and where  $W_{k+1} \approx Y_{k+1} Y_{k+1}^\top$ .

In both cases, this central step of the algorithm requires the solve of a standard (semi-definite) Riccati equation with low-rank factorized quadratic and constant terms. Accordingly, a low-rank factorized approximation to  $X_{k+1}$  can be computed by established Riccati equation solvers, like the LR-Newton-ADI method [\[20, 62\]](#), RADi [\[11\]](#), ILRSI [\[45\]](#), or projection-based methods [\[35, 56\]](#).

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**Algorithm 2:** Low-rank Riccati iteration (LR-RI).

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**Input:**  $A, B_1, B_2, C, E$  from (1), convergence tolerance  $\tau$ .

**Output:** Low-rank factor  $Z_k$  s.t.  $Z_k Z_k^\top \approx X_\infty$  is the stabilizing solution of (1).

1 Solve the Riccati equation with negative semi-definite quadratic term

$$A^\top W_0 E + E^\top W_0 A - E^\top W_0 B_2 B_2^\top W_0 E + C^\top C = 0,$$

for the low-rank factor  $Y_0$  such that  $Y_0 Y_0^\top \approx W_0$ .

2 Initialize  $Z_0 = Y_0$ ,  $U = [B_1 \quad -B_2]$  and  $k = 0$ .

3 **while**  $\|B_1^\top Y_k Y_k^\top E\|_2 > \tau$  **do**

4     Set  $A_k = A - UV_k^\top$  with the updated low-rank factor

$$V_k = [E^\top Z_k Z_k^\top B_1 \quad E^\top Z_k Z_k^\top B_2].$$

5     Solve the Riccati equation with negative semi-definite quadratic term

$$A_k^\top W_{k+1} E + E^\top W_{k+1} A_k - E^\top W_{k+1} B_2 B_2^\top W_{k+1} E + (E^\top W_k B_1)(E^\top W_k B_1)^\top = 0,$$

for the low-rank factor  $Y_{k+1}$  with  $Y_{k+1} Y_{k+1}^\top \approx W_{k+1}$ .

6     Update  $Z_{k+1} = [Z_k \quad Y_{k+1}]$  and increment  $k \leftarrow k + 1$ .

7 **end**

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**Remark 2** (Low-rank solutions). By the low-rank structure of the right hand-side and the quadratic terms,  $m_1, m_2, p \ll n$ , we can also expect the solution and update terms to be well approximated by low-rank factors such that  $r_k, q_k \ll n$ ; see [10, 58].

Combining all these ideas leads now to the *low-rank Riccati iteration (LR-RI)* method in Algorithm 2. Algorithm 2 has been implemented for dense coefficients in the MORLAB toolbox [25, 26] and for the large-scale sparse case in the M-M.E.S.S. toolbox [17, 53].

Another difference of Algorithm 2 compared to Algorithm 1 is the stabilizability test for  $(A + B_1 B_1^\top X_{k+1} E, B_2, E)$ . This additional test is expensive in the large-scale setting but can, in principle, be omitted. In case of  $(A + B_1 B_1^\top X_{k+1} E, B_2, E)$  not being stabilizable in intermediate steps, the iteration will diverge and there might be no stabilizing solutions for the intermediate Riccati equations with negative semi-definite quadratic terms anymore. The following remarks state further computational features of the new low-rank iteration method in Algorithm 2.

**Remark 3** (Unstable closed-loop matrix pencils). The intermediate closed-loop matrix pencils  $sE - A_k$  can potentially have unstable eigenvalues [41]. However, in case there exists a stabilizing solution  $X_\infty = X_\infty^\top \geq 0$  of (1),  $(A_k, B_2, E)$  will be stabilizable. Many of the low-rank solvers for definite Riccati equations need a stabilizing initial solution or a corresponding feedback matrix. In the large-scale sparse case, this can be computed using a sparse eigenvalue solver to compute the eigenvectors corresponding to the unstable eigenvalues of  $sE - A_k$  and to apply a partial stabilization approach [9] on the projected problem using the computed eigenvalue basis [5]. Note that if  $sE - A_{k_0}$  is stable for some  $k_0$  and the iteration converges, then  $sE - A_k$  will also be stable for all  $k \geq k_0$ ; cf. Proposition 1.

**Remark 4** (Solution of Riccati equations with positive semi-definite quadratic terms). The Riccati iteration method can be used as an alternative approach to solve Riccati

equations with positive semi-definite quadratic terms by setting  $B_2 = 0$ :

$$A^\top X E + E^\top X A + E^\top X B_1 B_1^\top X E + C^\top C = 0,$$

which occur, for example, in [3, 28, 50]. A necessary condition for applying the Riccati iteration method is in this case the stability of the matrix pencil  $sE - A$ , since otherwise the stabilizability of  $(A, B_2, E)$  will never be fulfilled. This problem does not occur in the Newton iteration from [8, 22, 61], where only a stabilizing initial feedback is needed.

### 2.3 Realization of linear solves in factored form

In general, the coefficients  $A_k = A + (B_1 B_1^\top - B_2 B_2^\top) X_k E$ , for  $k = 1, 2, \dots$ , of the intermediate Riccati equations in Step 4 of Algorithm 2 are  $n \times n$ -matrices without sparsity structures such that an explicit realization would make the approach infeasible in the large-scale setting. In this case, the coefficients need to be rewritten as the sparse system matrix  $A$  plus low-rank update:

$$A_k = A + (B_1 B_1^\top - B_2 B_2^\top) X_k E = A + \begin{bmatrix} B_1 & -B_2 \end{bmatrix} \begin{bmatrix} B_1^\top X_k E \\ B_2^\top X_k E \end{bmatrix} =: A + UV_k^\top, \quad (5)$$

with  $U, V_k \in \mathbb{R}^{n \times (m_1 + m_2)}$ , for all  $k = 1, 2, \dots$ . Thereby, matrix-vector multiplications with  $A_k$  can be performed without resorting to an explicit formation of  $A_k$ .

For the use within sparse direct solvers, this factored representation of  $A_k$  can be exploited as follows. Consider the shifted linear system

$$(\sigma E^\top - A_k^\top) X = (\sigma E^\top - A^\top - V_k U^\top) X = F, \quad (6)$$

for a *slim* right-hand side  $F \in \mathbb{R}^{n \times \ell}$ , with  $\ell \ll n$ , and a shift  $\sigma \in \mathbb{C}$ . The solution of such linear systems (6) are the backbone of standard iterative solvers for large-scale Riccati equations. By the *Sherman-Morrison-Woodbury formula* for matrix inversion [31] and with the abbreviation  $\Phi(\sigma) := (\sigma E^\top - A^\top)$ , the inverse of the shifted matrix in (6) is given by

$$(\Phi(\sigma) - V_k U^\top)^{-1} = \Phi(\sigma)^{-1} + \Phi(\sigma)^{-1} V_k (I_{m_1 + m_2} - U^\top \Phi(\sigma)^{-1} V_k)^{-1} U^\top \Phi(\sigma)^{-1}, \quad (7)$$

which actually amounts to  $2\ell$  sparse linear solves with  $\sigma E^\top - A^\top$  and one solve with an  $(m_1 + m_2) \times (m_1 + m_2)$ -dimensional matrix. Thus, in a practical realization of (7) for (6), one would compute

$$X = Z_1 + Z_2 (I_{m_1 + m_2} - U^\top Z_2)^{-1} U^\top Z_1,$$

with  $(\sigma E^\top - A^\top) Z_1 = F$ , and  $(\sigma E^\top - A^\top) Z_2 = V_k$ .

An alternative to the Sherman-Morrison-Woodbury formula is the augmented matrix approach, which makes use of the *block matrix inversion formula*. This approach is, often times, more stable than the Sherman-Morrison-Woodbury formula. Thereby, the solution of (6) is given as the solution of an augmented system of linear equations with

$$\begin{bmatrix} \sigma E^\top - A^\top & V_k \\ U^\top & I_{m_1 + m_2} \end{bmatrix} \begin{bmatrix} X \\ X^\perp \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \quad (8)$$

where the lower block of the solution,  $X^\perp$ , is an auxiliary variable with no relevance for the solution. Under the assumption that  $U$  and  $V_k$  have much fewer columns than  $n$ , systems of the form (8) can still be solved with standard sparse solvers.

## 2.4 Singular $E$ matrices and projected Riccati equations

A regular occurring case in applications involves singular  $E$  matrices. These especially occur in control problems with differential-algebraic equations (DAEs). In general, the presence of singular  $E$  matrices changes the concepts of stabilizability of matrix pencils and solvability of Riccati equations; see, e.g., [15, 32, 48] and references therein. However, in many cases, the singular part of  $E$  does not play any substantial role and it is actually enough to consider the solution of Riccati equations restricted to subspaces corresponding to the finite eigenvalues of the matrix pencil  $\lambda E - A$ , i.e., a restriction to the invertible part of  $E$ . This can be realized in two different concepts:

1. consider a projected version of the Riccati equation (1) with additional constraints on the stabilizing solution,
2. consider a truncated version of the Riccati equation (1).

In the following subsections, we consider first the general ideas for projected and truncated Riccati equations and, afterwards, the setup of incompressible flows as a particular example.

### 2.4.1 Projected and truncated equations

In the first concept, appropriate spectral projections  $\mathcal{P}_r, \mathcal{P}_\ell \in \mathbb{R}^{n \times n}$  onto the right and left deflating subspaces corresponding to the finite eigenvalues of  $\lambda E - A$  are explicitly introduced in (1) such that

$$A^\top X E + E^\top X A + E^\top X \left( B_1 B_1^\top - B_2 B_2^\top \right) X E + \mathcal{P}_r^\top C^\top C \mathcal{P}_r = 0, \quad (9a)$$

$$\mathcal{P}_\ell^\top X \mathcal{P}_\ell = X, \quad (9b)$$

needs to be solved instead. While in general, such projections  $\mathcal{P}_r, \mathcal{P}_\ell$  can be constructed using decompositions of the matrix pencil  $\lambda E - A$  that resemble the *Weierstrass canonical form*, they are in fact known for several specially structured problems that arise in the large-scale sparse setting; see [59] and the examples in [23, 60]. Note that the  $E$  matrix in (9) is still singular, but the equation and its solutions are restricted to the appropriate underlying subspace. Iterative methods like Algorithm 2 can be used to solve (9), where only the application of the spectral projections  $\mathcal{P}_r$  and  $\mathcal{P}_\ell$  is needed in the intermediate computational steps. This is also the common drawback of solving (9), since the repeated application of the projections can quickly become expensive, especially in the large-scale sparse case.

The second concept of truncated Riccati equations is more commonly used in practice. Thereby, we consider in general Riccati equations of the form

$$\widehat{A}^\top \widehat{X} \widehat{E} + \widehat{E}^\top \widehat{X} \widehat{A} + \widehat{E}^\top \widehat{X} \left( \widehat{B}_1 \widehat{B}_1^\top - \widehat{B}_2 \widehat{B}_2^\top \right) \widehat{X} \widehat{E} + \widehat{C}^\top \widehat{C} = 0, \quad (10)$$

where the truncated matrices are constructed by

$$\widehat{A} = W^\top A V, \quad \widehat{E} = W^\top E V, \quad \widehat{B} = W^\top B, \quad \widehat{C} = C V, \quad (11)$$

with the coefficient matrices from (1) and  $V, W \in \mathbb{R}^{n \times r}$ , basis matrices of the right and left deflating subspaces corresponding to the finite eigenvalues of  $\lambda E - A$ . By construction, the  $E$  matrix in (10) is nonsingular and the solution techniques mentioned so far for (1) can also be applied to (10). These basis matrices  $V$  and  $W$  can be computed and



applied explicitly to (1) up to medium-scale sized coefficient matrices. This is backbone of the implementation of model reduction methods for descriptor systems in the MORLAB toolbox [25, 26] and explained, e.g., in [24].

However, the explicit computation of the basis matrices  $V$  and  $W$  and of the resulting truncated coefficient matrices (11) is usually not possible in the large-scale sparse setting because of computation time and memory limitations. In this case, for many special sparse structures, the truncated Riccati equation (10) can be realized implicitly during the computations, i.e., instead of applying Algorithm 2 directly to (10), the original sparse coefficients from (1) are used and, by means of their structure,  $V$  and  $W$  are implicitly applied during the computational steps. Examples for this implicit truncation are given for different sparsity structures in [5, 30, 34, 54], which are also implemented in the function handle framework of the M-M.E.S.S. toolbox [17, 53]. Consequently, the implementation of Algorithm 2 in the M-M.E.S.S. toolbox can make use of the implicit truncation in case of singular  $E$  matrices. As a particular example, the following subsection considers the case of structured coefficient matrices arising from incompressible flows.

## 2.4.2 Implicit realization of truncations in case of flow problems

Riccati equations with indefinite quadratic terms (1) are of particular interest in the design of robust controllers for the stabilization of incompressible flows modeled by linearization of the Navier-Stokes equations [16]. We briefly touch this particular case as an example for implicit truncation since we will consider such a numerical example later. In this case, the coefficient matrices of (1) are structured and given by

$$A = \begin{bmatrix} \tilde{A} & J^T \\ J & 0 \end{bmatrix}, \quad E = \begin{bmatrix} \tilde{E} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \tilde{B}_2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}, \quad (12)$$

with  $\tilde{E}$  symmetric and invertible. The key to implicitly truncate and project Riccati equations with coefficients like (12) is the *discrete Leray projection*

$$\Pi = I_{\tilde{n}} - \tilde{E}^{-1} J^T (J \tilde{E}^{-1} J^T)^{-1} J; \quad (13)$$

see [34] for the most general case. Then, the truncated and projected Riccati equation that needs to be solved has the form

$$\Pi^T \tilde{A}^T \Pi \hat{X} \tilde{E} + \tilde{E}^T \hat{X} \Pi^T \tilde{A} \Pi + \tilde{E}^T \hat{X} \Pi^T (\tilde{B}_1 \tilde{B}_1^T - \tilde{B}_2 \tilde{B}_2^T) \Pi \hat{X} \tilde{E} + \Pi^T \tilde{C}^T \tilde{C} \Pi = 0, \quad (14)$$

where  $\hat{X} = \Pi \tilde{X} \Pi^T$ ; see [16, Sec. 3.1] and [14].

As mentioned in Section 2.4.1, we can neither explicitly compute the matrices in (14) nor the projection (13). However, instead one can use the original coefficient matrices with their special structure (12). As mentioned in Section 2.3, we practically need to solve linear systems like (6) in every step of the low-rank Riccati iteration. If we consider the coefficients of (14) in this setting, the low-rank update matrix (5) has the form

$$\Pi^T \tilde{A}^T \Pi + \tilde{E}^T \hat{X}_k \Pi^T (\tilde{B}_1 \tilde{B}_1^T - \tilde{B}_2 \tilde{B}_2^T) \Pi = \Pi^T (\tilde{A}^T + \tilde{E}^T \hat{X}_k (\tilde{B}_1 \tilde{B}_1^T - \tilde{B}_2 \tilde{B}_2^T)) \Pi = \Pi^T \tilde{A}_k \Pi,$$

where we also used that  $\tilde{E} \Pi = \Pi^T \tilde{E}$  and  $\Pi^2 = \Pi$ . By observing that all occurring right-hand sides will satisfy  $F = \Pi F$ , the solution of  $(\sigma \tilde{E}^T - \Pi^T \tilde{A}_k^T \Pi) X = F$  is alternatively given by

$$\begin{bmatrix} \sigma \tilde{E}^T - \tilde{A}^T + \tilde{E}^T \hat{X}_k (\tilde{B}_1 \tilde{B}_1^T - \tilde{B}_2 \tilde{B}_2^T) & -J^T \\ -J & 0 \end{bmatrix} \begin{bmatrix} X \\ X^\perp \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \quad (15)$$

Table 1: Results for the aircraft and cable mass benchmarks.

	$n$	$m_1$	$m_2$	$p$	$\#\lambda_{\text{unstab}}$	data	$\gamma$
AC10	55	3	2	5	2	dense	2.00
CM6	960	1	1	3	76	dense	2.00
rand512	512	2	6	5	8	dense	8.00
rand1024	1024	4	2	3	6	dense	8.00
rand2048	2048	3	3	1	2	dense	7.00
rand4096	4096	4	3	6	1	dense	10.00
rail	79 841	3	3	6	0	sparse	2.00
cylinderwake	47 136	1	1	6	4	sparse	50.00

where  $X^\perp$  is an auxiliary variable that does not play any role. This ensures that  $X = \Pi X$  as required during the iteration; see [34, Lem. 5.2]; and respects the sparsity structure of (12), since (15) is actually given by

$$\left( \sigma E^\top - A^\top + E^\top \begin{bmatrix} \widehat{X}_k & 0 \\ 0 & 0 \end{bmatrix} (B_1 B_1^\top - B_2 B_2^\top) \right) \begin{bmatrix} X \\ X^\perp \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix},$$

using the matrices from (12).

### 3 Numerical experiments

The numerical experiments reported in this section have been executed on a machine with 2 Intel(R) Xeon(R) Silver 4110 CPU processors running at 2.10 GHz and equipped with 192 GB total main memory. The computer runs on CentOS Linux release 7.5.1804 (Core) with MATLAB 9.9.0.1467703 (R2020b).

The benchmark data used can be found in Table 1, where the first four columns state the dimensions of the considered problem; cf. (1);  $\#\lambda_{\text{unstab}}$  denotes the number of unstable eigenvalues of the matrix pencil  $sE - A$ , in the data column the example type is denoted with either dense or sparse, and  $\gamma$  is a constant multiplied with the disturbance term,  $\frac{1}{\gamma}B_1$ , such that the considered Riccati equations have symmetric positive semi-definite stabilizing solutions.

For the comparison of results between different algorithms and benchmarks, we will report different information about the performance of the applied solvers in upcoming tables such as the used iteration steps, the overall runtime in seconds, the rank of the resulting solution factor, the final normalized residual of the Riccati iteration as given by  $\|B_1 Y_k Y_k^\top E\|_2^2 / \|CC^\top\|_2$ , the relative residual of the solution factor given by  $\|\mathcal{R}(Z_k Z_k^\top)\|_2 / \|Z_k^\top Z_k\|_2$  and the normalized residual of the solution factor computed as  $\|\mathcal{R}(Z_k Z_k^\top)\|_2 / \|CC^\top\|_2$  using the Riccati operator  $\mathcal{R}$  from (2), and lastly the norm of the full solution  $\|Z_k^\top Z_k\|_2$ .

#### Code and data availability

The source code of the implementations used to compute the presented results, the used data and computed results are available at

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Table 2: Results for the aircraft and cable mass benchmarks.

	AC10			CM6		
	LRRI	ICARE	SIGN	LRRI	ICARE	SIGN
Iteration steps	5	—	19	4	—	23
Runtime (s)	0.92801	0.47971	0.07705	29.9208	140.136	7.24722
Rank $Z_k$	53	55	55	569	758	781
Final res.	5.545e-25	—	—	1.873e-15	—	—
Relative res.	2.599e-07	9.617e-10	1.183e-09	1.910e-09	5.019e-08	2.125e-07
Normalized res.	1.554e-03	5.752e-06	7.074e-06	1.667e-05	4.381e-04	1.855e-03
$\ Z_k^T Z_k\ _2$	1.457e+01	1.457e+01	1.457e+01	1.253e+04	1.253e+04	1.253e+04

### 3.1 Factorized method for dense examples

Before we actually consider the large-scale sparse case, we test the new method from [Algorithm 2](#) on medium-scale dense benchmarks and compare the results with two other commonly known solvers that can be applied to [\(1\)](#). We use the dense low-rank factor version of [Algorithm 2](#) as implemented in `ml_icare_ric_fac` of [\[25\]](#), which we denote further on as LRRI, but we replace the internal Riccati equation solver by the factorized sign function approach [\[13\]](#) implemented in the very same toolbox [\[25\]](#) as `ml_cared1_sgn_fac`. For comparison, we use the Hamiltonian eigenvalue approach from [\[4\]](#) as it is implemented in the new MATLAB function `icare` from the Control System Toolbox™, further denoted by ICARE, and the classical (unfactorized) sign function solver [\[51\]](#) from the MORLAB toolbox [\[25\]](#) in `ml_cared1_sgn`, further on as SIGN. Since SIGN and ICARE can only compute unfactorized solutions of [\(1\)](#), we perform eigendecompositions of the computed solutions to check the positive semi-definiteness and to compute low-rank approximations of the solutions by truncating all components corresponding to non-positive eigenvalues.

Considering the benchmarks, the first two data sets in [Table 1](#) are practical examples taken from [\[43\]](#), where AC10 is an aircraft model and CM6 a cable mass model with low damping. The data was taken over with exactly the same naming as given in [\[43\]](#), where we have set  $C = C_1$ . The results of the different algorithms on these two data sets are shown in [Table 2](#). First, we recognize that for AC10, LRRI performs visibly worse than the other two approaches in terms of accuracy, since we are losing three orders of magnitude in the relative and normalized residuals. We assume this comes from the bad conditioning of the  $A$  matrix in this example and the repeated solution of Riccati equations with the matrix. However, for the CM6 example, this turned around, as LRRI performs now an order of magnitude better than ICARE and two orders better than SIGN. Concerning the computation times we see that LRRI performs reasonably well thanks to the efficient inner Riccati equation solver. While on the smaller data set AC10, the runtime of LRRI is in the same order of magnitude as ICARE, LRRI is already four times faster than ICARE for CM6. We are not able to outperform SIGN with LRRI in these examples.

To further investigate accuracy and performance of LRRI, we created random examples denoted by `rand*` in [Table 1](#) using the `randn` function in MATLAB. The results of the computations can be found in [Tables 3](#) and [4](#). Overall, LRRI competes very well against ICARE and SIGN in terms of accuracy. For all four presented random examples, the residuals of the solution factors lie in the same order of magnitude for all methods with only minor exceptions. In terms of runtimes, we see the same relations that we already recognized in [Table 2](#). LRRI easily outperforms ICARE due to its efficient inner factorized

Table 3: Results for the smaller random examples.

	rand512			rand1024		
	LRRi	ICARE	SIGN	LRRi	ICARE	SIGN
Iteration steps	8	—	14	7	—	8
Runtime (s)	6.94609	23.0720	1.39393	10.3911	217.230	2.96977
Rank $Z_k$	96	298	294	89	522	512
Final res.	7.889e-23	—	—	2.600e-14	—	—
Relative res.	7.544e-10	2.926e-11	3.508e-11	6.737e-11	6.980e-11	1.074e-10
Normalized res.	3.930e-10	1.524e-11	1.827e-11	6.164e-12	6.387e-12	9.830e-12
$\ Z_k^T Z_k\ _2$	3.202e+02	3.202e+02	3.202e+02	1.058e+02	1.058e+02	1.058e+02

Table 4: Results for the larger random examples.

	rand2048			rand4096		
	LRRi	ICARE	SIGN	LRRi	ICARE	SIGN
Iteration steps	5	—	11	4	—	9
Runtime (s)	34.8321	2043.78	21.4784	139.670	16042.8	125.835
Rank $Z_k$	59	992	967	72	1984	1961
Final res.	2.450e-21	—	—	2.000e-14	—	—
Relative res.	4.997e-10	2.977e-10	1.628e-10	1.811e-10	1.804e-09	1.813e-10
Normalized res.	3.402e-11	2.027e-11	1.108e-11	2.812e-11	2.803e-10	2.816e-11
$\ Z_k^T Z_k\ _2$	1.642e+02	1.642e+02	1.642e+02	6.760e+02	6.760e+02	6.760e+02

sign function solver. For increasing problem size also this speed-up tremendously increases further up to the largest example **rand4096**, where LRRi is 113 times faster than ICARE. Compared to SIGN, LRRi is still not able to outperform the classical sign function iteration method. However, we can observe that the difference in runtimes is getting smaller and smaller, i.e., we can expect LRRi to outperform SIGN for larger problems. Also, these results reveal the strong dependence of LRRi on the inner Riccati equation solver and the power that comes from the ability to use sophisticated implementations for the inner Riccati equations with negative semi-definite quadratic terms.

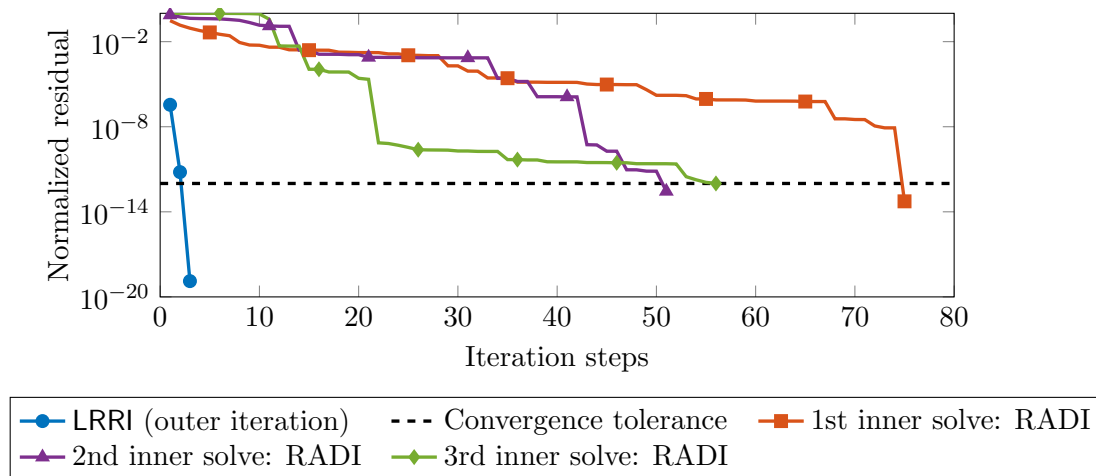
### 3.2 Low-rank approach for large-scale sparse examples

Now we come to the case of large-scale sparse Riccati equations with indefinite quadratic terms. The LRRi method from [Algorithm 2](#) is to our knowledge the only method suited to solve equations like [\(1\)](#) in the large-scale sparse setting. Therefore, we cannot compare our results to other methods. However, from the previous section we expect LRRi to yield reasonable accurate results in comparison to alternative Riccati equation solvers and the runtimes to be mainly depend on the inner Riccati equation solver. We use the implementation of LRRi from [\[53\]](#) with the option to have RADI [\[11\]](#) or LR-Newton-ADI [\[20\]](#) as inner solvers or to switch between them during runtime if suitable.

As first example, we consider the optimal cooling problem of a steel profile; see, e.g., [\[52\]](#); with the data available in [\[53\]](#). For the different matrices in the quadratic term of [\(1\)](#), we consider the boundary control of the three lower segments of the profile edges as

Table 5: Results of LRRI for the large-scale sparse examples.

	rail	cylinderwake
Iteration steps	3	3
Runtime (s)	72.7378	3469.59
Rank $Z_k$	169	418
Final res.	1.297e-19	2.184e-21
Relative res.	2.125e-21	1.996e-14
Normalized res.	9.766e-11	1.622e-03
$\ Z_k^T Z_k\ _2$	6.866e+11	5.056e+08

Figure 1: Convergence of Riccati iteration and inner solvers for the `rail` example.

disturbances to give us  $B_1$  and the rest to be control inputs in  $B_2$ . The resulting dimensions are given in Table 1. For LRRI, we use only the RADI method as inner Riccati equation solver and the results of the iteration can be seen in the `rail` column of Table 5. The iteration converges quickly and yields a very accurate solution of low rank. The behavior of the outer and inner iteration methods is shown in terms of normalized residuals in Figure 1. For all solvers, the same convergence tolerance has been used.

As second example, we consider the laminar flow in a wake with a cylinder obstacle at Reynolds number 60, as described in [16] and with the data available at [33]. The example has two inputs modeling the suction and injection of fluid at the back of the cylinder placed in the beginning of the wake. We consider the case that one of the outlets is defect and produces only noise that needs to be compensated. Therefore, the defect outlet gives us the  $B_1$  matrix and the control outlet gives us the matrix  $B_2$ . Also, the considered Riccati equation has exactly the structure as in (12) resulting from the underlying dynamical system of differential-algebraic equations. Therefore, we use the implicit truncation approach mentioned in Section 2.4.2, which is implemented in the `dae.2` function handles in [53]. Also, we note that the matrix pencil of this example has unstable eigenvalues. Since RADI is difficult to use for such a problem since it needs a stabilizing initial solution that produces a positive semi-definite residual in the Riccati operator, we switch only for the first iteration step of LRRI to the LR-Newton-ADI method and use a stabilizing Bernoulli feedback the same way as in [5]. The results of LRRI can be seen in the `cylinderwake`

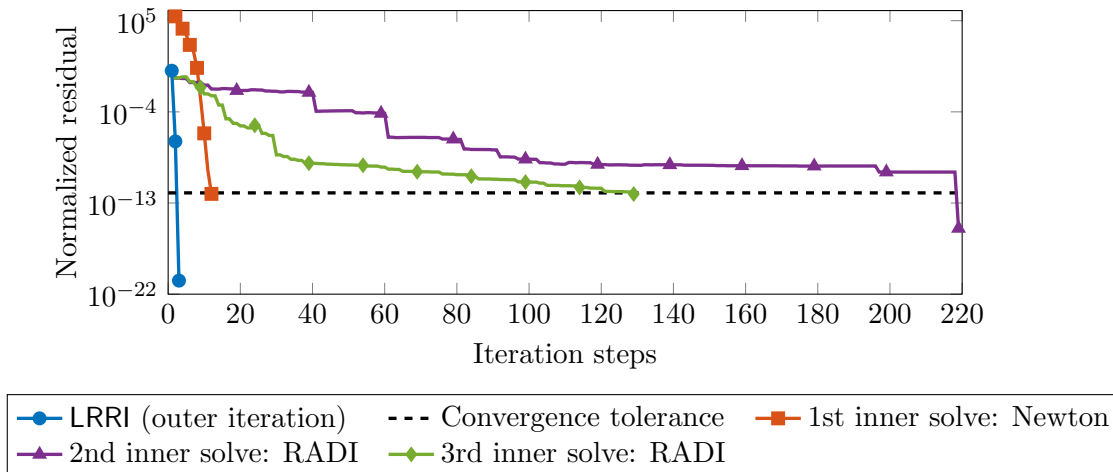


Figure 2: Convergence of Riccati iteration and inner solvers for the cylinder wake example.

column of Table 5 and the normalized residuals of the outer and inner iterations are shown in Figure 2. While the relative residual is comfortably small again for this example due to the very large norm of the stabilizing solution, the normalized residual is still quite high. This likely comes from the general bad conditioning of the problem and the small norm of the right-hand side matrix. But overall, the computed solution has been obtained with reasonable accuracy and in a reasonable amount of time.

## 4 Conclusions

We have developed a low-rank iterative method for solving large-scale sparse Riccati equations with indefinite quadratic terms, which is based on solutions of Riccati equations with negative semi-definite quadratic terms. Numerical examples have illustrated that, in the dense case, we can expect a similar accuracy and good performance in comparison to other established Riccati equation solvers, and that for large-scale sparse equations, the method also yields reasonably good results. We have also extended the LR-RI approach to indefinite algebraic Riccati equations related to descriptor systems with singular  $E$  matrix.

To our knowledge, the low-rank Riccati iteration is currently the only approach to solve Riccati equations with indefinite quadratic terms in the large-scale sparse case. Another idea that might be directly extendable to this problem are projection-based methods. However, a problem occurring already in the case of classical Riccati equations with semi-definite quadratic terms is the solvability of the projected equations. Only recently, new results have been obtained under which conditions constructed projection spaces preserve the existence of stabilizing solutions in projected Riccati equations with negative semi-definite quadratic terms [63]. This problem becomes even more complicated when dealing with Riccati equations with indefinite quadratic terms, and so far, there are neither theoretical nor numerical results available considering this.

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