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Automorphism groups of cubic fourfolds and K3 categories

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Abstract

In this paper, we study relations between automorphism groups of cubic fourfolds and Kuznetsov components. Firstly, we characterize automorphism groups of cubic fourfolds as subgroups of autoequivalence groups of Kuznetsov components using Bridgeland stability conditions. Secondly, we compare automorphism groups of cubic fourfolds with automorphism groups of their associated K3 surfaces. Thirdly, we note that the existence of a non-trivial symplectic automorphism on a cubic fourfold is related to the existence of associated K3 surfaces.

1. Introduction

1.1 K3 surfaces and sporadic finite groups

Finite symmetries of K3 surfaces are related to sporadic finite groups such as the Mathieu groups M_{23} and M_{24} and the Conway groups Co_0 and Co_1 in both mathematics and physics. Mukai [Muk88] proved that finite groups of symplectic automorphisms of K3 surfaces are certain subgroups of the Mathieu group M_{23}. In the context of physics, Eguchi, Ooguri and Tachikawa [EOT11] found Mathieu moonshine phenomena for the elliptic genera of K3 surfaces. This is the mysterious relation between the elliptic genera of K3 surfaces and the Mathieu group M_{24}. After [EOT11], Gaberdiel, Hohenegger and Volpato [GHV12] studied symmetries of K3 sigma models. From the mathematical point of view, they compared symmetries of the Mukai lattice with the symmetries of the Leech lattice N. For a K3 surface S, let $\mathrm{Stab}^*(S)$ be the distinguished connected component of the space of stability conditions on S as in [Bri08]. Huybrechts [Huy16b] interpreted the symmetries of the Mukai lattice in [GHV12] as the subgroup $\mathrm{Aut}_{\mathrm{s}}\left(D^b(S),\sigma\right)$ of the autoequivalence group of the derived category $D^b(S)$ of a K3 surface S, where σ is a stability condition in $\mathrm{Stab}^*(S)$ (see Definition 4.2). Huybrechts [Huy16b] proved the analogue of Mukai's theorem in [Muk88].

Theorem 1.1 ([Huy16b, Theorem 0.1]). *Let G be a group. The following are equivalent:*

(1) *There exist a K3 surface S and a stability condition $\sigma \in \mathrm{Stab}^*(S)$ such that G can be embedded into $\mathrm{Aut}_{\mathrm{s}}\left(D^b(S),\sigma\right)$.*

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(2) *The group G can be embedded into the Conway group $\mathrm{Co}_0 = \mathrm{O}(N)$ in such a way that* $\mathrm{rk}\left(N^G\right) \geqslant 4$.

1.2 Results

Laza and Cheng [LZ19] classified automorphism groups of cubic fourfolds. They obtained the analogue of Theorem 1.1.

Theorem 1.2 ([LZ19, Theorem 4.5]). *Let G be a group. The following are equivalent:*

(1) *There is a cubic fourfold X such that G can be embedded into the symplectic automorphism group $\mathrm{Aut}_{\mathrm{s}}(X)$ of X.*

(2) *The group G can be embedded into the Conway group Co_0 in such a way that there exists a primitive embedding of E_6 into $L^G \oplus U^2$, where U is the hyperbolic plane lattice.*

In this paper, we interpret symplectic automorphism groups of cubic fourfolds as subgroups of autoequivalence groups of K3 categories associated with cubic fourfolds. Then we compare symplectic automorphism groups of cubic fourfolds with groups of the form $\mathrm{Aut}_{\mathrm{s}}\left(D^b(S), \sigma\right)$ in Theorem 1.1. The key ingredient is K3 categories associated with cubic fourfolds [Kuz10]. Let X be a cubic fourfold. Then there is a semi-orthogonal decomposition

$$D^b(X) = \langle \mathcal{D}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle .$$

The admissible subcategory $\mathcal{D}_X$ is called the Kuznetsov component of X. The Serre functor of the Kuznetsov component $\mathcal{D}_X$ of X is isomorphic to the double shift functor $[2]\colon \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}_X$. Bayer, Lahoz, Macrì and Stellari [BLMS17] constructed stability conditions on Kuznetsov components of cubic fourfolds. Denote the stability condition on $\mathcal{D}_X$ constructed in [BLMS17] by σ (see Theorem 3.8). As in [Huy16b], we consider the group $\mathrm{Aut}(\mathcal{D}_X, \sigma)$ (respectively, $\mathrm{Aut}_{\mathrm{s}}(\mathcal{D}_X, \sigma)$) of autoequivalences (respectively, symplectic autoequivalences) of Fourier–Mukai type on $\mathcal{D}_X$ (Definitions 3.9 and 5.3). The first result in our paper is as follows.

Theorem 1.3. *The group homomorphism*

$$\mathrm{Aut}(X) \to \mathrm{Aut}(\mathcal{D}_X) , \quad f \mapsto f_*$$

induces the isomorphisms of groups

$$\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{D}_X, \sigma) , \quad \mathrm{Aut}_{\mathrm{s}}(X) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{s}}(\mathcal{D}_X, \sigma) ,$$

where $\mathrm{Aut}_{\mathrm{s}}(X)$ is the symplectic automorphism group of X.

The second result of our paper is the comparison of automorphisms of cubic fourfolds with automorphisms of K3 surfaces. Relations between cubic fourfolds and K3 surfaces have been studied via Hodge theory [Has00] and derived categories [Kuz10]. From the point of view of Hodge theory, labelled cubic fourfolds are related to polarized K3 surfaces (Definition 2.4). For a labelled cubic fourfold (X, K) and an associated K3 surface (S, h), we will construct the isomorphism between the labelled automorphism group $\mathrm{Aut}(X, K)$ of the cubic fourfold X and the polarized automorphism group $\mathrm{Aut}(S, h)$ of the K3 surface S via derived categories (Definitions 7.1 and 4.5). Hassett [Has00] introduced two arithmetic conditions on an integer d as follows:

($*$) $d > 6$ and $d \equiv 0$ or $2 \pmod 6$,

($**$) d is not divisible by 4, 9 or any odd prime $p \equiv 2 \pmod 3$.

Hassett [Has00] proved that for a labelled cubic fourfold (X, K) of discriminant d, the integer d satisfies $(*)$ and $(**)$ if and only if there is a polarized K3 surface (S, h) of degree d such that a Hodge isometry $K^{\perp}(-1) \simeq H^2_{\mathrm{prim}}(S, \mathbb{Z})$ exists (Theorem 2.6). We introduce certain lattices associated with labelled cubic fourfolds and polarized K3 surfaces following [AT14]. For a labelled cubic fourfold (X, K), there is the rank 3 primitive sublattice L_K of the Mukai lattice $H^*(\mathcal{D}_X, \mathbb{Z})$ of the Kuznetsov component $\mathcal{D}_X$ such that the orthogonal complement $L_K^{\perp}$ in $H^*(\mathcal{D}_X, \mathbb{Z})$ is Hodge isometric to $K^{\perp}(-1)$ (see Remark 2.5). For a polarized K3 surface (S, h), there is the rank 3 primitive sublattice L_h of the Mukai lattice $H^*(S, \mathbb{Z})$ of S such that the orthogonal complement $L_h^{\perp}$ in $H^*(S, \mathbb{Z})$ is Hodge isometric to the primitive cohomology $H^2_{\mathrm{prim}}(S, \mathbb{Z})$ of S (see Section 7). The following is the second result of our paper.

Theorem 1.4. *Let d be an integer satisfying $(*)$ and $(**)$. For a labelled cubic fourfold (X, K) of discriminant d, there is a polarized K3 surface (S, h) of degree d such that the following hold:*

(1) *There exists an object $\mathcal{E} \in D^b(S \times X)$ such that the Fourier–Mukai functor $\Phi_{\mathcal{E}} \colon D^b(S) \to \mathcal{D}_X$ associated with the Fourier–Mukai kernel $\mathcal{E}$ is an equivalence. The cohomological Fourier–Mukai transform $\Phi^H_{\mathcal{E}} \colon H^*(S, \mathbb{Z}) \xrightarrow{\sim} H^*(\mathcal{D}_X, \mathbb{Z})$ induces the isometry*

$$\Phi^H_{\mathcal{E}}|_{L_h} \colon L_h \xrightarrow{\sim} L_K$$

and the Hodge isometry

$$\Phi^H_{\mathcal{E}}|_{L_h^{\perp}} \colon L_h^{\perp} \xrightarrow{\sim} L_K^{\perp} .$$

(2) *There is a stability condition $\sigma_X \in \mathrm{Stab}^*(S)$ such that the group homomorphism*

$$(-)_{\mathcal{E}} \colon \mathrm{Aut}(X) \to \mathrm{Aut}\left(D^b(S), \sigma_X\right), \quad f \mapsto f_{\mathcal{E}} := \Phi_{\mathcal{E}}^{-1} \circ f_* \circ \Phi_{\mathcal{E}}$$

is an isomorphism of groups. Moreover, the restriction of $(-)_{\mathcal{E}}$ induces the isomorphisms of groups

$$(-)_{\mathcal{E}} \colon \mathrm{Aut}(X, K) \xrightarrow{\sim} \mathrm{Aut}(S, h) ,$$
$$(-)_{\mathcal{E}} \colon \mathrm{Aut}_{\mathrm{s}}(X, K) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{s}}(S, h) .$$

In particular, for any automorphism $f \in \mathrm{Aut}(X, K)$, there is an unique isomorphism $f_{\mathcal{E}} \in \mathrm{Aut}(S, h)$ such that the following diagram commutes:

$$\begin{array}{ccc} D^b(S) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathcal{D}_X \\ {\scriptstyle f_{\mathcal{E}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ D^b(S) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathcal{D}_X . \end{array}$$

The key of the proof of Theorem 1.4 is the construction of a polarized K3 surface (S, h) in terms of moduli spaces of stable objects in the Kuznetsov component $\mathcal{D}_X$ as in [BLM+19]. The stability condition σ_X in Theorem 1.4 is induced by the stability condition on $\mathcal{D}_X$ constructed in [BLMS17]. The first group isomorphism in Theorem 1.4(2) is deduced from Theorem 1.3. Labelled automorphisms of a labelled cubic fourfold (X, K) induce autoequivalences of the Kuznetsov component $\mathcal{D}_X$. Such autoequivalences induce automorphisms of the moduli space S of stable objects in $\mathcal{D}_X$ and fix the polarization.

The third result in our paper is about the relation between the existence of non-trivial symplectic automorphisms of cubic fourfolds and the existence of associated K3 surfaces.

Theorem 1.5. *Let X be a cubic fourfold. If the symplectic automorphism group $\mathrm{Aut}_{\mathrm{s}}(X)$ of X is not isomorphic to the trivial group 1 or the cyclic group $\mathbb{Z}/2\mathbb{Z}$ of order 2, then there is a K3 surface S such that $\mathcal{D}_X \simeq D^b(S)$.*

See also [Laz18, Proposition 2.5, Corollary 2.9]. The proof relies on the classification of symplectic automorphism groups of cubic fourfolds in [LZ19] and the lattice-theoretic technique in [Nik79] and [Mor84]. Using Theorem 1.4, we can find examples of finite symplectic autoequivalences of K3 surfaces which are not conjugate to symplectic automorphisms of K3 surfaces. For example, there is the K3 surface and the symplectic autoequivalence of order 9 from the symplectic automorphism of order 9 on the Fermat cubic fourfold (Example 8.4). From the Klein cubic fourfold, we can construct the K3 surface and the symplectic autoequivalence of order 11 (Example 8.5). Typical examples of autoequivalences of derived categories of K3 surfaces are shifts, tensoring line bundles, automorphisms and spherical twists. Since shifts, tensoring line bundles and spherical twists have infinite order, it is not easy to give non-trivial examples of finite symplectic autoequivalences in terms of K3 surfaces. We can construct other examples using [LZ19, Theorems 1.2 and 1.8].

1.3 Notation

We work over the complex numbers field $\mathbb{C}$. For a smooth projective variety X and an object $E \in D^b(X)$, we define the Mukai vector $v_X(E)$ of E by $v_X(E) := \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(X)}$. For smooth projective varieties X and Y and an object $\mathcal{E} \in D^b(X \times Y)$, the Fourier–Mukai functor $\Phi_{\mathcal{E}} \colon D^b(X) \to D^b(Y)$ is defined by $\Phi_{\mathcal{E}}(E) := \mathbf{R}p_*(q^*E \otimes^{\mathbf{L}} \mathcal{E})$, where $p \colon X \times Y \to Y$ and $q \colon X \times Y \to X$ are projections and $E \in D^b(X)$. Then the cohomological Fourier–Mukai transform $\Phi^H_{\mathcal{E}} \colon H^{2*}(X, \mathbb{Q}) \to H^{2*}(Y, \mathbb{Q})$ is the linear map defined by $\Phi^H_{\mathcal{E}}(\alpha) := p_*(q^*\alpha \cdot v_{X\times Y}(\mathcal{E}))$. For a triangulated category $\mathcal{D}$ and an exceptional object $E \in \mathcal{D}$, the left mutation functor $\mathbf{L}_E \colon \mathcal{D} \to E^{\perp}$ and the right mutation functor $\mathbf{R}_E \colon \mathcal{D} \to {}^{\perp}E$ with respect to E fit into the exact triangles

$$\mathbf{R}\,\mathrm{Hom}(E,F) \otimes E \to F \to \mathbf{L}_E(F)\,, \quad \mathbf{R}_E(F) \to F \to \mathbf{R}\,\mathrm{Hom}(F,E)^* \otimes E$$

for $F \in \mathcal{D}$. Denote the hyperbolic plane lattice by U.

2. Cubic fourfolds and K3 surfaces

In this section, we review relations between cubic fourfolds and K3 surfaces via derived categories and Hodge theory.

2.1 Mukai lattice of K3 surfaces

Let S be a K3 surface. The Mukai lattice $(H^*(S,\mathbb{Z}), (-,-))$ of S is isometric to the even unimodular lattice $U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$ of signature $(4,20)$. Moreover, the Mukai lattice $H^*(S,\mathbb{Z})$ of S has a weight 2 Hodge structure $\widetilde{H}(S)$ given by

$$\widetilde{H}^{2,0}(S) := H^{2,0}(S)\,, \quad \widetilde{H}^{1,1}(S) := \bigoplus_{p=0}^{2} H^{p,p}(S)\,, \quad \widetilde{H}^{0,2}(S) := H^{0,2}(S)\,.$$

The integral part $\widetilde{H}^{1,1}(S,\mathbb{Z}) := \widetilde{H}^{1,1}(S) \cap H^*(S,\mathbb{Z})$ is called the algebraic Mukai lattice of S. Then the transcendental lattice T_S of S is the orthogonal complement of $\widetilde{H}^{1,1}(S,\mathbb{Z})$. Taking Mukai vectors, we have the homomorphism $v := v_S \colon K_0(S) \to \widetilde{H}^{1,1}(S,\mathbb{Z})$.

2.2 Mukai lattices for cubic fourfolds

In this subsection, we introduce Mukai lattices for cubic fourfolds following Addington and Thomas [AT14].

Let X be a cubic fourfold. We denote the topological K-group of X by $K_{\text{top}}(X)$. For an class $\alpha \in K_{\text{top}}(X)$, we define the Mukai vector $v_X(\alpha)$ of α as $v_X(\alpha) := \text{ch}(\alpha)\sqrt{\text{td}(X)} \in H^*(X, \mathbb{Q})$. For elements $\alpha, \beta \in K_{\text{top}}(X)$, we have the topological Euler characteristic $\chi_{\text{top}}(\alpha, \beta) \in \mathbb{Z}$. Addington and Thomas introduced the Mukai lattice of $\mathcal{D}_X$.

Definition 2.1 ([AT14, Definition 2.2]). We define the cohomology group of $\mathcal{D}_X$ as

$$H^*(\mathcal{D}_X, \mathbb{Z}) := \left\{\alpha \in K_{\text{top}}(X) \mid \chi_{\text{top}}([\mathcal{O}_X(k)], \alpha) = 0 \text{ for } k = 0, 1, 2\right\}.$$

Let $(-,-)$ be the restriction of $-\chi_{\text{top}}(-,-)$ to $H^*(\mathcal{D}_X, \mathbb{Z})$. The pair $(H^*(\mathcal{D}_X, \mathbb{Z}), (-,-))$ is a lattice isometric to the even unimodular lattice $U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$ of signature $(4, 20)$. The lattice $(H^*(\mathcal{D}_X, \mathbb{Z}), (-,-))$ is called the Mukai lattice of $\mathcal{D}_X$. Moreover, $H^*(\mathcal{D}_X, \mathbb{Z})$ has a weight 2 Hodge structure $\widetilde{H}(\mathcal{D}_X)$ given by

$$\widetilde{H}^{2,0}(\mathcal{D}_X) := v_X^{-1}\left(H^{3,1}(X)\right), \quad \widetilde{H}^{1,1}(\mathcal{D}_X) := v_X^{-1}\left(\bigoplus_{p=0}^{4} H^{p,p}(X)\right), \quad \widetilde{H}^{0,2}(\mathcal{D}_X) := v_X^{-1}\left(H^{1,3}(X)\right).$$

The integral part $\widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z}) := \widetilde{H}^{1,1}(\mathcal{D}_X) \cap H^*(\mathcal{D}_X, \mathbb{Z})$ is called the algebraic Mukai lattice of $\mathcal{D}_X$. The transcendental lattice $T_{\mathcal{D}_X}$ of $\mathcal{D}_X$ is defined by the orthogonal complement $T_{\mathcal{D}_X} := \widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})^{\perp}$ in $H^*(\mathcal{D}_X, \mathbb{Z})$.

We consider Mukai vectors for objects in the Kuznetsov component.

Remark 2.2. We denote the natural map $K_0(\mathcal{D}_X) \to H^*(\mathcal{D}_X, \mathbb{Z})$ by v. By [AT14, Proposition 2.4], the image of v is equal to the algebraic Mukai lattice $\widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$. For an object $E \in \mathcal{D}_X$, we define the Mukai vector $v(E)$ of E as $v(E) := v([E])$. The map $v \colon K_0(\mathcal{D}_X) \to \widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$ is different from the map $v_X \colon K_{\text{top}}(X) \to H^*(X, \mathbb{Q})$.

Let $i \colon \mathcal{D}_X \to D^b(X)$ be the inclusion functor and $i^* \colon D^b(X) \to \mathcal{D}_X$ the left adjoint functor of i. For an integer $k \in \mathbb{Z}$, we define an element $\lambda_k := [i^*\mathcal{O}_{\text{line}}(k)] \in \widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$. The Gram matrix of the sublattice $\langle \lambda_1, \lambda_2 \rangle$ of $\widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$ is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Denote the sublattice $\langle \lambda_1, \lambda_2 \rangle$ of $\widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$ by A_2. The sublattice A_2 is related to the primitive cohomology lattice $H^4_{\text{prim}}(X, \mathbb{Z})$ of X.

Proposition 2.3 ([AT14, Proposition 2.3]). *Let $A_2^{\perp}$ be the orthogonal complement of A_2 in $H^*(\mathcal{D}_X, \mathbb{Z})$. Then we have the Hodge isometry $v_X \colon A_2^{\perp} \xrightarrow{\sim} H^4_{\text{prim}}(X, \mathbb{Z})(-1)$.*

By Proposition 2.3, we have $\text{rk}\, \widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z}) = \text{rk}\, H^{2,2}(X, \mathbb{Z}) + 1$.

2.3 Special cubic fourfolds

In this subsection, we recall the notion of special cubic fourfolds following Hassett [Has00].

Definition 2.4 ([Has00, Definition 3.1.3]). For a positive integer d, a labelled cubic fourfold (X, K) of discriminant d is a pair of a cubic fourfold X and a rank 2 primitive sublattice $K \subset$

$H^{2,2}(X,\mathbb{Z})$ such that K contains H^2 and $\operatorname{disc}(K) = d$. A cubic fourfold X is a special cubic fourfold of discriminant d if there is a rank 2 primitive sublattice $K \subset H^{2,2}(X,\mathbb{Z})$ such that (X,K) is a labelled cubic fourfold of discriminant d.

We interpret labelled cubic fourfolds of discriminant d via Mukai lattices.

Remark 2.5 ([AT14, Section 2.4]). Let (X,K) be a labelled cubic fourfold of discriminant d. We define the rank 3 primitive sublattice L_K of the algebraic Mukai lattice $\widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z})$ such that $\operatorname{disc} L_K = d$ as follow. There is a class $T \in H^{2,2}_{\text{prim}}(X,\mathbb{Z})$ such that $K \cap H^{2,2}_{\text{prim}}(X,\mathbb{Z}) = \mathbb{Z}\cdot T$. Using Proposition 2.3, we define the class $\kappa_T \in A_2^{\perp}$ by $\kappa_T := v_X^{-1}(T)$. Let $L_K \subset \widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z})$ be the saturation of the sublattice generated by A_2 and κ_T. By Proposition 2.3 and the definition of L_K, we have the Hodge isometry $v_X \colon L_K^{\perp} \xrightarrow{\sim} K^{\perp}(-1)$, where we take orthogonal complements in $H^*(\mathcal{D}_X,\mathbb{Z})$ and $H^4(X,\mathbb{Z})(-1)$, respectively. Since K and L_K are primitive sublattices of the unimodular lattices $H^4(X,\mathbb{Z})$ and $H^*(\mathcal{D}_X,\mathbb{Z})$, respectively, we have

$$d = \operatorname{disc}(K) = \operatorname{disc}\bigl(K(-1)\bigr) = \operatorname{disc}\bigl(K^{\perp}(-1)\bigr) = \operatorname{disc}(L_K)\,.$$

The moduli space $\mathcal{C}$ of cubic fourfolds is a 20-dimensional quasi-projective variety. For a positive integer d, denote the subset of special cubic fourfolds of discriminant d by $\mathcal{C}_d$. See Subsection 1.2 to recall the arithmetic conditions $(*)$ and $(**)$ on d. An integer d satisfies the condition $(*)$ if and only if $\mathcal{C}_d \neq \emptyset$ (see [Has00, Theorem 4.3.1]). For labelled cubic fourfolds and polarized K3 surfaces, Hassett [Has00] proved the following theorem.

Theorem 2.6 ([Has00, Theorem 1.0.2]). *Assume that an integer satisfies $(*)$. Let (X,K) be a labelled cubic fourfold of discriminant d. The integer d satisfies $(**)$ if and only if there is a polarized* K3 *surface (S,h) of degree d such that we have a Hodge isometry $K^{\perp}(-1) \simeq h^{\perp}$. Here, we take orthogonal complements of K and h in $H^4(X,\mathbb{Z})$ and $H^2(S,\mathbb{Z})$, respectively.*

In the context of derived categories, the following is known.

Theorem 2.7 ([AT14, Theorem 3.1], [BLM+19, Corollary 29.7]). *Let X be a cubic fourfold. The following are equivalent:*

(1) *There is a K3 surface S such that $\mathcal{D}_X \simeq D^b(S)$.*
(2) *There is an integer d satisfying $(*)$ and $(**)$ such that $X \in \mathcal{C}_d$.*
(3) *There is an embedding of the hyperbolic lattice U into the algebraic Mukai lattice $\widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z})$ of $\mathcal{D}_X$.*

3. Stability conditions

In this section, we review facts about stability conditions on Kuznetsov components of cubic fourfolds and derived categories of K3 surfaces.

3.1 Weak stability conditions

In this subsection, we introduce the notion of (weak) stability conditions following [Bri07] and [BLMS17].

Let $\mathcal{D}$ be a triangulated category over $\mathbb{C}$. The definition of weak stability conditions and stability conditions on $\mathcal{D}$ is the following.

Definition 3.1. Fix a finitely generated free abelian group Λ and a surjective group homomorphism $\mathrm{cl}\colon K_0(\mathcal{D}) \to \Lambda$. A *weak stability condition on* $\mathcal{D}$ (*with respect to* Λ) is a pair $\sigma = (Z, \mathcal{A})$ of a group homomorphism $Z\colon \Lambda \to \mathbb{C}$ (called a *central charge*) and the *heart* of a bounded t-structure $\mathcal{A}$ in $\mathcal{D}$ such that the following three properties hold:

(1) For any object $E \in \mathcal{A}$, we have $\operatorname{Im} Z(\mathrm{cl}(E)) \geqslant 0$, and if $\operatorname{Im} Z(\mathrm{cl}(E)) = 0$, then $Z(\mathrm{cl}(E)) \in \mathbb{R}_{\leqslant 0}$ holds. For simplicity, we will denote $Z(\mathrm{cl}(E))$ by $Z(E)$ for an object $E \in \mathcal{D}$.

We prepare terminology to state properties (2) and (3). For $E \in \mathcal{A}$ with $\operatorname{Im} Z(E) > 0$, we define the slope $\mu_\sigma(E)$ of E with respect to σ as

$$\mu_\sigma(E) := -\frac{\operatorname{Re} Z(E)}{\operatorname{Im} Z(E)}\,.$$

For $E \in \mathcal{A}$ with $\operatorname{Im} Z(E) = 0$, we put $\mu_\sigma(E) := \infty$. A non-zero object $E \in \mathcal{A}$ is σ-*semistable* if for any subobject F of E in $\mathcal{A}$, we have $\mu_\sigma(F) \leqslant \mu_\sigma(E)$.

(2) For any $E \in \mathcal{A}$, there exists a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

in $\mathcal{A}$ such that for any $1 \leqslant k \leqslant n$, the quotient $F_k := E_k/E_{k-1}$ is σ-semistable with

$$\mu_\sigma(F_1) > \mu_\sigma(F_2) > \cdots > \mu_\sigma(F_n)\,.$$

This filtration is called a *Harder–Narasimhan filtration* of E with respect to σ.

(3) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\operatorname{Ker} Z}$ is negative definite and $Q(\mathrm{cl}(E)) \geqslant 0$ for any σ-semistable object $E \in \mathcal{A}$. This property is called the *support property*.

A *weak stability condition* $\sigma = (Z, \mathcal{A})$ on $\mathcal{D}$ is a stability condition on $\mathcal{D}$ if $Z(E) \neq 0$ holds for any non-zero object $E \in \mathcal{A}$.

Bridgeland [Bri07] proved that the set $\mathrm{Stab}(\mathcal{D})$ of stability conditions on $\mathcal{D}$ with respect to Λ has the structure of a complex manifold.

Remark 3.2 ([Bri07, Lemma 5.2, Proposition 5.3]). Let $\sigma = (Z, \mathcal{A})$ be a stability condition on $\mathcal{D}$ with respect to Λ. For an object $E \in \mathcal{A} \setminus \{0\}$, we define the phase $\phi_\sigma(E)$ of E with respect to σ by

$$\phi_\sigma(E) := \frac{1}{\pi} \arg Z(E) \in (0, 1]\,.$$

For a real number $\phi \in (0, 1]$, we define the full subcategory $\mathcal{P}_\sigma(\phi)$ in $\mathcal{A}$ by

$$\mathcal{P}_\sigma(\phi) := \{E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \phi_\sigma(E) = \phi\} \cup \{0\}\,.$$

For a real number $\phi \in (0, 1]$, the full subcategory $\mathcal{P}_\sigma(\phi)$ is an abelian subcategory of $\mathcal{A}$.

We use weak stability conditions to construct new hearts of bounded t-structures. By the existence of Harder–Narasimhan filtrations, we have the following torsion pairs, and they produce new hearts of bounded t-structures.

Definition 3.3. Let $\sigma = (Z, \mathcal{A})$ be a weak stability condition on $\mathcal{D}$. For $\mu \in \mathbb{R}$, we define a torsion pair $(\mathcal{T}_\sigma^\mu, \mathcal{F}_\sigma^\mu)$ on $\mathcal{A}$ as

$$\mathcal{T}_\sigma^\mu := \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) > \mu \rangle\,,$$
$$\mathcal{F}_\sigma^\mu := \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) \leqslant \mu \rangle\,,$$

where $\langle - \rangle$ is the extension closure. We define the heart $\mathcal{A}_\sigma^\mu$ of a bounded t-structure as

$$\mathcal{A}_\sigma^\mu := \left\langle \mathcal{F}_\sigma^\mu[1], \mathcal{T}_\sigma^\mu \right\rangle .$$

We say that $\mathcal{A}_\sigma^\mu$ is obtained by the tilting of $\mathcal{A}$ with respect to the torsion pair $\left(\mathcal{T}_\sigma^\mu, \mathcal{F}_\sigma^\mu\right)$.

3.2 Stability conditions on K3 surfaces

In this subsection, we recall examples of stability conditions on derived categories of K3 surfaces.

Let S be a K3 surface. Using the group homomorphism $v\colon K_0(S) \to \widetilde{H}^{1,1}(S,\mathbb{Z})$, we consider only stability conditions on $D^b(S)$ with respect to $\widetilde{H}^{1,1}(S,\mathbb{Z})$. Take $\mathbb{R}$-divisors $\beta, \omega \in \mathrm{NS}(S)_\mathbb{R}$ such that ω is an ample class. The weak stability condition $\sigma_\omega = (Z_\omega, \mathrm{Coh}(S))$ is given by

$$Z_\omega(E) := i\,\mathrm{ch}_0(E) - \mathrm{ch}_1(E)\cdot\omega$$

for an object $E \in D^b(S)$. It is nothing but the slope stability on $\mathrm{Coh}(S)$. Using Definition 3.3, we define the heart $\mathcal{A}_{\beta,\omega}$ of a bounded t-structure on $D^b(S)$ by $\mathcal{A}_{\beta,\omega} := \mathrm{Coh}(S)_{\sigma_\omega}^{\beta\cdot\omega}$. Let $Z_{\beta,\omega}\colon \widetilde{H}^{1,1}(S,\mathbb{Z}) \to \mathbb{C}$ be the group homomorphism defined by $Z_{\beta,\omega}(w) := \left(e^{\beta+i\omega}, w\right)$, where $w \in \widetilde{H}^{1,1}(S,\mathbb{Z})$. Bridgeland [Bri08] proved the following theorem.

Theorem 3.4 ([Bri08, Lemma 6.2]). *If $Z_{\beta,\omega}(E) \notin \mathbb{R}_{<0}$ holds for any spherical sheaf E on S, then the pair $\sigma_{\beta,\omega} := (Z_{\beta,\omega}, \mathcal{A}_{\beta,\omega})$ is a stability condition on $D^b(S)$. If $\omega^2 > 2$, we have $Z_{\beta,\omega}(E) \notin \mathbb{R}_{<0}$ for any spherical sheaf E on S.*

3.3 Spaces of stability conditions on K3 surfaces

In this subsection, we recall structures of spaces of stability conditions on K3 surfaces.

Let S be a K3 surface. Let $\mathrm{Stab}(S)$ be the space of stability conditions on $D^b(S)$ with respect to $\widetilde{H}^{1,1}(S,\mathbb{Z})$. We define the subset $\mathcal{P}(S)$ of $\widetilde{H}^{1,1}(S,\mathbb{Z})\otimes\mathbb{C}$ as

$$\mathcal{P}(S) := \left\{\Omega \in \widetilde{H}^{1,1}(S,\mathbb{Z})\otimes\mathbb{C} \mid \langle \mathrm{Re}(\Omega), \mathrm{Im}(\Omega)\rangle_\mathbb{R} \text{ is a positive-definite plane}\right\}.$$

Let $\mathcal{P}^+(S)$ be the connected component of $\mathcal{P}(S)$ containing $e^{i\omega}$, where ω is an ample divisor on S. Let Δ_S be the set of (-2)-classes in $\widetilde{H}^{1,1}(S,\mathbb{Z})$. We define the subset $\mathcal{P}_0^+(S)$ of $\mathcal{P}^+(S)$ as

$$\mathcal{P}_0^+(S) := \mathcal{P}^+(S) \setminus \bigcup_{\delta\in\Delta_S} \delta^\perp .$$

We consider the action of the autoequivalence group $\mathrm{Aut}\left(D^b(S)\right)$ on $\mathrm{Stab}(S)$.

Definition 3.5. For an autoequivalence $\Phi \in \mathrm{Aut}\left(D^b(S)\right)$ and a stability condition $\sigma = (Z,\mathcal{A}) \in \mathrm{Stab}(S)$, we define the stability condition $\Phi\sigma$ by

$$\Phi\sigma := \left(Z \circ \left(\Phi^H\right)^{-1}, \Phi(\mathcal{A})\right) \in \mathrm{Stab}(S).$$

Since the Mukai pairing $(-,-)$ on $\widetilde{H}^{1,1}(S,\mathbb{Z})$ is non-degenerate, for a stability condition $\sigma = (Z,\mathcal{A}) \in \mathrm{Stab}(S)$, there is a unique element $\Omega_Z \in \widetilde{H}^{1,1}(S,\mathbb{Z})\otimes\mathbb{C}$ such that $Z(-) = (\Omega_Z, -)$. Let $\mathrm{Stab}^*(S) \subset \mathrm{Stab}(S)$ be the connected component containing stability conditions in Theorem 3.4. Bridgeland [Bri08] proved the following theorem.

Theorem 3.6 ([Bri08, Theorem 1.1]). *For a stability condition $\sigma = (Z,\mathcal{A}) \in \mathrm{Stab}^*(S)$, put $\pi_S(\sigma) := \Omega_Z$. Then π_S induces the covering map $\pi_S\colon \mathrm{Stab}^*(S) \to \mathcal{P}_0^+(S)$. We define*

$$\mathrm{Aut}^0\left(D^b(S)\right) := \left\{\Phi \in \mathrm{Aut}\left(D^b(S)\right) \mid \Phi^H = \mathrm{id},\ \Phi(\mathrm{Stab}^*(S)) \subset \mathrm{Stab}^*(S)\right\}.$$

Then the natural homomorphism $\mathrm{Aut}^0\left(D^b(S)\right) \to \mathrm{Deck}(\pi_S)$ is an isomorphism, where $\mathrm{Deck}(\pi_S)$ is the group of deck transformations of the covering map π_S.

3.4 Clifford algebra associated with a line on a cubic fourfold

In this subsection, we recall the Clifford algebra on the projective space $\mathbb{P}^3$ associated with a line on a cubic fourfold in [Kuz08] and [BLMS17, Section 7].

Let X be a cubic fourfold. Take a line $l \subset X$. Consider the blowing up $p_l \colon \mathrm{Bl}_l X \to X$ of X along the line l. We have the embedding $j_l \colon \mathrm{Bl}_l X \to \mathrm{Bl}_l \mathbb{P}^5$, where $\mathrm{Bl}_l \mathbb{P}^5 \to \mathbb{P}^5$ is the blowing up of $\mathbb{P}^5$ along the line l. The linear projection from l induces the following commutative diagram:

$$\begin{array}{ccccccc} & & \mathrm{Bl}_l X & \xhookrightarrow{j_l} & \mathrm{Bl}_l \mathbb{P}^5 & & \\ & & \downarrow{\scriptstyle p_l} & & \downarrow & \searrow^{q_l} & \\ l & \hookrightarrow & X & \hookrightarrow & \mathbb{P}^5 & \dashrightarrow & \mathbb{P}^3 . \end{array}$$

Set $h := \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(1))$. Note that $q_l \colon \mathrm{Bl}_l \mathbb{P}^5 \to \mathbb{P}^3$ is a $\mathbb{P}^2$-bundle via the isomorphism $\mathrm{Bl}_l \mathbb{P}^5 \simeq \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-h)\big)$. The composition $\pi_l := q_l \circ j_l \colon \mathrm{Bl}_l X \to \mathbb{P}^3$ is a conic fibration. Let $\mathcal{B}_0^l$ (respectively, $\mathcal{B}_1^l$) be the even part (respectively, the odd part) of the sheaf of Clifford algebras on $\mathbb{P}^3$ associated with π_L. For $k \in \mathbb{Z}$, we define the $\mathcal{B}_0^l$-bimodule $\mathcal{B}_k^l$ by $\mathcal{B}_k^l = \mathcal{B}_{k-2}^l \otimes \mathcal{O}_{\mathbb{P}^3}(h)$. Let $\mathrm{Coh}\left(\mathbb{P}^3, \mathcal{B}_0^l\right)$ be the category of coherent right $\mathcal{B}_0^l$-modules, and define $D^b\big(\mathbb{P}^3, \mathcal{B}_0^l\big) := D^b\big(\mathrm{Coh}\left(\mathbb{P}^3, \mathcal{B}_0^l\right)\big)$. Note that $\mathbf{L}p_l^* \colon D^b(X) \to D^b(\mathrm{Bl}_l X)$ is a fully faithful functor. There exists a coherent sheaf $\mathcal{E}_l$ of right $\pi_l^* \mathcal{B}_0^l$-modules on $\mathrm{Bl}_l X$ such that $\Psi_l(-) := \mathbf{R}\pi_{l*}\big(- \otimes \mathcal{O}_{\mathrm{Bl}_l X}(h) \otimes^{\mathbf{L}} \mathcal{E}_l\big)[1]$ is a fully faithful functor from $\mathbf{L}p_l^* \mathcal{D}_X$ to $D^b\big(\mathbb{P}^3, \mathcal{B}_0^l\big)$ and there is a semi-orthogonal decomposition $D^b\big(\mathbb{P}^3, \mathcal{B}_0^l\big) = \big\langle \Psi_l(\mathbf{L}p_l^* \mathcal{D}_X), \mathcal{B}_1^l, \mathcal{B}_2^l, \mathcal{B}_3^l \big\rangle$. (See [BLMS17, Proposition 7.7].)

Remark 3.7. By [Kuz08, Section 4], the coherent right $\mathcal{B}_0^L$-module $\mathcal{E}_L$ fits into the exact sequence

$$0 \to q_l^* \mathcal{B}_{-1}^l(-2H) \to q_l^* \mathcal{B}_0^l(-H) \to j_{l*}\mathcal{E}_l \to 0 .$$

3.5 Stability conditions on Kuznetsov components

In this subsection, we recall the examples of stability conditions on Kuznetsov components of cubic fourfolds in [BLMS17].

Let X be a cubic fourfold. Using the group homomorphism $v \colon K_0(\mathcal{D}_X) \to \widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$ in Remark 2.2, we consider only stability conditions on $\mathcal{D}_X$ with respect to $\widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$. Fix a line l on X. We define the Chern character map $\mathrm{ch}_{\mathcal{B}_0^l} \colon K_0\big(D^b\big(\mathbb{P}^3, \mathcal{B}_0^l\big)\big) \to H^*\big(\mathbb{P}^3, \mathbb{Q}\big)$ as

$$\mathrm{ch}_{\mathcal{B}_0^l}(E) := \mathrm{ch}(\mathrm{forg}(E)) \left(1 - \tfrac{11}{32} h^2\right),$$

where $E \in D^b\big(\mathbb{P}^3, \mathcal{B}_0^l\big)$ and $\mathrm{forg} \colon D^b\big(\mathbb{P}^3, \mathcal{B}_0^l\big) \to D^b\big(\mathbb{P}^3\big)$ is the forgetful functor. For $\beta \in \mathbb{R}$, we define the twisted Chern character map $\mathrm{ch}^\beta_{\mathcal{B}_0^l} \colon K_0\big(D^b\big(\mathbb{P}^3, \mathcal{B}_0^l\big)\big) \to H^*\big(\mathbb{P}^3, \mathbb{R}\big)$ as $\mathrm{ch}^\beta_{\mathcal{B}_0^l} := e^{-\beta h} \mathrm{ch}_{\mathcal{B}_0^l}$. The Chern character map $\mathrm{ch}_{\mathcal{B}_0^l}$ is same as $\mathrm{ch}^0_{\mathcal{B}_0^l}$. Using the isomorphism

$$\mathbb{R}^4 \xrightarrow{\sim} H^*\big(\mathbb{P}^3, \mathbb{R}\big), \quad (x_1, x_2, x_3, x_4) \mapsto \big(x_1, x_2 h, x_3 h^2, x_4 h^3\big),$$

we regard $\mathrm{ch}^\beta_{\mathcal{B}_0^l}(E) = \big(\mathrm{ch}^\beta_{\mathcal{B}_0^l,0}(E), \mathrm{ch}^\beta_{\mathcal{B}_0^l,1}(E), \mathrm{ch}^\beta_{\mathcal{B}_0^l,2}(E), \mathrm{ch}^\beta_{\mathcal{B}_0^l,3}(E)\big) \in \mathbb{R}^4$ for $E \in D^b\big(\mathbb{P}^3, \mathcal{B}_0^l\big)$. For $j = 1, 2$, we define the finitely generated free abelian group $\Lambda^j_{\mathcal{B}_0^l}$ as

$$\Lambda^j_{\mathcal{B}_0^l} := \big\langle \mathrm{ch}_{\mathcal{B}_0^l,k}(E) \mid E \in D^b\big(\mathbb{P}^3, \mathcal{B}_0^l\big),\ 0 \leqslant k \leqslant j \big\rangle_{\mathbb{Z}} \subset \mathbb{R}^4 .$$

Note that $\mathrm{rk}\, \Lambda^j_{\mathcal{B}_0^l} = 1 + j$. We define a weak stability condition $\sigma_{\mathrm{slope}} = \big(Z_{\mathrm{slope}}, \mathrm{Coh}\left(\mathbb{P}^3, \mathcal{B}_0^l\right)\big)$

on $D^b(\mathbb{P}^3, \mathcal{B}_0^l)$ with respect to $\Lambda^j_{\mathcal{B}_0^l}$ as

$$Z_{\mathrm{slope}}(E) := i\,\mathrm{ch}_{\mathcal{B}_0^l,0}(E) - \mathrm{ch}_{\mathcal{B}_0^l,1}(E)$$

for $E \in D^b(\mathbb{P}^3, \mathcal{B}_0^l)$. It is nothing but the slope stability. As in Definition 3.3, for $\beta \in \mathbb{R}$, consider the heart $\mathrm{Coh}^\beta\left(\mathbb{P}^3, \mathcal{B}_0^l\right)$ of a bounded t-structure in $D^b(\mathbb{P}^3, \mathcal{B}_0^l)$ defined by

$$\mathrm{Coh}^\beta\left(\mathbb{P}^3, \mathcal{B}_0^l\right) := \mathrm{Coh}\left(\mathbb{P}^3, \mathcal{B}_0^l\right)^\beta_{\sigma_{\mathrm{slope}}}.$$

For $\alpha > 0$ and $\beta \in \mathbb{R}$, we can define a weak stability condition $\sigma_{\alpha,\beta} = \left(Z_{\alpha,\beta}, \mathrm{Coh}^\beta\left(\mathbb{P}^3, \mathcal{B}_0^l\right)\right)$ with respect to $\Lambda^2_{\mathcal{B}_0^l}$ by

$$Z_{\alpha,\beta}(E) := i\,\mathrm{ch}^\beta_{\mathcal{B}_0^l,1}(E) + \tfrac{1}{2}\alpha^2\,\mathrm{ch}^\beta_{\mathcal{B}_0^l,0}(E) - \mathrm{ch}^\beta_{\mathcal{B}_0^l,2}(E)$$

for $E \in D^b(\mathbb{P}^3, \mathcal{B}_0^l)$. (See [BLMS17, Proposition 9.3].) Consider the heart $\mathrm{Coh}^0_{\alpha,\beta}\left(\mathbb{P}^3, \mathcal{B}_0^l\right)$ of a bounded t-structure on $D^b(\mathbb{P}^3, \mathcal{B}_0^l)$, which is defined by

$$\mathrm{Coh}^0_{\alpha,\beta}\left(\mathbb{P}^3, \mathcal{B}_0^l\right) := \mathrm{Coh}^\beta\left(\mathbb{P}^3, \mathcal{B}_0^l\right)^0_{\sigma_{\alpha,\beta}}.$$

Let $\widetilde{\Psi_l} \colon \mathcal{D}_X \xrightarrow{\sim} \Psi(\mathbf{L}p_l^*\mathcal{D}_X)$ be the equivalence induced by the fully faithful functor $\Psi_l \circ \mathbf{L}p_l^* \colon \mathcal{D}_X \to D^b(\mathbb{P}^3, \mathcal{B}_0^l)$.

Theorem 3.8 ([BLMS17, Theorem 1.2], [LPZ18, Proposition 2.6]). *For $0 < \alpha < 1/4$, we define*

$$\mathcal{A}^l_\alpha := \widetilde{\Psi_l}^{-1}\left(\mathrm{Coh}^0_{\alpha,-1}\left(\mathbb{P}^3, \mathcal{B}_0^l\right) \cap \widetilde{\Psi_l}(\mathcal{D}_X)\right),$$
$$Z^l_\alpha(E) := -iZ_{\alpha,-1}\left(\widetilde{\Psi_l}(E)\right), \quad E \in \mathcal{D}_X.$$

Then $\sigma^l_\alpha = \left(Z^l_\alpha, \mathcal{A}^l_\alpha\right)$ is a stability condition on $\mathcal{D}_X$ with respect to $\widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$. For any lines l and l' on X, we have $\sigma^l_\alpha = \sigma^{l'}_\alpha$. Denote the stability condition σ^l_α on $\mathcal{D}_X$ by $\sigma_\alpha = (Z_\alpha, \mathcal{A}_\alpha)$.

3.6 Spaces of stability conditions on Kuznetsov components

In this subsection, we recall structures of spaces of stability conditions on Kuznetsov components.

Let X be a cubic fourfold. Let $\mathrm{Stab}(\mathcal{D}_X)$ be the space of stability conditions on $\mathcal{D}_X$ with respect to $\widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$. We define the subset $\mathcal{P}(\mathcal{D}_X)$ of $\widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z}) \otimes \mathbb{C}$ as

$$\mathcal{P}(\mathcal{D}_X) := \left\{\Omega \in \widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z}) \otimes \mathbb{C} \mid \langle \mathrm{Re}(\Omega),\ \mathrm{Im}(\Omega)\rangle_{\mathbb{R}} \text{ is a positive-definite plane}\right\}.$$

Since the Mukai pairing $(-,-)$ on $H^*(\mathcal{D}_X, \mathbb{Z})$ is non-degenerate, for a stability condition $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D}_X)$, there is the unique element $\Omega_Z \in \widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z}) \otimes \mathbb{C}$ such that $Z(-) = (\Omega_Z, -)$. Let $\mathcal{P}^+(\mathcal{D}_X)$ be the connected component of $\mathcal{P}(\mathcal{D}_X)$ containing Ω_{Z_α} for $0 < \alpha < 1/4$. Here, σ_α is the stability condition in Theorem 3.8. Let Δ_X be the set of (-2)-classes in $\widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$. We define the subset $\mathcal{P}^+_0(\mathcal{D}_X)$ of $\mathcal{P}^+(\mathcal{D}_X)$ as

$$\mathcal{P}^+_0(\mathcal{D}_X) := \mathcal{P}^+(\mathcal{D}_X) \setminus \bigcup_{\delta \in \Delta_X} \delta^\perp.$$

We will consider only autoequivalences of Fourier–Mukai type.

Definition 3.9. An autoequivalence $\Phi\colon \mathcal{D}_X \to \mathcal{D}_X$ is called of Fourier–Mukai type if there exists an $\mathcal{E} \in D^b(X \times X)$ such that the following diagram commutes:

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi_{\mathcal{E}}} & D^b(X) \\ {\scriptstyle i^*}\downarrow & & \uparrow{\scriptstyle i} \\ \mathcal{D}_X & \xrightarrow{\Phi} & \mathcal{D}_X\,. \end{array}$$

Here, $\Phi_{\mathcal{E}}$ is the Fourier–Mukai functor with the Fourier–Mukai kernel $\mathcal{E}$. We denote the group of autoequivalences of Fourier–Mukai type by $\mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$. For $\Phi \in \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$, the linear map $\Phi^H\colon H^*(\mathcal{D}_X, \mathbb{Z}) \to H^*(\mathcal{D}_X, \mathbb{Z})$ is the Hodge isometry induced by Φ.

The definition of the action of the group $\mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$ on the space $\mathrm{Stab}(\mathcal{D}_X)$ is same as Definition 3.5.

Let $\mathrm{Stab}^*(\mathcal{D}_X) \subset \mathrm{Stab}(\mathcal{D}_X)$ be the connected component containing stability conditions in Theorem 3.8. Then the following holds.

Theorem 3.10 ([BLM+19, Theorem 29.1]). *For a stability condition $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}^*(\mathcal{D}_X)$, put $\pi_X(\sigma) := \Omega_Z$. Then π_X induces the covering map $\pi_X\colon \mathrm{Stab}^*(\mathcal{D}_X) \to \mathcal{P}_0^+(\mathcal{D}_X)$.*

4. Autoequivalences of K3 surfaces and stability conditions

In this section, we introduce certain subgroups of autoequivalence groups of K3 surfaces related to stability conditions following [Huy16b]. We will study polarized automorphisms of K3 surfaces and their relation with stability conditions.

4.1 Subgroups of autoequivalence groups of K3 surfaces

In this subsection, we see the definitions of the groups that we are interested in.

Let S be a K3 surface. First, we recall the notions of the symplectic automorphism group $\mathrm{Aut}_{\mathrm{s}}(S)$ of S and the symplectic autoequivalence group $\mathrm{Aut}_{\mathrm{s}}\left(D^b(S)\right)$.

Definition 4.1. An automorphism $f \in \mathrm{Aut}(S)$ of S is a symplectic automorphism of S if the pullback f^* of f acts trivially on $H^{2,0}(S)$. The group $\mathrm{Aut}_{\mathrm{s}}(S)$ of symplectic automorphisms of S is called the symplectic automorphism group of S. An autoequivalence $\Phi \in \mathrm{Aut}\left(D^b(S)\right)$ is a symplectic autoequivalence of $D^b(S)$ if the cohomological Fourier–Mukai transform Φ^H acts trivially on $\widetilde{H}^{2,0}(S)$. The group $\mathrm{Aut}_{\mathrm{s}}\left(D^b(S)\right)$ of symplectic autoequivalences of $D^b(S)$ is called the symplectic autoequivalence group of $D^b(S)$.

Huybrechts [Huy16b] studied the following subgroups of $D^b(S)$.

Definition 4.2. For a stability condition $\sigma \in \mathrm{Stab}^*(S)$, we define the group $\mathrm{Aut}\left(D^b(S), \sigma\right)$ of autoequivalences fixing σ by

$$\mathrm{Aut}\left(D^b(S), \sigma\right) := \left\{\Phi \in \mathrm{Aut}\left(D^b(S)\right) \mid \Phi\sigma = \sigma\right\}.$$

Denote the intersection of $\mathrm{Aut}\left(D^b(S), \sigma\right)$ and $\mathrm{Aut}_{\mathrm{s}}\left(D^b(S)\right)$ by $\mathrm{Aut}_{\mathrm{s}}\left(D^b(S), \sigma\right)$.

The groups in Definition 4.2 can be described in terms of Mukai latices.

Definition 4.3. For a linear subspace $W \subset H^*(S, \mathbb{R})$, we define

$$\mathrm{O}_{\mathrm{Hodge}}(H^*(S, \mathbb{Z}), W) := \{\varphi \in \mathrm{O}_{\mathrm{Hodge}}(H^*(S, \mathbb{Z})) \mid \varphi|_W = \mathrm{id}_W\}.$$

The weight 2 Hodge structure on $H^*(S,\mathbb{Z})$ defines the positive-definite plane P_S given by $P_S := \big(\widetilde{H}^{2,0}(S) \oplus \widetilde{H}^{0,2}(S)\big) \cap H^*(S,\mathbb{R})$. For a stability condition $\sigma = (Z,\mathcal{A}) \in \mathrm{Stab}^*(S)$, we have the positive-definite plane P_σ defined by $P_\sigma := \langle \mathrm{Re}\big(\pi_S(\sigma)\big), \mathrm{Im}\big(\pi_S(\sigma)\big)\rangle_{\mathbb{R}}$. For a stability condition $\sigma \in \mathrm{Stab}^*(S)$, we put $\Pi_\sigma := P_S \oplus P_\sigma$.

Huybrechts proved the following proposition.

Proposition 4.4 ([Huy16b, Proposition 1.4]). *Let $\sigma \in \mathrm{Stab}^*(S)$ be a stability condition on $D^b(S)$. Then we have the isomorphism*

$$(-)^H\colon\ \mathrm{Aut}\left(D^b(S),\sigma\right) \xrightarrow{\sim} \mathrm{O}_{\mathrm{Hodge}}(H^*(S,\mathbb{Z}),P_\sigma)\,, \quad \Phi \mapsto \Phi^H$$

of the groups $\mathrm{Aut}\left(D^b(S),\sigma\right)$ and $\mathrm{O}_{\mathrm{Hodge}}(H^*(S,\mathbb{Z}),P_\sigma)$. The restriction of this isomorphism to $\mathrm{Aut}_{\mathrm{s}}\left(D^b(S),\sigma\right)$ induces the isomorphism

$$(-)^H\colon\ \mathrm{Aut}_{\mathrm{s}}\left(D^b(S),\sigma\right) \xrightarrow{\sim} \mathrm{O}_{\mathrm{Hodge}}(H^*(S,\mathbb{Z}),\Pi_\sigma)$$

of the groups $\mathrm{Aut}_{\mathrm{s}}\left(D^b(S),\sigma\right)$ and $\mathrm{O}_{\mathrm{Hodge}}(H^*(S,\mathbb{Z}),\Pi_\sigma)$.

4.2 Polarized automorphisms of K3 surfaces

In this subsection, we study stability conditions on K3 surfaces fixed by polarized automorphisms. First, we define polarized automorphisms of K3 surfaces.

Definition 4.5. For a polarized K3 surface (S,h), we define the group $\mathrm{Aut}(S,h)$ of automorphisms of (S,h) by

$$\mathrm{Aut}(S,h) := \{f \in \mathrm{Aut}(S) \mid f^*h = h\}\,.$$

An automorphism $f \in \mathrm{Aut}(S)$ of a K3 surface S is called a polarized automorphism of S if there is a primitive ample divisor $h \in \mathrm{NS}(S)$ such that $f \in \mathrm{Aut}(S,h)$. Denote the intersection of $\mathrm{Aut}(S,h)$ and $\mathrm{Aut}_{\mathrm{s}}(S)$ by $\mathrm{Aut}_{\mathrm{s}}(S,h)$.

Let (S,h) be a polarized K3 surface. Using the polarization h, we consider the following stability conditions on $D^b(S)$.

Definition 4.6. Take real numbers $\alpha,\beta \in \mathbb{R}$ such that $\alpha > 0$ and $e^{\beta h + i\alpha h} \in \mathcal{P}_0^+(S)$. We define the stability condition $\sigma_{\alpha,\beta} = (Z_{\alpha,\beta},\mathcal{A}_{\alpha,\beta})$ by $\sigma_{\alpha,\beta} := \sigma_{\beta h,\alpha h}$ as in Theorem 3.4.

The stability conditions in Definition 4.6 is fixed by polarized automorphisms in $\mathrm{Aut}(S,h)$.

Proposition 4.7. *Take real numbers $\alpha,\beta \in \mathbb{R}$ such that $\alpha > 0$ and $e^{\beta h + i\alpha h} \in \mathcal{P}_0^+(S)$. For any automorphism $f \in \mathrm{Aut}(S,h)$, we have $f_*\sigma_{\alpha,\beta} = \sigma_{\alpha,\beta}$. In particular, we have the natural inclusions*

$$\mathrm{Aut}(S,h) \hookrightarrow \mathrm{Aut}\left(D^b(S),\sigma_{\alpha,\beta}\right)\,,$$
$$\mathrm{Aut}_{\mathrm{s}}(S,h) \hookrightarrow \mathrm{Aut}_{\mathrm{s}}\left(D^b(S),\sigma_{\alpha,\beta}\right)\,.$$

Proof. Let $f \in \mathrm{Aut}(S,h)$ be a polarized automorphism of S. Note that $f_*h = h$. Recall that $Z_{\alpha,\beta}(v) = \left(e^{\beta h + i\alpha h}, v\right)$ for an object $v \in \widetilde{H}^{1,1}(S,\mathbb{Z})$. We have

$$Z_{\alpha,\beta}(f^*(v)) = (e^{\alpha h + i\beta h}, f^*(v)) = (f_*(e^{\alpha h + i\beta h}), v) = (e^{\alpha h + i\beta h}, v) = Z_{\alpha,\beta}(v)\,.$$

Moreover, f_* preserves the torsion pair on the abelian category $\mathrm{Coh}(S)$ in Definition 3.3 and Theorem 3.4. So f_* preserves the heart $\mathcal{A}_{\alpha,\beta}$ of the bounded t-structure on $D^b(S)$. □

5. Automorphisms of cubic fourfolds and stability conditions

Section 5 is almost parallel to Section 4. In this section, we introduce certain subgroups of autoequivalence groups of Kuznetsov components of cubic fourfolds related to stability conditions. We will study automorphisms of cubic fourfolds and their relation with stability conditions.

5.1 Subgroups of autoequivalence groups of Kuznetsov components

In this subsection, we introduce subgroups of autoequivalence groups of Kuznetsov components as in Subsection 4.1.

Let X be a cubic fourfold. First, we compare the automorphism group $\mathrm{Aut}(X)$ with the autoequivalence group $\mathrm{Aut}(\mathcal{D}_X)$ of $\mathcal{D}_X$. For an automorphism $f \in \mathrm{Aut}(X)$, we have the autoequivalence $f_* \in \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$ since $f_*\mathcal{O}_X(k) \simeq \mathcal{O}_X(k)$ for any integer $k \in \mathbb{Z}$. So we obtain the group homomorphism $\rho_1\colon \mathrm{Aut}(X) \to \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$. Note that the admissible subcategory $\mathcal{D}_X$ fits into the semi-orthogonal decomposition

$$D^b(X) = \langle \mathcal{O}_X(-1), \mathcal{D}_X, \mathcal{O}_X, \mathcal{O}_X(1)\rangle$$

by the Serre duality on X. We define the projection functor $\mathrm{pr}\colon D^b(X) \to \mathcal{D}_X$ by $\mathrm{pr} := \mathbf{R}_{\mathcal{O}_X(-1)}\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{O}_X(1)}$.

Proposition 5.1. *The homomorphism ρ_1 is injective.*

Proof. Since $f_*\mathcal{O}_X(k) \simeq \mathcal{O}_X(k)$ for $k \in \mathbb{Z}$, for an automorphism $f \in \mathrm{Aut}(X)$, we have $f_* \circ \mathrm{pr} \simeq \mathrm{pr} \circ f_*$. So we obtain $f_* \circ \mathrm{pr}(\mathcal{O}_x) \simeq \mathrm{pr}(\mathcal{O}_{f(x)})$ for any point $x \in X$. By [Ouc17, Proposition 1.4], the homomorphism ρ_1 is injective. □

We define the symplectic automorphism group of X and the symplectic autoequivalence group of $\mathcal{D}_X$.

Definition 5.2. An automorphism $f \in \mathrm{Aut}(X)$ is a symplectic automorphism of X if f^* acts trivially on $H^{3,1}(X)$. The group $\mathrm{Aut}_{\mathrm{s}}(X)$ of symplectic automorphisms of X is called the symplectic automorphism group of X. An autoequivalence $\Phi \in \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$ is a symplectic autoequivalence if Φ^H acts trivially on $\widetilde{H}^{2,0}(\mathcal{D}_X)$. The group $\mathrm{Aut}_{\mathrm{s}}^{\mathrm{FM}}(\mathcal{D}_X)$ of symplectic autoequivalences of $\mathcal{D}_X$ is called the symplectic autoequivalence group of $\mathcal{D}_X$.

Let $f \in \mathrm{Aut}(X)$ be an automorphism of X. By Definition 2.1, we have $f \in \mathrm{Aut}_{\mathrm{s}}(X)$ if and only if $f_* \in \mathrm{Aut}_{\mathrm{s}}^{\mathrm{FM}}(\mathcal{D}_X)$. As in Definition 4.2, we introduce the following groups.

Definition 5.3. For a stability condition $\sigma \in \mathrm{Stab}^*(\mathcal{D}_X)$, we define the group $\mathrm{Aut}(\mathcal{D}_X, \sigma)$ of autoequivalences fixing σ by

$$\mathrm{Aut}(\mathcal{D}_X, \sigma) := \left\{\Phi \in \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X) \mid \Phi\sigma = \sigma\right\}.$$

Denote the intersection of $\mathrm{Aut}(\mathcal{D}_X, \sigma)$ and $\mathrm{Aut}_{\mathrm{s}}^{\mathrm{FM}}(\mathcal{D}_X)$ by $\mathrm{Aut}_{\mathrm{s}}(\mathcal{D}_X, \sigma)$.

As in Definition 4.3, we define the following groups in terms of Hodge theory.

Definition 5.4. For a linear subspace $W \subset H^*(\mathcal{D}_X, \mathbb{R})$, we define

$$\mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z}), W) := \{\varphi \in \mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z})) \mid \varphi|_W = \mathrm{id}_W\}.$$

The weight 2 Hodge structure on $H^*(\mathcal{D}_X, \mathbb{Z})$ defines the positive-definite plane P_X given by $P_X := \big(\widetilde{H}^{2,0}(\mathcal{D}_X) \oplus \widetilde{H}^{0,2}(\mathcal{D}_X)\big) \cap H^*(\mathcal{D}_X, \mathbb{R})$. For a stability condition $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}^*(\mathcal{D}_X)$, we have the positive-definite plane P_σ defined by $P_\sigma := \langle \mathrm{Re}\big(\pi_X(\sigma)\big), \mathrm{Im}\big(\pi_X(\sigma)\big)\rangle_{\mathbb{R}}$. For a stability condition $\sigma \in \mathrm{Stab}^*(\mathcal{D}_X)$, we put $\Pi_\sigma := P_X \oplus P_\sigma$.

For the stability condition in Theorem 3.8, the corresponding positive-definite planes come from A_2.

PROPOSITION 5.5 ([BLMS17, Proposition 9.11]). *For a real number $0 < \alpha < 1/4$, we have $P_{\sigma_\alpha} = A_2 \otimes \mathbb{R}$.*

5.2 Automorphisms of cubic fourfolds

In this subsection, we prove that automorphisms of cubic fourfolds fix the stability conditions in Theorem 3.8. The goal of this subsection is the following proposition.

PROPOSITION 5.6. *Fix a real number $0 < \alpha < 1/4$. For an automorphism $f \in \mathrm{Aut}(X)$ of X, we have $f_*\sigma_\alpha = \sigma_\alpha$. In particular, we have the natural inclusions*

$$\mathrm{Aut}(X) \hookrightarrow \mathrm{Aut}(\mathcal{D}_X, \sigma_\alpha)\,, \quad \mathrm{Aut}_{\mathrm{s}}(X) \hookrightarrow \mathrm{Aut}_{\mathrm{s}}(\mathcal{D}_X, \sigma_\alpha)\,.$$

In Section 7, we will see the above inclusions are isomorphisms. To prove Proposition 5.6, we study relations between automorphisms of cubic fourfolds and sheaves of Clifford algebras. Fix a line l on X. An automorphism $f \in \mathrm{Aut}(X)$ of X induces $f_l \colon \mathrm{Bl}_l X \xrightarrow{\sim} \mathrm{Bl}_{f(l)} X$ and $\tilde{f}_l \in \mathrm{Aut}\left(\mathbb{P}^3\right)$ such that we have the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{Bl}_l X & \xrightarrow{f_l} & \mathrm{Bl}_{f(l)} X \\ {\scriptstyle p_l}\downarrow & & \downarrow{\scriptstyle p_{f(l)}} \\ X & \xrightarrow{f} & X\,, \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Bl}_l X & \xrightarrow{f_l} & \mathrm{Bl}_{f(l)} X \\ {\scriptstyle \pi_l}\downarrow & & \downarrow{\scriptstyle \pi_{f(l)}} \\ \mathbb{P}^3 & \xrightarrow{\tilde{f}_l} & \mathbb{P}^3\,. \end{array}$$

The following is the relation between automorphisms of X and the sheaves of Clifford algebras on $\mathbb{P}^3$.

LEMMA 5.7 ([Kuz08, Lemma 3.2], [BLMS17, Lemma 7.2]). *Let $k \in \mathbb{Z}$ be an integer. For an automorphism $f \in \mathrm{Aut}(X)$ of X, we have $\tilde{f}_{l*}\mathcal{B}^l_k \simeq \mathcal{B}^{f(l)}_k$.*

Proof. In the proof of [Kuz08, Lemma 3.2], this is observed in a more general situation. (Cf. Lemma 7.2 of [BLMS17].) □

The fully faithful functor in Subsection 3.4 is compatible with automorphisms of X.

PROPOSITION 5.8. *Let $f \in \mathrm{Aut}(X)$ be an automorphism of X. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{D}_X & \xrightarrow{f_*} & \mathcal{D}_X \\ {\scriptstyle \Psi_l \circ \mathbf{L}p_l^*}\downarrow & & \downarrow{\scriptstyle \Psi_{f(l)} \circ \mathbf{L}p_{f(l)}^*} \\ D^b\left(\mathbb{P}^3, \mathcal{B}^l_0\right) & \xrightarrow{\tilde{f}_{l*}} & D^b\left(\mathbb{P}^3, \mathcal{B}^{f(l)}_0\right)\,. \end{array}$$

Proof. By Lemma 5.7, we have the equivalence $\tilde{f}_{l*} \colon D^b\left(\mathbb{P}^3, \mathcal{B}^l_0\right) \xrightarrow{\sim} D^b\left(\mathbb{P}^3, \mathcal{B}^{f(l)}_0\right)$. By the definition of f_l and $\tilde{f}_l$, Lemma 5.7 and the exact sequence in Remark 3.7, we have $f_{l*}\mathcal{E}_l \simeq \mathcal{E}_{f(l)}$. So we obtain the desired commutative diagram. □

We now prove Proposition 5.6.

Proof of Proposition 5.6. Let $f \in \mathrm{Aut}(X)$ be an automorphism of X. Note that f^* acts trivially on A_2. By Proposition 5.5, we have $Z_\alpha \circ f^* = Z_\alpha$. By Lemma 5.7 and Proposition 5.8, the equivalence $\tilde{f}_{l*}\colon D^b(\mathbb{P}^3, \mathcal{B}_0^l) \xrightarrow{\sim} D^b(\mathbb{P}^3, \mathcal{B}_0^{f(l)})$ induces the equivalence

$$\tilde{f}_{l*}\colon \Psi_l \circ \mathbf{L}p_l^*(\mathcal{D}_X) \xrightarrow{\sim} \Psi_{f(l)} \circ \mathbf{L}^*_{f(l)}(\mathcal{D}_X)\,.$$

Since $\tilde{f}_{l*}$ is compatible with tilting (cf. Definition 3.3) in the construction of $\mathcal{A}^l_\alpha$, we have $\tilde{f}_{l*}\mathcal{A}^l_\alpha = \mathcal{A}^{f(l)}_\alpha$. So we obtain $f_*\sigma^l_\alpha = \sigma^{f(l)}_\alpha$. By Theorem 3.8, we have $f_*\sigma_\alpha = \sigma_\alpha$. □

6. Automorphism groups of cubic fourfolds and Kuznetsov components

In this section, we characterize automorphism groups of cubic fourfolds as subgroups of autoequivalence groups of Kuznetsov components.

6.1 Automorphisms of cubic fourfolds as autoequivalences

In this subsection, we give the statement of the theorem which is main in this section. Let X be a cubic fourfold. Fix a real number $0 < \alpha < 1/4$. Put $\sigma := \sigma_\alpha \in \mathrm{Stab}^*(\mathcal{D}_X)$. By Propositions 5.1 and 5.6, the homomorphism $\rho_1\colon \mathrm{Aut}(X) \to \mathrm{Aut}(\mathcal{D}_X, \sigma)$ is injective. We define the homomorphism $\rho_2\colon \mathrm{Aut}(\mathcal{D}_X, \sigma) \to \mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z}), P_\sigma)$ by $\rho_2(\Phi) := \Phi^H$. The goal of this section is the following theorem.

Theorem 6.1. *The homomorphisms*

$$\rho_1\colon \mathrm{Aut}(X) \to \mathrm{Aut}(\mathcal{D}_X, \sigma)\,,$$
$$\rho_2\colon \mathrm{Aut}(\mathcal{D}_X, \sigma) \to \mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z}), P_\sigma)$$

are isomorphisms. In particular, we have isomorphisms

$$\rho_1\colon \mathrm{Aut_s}(X) \to \mathrm{Aut_s}(\mathcal{D}_X, \sigma)\,,$$
$$\rho_2\colon \mathrm{Aut_s}(\mathcal{D}_X, \sigma) \to \mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z}), \Pi_\sigma)\,.$$

By Proposition 5.1, it is enough to show that $\rho_2 \circ \rho_1$ is surjective and ρ_2 is injective.

First, we prove the surjectivity of $\rho_2 \circ \rho_1$.

Proposition 6.2. *The composition $\rho_2 \circ \rho_1\colon \mathrm{Aut}(X) \to \mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z}), P_\sigma)$ of ρ_1 and ρ_2 is surjective.*

Proof. By Proposition 5.5, we have $\mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z}), P_\sigma) = \mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z}), A_2 \otimes \mathbb{R})$. By Proposition 2.3, there is the isomorphism

$$v\colon \mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z}), A_2 \otimes \mathbb{R}) \xrightarrow{\sim} \mathrm{O}_{\mathrm{Hodge}}(H^*_{\mathrm{prim}}(X, \mathbb{Z}))\,.$$

Under this isomorphism, the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{O}_{\mathrm{Hodge}}(H^4_{\mathrm{prim}}(X, \mathbb{Z}))$ is compatible with $\rho_2 \circ \rho_1$. By the Torelli theorem for cubic fourfolds, we obtain the statement. □

We will prove the following proposition.

Proposition 6.3. *The homomorphism $\rho_2\colon \mathrm{Aut}(\mathcal{D}_X, \sigma) \to \mathrm{O}_{\mathrm{Hodge}}(H^*(\mathcal{D}_X, \mathbb{Z}), P_\sigma)$ is injective. Equivalently, if $\Phi \in \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$ satisfies $\Phi\sigma = \sigma$ and $\Phi^H = \mathrm{id}$, we have $\Phi = \mathrm{id}_{\mathcal{D}_X}$ in $\mathrm{Aut}(\mathcal{D}_X, \sigma)$.*

To prove Proposition 6.3, we study the Fano scheme $F(X)$ of lines on X.

6.2 Fano schemes of lines

In this subsection, we recall the description of Fano schemes of lines on cubic fourfolds in terms of moduli spaces of stable objects in Kuznetsov components.

Let X be a cubic fourfold. The Fano scheme $F(X)$ of lines on X can be described by the moduli theory on the Kuznetsov component $\mathcal{D}_X$. For a line l on X, there is the exact sequence

$$0 \to F_l \to \mathcal{O}_X^{\oplus 4} \to I_l(1) \to 0\,,$$

where I_l is the ideal sheaf of l in X. Then we have $i^*(\mathcal{O}_l(1)) \simeq F_l[2]$ (see [KM09, Lemma 5.1]). We consider the autoequivalence $\Xi := \mathbf{R}_{\mathcal{O}_X(-1)}(- \otimes \mathcal{O}_X(-1)) \in \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$ on $\mathcal{D}_X$. For a line $l \in F(X)$, we define the object $E_l \in \mathcal{D}_X$ as $E_l := \Xi(F_l)$. For a line $l \in F(X)$, the object E_l fits into the exact triangle $\mathcal{O}_X(-1)[1] \to E_l \to I_l$ (see [MS12, Section 2.3]). Li, Pertusi and Zhao proved the following theorem.

Theorem 6.4 ([LPZ18, Theorem 1.1]). *Take a real number $0 < \alpha < 1/4$. Then we have an isomorphism $u\colon F(X) \xrightarrow{\sim} M_{\sigma_\alpha}(v)$ given by $u([l]) := E_l$, where $v := \lambda_1 + \lambda_2$.*

Denote the Fano scheme of lines on X by $F(X)$. For the universal line

$$\mathcal{F}(X) := \{(x,l) \in X \times F(X) \mid x \in l\}$$

on X, consider the natural projections $p\colon \mathcal{F}(X) \to F(X)$ and $q\colon \mathcal{F}(X) \to X$. We define an exact functor $\Sigma\colon \mathcal{D}_X \to D^b(F(X))$ as $\Sigma := \mathbf{R}p_* \circ \mathbf{L}q^* \circ i$. Denote the right adjoint functor of Σ by $\Sigma_R\colon D^b(F(X)) \to \mathcal{D}_X$. Addington proved the following proposition.

Proposition 6.5 ([Add16, Section 5.1]). *For $[l] \in F(X)$, we have $\Sigma_R(\mathcal{O}_l(1)) \simeq F_l[1]$.*

We modify the adjoint functors Σ and Σ_R for this situation.

Definition 6.6. Take a real number $0 < \alpha < 1/4$, and put $\sigma := \sigma_\alpha$. We define the exact functors $P^*\colon \mathcal{D}_X \to D^b(M_\sigma(v))$ and $P_*\colon D^b(M_\sigma(v)) \to \mathcal{D}_X$ by

$$P^* := u_* \circ \Sigma \circ \Xi^{-1} \circ [1]\colon \mathcal{D}_X \to D^b(M_\sigma(v))\,,$$
$$P_* := [-1] \circ \Xi \circ \Sigma_R \circ u^*\colon D^b(M_\sigma(v)) \to \mathcal{D}_X\,.$$

Then P_* is the right adjoint functor of P^*.

The above functors are related to a universal family of the moduli space $M_\sigma(v)$.

Remark 6.7. Since $\Xi \in \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$, the composition $i \circ P_*\colon D^b(M_\sigma(v)) \to D^b(X)$ is a Fourier–Mukai functor; that is, there is an object $\mathcal{U} \in D^b(M_\sigma(v) \times X)$ such that $i \circ P_* \simeq \Phi_{\mathcal{U}}$. By Proposition 6.5, the object $\mathcal{U}$ is a universal family of the moduli space $M_\sigma(v)$ over X.

We will need the following result about kernels of actions of automorphism groups on cohomology groups for irreducible holomorphic symplectic manifolds.

Theorem 6.8 ([Bea83, Theorem 2.1], [HT13, Proposition 10]). *Take a positive integer $n > 0$. Let M be an irreducible holomorphic symplectic manifold deformation equivalent to the Hilbert scheme of n-points on K3 surfaces. Consider the group homomorphism*

$$\rho\colon\ \mathrm{Aut}(M) \to \mathrm{O}\big(H^2(M,\mathbb{Z})\big)\,, \quad f \mapsto f^*\,.$$

Then we have $\mathrm{Ker}(\rho) = 1$.

Let $\Sigma^H\colon H^*(\mathcal{D}_X, \mathbb{Q}) \to H^*(F(X), \mathbb{Q})$ be the linear map induced by $\Sigma\colon \mathcal{D}_X \to D^b(F(X))$.

Proposition 6.9. *The restriction* $\Sigma^H|_{A_2^\perp} A_2^\perp \xrightarrow{\sim} H^2_{\mathrm{prim}}(F(X),\mathbb{Z})$ *is a Hodge isometry, where* $H^2_{\mathrm{prim}}(F(X),\mathbb{Z})$ *is the primitive cohomology group of* $F(X)$ *with respect to the Plücker polarization of* $F(X)$.

Proof. The Abel–Jacobi map $p_*q^*\colon H^4_{\mathrm{prim}}(X,\mathbb{Z})(-1) \xrightarrow{\sim} H^2_{\mathrm{prim}}(F(X),\mathbb{Z})$ is the Hodge isometry. By Proposition 2.3 and the definition of Σ, we have $\Sigma^H_{A_2{}^\perp} = p_*q^*v$. ☐

Lemma 6.10. *For an autoequivalence* $\Phi \in \mathrm{Ker}(\rho_2)$, *we have isomorphisms* $\Phi \circ P_* \simeq P_*$ *and* $\mathcal{P}^* \circ \Phi \simeq P^*$.

Proof. Let $\Phi \in \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X)$ be an autoequivalence satisfying $\Phi\sigma = \sigma$ and $\Phi^H = \mathrm{id}$. Then Φ induces an automorphism $\Phi_{\sigma,v}\colon M_\sigma(v) \xrightarrow{\sim} M_\sigma(v)$, $[E] \mapsto [\Phi(E)]$ such that the following diagram commutes:

$$\begin{array}{ccc} v^\perp & \xrightarrow{\Phi^H} & v^\perp \\ \downarrow{\scriptstyle \theta_{\sigma,v}} & & \downarrow{\scriptstyle \theta_{\sigma,v}} \\ H^2(M_\sigma(v),\mathbb{Z}) & \xrightarrow{\Phi_{\sigma,v*}} & H^2(M_\sigma(v),\mathbb{Z})\,. \end{array}$$

Since $\Phi^H = \mathrm{id}$, we have $\Phi_{\sigma,v_*} = \mathrm{id}$. By Theorem 6.8, we obtain $\Phi_{\sigma,v} = \mathrm{id}_{M_\sigma(v)}$. Since Φ is of Fourier–Mukai type, there is an object $\mathcal{E} \in D^b(X \times X)$ such that $\Phi \simeq i \circ \Phi_{\mathcal{E}} \circ i^*$. By Remark 6.7, we have $\Phi_{\mathcal{U}} \simeq i \circ P_*$, where $\mathcal{U} \in D^b(M_\sigma(v) \times X)$ is a universal family of $M_\sigma(v)$. For $[E] \in M_\sigma(v)$, we have

$$\Phi_{\mathcal{E}} \circ \Phi_{\mathcal{U}}(\mathcal{O}_{[E]}) \simeq i^* \circ \Phi \circ i(E) \simeq E\,.$$

The convolution product $\mathcal{E} \circ \mathcal{U}$, which is a Fourier–Mukai kernel of the composition $\Phi_{\mathcal{E}} \circ \Phi_{\mathcal{U}}$, is also a universal family of $M_\sigma(v)$. So there is a line bundle $L \in \mathrm{Pic}(M_\sigma(v))$ such that $\mathcal{E} \circ \mathcal{U} \simeq \mathcal{U} \otimes p_M^* L$, where $p_M\colon M_\sigma(v) \times X \to M_\sigma(v)$ is the projection. Since $i^* \circ i \simeq \mathrm{id}_{\mathcal{D}_X}$ and $\Phi_{\mathcal{E}} \circ \Phi_{\mathcal{U}} \simeq \Phi_{\mathcal{U}} \circ (- \otimes L)$, we obtain $\Phi \circ P_* \simeq P_* \circ (- \otimes L)$. By the uniqueness of left adjoint functors, we have $P^* \circ \Phi^{-1} \simeq \big(- \otimes L^{-1}\big) \circ P^*$, and the isomorphism induces $(- \otimes L) \circ P^* \simeq P^* \circ \Phi$. Let $(P^*)^H\colon H^*(\mathcal{D}_X,\mathbb{Q}) \to H^*(M_\sigma(v),\mathbb{Q})$ be the linear map induced by $P^*\colon \mathcal{D}_X \to D^b(M_\sigma(v))$. Then we have $e^L \circ (P^*)^H = (P^*)^H$. Take a non-zero divisor $D \in \mathrm{NS}(M_\sigma(v))$. By Proposition 6.9, there is an element $w \in \widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z})$ such that $D = (P^*)^H(w)$. Then we have

$$\begin{aligned} D &= (P^*)^H(w) \\ &= e^L \cdot (P^*)^H(w) \\ &= D + L \cdot D + \tfrac{1}{2}L^2 \cdot D + \tfrac{1}{6}L^3 \cdot D\,. \end{aligned}$$

The natural map $\mathrm{Sym}^2\big(H^2(M_\sigma(v),\mathbb{C})\big) \to H^4(M_\sigma(v),\mathbb{C})$ is injective by Verbitsky's result [Ver96, Theorem 1.3]. So we have $L = 0$. Therefore, we have $\Phi \circ P_* \simeq P_*$ and $P^* \circ \Phi \simeq P^*$. ☐

6.3 Comonads

In this subsection, we collect definitions and basic properties of comodules over comonads following [Ela11]. Let $\mathcal{C}$ be a category.

Definition 6.11. A *comonad* $\mathbb{T} = (T, \varepsilon, \delta)$ on $\mathcal{C}$ consists of an endo-functor $T\colon \mathcal{C} \to \mathcal{C}$ and morphisms $\varepsilon\colon T \to \mathrm{id}_{\mathcal{C}}$ and $\delta\colon T \to T^2$ of functors such that the following diagrams commute:

$$\begin{array}{ccc} T & \xrightarrow{\delta} & T^2 \\ \downarrow{\scriptstyle \delta} & \searrow^{\mathrm{id}_T} & \downarrow{\scriptstyle T\varepsilon} \\ T^2 & \xrightarrow{\varepsilon T} & T\,, \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{\delta} & T^2 \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle T\delta} \\ T^2 & \xrightarrow{\delta T} & T^3\,. \end{array}$$

From adjoint functors, we can construct comonads.

Example 6.12. Let $P = (P^* \dashv P_*)$ be adjoint functors $P^*\colon \mathcal{C}' \to \mathcal{C}$ and $P_*\colon \mathcal{C} \to \mathcal{C}'$. Denote the unit and the counit by $\eta_P\colon \mathrm{id}_{\mathcal{C}'} \to P_* \circ P^*$ and $\varepsilon_P\colon P^* \circ P_* \to \mathrm{id}_{\mathcal{C}}$, respectively. We have an endo-functor $T_P := P^* \circ P_*$ and a morphism $\delta_P := P^*\eta_P P_*$ of functors. Then the triple $\mathbb{T}(P) := (T_P, \varepsilon_P, \delta_P)$ is a comonad on $\mathcal{C}$.

For a comonad, we have the notion of comodules over the comonad.

Definition 6.13. Let $\mathbb{T} = (T, \varepsilon, \delta)$ be a comonad on $\mathcal{C}$. A *comodule* over $\mathbb{T}$ is a pair (C, θ_C) of an object $C \in \mathcal{C}$ and a morphism $\theta_C\colon C \to T(C)$ such that

(1) $\varepsilon(C) \circ \theta_C = \mathrm{id}_C$, and
(2) the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\theta_C} & T(C) \\ {\scriptstyle \theta_C}\downarrow & & \downarrow{\scriptstyle T(\theta_C)} \\ T(C) & \xrightarrow{\delta(C)} & T^2(C)\,. \end{array}$$

Definition 6.14. Let $\mathbb{T} = (T, \varepsilon, \delta)$ be a comonad on $\mathcal{C}$. We define the category $\mathcal{C}_{\mathbb{T}}$ of comodules over $\mathbb{T}$ as follows:

(1) The set $\mathrm{Ob}(\mathcal{C}_{\mathbb{T}})$ of objects in $\mathcal{C}_{\mathbb{T}}$ consists of comodules over $\mathbb{T}$.
(2) For comodules $(C_1, \theta_{C_1}), (C_2, \theta_{C_2}) \in \mathrm{Ob}(\mathcal{C}_{\mathbb{T}})$, we have

$$\mathrm{Hom}_{\mathcal{C}_{\mathbb{T}}}((C_1, \theta_{C_1}), (C_2, \theta_{C_2})) := \{f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \mid T(f) \circ \theta_{C_1} = \theta_{C_2} \circ f\}\,.$$

We have the following natural functors.

Definition 6.15. Let $\mathbb{T} = (T, \varepsilon, \delta)$ be a comonad on $\mathcal{C}$. We define a functor $Q_*\colon \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ as follows:

(1) For an object $C \in \mathcal{C}$, set $Q_*(C) := (T(C), \delta(C))$.
(2) For a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$, set $Q_*(f) := T(f)$.

We define a functor $Q^*\colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ as the forgetful functor. Then we have adjoint functors $Q = (Q^* \dashv Q_*)$.

Theorem 6.16 ([BW05, 3.2.3]). *Let $P = (P^* \dashv P_*)$ be adjoint functors $P^*\colon \mathcal{C} \to \mathcal{C}'$ and $P_*\colon \mathcal{C}' \to \mathcal{C}$. Then there exists a functor $\Gamma_P\colon \mathcal{C}' \to \mathcal{C}_{\mathbb{T}(P)}$, unique up to an isomorphism, such that $\Gamma_P \circ P_* \simeq Q_*$ and $Q^* \circ \Gamma_P \simeq P^*$. The functor $\Gamma_P\colon \mathcal{C}' \to \mathcal{C}_{\mathbb{T}(P)}$ is called the comparison functor.*

The following proposition gives sufficient conditions for a comparison functor to be an equivalence.

Proposition 6.17 ([Ela11, Theorem 3.9, Corollary 3.11]). *Let $P = (P^* \dashv P_*)$ be adjoint functors $P^*\colon \mathcal{C} \to \mathcal{C}'$ and $P_*\colon \mathcal{C}' \to \mathcal{C}$. If $\mathcal{C}$ is idempotent complete and the functor morphism $\eta_P\colon \mathrm{id}_{\mathcal{C}} \to P_* \circ P^*$ is a split mono, that is, there exists a functor morphism $\zeta\colon P_* \circ P^* \to \mathrm{id}_{\mathcal{C}}$ such that $\zeta \circ \eta = \mathrm{id}$, then $\Gamma_P\colon \mathcal{C} \to \mathcal{D}_{\mathbb{T}(P)}$ is an equivalence.*

6.4 Application of comonads

In this subsection, we prove that ρ_2 is injective as an application of comonads. Let X be a cubic fourfold. Fix a real number $0 < \alpha < 1/4$ and put $v := [E_l]$, where l is a line on X. Addington [Add16] proved the following theorem.

Theorem 6.18 ([Add16, Section 5.1]). *The unit* $\eta\colon \mathrm{id}_{\mathcal{D}_X} \to \Sigma_R \circ \Sigma$ *is a split mono, and* $\Sigma_R \circ \Sigma \simeq \mathrm{id}_{\mathcal{D}_X} \oplus [-2]$.

As a consequence of Theorem 6.18, we have the following statement.

Corollary 6.19. *The unit* $\eta\colon \mathrm{id}_{\mathcal{D}_X} \to P_* \circ P^*$ *is a split mono, and* $P_* \circ P^* \simeq \mathrm{id}_{\mathcal{D}_X} \oplus [-2]$.

We now prove Proposition 6.3.

Proof of Proposition 6.3. As in Example 6.12, let $\mathbb{T}(P)$ be the comonad on $D^b(M_\sigma(v))$ determined by the adjoint pair $P = (P^* \dashv P_*)$. By Theorem 6.16, there is a comparison functor $\Gamma_P\colon \mathcal{D}_X \to D^b(M_\sigma(v))_{\mathbb{T}(P)}$, unique up to an isomorphism, such that $\Gamma_P \circ P_* \simeq Q_*$ and $Q^* \circ \Gamma_P \simeq P^*$. Since $\Phi \circ P_* \simeq P_*$ and $P^* \circ \Phi \simeq P^*$, we have $\Gamma_P \circ \Phi \circ P_* \simeq Q_*$ and $Q^* \circ \Gamma_P \circ \Phi \simeq P^*$. So the composition $\Gamma_P \circ \Phi$ is also a comparison functor. Due to the uniqueness of comparison functors, we have $\Gamma_P \simeq \Gamma_P \circ \Phi$. By Corollary 6.19 and Theorem 6.17, the comparison functor Γ_P is an equivalence. Therefore, we obtain $\Phi \simeq \mathrm{id}_{\mathcal{D}_X}$. □

As a corollary of Proposition 6.3, we have the following.

Corollary 6.20. *We define*

$$\mathrm{Aut}^0(\mathcal{D}_X) := \left\{ \Phi \in \mathrm{Aut}^{\mathrm{FM}}(\mathcal{D}_X) \mid \Phi^H = \mathrm{id}, \Phi(\mathrm{Stab}^*(\mathcal{D}_X)) \subset \mathrm{Stab}^*(\mathcal{D}_X) \right\}.$$

Then the natural homomorphism $\mathrm{Aut}^0(\mathcal{D}_X) \to \mathrm{Deck}(\pi)$ *is injective, where* $\mathrm{Deck}(\pi)$ *is the group of deck transformations of the covering* π.

Proof. Take $\Phi \in \mathrm{Ker}\big(\mathrm{Aut}^0(\mathcal{D}_X) \to \mathrm{Deck}(\pi)\big)$. Then $\Phi\sigma = \sigma$ holds. By Proposition 6.3, we have $\Phi = \mathrm{id}_{\mathcal{D}_X}$. □

7. Automorphisms of cubic fourfolds and K3 surfaces

In this section, we compare automorphisms of cubic fourfolds and autoequivalences of derived categories of K3 surfaces. For labelled automorphisms of cubic fourfolds, they induce polarized automorphisms of K3 surfaces. We introduce the notion of labelled automorphisms of cubic fourfolds.

Definition 7.1. For a labelled cubic fourfold (X, K), we define the labelled automorphism group $\mathrm{Aut}(X, K)$ of (X, K) by

$$\mathrm{Aut}(X, K) := \{ f \in \mathrm{Aut}(X) \mid f^*|_K = \mathrm{id}_K \}.$$

An automorphism $f \in \mathrm{Aut}(X)$ is called a labelled automorphism of X if there is a rank 2 primitive sublattice $K \subset H^{2,2}(X, \mathbb{Z})$ such that (X, K) is a labelled cubic fourfold and $f \in \mathrm{Aut}(X, K)$.

We study polarized K3 surfaces associated with labelled cubic fourfolds from the point of view of moduli spaces of stable objects in Kuznetsov components.

Take an integer d that satisfies conditions $(*)$ and $(**)$. Let (X, K) be a labelled cubic fourfold of discriminant d. By Remark 2.5 and Theorem 2.6, there is a polarized K3 surface

(S,h) of degree d such that there is a Hodge isometry $\varphi\colon L_K^\perp \xrightarrow{\sim} L_h^\perp$, where L_h is the sublattice $H^0(S,\mathbb{Z}) \oplus \mathbb{Z}\cdot h \oplus H^4(S,\mathbb{Z})$ of the Mukai lattice $H^*(S,\mathbb{Z})$ of S. By [Nik79, Theorem 1.14.4], there is a Hodge isometry $\tilde{\varphi}\colon H^*(\mathcal{D}_X,\mathbb{Z}) \xrightarrow{\sim} H^*(S,\mathbb{Z})$ such that $\tilde{\varphi}|_{L_K^\perp} = \mathrm{id}_{L_K^\perp}$. Then $\tilde{\varphi}$ induces the isometry $\tilde{\varphi}|_{L_K}\colon L_K \xrightarrow{\sim} L_h$. We will reconstruct the polarized K3 surface (S,h) in terms of moduli spaces of stable objects in $\mathcal{D}_X$. Then we will obtain the nice equivalence $\mathcal{D}^b(S) \xrightarrow{\sim} \mathcal{D}_X$ in the context of both Hodge theory and moduli theory. First, we choose a Mukai vector and a stability condition on $\mathcal{D}_X$ to specify the moduli space. We put $v := \tilde{\varphi}^{-1}(0,0,1) \in \widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z})$. Note that the Mukai vector v is a primitive isotropic. Replacing $\tilde{\varphi}$ with $(\mathrm{id}_{H^0}\oplus -\mathrm{id}_{H^2}\oplus \mathrm{id}_{H^4})\circ\tilde{\varphi}$ if necessary, we may assume that $\Omega_1 := \tilde{\varphi}^{-1}(e^{ih})$ is contained in $\mathcal{P}_0^+(\mathcal{D}_X)$. Here, we use $d > 6$ and Theorems 3.4 and 3.6. We define $\Omega_n := \tilde{\varphi}^{-1}(e^{inh}) \in \mathcal{P}_0^+(\mathcal{D}_X)$ for a positive integer n.

Definition 7.2. For a positive integer n, we define the group homomorphism

$$Z_n\colon \widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z}) \to \mathbb{C}\,, \quad w \mapsto (\Omega_n, w)\,.$$

The following lemma will be used in the proof of Lemma 7.4.

Lemma 7.3 ([Bri08, Lemma 8.2]). *Fix $C > 0$. For $\Omega \in \mathcal{P}_0^+(\mathcal{D}_X)$, there are finitely many elements $w \in \widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z})$ such that $w^2 \geqslant -2$ and $|(\Omega,w)| \leqslant C$.*

Recall that a stability condition $\sigma \in \mathrm{Stab}^*(\mathcal{D}_X)$ is v-generic if and only if a σ-semistable object $E \in \mathcal{D}_X$ with $v(E) = v$ is σ-stable, by the primitivity of v.

Lemma 7.4. *For a positive integer n, let $\sigma_n = (Z_n, \mathcal{A}_n) \in \mathrm{Stab}^*(S)$ be a stability condition on S. Then there is a positive integer N such that σ_n is v-generic for any integer $n \geqslant N$.*

Proof. By Lemma 7.3, the set

$$\Gamma := \{w \in \widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z}) \mid w^2 \geqslant -2,\ |(\Omega_1,w)| \leqslant 1\}$$

is a finite set. For $w \in \widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z})$, let w_0 be the H^0-part of $\tilde{\varphi}(w) \in \widetilde{H}^{1,1}(S,\mathbb{Z})$. We define $N := \max\{|w_0| \mid w \in \Gamma\} + 1$. Take an integer $n \geqslant N$. Let E be a σ_n-stable object with $v(E) = v$ and $\phi_{\sigma_n}(E) = 1$. By the definitions of Ω_n and v, we have $Z_n(v(E)) = -1$. Assume that there is a subobject A of E in $\mathcal{A}_n$ such that A is σ_n-stable and $\phi_{\sigma_n}(A) = 1$. Due to $\phi_{\sigma_n}(A) = 1$, we have $h\cdot c = 0$. Since h is ample and $\mathrm{sign}(\mathrm{NS}(S)) = (1,\rho(S))$, we have $c^2 < -2$. If $r = 0$, then $v(A)^2 = c^2 < -2$ holds, and this gives a contradiction. So the integer r is not zero. Since $Z_n(A)$ and $Z_n(E/A)$ are negative real numbers and $Z_n(A) + Z_n(E/A) = -1$, we have $|Z_n(v(A))| \leqslant 1$. Set $(r,c,m) := \tilde{\varphi}(v(A))$. Then we obtain

$$\begin{aligned} Z_n(v(A)) &= (\Omega_n, v(A)) = (e^{inh},(r,c,m)) = -m + \tfrac{1}{2}ndr \\ &= (e^{ih},(rn,c,m)) = (\Omega_1, \tilde{\varphi}(rn,c,m)). \end{aligned}$$

So $\tilde{\varphi}(rn,c,m) \in \Gamma$ holds. Note that $|rn| \geqslant N$ holds. By the definition of N, this gives a contradiction. Therefore, E is a σ_n-stable object. □

We choose a v-generic stability condition σ_n such that σ_n is fixed by the action of $\mathrm{Aut}(X,K)$.

Proposition 7.5. *Let n be a positive integer. There is a stability condition $\sigma_n = (Z_n,\mathcal{A}_n) \in \mathrm{Stab}^*(\mathcal{D}_X)$ such that $f_*\sigma_n = \sigma_n$ holds for any automorphism $f \in \mathrm{Aut}(X,K)$.*

Proof. Recall that we have the isometry $\tilde{\varphi}|_{L_K}\colon L_K \xrightarrow{\sim} L_h$. Since $e^{inh} \in L_h \otimes \mathbb{C}$, the class Ω_n is contained in $L_K \otimes \mathbb{C}$. Since $A_2 \subset L_K$, there are no (-2)-classes in $L_K^\perp$ by Theorem 3.8 and Proposition 5.5. So the intersection $\mathcal{P}_0^+(\mathcal{D}_X)\cap(L_K\otimes\mathbb{C})$ is path-connected. Fix $0 < \alpha < 1/4$. Take a path γ from $\pi_X(\sigma_\alpha)$ to Ω_n. By Theorem 3.6, the path γ has the unique lift $\tilde{\gamma}\colon [0,1] \to \mathrm{Stab}^*(S)$ such that $\tilde{\gamma}(0) = \sigma_\alpha$. By the definition of L_K and $\mathrm{Aut}(X,K)$, the automorphism f^* acts trivially on the lattice L_K. By Proposition 5.6, the morphism $f_* \circ \tilde{\gamma}$ is also a lift of γ starting from σ_α. We define the stability condition $\sigma_n = (Z_n, \mathcal{A}_n) \in \mathrm{Stab}^*(\mathcal{D}_X)$ by $\sigma_n := \tilde{\gamma}(1)$. By the uniqueness of lifts, we have $f_*\sigma_n = \sigma_n$ for any automorphism $f \in \mathrm{Aut}(X,K)$. □

Fix a sufficiently large integer $n > 0$ as in Lemma 7.4. Take a stability condition $\sigma = (Z, \mathcal{A}) := \sigma_n$ as in Proposition 7.5. Since v is a primitive isotropic and σ is v-generic, the moduli space $M_\sigma(v)$ of σ-stable objects with Mukai vector v is a K3 surface. There is an isotropic $v' \in \widetilde{H}^{1,1}(\mathcal{D}_X, \mathbb{Z})$ such that $(v, v') = -1$. Hence, $M_\sigma(v)$ is the fine moduli space. Take a universal family $\mathcal{U} \in D^b(M_\sigma(v) \times X)$ of $M_\sigma(v)$ over X. Then we have the equivalence $\Phi_{\mathcal{U}}\colon D^b(M_\sigma(v)) \xrightarrow{\sim} \mathcal{D}_X$. By [BM14, Theorem 1.3], we have an ample divisor l_σ on $M_\sigma(v)$ such that $l_\sigma \cdot C = \mathrm{Im}\, Z(\Phi_{\mathcal{U}}(\mathcal{O}_C))$ holds for any curve C on $M_\sigma(v)$. We denote the ample divisor l_σ/n by ω.

Proposition 7.6. *There are an isomorphism $t\colon M_\sigma(v) \xrightarrow{\sim} S$ and a universal family $\mathcal{U} \in D^b(M_\sigma(v) \times X)$ of $M_\sigma(v)$ over X such that $t^*h = \omega$ and $t_* = \tilde{\varphi} \circ \Phi^H_{\mathcal{U}}$.*

Proof. By the definition of the Mukai vector v, we have $\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}(0,0,1) = (0,0,1)$. Since $\big(\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}\big)^{-1}(1,0,0)$ is an isotropic and $\big(\big(\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}\big)^{-1}(1,0,0), (0,0,1)\big) = -1$, there is a line bundle L on $M_\sigma(v)$ such that $\big(\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}\big)^{-1}(1,0,0) = \big(1, L, L^2/2\big)$. Equivalently, we have $\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}\big(1, L, L^2/2\big) = (1,0,0)$. Let $p_M\colon M_\sigma(v) \times X \to M_\sigma(v)$ be the projection. Replacing $\mathcal{U}$ with $\mathcal{U} \otimes p_M^* L^{-1}$, we may assume that $\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}(1,0,0) = (1,0,0)$ holds. The restriction of $\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}$ to the second cohomology group $H^2(M_\sigma(v), \mathbb{Z})$ of $M_\sigma(v)$ is the Hodge isometry $\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}|_{H^2(M_\sigma(v),\mathbb{Z})}\colon H^2(M_\sigma(v),\mathbb{Z}) \xrightarrow{\sim} H^2(S,\mathbb{Z})$. Take a curve C on $M_\sigma(v)$. We have

$$\begin{aligned} l_\sigma \cdot C = Z(\Phi_{\mathcal{U}}(\mathcal{O}_C)) &= (\mathrm{Im}(\Omega_n), v(\Phi_{\mathcal{U}}(\mathcal{O}_C))) = (\tilde{\varphi}^{-1}(0,nh,0), \Phi^H_{\mathcal{U}}(v(\mathcal{O}_C))) \\ &= ((\tilde{\varphi} \circ \Phi^H_{\mathcal{U}})^{-1}(0,nh,0), (0, C, \chi(\mathcal{O}_C)) = (\tilde{\varphi} \circ \Phi^H_{\mathcal{U}})^{-1}(0,nh,0) \cdot C\,. \end{aligned}$$

So we have $\big(\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}\big)^{-1}(0,h,0) = (0,\omega,0)$ or, equivalently, $\tilde{\varphi} \circ \Phi^H_{\mathcal{U}}(0,\omega,0) = (0,h,0)$. By the Torelli theorem for K3 surfaces, there is an isomorphism $t\colon M_\sigma(v) \xrightarrow{\sim} S$ such that $t_* = \tilde{\varphi} \circ \Phi^H_{\mathcal{U}}$. In particular, we have $t^*h = \omega$. □

From now on, take the universal family $\mathcal{U}$ as in Proposition 7.6. By Proposition 7.6, we obtain the following.

Remark 7.7. Let L_ω be the sublattice $H^0(M_\sigma(v),\mathbb{Z}) \oplus \mathbb{Z}\cdot\omega \oplus H^4(M_\sigma(v),\mathbb{Z})$ of $H^*(\mathcal{M}_\sigma(v),\mathbb{Z})$. The Hodge isometry $\Phi^H_{\mathcal{U}}\colon H^*(M_\sigma(v),\mathbb{Z}) \xrightarrow{\sim} H^*(\mathcal{D}_X,\mathbb{Z})$ induces the isometry

$$\Phi^H_{\mathcal{U}}|_{L_\omega}\colon L_\omega \xrightarrow{\sim} L_K$$

and the Hodge isometry

$$\Phi^H_{\mathcal{U}}|_{L_\omega^\perp}\colon L_\omega^\perp \xrightarrow{\sim} L_K^\perp\,.$$

The equivalence $\Phi_{\mathcal{U}}\colon D^b(M_\sigma(v)) \xrightarrow{\sim} \mathcal{D}_X$ induces the isomorphism between the distinguished connected components of the spaces of stability conditions.

Proposition 7.8. *For a stability condition $\tau := (W, \mathcal{B}) \in \mathrm{Stab}(\mathcal{D}_X)$, we define the stability condition $\Phi^*_{\mathcal{U}}\tau \in \mathrm{Stab}(M_\sigma(v))$ by*

$$\Phi^*_{\mathcal{U}}\tau := \big(W \circ \Phi^H_{\mathcal{U}}, \Phi^{-1}_{\mathcal{U}}(\mathcal{B})\big)\,.$$

Then we have the isomorphism

$$\Phi^*_{\mathcal{U}}\colon \mathrm{Stab}^*(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Stab}^*(M_\sigma(v))\,.$$

Proof. Let σ be the stability condition as in Lemma 7.4. By the construction of σ and Proposition 7.6, we have $Z \circ \Phi_{\mathcal{U}}^{H}(-) = \left(e^{in\omega}, -\right)$. By Lemma 7.4, for any point $[E] \in M_\sigma(v)$, we have the σ-stable object $\Phi_{\mathcal{U}}(\mathcal{O}_{[E]}) \simeq E$. By [Bri08, Proposition 10.3], we have $\Phi_{\mathcal{U}}^{*}\sigma = \sigma_{0,n\omega}$ as in Theorem 3.4. So we obtain the isomorphism $\Phi_{\mathcal{U}}^{*}\colon \mathrm{Stab}^*(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Stab}^*(M_\sigma(v))$. □

Using results in Section 6 and this subsection, we obtain the following theorem.

Theorem 7.9. *There is a stability condition $\sigma_X \in \mathrm{Stab}^*(M_\sigma(v))$ such that we have the isomorphism of groups*

$$(-)_{\mathcal{U}}\colon \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}\left(D^b(M_\sigma(v)), \sigma_X\right), \quad f \mapsto f_{\mathcal{U}} := \Phi_{\mathcal{U}}^{-1} \circ f_* \circ \Phi_{\mathcal{U}}\,.$$

Moreover, the restriction of $(-)_{\mathcal{U}}$ to the labelled automorphism group $\mathrm{Aut}(X, K)$ of (X, K) induces the isomorphism of groups

$$(-)_{\mathcal{U}}\colon \mathrm{Aut}(X, K) \xrightarrow{\sim} \mathrm{Aut}(M_\sigma(v), \omega)\,.$$

Proof. Fix a real number $0 < \alpha < 1/4$. Put $\sigma_X := \Phi_{\mathcal{U}}^{*}\sigma_\alpha \in \mathrm{Stab}(\mathcal{D}_X)$. By Proposition 7.8, the stability condition σ_X is in the distinguished connected component $\mathrm{Stab}^*(M_\sigma(v))$. By Proposition 4.4 and Theorem 6.1, we have the isomorphism

$$(-)_{\mathcal{E}}\colon \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}\left(D^b(M_\sigma(v)), \sigma_X\right), \quad f \mapsto f_{\mathcal{U}} := \Phi_{\mathcal{U}}^{-1} \circ f_* \circ \Phi_{\mathcal{U}}\,.$$

Let $f \in \mathrm{Aut}(X, K)$ be an automorphism. By Proposition 7.5, since $v \in L_K$, we obtain the automorphism

$$f_{\sigma,v}\colon M_\sigma(v) \xrightarrow{\sim} M_\sigma(v)\,, \quad [E] \mapsto [f_*E]\,.$$

Take a point $[E] \in M_\sigma(v)$. By Proposition 7.5, since $v \in L_K$, we have

$$\Phi_{\mathcal{U}}^{-1} \circ f_* \circ \Phi_{\mathcal{U}}(\mathcal{O}_{[E]}) \simeq \mathcal{O}_{[f_*E]}\,.$$

There is a line bundle L on $M_\sigma(v)$ such that $\Phi_{\mathcal{U}} \circ f_{\sigma,v*} \circ (- \otimes L) \simeq f_* \circ \Phi_{\mathcal{U}}$. Since $\Phi_{\mathcal{U}}^{H}(1,0,0) = \tilde{\varphi}(1,0,0)$ in L_K, we have $\Phi_{\mathcal{U}}^{H}\left(1, L, L^2/2\right) = \Phi_{\mathcal{U}}^{H}(1,0,0)$. So $L = 0$ holds. Due to $\Phi_{\mathcal{U}}^{H}(0,\omega,0) = \tilde{\varphi}(0,h,0)$ in L_K, we obtain $f_{\sigma,v}^{*}\omega = \omega$. Hence, we have $f_{\sigma,v*} = f_{\mathcal{U}}$. We shall prove that $(-)_{\mathcal{U}}\colon \mathrm{Aut}(X,K) \hookrightarrow \mathrm{Aut}(M_\sigma(v),\omega)$ is surjective. Take an automorphism $g \in \mathrm{Aut}(M_\sigma(v),\omega)$. By Proposition 2.3 and Remark 7.7, the composition $\Phi_{\mathcal{U}}^{H} \circ g_* \circ \left(\Phi_{\mathcal{U}}^{-1}\right)^{H}$ induces the Hodge isometry $\Phi_{\mathcal{U}}^{H} \circ g_* \circ \left(\Phi_{\mathcal{U}}^{-1}\right)^{H}|_{A_2^\perp}\colon H^4_{\mathrm{prim}}(X,\mathbb{Z}) \xrightarrow{\sim} H^4_{\mathrm{prim}}(X,\mathbb{Z})$. By the Torelli theorem for cubic fourfolds [Voi86], there is an automorphism $f \in \mathrm{Aut}(X)$ such that $f_* = \Phi_{\mathcal{U}}^{H} \circ g_* \circ \left(\Phi_{\mathcal{U}}^{-1}\right)^{H}|_{A_2^\perp}$. By Remark 7.7, we have $f \in \mathrm{Aut}(X,K)$. Then the equality $f_{\sigma,v*} = g_*$ holds. By the Torelli theorem for K3 surfaces, we obtain $f_{\sigma,v} = g$. Therefore, the homomorphism $(-)_{\mathcal{U}}\colon \mathrm{Aut}(X,K) \hookrightarrow \mathrm{Aut}(M_\sigma(v),\omega)$ is surjective. □

Therefore, we have obtained Theorem 1.4 from Proposition 7.6, Remark 7.7 and Theorem 7.9.

8. Symplectic automorphisms of cubic fourfolds and associated K3 surfaces

In this section, we study relations between symplectic automorphisms of cubic fourfolds and autoequivalences of derived categories of K3 surfaces.

Let X be a cubic fourfold, and put $G := \mathrm{Aut}_{\mathrm{s}}(X)$. We define the coinvariant sublattice $S_G(X)$ of $H^4(X,\mathbb{Z})$ by $S_G(X) := \left(H^4(X,\mathbb{Z})^G\right)^\perp$. Let T_X be the orthogonal complement of $H^{2,2}(X,\mathbb{Z})$ in $H^4(X,\mathbb{Z})$. By Proposition 2.3, we have $T_{\mathcal{D}_X} \simeq T_X(-1)$.

Lemma 8.1. *For a symplectic automorphism $f \in G$ of X, we have $f^*|_{T_X} = \mathrm{id}_{T_X}$.*

Proof. The proof is the same as in the case of K3 surfaces. See [Huy16a, Remark 3.3 in Section 3 and Remark 1.2 in Section 15]. ☐

We note the following.

Theorem 8.2. *If G is not isomorphic to the trivial group 1 or $\mathbb{Z}/2\mathbb{Z}$, then there exists a unique K3 surface S such that $\mathcal{D}_X \simeq D^b(S)$.*

Proof. Since the signature of $H^4_{\mathrm{prim}}(X,\mathbb{Z})$ is $(20,2)$, the signature of $T_{\mathcal{D}_X}$ is $(2, 20-\rho)$, where ρ is some non-negative integer. By Lemma 8.1, the sublattices T_X and $S_G(X)$ are orthogonal in $H^4_{\mathrm{prim}}(X,\mathbb{Z})$. By [LZ19, Theorem 1.2], we have $\operatorname{rk} S_G(X) \geqslant 12$. Therefore, we obtain $\rho \geqslant 12$. By [Mor84, Corollary 2.10], there is a unique K3 surface S of Picard number ρ such that $T_{\mathcal{D}_X}$ has a unique primitive embedding $T_{\mathcal{D}_X} \hookrightarrow H^2(S,\mathbb{Z})$ up to isometries of $H^2(S,\mathbb{Z})$ such that $T_{\mathcal{D}_X} = T_S$. Now, we have two primitive embeddings

$$T_{\mathcal{D}_X} \hookrightarrow H^2(S,\mathbb{Z}) \hookrightarrow H^*(S,\mathbb{Z}), \quad T_{\mathcal{D}_X} \hookrightarrow H^*(\mathcal{D}_X,\mathbb{Z}).$$

For the primitive embedding $T_{\mathcal{D}_X} \hookrightarrow H^2(S,\mathbb{Z}) \hookrightarrow H^*(S,\mathbb{Z})$, we have $U \simeq H^0(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}) \subset T^{\perp}_{\mathcal{D}_X}$. By [Nik79, Theorem 1.14.4], there is a Hodge isometry $H^*(S,\mathbb{Z}) \simeq H^*(\mathcal{D}_X,\mathbb{Z})$. So we have $U \subset \widetilde{H}^{1,1}(\mathcal{D}_X,\mathbb{Z})$. By Theorem 2.7 and the uniqueness of S, we obtain $\mathcal{D}_X \simeq D^b(S)$. ☐

The following is the list of orders of finite symplectic automorphisms of K3 surfaces.

Remark 8.3 ([Nik80, Section 5]). Let S be a K3 surface. If f is a symplectic automorphism of S with the finite order, then we have $1 \leqslant \operatorname{ord}(f) \leqslant 8$.

We can find examples of finite symplectic autoequivalences of K3 surfaces via symplectic automorphisms of cubic fourfolds. In some cases, they are not conjugate to symplectic automorphisms of K3 surfaces.

Example 8.4. Let X be the Fermat cubic fourfold, which is defined by the equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = 0.$$

By [LZ19, Theorem 1.8(1)], the Fermat cubic fourfold X is the unique cubic fourfold with a symplectic automorphism f_9 of order 9. By Theorem 8.2, there is a unique K3 surface S such that $\mathcal{D}_X \simeq D^b(S)$. Then f_9 induces the symplectic autoequivalence $\Phi_9 \in \operatorname{Aut}_{\mathrm{s}}\left(D^b(S)\right)$ of order 9. By Remark 8.3, the automorphism Φ_9 is not conjugate to symplectic automorphisms of S.

Example 8.5. Let X be the Klein cubic fourfold, which is defined by the equation

$$x_1^3 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_2 = 0.$$

This is the triple cover of $\mathbb{P}^4$ branched along the Klein cubic threefold in [Adl78]. By [LZ19, Theorem 1.8(5)], the Klein cubic fourfold X is the unique cubic fourfold with a symplectic automorphism f_{11} of order 11, and its symplectic automorphism group is the finite simple group $L_2(11)$ (cf. [Adl78]). By Theorem 8.2, there is a unique K3 surface S such that $\mathcal{D}_X \simeq D^b(S)$. Then f_{11} induces the symplectic autoequivalence $\Phi_{11} \in \operatorname{Aut}_{\mathrm{s}}\left(D^b(S)\right)$ of order 11. By Remark 8.3, the morphism Φ_{11} is not conjugate to symplectic automorphisms of S.

See [LZ19, Theorems 1.2 and 1.8] for other interesting cubic fourfolds.

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