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Quantizations of local surfaces and rebel instantons

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Abstract. We construct explicit deformation quantizations of the noncompact complex surfaces $Z_k := \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ and describe their effect on moduli spaces of vector bundles and instanton moduli spaces. We introduce the concept of rebel instantons, as being those which react badly to some quantizations, misbehaving by shooting off extra families of noncommutative instantons. We then show that the quantum instanton moduli space can be viewed as the étale space of a constructible sheaf over the classical instanton moduli space with support on rebel instantons.

1. Motivation and results

In this work, we clarify a specific aspect of the quantization of SU(2) instantons, by describing explicitly the effect that deformation quantization has on moduli spaces of instantons over noncompact complex surfaces. Our main contribution is identifying those classical instantons which react badly to certain choices of quantization, by shooting off extra families of noncommutative instantons. We call them rebel instantons. Thus, rebel instantons cause the quantum moduli spaces to become larger than the corresponding moduli on the classical limit.

Our complex surfaces of choice are total spaces of negative line bundles on the complex projective line $\mathbb{P}^1$, namely, the surfaces $Z_k := \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ for $k \geq 1$. Our interest in them originates in the wish to understand how the mathematical physics on them reacts to contraction of the complex curve $\mathbb{P}^1$ to a point. This can only be done in the case of negative normal bundle, hence the choice of $-k$ neighborhoods. Contracting this curve produces a singular surface whenever $k > 1$. In fact, the case $k = 1$—where the contraction produces the smooth surface $\mathbb{C}^2$—allows for quantization without any rebel instantons, in contrast to what happens when $k > 1$.

For each surface Z_k, we compute all possible holomorphic Poisson structures and then calculate explicitly corresponding deformation quantizations, writing out star products. We study instantons and their moduli via the Kobayashi–Hitchin correspondence, that is, by constructing holomorphic vector bundles over these surfaces. We describe vector bundles concretely using matrices, in the spirit of the classical ADHM construction [1], but in a somewhat further simplified manner which is made possible by the filtrability of bundles on Z_k. Accordingly, an instanton or vector bundle on Z_k can be described by a single upper-triangular matrix with polynomial entries. Proving an analogous filtrability

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result in the noncommutative case allows us to describe vector bundles over noncommutative deformations and to describe explicitly how their moduli spaces change after deforming the surface, a result which may be of independent interest.

Our main result is the following.

Theorem (Theorem 9.6). *The quantum instanton moduli space $\mathbb{QI}_j^{(1)}(\mathcal{Z}_k(\sigma))$ can be viewed as the étale space of a constructible sheaf over the classical instanton moduli space $\mathbb{MI}_j(Z_k)$, which is supported on a closed subvariety, being trivial over*

$$S_0 := \{P \in \mathbb{MI}_j(Z_k) \mid p_{1,k-j+1} \neq 0\}$$

and having stalk of dimension i over

$$S_i := \{P \in \mathbb{MI}_j(Z_k) \mid p_{1,k-j+1} = \cdots = p_{1,k-j+1+i} = 0,\ p_{1,k-j+2+i} \neq 0\}.$$

Since we are dealing with moduli spaces of bundles on noncompact varieties, the topology of moduli spaces of bundles is rather subtle, already for the classical moduli space [5]. A detailed analysis of the topology of the quantum moduli space would certainly be important and might provide insight into further interesting phenomena, such as studying the effect of evaluating the formal deformation parameter $\hbar$ to a nonzero constant where possible. However, these considerations would deserve a separate treatment and we thus choose to sideline these topological issues for the present article by simply viewing the quantum moduli space as the étale space (or sheaf space) of a constructible sheaf, constant over each stratum, making the map to the classical moduli space continuous.

Nonetheless, Theorem 9.6 already shows that the effect of noncommutative deformations on instantons is radically different from the effect of commutative ones. In fact, [8, Thm. 7.3] showed that (nontrivial) classical deformations of Z_k admit *no* instantons, although the surfaces Z_k have rich instanton moduli spaces. (See Definition 9.2 for the representation of instantons in canonical coordinates.)

We now describe some of the literature on the subject and next the structure of this paper. Instantons on noncommutative spaces have been considered from various points of view, starting with instantons on noncommutative $\mathbb{R}^4$ and noncommutative tori [32] and their relations with string theory [35]. Nekrasov–Schwarz [32] proposed a modification of the ADHM equations, which describe instantons on a noncommutative $\mathbb{R}^4$. Kapustin–Kuznetsov–Orlov [20] showed that the complex point of view can also be generalized to the noncommutative setting, identifying these noncommutative instantons with algebraic vector bundles on a noncommutative projective plane $\mathbb{P}^2_\hbar$ framed at a line at infinity. The space of solutions to these modified ADHM equations turns out to yield a smooth compactification of the moduli space of instantons on the (commutative) $\mathbb{R}^4$, so that the noncommutative viewpoint also sheds light onto classical instantons. (Other interesting features and generalizations are as follows. This compactification can be viewed as the moduli space of torsion-free sheaves on $\mathbb{P}^2$ framed at a fixed line at infinity; see Nakajima [30]. Furthermore, from the point of view of instanton counting, moduli of torsion-free sheaves were extensively used as partial compactifications of moduli of instantons, in the very successful instanton partition function defined by Nekrasov, and explored in [19, 31, 33]. There are also approaches to noncommutative instantons from the point of view of noncommutative geometry as, for example, in [10, 11]. We shall not pursue these aspects here.)

Classical SU(2) instantons on Z_k can be identified with holomorphic rank 2 bundles on Z_k via the Kobayashi–Hitchin correspondence, which for $k = 1$ was proven by King [22] and for $k \geq 2$ in [18]. Considering deformation quantizations $\mathcal{A}$ of the sheaf $\mathcal{O}_{Z_k}$, we obtain instantons on noncommutative deformations of Z_k as locally free sheaves of $\mathcal{A}$-modules, generalizing the concept of holomorphic vector bundle to these noncommutative spaces. In general, the surfaces Z_k do not admit locally constant holomorphic Poisson structures, and instead of the Moyal product, we shall thus consider the Kontsevich star product extended to these surfaces.

The paper is organized as follows. In Sections 2–4, we study the noncommutative deformation theory of the surfaces Z_k. In Section 5, we review the theory of vector bundles on Z_k and their commutative deformations and discuss vector bundles on noncommutative deformations of Z_k in Section 6. We use the explicit expressions of star products obtained in Section 4 to show in Section 7 that purely noncommutative deformations have nontrivial moduli of vector bundles. Applications to (noncommutative) instantons are described in Sections 8–9.

2. Poisson geometry

In this section, we study the holomorphic Poisson geometry of the surfaces $Z_k = \operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-k)$ for $k \geq 1$, which will be the starting point for quantizations in subsequent sections.

Definition 2.1 ([26]). A *holomorphic Poisson bracket* on a complex manifold or smooth complex algebraic variety X is a $\mathbb{C}$-bilinear map

$$\{\cdot,\cdot\}\colon \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

satisfying

$$\begin{aligned} \{f,g\} &= -\{g,f\} && (\textit{skew-symmetry}),\\ \{fg,h\} &= f\{g,h\} + \{f,h\}g && (\textit{Leibniz rule}),\\ \big\{\{f,g\},h\big\} &= \big\{f,\{g,h\}\big\} + \big\{\{f,h\},g\big\} && (\textit{Jacobi identity}) \end{aligned}$$

for all $f, g, h \in \mathcal{O}_X$.[1]

Equivalently, a holomorphic Poisson structure may be described by a holomorphic bivector field $\sigma \in \mathrm{H}^0(X, \Lambda^2\mathcal{T}_X)$ whose Schouten–Nijenhuis bracket $[\sigma,\sigma] \in \mathrm{H}^0(X, \Lambda^3\mathcal{T}_X)$

[1] Here we write $f, g, h \in \mathcal{O}_X$ for sections of $\mathcal{O}_X$ over some open set. More precisely, a holomorphic Poisson bracket on $\mathcal{O}_X$ is a family of Poisson brackets indexed by the open sets of X and compatible with the restriction morphisms of the sheaf $\mathcal{O}_X$.

is zero. The associated Poisson bracket is then given by the pairing $\langle \cdot, \cdot \rangle$ between vector fields and forms

$$\{f, g\} = \langle \sigma, \mathrm{d}f \wedge \mathrm{d}g \rangle.$$

Remark 2.2. On a smooth *surface* X, the condition $[\sigma, \sigma] = 0$ is satisfied for any bivector field $\sigma \in \mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X)$ since $\Lambda^3 \mathcal{T}_X = 0$, and thus, $\mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X)$ may be identified with the space of holomorphic Poisson structures.

We now focus on the surfaces Z_k and their Poisson structures, which we shall describe explicitly in canonical coordinates.

Notation 2.3. We fix coordinate charts U, V on Z_k, which we will refer to as *canonical coordinates*, where

$$U = \mathbb{C}^2_{z,u} = \{(z, u) \in \mathbb{C}^2\} \quad \text{and} \quad V = \mathbb{C}^2_{\xi,v} = \{(\xi, v) \in \mathbb{C}^2\} \tag{2.4}$$

such that on $U \cap V = \mathbb{C}^* \times \mathbb{C}$ we identify

$$(\xi, v) = (z^{-1}, z^k u) \tag{2.5}$$

so that z and z^{-1} are the local coordinates on $\mathbb{P}^1$ which is the zero section of $Z_k = \operatorname{Tot} \mathcal{O}_{\mathbb{P}^1}(-k)$.

We denote by ℓ the ideal of the zero section of Z_k regarded as a divisor. Hence, ℓ is generated by u on the U chart and by v on the V chart.

In canonical coordinates, a holomorphic bivector field $\sigma \in \mathrm{H}^0(Z_k, \Lambda^2 \mathcal{T}_{Z_k})$ is thus of the form $\sigma_U \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u}$ on U and $\sigma_V \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial v}$ on V, where σ_U (resp. σ_V) is a holomorphic function in z and u (resp. ξ and v) whose precise form will be given in Lemma 2.8.

Given a bivector field σ on Z_k, the corresponding *Poisson bracket* $\{f, g\}_\sigma$ of two global functions $f, g \in \mathrm{H}^0(Z_k, \mathcal{O})$ may be written in canonical coordinates as

$$\{f, g\}_\sigma|_U = \left\langle \sigma_U \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u}, \mathrm{d}f_U \wedge \mathrm{d}g_U \right\rangle = \sigma_U \left(\frac{\partial f_U}{\partial z} \frac{\partial g_U}{\partial u} - \frac{\partial f_U}{\partial u} \frac{\partial g_U}{\partial z} \right),$$
$$\{f, g\}_\sigma|_V = \left\langle \sigma_V \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial v}, \mathrm{d}f_V \wedge \mathrm{d}g_V \right\rangle = \sigma_V \left(\frac{\partial f_V}{\partial \xi} \frac{\partial g_V}{\partial v} - \frac{\partial f_V}{\partial v} \frac{\partial g_V}{\partial \xi} \right),$$

where d is the exterior derivative and f_U, g_U denote the restrictions of f, g to U written in (z, u)-coordinates and similarly for f_V, g_V.

Notation 2.6. When referring to a bivector field $\sigma \in \mathrm{H}^0(Z_k, \Lambda^2 \mathcal{T}_{Z_k})$ or its corresponding Poisson structure, we will work in canonical coordinates in the basis $\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u}$ and $\frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial v}$ and only write its coefficient functions as a pair (σ_U, σ_V)—in fact, we often just write σ_U in (z, u)-coordinates, as σ_V can be recovered by writing $-z^{k-2} \sigma_U$ as a function of ξ and v via the change of variables (2.5).

An application of the exponential sheaf sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

shows that any line bundle on Z_k can be identified with the pullback $\pi^*\mathcal{O}_{\mathbb{P}^1}(n)$ of the projection $\pi\colon Z_k \to \mathbb{P}^1$ to the zero section of Z_k. We write $\mathcal{O}_{Z_k}(n)$ or $\mathcal{O}(n)$ for $\pi^*\mathcal{O}_{\mathbb{P}^1}(n)$, whose change of coordinates from U to V can be given by the transition function z^{-n}. Here, $n \in \mathbb{Z}$ is the first Chern class of $\mathcal{O}(n)$.

To calculate Poisson structures, note that $\Lambda^2\mathcal{T}_{Z_k}$ is the anticanonical line bundle. The transition matrix for the tangent bundle of Z_k is given in canonical coordinates by the Jacobian matrix of the change of coordinates $(z,u) \mapsto (z^{-1}, z^k u)$ of the manifold. For Z_k,

$$\mathrm{Jac}_{UV} = \begin{pmatrix} \frac{\partial}{\partial z} z^{-1} & \frac{\partial}{\partial u} z^{-1} \\ \frac{\partial}{\partial z} z^k u & \frac{\partial}{\partial u} z^k u \end{pmatrix} = \begin{pmatrix} -z^{-2} & 0 \\ k\, z^{k-1} u & z^k \end{pmatrix}. \tag{2.7}$$

The transition function for the anticanonical line bundle is then given by the determinant of this Jacobian, which is $-z^{k-2}$. We can thus identify $\mathrm{H}^0(Z_k, \Lambda^2\mathcal{T}_{Z_k}) \simeq \mathrm{H}^0(Z_k, \mathcal{O}(-k+2))$ as the space of Poisson structures.

Lemma 2.8. *A general Poisson structure $\sigma \in \mathrm{H}^0(Z_k, \Lambda^2\mathcal{T}_{Z_k})$ on Z_k is given in canonical coordinates by*

$$\left(\sigma_U \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u}, \sigma_V \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial v}\right),$$

where (σ_U, σ_V) is of the form

(1) $(f_U + z\, g_U, -\xi\, f_V - g_V)$ *for* $k = 1$,

(2) $(f_U, -f_V)$ *for* $k = 2$,

(3) $(u\, f_U + zu\, g_U + z^2 u\, h_U, -\xi^2 v\, f_V - \xi v\, g_V - v\, h_V)$ *for* $k \geq 3$,

for any global functions $(f_U, f_V), (g_U, g_V), (h_U, h_V) \in \mathrm{H}^0(Z_k, \mathcal{O}_{Z_k})$.

In the basis $(\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u}, \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial v})$, the space of Poisson structures $\mathrm{H}^0(Z_k, \Lambda^2\mathcal{T}_{Z_k})$ is thus generated by

(1) $(1, -\xi), (z, -1)$ *for* $k = 1$,

(2) $(1, -1)$ *for* $k = 2$,

(3) $(u, -\xi^2 v), (zu, -\xi v), (z^2 u, -v)$ *for* $k \geq 3$,

as a module over global functions.

Proof. For (1), we need to calculate $\mathrm{H}^0(Z_1, \mathcal{O}(1))$, which is Lemma A.1. For (2) and (3), we need to calculate $\mathrm{H}^0(Z_k, \mathcal{O}(-k+2))$, which is Lemma A.2. ∎

Remark 2.9. The space of Poisson structures on Z_k is of infinite dimension over $\mathbb{C}$, but restricted to the nth infinitesimal neighborhood $\ell^{(n)}$, the space of Poisson structures is of dimension

$$\begin{cases} \frac{(n+1)(n+4)}{2} & \text{for } k = 1, \\ n^2 & \text{for } k = 2, \\ \frac{n((n-1)k+4)}{2} & \text{for } k \geq 3. \end{cases}$$

Notation 2.10. Write $Z_{\geq m}$ for Z_k with $k \geq m$.

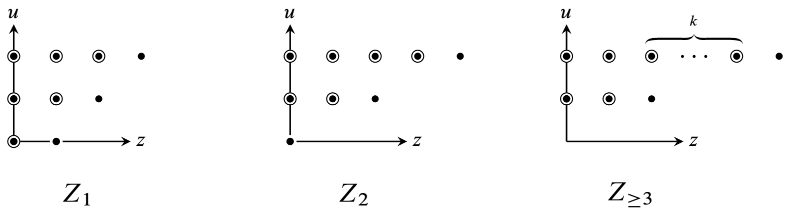


Figure 1. Monomials of Poisson structures on Z_k.

To describe explicit quantizations of Poisson structures on non-affine varieties, the following property will be useful.

Definition 2.11. A bivector field $\sigma \in \mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X)$ is *tangent* to a divisor $D \subset X$ if

$$\{\mathcal{O}_X, \mathcal{I}_D\} \subset \mathcal{I}_D,$$

i.e., if the ideal sheaf $\mathcal{I}_D$ of D is a Poisson ideal. Geometrically, σ is tangent to D if, for every function $f \in \mathcal{O}_X$, the restriction of the Hamiltonian vector field $X_f = \{f, \cdot\}_\sigma$ to the divisor D is tangent to D.

In Section 4, we shall quantize Poisson structures tangent to the complement of an affine coordinate chart, so we record their particular form.

Proposition 2.12. *Consider the open immersion $\mathbb{C}^2 \simeq U \subset Z_k$ with complementary divisor $D = Z_k \setminus U = \{(0, v) \in V\} \simeq \mathbb{C}$. Then the space of Poisson structures tangent to D is generated by*

(1) $(1, -\xi)$ *for* $k = 1$,

(2) $(u, -\xi^2 v), (zu, -\xi v)$ *for* $k \geq 2$,

as a module over global functions.

Proof. This follows from Lemma 2.8 and the observation that since $\mathcal{I}_D|_V = (\xi)$, the coefficient function σ_V of a Poisson structure (σ_U, σ_V) which is tangent to D should be a multiple of ξ in V-coordinates. ■

Poisson structures on Z_k are depicted in Figure 1, where dots represent the monomials (in U-coordinates) which appear in the expression of a general Poisson structure on Z_k and circled dots those of a Poisson structure tangent to $D = Z_k \setminus U$.

Recall that the *rth degeneracy locus* of a holomorphic Poisson structure on a complex manifold or algebraic variety X is defined as

$$D_{2r}(\sigma) = \{x \in X \mid \operatorname{rank} \sigma(x) \leq 2r\},$$

where σ is viewed as a map $\mathcal{T}_X^* \to \mathcal{T}_X$ by contracting a 1-form with the bivector field σ. At a given point on a complex *surface*, a holomorphic Poisson structure has either full rank or rank 0. Thus, for the surfaces Z_k, we call $\operatorname{dgn}(\sigma) := D_0(\sigma)$ *the degeneracy locus* of σ and this degeneracy locus is given by the zeros of the coefficient functions σ_U and σ_V.

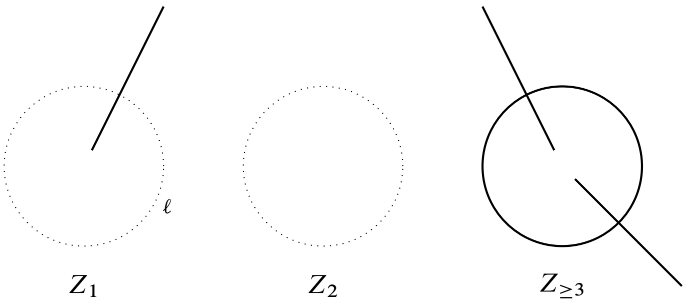


Figure 2. The (real part of the) degeneracy loci of the Poisson structures of Example 2.13.

A non-degenerate holomorphic Poisson structure σ is called a *holomorphic symplectic* structure as σ determines a non-degenerate closed holomorphic 2-form ω by

$$\omega(X_f, X_g) = \{f, g\}_\sigma,$$

where X_f denotes the Hamiltonian vector field associated to a function f.

As we shall see in Proposition 2.16, Z_2 is the only one of the surfaces Z_k that admits a holomorphic symplectic structure and when this structure is algebraic, it is unique up to scaling.

Example 2.13. If σ is as in Lemma 2.8 with global functions (f_U, f_V), (g_U, g_V), (h_U, h_V) simply (nonzero) *constants*, we have

$$\operatorname{dgn}(\sigma) = \begin{cases} \pi^{-1}(x) & \text{for } k = 1, \\ \emptyset & \text{for } k = 2, \\ \pi^{-1}(x) \cup \pi^{-1}(y) \cup \ell & \text{for } k \geq 3, \end{cases}$$

where $\ell \subset Z_k$ is the zero section and $x, y \in \mathbb{P}^1$ are two points, possibly equal. These degeneracy loci are depicted in Figure 2.

For $k = 2$, the Poisson structure is non-degenerate and defines a holomorphic symplectic form.

If σ is a Poisson structure which is tangent to a fibre $D = \pi^{-1}(x) \simeq \mathbb{C}$ of the bundle projection $\pi\colon Z_k \to \mathbb{P}^1$, the picture of Figure 2 reduces to two cases

$$\operatorname{dgn}(\sigma) = \begin{cases} \pi^{-1}(x) & \text{for } k = 1, \\ \pi^{-1}(x) \cup \ell & \text{for } k \geq 2. \end{cases} \tag{2.14}$$

We can set $U = Z_k \setminus D$ so that $x \in \mathbb{P}^1$ is given in canonical coordinates by $\xi = 0$ and in these coordinates the form of σ is given in Proposition 2.12.

Equation (2.14) describes the degeneracy locus for a nonzero linear combination of generators of those Poisson structures tangent to a fibre $D = \pi^{-1}(x)$. The degeneracy locus of a general Poisson structure σ tangent to D may be more complicated. However, to construct quantizations of σ, it is only necessary to require that $\mathrm{dgn}(\sigma)$ contain a fibre of the projection π. The effect of quantizations on moduli then only depends on whether $\mathrm{dgn}(\sigma)$ also contains all of ℓ or not and so we introduce the following notation.

Notation 2.15. Write σ_0, respectively σ, for Poisson structures on Z_k such that

$$\ell \not\subset \mathrm{dgn}(\sigma_0) \supset \pi^{-1}(x),$$
$$\ell \subset \mathrm{dgn}(\sigma) \supset \pi^{-1}(x),$$

respectively.

Note that σ_0 is a "minimally degenerate" Poisson structure on Z_1, whereas σ denotes Poisson structures which are degenerate on all of ℓ, which can occur on Z_k for all $k \geq 1$.

Proposition 2.16. *Z_k admits a holomorphic symplectic structure if and only if $k = 2$. Moreover, if the holomorphic symplectic structure on Z_2 is algebraic, then it is a constant multiple of the canonical symplectic structure on $Z_2 \simeq T^*\mathbb{P}^1$.*

Proof. A holomorphic symplectic structure corresponds to a nowhere vanishing global holomorphic section of $\Lambda^2 \mathcal{T}_{Z_k} \simeq \mathcal{O}(-k+2)$, which exists only when this bundle is trivial, i.e., when $k = 2$.

For Z_2, the canonical non-degenerate Poisson structure is given by $(1, -1)$, so that in coordinates the bracket of two functions f, g is given by

$$\{f, g\}|_U = \left\langle \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u}, \mathrm{d} f_U \wedge \mathrm{d} g_U \right\rangle,$$
$$\{f, g\}|_V = -\left\langle \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial v}, \mathrm{d} f_V \wedge \mathrm{d} g_U \right\rangle.$$

If a non-degenerate Poisson structure on Z_2 is algebraic, i.e., an algebraic section of $\Lambda^2 \mathcal{T}_{Z_2}$, it is a constant multiple of the canonical non-degenerate Poisson structure, as every non-constant polynomial in z, u has at least one zero. (This does not hold in the analytic category, as there are non-constant non-vanishing complex analytic sections.) ■

Alternative proof. We shall see in Section 9 that Z_k is the minimal resolution of the $\frac{1}{k}(1,1)$ surface singularity $X_k \simeq \mathbb{C}^2/\Gamma$, where Γ is generated by $\gamma = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix}\right)$ for ω a primitive kth root of unity. A resolution of $\mathbb{C}^2/\Gamma$ admits a holomorphic symplectic form if and only if $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$. However, $\det \gamma = \omega^2 = 1$ if and only if $k = 2$. ■

Remark 2.17. Here, we discuss the case of *degenerate* Poisson structures on Z_k, leaving the holomorphic symplectic case on Z_2 for future work. More precisely, our construction works for Poisson structures tangent to a fibre of $\pi \colon Z_k \to \mathbb{P}^1$ (see Definition 2.11 and Proposition 4.1), which the holomorphic symplectic structure on Z_2 does not satisfy (cf. Remark 4.9).

3. Deformation quantization and star products

Let $\hbar$ be a formal variable and denote by $\mathcal{O}_X[\![\hbar]\!]$ the completed tensor product $\mathcal{O}_X \mathbin{\hat{\otimes}} \mathbb{C}[\![\hbar]\!]$, viewed as a sheaf of $\mathbb{C}[\![\hbar]\!]$-vector spaces on X, where a "section" over an open set U is given by a formal power series $f = \sum_{n=0}^{\infty} f_n \hbar^n$ with each $f_n \in \mathcal{O}_X(U)$. We shall turn $\mathcal{O}_X[\![\hbar]\!]$ into a sheaf of associative $\mathbb{C}[\![\hbar]\!]$-algebras by formally deforming the usual commutative product on functions. The augmentation $\mathbb{C}[\![\hbar]\!] \to \mathbb{C}$ induces an augmentation $\mathcal{O}_X[\![\hbar]\!] \to \mathcal{O}_X$.

Definition 3.1. A *star product* on a complex manifold (or smooth complex algebraic variety) X is a $\mathbb{C}[\![\hbar]\!]$-bilinear associative product

$$\star\colon \mathcal{O}_X[\![\hbar]\!] \times \mathcal{O}_X[\![\hbar]\!] \to \mathcal{O}_X[\![\hbar]\!]$$

which is of the form

$$(f, g) \mapsto fg + \sum_{n=1}^{\infty} B_n(f, g)\hbar^n,$$

where B_n are bidifferential operators, i.e., bilinear operators which are differential operators in both arguments.

Definition 3.2 ([36, §0.1]). Two star products $\star$, $\star'$ on X are said to be *gauge equivalent*, if there exists an isomorphism $(\mathcal{O}_X[\![\hbar]\!], \star) \simeq (\mathcal{O}_X[\![\hbar]\!], \star')$ which commutes with the augmentations $\mathcal{O}_X[\![\hbar]\!] \to \mathcal{O}_X$.

Definition 3.3. Let (X, σ) be a holomorphic Poisson manifold with associated Poisson bracket $\{\cdot\,,\cdot\}_\sigma$. A *deformation quantization* of (X, σ) is a pair $(X, \star_\sigma)$, where $\star_\sigma$ is a star product on X with $B_1(f, g) = \{f, g\}_\sigma$, that is,

$$f \star_\sigma g = fg + \{f, g\}_\sigma \hbar + \cdots .$$

We set $\mathcal{A}^\sigma := (\mathcal{O}[\![\hbar]\!], \star_\sigma)$ the sheaf of formal functions with holomorphic coefficients on the quantization $(X, \star_\sigma)$.

We call $\mathcal{Z}_k(\sigma) = (Z_k, \mathcal{A}^\sigma)$ a *noncommutative deformation* of Z_k. As we usually work with a specified fixed Poisson structure, we usually use the abbreviated notations $\mathcal{A}$, $\{\cdot\,,\cdot\}$ and $\star$.

The existence of star products for arbitrary possibly degenerate Poisson structures was first proved in the C^∞ setting in Kontsevich's seminal paper [24] and later generalized to the algebro-geometric setting [23, 36]. We quote this generalization in a simplified form.

Theorem 3.4 ([36, Cor. 11.2]). *Let X be a complex algebraic variety with structure sheaf $\mathcal{O}_X$ and assume that $\mathrm{H}^1(X, \mathcal{O}_X)$ and $\mathrm{H}^2(X, \mathcal{O}_X)$ vanish. Then there is a bijection*

$$\{\textit{Poisson deformations of } \mathcal{O}_X\}/\!\sim\ \longleftrightarrow\ \{\textit{associative deformations of } \mathcal{O}_X\}/\!\sim,$$

where $\sim$ denotes gauge equivalence.

We note that the surfaces Z_k satisfy the hypotheses of Theorem 3.4.

Remark 3.5. Working with the sheaf of algebras $(\mathcal{O}_X[\![\hbar]\!], \star)$, we are restricting ourselves to deformations which are in some sense "purely noncommutative." More generally [21,23,36], in the non-affine setting, one may consider formal deformations of $\mathcal{O}_X$, which do not necessarily have $\mathcal{O}_X[\![\hbar]\!]$ as underlying sheaf of $\mathbb{C}[\![\hbar]\!]$-vector spaces, but could simultaneously deform $\mathcal{O}_X$ in some commutative direction, corresponding to simultaneously deforming the restriction morphisms of the sheaf $\mathcal{O}_X$. Although Z_k does admit commutative deformations, we are restricting ourselves to these purely noncommutative directions for two reasons. On the one hand, we expect that turning on a commutative direction of deformation would imply that the moduli spaces of vector bundles become very small or even trivial, as is the case for the purely commutative deformations [8]. On the other hand, it seems difficult to obtain explicit simultaneous deformations in the generality we achieve in this article, as it was shown in [7, §5.3] that simultaneous commutative and noncommutative deformations of Z_k may be obstructed.

3.1. The Kontsevich star product on $\mathbb{C}^d$

As part of an explicit quasi-isomorphism of L_∞ algebras in the proof of his formality theorem, Kontsevich [24] gave an explicit construction for a star product on $(\mathbb{R}^d, \pi)$ for some (real) Poisson structure π. The formula applies without change to the complex setting for a (holomorphic) Poisson structure on $\mathbb{C}^d$, which we will use in Section 4 to give explicit star products on Z_k.

We briefly recall the definition of this star product and refer to [23,24] for details.

Definition 3.6. Let σ be a Poisson structure on $\mathbb{C}^d$. The *Kontsevich star product* $\star_\sigma^{\mathrm{K}}$ on $\mathbb{C}^d$ is given by

$$f \star_\sigma^{\mathrm{K}} g = fg + \sum_{n=1}^{\infty} \hbar^n \sum_{\Gamma \in \mathfrak{G}_{n,2}} w_\Gamma \, B_\Gamma(f, g), \tag{3.7}$$

where

- $\mathfrak{G}_{n,2}$ is the set of *admissible* graphs with n filled ("first type") and 2 unfilled ("second type") vertices,
- B_Γ is the bidifferential operator for σ associated to the graph Γ, and
- w_Γ is the *weight* of the graph Γ obtained as the integral over a certain configuration space.

An admissible graph $\Gamma \in \mathfrak{G}_{n,2}$ has two filled vertices representing the two entries of B_Γ and n filled vertices. Arrows, which always start at filled vertices, represent derivatives of the target of the arrow. We shall use the notation ∂_i for the derivative with respect to the ith coordinate of $\mathbb{C}^d$ and write σ^{ij} for the coefficient function of $\partial_i \wedge \partial_j$. The σ^{ij} are holomorphic functions on $\mathbb{C}^d$ and define a skew-symmetric $d \times d$ matrix.

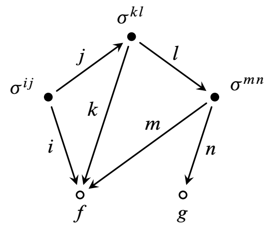


Figure 3. The graph Γ of Example 3.8.

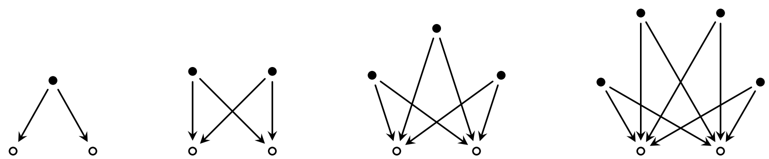


Figure 4. The graphs in $\mathfrak{G}_{n,2}$ for $n = 1, 2, 3, 4$ contributing to B_n in the Moyal product.

Example 3.8. The bidifferential operator for the graph Γ of Figure 3 is given by

$$B_\Gamma(f, g) = \sum_{1 \le i,j,k,l,m,n \le d} \sigma^{ij}\, \partial_j(\sigma^{kl})\, \partial_l(\sigma^{mn})\, \partial_i \partial_k \partial_m(f)\, \partial_n(g).$$

The graphs with arrows ending in filled vertices represent bidifferential operators which take derivatives of σ^{ij}. When the Poisson structure σ is constant and non-degenerate, i.e.,

$$\sigma = \sum_{i,j} \sigma^{ij}\, \partial_i \wedge \partial_j \quad \sigma^{ij} = -\sigma^{ji} \in \mathbb{C},$$

graphs with arrows ending in filled vertices represent the zero operator and thus, for each n, there is only one graph in $\mathfrak{G}_{n,2}$ which contributes to B_n (see Figure 4). For σ constant, the Kontsevich star product $\star_\sigma^{\mathrm{K}}$ then coincides with the *Moyal product* $\star_\sigma^{\mathrm{M}}$ given by

$$\begin{aligned} f \star_\sigma^{\mathrm{M}} g &= fg + \hbar \sum_{i,j} \sigma^{ij}\, \partial_i(f)\, \partial_j(g) + \frac{\hbar^2}{2} \sum_{i,j,k,l} \sigma^{ij}\sigma^{kl}\, \partial_i\partial_k(f)\, \partial_j\partial_l(g) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{i_1,\ldots,i_n,j_1,\ldots,j_n} \Big(\prod_{k=1}^{n} \sigma^{i_k j_k}\Big) \times \Big(\prod_{k=1}^{n} \partial_{i_k}\Big)(f) \times \Big(\prod_{k=1}^{n} \partial_{j_k}\Big)(g), \end{aligned} \tag{3.9}$$

where the symbol $\times$ denotes the usual product.

In general, a holomorphic Poisson structure σ will not be constant, not even locally. (For the surfaces Z_k, see Lemma 2.8 and Proposition 2.16.) In this case, the formula for the Kontsevich star product also involves derivatives of the coefficient functions σ^{ij}.

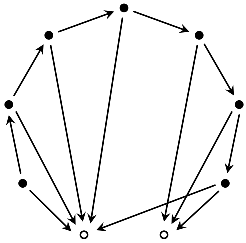
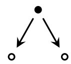


Figure 5. The graph of [14] contributing to B_7 with weight $\zeta(3)^2/\pi^6$ up to rationals.

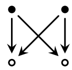
Lemma 3.10 ([12]). *Up to second order in $\hbar$, the Kontsevich star product for σ on $\mathbb{C}^d$ is given by*

$$f \star_\sigma^{\mathrm{K}} g = fg$$

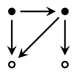
$$+ \hbar \sum_{i,j} \sigma^{ij}\, \partial_i(f)\, \partial_j(g)$$



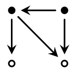
$$+ \frac{\hbar^2}{2} \sum_{i,j,k,l} \sigma^{ij} \sigma^{kl}\, \partial_i \partial_k(f)\, \partial_j \partial_l(g)$$



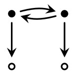
$$+ \frac{\hbar^2}{3} \sum_{i,j,k,l} \sigma^{ij}\, \partial_i(\sigma^{kl})\, \partial_j \partial_l(f)\, \partial_k(g)$$



$$+ \frac{\hbar^2}{3} \sum_{i,j,k,l} \sigma^{kl}\, \partial_k(\sigma^{ij})\, \partial_i(f)\, \partial_j \partial_l(g)$$



$$- \frac{\hbar^2}{6} \sum_{i,j,k,l} \partial_l(\sigma^{ij})\, \partial_j(\sigma^{kl})\, \partial_i(f)\, \partial_k(g)$$



$$+ \cdots .$$

To give the Kontsevich star product explicitly to higher orders, one would have to compute the weights of all admissible graphs. Already for a single graph, this computation is nontrivial, yet necessary even if one were to restrict oneself to, say, linear Poisson structures. For example, Felder–Willwacher [14] computed the weight of the graph shown in Figure 5 to be $\zeta(3)^2/\pi^6$ up to rationals, omitting 50 terms known to be rational. (The precise weight was recently given as $\frac{13}{2903040} + \frac{\zeta(3)^2}{256\pi^6}$ in [6], which relates the weights to multiple zeta values with integer coefficients.) As each filled vertex in this graph has only one ingoing arrow, the associated bidifferential operator is nonzero even for linear Poisson structures, and thus it will contribute to the expression of B_7 for any non-constant Poisson structure.

For the computations of cohomology and Ext^i groups in Section 6.2, we will not need the precise weights in the Kontsevich star product, only the possible bidifferential operators that appear in its expression—and the highly nontrivial fact that there exists a choice of weights making the star product into an associative product.

3.2. Star products on Z_k

The cohomology of Z_k can be described by the cohomology of the commutative diagram

$$\begin{array}{ccccc} & & Z_k & & \\ & \nearrow & & \nwarrow & \\ U & & & & V \\ & \nwarrow & & \nearrow & \\ & & U \cap V & & \end{array} \tag{3.11}$$

where the arrows are inclusions of open sets.

Dual to (3.11), we have a commutative diagram of (commutative) algebras

$$\begin{array}{ccccc} & & \mathcal{O}(Z_k) & & \\ & \swarrow & & \searrow & \\ \mathcal{O}(U) & & & & \mathcal{O}(V) \\ & \searrow & & \swarrow & \\ & & \mathcal{O}(U \cap V) & & \end{array}$$

where the arrows are the restriction morphisms.

To define a star product on Z_k, one has to find star products on $\mathcal{O}[\![\hbar]\!](W)$, where $W \in \{Z_k, U, V, U \cap V\}$ making

$$\begin{array}{ccccc} & & \mathcal{O}[\![\hbar]\!](Z_k) & & \\ & \swarrow & & \searrow & \\ \mathcal{O}[\![\hbar]\!](U) & & & & \mathcal{O}[\![\hbar]\!](V) \\ & \searrow & & \swarrow & \\ & & \mathcal{O}[\![\hbar]\!](U \cap V) & & \end{array} \tag{3.12}$$

a commutative diagram of associative algebras.

We now describe quantizations of Poisson structures explicitly.

4. Quantizing degenerate Poisson structures

To obtain explicit examples of deformation quantizations of Poisson structures on Z_k, we adapt the construction given in Kontsevich [23] of a quantizable compactification of a smooth affine Poisson variety. We show that the open immersion of a coordinate chart $U \subset Z_k$ is quantizable in case the Poisson structure is tangent to the divisor $D = Z_k \setminus U$; see Definition 2.11.

Proposition 4.1. *Let σ be a Poisson structure on Z_k such that it is tangent to $D = Z_k \setminus U$. Then $U \subset Z_k$ is a quantizable open immersion; i.e., the Kontsevich star product on U extends to a global star product on Z_k.*

Proof. Let $D = Z_k \setminus U$ and write the algebras $\mathcal{O}(U \cap V), \mathcal{O}(U), \mathcal{O}(V), \mathcal{O}(Z_k)$ in the $\mathbb{C}$-vector space basis $\{z^l u^i\}$, where $i \geq 0$ and

$$\begin{cases} -\infty < l < \infty & \text{on } U \cap V, \\ 0 \leq l < \infty & \text{on } U, \\ -\infty < l \leq ki & \text{on } V, \\ 0 \leq l \leq ki & \text{on } U \cup V = Z_k. \end{cases} \tag{4.2}$$

The case $k = 1$. As seen in Figure 1, a Poisson structure on Z_1 tangent to D is generated by the monomial 1 in U-coordinates; i.e., such a Poisson structure is of the form $\sigma_U = f$ for some global function $f \in \mathrm{H}^0(Z_1, \mathcal{O})$.

First assume that f is constant. The Kontsevich star product on $U \simeq \mathbb{C}^2$ thus coincides with the Moyal product (3.9). Denoting by $\star$ the restriction of this star product to $U \cap V$, the star product of two arbitrary monomials $z^{l_1}u^{i_1}, z^{l_2}u^{i_2} \in \mathcal{O}(U \cap V)$ is then given by

$$z^{l_1}u^{i_1} \star z^{l_2}u^{i_2} = \sum_{n \geq 0} a_n \, z^{l_1+l_2-n} u^{i_1+i_2-n} \hbar^n \tag{4.3}$$

for some constant coefficients $a_n \in \mathbb{Q}$ depending only on l_1, l_2, i_1, i_2. In particular,

$$\begin{aligned} a_0 &= 1, \\ a_1 &= l_1 i_2 - l_2 i_1, \\ a_2 &= \frac{1}{2}\big(l_1(l_1-1)i_2(i_2-1) - 2l_1 l_2 i_1 i_2 + l_2(l_2-1)i_1(i_1-1)\big). \end{aligned}$$

If both (l_1, i_1) and (l_2, i_2) satisfy one and the same bound of (4.2), then so does $(\max\{0, l_1 + l_2 - n\}, i_1 + i_2 - n)$. Hence, $\star$ preserves the subalgebras $\mathcal{O}(Z_1), \mathcal{O}(U)$, and $\mathcal{O}(V)$ of $\mathcal{O}(U \cap V)$ and thus defines a global star product.

If f is not constant, its power series expansion in U-coordinates is of the form $f = \sum_{i=0}^{\infty} \sum_{l=0}^{i} f_{il} z^l u^i$. (Note that f being a global function means that the monomials appearing in the power series expansion of f satisfy *all* of the bounds (4.2).)

Now, the Kontsevich star product also has contributions coming from graphs $\Gamma \in \mathfrak{G}_{n,2}$ with filled vertices representing a copy of the Poisson structure $\sigma_U = f$. However, a filled vertex lowers the powers of z and u each by 1 but simultaneously multiplies by $\sigma_U = f$, so that the exponents still satisfy the bounds (4.2).

The case $k \geq 2$. Poisson structures tangent to D are generated by the monomials u, zu over global functions (see Proposition 2.12 and Figure 1). The proof is now the same as for $k = 1$ with σ_U having vanishing constant term. ∎

Remark 4.4. The open immersion $U \subset Z_k$ satisfies some of the hypotheses of a quantizable compactification in the sense of [23, Def. 4], except for Z_k not being compact and the divisor $D = Z_k \setminus U$ not being ample. However, the complement of D is affine and even isomorphic to $\mathbb{C}^2$, which is the deciding factor for the construction. So, even though we do not have a compactification, we still obtain a quantization.

Given a quasi-projective variety X and an affine subvariety $U \subset X$, one could of course consider a quantizable smooth compactification $U \subset X \subset \overline{X}$ and consider quantizations of Poisson structures which are tangent to $\overline{X} \setminus U$. However, if one is interested in quantizations of noncompact X, this strategy is more restrictive. For the case at hand, we have a smooth compactification $U \subset Z_k \subset F_k$ to the kth Hirzebruch surface, but the space of Poisson structures which are tangent to the divisor $F_k \setminus U$ is of dimension 3 for $k = 1$ and of dimension 2 for $k \geq 2$, whereas the space of quantizable Poisson structures for the open immersion $U \subset Z_k$ is infinite dimensional.

Example 4.5. Let σ be the Poisson structure $(1, -\xi)$ on Z_1 and consider the quantizable open immersion $U \subset Z_1$. Proposition 2.12 shows that $(1, -\xi)$ is tangent to $D = Z_1 \setminus U = \{\xi = 0\}$ and Proposition 4.1 shows that there exists a global star product on Z_1, which can be computed in canonical coordinates by the Moyal product on U.

The corresponding star product on the V chart is then given, up to second order, by

$$\begin{aligned} f \star g = fg &- \hbar\big(\xi\, \partial_\xi(f)\, \partial_v(g) - \xi\, \partial_v(f)\, \partial_\xi(g)\big) \\ &+ \hbar^2 \Big(\frac{1}{2}\xi^2\, \partial_\xi^2(f)\, \partial_v^2(g) - \xi^2\, \partial_\xi\partial_v(f)\, \partial_\xi\partial_v(g) + \frac{1}{2}\xi^2\, \partial_v^2(f)\, \partial_\xi^2(g) \\ &\qquad + \xi\, \partial_\xi(f)\, \partial_v^2(g) + \xi\, \partial_v(f)\, \partial_\xi\partial_v(g) + \xi\, \partial_\xi\partial_v(f)\, \partial_v(g) \\ &\qquad + \xi\, \partial_v^2(f)\, \partial_\xi(g) - \partial_v(f)\, \partial_v(g) - v\, \partial_v(f)\, \partial_v^2(g) - v\, \partial_v^2(f)\, \partial_v(g)\Big). \end{aligned}$$

To verify this, rewrite the Moyal product on U using the identities

$$\begin{aligned} \partial_z &= -\xi^2\partial_\xi + \xi v \partial_v, \\ \partial_u &= \xi^{-1}\partial_v \end{aligned}$$

obtained by the change of coordinates and the commutation relations

$$[\partial_\xi, \xi] = \mathrm{id}, \quad [\partial_v, v] = \mathrm{id} \tag{4.6}$$

where ξ, v in (4.6) are thought of as differential operators of order 0.

The following proposition shows that certain Poisson structures which in canonical coordinates do not satisfy the hypotheses of Proposition 4.1 may still be quantized by the same method after performing a linear change of coordinates.

Proposition 4.7. *Let $\sigma = (\sigma_U, \sigma_V)$ be a Poisson structure on Z_k which on U is of the form $\sigma_U = u^d P(z)$ for $d \geq 1$ and $P(z)$ a polynomial in z. Then there exists a linear change of coordinates $U \simeq U'$ such that $U' \subset Z_k$ is a quantizable open immersion.*

Proof. First note that, by Lemma 2.8, $P(z)$ is a polynomial of degree $n = (d-1)k + 2$. We can thus write

$$\sigma_U = u^d \prod_{1 \le i \le n} (z - \lambda_i).$$

Now, choose any λ_i and consider the linear change of coordinates $U \simeq U'$ given by $(z, u) \mapsto (z', u)$ for $z' = z + \lambda_i$. In these coordinates, we have that

$$\sigma_U = z' u^d \prod_{i \ne j} (z' - \lambda_j).$$

Then σ is tangent to the divisor $\{z' = 0\}$ and $U' \subset Z_k$ is a quantizable open immersion. ■

Corollary 4.8. *For $k \ne 2$, any linear combination of the generators of Poisson structures on Z_k can be quantized via an open immersion $\mathbb{C}^2 \simeq U' \subset Z_k$.*

For the holomorphic symplectic structure on Z_2, the same method does not apply (see Kontsevich [24, §3.5] for more details).

Remark 4.9. The Moyal product on $U \simeq \mathbb{C}^2$ does not extend naively to a star product on Z_2. In fact, the Moyal product on $\mathcal{O}[\![\hbar]\!](U)$ does not preserve the subalgebra $\mathcal{O}[\![\hbar]\!](Z_k)$. Recall from (3.9) that the bidifferential operators B_n in the expression of the Moyal product on $\mathbb{C}^2_{z,u}$ are given by

$$B_n = \frac{1}{n!} \sum_{0 \le i \le n} (-1)^i \binom{n}{i} \partial_z^{n-i} \partial_u^i (f)\, \partial_z^i \partial_u^{n-i} (g).$$

Now, consider the star product of the functions $zu, z^2u \in \mathcal{O}(Z_2) \subset \mathcal{O}(U)$. Then

$$\begin{aligned} z^2u \star zu &= z^3u^2 + B_1(z^2u, zu)\hbar + B_2(z^2u, zu)\hbar^2 + \cdots \\ &= z^3u^2 + \qquad 2z^2u\hbar + \qquad\quad 2z\hbar^2 + \cdots. \end{aligned}$$

But $2z \notin \mathcal{O}(Z_2)$, so $z^2u \star zu \notin \mathcal{O}[\![\hbar]\!](Z_k)$.

5. Geometry of the commutative local surfaces

In this section, we summarize the known results about vector bundles and moduli for Z_k which will be used to construct the noncommutative counterparts in Section 6.

As seen in Section 2, holomorphic line bundles on Z_k are classified by their first Chern class and we denote by $\mathcal{O}_{Z_k}(n)$ the line bundle with first Chern class n, omitting the subscript when it is clear from the context. Recall that a rank r bundle E on X is called *filtrable* if there exists an increasing filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = E$ of subbundles such that $E_i/E_{i-1} \in \operatorname{Pic} X$, where $1 \le i \le r$.

Theorem 5.1 ([16, Lem. 3.1, Thm. 3.2]). *Holomorphic vector bundles on Z_k are algebraic and filtrable.*

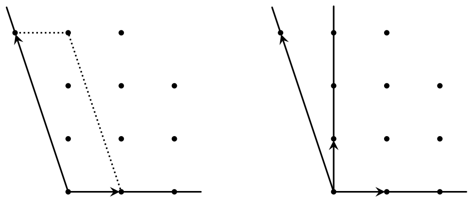


Figure 6. Z_k as toric resolution of X_k.

Definition 5.2 ([2]). Let E be a rank r holomorphic vector bundle on Z_k. The restriction of E to the zero section $\ell \simeq \mathbb{P}^1$ is a rank r bundle on $\mathbb{P}^1$, which by Grothendieck's lemma splits as a direct sum of line bundles. Thus, $E|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(j_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(j_r)$. We call $(j_1, \ldots, j_r)$ the *splitting type* of E. When E is a rank 2 bundle with first Chern class 0, the splitting type is $(j, -j)$ for some $j \geq 0$ and we say for short that E has *splitting type* j.

Remark 5.3. Filtrability of vector bundles on Z_k implies that moduli of rank 2 vector bundles are parametrized by classes in $\mathrm{Ext}^1(\mathcal{O}(j_2), \mathcal{O}(j_1))$ and algebraicity implies that such spaces of extensions are finite dimensional. For suitable numerical invariants or a suitable notion of stability, one may extract finite-dimensional moduli spaces from the naive quotient of the vector spaces $\mathrm{Ext}^1(\mathcal{O}(j_2), \mathcal{O}(j_1))$ modulo bundle isomorphisms.

For applications to SU(2) instantons, one considers bundles with vanishing first Chern class. Thus, we study the quotient $\mathrm{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))/\sim$, where $\sim$ denotes bundle isomorphism. However, this quotient has an extremely complicated structure; in particular, it is non-Hausdorff in the analytic topology. A stratification into Hausdorff components was presented in [5, Thm. 4.15] by using two numerical invariants, whose sum makes up the local second Chern class (see Remark 5.7).

To define the local Chern class, consider the (affine) surface X_k obtained by contracting the zero section $\ell \subset Z_k$ to a point. Then $X_1 \simeq \mathbb{C}^2$ and for $k \geq 2$, one obtains the $\frac{1}{k}(1,1)$ surface singularity and $\tau\colon Z_k \to X_k$ is its (toric) resolution, which is given by inclusion of fans as shown in Figure 6.

Remark 5.4. Observe that $X_k \simeq \mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$ is a cyclic group of order k with a generator acting on $\mathbb{C}^2$ via multiplication by $\gamma = \left(\begin{smallmatrix}\omega & 0\\ 0 & \omega\end{smallmatrix}\right)$ for ω a primitive kth root of unity. Thus, X_2 is the A_1 surface singularity, but for $k \geq 3$, we have that $\det\gamma = \omega^2 \neq 1$. In particular, $\Gamma \not\subset \mathrm{SU}(2)$ so that $Z_{\geq 3}$ is not an ALE space; see [25, Thm. 1.2].

Definition 5.5. Let E be a holomorphic rank 2 bundle on Z_k and let $\tau\colon Z_k \to X_k$ be the contraction map. The *local second Chern class* of E is the local holomorphic Euler characteristic of E around ℓ, that is,

$$\chi(\ell, E) = h^0\big(X_k, (\tau_* E)^{\vee\vee}/\tau_* E\big) + h^0(X_k, \mathrm{R}^1\tau_* E). \tag{5.6}$$

Remark 5.7. The terms on the right-hand side of (5.6) define two independent holomorphic invariants of the vector bundle E. In [5], these invariants were called the *width* of E, since $h^0(X_k, (\tau_* E)^{\vee\vee}/\tau_* E)$ measures the default of the direct image from being locally free, and the *height* of E, since $h^0(X_k, \mathrm{R}^1\tau_* E)$ measures how far E is from being a split extension. These are two independent numerical invariants, and the pair stratifies the moduli spaces $\mathfrak{M}_j(Z_k)$ into Hausdorff components [5, Thm. 4.15].

To describe actual moduli spaces, [18, Def. 5.2] gives an ad hoc definition of stability, calling a rank 2 vector bundle on Z_k *(framed) stable* when it is holomorphically trivial (and framed) on $Z_k \setminus \ell$.

Notation 5.8. Denote by $\mathfrak{M}_j(Z_k)$ the subspace of $\mathrm{Ext}^1(\mathcal{O}_{Z_k}(j), \mathcal{O}_{Z_k}(-j))/\sim$ consisting of those classes corresponding to stable vector bundles, where $\sim$ denotes bundle isomorphism.

Moduli spaces of rank 2 bundles on Z_k were studied in [17, Thm. 3.5] for the case $k = 1$ and in [5, Thm. 4.11] for the cases $k \geq 1$. The moduli spaces of stable bundles with splitting type j on Z_k turn out to be smooth quasi-projective varieties of dimension $2j - k - 2$ [5, Thm. 4.11]. In fact, we have the following.

Theorem 5.9 ([5, Thm. 4.11]). *The moduli space of rank 2 holomorphic bundles on Z_k with vanishing first Chern class and splitting type j contains an open dense subset isomorphic to $\mathbb{P}^{2j-k-2}$ minus a closed subvariety of codimension at least $k+1$.*

6. Geometry of noncommutative deformations

We now study vector bundles over noncommutative deformations $\mathcal{Z}_k(\sigma) = (Z_k, \mathcal{A}^\sigma)$ of Z_k (see Definition 3.3).

Definition 6.1. We call a locally free sheaf of $\mathcal{A}^\sigma$-modules of rank r over $\mathcal{Z}_k(\sigma)$ a *vector bundle of rank r over $\mathcal{Z}_k(\sigma)$*. A vector bundle of rank 1 over $\mathcal{Z}_k(\sigma)$ is called a *line bundle*.

We recall some properties of deformation quantizations and refer to Kashiwara and Schapira [21] for more details.

Proposition 6.2 ([21]). *Let (X, σ) be a holomorphic Poisson manifold and let $\mathcal{A} = \mathcal{A}^\sigma$ be a deformation quantization of $\mathcal{O}_X$.*

(i) *Let $\mathcal{E}$ be a coherent sheaf of $\mathcal{A}$-modules without $\hbar$-torsion. If $\mathcal{E}/\hbar\mathcal{E}$ is locally free of rank r as a sheaf of $\mathcal{O}_X$-modules, then $\mathcal{E}$ is locally free of rank r as a sheaf of $\mathcal{A}$-modules.*

(ii) *If $U \subset X$ is affine, i.e., $\mathrm{H}^k(U, \mathcal{F}|_U) = 0$ for any $k > 0$ and for any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, then $\mathrm{H}^k(U, \mathcal{G}) = 0$ for any coherent sheaf $\mathcal{G}$ of $\mathcal{A}$-modules.*

Here, (ii) states that a Leray cover for $(X, \mathcal{O}_X)$ is also a Leray cover for $(X, \mathcal{A})$. In particular, we may use the canonical coordinate charts on Z_k to calculate cohomology or extension groups of coherent (or locally free) sheaves of $\mathcal{A}$-modules. After showing that rank 2 bundles are extensions of line bundles (Theorem 6.9), we will calculate these extension groups and use them to obtain moduli of vector bundles over noncommutative deformations in Section 7.

In terms of the canonical coordinate charts for Z_k, Definition 6.1 implies that a rank r vector bundle over a noncommutative $\mathcal{Z}_k(\sigma)$ is given by two free rank r modules over U and over V, respectively, with a global structure defined by a *transition matrix*, i.e., a $\star$-invertible $r \times r$ matrix with entries in $\mathcal{A}^\sigma(U \cap V)$, determining the vector bundle uniquely up to isomorphism, where an isomorphism of vector bundles on $\mathcal{Z}_k(\sigma)$ is an isomorphism of $\mathcal{A}^\sigma$-modules, which can be phrased using the coordinate charts of Z_k as follows.

Definition 6.3. Let E and E' be vector bundles over $\mathcal{Z}_k(\sigma)$ defined by transition matrices T and T', respectively. An *isomorphism* between E and E' is given by a pair of matrices A_U and A_V with entries in $\mathcal{A}^\sigma(U)$ and $\mathcal{A}^\sigma(V)$, respectively, which are invertible with respect to $\star$ and such that

$$T' = A_V \star T \star A_U.$$

Definition 6.4. Let X be a complex manifold (resp. smooth algebraic variety) and let $\mathcal{A}$ be a noncommutative associative deformation of $\mathcal{O}_X$ over $\mathbb{C}[\![\hbar]\!]$. The augmentation $\mathcal{A} \to \mathbb{C} \otimes_{\mathbb{C}[\![\hbar]\!]} \mathcal{A} \simeq \mathcal{O}_X$ is given by $\mathcal{A} \to \mathcal{A}/\hbar\mathcal{A}$. Augmentation induces a map on quasi-coherent sheaves of $\mathcal{A}$-modules and the image of a sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is called the *classical limit* of $\mathcal{F}$. The image of any cocycle or cohomology class $\alpha \in \mathrm{H}^i(X, \mathcal{F}(U))$, for $U \subset X$ open, is called the *classical limit* of α.

6.1. Line bundles

Lemma 6.5. *Let $\mathcal{A}$ be a deformation quantization of $\mathcal{O}$. Then an $\mathcal{A}$-module $\mathcal{S}$ is acyclic if and only if $S = \mathcal{S}/\hbar\mathcal{S}$ is acyclic.*

Proof. Consider the short exact sequence

$$0 \to \mathcal{S} \xrightarrow{\hbar} \mathcal{S} \to S \to 0.$$

It gives for $j > 0$ surjections

$$\mathrm{H}^j(X, \mathcal{S}) \xrightarrow{\hbar} \mathrm{H}^j(X, \mathcal{S}) \to 0.$$

This immediately implies that $\mathrm{H}^j(X, \mathcal{S}) = 0$ for $j > 0$. The converse is immediate. ∎

Definition 6.6. Let $\mathcal{Z}_k(\sigma)$ be a noncommutative deformation of Z_k. Denote by $\mathcal{A}(j)$ the line bundle over $\mathcal{Z}_k(\sigma)$ with transition function z^{-j}.

Proposition 6.7. *Any line bundle on $\mathcal{Z}_k(\sigma)$ is isomorphic to $\mathcal{A}(j)$ for some $j \in \mathbb{Z}$, i.e.,* $\mathrm{Pic}(\mathcal{Z}_k(\sigma)) \simeq \mathbb{Z}$.

Proof. Let $f = f_0 + \sum_{n=1}^{\infty} \tilde{f}_n \hbar^n \in \mathcal{A}^*(U \cap V)$ be the transition function for $\mathcal{L}$. Then there exist functions $a_0 \in \mathcal{O}^*(U)$ and $\alpha_0 \in \mathcal{O}^*(V)$ such that $\alpha_0 f_0 a_0 = z^{-j}$ and viewing a_0, resp. α_0, as elements in $\mathcal{A}^*(U)$, resp. $\mathcal{A}^*(V)$, one has $\alpha_0 \star f \star a_0 = z^{-j} + \sum_{n=1}^{\infty} f_n \hbar^n$ for some $f_n \in \mathcal{O}(U \cap V)$. We may thus assume that the transition function of $\mathcal{L}$ is $z^{-j} + \sum_{n=1}^{\infty} f_n \hbar^n$.

To give an isomorphism $\mathcal{L} \simeq \mathcal{A}(j)$, it suffices to define functions $a_n \in \mathcal{O}(U)$ and $\alpha_n \in \mathcal{O}(V)$ for $n \geq 1$ satisfying

$$\left(1 + \sum_{n=1}^{\infty} \alpha_n \hbar^n\right) \star \left(z^{-j} + \sum_{n=1}^{\infty} f_n \hbar^n\right) \star \left(1 + \sum_{n=1}^{\infty} a_n \hbar^n\right) = z^{-j}. \tag{6.8}$$

Collecting terms by powers of $\hbar$, (6.8) is equivalent to the system of equations

$$S_n + z^{-j} a_n + z^{-j} \alpha_n = 0 \quad n = 1, 2, \ldots,$$

where S_n is a finite sum involving f_i, B_i for $i \leq n$ but only a_i, α_i for $i < n$. The first terms are

$$\begin{aligned} S_1 &= f_1, \\ S_2 &= f_2 + \alpha_1 f_1 + a_1 f_1 + B_1(\alpha_1, z^{-j}) + B_1(z^{-j}, a_1) + \alpha_1 z^{-j} a_1, \\ S_3 &= f_3 + B_2(\alpha_1, z^{-j}) + B_2(z^{-j}, a_1) + B_1(\alpha_2, z^{-j}) + B_1(z^{-j}, a_2) \\ &\quad + B_1(\alpha_1, f_1) + B_1(\alpha_1, z^{-j} a_1) + B_1(z^{-j}, a_1) + \alpha_2 f_1 + \alpha_2 z^{-j} a_1 \\ &\quad + \alpha_1 f_2 + \alpha_1 f_1 a_1 + \alpha_1 z^{-j} a_2 + f_2 a_1 + f_1 a_2. \end{aligned}$$

Since $\mathrm{H}^1(Z_k, \mathcal{O}) = 0$, we can solve these equations recursively, for example, by defining a_n to cancel all terms of $z^j S_n$ having positive powers of z and setting $\alpha_n = z^j S_n - a_n$. ∎

6.2. Vector bundles

Generalizing Theorem 5.1 to the noncommutative setting, we prove filtrability and formal algebraicity for bundles over deformation quantizations of Z_k. Let $\mathcal{Z}_k(\sigma)$ be a noncommutative deformation of Z_k.

Theorem 6.9. *Vector bundles over $\mathcal{Z}_k(\sigma)$ are filtrable.*

First proof. Let $\mathcal{E}$ be the rank 2 bundle given by transition matrix $T = T_0 + \sum_{n=1}^{\infty} T_n \hbar^n$. Let (A_U, A_V) be an isomorphism of the classical limit $\mathcal{E}/\hbar\mathcal{E}$ (with transition matrix T_0) with a filtered bundle, i.e.,

$$A_V T_0 A_U = \begin{pmatrix} z^{j_1} & b_0 \\ 0 & z^{j_2} \end{pmatrix}$$

with $j_1 \geq j_2$. Then (A_U, A_V) gives an isomorphism with the bundle given by

$$A_V \star T \star A_U = \begin{pmatrix} z^{j_1} + O(\hbar) & b_0 + O(\hbar) \\ O(\hbar) & z^{j_2} + O(\hbar) \end{pmatrix} = T'. \tag{6.10}$$

Now, choose α_n, δ_n and a_n, d_n as in the proof of Proposition 6.7 such that the $(1,1)$ and $(2,2)$ entries of (6.10) are taken to the transition functions of $\mathcal{A}(-j_1)$ and $\mathcal{A}(-j_2)$, respectively. Then for

$$A'_V = \begin{pmatrix} 1+\sum_{n=1}^{\infty}\alpha_n\hbar^n & 0 \\ 0 & 1+\sum_{n=1}^{\infty}\delta_n\hbar^n \end{pmatrix},$$

$$A'_U = \begin{pmatrix} 1+\sum_{n=1}^{\infty}a_n\hbar^n & 0 \\ 0 & 1+\sum_{n=1}^{\infty}d_n\hbar^n \end{pmatrix},$$

we have that

$$A'_V \star T' \star A'_U = \begin{pmatrix} z^{j_1} & b_0 + O(\hbar) \\ O(\hbar^N) & z^{j_2} \end{pmatrix} = \begin{pmatrix} z^{j_1} & b_0 + O(\hbar) \\ \sum_{n=N}^{\infty} c_n\hbar^n & z^{j_2} \end{pmatrix} = T''$$

for some integer $N \geq 1$. Now, choose c'_N and γ'_N such that

$$z^{j_1}\gamma'_N + c_N + z^{j_2}c'_N = 0$$

which is possible since $\mathrm{H}^1(Z_k, \mathcal{O}(j_1 - j_2)) = 0$, as $j_1 \geq j_2$. Then

$$\begin{pmatrix} 1 & 0 \\ \gamma'_N\hbar^N & 1 \end{pmatrix} \star T'' \star \begin{pmatrix} 1 & 0 \\ c'_N\hbar^N & 1 \end{pmatrix} = \begin{pmatrix} z^{j_1} + O(\hbar) & b_0 + O(\hbar) \\ O(\hbar^{N+1}) & z^{j_2} + O(\hbar) \end{pmatrix} = T'''.$$

As for the isomorphism between the bundles defined by T' and T'', find matrices taking the $(1,1)$ and $(2,2)$ entries of T''' to z^{j_1} and z^{j_2} as above. We thus get that an isomorphic bundle may be given by a transition matrix of the form

$$\begin{pmatrix} z^{j_1} & b_0 + O(\hbar) \\ O(\hbar^{N+1}) & z^{j_2} \end{pmatrix}.$$

Applying the principle of (strong) induction, we conclude that $\mathcal{E}$ is isomorphic to the bundle given by a transition matrix

$$\begin{pmatrix} z^{j_1} & b_0 + \sum_{n=1}^{\infty} b_n\hbar^n \\ 0 & z^{j_2} \end{pmatrix}.$$

In particular, any rank 2 bundle is an extension of line bundles. The calculation for rank r is similar. ■

Second proof. This is a generalization of Ballico–Gasparim–Köppe [4, Thm. 3.2] to the noncommutative case. Let $\mathcal{E}$ be a sheaf of $\mathcal{A}$-modules. Lemma 6.5 gives that the classical limit $\mathcal{E}_0 = \mathcal{E}/\hbar\mathcal{E}$ is acyclic as a sheaf of $\mathcal{A}$-modules (and equivalently as a sheaf of $\mathcal{O}$-modules) if and only if $\mathcal{E}$ is acyclic as a sheaf of $\mathcal{A}$-modules.

Filtrability for a bundle E over Z_k is obtained due to the vanishing of cohomology groups $\mathrm{H}^i(Z_k, E \otimes \mathrm{S}^n N^*)$ for $i = 1, 2$, where N^* is the conormal bundle of $\ell \subset Z_k$ and $n > 0$ are integers; the proof proceeds by induction on n. In the noncommutative case, let $\mathcal{S}$ denote the kernel of the projection $\mathcal{A}^{(n)} \to \mathcal{A}^{(n-1)}$. By construction, we have that $\mathcal{S}/\hbar\mathcal{S} = \mathrm{S}^n N^*$ and the required vanishing of cohomologies is guaranteed by Lemma 6.5. ■

As for Z_k, rank 2 vector bundles on $\mathcal{Z}_k(\sigma)$ are thus extensions of line bundles. Hence, moduli spaces of rank 2 bundles may be built out of quotients of Ext^1's.

Definition 6.11. Theorem 3.3 of [16] showed that a rank 2 bundle E on Z_k with first Chern class $c_1(E) = 0$ can be given by a *canonical transition matrix*

$$T_0 = \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \quad \text{with } p = \sum_{i=0}^{\lfloor \frac{2j-2}{k} \rfloor} \sum_{l=ki-j+1}^{j-1} p_{il} z^l u^i \in \mathrm{Ext}^1\big(\mathcal{O}(j), \mathcal{O}(-j)\big).$$

Accordingly, for a noncommutative deformation $\mathcal{Z}_k(\sigma)$, we define the corresponding notion of *canonical transition matrix* as

$$T = \begin{pmatrix} z^j & \mathbf{p} \\ 0 & z^{-j} \end{pmatrix} \quad \text{with } \mathbf{p} = \sum_{n=0}^{\infty} p_n \hbar^n \in \mathrm{Ext}^1\big(\mathcal{A}(j), \mathcal{A}(-j)\big).$$

We will now see that each p_n can be given the canonical form of the classical case.

Lemma 6.12. *Let $\mathcal{A}$ be a deformation quantization of $\mathcal{O}_{Z_k}$. There is an injective map of $\mathbb{C}$-vector spaces*

$$\mathrm{Ext}^1_{\mathcal{A}}\big(\mathcal{A}(j), \mathcal{A}(-j)\big) \to \prod_{n=0}^{\infty} \mathrm{Ext}^1_{\mathcal{O}}\big(\mathcal{O}(j), \mathcal{O}(-j)\big)\hbar^n \simeq \mathrm{Ext}^1_{\mathcal{O}}\big(\mathcal{O}(j), \mathcal{O}(-j)\big)[\![\hbar]\!]$$

$$\mathbf{p} = p_0 + \sum_{n=1}^{\infty} p_n \hbar^n \mapsto (p_0, p_1\hbar, p_2\hbar^2, \ldots),$$

where $p_i \in \mathrm{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))$.

Short proof. $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{A}(j), \mathcal{A}(-j))$ is the quotient of $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(j), \mathcal{O}(-j))[\![\hbar]\!]$ by the relations $q_n \simeq q_n + \sum p_i p_{n-i}$. ∎

Proof. Two extension classes $\mathbf{p} = p_0 + \sum_{n=1}^{\infty} p_n \hbar^n$ and $\mathbf{p}' = p'_0 + \sum_{n=1}^{\infty} p'_n \hbar^n$ are equivalent if there exist two functions

$$b = b_0 + \sum_{n=1}^{\infty} b_n \hbar^n \in \mathcal{A}(U) \quad \text{and} \quad \beta = \beta_0 + \sum_{n=1}^{\infty} \beta_n \hbar^n \in \mathcal{A}(V) \tag{6.13}$$

such that

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \star \begin{pmatrix} z^j & \mathbf{p} \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} z^j & \mathbf{p}' \\ 0 & z^{-j} \end{pmatrix} \star \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

that is,

$$\mathbf{p} + \beta \star z^{-j} = \mathbf{p}' + z^j \star b \tag{6.14}$$

or equivalently

$$p_n + \beta_n z^{-j} + \sum_{i=1}^{n} B_i(\beta_{n-i}, z^{-j}) = p'_n + z^j b_n + \sum_{i=1}^{n} B_i(z^j, b_{n-i}) \quad \text{for } n \in \mathbb{N}.$$

For each n, we may choose b_n and β_n reducing p_n to the canonical form

$$p_n = \sum_{i=0}^{\lfloor \frac{2j-2}{k} \rfloor} \sum_{l=ik-j+1}^{j-1} p_{il}^n z^l u^i \tag{6.15}$$

as in (6.11). ∎

Lemma 6.16. *Let $\mathcal{Z}_1(\sigma)$ be the deformation quantization of Z_1 with Poisson structure $\sigma = (1, -\xi)$. Then any $\mathbf{p} \in \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{A}(j), \mathcal{A}(-j))$ is represented in the canonical form $\mathbf{p} = \sum_{n=0}^{\infty} p_n \hbar^n$, where p_n is of the form $p_n = \sum_{i=0}^{2j-2} \sum_{l=i-j+1}^{j-1} p_{il}^n z^l u^i$.*

Proof. On the U chart, the holomorphic Poisson structure σ is induced by the bivector $\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u}$, and the corresponding star product is given by the Moyal product (3.9).

As in the proof of Lemma 6.12, elements $\mathbf{p} = p_0 + \sum_{n=1}^{\infty} p_n \hbar^n$ and $\mathbf{p}' = p'_0 + \sum_{n=1}^{\infty} p'_n \hbar^n$ in $\mathrm{Ext}^1(\mathcal{A}(j), \mathcal{A}(-j))$ are equivalent if there exist two functions $b \in \mathcal{A}(U)$ and $\beta \in \mathcal{A}(V)$ of the form (6.13) such that

$$\mathbf{p}' = \mathbf{p} + \beta \star z^{-j} - z^j \star b.$$

We carry out calculations on the U chart, hence using the Moyal product. For $n = 1, 2$, this gives

$$p'_1 = p_1 + \beta_1 z^{-j} - \left(\frac{\partial}{\partial u}\beta_0\right)\left(\frac{\partial}{\partial z} z^{-j}\right) - z^j b_1 + \left(\frac{\partial}{\partial z} z^j\right)\left(\frac{\partial}{\partial u} b_0\right),$$

$$\begin{aligned} p'_2 = p_2 + \beta_2 z^{-j} &- \left(\frac{\partial}{\partial u}\beta_1\right)\left(\frac{\partial}{\partial z} z^{-j}\right) - \left(\frac{\partial^2}{\partial u^2}\beta_0\right)\left(\frac{\partial^2}{\partial z^2} z^{-j}\right) \\ &- z^j b_2 + \left(\frac{\partial}{\partial z} z^j\right)\left(\frac{\partial}{\partial u} b_1\right) + \left(\frac{\partial^2}{\partial z^2} z^j\right)\left(\frac{\partial^2}{\partial u^2} b_0\right), \end{aligned}$$

etc. But we then find out that ultimately the calculations repeat at each step the same type of calculation done on neighborhood zero; thus, at each step, the shape of the polynomial is the same as what we have in the classical limit. ∎

Theorem 6.17. *Let σ be a Poisson structure on Z_k tangent to the divisor $Z_k \setminus U$ and let $\mathcal{A}$ be the quantization of the open immersion $U \subset Z_k$. Then*

$$\mathrm{Ext}^1_{\mathcal{A}}\left(\mathcal{A}(j), \mathcal{A}(-j)\right) \simeq \mathrm{Ext}^1_{\mathcal{O}}\left(\mathcal{O}(j), \mathcal{O}(-j)\right)[\![\hbar]\!].$$

Proof. Recall the proof of Lemma 6.12 where it was shown that an extension class $\mathbf{p} \in \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{A}(j), \mathcal{A}(-j))$ may be reduced to the form $\mathbf{p} = p_0 + \sum_{n=1}^{\infty} p_n \hbar^n$, where $p_i \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(j), \mathcal{O}(-j))$. It remains to show that this form cannot be reduced further.

For Z_1, this is the content of Lemma 6.16. For $Z_{\geq 2}$, Proposition 2.12 shows that, for σ tangent to D, $\sigma_U = u\, f_U + zu\, g_U$ for some global functions f_U, g_U. Since σ_U is a multiple of u, it follows that the bilinear operators B_n in the expression of the star product never lower the exponents of u; cf. the proof of Proposition 4.1. Thus, the star products in (6.14) never contain terms with exponents of u low enough to reduce the general form (6.15) of p_n any further. ∎

Definition 6.18. We say that $\mathbf{p} = \sum p_n \hbar^n \in \mathcal{O}[\![\hbar]\!]$ is formally algebraic if p_n is a polynomial for every n.

We say that a vector bundle over $\mathcal{Z}_k(\sigma)$ is *formally algebraic* if it is isomorphic to a vector bundle given by formally algebraic transition functions. In addition, if there exists N such that $p_n = 0$ for all $n > N$, we then say that $\mathbf{p}$ is *algebraic*.

Corollary 6.19. *Vector bundles on noncommutative deformations of Z_k are formally algebraic.*

Proof. Lemma 6.12 shows that rank 2 vector bundles over $\mathcal{Z}_k(\sigma)$ are formally algebraic, and in light of Theorem 6.9, we obtain the result for all ranks. ∎

Remark 6.20. A quantization of Z_k will also give a quantization of the singular affine surface $X_k = \operatorname{Spec}(\mathrm{H}^0(Z_k, \mathcal{O}))$, obtained from Z_k by contracting the zero section ℓ to a point. (X_k contains an isolated $\frac{1}{k}(1,1)$ singularity; see Section 9 and Figure 6.) Indeed, a quantization of Z_k gives a star product on $\mathcal{O}_{Z_k}(U)$ for all open sets U and thus in particular a star product on the algebra of global functions $\mathrm{H}^0(Z_k, \mathcal{O}) \simeq \mathrm{H}^0(X_k, \mathcal{O})$.

Note that Poisson structures on singular affine *toric* varieties (of which X_k is a particular case) are known to always admit quantizations (see Filip [15]), but in general, there are also counterexamples to the quantization of singular Poisson algebras (see Mathieu [29] and also Schedler [34]).

Since $\operatorname{Ext}^1_{Z_k}(\mathcal{O}(j), \mathcal{O}(-j))$ is finite dimensional, it follows that whenever $\mathcal{Z}_k(\sigma)$ is a noncommutative deformation having finite order in $\hbar$, then the space of extensions $\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{A}(j), \mathcal{A}(-j))$ is also finite dimensional. For moduli of vector bundles on $\mathcal{Z}_k(\sigma)$ up to all orders of $\hbar$, see Remark 7.18.

7. Moduli of bundles on noncommutative deformations

We now calculate moduli spaces for vector bundles on deformation quantizations of (Z_k, σ) for σ a holomorphic Poisson structure.

Remark 7.1. Analogous to the situation of Remark 5.3, one has filtrability of vector bundles also in the noncommutative case (Lemma 6.9) so that moduli of rank 2 vector bundles can be described as quotients of Ext^1 of line bundles. Furthermore, we have formal algebraicity (Corollary 6.19) as the extension groups are $\mathbb{N}$-graded by powers of $\hbar$ with finite-dimensional graded components or indeed finite dimensional if we take $\hbar$ only up to a fixed finite power. We then obtain finite-dimensional quotients taking Ext^1 modulo bundle isomorphisms, where here isomorphisms are defined in Definition 6.3 using the (noncommutative) star product.

We thus may proceed as in the classical (commutative) setting and extract moduli spaces from extension groups of line bundles, by considering extension classes up to bundle isomorphism.

Notation 7.2. We denote by $\mathfrak{M}_j(\mathcal{Z}_k(\sigma))$ the subspace of $\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{A}(j), \mathcal{A}(-j))/{\sim}$ consisting of those classes of formally algebraic vector bundles, whose classical limit is a stable vector bundle of charge j. Here, $\sim$ denotes bundle isomorphism as in Definition 6.3. We denote by $\mathfrak{M}_j^{(n)}(\mathcal{Z}_k(\sigma))$ the moduli of bundles obtained by imposing the cutoff $\hbar^{n+1} = 0$; that is, the superscript $^{(n)}$ means quantized to level n. Accordingly, $\mathfrak{M}_j^{(1)}(\mathcal{Z}_k(\sigma))$ stands for first-order quantization and $\mathfrak{M}_j^{(0)}(\mathcal{Z}_k(\sigma)) = \mathfrak{M}_j(Z_k)$ recovers the classical moduli space of Definition 5.8 obtained when $\hbar = 0$.

Definition 7.3. The *splitting type* of a vector bundle E on $\mathcal{Z}_k(\sigma)$ is defined to be the splitting type of its classical limit as in Definition 5.2. Hence, when the classical limit is an $\mathrm{SL}(2,\mathbb{C})$ bundle, the splitting type of E is the smallest integer j such that E can be written as an extension of $\mathcal{A}(j)$ by $\mathcal{A}(-j)$.

We will look at rank 2 bundles of a fixed splitting type j on the first formal neighborhood $\ell^{(1)}$ of $\ell \subset Z_k$. In order for the relevant space of extensions to be nonzero, one should assume $k \leq 2j - 2$. Note that whenever $j \leq k \leq 2j - 2$ the *full* moduli space of splitting type j bundles on Z_k is supported on $\ell^{(1)}$. For general k, the moduli spaces of bundles on $\ell^{(1)}$ are dense open subspaces of the full moduli spaces of bundles on Z_k, so even by restricting to bundles on $\ell^{(1)}$, one obtains a partial description of the moduli space of bundles on all of Z_k. We thus refrain from introducing new notation, keeping the same notation as in Notation 7.2.

Moreover, we present the calculation only up to first order in $\hbar$, although we note that the explicit formulas of the star products given in Section 3.2 enable one to determine the moduli also to higher orders in $\hbar$ (see Remark 7.18).

Let $p + p'\hbar$ and $q + q'\hbar$ be two extension classes in $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{A}(j), \mathcal{A}(-j))$ which are of splitting type j; i.e., in canonical U-coordinates p, p', q, q' are multiples of u.

The bundles defined by $p + p'\hbar$ and $q + q'\hbar$ are isomorphic, if there exist invertible matrices

$$\begin{pmatrix} a + a'\hbar & b + b'\hbar \\ c + c'\hbar & d + d'\hbar \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha + \alpha'\hbar & \beta + \beta'\hbar \\ \gamma + \gamma'\hbar & \delta + \delta'\hbar \end{pmatrix}$$

whose entries are holomorphic on U and V, respectively, such that

$$\begin{pmatrix} \alpha + \alpha'\hbar & \beta + \beta'\hbar \\ \gamma + \gamma'\hbar & \delta + \delta'\hbar \end{pmatrix} \star \begin{pmatrix} z^j & q + q'\hbar \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} z^j & p + p'\hbar \\ 0 & z^{-j} \end{pmatrix} \star \begin{pmatrix} a + a'\hbar & b + b'\hbar \\ c + c'\hbar & d + d'\hbar \end{pmatrix}. \tag{7.4}$$

We wish to determine the constraints such an isomorphism imposes on the coefficients of q and q'. This is more convenient if rewritten by right-multiplying (7.4) with the inverse of $\left(\begin{smallmatrix} z^j & q+q'\hbar \\ 0 & z^{-j} \end{smallmatrix}\right)$ or more precisely by the right inverse with respect to $\star$, which (modulo $\hbar^2$) is

$$\begin{pmatrix} z^{-j} & -q - q'\hbar + 2z^{-j}\{z^j, q\} \\ 0 & z^j \end{pmatrix}.$$

On $\ell^{(1)}$, we have that $u^2 = 0$ and therefore $a = a_0 + a_1 u$, $\alpha = \alpha_0 + \alpha_1 u$, etc., where a_1, α_1, etc. are holomorphic functions in z.

We first observe that for the classical limit the calculations are given in [16, §3.1]. In particular, following the details of the proof of [16, Prop. 3.3], we may assume that

$a_0 = \alpha_0$ are constant, $d_0 = \delta_0$ are constant, and $b = \beta = 0$. Since we already know that on the classical limit the only equivalence on $\ell^{(1)}$ is given projectivization, we may assume that $p = q$ keeping in mind that there is a projectivization to be done in the end. We may also assume that the determinants of the changes of coordinates on the classical limit are 1. Accordingly, we may simplify (7.4) to

$$\begin{pmatrix} \alpha + \alpha'\hbar & \beta'\hbar \\ \gamma + \gamma'\hbar & \delta + \delta'\hbar \end{pmatrix}$$
$$= \begin{pmatrix} z^j & p + p'\hbar \\ 0 & z^{-j} \end{pmatrix} \star \begin{pmatrix} a + a'\hbar & b'\hbar \\ c + c'\hbar & d + d'\hbar \end{pmatrix} \star \begin{pmatrix} z^{-j} & -p - \big(q' - 2\{z^j, p\}z^{-j}\big)\hbar \\ 0 & z^j \end{pmatrix}, \quad (7.5)$$

where $a_0 = d_0 = \alpha_0 = \delta_0 = 1$.

Since we already know the moduli on the classical limit, we only need to study the terms containing $\hbar$, which after multiplying are

$$(1,1) = a' + \{z^j a, z^{-j}\} + \{z^j, a\}z^{-j} + \{pc, z^{-j}\} + \{p, c\}z^{-j} + (pc' + p'c)z^{-j},$$
$$(2,1) = z^{-2j}c',$$
$$(1,2) = -\{a, p\}z^j - \{z^j, a\}p + \{z^j, p\}a + \{pd, z^j\} + \{p, d\} + 2z^{-j}\{z^j, p\}pc$$
$$+ z^{2j}b' - (pa' + q'a)z^j + (pd' + p'd)z^j - (pc' + p'c + q'c)p,$$
$$(2,2) = d' + \{z^{-j}d, z^j\} + \{z^{-j}, d\}z^j - \{z^{-j}c, p\} - \{z^{-j}, c\}p - (pc' + q'c)z^{-j}$$
$$+ 2\{z^j, p\}z^{-2j}c.$$

All four terms must be adjusted by using free variables to only contain expressions which are holomorphic on V in order to satisfy (7.5). For example, the $(2,1)$ term shows that this condition is satisfied precisely when c' is a section of $\mathcal{O}(2j)$. A simple verification by computing the Poisson brackets shows that the $(1,1)$ and $(2,2)$ terms can always be adjusted by choosing, say, c and d' appropriately, leaving the coefficients of a' free.

It remains to analyze the term $(1,2)$. Because we are working on the first formal neighborhood of ℓ, we may drop terms in u^2 (recall that we assume that p, p', q' are multiples of u), giving

$$(1,2) = -\{a, p\}z^j - \{z^j, a\}p + \{z^j, p\}a + \{pd, z^j\} + \{p, d\}$$
$$+ 2\{z^j, p\}pc + z^{2j}b' - (pa' + q'a)z^j + (pd' + p'd)z^j. \quad (7.6)$$

Since $z^{2j}b'$ is there to cancel out any possible terms having power of z greater than or equal to $2j$, we only need to consider the coefficients of the monomials

$$z, z^2, \dots, z^{2j-1} \quad \text{and} \quad z^{k+1}u, \dots, z^{2j-1}u.$$

7.1. Examples

We now calculate moduli spaces for particular values of k and j and for different choices of Poisson structure. We will use the notation $P \in \mathfrak{M}_j(Z_k)$ to refer to a point in the moduli space, which we write in canonical coordinates as

$$P = [p_{1,k-j+1} : p_{1,k-j+2} : \cdots : p_{1,j-1}] \quad (7.7)$$

so that P corresponds to the isomorphism class of the bundle defined by

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$$

for $p = p_{1,k-j+1}z^{k-j+1}u + \cdots + p_{1,j-1}z^{j-1}u$.

Example 7.8. Let $(k, j, \sigma_U) = (1, 2, 1)$ and consider the Moyal product on $U \subset Z_1$, which by Proposition 4.1 extends to all of Z_1. We calculate the moduli space of vector bundles of splitting type 2 on $\ell^{(1)}$.

Recall that for these values of k and j the terms appearing in (7.6) take the form

$$\begin{aligned} a &= 1 + a_1(z)u, & p &= (p_{10} + p_{11}z)u, \\ c &= \sum_{l=0}^{4} c_{0,l}z^l + \Big(\sum_{l=0}^{5} c_{1,l}z^l\Big)u, & p' &= (p'_{10} + p'_{11}z)u, \\ d &= 1 + d_1(z)u, & q' &= (q'_{10} + p'_{11}z)u. \end{aligned}$$

After calculating the Poisson brackets appearing in (7.6), one finds that the coefficients of z, z^2, z^3 vanish. Thus, we only need to analyze the coefficients of z^2u and z^3u. The required vanishing of the coefficient of z^2u imposes the condition

$$\begin{aligned} p'_{10} - q'_{10} = {} & (a'_{00} - d'_{00} + a_{11} + 5d_{11})p_{10} - (a_{10} - 3d_{11})p_{11} \\ & - 4(c_{01}p_{11}^2 + 2c_{02}p_{10}p_{11} + c_{04}p_{10}^2). \end{aligned} \tag{7.9}$$

Similarly, the vanishing of the coefficient of z^3u imposes

$$\begin{aligned} p'_{11} - q'_{11} = {} & (a'_{01} - d'_{01} + 4a_{12})p_{10} - (-a'_{00} + d'_{00} - d_{11})p_{11} \\ & - 4(c_{02}p_{11}^2 + 2c_{03}p_{10}p_{11} + c_{04}p_{10}^2). \end{aligned} \tag{7.10}$$

Since for $P \in \mathfrak{M}_2(Z_1)$ its coefficients p_{10} and p_{11} do not vanish simultaneously, we can choose coefficients of a, d, or c to solve both (7.9) and (7.10). We observe that two out of a, d, c will already have been fixed on a previous step, when canceling coefficients in the $(1, 1)$ and $(2, 2)$ terms, but there remains always one of them free to be chosen. Hence, we can solve both equations for any values of q'_{10} and q'_{11}, so that for Poisson structure $\sigma_0 = (1, -\xi)$ two bundles over $\mathcal{Z}_1(\sigma_0)$ are isomorphic whenever their classical limits are, giving an isomorphism of moduli spaces

$$\mathfrak{M}_2^{(1)}\big(\mathcal{Z}_1(\sigma_0)\big) \simeq \mathfrak{M}_2(Z_1).$$

Example 7.11. Let $(k, j, \sigma_U) = (1, 2, u)$. For $\sigma = (u, -\xi^2 v)$, which by Proposition 4.1 defines a global star product on Z_1, (7.9) and (7.10) simplify to

$$\begin{aligned} p'_{10} - q'_{10} &= (a'_{00} - d'_{00})p_{10}, \\ p'_{11} - q'_{11} &= (a'_{01} - d'_{01})p_{10} + (a'_{00} - d'_{00})p_{11} \end{aligned}$$

which can be written as the Toeplitz system

$$\begin{pmatrix} p'_{11} - q'_{11} \\ p'_{10} - q'_{10} \end{pmatrix} = \begin{pmatrix} p_{10} & p_{11} \\ 0 & p_{10} \end{pmatrix} \begin{pmatrix} a'_{01} - d'_{01} \\ a'_{00} - d'_{00} \end{pmatrix}. \tag{7.12}$$

It then follows that the moduli space behavior is quite different from the case studied in Example 7.8. Here, if $p_{10} \neq 0$, the matrix $\left(\begin{smallmatrix} p_{10} & p_{11} \\ 0 & p_{10} \end{smallmatrix}\right)$ is invertible, so we can solve (7.12) for any q', giving the equivalence relation

$$(p_{10}, p_{11}, p'_{10}, p'_{11}) \sim (\lambda p_{10}, \lambda p_{11}, q'_{10}, q'_{11}) \quad \text{if } p_{10} \neq 0,$$

so that the fibre of the projection

$$\mathfrak{M}_2^{(1)}(\mathcal{Z}_1(\sigma)) \xrightarrow{\pi} \mathfrak{M}_2(Z_1)$$

to the classical limit is just a point provided $p_{10} \neq 0$.

However, if $p_{10} = 0$, that is, over the single point $P = [0:1]$ in the classical moduli space $\mathfrak{M}_2(Z_1)$, (7.12) imposes the additional constraint that $q'_{10} = p'_{10}$. (The coefficient q'_{11} remains arbitrary because p_{11} must be nonzero in this case.)

We thus obtain the following equivalence relation:

$$(p_{10}, p_{11}, p'_{10}, p'_{11}) \sim (\lambda p_{10}, \lambda p_{11}, p'_{10}, q'_{11}) \quad \text{if } p_{10} = 0.$$

Here, we observe that both p'_{11} and q'_{11} are arbitrary, so that the resulting equivalence relation on the fibre over $P = [0:1] \in \mathfrak{M}_2(Z_1)$ can equivalently be represented by

$$(0, p_{11}, p'_{10}, *) \sim (0, \lambda p_{11}, p'_{10}, *)$$

(where $*$ denotes an arbitrary complex number). The points on the fibre are thus parametrized by the different values of $p'_{10} \in \mathbb{C}$. Therefore, we have obtained that the fibre $L = \pi^{-1}([0:1]) = [0:1] \times \mathbb{C}$ is an affine line. Equivalently, the collection of points $L \subset \mathfrak{M}_2^{(1)}(\mathcal{Z}_1(\sigma))$ that have P as its classical limit is $L = [0:1] \times \mathbb{C}$. We can thus view $\mathfrak{M}_2^{(1)}(\mathcal{Z}_1(\sigma))$ as the étale space of a skyscraper sheaf supported at P, giving a continuous surjection

$$\begin{array}{c} \mathfrak{M}_2^{(1)}(\mathcal{Z}_1(\sigma)) \\ \downarrow \\ \mathfrak{M}_2(Z_1). \end{array}$$

Example 7.13. Let $(k, j, \sigma_U) = (1, 3, 1)$. Here, the imposed constraints are that the coefficients of z^2u, z^3u, z^4u, z^5u must vanish. To illustrate the calculation, we list the first two. The analogues of (7.9) and (7.10), corresponding to the vanishing of the coefficients of z^2u and z^3u, are

$$\begin{aligned} & p'_{1-1} - q'_{1-1} \\ & \quad = (a'_{00} - d'_{00} + 2a_{11} + 8d_{11})p_{1-1} + 6d_{10}p_{10} \\ & \qquad - 6\big(c_{00}(p_{11}^2 + 2p_{10}p_{12}) + 2c_{01}(p_{10}p_{11} + p_{1-1}p_{12}) \\ & \qquad\quad + c_{02}(p_{10}^2 + 2p_{1-1}p_{11}) + 2c_{03}p_{1-1}p_{10} + c_{04}p_{1-1}^2\big), \end{aligned}$$

$$
\begin{aligned}
&p'_{10} - q'_{10} \\
&\quad = (a'_{01} - d'_{01} + 3a_{12} + 9d_{12})p_{1-1} + (a'_{00} - d'_{00} + a_{11} + 7d_{11})p_{10} - (a_{10} - 5d_{10})p_{11} \\
&\qquad - 6\big(2c_{00}p_{11}p_{12} + c_{01}(p_{11}^2 + 2p_{10}p_{12}) + 2c_{02}(p_{10}p_{11} + p_{1-1} + p_{12}) \\
&\qquad\qquad + c_{03}(p_{10}^2 + 2p_{1-1}p_{11}) + 2c_{04}p_{1-1}p_{10} + c_{05}p_{1-1}^2\big)
\end{aligned}
$$

and similarly for the coefficients of z^4u and z^5u. It then turns out that over the points belonging to the moduli space $\mathfrak{M}_3(Z_1)$ we can solve all the constraint equations, so that q is arbitrary, and once again as in Example 7.8 we obtain an isomorphism between the quantum and the classical moduli

$$\mathfrak{M}_3^{(1)}\big(\mathcal{Z}_1(\sigma_0)\big) \simeq \mathfrak{M}_3(Z_1).$$

Generalizing Examples 7.8 and 7.13, we obtain the following theorem.

Theorem 7.14. *Let σ_0 be the Poisson structure which in canonical coordinates is $\sigma_0 = (1, -\xi)$. Then for each j, we obtain an isomorphism*

$$\mathfrak{M}_j^{(1)}\big(\mathcal{Z}_1(\sigma_0)\big) \simeq \mathfrak{M}_j(Z_1).$$

Example 7.15. For $(k, j, \sigma_U) = (1, 3, u)$, the relevant monomials of (7.6) are $z^2u, \dots, z^5u$ and the vanishing condition can be written as the Toeplitz system

$$
\begin{pmatrix} p'_{12} - q'_{12} \\ p'_{11} - q'_{11} \\ p'_{10} - q'_{10} \\ p'_{1-1} - q'_{1-1} \end{pmatrix}
=
\begin{pmatrix} p_{1-1} & p_{10} & p_{11} & p_{12} \\ 0 & p_{1-1} & p_{10} & p_{11} \\ 0 & 0 & p_{1-1} & p_{10} \\ 0 & 0 & 0 & p_{1-1} \end{pmatrix}
\begin{pmatrix} a'_{03} - d'_{03} \\ a'_{02} - d'_{02} \\ a'_{01} - d'_{01} \\ a'_{00} - d'_{00} \end{pmatrix}. \tag{7.16}
$$

As in Example 7.11, the fibres of the projection

$$
\begin{array}{c}
\mathfrak{M}_3^{(1)}\big(\mathcal{Z}_1(\sigma)\big) \\
\pi \big\downarrow \\
\mathfrak{M}_3(Z_1)
\end{array}
$$

vary depending on the coordinates of the point $[p_{1-1} : p_{10} : p_{11} : p_{12}] \in \mathfrak{M}_3(Z_1) \subset \mathbb{P}^3$. We have the following.

- If $p_{1-1} \neq 0$, we can solve all the equations by choosing a' and d' appropriately. Hence, there exists an isomorphism for any value of q' and consequently the fibre in this case is only a point. Thus,

  $$S_0 := \big\{P \in \mathfrak{M}_3(Z_1) \mid p_{1-1} \neq 0\big\}$$

  is an open set of the classical moduli space over which the map π is an isomorphism.

- Now, assume $p_{1-1} = 0$ but $p_{10} \neq 0$. Then the fourth equation imposes the condition $q'_{1-1} = p'_{1-1}$ but the other equations can be solved for any values of $q'_{10}, q'_{11}, q'_{12}$, so that isomorphism gives the equivalence relation

$$[0:1:p_{11}:p_{12}](p'_{1-1}, p'_{10}, p'_{11}, p'_{12}) \sim [0:1:p_{11}:p_{12}](p'_{1-1}, *, *, *),$$

  where $*$ stands for an arbitrary complex value. Thus, the fibre of π over a point $[0:1:p_{11}:p_{12}]$ is a copy of $\mathbb{C}$ parametrized by the different values of p'_{1-1}. In other words, setting

$$S_1 := \{P \in \mathfrak{M}_3(Z_1) \mid p_{1-1} = 0,\ p_{10} \neq 0\},$$

  we have that if $P \in S_1$ then $\pi^{-1}(P) \simeq \mathbb{C}$.
- Next assume that $p_{1-1} = p_{10} = 0$ but $p_{11} \neq 0$; then the third and fourth equations of (7.16) impose the conditions $q'_{1-1} = p'_{1-1}$ and $q'_{10} = p'_{10}$ but the remaining two equations can be solved for any values of q'_{11}, q'_{12}. Hence, in this case, isomorphism imposes the equivalence relation

$$[0:0:1:p_{12}](p'_{1-1}, p'_{10}, p'_{11}, p'_{12}) \sim [0:0:1:p_{12}](p'_{1-1}, p'_{10}, *, *)$$

  with the fibre of π over such a point being a copy of $\mathbb{C}^2$ parametrized by the values of (p'_{1-1}, p'_{10}). Thus, setting

$$S_2 := \{P \in \mathfrak{M}_3(Z_1) \mid p_{1-1} = p_{10} = 0,\ p_{11} \neq 0\},$$

  we have that if $P \in S_2$ then $\pi^{-1}(P) \simeq \mathbb{C}^2$.
- Lastly, there would be the point $[0:0:0:1]$ to be considered, but it does not belong to $\mathfrak{M}_3(Z_1)$ because it corresponds to a vector bundle with charge 5 (see Definition 5.5 and Table 1), so we are already done.

In conclusion, we have a continuous surjection

$$\begin{array}{c} \mathfrak{M}_3^{(1)}(\mathcal{Z}_1(\sigma)) \\ \pi \big\downarrow \\ \mathfrak{M}_3(Z_1) \end{array}$$

where $\mathfrak{M}_3^{(1)}(\mathcal{Z}_1(\sigma))$ is viewed as the étale space of a constructible sheaf, with stalks of dimension i over the strata S_i for $i = 0, 1, 2$.

7.2. General case

Let σ be a holomorphic Poisson structure on Z_k such that σ_U is a multiple of u and let $\mathcal{Z}_k(\sigma)$ be any deformation quantization of Z_k. (Note that by Lemma 2.8 *any* Poisson structure on $Z_{\geq 3}$ is of this form.)

Monomial	Width	Height	Charge
$z^{-1}u$	3	2	5
u	1	2	3
zu	1	2	3
z^2u	3	2	5
u^2	3	3	6
zu^2	2	3	5
z^2u^2	3	3	6
zu^3	5	3	7
z^2u^3	5	3	7
z^2u^4	5	3	8
zero	6	3	9

Table 1. Numerical invariants for rank 2 vector bundles of splitting type $j = 3$ on Z_1.

Then for any j, the moduli space $\mathfrak{M}_j^{(1)}(Z_k(\sigma))$ may be described by the Toeplitz system

$$\begin{pmatrix} p'_{1,j-1} - q'_{1,j-1} \\ p'_{1,j-2} - q'_{1,j-2} \\ \vdots \\ p'_{1,k-j+2} - q'_{1,k-j+2} \\ p'_{1,k-j+1} - q'_{1,k-j+1} \end{pmatrix} = \begin{pmatrix} p_{1,k-j+1} & p_{1,k-j+2} & p_{1,k-j+3} & \cdots & p_{1,j-1} \\ 0 & p_{1,k-j+1} & p_{1,k-j+2} & \ddots & \vdots \\ 0 & 0 & p_{1,k-j+1} & \ddots & p_{1,k-j+3} \\ \vdots & \ddots & \ddots & \ddots & p_{1,k-j+2} \\ 0 & \cdots & 0 & 0 & p_{1,k-j+1} \end{pmatrix} \begin{pmatrix} a'_{0,2j-k-2} - d'_{0,2j-k-2} \\ a'_{0,2j-k-3} - d'_{0,2j-k-3} \\ \vdots \\ a'_{0,1} - d'_{0,1} \\ a'_{0,0} - d'_{0,0} \end{pmatrix}.$$

Examples 7.11 and 7.15 thus readily generalize to give the following.

Theorem 7.17. *Let σ be a holomorphic Poisson structure on Z_k such that σ_U is a multiple of u. Then the quantum moduli space $\mathfrak{M}_j^{(1)}(\mathcal{Z}_k(\sigma))$ can be viewed as the étale space of a constructible sheaf over the classical moduli space $\mathfrak{M}_j(Z_k)$, the sheaf being trivial over the open set*

$$S_0 := \{P \in \mathfrak{M}_j(Z_k) \mid p_{1,k-j+1} \neq 0\}$$

and with stalk of dimension i over the locally closed subvarieties

$$S_i := \{P \in \mathfrak{M}_j(Z_k) \mid p_{1,k-j+1} = \cdots = p_{1,k-j+i} = 0,\ p_{1,k-j+1+i} \neq 0\}.$$

Proof. The classical moduli space gets stratified into disjoint subsets S_i over which the fibre of the projection

$$\begin{array}{c} \mathfrak{M}_j^{(1)}(\mathcal{Z}_k(\sigma)) \\ \pi \big\downarrow \\ \mathfrak{M}_j(Z_k) \end{array}$$

has dimension i. The strata are the dense open subset

$$S_0 := \{P \in \mathfrak{M}_j(Z_k) \mid p_{1,k-j+1} \neq 0\}$$

and locally closed subsets of decreasing dimension

$$S_i := \{P \in \mathfrak{M}_j(Z_k) \mid p_{1,k-j+1} = \cdots = p_{1,k-j+i} = 0,\ p_{1,k-j+1+i} \neq 0\}. \qquad \blacksquare$$

Remark 7.18. *Higher powers of $\hbar$.* Repeating the calculation for higher powers of $\hbar$, one finds that $\mathfrak{M}_2^{(2)}(\mathcal{Z}_1(\sigma)) \simeq \mathfrak{M}_2^{(1)}(\mathcal{Z}_1(\sigma))$ so that, for splitting type 2, considering terms up to $\hbar^2$ does not give any more vector bundles. We thus expect that for arbitrary powers of $\hbar$ one obtains an isomorphism $\mathfrak{M}_2(\mathcal{Z}_1(\sigma)) \simeq \mathfrak{M}_2^{(1)}(\mathcal{Z}_1(\sigma))$ with all splitting type 2 vector bundles supported on the first neighborhood of $\hbar$. In particular, this implies that bundles of splitting type 2 on $\mathcal{Z}_1(\sigma)$ are *algebraic* in the sense of Definition 6.18.

Similarly, Theorem 7.14 would imply

$$\mathfrak{M}_j(\mathcal{Z}_1(\sigma_0)) \simeq \mathfrak{M}_j(Z_1)$$

for the minimally degenerate Poisson structure σ_0 and arbitrary j, which corresponds to the statement that bundles of arbitrary splitting type on $\mathcal{Z}_1(\sigma_0)$ are algebraic.

On the other hand, the calculation of $\mathfrak{M}_2^{(2)}(\mathcal{Z}_1(\sigma))$ also suggests that for $j \geq 3$ and Poisson structures which are degenerate on all of ℓ, for example, any Poisson structure on $Z_{k\geq 3}$, the moduli space $\mathfrak{M}_j(\mathcal{Z}_k(\sigma))$ should also contain bundles whose presentation requires nontrivial coefficients of $\hbar^n$ for $n > 1$. In particular, this implies that the moduli space $\mathfrak{M}_j^{(n)}(\mathcal{Z}_k(\sigma))$ is larger than $\mathfrak{M}_j^{(1)}(\mathcal{Z}_k(\sigma))$.

The calculation of $\mathfrak{M}_2^{(2)}(\mathcal{Z}_1(\sigma))$ is analogous to the calculations of Examples 7.8 and 7.11 and can be reproduced by using the Kontsevich star product extended to Z_k as given in Proposition 4.1, whose terms up to second order can be obtained from Lemma 3.10. However, the calculation is considerably longer and we do not include it here, leaving a more general investigation for future work.

8. Classical instantons

By definition, an *instanton* on a 4-manifold X is a connection A minimizing the Yang–Mills functional

$$\mathrm{YM}(A) = \int_X \operatorname{tr} F \wedge F,$$

where F is the curvature of A. The Euler–Lagrange equations for the Yang–Mills functional

$$D(*F) = 0 \tag{8.1}$$

are called the *Yang–Mills equations* (here $*$ denotes Hodge dual). Thus, an instanton is a solution of the Yang–Mills equations. A linearized version of these equations is given by the anti-self-duality (ASD) equations

$$F^+ = \frac{1}{2}(F + *F) = 0. \tag{8.2}$$

Equation (8.2) is sometimes called the *instanton equation* because its solutions also satisfy (8.1).

The translation from gauge theory to complex geometry is made via the so-called Kobayashi–Hitchin correspondence, which relates instantons and vector bundles as well as their moduli spaces [28]. If X is a complex Kähler surface and E is an $\mathrm{SU}(2)$ bundle on X, then the moduli space M_E of irreducible ASD connections on E is a complex analytic space and each point in M_E has a neighborhood which is the base of a universal deformation of the corresponding stable vector bundle [13, Prop. 6.4.4]. A version of such a correspondence for noncompact surfaces states the following.

Lemma 8.3 ([18, Cor. 5.5]). *A holomorphic* $\mathrm{SL}(2,\mathbb{C})$ *vector bundle on* Z_k *corresponds to an* $\mathrm{SU}(2)$ *instanton if and only if its splitting type is a multiple of* k.

The surfaces Z_k have rich moduli spaces of instantons, which unfortunately disappear under any small commutative deformation of Z_k. In fact, for $j \equiv 0 \bmod k$, we have the following.

Theorem 8.4 ([5, Thm. 4.11]). *The moduli space of irreducible* $\mathrm{SU}(2)$ *instantons on* Z_k *with charge (and splitting type)* j *is a quasi-projective variety of dimension* $2j - k - 2$.

In contrast, we have the following.

Theorem 8.5 ([8, Thm. 7.3]). *Let* $\mathcal{Z}_k(\tau)$ *be a nontrivial commutative deformation of* Z_k. *Then the moduli spaces of irreducible* $\mathrm{SU}(2)$ *instantons on* $\mathcal{Z}_k(\tau)$ *are empty.*

Disappearance of instantons under classical deformations gave us a strong motivation to explore noncommutative directions of deformations. From the point of view of algebraic deformation theory, both classical and noncommutative deformations can be regarded on equal footing—as components in the Hochschild–Kostant–Rosenberg decomposition of the Hochschild cohomology

$$\mathrm{HH}^2(Z_k) \simeq \mathrm{H}^1(Z_k, \mathcal{T}_{Z_k}) \oplus \mathrm{H}^0(Z_k, \Lambda^2 \mathcal{T}_{Z_k})$$

which parametrizes deformations of the Abelian of category of (quasi)coherent sheaves in the sense of [27]. (Note that simultaneous deformations of Z_k in commutative and noncommutative directions may be obstructed. The obstruction calculus was studied in [7].) However, from a physics point of view, we hoped, and intuitively expected, that moduli of instantons appear again on directions of noncommutative deformations. We will see that this is indeed the case. Furthermore, we discover that some instantons react wildly to certain types of quantization, causing quantum moduli spaces to become larger than the classical ones.

The geometry underlying the disappearance of instantons on commutative deformations is the fact that if $\mathcal{Z}_k(\tau)$ is any nontrivial commutative deformation of Z_k, then every holomorphic vector bundle on $\mathcal{Z}_k(\tau)$ splits as a direct sum of line bundles [8, Thm. 6.10]. The key issue here is that for $\tau \neq 0$ the deformations $\mathcal{Z}_k(\tau)$ are affine [8, Thm. 6.18]. In particular, by [8, Thm. 6.6] such a nontrivial deformation contains no compact complex curves. In physics language, we may say that there is no compact manifold which can hold the instanton charge.

In Yang–Mills theory, instantons are well known to carry topological charges. Under the Kobayashi–Hitchin correspondence, the charge of an SU(2) instanton on a complex surface translates into the second Chern class of its corresponding $\mathrm{SL}(2,\mathbb{C})$ holomorphic vector bundle. Here, we use instead the concept of *local charge*, given that the second Chern class of a bundle on Z_k vanishes, since $\mathrm{H}^4(Z_k,\mathbb{Z}) \simeq \mathrm{H}^4(S^2,\mathbb{Z}) = 0$. The terminology "local charge" is motivated by the fact that it provides a local contribution to the second Chern class when we consider a compact surface containing an embedded Z_k.

Definition 8.6. We define the *normalized charge* of an instanton on Z_k to be the sum of the local second Chern class (Definition 5.5) of the bundle to which it corresponds and ε, where $\varepsilon = 1$ if $k \geq 2$ and 0 when $k = 1$. In what follows, we will simply refer to this normalized charge as the *charge* of the instanton for brevity.

Remark 8.7. The reason for this normalization is that, once normalized, the minimal charge of an SU(2) instanton with splitting type j equals j, and this allows us to express the theorems that follow in a much simpler way. We observe that the addition of $\varepsilon = 1$ in the cases $k \geq 2$ corresponds to the fact that the surface X_k obtained by contracting the $\mathbb{P}^1$ to a point has one singularity. In case we considered more general surfaces containing other contractible curves, the correct normalization should most likely be to count all singularities obtained by their contraction. For bounds on the values of instanton charges, see [5, 18].

We note also that there exist more than one notion of Chern classes for sheaves on singular varieties. A particularly useful one, presented by Blache [9], is the concept of orbifold Chern class, defined using the familiar integration formula $\int_X \mathrm{ch}(E)\,\mathrm{td}(X)$ appearing in Hirzebruch–Riemann–Roch formulas, but which in the case of singular varieties differs from the Euler characteristic by a weighted counting of singularities. In fact, Blache proves the following formula:

$$\chi(X,E)=\int_X \operatorname{ch}(E)\operatorname{td}(X)+\sum_{x\in \operatorname{Sing}(X)} \mu_{x,X}(E).$$

Here, our ε may be regarded as keeping track of when the defect term μ is nonzero.

As an illustration, we give instanton invariants from [3] in Table 1.

9. Rebel instantons

We wish to investigate the effect that noncommutative deformations have on instantons—these effects can already be observed at first order in $\hbar$. We thus reinterpret the results of Section 6 into the language of instantons.

Noncommutative versions of (8.1) and (8.2) and instanton solutions for the cases of noncommutative $\mathbb{R}^4$ are presented in [32], with self-duality being given by the generalized ASD equations

$$F^+_{\mu\nu}=0,$$

where the curvature of the connection is calculated by the generalized formula

$$F^i_{\mu\nu,j}=\partial_\mu A^i_{\nu,j}-\partial_\nu A^i_{\mu,j}+A^i_{\mu,k}\star A^k_{\nu,j}-A^i_{\nu,k}\star A^k_{\mu,j}.$$

ASD connections are then automatically solutions to the deformed Yang–Mills equations

$$\partial_\mu F_{\mu\nu}-A_\mu\star F_{\mu\nu}=0.$$

Instantons on noncommutative $\mathbb{R}^4$ in this sense were further studied in [35] for their relations with string theory and in [20] on noncommutative projective planes. For our noncommutative deformations of Z_k, we have obtained global star products and one could also approach the study of instantons using such deformed equations. However, it is more convenient to work directly in the language of vector bundles.

Definition 9.1. Based on Definition 7.3 and the result of Lemma 8.3, a formally algebraic bundle on $\mathcal{Z}_k(\sigma)$ of rank 2 and splitting type nk is called a *formal instanton*. The *charge* of a formal instanton on $\mathcal{Z}_k(\sigma)$ is defined as the charge of its classical limit (Definition 8.6), i.e., obtained by setting $\hbar=0$.

Definition 9.2. According to Definition 6.11, instantons on Z_k and on deformations of Z_k can be represented in canonical coordinate charts by a pair $(j,\mathbf{p})$ of an integer and a formal expression $\mathbf{p}=\sum p_n\hbar^n$, where p_n are polynomials. For a fixed splitting type j, the instanton is thus determined by the coefficients of the corresponding polynomials, and in this case, we denote the point on the corresponding moduli space simply by P as in (7.7).

Notation 9.3. We will denote by $\mathbb{QI}_j(\mathcal{Z}_k(\sigma))$ the moduli space of formal instantons of charge j on $\mathcal{Z}_k(\sigma)$ and by $\mathbb{QI}_j^{(n)}(\mathcal{Z}_k(\sigma))$ the subspace obtained by imposing the cut $\hbar^{n+1}=0$. Hence, $\mathbb{QI}_j^{(0)}(\mathcal{Z}_k(\sigma))$ equals the classical moduli space $\mathbb{MI}_j(Z_k)$ of instantons of charge j.

Therefore, by definition we have the following.

Lemma 9.4. *The classical limit of any formal instanton on $\mathcal{Z}_k(\sigma)$ is an instanton on Z_k.*

Proof. The correspondence between vector bundles and instantons may be applied to both quantum and classical moduli. The projection onto classical moduli translates as follows.

$$\begin{array}{ccc} \mathfrak{M}_j^{(1)}(\mathcal{Z}_k(\sigma)) & & \mathbb{QI}_j^{(1)}(\mathcal{Z}_k(\sigma)) \\ \downarrow{\scriptstyle \pi} & \longleftrightarrow & \downarrow{\scriptstyle \pi} \\ \mathfrak{M}_j(Z_k) & & \mathbb{MI}_j(Z_k). \end{array}$$

■

We now interpret the statements of Theorems 7.14 and 7.17 in terms of instantons. These get rephrased as:

Theorem 9.5. *Let σ_0 be the Poisson structure on Z_1 which in canonical coordinates is described by $\sigma_0 = (1, -\xi)$; hence σ_0 is degenerate at a single point of the line $\ell \subset Z_1$. Then for each value of the charge j, we obtain an isomorphism between the quantum and the classical instanton moduli spaces*

$$\mathbb{QI}_j^{(1)}(\mathcal{Z}_1(\sigma_0)) \simeq \mathbb{MI}_j(Z_1).$$

Theorem 9.6. *Let σ be a holomorphic Poisson structure such that σ_U is a multiple of u. Then the quantum instanton moduli space $\mathbb{QI}_j^{(1)}(\mathcal{Z}_k(\sigma))$ can be viewed as the étale space of a constructible sheaf over the classical instanton moduli space $\mathbb{MI}_j(Z_k)$ which is supported on a closed subvariety, being trivial over*

$$S_0 := \{P \in \mathbb{MI}_j(Z_k) \mid p_{1,k-j+1} \neq 0\}$$

and having stalk of dimension i over

$$S_i := \{P \in \mathbb{MI}_j(Z_k) \mid p_{1,k-j+1} = \cdots = p_{1,k-j+i} = 0,\ p_{1,k-j+1+i} \neq 0\}.$$

Notation 9.7. When regarding the quantum moduli space $\mathbb{QI}_j(\mathcal{Z}_k(\sigma))$ as the étale space of a sheaf over the classical moduli space $\mathbb{MI}_j(Z_k)$, we will use the notation $\mathcal{Q}^\sigma$ and call it the *quantizing sheaf*.

Definition 9.8. A (classical) instanton A is called a *rebel instanton* for σ if the stalk $\mathcal{Q}^\sigma_A$ of the quantizing sheaf $\mathcal{Q}^\sigma$ at A is nontrivial. The dimension of the stalk at A is called the *level of rebelliousness* of A. Hence, if the stalk $\mathcal{Q}^\sigma_A$ has rank n, then A is said to present level n rebelliousness, or equivalently, A is called *n-rebel instanton*. For brevity, the vocabulary *rebel* will be used to denote rebelliousness of any level $n \geq 1$.

An instanton that is not rebel for σ is called *σ-tame*.

In particular, in the language of Definition 9.8 (see also Notation 2.15), we can rephrase Theorem 9.5 as follows.

Theorem 9.9. *Let σ_0 be a minimally degenerate Poisson structure on Z_1. Then all instantons on Z_1 are σ_0-tame.*

Note that the zero section ℓ of Z_1 contracts to a smooth point and Theorem 9.9 stands in stark contrast to the case of $k \geq 3$ where the zero section ℓ of Z_k contracts to a $\frac{1}{k}(1,1)$ singularity.

Theorem 9.10. *Let $k \geq 3$. Then Z_k produces rebel instantons for any holomorphic Poisson structure σ.*

To be interesting from a physical perspective, it is important that the first-order deformations parametrized by a holomorphic Poisson structure can be continued to all higher orders, which was shown in Section 4, at least for Poisson structures tangent to a fibre of the projection to $\mathbb{P}^1$. Theorems 9.9 and 9.10 now exhibit a phenomenon that can already be observed at first order in $\hbar$ and for higher orders in $\hbar$, we expect Theorems 9.9 and 9.10 to generalize as follows.

Phrased in the language of (rebel) instantons, Remark 7.18 suggests that, for the minimally degenerate Poisson structure σ_0 on Z_1, we expect an isomorphism $\mathbb{QI}_j(\mathcal{Z}_1(\sigma_0)) \simeq \mathbb{MI}_j(Z_1)$; i.e., the absence of rebel instantons implies that the moduli of noncommutative and classical instantons are isomorphic for arbitrary powers of $\hbar$.

On the other hand, we expect the existence of n-rebel instantons (as, for example, in Theorem 9.10) to imply the existence of noncommutative instantons whose minimal representatives $\mathbf{p} = \sum_{n\geq 0} p_n \hbar^n$ contain non-vanishing terms p_n; i.e., n-rebel instantons produce noncommutative instantons up to $\hbar^n$.

A. Cohomology

In this appendix, we determine global sections of Z_k with line bundle coefficients. These are used in the proof of Lemma 2.8 to determine the space of Poisson structures on Z_k.

The coordinate ring of Z_k is

$$R = \mathrm{H}^0(Z_k, \mathcal{O}) = \mathbb{C}[x_0, x_1, \ldots, x_k]/(x_i x_{j+1} - x_{i+1} x_j)_{0\leq i<j\leq k-1}.$$

The contraction map $Z_k \to \operatorname{Spec} R$ is given in (z,u)-coordinates by $z^i u \mapsto x_i$. We compute $\mathrm{H}^0(Z_k, \mathcal{O}(j))$.

Lemma A.1. *The cohomology $\mathrm{H}^0(Z_k, \mathcal{O}(j))$ for $j \geq 0$ is generated as an R-module by the monomials $\beta_i = z^i$ for $0 \leq i \leq j$ with relations*

$$\beta_i x_{l-1} - \beta_{i-1} x_l = 0,$$

where $1 \leq l \leq k$ and $1 \leq i \leq j$.

Proof. Let $\mathfrak{U} = \{U, V\}$ be our canonical coordinates on Z_k and let $\sigma \in \check{\mathrm{C}}^0(\mathfrak{U}, \mathcal{O}(j))$ be a 0-cochain; thus σ consists of a pair of holomorphic sections (σ_U, σ_V). Writing out the general form of such a cochain, we have $\sigma_U = \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \sigma_{il} z^l u^i$ as this is just an arbitrary holomorphic map on U, where the line bundle trivializes. As we know, H^0

computes global holomorphic sections, so to find the cohomology class of σ, we just need to find the most general such σ that extends holomorphically to V. The transition function for the line bundle $\mathcal{O}(j)$ is $T = z^{-j}$. Changing coordinates then gives $T\sigma = z^{-j}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sigma_{il}z^l u^i$ which needs to be holomorphic on V. This gives the condition that $T\sigma$ must contain only terms $z^r u^s$ with $r \leq ks$. Terms that do not satisfy this condition are not cocycles and must be removed from the expression of $T\sigma$. We are thus left with $T\sigma = \sum_{i=0}^{\infty}\sum_{l=0}^{ki+j}\sigma_{il}z^{l-j}u^i$ or equivalently $\sigma = \sum_{i=0}^{\infty}\sum_{l=0}^{ki+j}\sigma_{il}z^l u^i$ which can be rewritten as

$$\sigma = \sum_{l=0}^{j}\sigma_{0l}z^l + \sum_{i=1}^{\infty}\sum_{l=0}^{ki+j}\sigma_{il}z^{l-ki}(z^k u)^i.$$

Now notice that on the right-hand side every term on the second sum can be obtained from a term on the first sum by multiplying by $(z^k u)^i$ which are global holomorphic functions on Z_k; thus, $\mathrm{H}^0(Z_k, \mathcal{O}(j))$ with $j \geq 0$ is generated as an R-module by the monomials $\beta_i = z^i$ for $0 \leq i \leq j$. Since $j \geq 0$, there is at least one such β_i. These satisfy the equalities

$$\begin{aligned} \beta_1 x_0 &= \beta_0 x_1 = zu, \\ \beta_2 x_0 = \beta_1 x_1 &= \beta_0 x_2 = z^2 u, \\ &\vdots \\ \beta_j x_{k-1} &= \beta_{j-1} x_k = z^{k+j-1}u. \end{aligned}$$ ∎

To compute $\mathrm{H}^0(Z_k, \mathcal{O}(-j))$, set $\nu = -j \mod k$, so that

$$-j = -qk + \nu$$

with $0 \leq \nu < k$.

Lemma A.2. *The cohomology group* $\mathrm{H}^0(Z_k, \mathcal{O}(-j))$ *for* $j > 0$ *is generated by the monomials* $\alpha_i = z^i u^q$, *for* $0 \leq i \leq \nu$, *with relations*

$$\alpha_i x_{l-1} - \alpha_{i-1} x_l = 0$$

for $1 \leq i \leq \nu$ *and* $1 \leq l \leq k$.

Proof. As in the proof of Lemma A.1, we start with a 0-cochain (σ_U, σ_V) having $\sigma_U = \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sigma_{il}z^l u^i$, and we look for the corresponding σ_V making this a 0-cocycle. Changing coordinates, we have

$$T\sigma = z^j \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sigma_{il}z^l u^i.$$

Here, since $j > 0$, terms on u^0 are not holomorphic on V. To have holomorphicity, we need only terms $z^r u^s$ with $r \leq ks$; we are thus looking to satisfy the condition $j + l \leq ki$. Thus, we arrive at the expression $T\sigma = \sum_{i=0}^{\infty}\sum_{l=0}^{ki-j}\sigma_{il}z^{j+l}u^i$ whose terms are nonzero only when $ki - j \geq 0$, that is, $i \geq \lfloor j/k \rfloor$. Using the notation set up just above for the

Euclidean algorithm, we have $q = \lfloor j/k \rfloor$, and we are searching for holomorphic terms in the expression $T\sigma = \sum_{i=q}^{\infty} \sum_{l=0}^{ki-j} \sigma_{il} z^{j+l} u^i$. Note that for $i = q$ we have $0 \le l \le kq - j = \nu$; thus, we may rewrite

$$T\sigma = \sum_{l=0}^{\nu} \sigma_{ql} z^{j+l} u^q + \sum_{i=q+1}^{\infty} \sum_{l=0}^{ki-j} \sigma_{il} z^{j+l} u^i$$

or equivalently

$$T\sigma = \sum_{l=0}^{\nu} \sigma_{ql} z^{j+l} u^q + \sum_{i=1}^{\infty} \sum_{l=0}^{\nu+ki} \sigma_{q+i,l} z^{j+l} u^{q+i}.$$

Thus,

$$\sigma = \sum_{l=0}^{\nu} \sigma_{ql} z^{l} u^q + \sum_{i=1}^{\infty} \sum_{l=0}^{\nu+ki} \sigma_{q+i,l} z^{l} u^{q+i}$$

and the second sum on the right-hand side has only terms that can be obtained from the first sum via multiplying by $(z^{\le k} u)^i$ which are global holomorphic functions on Z_k. Hence, $\mathrm{H}^0(Z_k, \mathcal{O}(-j))$ is generated by the monomials $\alpha_i = z^i u^q$, for $0 \le i \le \nu$. These satisfy the relations

$$\begin{aligned} \alpha_1 x_0 &= \alpha_0 x_1 = z u^q, \\ \alpha_2 x_0 = \alpha_1 x_1 &= \alpha_0 x_2 = z^2 u^q, \\ \vdots \qquad & \vdots \qquad \vdots \\ \alpha_\nu x_{k-1} &= \alpha_{\nu-1} x_k = z^{k+\nu-1} u^q. \end{aligned}$$ ∎

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