

# Tournaments, Johnson Graphs, and NC-Teaching

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## Abstract

Quite recently a teaching model, called “No-Clash Teaching” or simply “NC-Teaching”, had been suggested that is provably optimal in the following strong sense. First, it satisfies Goldman and Matthias’ collusion-freeness condition. Second, the NC-teaching dimension (= NCTD) is smaller than or equal to the teaching dimension with respect to any other collusion-free teaching model. It has also been shown that any concept class which has NC-teaching dimension  $d$  and is defined over a domain of size  $n$  can have at most  $2^d \binom{n}{d}$  concepts. The main results in this paper are as follows. First, we characterize the maximum concept classes of NC-teaching dimension 1 as classes which are induced by tournaments (= complete oriented graphs) in a very natural way. Second, we show that there exists a family  $(\mathcal{C}_n)_{n \geq 1}$  of concept classes such that the well known recursive teaching dimension (= RTD) of  $\mathcal{C}_n$  grows logarithmically in  $n = |\mathcal{C}_n|$  while, for every  $n \geq 1$ , the NC-teaching dimension of  $\mathcal{C}_n$  equals 1. Since the recursive teaching dimension of a finite concept class  $\mathcal{C}$  is generally bounded  $\log |\mathcal{C}|$ , the family  $(\mathcal{C}_n)_{n \geq 1}$  separates RTD from NCTD in the most striking way. The proof of existence of the family  $(\mathcal{C}_n)_{n \geq 1}$  makes use of the probabilistic method and random tournaments. Third, we improve the afore-mentioned upper bound  $2^d \binom{n}{d}$  by a factor of order  $\sqrt{d}$ . The verification of the superior bound makes use of Johnson graphs and maximum subgraphs not containing large narrow cliques.

## 1 Introduction

Learning from examples that were carefully chosen by a teacher (e.g. a human expert) presents an alternative to the commonly used model of learning from randomly chosen examples. A model of teaching should be sufficiently restrictive to rule out collusion between the learner and the teacher. For instance, the teacher should not be allowed to encode a direct representation of the target concept (such as a Boolean formula or a neural network) within the chosen sequence of examples. [7] suggested to consider a learner-teacher pair as collusion-free if it satisfies the following condition: if the learner is in favor of concept  $C$  after having seen the labeled teaching set  $\mathcal{T}$  chosen by the teacher, the learner should again be in favor of  $C$  after having seen a superset  $\mathcal{S}$  of  $\mathcal{T}$  as long as the label assignment in  $\mathcal{S}$  still coincides with the label assignment induced by  $C$ . In other words: the learners guess  $C$  for the target concept should not be altered when the data give even more support to  $C$  than the original labeled teaching set  $\mathcal{T}$  is giving. Most existing abstract models of teaching are collusion-free in this sense. Quite recently, [10] introduced a new model, called no-clash teaching or simply NC-teaching, that is collusion-free and furthermore optimal in the following strong sense: For any model  $M$ , let  $\text{M-TD}(\mathcal{C})$  denote the corresponding teaching dimension of concept class  $\mathcal{C}$  (= smallest number that upper-bounds the size of any of the employed teaching sets provided that learner and teacher interact as prescribed by model  $M$ ). Then  $\text{NCTD}(\mathcal{C}) \leq \text{M-TD}(\mathcal{C})$  holds for any model  $M$  that satisfies Goldman and Mathias’ collusion-freeness criterion.

In this paper, we pursue the following questions:

- What is the maximum size of a concept class which has NC-dimension  $d$  and is defined over a domain of size  $n$ ?
- How do the classes of maximum size look like?

- How does the NC-model of teaching relate to well known model of recursive teaching?

Before we outline the structure of this paper, we put the first two of these questions into a more general context.

## 1.1 Bounds of the Sauer-Shelah Type

A well-known lemma of Sauer [14] and Shelah [15] states that a concept class of VC-dimension  $d$  can induce at most

$$\Phi_d(m) = \sum_{i=0}^d \binom{m}{i}$$

distinct binary label patterns on  $m \geq d$  instances (taken from the underlying domain). This implies that a concept class of VC-dimension  $d$  that is defined over a domain of size  $n$  contains at most  $\Phi_d(n)$  distinct concepts. More results of the Sauer-Shelah type are known in the literature. Here we focus on results that are related to teaching. Consider, for instance, the model of recursive teaching (introduced by [18]). As shown by [13],  $\Phi_d(n)$  also upper-bounds the size of any concept class which has recursive teaching dimension  $d$  and is defined over a domain of size  $n$ . As shown by [10],  $2^d \binom{n}{d}$  upper-bounds the size of any concept class of NC-teaching dimension  $d$  that is defined over a domain of size  $n$ . While the upper bound  $\Phi_d(n)$  is tight if  $d$  equals the VC-dimension or the recursive teaching dimension, the corresponding bound  $2^d \binom{n}{d}$  in case of  $d = \text{NCTD}(\mathcal{C})$  is tight only for  $d = 1$  (as we will show in this paper).

## 1.2 Maximum Concept Classes

Concept classes of VC-dimension  $d$  inducing  $\Phi_d(m)$  distinct binary label patterns on any sequence of  $m$  distinct instances are called “maximum classes” (a notion that dates back to early work of [17]). The theoretical study of maximum classes had been fruitful for several reasons:

- Although there is a wide variety of maximum classes, they have much structure in common. This structure can be uncovered by exploiting the general definition of a maximum class (which abstracts away the peculiarities of specific maximum classes).
- The investigation of maximum classes and their structural properties often leads to problems with a combinatorial flavor that may be considered interesting in their own right.
- The validity of conjectures, believed to hold for arbitrary concept classes (like, for instance, the Sample-Compression conjecture of [16]) can be tested by showing their validity for maximum classes (as it has been done successfully by [5] and [1]).

Given the teaching-related bounds of the Sauer-Shelah type, it is a natural idea to define and examine maximum classes in that context too. Here we are particularly interested in “NC-maximum classes”, i.e., concept classes of maximum size among the ones having NC-dimension  $d$  and being defined over a domain of size  $n$ .

## 1.3 Structure of the Paper

In Section 2, we call into mind the definition of various teaching models and the corresponding teaching dimensions. Section 3 contains the results which are related to tournaments. We first define two concept classes, the class  $\mathcal{C}^1[G]$  of size  $n$  and the class  $\mathcal{C}^2[G]$  of size  $2n$ , both of which are induced by a tournament  $G$  with  $n$  vertices. Then we show the following results:

- A concept class over domain  $[n]$  is an NC-maximum class of NC-dimension 1 if and only if there exists a tournament  $G$  with  $n$  vertices such that  $\mathcal{C} = \mathcal{C}^2[G]$ .
- For every tournament  $G$ , the the class  $\mathcal{C}^1[G]$  has NC-teaching dimension 1.

- There is a strictly positive probability for the event that a random tournament  $G$  induces a class  $\mathcal{C}^1[G]$  whose recursive teaching dimension is at least  $\log(n) - O(\log \log(n))$ .

The last two results establish an RTD-NCTD ratio of order  $\log n$ . This is particularly remarkable since  $\text{RTD}(\mathcal{C})$  is upper-bounded by  $\log |\mathcal{C}|$  for every finite concept class  $\mathcal{C}$ . In Section 4, one finds the results which are related to Johnson-graphs. It is shown that a concept class which has NC-dimension  $d$  and is defined over a domain of size  $n$  contains at most  $\left(2\sqrt{\frac{2}{d+1}} - \frac{2}{d+1}\right) \cdot 2^d \binom{n}{d}$  concepts. This improves the best previously known upper bound,  $2^d \binom{n}{d}$ , by a factor of order  $\sqrt{d}$ . It also shows that the size of an NC-maximum class of NC-dimension  $d \geq 2$  is strictly smaller than  $2^d \binom{n}{d}$ . The key lemma behind this result is Lemma 4.6, which relates the NC-teaching sets for a concept class  $\mathcal{C}$  to subgraphs of a Johnson graph which do not contain large narrow cliques. The final Section 5 mentions some open problems.

## 2 Definitions, Notations and Facts

As usual a *concept over domain*  $\mathcal{X}$  is a function from  $\mathcal{X}$  to  $\{0, 1\}$  or, equivalently, a subset of  $\mathcal{X}$ . A set whose elements are concepts over domain  $\mathcal{X}$  is referred to as a *concept class over*  $\mathcal{X}$ . The elements of  $\mathcal{X}$  are called *instances*. The powerset of  $\mathcal{X}$  is denoted by  $\mathcal{P}(\mathcal{X})$ . The set of all subsets of size  $d$  of  $\mathcal{X}$  is denoted by  $\mathcal{P}_d(\mathcal{X})$ . We refer to elements of  $\mathcal{P}_d(\mathcal{X})$  as *d-subsets* of  $\mathcal{X}$ .

**Definition 2.1** (Teaching Models [6, 18, 10]). *Let  $\mathcal{C}$  be a concept class over  $\mathcal{X}$ .*

1. *A teaching set for  $C \in \mathcal{C}$  is a subset  $D \subseteq \mathcal{X}$  which distinguishes  $C$  from any other concept in  $\mathcal{C}$ , i.e., for every  $C' \in \mathcal{C} \setminus \{C\}$ , there exists some  $x \in D$  such that  $C(x) \neq C'(x)$ . The size of the smallest teaching set for  $C \in \mathcal{C}$  is denoted by  $\text{TD}(C, \mathcal{C})$ . The teaching dimension of  $\mathcal{C}$  in the Goldman-Kearns model of teaching is then given by*

$$\text{TD}(\mathcal{C}) = \max_{C \in \mathcal{C}} |T(C, \mathcal{C})| .$$

*A related quantity is*

$$\text{TD}_{\min}(\mathcal{C}) = \min_{C \in \mathcal{C}} |T(C, \mathcal{C})| .$$

2. *Let  $T : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{X})$  be a mapping that assigns to every concept in  $\mathcal{C}$  a set of instances.  $T$  is called admissible for  $\mathcal{C}$  in the NC-model<sup>1</sup> of teaching, or simply an NC-teacher for  $\mathcal{C}$ , if, for every  $C \neq C' \in \mathcal{C}$ , there exists  $x \in T(C) \cup T(C')$  such that  $C(x) \neq C'(x)$ . The teaching dimension of  $\mathcal{C}$  in the NC-model of teaching is given by*

$$\text{NCTD}(\mathcal{C}) = \min_{C \in \mathcal{C}} \{ \max_{C' \in \mathcal{C}} |T(C, C')| : T \text{ is an NC-teacher for } \mathcal{C} \} .$$

3. *Let  $\mathcal{C}_{\min} \subseteq \mathcal{C}$  be the easiest-to-teach concepts in  $\mathcal{C}$ , i.e.,*

$$\mathcal{C}_{\min} = \{C \in \mathcal{C} : \text{TD}(C, \mathcal{C}) = \text{TD}_{\min}(\mathcal{C})\} .$$

*The recursive teaching dimension of  $\mathcal{C}$  is then given by*

$$\text{RTD}(\mathcal{C}) = \begin{cases} \text{TD}_{\min}(\mathcal{C}) & \text{if } \mathcal{C} = \mathcal{C}_{\min} \\ \max\{\text{TD}_{\min}(\mathcal{C}), \text{RTD}(\mathcal{C} \setminus \mathcal{C}_{\min})\} & \text{otherwise} \end{cases} .$$

*In all three models, the set  $T(C)$  is referred to as the teaching set for  $C$ . The order of  $T$  is defined as the size of the largest of  $T$ 's teaching sets, i.e.,  $\text{order}(T) = \max_{C \in \mathcal{C}} |T(C)|$ .*

Some remarks are in place here:

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<sup>1</sup>NC = No-Clash.

1. It was shown in [3] that

$$\text{RTD}(\mathcal{C}) = \max_{\mathcal{C}' \subseteq \mathcal{C}} \text{TD}_{\min}(\mathcal{C}') . \quad (1)$$

2. The set  $T(C)$  in Definition 2.1 is an *unlabeled* set of instances. Intuitively, one should think of the learner as receiving the correctly *labeled teaching set*, i.e., the learner receives  $T(C)$  *plus* the corresponding  $C$ -labels where  $C$  is the concept that is to be taught.
3. We say that two concepts  $C$  and  $C'$  *clash* (with respect to  $T : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{X})$ ) if they agree on  $T(C) \cup T(C')$ , i.e, if they assign the same 0, 1-label to all instances in  $T(C) \cup T(C')$ . NC-teachers for  $\mathcal{C}$  are teachers who avoid clashes between any pair of distinct concepts from  $\mathcal{C}$ .

As already observed by [10]), NC-Teachers can be normalized:

- We may assume without loss of generality that  $|T(C)| = d$  for every  $C \in \mathcal{C}$  where  $d$  denotes the order of  $T$ .

This will be henceforth assumed. Let  $n = |\mathcal{X}|$  and  $0 \leq d \leq n$ . An NC-teacher  $T$  for  $\mathcal{C}$  of order  $d$  then assigns to every concept  $C \in \mathcal{C}$  a set taken from  $\mathcal{P}_d(\mathcal{X})$ . Clearly  $|\mathcal{P}_d(\mathcal{X})| = \binom{n}{d}$  and, for each fixed set  $S \in \mathcal{P}_d(\mathcal{X})$ , there can be at most  $2^d$  distinct concepts in  $\mathcal{C}$  with NC-teaching set  $S$ . For this simple reason, the following holds:

**Theorem 2.2** ([10]). *Any concept class which has NC-teaching dimension  $d$  and is defined over a domain of size  $n$  contains at most  $2^d \binom{n}{d}$  concepts.*

An *NC-maximum class* (with respect to parameters  $n$  and  $d$ ) is a concept class of largest size among all classes having NC-dimension  $d$  and being defined over a domain  $\mathcal{X}$  of size  $n$ , say  $\mathcal{X} = [n]$ . The size of such a class will be denoted by  $M_{NC}(n, d)$  throughout this paper. According to Theorem 2.2,  $M_{NC}(n, d) \leq 2^d \binom{n}{d}$ . We will see in the course of this paper that this upper bound on  $M_{NC}(n, d)$  is tight only for  $d = 1$  and, for  $d \geq 2$ , it can be improved (at least) by a factor of order  $\sqrt{d}$ .

### 3 Results on Concept Classes Induced by Tournaments

The following notion will play a central role in this section:

**Definition 3.1** (concept class induced by a tournament). *Let  $G = ([n], E)$  be a tournament with  $n$  players, i.e.,  $G$  is a directed graph obtained from the complete graph with vertices  $1, \dots, n$  by giving every edge  $\{i, j\}$  an orientation (either  $(i, j)$  or  $(j, i)$ ).<sup>2</sup> The concept classes  $\mathcal{C}^1[G]$  and  $\mathcal{C}^2[G]$  are given by*

$$\mathcal{C}^1[G] = \{\overline{C_1}, \dots, \overline{C_n}\} \quad \text{and} \quad \mathcal{C}^2[G] = \{C_1, \dots, C_n\} \cup \{\overline{C_1}, \dots, \overline{C_n}\}$$

where, for  $j = 1, \dots, n$ , we set

$$C_j = \{i \in [n] : (i, j) \in E\} \quad \text{and} \quad \overline{C_j} = [n] \setminus C_j . \quad (2)$$

We will refer to  $\mathcal{C}^1[G]$  (resp. to  $\mathcal{C}^2[G]$ ) as the first (resp. the second) concept class induced by  $G$ .

Intuitively, we may think of  $C_j$  as consisting of all  $i$  who have won against  $j$  in the tournament  $G$ . Note that  $j \in \overline{C_j}$ .

**Example 3.2.** *Consider the tournament  $G_n$  with vertices  $1, \dots, n$  and, with edges  $(i, j)$  for all  $1 \leq i < j \leq n$  (i.e., edges are always directed from smaller to larger numbers). Let  $\mathcal{C}_n^2 = \mathcal{C}^2[G_n]$  denote the second concept class induced by  $G_n$ . Then the concepts in  $\mathcal{C}_n^2$  are  $C_j = \{1, \dots, j-1\}$  and  $\overline{C_j} = \{j, \dots, n\}$  for  $j = 1, \dots, n$ . In other words,  $\mathcal{C}_n^2$  contains all left half-intervals over domain  $[n]$  (including  $\emptyset$  but excluding  $[n]$ ) and all right half-intervals (including  $[n]$  but excluding  $\emptyset$ ).*

<sup>2</sup>Intuitively, edge  $(i, j)$  represents the event that  $j$  has lost against  $i$  in the tournament.

We will show in Section 3.1 that a concept class  $\mathcal{C}$  over  $[n]$  is an NC-maximum class of NC-teaching dimension 1 if and only if there exists some tournament  $G$  with  $n$  players such that  $\mathcal{C} = \mathcal{C}^2[G]$ . In Section 3.2, we show that there exists a family  $(\mathcal{C}_n)_{n \geq 1}$  of concept classes such that  $\text{RTD}(\mathcal{C}_n)$  grows logarithmically with  $n = |\mathcal{C}_n|$  while  $\text{NCTD}(\mathcal{C}_n) = 1$  for all  $n \geq 1$ . This is proven by the probabilistic method which deals here with the concept class  $\mathcal{C}^1(G_n)$  for a random tournament  $G_n$  with  $n$  players.

### 3.1 NC-Maximum Classes of Dimension 1

Here is the main result of this section:

**Theorem 3.3.** *A concept class  $\mathcal{C}$  over  $[n]$  is an NC-maximum class of NC-teaching dimension 1 if and only if  $\mathcal{C} = \mathcal{C}^2[G]$  for some tournament  $G$  with  $n$  players.*

*Proof.* Suppose first that  $\mathcal{C} = \mathcal{C}^2[G]$  for some tournament  $G = ([n], E)$ . Then  $\mathcal{C}^2[G]$  contains the  $2n$  concepts that are given by (2). Consider the mapping  $T : \mathcal{C}^2[G] \rightarrow \mathcal{P}_1([n])$  that assigns  $\{j\}$  to the concepts  $C_j$  and  $\overline{C_j}$  for  $j = 1, \dots, n$ .  $T$  avoids clashes between any pair of distinct concepts, which can be seen as follows:

- Since  $j \notin C_j$  and  $j \in \overline{C_j}$  holds for every  $j \in [n]$ , there is no clash between  $C_j$  and  $\overline{C_j}$ .

Assume now that  $i$  and  $j$  are two arbitrary but distinct indices from  $[n]$ . The following observations show that neither  $C_i$  and  $C_j$  nor  $\overline{C_i}$  and  $\overline{C_j}$  agree on  $\{i, j\}$ :

- If  $C_j$  agrees with  $C_i$  on  $\{i\}$ , then  $i \notin C_j$ . It follows that  $(i, j) \notin E$ . Thus  $(j, i) \in E$  so that  $j \in C_i$ , which means that  $C_i$  disagrees with  $C_j$  on  $\{j\}$ . Hence there is no clash between  $C_i$  and  $C_j$ .
- By symmetry, it follows that there is no clash between  $\overline{C_i}$  and  $\overline{C_j}$ .
- If  $C_j$  agrees with  $\overline{C_i}$  on  $\{i\}$ , then  $i \in C_j$  so that  $(i, j) \in E$ . It follows that  $(j, i) \notin E$  so that  $j \notin C_i$ . Thus  $j \in \overline{C_i}$ , which means that  $\overline{C_i}$  disagrees with  $C_j$  on  $\{j\}$ . Hence there is no clash between  $\overline{C_i}$  and  $C_j$ .

The above discussion shows that  $T$  avoids clashes and is therefore an NC-teacher for  $\mathcal{C}^2[G]$ . Since all teaching sets are of size 1, we get  $\text{NCTD}(\mathcal{C}^2[G]) = 1$ . In view of Theorem 2.2, the class  $\mathcal{C}^2[G]$  with  $2n = 2^1 \binom{n}{1}$  concepts is an NC-maximum class.

Suppose now that  $\mathcal{C}$  is an NC-maximum class of NC-teaching dimension 1 over  $[n]$ . The first part of the proof in combination with Theorem 2.2 implies that  $|\mathcal{C}| = 2n$ . Let  $T : \mathcal{C} \rightarrow \mathcal{P}_1([n])$  be an NC-teacher for  $\mathcal{C}$ . It follows that each set  $\{j\} \in \mathcal{P}_1([n])$  is assigned to exactly two concepts. Moreover these two concepts must disagree on  $\{j\}$ . We denote the concept with NC-teaching set  $\{j\}$  that contains  $j$  (resp. does not contain  $j$ ) by  $\overline{C_j}$  (resp. by  $C_j$ ). Fix two indices  $i \neq j$  and consider the following assertions:

1.  $C_j$  disagrees with  $C_i$  on  $\{i\}$ .
2.  $C_j$  agrees with  $\overline{C_i}$  on  $\{i\}$ .
3.  $\overline{C_i}$  disagrees with  $C_j$  on  $\{j\}$ .
4.  $\overline{C_i}$  agrees with  $\overline{C_j}$  on  $\{j\}$ .
5.  $\overline{C_j}$  disagrees with  $\overline{C_i}$  on  $\{i\}$ .
6.  $\overline{C_j}$  agrees with  $C_i$  on  $\{i\}$ .
7.  $C_i$  disagrees with  $\overline{C_j}$  on  $\{j\}$ .
8.  $C_i$  agrees with  $C_j$  on  $\{j\}$ .

Since the assignment of NC-teaching sets to concepts avoids clashes, it is easily seen that the following holds:

- Any assertion is an immediate logical consequence of the preceding one.
- The first assertion is an immediate logical consequence of the last one.

It follows that these eight assertions are equivalent. An inspection of the second and the fifth assertion reveals that  $\overline{C}_j = [n] \setminus C_j$ . Consider now the directed graph  $G = ([n], E)$  with

$$E = \{(i, j) : C_j \text{ agrees with } \overline{C}_i \text{ on } \{i\}\} .$$

An inspection of the second and the seventh assertion reveals that exactly one of the edges  $(i, j)$  and  $(j, i)$  belongs to  $E$ . It follows that  $G$  is a tournament. Moreover, the above definition of  $E$  makes sure that, for every  $j \in [n]$ ,  $C_j = \{i \in [n] : (i, j) \in E\}$ . We may therefore conclude that  $\mathcal{C} = \mathcal{C}^2[G]$ .  $\square$

The following result, which shows that the general inequality  $M_{NC}(n, d) \leq 2^d \binom{n}{d}$  holds with equality for  $d = 1$ , is a direct consequence of Theorem 3.3:

**Corollary 3.4.** *For all  $n \geq 1$ :  $M_{NC}(n, 1) = 2n$ .*

Here is another direct consequence of Theorem 3.3:

**Corollary 3.5.** *For  $n \geq 1$  and every tournament  $G$  with  $n$  players:  $\text{NCTD}(\mathcal{C}^1[G]) = 1$ .*

*Proof.* Since  $\mathcal{C}^1[G] \subseteq \mathcal{C}^2[G]$ , a simple monotonicity argument shows that  $\text{NCTD}(\mathcal{C}^1[G]) \leq \text{NCTD}[\mathcal{C}^2[G]$ . Clearly  $\text{NCTD}(\mathcal{C}^1[G]) \geq 1$ . Theorem 3.3 implies that  $\text{NCTD}(\mathcal{C}^2[G]) = 1$ . Thus  $\text{NCTD}(\mathcal{C}^1[G]) = 1$ .  $\square$

## 3.2 Classes with NCTD 1 and Logarithmic RTD

In this section, we make use of the following version of the Chernoff bound:

**Lemma 3.6** ([2, 9]). *Let  $X_1, \dots, X_m$  be a sequence of  $m$  independent Bernoulli trials, each with probability  $p$  of success. Let  $Z = X_1 + \dots + X_m$  be the random variable that counts the total number of successes (so that  $\mathbb{E}[Z] = pm$ ). Then, for  $0 \leq \gamma \leq 1$ , the following holds:*

$$\Pr[Z < (1 - \gamma)pm] \leq \exp\left(\frac{-pm\gamma^2}{2}\right) . \quad (3)$$

Here is the main result of this section:

**Theorem 3.7.** *For all sufficiently large  $n$ , there exists a concept class  $\mathcal{C}$  of size  $n$  which satisfies  $\text{RTD}(\mathcal{C}) \geq \text{TD}_{\min}(\mathcal{C}) \geq \lfloor \log(n) - 2 \log \log(2n) \rfloor - 4$  and  $\text{NCTD}(\mathcal{C}) = 1$ .*

*Proof.* We know from Corollary 3.5 that, for every tournament  $G$  with  $n$  players, the concept class  $\mathcal{C}^1[G]$  (which is of size  $n$ ) has NC-dimension 1. We know already from (1) that RTD is lower-bounded by  $\text{TD}_{\min}$ . The proof of the theorem can therefore be accomplished by showing that, for all sufficiently large  $n$ , there exists a tournament  $G$  with  $n$  players such that  $\text{TD}_{\min}(\mathcal{C}^1[G]) \geq \lfloor \log(n) - 2 \log \log(2n) \rfloor - 4$ . For this purpose, we make use of the probabilistic method. Details follow.

Suppose that  $\hat{G} = ([n], \hat{E})$  is a *random tournament*, i.e., for every  $1 \leq i < j \leq n$ , we decide by means of a fair coin whether  $(i, j)$  or  $(j, i)$  is included into  $\hat{E}$ . Consider the class  $\mathcal{C}^1[\hat{G}]$  (the first concept class induced by  $\hat{G}$ ). Let  $k \geq 1$  be a parameter whose precise definition (as a function in  $n$ ) is postponed to a later stage. For every set  $S \subseteq [n]$  of size  $k$  and every  $b : S \rightarrow \{0, 1\}$ , let  $Z_{S,b}$  be the random variable which counts how many concepts  $C \in \mathcal{C}^1[\hat{G}]$  satisfy

$$\forall s \in S : C(s) = b(s) . \quad (4)$$

Each concept  $C_i$  with  $i \in [n] \setminus S$  satisfies Condition (4) with a probability of exactly  $2^{-k}$ . Therefore

$$\mathbb{E}[Z_{S,b}] \geq 2^{-k} \cdot (n - k) .$$

An application of (3) with  $p = 2^{-k}$ ,  $\gamma = 1/2$  and  $m = n - k$  yields that, for every fixed choice of  $S$  and  $b$ , we have

$$\Pr[Z_{S,b} < 2^{-(k+1)}(n-k)] \leq \exp\left(\frac{-2^{-k}(n-k)}{8}\right) = \exp\left(2^{-(k+3)}(n-k)\right) .$$

As there are  $\binom{n}{k}2^k$  possible choices for  $(S, b)$ , an application of the union bound yields

$$\Pr[\exists(S, b) : Z_{S,b} < 2^{-(k+1)}(n-k)] \leq \binom{n}{k}2^k \cdot \exp\left(-2^{-(k+3)}(n-k)\right) .$$

We now set  $k' = \log(n) - 2 \log \log(2n) - 4$  and  $k = \lfloor k' \rfloor$ .

**Claim 3.8.** *For all sufficiently large  $n$ , we have*

$$2^{-(k+1)}(n-k) \geq 2 \quad \text{and} \quad \binom{n}{k}2^k \cdot \exp\left(-2^{-(k+3)}(n-k)\right) < 1 .$$

**Proof of the Claim:** For almost all  $n$ , we have  $n - k \geq n/2$ . It therefore suffices to show that, for all sufficiently large  $n$ , we have

$$2^{-(k+2)}n \geq 2 \quad \text{and} \quad \binom{n}{k}2^k \cdot \exp\left(-2^{-(k+4)}n\right) < 1 .$$

The first inequality is valid because  $k \leq k' < \log(n) - 3$ . After replacing  $\binom{n}{k}$  by  $n^k$  and taking the logarithm on both hand-sides, we obtain the following sufficient condition for the second inequality:

$$k \log(2n) - 2^{-(k+4)}n < 0 .$$

By the above choice of  $k$ , we have

$$k \log(2n) \leq k' \log(2n) < \log^2(2n) \quad \text{and} \quad 2^{-(k+4)}n \geq 2^{-(k'+4)}n \geq \log^2(2n) ,$$

which completes the proof of the claim.

It follows that there is a strictly positive probability for the event that, for each pair  $(S, b)$ , at least 2 concepts from  $\mathcal{C}^1[G]$  satisfy condition (4). Since  $S$  ranges over all subsets of  $[n]$  of size  $n$  and  $b$  ranges over all bit patterns from  $\{0, 1\}^k$ , we can draw the following conclusion: there exists a tournament  $G$  with  $n$  players such that none of the concepts in  $\mathcal{C} := \mathcal{C}^1[G]$  can be uniquely specified by  $k$  (or less) labeled examples. This clearly implies that  $\text{TD}_{\min}(\mathcal{C}) \geq k$ .  $\square$

With a little extra-effort, one can show that only an asymptotically vanishing fraction of tournaments induces concept classes whose  $\text{TD}_{\min}$  is upper bounded by  $\lfloor \log(n) - 2 \log \log(2n) \rfloor - 5$ . Hence almost all of these classes are hard to teach in the RTD-model. More precisely, the following holds:

**Corollary 3.9.** *Let  $\tau_n$  denote the fraction of tournaments  $G = ([n], E)$  such that  $\text{TD}_{\min}(\mathcal{C}^1[G]) \leq \lfloor \log(n) - 2 \log \log(2n) \rfloor - 5$ . Then, for all sufficiently large  $n$ , we have that*

$$\tau_n \leq \frac{1}{(2n)^{\log(2n)}} . \tag{5}$$

*Proof.* We use the probabilistic method thereby proceeding almost as in the proof of Theorem 3.7. In the sequel, we stress the differences to that proof:

- We set  $k' = \log(n) - 2 \log \log(2n) - 5$  and  $k = \lfloor k' \rfloor$ .

- We have to show that<sup>3</sup>

$$2^{-(k+2)}n \geq 2 \quad \text{and} \quad \binom{n}{k} 2^k \cdot \exp\left(-2^{-(k+4)}n\right) < \frac{1}{(2n)^{\log(2n)}} \quad (6)$$

holds for all sufficiently large  $n$ .

The first inequality is immediate from the choice of  $k$ . As for the second inequality<sup>4</sup>, it suffices to show that

$$k \log(2n) - 2^{-(k+4)}n < \log\left(\frac{1}{(2n)^{\log(2n)}}\right) = -\log^2(2n) .$$

By the above choice of  $k$ , we have

$$k \log(2n) \leq k' \log(2n) < \log^2(2n) \quad \text{and} \quad 2^{-(k+4)}n \geq 2^{-(k'+4)}n \geq 2 \log^2(2n) ,$$

which completes the proof of (6). From these findings, it is easy to deduce (5).<sup>5</sup> □

## 4 No-Clash Teaching Sets and their Relation to Johnson Graphs

In Section 4.1, we call into mind the definition of Johnson graphs (and related notions) along with some facts (all of which are well known and also easy to verify). In Section 4.2, the tools from Section 4.1 are used to improve the upper bound  $2^d \binom{n}{d}$  on  $M_{NC}(n, d)$  by a factor of order  $\sqrt{d}$ .

### 4.1 Johnson Graphs and their Subgraphs

**Definition 4.1** (Johnson graph). *Let  $J(n, k)$  denote the graph with vertex set  $\mathcal{P}_k([n])$  and an edge between  $A, B \in \mathcal{P}_k([n])$  iff  $|A \cap B| = k - 1$ . The graphs  $J(n, k)$  with  $1 \leq k \leq n$  are called Johnson graphs<sup>6</sup>. A clique  $\mathcal{K} \subseteq \mathcal{P}_k(n)$  in  $J(n, k)$  is said to be wide if the sets in  $\mathcal{K}$  have a common intersection of size  $k - 1$ . Analogously,  $\mathcal{K}$  is said to be narrow if the union of all sets in  $\mathcal{K}$  has size  $k + 1$ .*

**Warning:** The distinction between wide and narrow cliques would be blurred if we represented the  $N = \binom{n}{k}$  vertices simply by numbers  $1, \dots, N$ . In what follows, the representation of the  $N$  vertices by  $k$ -subsets of  $[n]$  is quite essential.

Note that  $J(n, 1)$  is isomorphic to the complete graph  $K_n$ . The vertices  $\{1\}, \dots, \{n\}$  of  $J(n, 1)$  form a wide clique.  $J(n, 2)$  is isomorphic to the line graph  $L(K_n)$ .  $J(k, k)$  is a graph with a single vertex  $[k]$  (and no edges).  $J(k+1, k)$  is isomorphic to  $K_{k+1}$ . The  $k+1$  vertices of  $J(k+1, k)$  form a narrow clique. Cliques of size 2 in  $J(n, k)$  are wide and narrow. Cliques of size 3 or more cannot be wide and narrow at the same time. A clique of size 3 is also called *triangle* in the sequel. Here are some more of the known (and easy-to-check) facts concerning Johnson graphs:

- Lemma 4.2.**
1. *Distinct sets in  $\mathcal{P}_k([n])$  with a common intersection of size  $k - 1$  (resp. a union of size  $k + 1$ ) must necessarily form a clique.*
  2. *Any clique in  $J(n, k)$  is wide or narrow.*
  3. *The mapping  $A \mapsto [n] \setminus A$  is a graph isomorphism between  $J(n, k)$  and  $J(n, n - k)$ . This isomorphism transforms narrow cliques into wide cliques, and vice versa.*

For any  $\mathcal{F} \subseteq \mathcal{P}_k(n)$ , we denote by  $\langle \mathcal{F} \rangle$  the subgraph of  $J(n, k)$  induced by  $\mathcal{F}$ . We denote the subgraph relation by “ $\leq$ ” (e.g.,  $\langle \mathcal{F} \rangle \leq J(n, k)$ ). The following observation is rather obvious:

<sup>3</sup>Compare with Claim 3.8.

<sup>4</sup>Compare with the proof of Claim 3.8.

<sup>5</sup>Compare with the end of the proof of Theorem 3.7.

<sup>6</sup>named after the former American mathematician Selmer M. Johnson



**Lemma 4.3.** 1. A graph with edge set  $\mathcal{F} \subseteq \mathcal{P}_2([n])$  contains a triangle iff  $\langle \mathcal{F} \rangle \leq J(n, 2)$  contains a narrow triangle.

2. A graph with edge set  $\mathcal{F} \subseteq \mathcal{P}_2([n])$  contains a vertex of degree  $c$  or more iff  $\langle \mathcal{F} \rangle \leq J(n, 2)$  contains a wide clique of size  $c$ .

We now fix some notation. The size of the largest  $\mathcal{F} \subseteq \mathcal{P}_k(n)$  such that  $\langle \mathcal{F} \rangle \leq J(n, k)$  does not contain a narrow  $(t+1)$ -clique is denoted by  $H_t(n, k)$ . Moreover, we set  $h_t(n, k) = \binom{n}{k}^{-1} \cdot H_t(n, k)$ . For any  $\mathcal{F} \subseteq \mathcal{P}_k([n])$  and  $I \subseteq [n]$ , we define

$$\mathcal{F}_{+I} = \{J \in \mathcal{F} \mid J \subseteq I\} .$$

Note that any (narrow or wide) clique in  $\mathcal{F}_{+I}$  would be a clique of the same type and size within  $\mathcal{F}$ . Hence, if  $\langle \mathcal{F} \rangle$  does not contain a narrow  $(t+1)$ -clique, then the same holds for  $\mathcal{F}_{+I}$ . Given these notations and observations, the following holds:

**Lemma 4.4.** For all  $1 \leq t \leq k \leq n-2$ :

$$h(k, k) = 1 \quad \text{and} \quad h_t(n, k) \leq h_t(n-1, k) \leq h_t(k+1, k) = \frac{t}{k+1} . \quad (7)$$

*Proof.*  $J(k, k)$  is a graph consisting of a single isolated vertex and  $J(k+1, k)$  is a narrow clique of size  $k+1$ . Hence  $H_t(k, k) = 1$  and  $H_t(k+1, k) = t$ , which implies that  $h_t(k, k) = 1$  and  $h_t(k+1, k) = \frac{t}{k+1}$ . The proof can now be accomplished by showing that  $h_t(n, k) \leq h_t(n-1, k)$ .<sup>7</sup> Fix a family  $\mathcal{F} \subseteq \mathcal{P}_k([n])$  of size  $H_t(n, k)$  such that  $\langle \mathcal{F} \rangle \leq J(n, k)$  does not contain a narrow  $(t+1)$ -clique. There are  $k \cdot H_t(n, k)$  occurrences of elements from  $[n]$  within the sets of  $\mathcal{F}$ . By the pigeon-hole principle, there exists an  $i \in [n]$  that occurs in at most  $\frac{k}{n} \cdot H_t(n, k)$  sets of  $\mathcal{F}$ . Set  $I = [n] \setminus \{i\}$ . It follows that

$$H_t(n-1, k) \geq |\mathcal{F}_{+I}| \geq \left(1 - \frac{k}{n}\right) H_t(n, k) .$$

Hence

$$h_t(n, k) \leq \frac{n}{n-k} \binom{n}{k}^{-1} \binom{n-1}{k} h_t(n-1, k) = h_t(n-1, k) .$$

□

We briefly note that the proof of the  $h(n, k) \leq h(n-1, k)$  made use only of the fact that the feature of avoiding a narrow  $(t+1)$ -clique is inherited from  $\mathcal{F}$  to  $\mathcal{F}_{+I}$ . Hence the same monotonicity is valid whenever this kind of inheritance is granted.

The parameter  $h_2(n, 2)$  can be determined exactly:

**Remark 4.5.** For every  $n \geq 2$ , we have  $h_2(n, 2) \leq \frac{n}{2(n-1)}$ . Moreover, this holds with equality if  $n$  is even.

*Proof.* According to Mantel's theorem ([11]) — in a more general form known as Turan's theorem ([12]) — any triangle-free graph has at most  $n^2/4$  edges. For even  $n$ , this bound is tight because the complete bipartite graph  $K_{n/2, n/2}$  is triangle free and has  $n^2/4$  edges. In combination with Lemma 4.3, we may conclude that  $H_2(n, 2) \leq n^2/4$ , and this holds with equality if  $n$  is even. Hence  $h_2(n, 2) \leq \binom{n}{2}^{-1} \cdot \frac{n^2}{4} = \frac{n}{2(n-1)}$ , again with equality if  $n$  is even. □

<sup>7</sup>This inequality is, in principle, known from [8]. The proof in [8] is written in Hungarian and it is formulated for hereditary properties of hypergraphs: if we view  $\mathcal{F} \subseteq \mathcal{P}_k(n)$  as a set of  $k$ -uniform hyperedges, then not containing  $t+1$  hyperedges whose union is of size  $k+1$  will become a hereditary hypergraph property. In our application of this result, it is however more intuitive to view the elements of  $\mathcal{F}$  as vertices of the Johnson graph. In order to make this paper more self-contained, we therefore included the short proof for  $h_t(n, k) \leq h_t(n-1, k)$ , which uses a simple averaging argument.

## 4.2 New Bounds on the Size of NC-Maximum Classes

Let  $\mathcal{C}$  be a concept class over  $[n]$  such that  $\text{NCTD}(\mathcal{C}) = d$ , as witnessed by an NC-teacher  $T : \mathcal{C} \rightarrow \mathcal{P}_d(n)$ . Let  $\mathcal{F} = \{T(C) : C \in \mathcal{C}\} \subseteq \mathcal{P}_d(n)$  be the family of all teaching sets assigned by  $T$  to the concepts of  $\mathcal{C}$ . For every  $F \in \mathcal{F}$ , let  $1 \leq m(F) \leq 2^d$  denote the number of concepts  $C \in \mathcal{C}$  with  $T(C) = F$ . Clearly  $|\mathcal{C}| = \sum_{F \in \mathcal{F}} m(F)$ . For every  $2 \leq t \leq d$ , we define

$$\mathcal{F}_t = \left\{ F \in \mathcal{F} : m(F) > \frac{2^{d+1}}{t+1} \right\}. \quad (8)$$

We view the sets in  $\mathcal{F}_t$  as vertices in the Johnson graph  $J(n, d)$  so that  $\langle \mathcal{F}_t \rangle$  denotes the subgraph of  $J(n, d)$  induced by  $\mathcal{F}_t$ . With these notations, the following holds:

**Lemma 4.6.** *The graph  $\langle \mathcal{F}_t \rangle \leq J(n, d)$  does not contain a narrow  $(t+1)$ -clique.*

*Proof.* Assume for contradiction that  $\langle \mathcal{F}_t \rangle$  does contain a narrow  $(t+1)$ -clique  $\mathcal{K}$ , say  $\mathcal{K} = \{F_1, \dots, F_{t+1}\} \subseteq \mathcal{F}_t$ . Set  $D = F_1 \cup \dots \cup F_{t+1} \subseteq [n]$ . The definition of a narrow clique in  $J(n, d)$  implies that  $|D| = d+1$ . From the definition of  $\mathcal{F}_t$ , we may infer that  $m(F_1) + \dots + m(F_{t+1}) > 2^{d+1}$ . Thus  $\mathcal{C}$  contains more than  $2^{d+1}$  concepts  $C$  whose teaching set  $T(C)$  belongs to  $\mathcal{K}$ . By the pigeon-hole principle, there must be two distinct concepts  $C_1$  and  $C_2$  such that  $T(C_1), T(C_2) \in \mathcal{K}$  and  $C_1$  and  $C_2$  coincide on  $D$ . But, since  $T(C_1) \cup T(C_2) \subseteq D$ , this means that  $C_1$  and  $C_2$  clash with respect to  $T$ . We arrived at a contradiction.  $\square$

We are now finally in the position to prove the (previously announced) improved upper bound on  $M_{NC}(n, d)$ :

**Theorem 4.7.** *For  $2 \leq t \leq d \leq n$ , the following holds:*

$$M_{NC}(n, d) \leq H_t(n, d)2^d + \left( \binom{n}{d} - H_t(n, d) \right) \frac{2^{d+1}}{t+1} = \left( h_t(n, d) + (1 - h_t(n, d)) \frac{2}{t+1} \right) \cdot 2^d \binom{n}{d}.$$

Moreover, for  $t = \lfloor \sqrt{2(d+1)} \rfloor$ , one gets

$$M_{NC}(n, d) \leq \left( 2\sqrt{\frac{2}{d+1}} - \frac{2}{d+1} \right) \cdot 2^d \binom{n}{d}. \quad (9)$$

*Proof.* Let  $\mathcal{C}$  be an NC-maximum class for parameters  $n$  and  $d$ . Then  $|\mathcal{C}| = M_{NC}(n, d)$ . Consider an NC-teacher  $T : \mathcal{C} \rightarrow \mathcal{P}_d(n)$  for  $\mathcal{C}$ . Let  $\mathcal{F} = \{T(C) : C \in \mathcal{C}\}$  and let  $\mathcal{F}_t \subseteq \mathcal{F}$  be as defined in (8). Lemma 4.6 implies that  $\mathcal{F}_t \subseteq \mathcal{P}_d(n)$  is of size at most  $H_t(n, d)$ . The size of  $\mathcal{C}$  can therefore be bounded as follows:

$$|\mathcal{C}| = \sum_{F \in \mathcal{F}} m(F) = \sum_{F \in \mathcal{F}_t} m(F) + \sum_{F \notin \mathcal{F}_t} m(F) \leq H_t(n, d)2^d + \left( \binom{n}{d} - H_t(n, d) \right) \frac{2^{d+1}}{t+1}.$$

From this and  $h_t(n, d) = \binom{n}{d}^{-1} \cdot H_t(n, d)$ , we immediately obtain

$$|\mathcal{C}| \leq \left( h_t(n, d) + (1 - h_t(n, d)) \frac{2}{t+1} \right) \cdot 2^d \binom{n}{d}.$$

According to Lemma 4.4, we have  $h_t(n, d) \leq \frac{t}{d+1}$ . Moreover, we may set  $t = \lfloor \sqrt{2(d+1)} \rfloor$  and can then proceed as follows:

$$h_t(n, d) + (1 - h_t(n, d)) \frac{2}{t+1} \leq \frac{t}{d+1} + \left( 1 - \frac{t}{d+1} \right) \frac{2}{t+1} \leq 2\sqrt{\frac{2}{d+1}} - \frac{2}{d+1}.$$

Putting everything together, we obtain (9).  $\square$

A simple computation shows that

$$2\sqrt{\frac{2}{d+1}} - \frac{2}{d+1} \leq 1$$

with equality for  $d = 1$  only. Hence the following holds:

**Corollary 4.8.** *For  $2 \leq d \leq n$ , we have that  $M_{NC}(n, d) < 2^d \binom{n}{d}$ .*

For  $d = 2$ , the upper bound on  $M_{NC}(n, d)$  from Theorem 4.7 can be slightly improved:

**Corollary 4.9.** *For every  $n \geq 2$ , the following holds:*

$$M_{NC}(n, 2) \leq \frac{(5n-4)n}{3} \approx \frac{5n^2}{3} .$$

*Proof.* We know from Theorem 4.7 that

$$M_{NC}(n, 2) \leq \left( h_2(n, 2) + (1 - h_2(n, 2)) \frac{2}{3} \right) \cdot 4 \binom{n}{2} .$$

We know from Remark 4.5 that  $h_2(n, 2) \leq \frac{n}{2(n-1)}$ . The assertion of the corollary now follows from a straightforward calculation.  $\square$

Similar slight improvements of the bound in Theorem 4.7 are possible for other small values of  $d$ .

## 5 Open Problems

According to Theorem 3.7, there exists a family  $(\mathcal{C}_n)_{n \geq 1}$  of concept classes such that  $\text{RTD}(\mathcal{C}_n)$  grows logarithmically with  $n = |\mathcal{C}_n|$  while  $\text{NCTD}(\mathcal{C}_n) = 1$  for all  $n \geq 1$ . The existence proof is based on the probabilistic method and therefore non-constructive. The best  $\text{RTD-NCTD}$  ratio, known so far for a concrete class, is the ratio for the class of parity functions in  $n$  Boolean variables (a class of size  $2^n$ ). It is shown in [4], that the  $\text{RTD}$  of the parity class equals  $n$  while the  $\text{NCTD}$  of this class is bounded by  $n/4$ .

**Open Problem 1:** Find a concrete class which establishes a large  $\text{RTD-NCTD}$  ratio (ideally a ratio of order  $\log |\mathcal{C}|$ ).

Theorem 3.3 characterizes  $\text{NC-maximum}$  classes of  $\text{NC-dimension}$  1 as classes of the form  $\mathcal{C}^2[G]$  for some tournament  $G$ . Hence the structure of  $\text{NC-maximum}$  classes of dimension 1 is now perfectly known, whereas the structure of  $\text{NC-maximum}$  classes of higher dimension is still unknown.

**Open Problem 2:** Find structural properties which are shared by all  $\text{NC-maximum}$  classes of  $\text{NC-dimension}$   $d \geq 2$ .

An obstacle for solving the second open problem is that we do not even know the size  $M_{NC}(n, d)$  of  $\text{NC-maximum}$  classes having  $\text{NC-dimension}$   $d \geq 2$  and being defined over a domain of size  $n$ . While we can conclude from Theorem 3.3 that  $M_{NC}(n, 1) = 2n$ , the quantity  $M_{NC}(n, d)$  with  $d \geq 2$  is still unknown to us (although Theorem 4.7 makes a first step towards finding non-trivial bounds on  $M_{NC}(n, d)$  for  $d \geq 2$ ).

**Open Problem 3:** Find better (upper and lower) bounds on  $M_{NC}(n, d)$  respectively, if possible, determine  $M_{NC}(n, d)$  exactly.

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