Metric Dimension Parameterized by Feedback Vertex Set and Other Structural Parameters∗†

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Abstract

For a graph $G$, a subset $S \subseteq V(G)$ is called a resolving set if for any two vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that $d(w, u) \neq d(w, v)$. The Metric Dimension problem takes as input a graph $G$ and a positive integer $k$, and asks whether there exists a resolving set of size at most $k$. This problem was introduced in the 1970s and is known to be NP-hard [GT 61 in Garey and Johnson’s book]. In the realm of parameterized complexity, Hartung and Nichterlein [CCC 2013] proved that the problem is W[2]-hard when parameterized by the natural parameter $k$. They also observed that it is FPT when parameterized by the vertex cover number and asked about its complexity under smaller parameters, in particular the feedback vertex set number. We answer this question by proving that Metric Dimension is W[1]-hard when parameterized by the feedback vertex set number. This also improves the result of Bonnet and Purohit [IPEC 2019] which states that the problem is W[1]-hard parameterized by the treewidth. Regarding the parameterization by the vertex cover number, we prove that Metric Dimension does not admit a polynomial kernel under this parameterization unless NP ⊆ coNP/poly. We observe that a similar result holds when the parameter is the distance to clique. On the positive side, we show that Metric Dimension is FPT when parameterized by either the distance to cluster or the distance to co-cluster, both of which are smaller parameters than the vertex cover number.

∗An extended abstract of this paper will appear in the proceedings of MFCS 2022.
†Research supported by the European Research Council (ERC) consolidator grant No. 725978 SYSTEMATICGRAPH.
1 Introduction

Problems dealing with distinguishing the vertices of a graph have attracted a lot of attention over the years, with the metric dimension problem being a classic one that has been vastly studied since its introduction in the 1970s by Slater [1], and independently by Harary and Melter [2]. Formally, given a graph $G$ and an integer $k \geq 1$, the Metric Dimension problem asks whether there exists a subset $S \subseteq V(G)$ of vertices of $G$ of size at most $k$ such that, for any two vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that $d(w, u) \neq d(w, v)$. If such a subset $S \subseteq V(G)$ exists, it is called a resolving set. The size of a smallest resolving set of a graph $G$ is the metric dimension of $G$, and is denoted by $MD(G)$.

There are many variants and problems associated to the metric dimension, with identifying codes [3], adaptive identifying codes [4], and locating dominating sets [5] asking for the vertices to be distinguished by their neighborhoods in the subset chosen. Other variants of note are the $k$-metric dimension, where each pair of vertices must be resolved by $k$ vertices in $S \subseteq V(G)$ instead of just one [6], and the truncated metric dimension, where the distance metric is the minimum of the distance in the graph and some integer $k$ [7]. Along similar lines, in the centroidal dimension problem, each vertex must be distinguished by its relative distances to the vertices in $S \subseteq V(G)$ [8]. The metric dimension has also been considered in digraphs, with [9] providing a summary of the related work in this area. Interestingly, there are many game-theoretic variants of the metric dimension, such as sequential metric dimension [10], the localization game [11, 12], and the centroidal localization game [13]. The metric dimension and its variants have been studied for both their theoretical interest and their numerous applications such as in network verification [14], fault-detection in networks [15], pattern recognition and image processing [16], graph isomorphism testing [17], chemistry [18, 19], and genomics [20]. For more on these variants and others, see [21] for the latest survey.

Much of the related work around the metric dimension problem focuses on its computational complexity. Metric Dimension was first shown to be $NP$-complete in general graphs in [22]. Later, it was also shown to be $NP$-complete in split graphs, bipartite graphs, co-bipartite graphs, and line graphs of bipartite graphs in [23], in bounded-degree planar graphs [24], and interval and permutation graphs of diameter 2 [25]. On the positive side, there are linear-time algorithms for Metric Dimension in trees [1], cographs [23], and cactus block graphs [26], and a polynomial-time algorithm for outerplanar graphs [24].

Since the problem is $NP$-hard even for very restricted cases, it is natural to ask for ways to confront this hardness. In this direction, the parameterized complexity paradigm allows for a more refined analysis of the problem's complexity. In this setting, we associate each instance $I$ with a parameter $\ell$, and are interested in an algorithm with running time $f(\ell) \cdot |I|^{O(1)}$ for some computable function $f$. Parameterized problems that admit such an algorithm are called fixed parameter tractable.
(FPT) with respect to the parameter under consideration. On the other hand, under standard complexity assumptions, parameterized problems that are hard for the complexity class $W[1]$ or $W[2]$ do not admit such fixed-parameter algorithms. A parameter may originate from the formulation of the problem itself (called natural parameters) or it can be a property of the input graph (called structural parameters).

Hartung and Nichterlein [27] proved that Metric Dimension is $W[2]$-hard when parameterized by the natural parameter the solution size $k$ even when the input graph is bipartite and has maximum degree 3. This motivated the study of the parameterized complexity of the problem under structural parameterizations. It was observed in [27] that the problem admits a simple FPT algorithm when parameterized by the vertex cover number. It took a considerable amount of work and/or meta-results to prove that there are FPT algorithms parameterized by the max leaf number [28], the modular width or treelength plus the maximum degree [29], and the treedepth [30]. In [23], they gave an XP algorithm parameterized by the feedback edge set number. Only recently, it was shown that Metric Dimension is $W[1]$-hard parameterized by the treewidth [31], answering an open question mentioned in [24, 28, 29]. This result was improved upon since, with it being shown that Metric Dimension is even $NP$-hard in graphs of treewidth 24 [32]. For more on the metric dimension, see [33] for a recent survey.

**Our contributions.** In this paper, we continue the analysis of structural parameterizations of Metric Dimension. See the Hasse diagram in Figure 1 for a summary of known results and our contributions. As mentioned before, it is known that Metric Dimension is $W[1]$-hard parameterized by the treewidth [31]. There are two natural directions to improve this result. One direction was to show that Metric Dimension is para-$NP$-hard parameterized by the treewidth, which was proven in [32]. Another direction is to prove that Metric Dimension is $W[1]$-hard for a higher parameter than treewidth, i.e., one for which the treewidth is upper bounded by a function of it. A parameter fitting this profile is the feedback vertex set number since the treewidth of a graph $G$ is upper bounded by the feedback vertex set number of $G$ plus one. Moreover, the complexity of Metric Dimension parameterized by the feedback vertex set number is left as an open problem in [27], the seminal paper on the parameterized complexity of Metric Dimension. We take this direction and answer this open question of [27] by proving that Metric Dimension is $W[1]$-hard parameterized by the feedback vertex set number (see Section 3). We then revisit the complexity of the problem when parameterized by the vertex cover number. Recall that the problem is known to admit an FPT algorithm, and hence, a kernel, under this parameterization. We prove that, however, Metric Dimension does not admit a polynomial kernel unless $NP \subseteq coNP/poly$ when parameterized by the vertex cover number (see Section 4)\(^\dagger\). On the positive side, we then show that

\(^\dagger\)After an extended abstract of this paper was short-listed for the proceedings of MFCS 2022, Florent Foucaud informed us of the paper of Gutin et al. [34], which contains a slightly stronger
Figure 1: Hasse diagram of graph parameters and associated results for METRIC DIMENSION. An edge from a lower parameter to a higher parameter indicates that the lower one is upper bounded by a function of the higher one. Colors correspond to the known hardness with respect to the highlighted parameter. The parameters for which the hardness remains an open question are not colored. The crossed bold circle in the upper-right corner means that METRIC DIMENSION does not admit a polynomial kernel when parameterized by the marked parameter unless $\text{NP} \subseteq \text{coNP}/\text{poly}$; the white one if a polynomial kernel exists. The bold borders highlight parameters that are covered in this paper. Also see Footnote 1.

METRIC DIMENSION is FPT for the structural parameters the distance to cluster and the distance to co-cluster both of which are smaller parameters than the vertex cover number (see Section 5). Note that the FPT algorithm for the distance to cluster parameter implies an FPT algorithm for the distance to clique parameter. With a slight modification of the reduction in Section 4, we establish the problem does not admit a polynomial kernel, under the same assumption, when the parameter is the distance to clique.

2 Preliminaries

Graph theory. We use standard graph-theoretic notation and refer the reader to [35] for any undefined notation. For an undirected graph $G$, sets $V(G)$ and $E(G)$ denote its set of vertices and edges, respectively. Two vertices $u, v \in V(G)$ are adjacent or neighbors if $uv \in E(G)$. The open neighborhood of a vertex $u \in V(G)$, denoted by $N(u) := N_G(u)$, is the set of vertices that are neighbors of $u$. The closed result.
neighborhood of a vertex $u \in V(G)$ is denoted by $N[u] := N_G[u] := N_G(u) \cup \{u\}$. For any $X \subseteq V(G)$ and $u \in V(G)$, $N_X(u) = N_G(u) \cap X$. Any two vertices $u, v \in V(G)$ are true twins if $N[u] = N[v]$, and are false twins if $N(u) = N(v)$. Observe that if $u$ and $v$ are true twins, then $uv \in E(G)$, but if they are only false twins, then $uv \notin E(G)$. For any $u, v \in V(G)$, we say that $u$ is connected to $v$ by a path $P$ of length $\ell$ if $P = w_0w_1 \ldots w_\ell$, where $w_0 = u$ and $v = w_\ell$. For a path $P$, we denote the length of $P$ by $\ell(P)$, and, for any $u, v \in V(P)$, we let $P[u, v]$ be the subpath of $P$ from $u$ to $v$.

The distance between two vertices $u, v \in V(G)$ in $G$, denoted by $d(u, v) := d_G(u, v)$, is the length of a $(u, v)$-shortest path in $G$. The distance between two given subsets $X, Y \subseteq V(G)$, denoted by $d(X, Y)$, is the minimum length of a shortest path from a vertex in $X$ to a vertex in $Y$, i.e., $d(X, Y) = \min_{x \in X, y \in Y} d(x, y)$. For a subset $S$ of $V(G)$, we denote the graph obtained by deleting $S$ from $G$ by $G - S$. We denote the subgraph of $G$ induced on the set $S$ by $G[S]$. The complement of a graph $G$ is a graph $H$ with the same vertex set, and such that any two vertices $u, v \in V(G)$ are adjacent in $H$ if and only if they are not adjacent in $G$.

A set of vertices $Y$ is said to be an independent set if no two vertices in $Y$ are adjacent. For a graph $G$, a set $X \subseteq V(G)$ is said to be a vertex cover if $V(G) \setminus X$ is an independent set. A vertex cover $X$ is a minimum vertex cover if for any other vertex cover $Y$ of $G$, we have $|X| \leq |Y|$. The vertex cover number of graph $G$ is the size of a minimum vertex cover of a graph $G$. For a graph $G$, a set $X \subseteq V(G)$ is said to be a feedback vertex set if $V(G) \setminus X$ is an acyclic graph. We define the notation of the feedback vertex set number in the analogous way. A set of vertices $Y$ is said to be a clique if any two vertices in $Y$ are adjacent. We say graph $G$ is a cluster graph if it is a disjoint union of cliques. Also, we say $G$ is a co-cluster graph if its complement is a cluster graph. For a graph class $\mathcal{F}$, we say a subset $X \subseteq V(G)$ is a $\mathcal{F}$-modulator of $G$ if $G - X \in \mathcal{F}$. A $\mathcal{F}$-modulator $X$ is a minimum $\mathcal{F}$-modulator if for any other $\mathcal{F}$-modulator $Y$ of $G$, we have $|X| \leq |Y|$. The distance to $\mathcal{F}$ of graph $G$ is the size of a minimum $\mathcal{F}$-modulator.

Metric Dimension. A subset of vertices $S \subseteq V(G)$ resolves a pair of vertices $u, v \in V(G)$ if there exists a vertex $w \in S$ such that $d(w, u) \neq d(w, v)$. A vertex $u \in V(G)$ is distinguished by a subset of vertices $S \subseteq V(G)$ if, for any $v \in V(G) \setminus \{u\}$, there exists a vertex $w \in S$ such that $d(w, u) \neq d(w, v)$. For an ordered subset of vertices $S = \{s_1, \ldots, s_k\} \subseteq V(G)$ and a single vertex $u \in V(G)$, the distance vector of $S$ with respect to $u$ is $r(S|u) := (d(s_1, u), \ldots, d(s_k, u))$. The following observation is throughout the paper.

Observation 2.1. Let $G$ be a graph. Then, for any (true or false) twins $u, v \in V(G)$ and any $w \in V(G) \setminus \{u, v\}$, $d(u, w) = d(v, w)$, and so, for any resolving set $S$ of $G$, $S \cap \{u, v\} \neq \emptyset$.

Proof. As $w \in V(G) \setminus \{u, v\}$ and $u$ and $v$ are (true or false) twins, the shortest $(u, w)$- and $(v, w)$-paths contain a vertex of $N := N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Thus,
Parameterized Complexity. An instance of a parameterized problem $\Pi$ comprises an input $I$, which is an input of the classical instance of the problem and an integer $\ell$, which is called as the parameter. A problem $\Pi$ is said to be fixed-parameter tractable or in FPT if given an instance $(I, \ell)$ of $\Pi$, we can decide whether or not $(I, \ell)$ is a Yes-instance of $\Pi$ in time $f(\ell) \cdot |I|^{O(1)}$, for some computable function $f$ whose value depends only on $\ell$.

A compression of a parameterized problem $\Pi_1$ into a (non-parameterized) problem $\Pi_2$ is a polynomial algorithm that maps each instance $(I_1, \ell_1)$ of $\Pi_1$ to an instance $I_2$ of $\Pi_2$ such that (1) $(I_1, \ell_1)$ is a Yes-instance of $\Pi_1$ if and only if $I_2$ is a Yes-instance of $\Pi_2$, and (2) the size of $I_2$ is bounded by $g(\ell)$ for a computable function $g$. The output $I_2$ is also called a compression and the function $g$ is said to be the size of the compression. A compression is polynomial if $g$ is polynomial. A compression is said to be kernel if $\Pi_1 = \Pi_2$. It is known that the problem is FPT if and only if it admits a kernel (See, for example, [36, Lemma 2.2]).

It is typical to describe a compression or kernelization algorithm as a series of reduction rules. A reduction rule is a polynomial time algorithm that takes as an input an instance of a problem and outputs another (usually reduced) instance. A reduction rule said to be applicable on an instance if the output instance is different from the input instance. A reduction rule is safe if the input instance is a Yes-instance if and only if the output instance is a Yes-instance. For more on parameterized complexity and related terminologies, we refer the reader to the recent book by Cygan et al. [36].

3 The Feedback Vertex Set Number

In this section, we prove that Metric Dimension is $\text{W}[1]$-hard parameterized by the feedback vertex set number. To prove this, we reduce from the NAE-INTEGER-3-SAT problem defined as follows. An instance of this problem consists of a set $X$ of variables, a set $C$ of clauses, and an integer $d$. Each variable takes a value in $\{1, \ldots, d\}$, and clauses are of the form $(x \leq a_x, y \leq a_y, z \leq a_z)$, where $a_x, a_y, a_z \in \{1, \ldots, d\}$. A clause is satisfied if not all three inequalities are true and not all are false. The goal is to find an assignment of the variables that satisfies all given clauses. This problem was shown to be $\text{W}[1]$-hard parameterized by the number of variables [37]. The remainder of this section is devoted to the proof of the following.

**Theorem 1.** Metric Dimension is $\text{W}[1]$-hard parameterized by the feedback vertex set number.

**Reduction.** We reduce from NAE-INTEGER-3-SAT: given an instance $(X, C, d)$ of this problem, we construct an instance $(G, k)$ of Metric Dimension as follows.
(a) The variable gadget $G_x$.

(b) The clause gadget $G_c$ is the disjoint union of $H_c$ (left) and $H_\overline{c}$ (right).

Figure 2: The gadgets in the proof of Theorem 1.

- For each variable $x \in X$, we introduce a cycle $G_x$ of length $2d + 2$ which has two distinguished anchor vertices $u_x^1$ and $u_x^2$ as depicted in Figure 2a; for convenience, we may also refer to $u_x^1$ as $v_x^0$ or $w_x^0$, and to $u_x^2$ as $v_x^{d+1}$ or $w_x^{d+1}$.

- For each clause $c = (x \leq a_x, y \leq a_y, z \leq a_z)$, we introduce the gadget $G_c$ depicted in Figure 2b consisting of two vertex-disjoint copies $H_c$ and $H_{\overline{c}}$ of the same graph. More precisely, for $\ell \in \{c, \overline{c}\}$, $H_\ell$ consists of a $K_{1,3}$ on the vertex set $\{\ell, v_\ell, p_\ell^1, p_\ell^2\}$, where $v_\ell$ has degree three, and a path $P_{\ell \ell}$ of length $d$ connects $\ell$ to $b_\ell$.

The subgraph of $G_c$ induced by $\{\ell, v_\ell, p_\ell^1, p_\ell^2 \mid \ell \in \{c, \overline{c}\}\}$ is referred to as the core of $G_c$.

- We further connect $G_c$ to $G_x, G_y, \text{ and } G_z$ as follows.

  - For every $t \in \{x, y, z\}$, we connect $b^c$ to $u_t^1$ by a path $P_{t,c}^1$ of length $4d - a_t$, and $v^c$ to $u_t^2$ by a path $P_{t,c}^2$ of length $4d + a_t - 1$. Furthermore, letting $w_{t,c}^c$ be the neighbor of $v^c$ on $P_{2,c}^1$, we attach a copy $W_{t,c}^c$ of $K_{1,3}$ to $w_{t,c}^c$ by identifying $w_{t,c}^c$ with one of the leaves.

  We denote by $t_{1,c}^c$ and $t_{2,c}^c$ the two remaining leaves and refer to $W_{t,c}^c$ as a pendant claw.

  - Similarly, for every $t \in \{x, y, z\}$, we connect $b^\overline{c}$ to $u_t^1$ by a path $P_{t,\overline{c}}^1$ of length $3d + a_t$, and $v^\overline{c}$ to $u_t^2$ by a path $P_{t,\overline{c}}^2$ of length $5d - a_t$. Furthermore, letting $w_{t,\overline{c}}^\overline{c}$ be the neighbor of $v^\overline{c}$ on $P_{2,\overline{c}}^1$, we attach a copy $W_{t,\overline{c}}^\overline{c}$ of $K_{1,3}$ to $w_{t,\overline{c}}^\overline{c}$ by identifying $w_{t,\overline{c}}^\overline{c}$ with one of the leaves.

  We denote by $t_{1,\overline{c}}^\overline{c}$ and $t_{2,\overline{c}}^\overline{c}$ the two remaining leaves and refer to $W_{t,\overline{c}}^\overline{c}$ as a pendant claw.

- Finally, we introduce a path $P = t_1pt_2$ which we connect to the clause gadgets as follows.
For every clause $c \in C$ and $\ell \in \{c, \overline{c}\}$, we connect $p$ to $v^\ell$ by a path $P_\ell$ of length $2d$.

Furthermore, letting $w^\ell$ be the neighbor of $p$ on $P_\ell$, we attach a copy $W^\ell$ of $K_{1,3}$ to $w^\ell$ by identifying $w^\ell$ with one of the leaves.

We denote by $t^\ell_1$ and $t^\ell_2$ the two remaining leaves and refer to $W^\ell$ as a pendant claw.

This concludes the construction of $G$ (see Figure 3). We set $k = |X| + 10|C| + 1$, and return $(G, k)$ as an instance of Metric Dimension.

Observe that the feedback vertex set number of $G$ is at most $2|X| + 1$: indeed, removing $\{p\} \cup \{u^x_1, u^x_2 \mid x \in X\}$ from $G$ results in a graph without cycles.

We next show that $(G, k)$ is a Yes-instance for Metric Dimension if and only if the instance $(X, C, d)$ is satisfiable.

**Lemma 1.** If $(G, k)$ is a Yes-instance for Metric Dimension then $(X, C, d)$ is a Yes-instance for NAE-Integer-3-Sat.

**Proof.** Assume that $(G, k)$ is a Yes-instance for Metric Dimension and let $S$ be a resolving set of size at most $k$. By Observation 2.1, for any clause $c \in C$,

$$|S \cap \{p^c_1, p^c_2\}| \geq 1, \ |S \cap \{p^\overline{c}_1, p^\overline{c}_2\}| \geq 1, \ |S \cap \{t^c_1, t^c_2\}| \geq 1, \ \text{and} \ |S \cap \{t^\overline{c}_1, t^\overline{c}_2\}| \geq 1. \ (1)$$

Also, by Observation 2.1, for any clause $c \in C$ and any variable $x \in X$ appearing in $c$,

$$|S \cap \{t^{x,c}_1, t^{x,c}_2\}| \geq 1 \ \text{and} \ |S \cap \{t^{x,\overline{c}}_1, t^{x,\overline{c}}_2\}| \geq 1. \ (2)$$

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For the same reason,
\[ |S \cap \{t_1, t_2\}| \geq 1. \] (3)

Consider now a variable \( x \in X \). Since any path from a vertex in \( V(G) \setminus V(G_x) \) to a vertex in \( \{v_i^x, w_i^x \mid i \in [d]\} \) contains \( u_i^x \) or \( u_i^x \), and, for any \( i \in [d] \) and \( u \in \{u_i^x, u_i^y\} \),
\[ d(u, v_i^x) = d(u, w_i^x), \] no vertex in \( V(G) \setminus \{v_i^x, w_i^x \mid i \in [d]\} \) can resolve \( v_i^x \) and \( w_i^x \) for any \( i \in [d] \). It follows that
\[ |S \cap \{v_i^x, w_i^x \mid i \in [d]\}| \geq 1. \] (4)

Now, note that \( S \) has size at most \( k = |X| + 10|C| + 1 \), and so, equality must in fact hold in every inequality of Equations (1) to (4). Without loss of generality, let us assume that \( t_1 \in S \) and that, for every clause \( c \in C \) and variable \( x \in X \) appearing in \( c \), we have that \( p_i^1, p_i^2, t_i^1, t_i^2, t_i^3, t_i^4 \in S \).

For every variable \( x \in X \), assume, without loss of generality, that \( S \cap \{v_i^x, w_i^x \mid i \in [d]\} = S \cap \{v_i^x \mid i \in [d]\} \), and let \( i_x \in [d] \) be the index of the vertex in \( S \cap \{v_i^x \mid i \in [d]\} \). We contend that the assignment which sets each variable \( c \) from Claim 3.1(i); and if \( a \) from Claim 3.2(i) and (ii), if \( \{c_i, c_j, c_k\} \) and \( \{c_i, c_j, c_k\} \). Indeed, consider a clause \( c = (x \leq a_x, y \leq a_y, z \leq a_z) \). We first aim to show that, for every \( w \in S \setminus \{V(G_x) \cup V(G_y) \cup V(G_z)\} \) and \( \ell \in \{c, \overline{c}\} \), \( d(w, \ell) = d(w, p_i^1) \). Note that it suffices to show that any shortest path from \( w \in S \setminus \{V(G_x) \cup V(G_y) \cup V(G_z)\} \) to \( \ell \in \{c, \overline{c}\} \) contains \( v_i^x \), as then \( d(w, \ell) = d(w, v_i^x) + 1 = d(w, p_i^1) \). Now, if \( w \in V(G_t) \) for some \( t \in \{c, \overline{c} \mid c \in C\} \) different from \( \ell \), then this readily follows from Claim 3.1(i); and if \( w \in V(G_t) \) for some \( t \in X \setminus \{x, y, z\} \), then this readily follows from Claim 3.1(iv). If \( w = t_i^r \) for some \( r \in X \) and \( q \in \{c, \overline{c} \mid c \in C\} \), then \( d(t_i^r, \ell) = d(t_i^r, v_i^x) + d(v_i^x, \ell) \), and so, by Claim 3.1(i), any path from \( w \) to \( \ell \) contains \( v_i^x \). Finally, if \( w \in \{t_i^1, t_i^2 \mid c \in C\} \), then clearly any shortest path from \( w \) to \( \ell \) contains \( v_i^x \).

Since \( S \) is a resolving set, it follows that, for every clause \( c \in C \), there exist \( t, f \in \{x, y, z\} \) such that \( d(v_i^x, c) \neq d(v_i^x, p_i^2) \) and \( d(v_i^x, c) \neq d(v_i^x, p_i^2) \). Now, by Claim 3.2(i) and (ii), if \( i_t > a_t \), then
\[ d(v_i^x, c) = 5d + 1 + a_t - i_t = d(v_i^x, v_i^x) + 1 = d(v_i^x, p_i^2), \] a contradiction to our assumption. Therefore, \( i_t \leq a_t \). Similarly, if \( i_f \leq a_f \), then by Claim 3.3(i) and (ii),
\[ d(v_i^x, \overline{c}) = 5d + 1 + i_f - a_f = d(v_i^x, v_i^x) + 1 = d(v_i^x, p_i^2), \]
a contradiction to our assumption. Therefore, \( i_f > a_f \), and so, the assignment constructed above indeed satisfies every clause in \( C \) which concludes the proof.

**Lemma 2.** If \((X, C, d)\) is a Yes-instance for NAE-INTEGER-3-SAT then \((G, k)\) is a Yes-instance for METRIC DIMENSION.
Proof. Assume that \((X, C, d)\) is satisfiable and let \(\phi : X \to \{1, \ldots, d\}\) be an assignment of the variables satisfying every clause in \(C\). We construct a resolving set \(S\) of \(G\) as follows. First, we add \(t_1\) to \(S\). For every variable \(x \in X\), we add \(v_{\phi(x)}^x\) to \(S\). For every clause \(c \in C\), we add \(p_{\ell c}^c, p_{\bar{\ell} c}^c, t_{\ell}^c, t_{\bar{\ell}}^c\) to \(S\) and further add, for every variable \(t\) appearing in \(c\), \(t_{\phi(t)}^c\) to \(S\). Note that \(|S| = k\) and that every vertex of \(S\) is distinguished by itself. Let us show that \(S\) is indeed a resolving set of \(G\). To this end, we first prove the following.

**Claim 3.1.** For any two distinct \(s, t \in \{c, \bar{c} \mid c \in C\}\) and any two distinct variables \(x, y \in X\), the following hold.

(i) The shortest path from \(H_s\) to \(H_t\) contains \(P_s\) and \(P_t\) as subpaths and has length \(4d\).

(ii) \(d(V(G_x), V(G_y)) \geq 6d\).

(iii) If \(x\) appears in the clause corresponding to \(s\), then \(d(V(G_x), V(H_s)) \geq 3d\).

(iv) If \(x\) does not appear in the clause corresponding to \(s\), then any shortest path from \(G_x\) to \(H_s\) contains \(P_s\) as a subpath and has length at least \(8d\).

Proof. Observe first that since any path \(P\) from \(G_x\) to a clause or another variable gadget contains, as a subpath, some path \(P_{x, \ell}^c\), where \(\ell \in \{2\}\) and \(\ell \in \{c, \bar{c}\}\) for some clause \(c\) containing \(x\), then

\[\text{lgt}(P) \geq \text{lgt}(P_{x, \ell}^c) \geq 3d.\]

This implies, in particular, that

\[\min_{c \in C} d(V(G_x), V(G_c)) \geq 3d\]  \hspace{1cm} (5)

and

\[d(V(G_x), V(G_y)) \geq \min_{c \in C} d(V(G_x), V(G_c)) + \min_{c \in C} d(V(G_y), V(G_c)) \geq 6d.\]

Thus, items (ii) and (iii) hold true. To prove item (i), let \(P\) be a shortest path from \(H_s\) to \(H_t\). Observe that \(\text{lgt}(P) \leq 4d\) since \(P_s\) together with \(P_t\) form a path from \(H_s\) to \(H_t\) of length \(4d\). If \(P\) does not contain \(P_s\) as a subpath, then there exist \(i \in \{2\}\) and a variable \(x\) contained in the clause corresponding to \(s\) such that \(P\) contains \(P_{x, i}^c\) as a subpath. But then,

\[\text{lgt}(P) \geq \text{lgt}(P_{x, i}^c) + d(V(G_x), V(H_t)) \geq 3d + 3d\]

by Equation (5), a contradiction to the above observation. Thus, \(P\) contains \(P_s\) as a subpath and we conclude, by symmetry, that \(P\) also contains \(P_t\) as a subpath. Finally,
to prove item (iv), let $P$ be a shortest path from $G_x$ to $H_s$, and let $c \in C$ be a clause containing $x$. Then,

$$\text{lgt}(P) \leq \text{lgt}(P_s) + \text{lgt}(P_c) + d(v^c, V(G_x)) \leq 4d + \min\{d + P_{1}^{x,c}, P_{2}^{x,c}\} < 4d + 5d.$$ 

Now, if $P$ does not contain $P_s$ as a subpath, then there exist $i \in [2]$ and a variable $z \in X$ contained in the clause corresponding to $s$ such that $P$ contains $P_{i}^{z,s}$; in particular, since $z \neq x$ by assumption,

$$\text{lgt}(P) \geq \text{lgt}(P_{i}^{z,s}) + d(V(G_z), V(G_x)) \geq 3d + 6d,$$

a contradiction to the above. Thus, $P$ contains $P_s$ as a subpath, and so,

$$\text{lgt}(P) = \text{lgt}(P_s) + d(p, V(G_x)) = 2d + 2d + \min_{\ell \in \{c, \overline{c} \mid c \in C\}} d(v^\ell, V(G_x)) \geq 4d + 4d,$$

which proves item (iv).

Claim 3.2. For every clause $c = (x \leq a_x, y \leq a_y, z \leq a_z)$ and every $t \in \{x, y, z\}$, the following hold.

(i) For every $i \in \{0, \ldots, d + 1\}$, if $i \leq a_i$, then the shortest path from $v^t_i$ to $c$ contains $P_{1}^{t,c}$ as a subpath and has length $5d + i - a_i$. Otherwise, the shortest path from $v^t_i$ to $c$ contains $P_{2}^{t,c}$ as a subpath and has length $5d + 1 + a_i - i$.

(ii) For every $i \in \{0, \ldots, d + 1\}$, if $i \leq a_i - 1$, then the shortest path from $v^t_i$ to $v^c$ contains $P_{1}^{t,c}$ as a subpath and has length $5d + 1 + i - a_i$. Otherwise, the shortest path from $v^t_i$ to $v^c$ contains $P_{2}^{t,c}$ as a subpath and has length $5d + a_i - i$.

(iii) For every $i \in \{0, \ldots, d + 1\}$, if $i \leq a_i - 2$, then the shortest path from $v^t_i$ to $t^c_{i-1}$ contains $P_{1}^{t,c}$ as a subpath and has length $5d + 4 + i - a_i$. Otherwise, the shortest path from $v^t_i$ to $t^c_{i-1}$ contains $P_{2}^{t,c}[u^t_i, w^t_c]$ as a subpath and has length $5d + 1 + a_i - i$.

Proof. Consider $i \in \{0, \ldots, d + 1\}$. Then,

$$d(v^t_i, c) \leq d(v^t_i, u^t_i) + \text{lgt}(P_{1}^{t,c}) + d(b^c, c) = i + 4d - a_i + d$$

and

$$d(v^t_i, c) \leq d(v^t_i, u^t_2) + \text{lgt}(P_{2}^{t,c}) + 1 = d - i + 1 + 4d + a_i - 1 + 1.$$

Now, if a shortest path $P$ from $v^t_i$ to $c$ does not contain $P_{1}^{t,c}$ nor $P_{2}^{t,c}$ as a subpath, then $P$ necessarily contains, as a subpath, a path $P_{j}^{t,\ell}$ for some $j \in [2]$ and $\ell \in \{c', \overline{c}, \overline{c'}\}$, where $c'$ is a clause containing $t$. In particular,

$$\text{lgt}(P) \geq \text{lgt}(P_{j}^{t,\ell}) + d(V(H_t), V(H_c)) \geq 3d + 3d.$$
But, \( \min\{5d + i - a_t, 5d + 1 + a_t - i\} < 6d \) as \( a_t \in [d] \) and \( i \in \{0, \ldots, d + 1\} \), a contradiction to the above. Thus, any shortest path from \( v_i^t \) to \( c \) contains \( P_{11}^{t,c} \) or \( P_{22}^{t,c} \) as a subpath, and so,

\[
d(v_i^t, c) = \min\{5d + i - a_t, 5d + 1 + a_t - i\}.
\]

Hence, since \( 5d + i - a_t \leq 5d + 1 + a_t - i \) if and only if \( i \leq a_t + 1/2 \), item (i) follows. Now, as above, we may argue that any shortest path from \( v_i^t \) to \( c \) contains \( P_{11}^{t,c} \) or \( P_{22}^{t,c} \) as a subpath, and so,

\[
d(v_i^t, c) = \min\{d(v_i^t, u_1^t) + \text{lgt}(P_{11}^{t,c}) + d(b^t, v^c), d(v_i^t, u_2^t) + \text{lgt}(P_{22}^{t,c})\}
\]

\[
= \min\{i + 4d - a_t + 1, d - i + 1 + 4d + a_t - 1\}.
\]

Hence, since \( 5d + 1 + i - a_t \leq 5d + 1 + a_t - i \) if and only if \( i \leq a_t + 3/2 \), item (ii) follows. Finally, by arguing as above, we can show that any shortest path from \( v_i^t \) to \( t_{11}^{t,c} \) contains \( P_{11}^{t,c} \) or \( P_{22}^{t,c}[u_2, w^{t,c}] \) as a subpath, and so,

\[
d(v_i^t, t_{11}^{t,c}) = \min\{d(v_i^t, u_1^t) + \text{lgt}(P_{11}^{t,c}) + d(b^t, t_{11}^{t,c}), d(v_i^t, u_2^t) + \text{lgt}(P_{22}^{t,c})[u_2, w^{t,c}] + 2\}
\]

\[
= \min\{i + 4d - a_t + 1, d - i + 1 + 4d + a_t + 2\}.
\]

Hence, since \( 5d + 4 + i - a_t \leq 5d + 4 + a_t - i \) if and only if \( i \leq a_t - 1/2 \), item (iii) follows. \( \square \)

Claim 3.3. For every clause \( c = (x \leq a_x, y \leq a_y, z \leq a_z) \) and every \( t \in \{x, y, z\} \), the following hold.

(i) For every \( i \in \{0, \ldots, d + 1\} \), if \( i \leq a_t \), then the shortest path from \( v_i^t \) to \( \overline{c} \) contains \( P_{11}^{t,\overline{c}} \) as a subpath and has length \( 5d + 1 + i - a_t \). Otherwise, the shortest path from \( v_i^t \) to \( \overline{c} \) contains \( P_{22}^{t,\overline{c}} \) as a subpath and has length \( 5d + 1 + a_t - i \).

(ii) For every \( i \in \{0, \ldots, d + 1\} \), if \( i \leq a_t + 1 \), then the shortest path from \( v_i^t \) to \( v^c \) contains \( P_{11}^{t,c} \) as a subpath and has length \( 5d + i - a_t \). Otherwise, the shortest path from \( v_i^t \) to \( v^c \) contains \( P_{22}^{t,c} \) as a subpath and has length \( 5d + 1 + a_t - i \).

(iii) For every \( i \in \{0, \ldots, d + 1\} \), if \( i \leq a_t + 2 \), then the shortest path from \( v_i^t \) to \( t_{11}^{t,c} \) contains \( P_{11}^{t,c}[u_1^t, w^{t,c}] \) as a subpath and has length \( 5d + 1 + i - a_t \). Otherwise, the shortest path from \( v_i^t \) to \( t_{11}^{t,c} \) contains \( P_{22}^{t,c} \) as a subpath and has length \( 5d + 5 + a_t - i \).

Proof. As in the proof of Claim 3.2, we may argue that for every \( i \in \{0, \ldots, d + 1\} \), any shortest path from \( v_i^t \) to \( \overline{c} \) (or \( v^c \)) contains \( P_{11}^{t,\overline{c}} \) or \( P_{22}^{t,\overline{c}} \) as a subpath. Similarly, any shortest path from \( v_i^t \) to \( t_{11}^{t,c} \) contains \( P_{11}^{t,c}[u_1^t, w^{t,c}] \) or \( P_{22}^{t,c} \) as a subpath. It follows that

\[
d(v_i^t, \overline{c}) = \min\{d(v_i^t, u_1) + \text{lgt}(P_{11}^{t,\overline{c}}) + 1, d(v_i^t, u_2) + \text{lgt}(P_{22}^{t,\overline{c}}) + \text{lgt}(P_{b,\overline{c}})\}
\]

\[
= \min\{i + 5d - a_t + 1, d - i + 1 + 3d + a_t + d\}.
\]
Hence, since $5d + 1 + i - a_t \leq 5d + 1 + a_t - i$ if and only if $i \leq a_t$, item (i) follows. Similarly,

$$d(v_i^t, \overline{v}) = \min\{d(v_i^t, u_1^t) + \text{lg}(P_1^t \overline{c}), d(v_i^t, u_2^t) + \text{lg}(P_2^t \overline{c}) + \text{lg}(P_{\overline{c}}) + 1\}$$

$$= \min\{i + 5d - a_t, d - i + 1 + 3d + a_t + d + 1\}.$$ 

Hence, since $5d + i - a_t \leq 5d + 2 + a_t - i$ and only if $i \leq a_t + 1$, item (ii) follows. Finally, letting $\overline{P_2^t} \overline{c}$ be the path obtained by concatenating $P_2^t \overline{c}$ and $P_{\overline{c}}$ (so that the next equation fits in one line), then

$$d(v_i^t, t_1^t \overline{c}) = \min\{d(v_i^t, u_1^t) + \text{lg}(P_1^t [u_1^t, w_1^t]), d(v_i^t, u_2^t) + \text{lg}(P_2^t \overline{c}) + \text{lg}(P_{\overline{c}}) + \text{lg}(t_1^t \overline{c})\}$$

$$= \min\{i + 5d - a_t - 1 + 2, d - i + 1 + 3d + a_t + d + 4\}.$$ 

Hence, since $5d + 1 + i - a_t \leq 5d + 5 + a_t - i$ if and only if $i \leq a_t + 2$, item (iii) follows.

Consider now two distinct vertices $u, v \in V(G)$ and let us show that $S$ resolves the pair $u, v$. To this end, we distinguish the following cases.

**Case 1.** At least one of $u$ and $v$ belongs to a pendant claw. Without loss of generality, assume first that $u \in V(W_{\phi^t})$, where $t \in \{c, \overline{c} \mid c \in C\}$. If $v \in V(G) \setminus V(W_{\phi^t})$, then $d(t_1^t, v) > 2 \geq d(t_1^t, u)$. Suppose therefore that $v \in V(W_{\phi^t})$ as well. If $\{u, v\} \neq \{w^t, t_2^t\}$, then $d(t_1^t, u) \neq d(t_1^t, v)$. If $\{u, v\} = \{w^t, t_2^t\}$, then $d(t_1^t, w^t) = 2 < 4 = d(t_1^t, t_2^t)$. Second, assume that $u \in V(W_{\phi^t})$, where $t \in \{c, \overline{c}\}$ for some clause $c \in C$ and $t$ is a variable appearing in clause $c$. If $v \in V(G) \setminus V(W_{\phi^t})$, then $d(t_1^t, v) > 2 \geq d(t_1^t, u)$. Suppose therefore that $v \in V(W_{\phi^t})$ as well. If $\{u, v\} \neq \{w^t, t_2^t\}$, then $d(t_1^t, u) \neq d(t_1^t, v)$. If $\{u, v\} = \{w^t, t_2^t\}$, then $d(p_1^t, w^t) = 2 < 4 = d(p_1^t, t_2^t)$.

**Case 2.** At least one of $u$ and $v$ belongs to the core of a clause gadget. Assume, without loss of generality, that $u \in \{t, v^t, p_1^t, p_2^t\}$, where $t \in \{c, \overline{c}\}$ for some clause $c = (x \leq a_x, y \leq a_y, z \leq a_z)$. If $v$ is not a neighbor of $v^t$, then $d(p_1^t, v) > 2 \geq d(p_1^t, u)$. If $v$ is the neighbor of $v^t$ on the path $P_t$, then $d(t_1^t, v) = d < d(t_1^t, u)$. Also, $v = w^t, t_1^t$ was covered by the previous case. So, consider $v \in \{t, v^t, p_1^t, p_2^t\}$. If $\{u, v\} \neq \{t, p_2^t\}$, then clearly $d(p_1^t, u) \neq d(p_1^t, v)$. Assume therefore that $\{u, v\} = \{t, p_2^t\}$. Since $\phi$ satisfies $c$, there exist $t, f \in \{x, y, z\}$ such that $\phi(t) \leq a_t$ and $\phi(f) > a_f$. Then, either $\phi(t) < a_t$, in which case, by Claim 3.2(i) and (ii),

$$d(v_i^t, c) = 5d + \phi(t) - a_t < 5d + \phi(t) - a_t + 2 = d(v_i^t, v^t) + 1 = d(v_i^t, p_2^t),$$

or $\phi(t) = a_t$, in which case, by Claim 3.2(i) and (ii),

$$d(v_i^t, c) = 5d + 1 = d(v_i^t, v^t) + 1 = d(v_i^t, p_2^t).$$

Similarly, either $\phi(f) = a_f + 1$, in which case, by Claim 3.3(i) and (ii),

$$d(v_i^f, c) = 5d + 2 = d(v_i^f, v^t) + 1 = d(v_i^f, p_2^t).$$
or $\phi(f) > a_f + 1$, in which case, by Claim 3.3(i) and (ii),
\[ d(v_{\phi(f)}^x, c) = 5d + 1 + a_f - \phi(f) < 5d + 3 + a_f - \phi(f) = d(v_{\phi(f)}^x, v^x) + 1 = d(v_{\phi(f)}^x, p_{\phi(f)}^2). \]

In all cases, we conclude that there exists $w \in S$ such that $d(w, \ell) \neq d(w, p_2^x)$.

**Case 3.** At least one of $u$ and $v$ belongs to a variable gadget. Assume, without loss of generality, that $u \in V(G_x)$ for some variable $x \in X$. By the previous cases, we may assume that $v$ does not belong to the core of a clause gadget or a pendant claw.

If $v \in V(G_y)$ for some variable $y \neq x$, then by Claim 3.1(ii),
\[ d(v_{\phi(x)}^x, u) \leq d + 1 < 6d \leq d(V(G_x), V(G_y)) \leq d(v_{\phi(x)}^x, v). \]

Now, suppose that $v \in V(G_x)$ as well. If \( \{ u, v \} = \{ v_i, w_1^x \} \) for some $i \in [d]$, then
\[ d(v_{\phi(x)}^x, v_i^x) = |\phi(x) - i| < d(v_{\phi(x)}^x, w_1^x) = \min \{ \phi(x) + i, 2d + 2 - \phi(x) - i \}. \]

Suppose next that $u = v_i^x$ and $v = v_j^x$ for two distinct $i, j \in \{ 0, \ldots, d + 1 \}$, say $i < j$, without loss of generality. Consider a clause $c = (x \leq a_x, y \leq a_y, z \leq a_z)$ containing $x$. If $j < a_x$, then by Claim 3.2(ii),
\[ d(p_i^c, v_i^x) = d(v_c, v_i^x) + 1 = 5d + 2 + i - a_x < 5d + 2 + j - a_x = d(v_c, v_j^x) + 1 = d(p_i^c, v_j^x). \]

Now, suppose that $i < a_x \leq j$. Then, by Claim 3.2(ii),
\[ d(p_i^c, v_i^x) = d(v_c, v_i^x) + 1 = 5d + 2 + i - a_x - (5d + a_x - j + 1) = i + j + 1 - 2a_x. \]

Thus, if $i + j + 1 - 2a_x \neq 0$, then $d(p_i^c, v_i^x) \neq d(p_i^c, v_j^x)$. Now, if $i + j + 1 - 2a_x = 0$, then either $j = a_x$ and $i = a_x - 1$, in which case, by Claim 3.2(iii),
\[ d(t_i^{c,x}, v_i^x) = 5d + 2 > 5d + 1 = d(t_i^{c,x}, v_j^x), \]
or $j > a_x$ and $i < a_x - 1$, in which case, by Claim 3.2(iii),
\[ d(t_i^{c,x}, v_j^x) = 5d + 1 + a_x - j = 5d + 2 + i - a_x < 5d + 4 + i - a_x = d(t_i^{c,x}, v_i^x). \]

Finally, if $a_x \leq i < j$, then by Claim 3.2(ii),
\[ d(p_i^c, v_j^x) = 5d + 1 + a_x - j < 5d + 1 + a_x - i = d(p_i^c, v_i^x). \]

Since for any $t \in V(G) \setminus V(G_x)$ and $k \in [d]$, $d(t, v_k^x) = d(t, w_k^x)$, we conclude similarly if either $u = v_i^x$ and $v = w_j^x$, or $u = w_i^x$ and $v = w_j^x$ for two distinct $i, j \in \{ 0, \ldots, d + 1 \}$.

Assume, henceforth, that $v \notin \bigcup_{x \in X} V(G_x)$. If $v$ does not belong to a path connecting $G_x$ to some clause gadget, then
\[ d(v_{\phi(x)}^x, v) \geq \min_{c \in C} d(V(G_x), V(G_c)) \geq 3d \geq d + 1 \geq d(v_{\phi(x)}^x, u) \]

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by Claim 3.1(iii) and (iv). Suppose therefore that \( v \in V(P_i^{x,c}) \), where \( i \in \{1, 2\} \) and \( \ell \in \{c, \overline{c}\} \) for some clause \( c = (x \leq a_x, y \leq a_y, z \leq a_z) \) containing \( x \). Without loss of generality, let us assume that \( u = v_j^c \) where \( j \in \{0, \ldots, d + 1\} \).

Assume first that \( \ell = c \) and \( i = 1 \). Let \( P_1^{x,c} = z_0 \ldots z_{4d-ax} \), where \( z_0 = b^{c} \) and \( z_{4d-ax} = u_i^c \). Let \( v = z_k \), where \( k \in [4d+ax-1] \). If \( j \leq a_x - 1 \), then by Claim 3.2(ii), the shortest path from \( p_1^c \) to \( v_j^c \) contains \( P_1^{x,c} \) as a subpath, which implies in particular that \( d(p_1^c, v) < d(p_1^c, u) \). Suppose therefore that \( j \geq a_x \). Then, by Claim 3.2(ii),

\[
d(p_1^c, u) - d(p_1^c, v) = 5d + 1 + a_x - j - (d + k + 2).
\]

Thus, if \( 5d + 1 + a_x - j - (d + k + 2) \neq 0 \), then \( d(p_1^c, u) \neq d(p_1^c, v) \). Now, if \( 5d + 1 + a_x - j - (d + k + 2) = 0 \), then by Claim 3.2(iii),

\[
d(t_1^{x,c}, u) = 5d + 1 + a_x - j = d + k + 2 < d + k + 4 = d(t_1^{x,c}, v).
\]

Second, assume that \( \ell = c \) and \( i = 2 \). Let \( P_2^{x,c} = z_0 \ldots z_{4d+ax-1} \), where \( z_0 = v^c \) and \( z_{4d+ax-1} = u_i^c \). Let \( v = z_k \), where \( k \in [4d + ax - 2] \) (note that since \( v \) does not belong to the core of a clause gadget or a pendant claw by assumption, in fact \( k \geq 2 \)). If \( j \geq a_x \), then by Claim 3.2(ii), the shortest path from \( p_1^c \) to \( u \) contains \( P_2^{x,c} \) as a subpath, which implies in particular that \( d(p_1^c, v) < d(p_1^c, u) \). Otherwise, \( j \leq a_x - 1 \), in which case

\[
d(p_1^c, u) - d(p_1^c, v) = 5d + 2 + j - a_x - (k + 1).
\]

Thus, if \( 5d + 2 + j - a_x - (k + 1) \neq 0 \), then \( d(p_1^c, u) \neq d(p_1^c, v) \). Now, if \( 5d + 2 + j - a_x - (k + 1) = 0 \), then \( j < a_x - 1 \) since \( k < 5d \), and so, by Claim 3.2(iii),

\[
d(t_1^{x,c}, u) = 5d + 4 + j - a_x = k + 3 > k + 1 = d(t_1^{x,c}, v).
\]

Third, assume that \( \ell = \overline{c} \) and \( i = 1 \). Let \( P_1^{x,\overline{c}} = z_0 \ldots z_{5d-ax} \), where \( z_0 = v^\overline{c} \) and \( z_{5d-ax} = u_i^\overline{c} \). Let \( v = z_k \), where \( k \in [5d + ax - 1] \) (note that since \( v \) does not belong to the core of a clause gadget or a pendant claw by assumption, in fact \( k \geq 2 \)). If \( j \leq a_x + 1 \), then by Claim 3.3(ii), the shortest path from \( p_1^{\overline{c}} \) to \( v_j^c \) contains \( P_1^{x,\overline{c}} \) as a subpath which implies in particular that \( d(p_1^{\overline{c}}, v) < d(p_1^{\overline{c}}, u) \). Suppose therefore that \( j \geq a_x + 2 \). Then, by Claim 3.3(ii),

\[
d(p_1^{\overline{c}}, v_j^c) - d(p_1^{\overline{c}}, z_k) = 5d + 3 + a_x - j - (k + 1).
\]

Thus, if \( 5d + 3 + a_x - j - (k + 1) \neq 0 \), then \( d(p_1^{\overline{c}}, v_j^c) \neq d(p_1^{\overline{c}}, z_k) \). Now, if \( 5d + 3 + a_x - j - (k + 1) = 0 \), then \( j > a_x + 2 \) since \( k < 5d \), and so, by Claim 3.3(iii),

\[
d(t_1^{x,\overline{c}}, v_j^c) = 5d + 5 + a_x - j = k + 3 > k + 1 = d(t_1^{x,\overline{c}}, z_k).
\]

Assume finally that \( \ell = \overline{c} \) and \( i = 2 \). Let \( P_2^{x,\overline{c}} = z_0 \ldots z_{3d+ax} \), where \( z_0 = b^\overline{c} \) and \( z_{3d+ax} = u_i^\overline{c} \). Let \( v = z_k \), where \( k \in [3d + ax - 1] \). If \( j \geq a_x + 2 \), then by Claim 3.3(ii),
the shortest path from \( p_i \) to \( u \) contains \( P_{2}^{z} \) as a subpath, which implies in particular that \( d(p_i, v) < d(p_i, u) \). Suppose therefore that \( j \leq a_x + 1 \). Then, by Claim 3.3(ii),
\[
d(p_i, v_j) - d(p_i, z_k) = 5d + 1 + j - a_x - (d + k + 2).
\]
Thus, if \( 5d + 1 + j - a_x - (d + k + 2) \neq 0 \), then \( d(p_i, v_j) \neq d(p_i, z_k) \). Now, if \( 5d + 1 + j - a_x - (d + k + 2) = 0 \), then \( j < a_x + 1 \) since \( k < 4d \), and so, by Claim 3.3(iii),
\[
d(t^x_{i}, v_j) = 5d + 1 + j - a_x = d + k + 2 < d + k + 4 = d(t^x_{i}, z_k).
\]
In all cases, we conclude that there exists \( w \in S \) such that \( d(w, u) \neq d(w, v) \).

**Case 4. None of the above.** First, note that \( p \) is distinguished by \( S \) since it is the unique vertex of \( G \) at distance 1 from \( t_1 \). Second, \( t_2 \) is distinguished by \( S \) since it is the unique vertex of \( G \) at distance 2 from \( t_1 \) and distance 4 from \( t_1^c \) and \( t_1^\ell \) for all \( c \in C \). Thus, in this last case, we can assume that both \( u \) and \( v \) belong to \( S \) or to some path connecting \( t_1 \) to \( S \) or to some path with \( \ell \) different from \( \ell \) or \( c \in C \).

Assume first that \( u \in V(P_1) \) for some \( \ell \in \{c, \overline{c} \mid c \in C\} \). If \( v \in V(P_2) \) as well, then surely \( d(t_1, u) \neq d(t_1, v) \). If \( v \in V(P_3) \) for some \( q \in \{c, \overline{c} \mid c \in C\} \) different from \( \ell \), then \( d(p_i, u) < d(p_i, v) \) since the unique shortest path from \( p_i \) to \( v \) contains \( P_1 \) as a subpath. Finally, if there exists \( q \in \{c, \overline{c} \mid c \in C\} \) such that \( v \) belongs to \( P_{q_2} \), or to some path connecting \( H \) to a variable gadget, then \( d(t_1, v) > d(t_1, v^q) \geq d(t_1, u) \).

Second, assume that \( u \in V(P_i^{x, \ell}) \), where \( i \in [2] \) and \( \ell \in \{c, \overline{c} \} \) for some clause \( c \) containing variable \( x \). Note that by the previous paragraph, we may assume that \( v \notin \bigcup_{c \in C} V(P_q) \cup V(P_{q_2}) \). Suppose first that \( v \in V(P_j^{y, q}) \), where \( j \in [2] \) and \( q \in \{c', \overline{c}' \} \) for some clause \( c' \) containing variable \( y \). Note that \( d(t_1, u) = d(t_1, u) \). So, if \( q \neq \ell \), then either \( d(t_1, u) \neq d(t_1, v) \), or
\[
d(t_1, v) - d(t_1, u) = d(t_1, p) + d(t_1, v) - 1 - d(t_1, u) = d(t_1, v) + 2 - d(t_1, u) = 2.
\]
Thus, assume that \( q = \ell \). Suppose first that \( x = y \). If \( i = j \), then surely \( d(p_i, u) \neq d(p_i, v) \). Otherwise, assume, without loss of generality, that \( u \) belongs to the path containing \( w^{x, \ell} \). Note that \( d(t^x_{1}, u) = d(p_i, u) \). Then, either \( d(p_i, u) \neq d(p_i, v) \), or \( d(t^x_{1}, v) - d(t^x_{1}, u) = d(t^x_{1}, v) + d(p_i, v) - 1 - d(t^x_{1}, u) = d(p_i, v) + 2 - d(p_i, u) = 2 \).

Second, suppose that \( x \neq y \). If \( u \) belongs to the path containing \( w^{x, \ell} \), then we argue as previously. By symmetry, we may also assume that \( v \) does not belong to the path containing \( w^{y, \ell} \). This implies, in particular, that \( i = j + b' \) is the endpoint in \( H \) of both \( P_i^{x, \ell} \) and \( P_j^{y, \ell} \). Thus, \( d(v_{\phi(x), u}^{x}) \leq d(v_{\phi(x), u}^{x}) + d(u^{x}, b') \leq d + 4d \). First, note that if a shortest path \( P \) from \( v_{\phi(x), u}^{x} \) to \( v \) contains \( P_{i}^{x, \ell} \) as a subpath, then since \( u \in V(P_i^{x, \ell}) \), it follows that \( d(v_{\phi(x), u}^{x}) < d(v_{\phi(x), v}^{x}) \). Hence, we may assume that \( P \)
contains a vertex in $G_y$ or both $v^\ell$ and $b^\ell$. By Claim 3.1(ii), if $P$ contains a vertex in $G_y$, then
\[ d(v^\ell_{\phi(x)}, v) > d(V(G_x), V(G_y)) \geq 6d > 5d = d(v^\ell_{\phi(x)}, u). \]
Otherwise, $P$ contains $v^\ell$ and $b^\ell$, and so, letting $t \in [2] \setminus \{i\}$, we get that
\[ \lgt(P) \geq d(v^\ell_{\phi(x)}, u^\ell_t) + d(u^\ell_t, v^\ell) + d(v^\ell, b^\ell) + d(b^\ell, v) \geq 1 + \frac{d}{2} + \frac{d}{2} + \frac{d}{2} > 5d \]
Suppose finally that $u \in V(P_{\phi'})$ for some $\ell \in \{\kappa, c | c \in C\}$. By the two previous paragraphs, we may assume that $v \in V(P_{\phi'})$ for some $q \in \{\kappa, c | c \in C\}$. If $q = \ell$, then surely $d(p^\ell_1, u) \neq d(p^\ell_1, v)$. Otherwise, by Claim 3.1(i),
\[ d(p^\ell_1, v) \geq d(V(H_\ell), V(H_q)) = 4d + d + 1 \geq d(p^\ell_1, u) \]
which concludes case 4.

By the above case analysis, we conclude that, for any $u, v \in V(G)$, there exists $w \in S$ such that $d(w, u) \neq d(w, v)$, that is, $S$ is a resolving set of $G$. Since $|S| = k$, the lemma follows.

\section{The Vertex Cover Number and the Distance to clique}

In this section, we prove that \textsc{Metric Dimension} parameterized by either the vertex cover number or the distance to clique does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP}/\text{poly}$. The two reductions are similar and start from the SAT problem, in which we are given a conjunctive normal form (CNF) formula $\phi$ on $n$ variables and $m$ clauses, and we are asked whether there exists an assignment of either true or false to each of the variables, such that $\phi$ is true (satisfied). The SAT problem is known to not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP}/\text{poly}$ \cite{38}. We first prove that \textsc{Metric Dimension} parameterized by the vertex cover number does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP}/\text{poly}$, and then prove the same result for the distance to clique by slightly modifying the latter reduction.

**Theorem 2.** \textsc{Metric Dimension} parameterized by the vertex cover number does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP}/\text{poly}$.

\textit{Proof.} By a reduction from SAT, we prove that \textsc{Metric Dimension} parameterized by the vertex cover number does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP}/\text{poly}$. Let $\phi$ be an instance of SAT, that is, $\phi$ is a SAT formula on $n$ variables $x_1, \ldots, x_n$ and $m$ clauses $C_1, \ldots, C_m$. Since any SAT formula on $n$ variables trivially has at most $3^n - 1$ unique clauses, we may assume that $m \leq 3^n - 1$. From $\phi$, we
construct an instance \((G, k)\) of \textsc{Metric Dimension} such that the instance \(\phi\) is satisfiable if and only if \((G, k)\) is a \textsc{Yes}-instance for \textsc{Metric Dimension}. Set \(k = n + \alpha + 1\).

\textbf{Reduction.} The construction of \(G\) is as follows (see Figure 4 for an illustration).

- For each \(i \in [n]\), construct a cycle \((t_ia_1^ib_1^if_1b_2^iI_i)\) on 6 vertices, and let \(I_i := \{a_1^i, a_2^i, b_1^i, b_2^i\}\).
- Construct a path \((g^1gg^2)\) on 3 vertices and for each \(i \in [n]\), make both \(f_i\) and \(t_i\) adjacent to \(g\).
- For each \(j \in [m]\), add a pair of vertices \(c_1^j\) and \(c_2^j\), and let \(C_j := \{c_1^j, c_2^j\}\).
- For each \(j \in [m]\), make \(c_2^j\) adjacent to both \(f_i\) and \(t_i\) for each \(i \in [n]\).
- For each \(j \in [m]\) and each \(i \in [n]\), if \(x_i = \text{True}\) does not satisfy the clause \(C_j\) in \(\phi\), then make \(c_1^j\) adjacent to \(t_i\), and if \(x_i = \text{False}\) does not satisfy \(C_j\), then make \(c_2^j\) adjacent to \(f_i\).
- Let \(\alpha = [n \cdot \log_2 3]\), and, for each \(\ell \in [\alpha]\), construct a path \((z_1^\ell z_2^\ell z_3^\ell)\) on 3 vertices.
- For each \(j \in [m]\), consider the binary representation \(\text{bin}(j)\) of \(j\), and connect both \(c_1^j\) and \(c_2^j\) with \(z_\ell\) if \(\text{bin}(j)[\ell] = 1\), where \([\ell]\) represents the \(\ell\)th bit of \(j\) in its binary representation going from right to left.
- Finally, construct a clique on the vertices \(z_1, \ldots, z_\alpha, g\).

This completes the construction of the graph \(G\), which is clearly achieved in polynomial time. To simplify notation for the proof, let \(T := \{t_1, \ldots, t_n\} \cup \{f_1, \ldots, f_m\}\) and \(I := I_1 \cup \cdots \cup I_n\).

\textbf{Claim 4.1.} If \(\phi\) is satisfiable then \((G, k)\) is a \textsc{Yes}-instance for \textsc{Metric Dimension}.

\textbf{Proof.} Assume that \(\phi\) is satisfiable and consider a truth assignment of the variables of \(\phi\) satisfying \(\phi\). We construct a resolving set \(R\) of \(G\) of size at most \(k = n + \alpha + 1\) as follows. Let \(R_1 := \{g^1, z_1^1, \ldots, z_\alpha^1\}\) be an ordered subset of vertices. Let \(R_2\) be an initially empty ordered subset of vertices. We add vertices to \(R_2\) in such a way that the \(i\)th added vertex is in the \(i\)th position of the ordered subset. For each \(i \in [n]\), if \(x_i = \text{True}\) (\(x_i = \text{False}\), respectively) in the considered truth assignment, then add \(a_1^i\) (\(b_1^i\), respectively) to \(R_2\). Let \(R := R_1 \cup R_2\) be the ordered subset of vertices with the vertices of \(R_1\) appearing in order before the vertices of \(R_2\) (also in order).

Let us show that \(R\) is indeed a resolving set of \(G\). Clearly, the vertices of \(R\) distinguish themselves. According to Table 1, the vertices of \(R_1\) resolve any two vertices \(u, v \in V(G)\) except in the following three cases: (1) \(\{u, v\} \subseteq T \cup \{g^2\}\); (2) \(\{u, v\} \subseteq I\) or (3) \(u \in C_j\) and \(v \in C_j \cup \{z_1^2, \ldots, z_m^2\}\) for some \(j \in [m]\). Thus, the remaining \(n\) vertices of \(R_2\) must resolve \(u\) and \(v\) in these cases; we consider below each case one by one.
Figure 4: Illustration of the graph $G$ constructed in the proof of Theorem 2. The vertices $z_1, \ldots, z_\alpha, g$ are in a clique that is not drawn. In this particular case, $\phi$ has a clause $(x_1 \vee x_2 \vee \overline{x_\alpha})$.

| Vertex set $v$ is in | $r(R_1|v)$ |
|----------------------|-------------|
| $\{g^2\}$           | $(2, 3, \ldots, 3)$ |
| $\{g\}$             | $(1, 2, \ldots, 2)$ |
| $T$                  | $(2, 3, \ldots, 3)$ |
| $I$                  | $(3, 4, \ldots, 4)$ |
| $C_j$ for some $j \in [m]$ | $(3, 3 - \text{bin}(j)[i], \ldots, 3 - \text{bin}(j)[\alpha])$ |
| $\{z^\ell\}$ for some $\ell \in [\alpha]$ | $(2, 2, \ldots, 2, 1, 2, \ldots, 2)$ |
| $\{z^2\}$ for some $\ell \in [\alpha]$ | $(3, 3, \ldots, 3, 2, 3, \ldots, 3)$ |

Table 1: Distance vector of $R_1$ with respect to $v$ for different cases of $v \in V(G)$.

**Case 1.** $\{u, v\} \subseteq T \cup \{g^2\}$. For each $i \in [n]$, there is a vertex $w_i \in \{a^1_i, b^1_i\} \cap R_2$ and so, $d(w_i, g^2) = 3$ and $d(w_i, t_s) = d(w_i, f_s) = 3$ for all $s \in [n]$ such that $s \neq i$. Moreover, if $w_i = a^1_i$ ($w_i = b^1_i$, respectively), then $d(w_i, t_i) = 1$ and $d(w_i, f_i) = 2$ ($d(w_i, t_i) = 2$ and $d(w_i, f_i) = 1$, respectively). Thus, the vertices $u$ and $v$ are resolved in this case.

**Case 2.** $\{u, v\} \subseteq I$. As in the previous case, for each $i \in [n]$, there is a vertex $w_i \in \{a^1_i, b^1_i\} \cap R_2$, and so, $d(w_i, q) = 4$ for all $q \in I \setminus I_i$. Moreover, if $w_i = a^1_i$ ($w_i = b^1_i$, respectively), then $d(w_i, b^1_i) = 1$, $d(w_i, a^2_i) = 2$, and $d(w_i, b^2_i) = 3$ ($d(w_i, a^1_i) = 1$, $d(w_i, b^1_i) = 2$, and $d(w_i, a^2_i) = 3$, respectively). Thus, the vertices $u$ and $v$ are resolved in this case.
Case 3. \( u \in C_j \) and \( v \in C_j \cup \{z_1^2, \ldots, z_m^2\} \) for some \( j \in [m] \). Let \( s \in [n] \) be such that \( x_s \) satisfies clause \( C_j \) in the considered truth assignment of the variables. By construction, there exists \( w \in \{a_s^1, b_s^1\} \cap R_2 \). We claim that \( d(w, c_j^3) = 3 \). Indeed, if \( w = a_s^1 \) (or \( w = b_s^1 \), respectively) then by construction, setting \( x_s \) to \text{True} (\text{False}, respectively) satisfies the clause \( C_j \) in \( \phi \), and hence, there is no edge between \( t_s \) (\( f_s \), respectively) and \( c_j^3 \) in \( G \). Note that for all \( i \in [n], j \in [m], \) and \( \ell \in [\alpha] \), for any vertex \( q \in I, d(q, c_j^2) = 2 \) and \( d(q, z_\ell^2) = 4 \). Thus, the vertices \( u \) and \( v \) are resolved in this case.

Since in every case, the vertices \( u \) and \( v \) are resolved by \( R \), we conclude that \( R \) is indeed a resolving set; and since \( |R| = n + \alpha + 1 \), the claim follows.

Claim 4.2. If \((G, k)\) is a Yes-instance for Metric Dimension then \( \phi \) is satisfiable.

Proof. Assume that \((G, k)\) is a Yes-instance for Metric Dimension and let \( R \) be a resolving set of size at most \( k \). By Observation 2.1, \( R \) contains at least one vertex in \( \{g_1, g_2\} \) and at least one vertex in \( \{z_1^2, z_2^2\} \) for every \( \ell \in [\alpha] \). Furthermore, since for every \( i \in [n], d(a_i^1, t_i) = d(a_i^2, t_i), d(a_i^1, f_i) = d(a_i^2, f_i), \) and \( t_i \) and \( f_i \) separate \( I_i \) from the rest of \( G \), necessarily \( R \cap I_i \neq \emptyset \). Since \(|R| \leq n + \alpha + 1 \), this implies, in particular, that \(|R \cap \{g_1, g_2\}| = 1, |R \cap \{z_1^2, z_2^2\}| = 1 \) for all \( \ell \in [\alpha] \), and \(|R \cap I_i| = 1 \) for every \( i \in [n] \).

Without loss of generality, let us assume that \( \{g_1, z_1^1, \ldots, z_\alpha^1\} \subset R \). Consider \( j \in [m] \). As shown in Table 1, no vertex in \( \{g_1, z_1^1, \ldots, z_\alpha^1\} \) can resolve two vertices of \( C_j \), and thus, there must exist \( w \in R \cap I \) such that \( d(w, c_j^1) \neq d(w, c_j^2) \). Since for every \( i \in [n] \) such that \( x_i \) does not appear in the clause \( C_j \), \( d(u, c_j^1) = d(u, c_j^2) \) for every \( u \in I_i \), there must exist \( i \in [n] \) such that \( x_i \) appears in the clause \( C_j \) and \( w \in R \cap I_i \). In particular, \( c_j^1 \) must be nonadjacent to one of \( t_i \) and \( f_i \). Now, if \( c_j^2 \) is nonadjacent to \( t_i \), then, by construction, \( c_j^3 \) is adjacent to \( f_i \), and so, \( w \in \{a_1^1, a_2^1\} \) as otherwise \( d(w, c_j^1) = d(w, c_j^2) \). Symmetrically, if \( c_j^2 \) is nonadjacent to \( f_i \), then, by construction, \( c_j^3 \) is adjacent to \( t_i \), and so, \( w \in \{b_1^1, b_2^1\} \) as otherwise \( d(w, c_j^1) = d(w, c_j^2) \). It follows that the truth assignment obtained by setting a variable \( x_i \) to \text{True} if \( R \cap I_i \subset \{a_1^1, a_2^1\} \), and to \text{False} otherwise, satisfies every clause.

Lastly, note that the vertex cover number of \( G \) is at most \( 4n + \alpha + 1 \) since \( \{g\} \cup T \cup \{a_1, \ldots, a_n\} \cup \{a_1^2, \ldots, a_n^2\} \cup \{z_1, \ldots, z_\alpha\} \) is a vertex cover of \( G \). Hence, the existence of a polynomial kernel for Metric Dimension parameterized by the vertex cover number of \( G \) would contradict the fact that there is no polynomial kernel for SAT parameterized by the number of its variables, which concludes the proof.

Theorem 3. Metric Dimension parameterized by the distance to clique does not admit a polynomial kernel unless \( \text{NP} \subseteq \text{coNP/poly} \).

Proof. We slightly modify the reduction given in the proof of Theorem 2: to the previously constructed graph \( G \) in the proof of Theorem 2, we simply add edges so that the vertices of \( C_1, \ldots, C_m \) induce a clique \( K \) on \( 2m \) vertices. Then the exact
same proof as that of Theorem 2 still applies here since all of the distances mentioned in the proof of Theorem 2 remain the same in this construction.

Finally, it is easy to see that the distance to a clique of $G$ is at most $6n + 3\alpha + 3$ since deleting every vertex in $G$ except for the vertices of $K$ results in a clique. Thus, the existence of a polynomial kernel for Metric Dimension parameterized by the distance to clique would contradict the fact that there is no polynomial kernel for SAT parameterized by the number of its variables, which concludes the proof.

\[ \square \]

5 The Distance to Cluster and the Distance to co-cluster

In this section, we prove that Metric Dimension is FPT parameterized by either the distance to cluster or the distance to co-cluster. In fact, we show that the problem admits an exponential kernel parameterized by the distance to cluster (or co-cluster). Since the main ideas for these two parameters are the same, we focus on the distance to cluster parameter. We remark that applying Reduction Rule 5.2 for false twins (instead of true twins) and defining equivalence classes over the independent sets (instead of cliques) for Reduction Rule 5.3, we indeed get the similar result for the distance to co-cluster. Recall that, for a graph $G$, the distance to cluster of $G$ is the minimum number of vertices of $G$ that need to be deleted so that the resulting graph is a cluster graph, that is, a disjoint union of cliques.

Theorem 4. Metric Dimension is FPT parameterized by the distance to cluster.

Proof. Let $(G, k)$ be an instance of Metric Dimension and let $X \subseteq V(G)$ be such that $G - X$ is a disjoint union of cliques. To obtain a kernel for the problem, we present a set of reduction rules. The safeness of the following reduction rule is trivial.

Reduction Rule 5.1. If $V(G) \neq \emptyset$ and $k \leq 0$ then return a trivial No-instance.

Reduction Rule 5.2. If there exist $u, v, w \in V(G)$ such that $u, v, w$ are true (or false) twins, then remove $u$ from $G$ and decrease $k$ by one.

Claim 5.1. Reduction Rule 5.2 is safe.

Proof. Assume that there exist $u, v, w \in V(G)$ such that $u, v, w$ are (true or false) twins. We first claim that there exists a minimum resolving set of $G$ that contains $u$. Indeed, let $S$ be a minimum resolving set of $G$ and suppose that $u \notin S$. Then, by Observation 2.1, $v, w \in S$. Let $S_u = (S \setminus \{v\}) \cup \{u\}$. We contend that $S_u$ is also a resolving set of $G$. Indeed, towards a contradiction, suppose that there exist $x, y \in V(G)$ such that no vertex in $S_u$ resolves $x$ and $y$. Since $S$ is a resolving set of $G$, it must then be that $v$ resolves $x$ and $y$. But, by Observation 2.1, $\text{dist}(u, x) = \text{dist}(v, x) \neq \text{dist}(v, y) = \text{dist}(u, y)$, and so, $u$ resolves $x$ and $y$, a contradiction to our assumption. Thus, $S_u$ is indeed a resolving set.
We next show that \( G \) has a resolving set of size at most \( k \) if and only if \( G - \{u\} \) has a resolving set of size at most \( k - 1 \). Assume first that \( G \) has a resolving set \( S \) of size at most \( k \). By the above paragraph, we may assume that \( u \in S \). Furthermore, by Observation 2.1, at least one \( v \) and \( w \) belongs to \( S \) as well, say \( v \in S \) without loss of generality. But then, \( S \setminus \{u\} \) is a resolving set of \( G - \{u\} \): indeed, for any \( x, y \in V(G) \) such that \( \text{dist}(u, x) \neq \text{dist}(u, y) \), we have that \( \text{dist}(v, x) \neq \text{dist}(v, y) \) by Observation 2.1. Conversely, if \( G - \{u\} \) has a resolving set \( S \) of size at most \( k - 1 \), then it is not difficult to see that \( S \cup \{u\} \) is a resolving set of \( G \).

We assume, henceforth, that Reduction Rule 5.2 has been exhaustively applied to \((G, k)\). This implies, in particular, that for every clique \( C \) of \( G - X \), there are at most two vertices in \( C \) with the same neighborhood in \( X \). Since the number of distinct neighborhoods in \( X \) is at most \( 2^{\vert X \vert} \), each clique in \( G - X \) has order at most \( 2^{\vert X \vert+1} \). We now aim to bound the number of cliques in \( G - X \). To this end, we define a notion of equivalence classes over the set of cliques of \( G - X \). It will easily be seen that the number of equivalence classes is at most \( 2^{2^{\vert X \vert+1}} \). The number of cliques in each equivalence class will then be bounded by using Reduction Rule 5.3.

For every clique \( C \) of \( G - X \), the signature \( \text{sign}(C) \) of \( C \) is the multiset containing the neighborhoods in \( X \) of each vertex of \( C \), that is, \( \text{sign}(C) = \{ N(u) \cap X : u \in C \} \). For any two cliques \( C_1, C_2 \) of \( G - X \), we say that \( C_1 \) and \( C_2 \) are identical, which we denote by \( C_1 \sim C_2 \), if and only if \( \text{sign}(C_1) = \text{sign}(C_2) \). It is not difficult to see that \( \sim \) is in fact an equivalence relation with at most \( 2^{2^{\vert X \vert+1}} \) equivalence classes: indeed, since the number of distinct neighborhoods in \( X \) is at most \( 2^{\vert X \vert} \), and at most two vertices of each clique have the same neighborhood in \( X \), the number of distinct signatures is at most \( 2^{2^{\vert X \vert+1}} \). Consider now an equivalence class \( C \) of \( \sim \). Note that since the signature of a clique is a multiset, the number of vertices in each \( C \in \mathcal{C} \) is equal to \( |\text{sign}(C)| \). For any \( C_1, C_2 \in \mathcal{C} \), we say that two vertices \( u \in C_1 \) and \( v \in C_2 \) are clones if \( N(u) \cap X = N(v) \cap X \) (in particular, if \( C_1 = C_2 \) and \( u \neq v \), then \( u, v \) are true twins). For any \( C_1, C_2 \in \mathcal{C} \) and any \( u \in C_1 \), we denote by \( c(u, C_2) \) the set of clones of \( u \) in \( C_2 \) (note that \( |c(u, C_2)| \leq 2 \)). Now observe that, for any two cliques \( C_1, C_2 \in \mathcal{C} \), the number of pairs of true twins in \( C_1 \) and \( C_2 \) is the same: we let \( t(C) \) be the number of pairs of true twins in each clique of \( \mathcal{C} \). We highlight that there are exactly \( 2t(C) \) vertices in each clique of \( \mathcal{C} \) that have true twins. The following claim for clones is the analog of Observation 2.1 for twins.

**Claim 5.2.** Let \( C_1 \) and \( C_2 \) be two cliques of an equivalence class \( \mathcal{C} \) of \( \sim \). Let \( u \in C_1 \) and \( v \in C_2 \) be clones. Then, for any \( w \in V(G) \setminus (V(C_1) \cup V(C_2)) \), \( d(u, w) = d(v, w) \), and so, for any resolving set \( S \) of \( G \), \( S \cap (V(C_1) \cup V(C_2)) \neq \emptyset \).

**Proof.** Let \( w \in V(G) \setminus (V(C_1) \cup V(C_2)) \). We show that \( d(w, u) = d(w, v) \). Towards a contradiction and without loss of generality, assume that \( d(w, u) < d(w, v) \). Let \( x \) be the first internal vertex on some shortest \((u, w)\)-path \( P \). Then, by definition, either \( x \in X \) or \( x \in C_1 \). Suppose first that \( x \in X \). Since \( v \) is a clone of \( u \), \( N_X(u) = N_X(v) \),
and so, \( x \in N(v) \). It follows that \( d(w, v) \leq 1 + d(x, w) = 1 + d(u, w) - 1 = d(u, w) \),
a contradiction. Second, suppose that \( x \in C_1 \) and let \( y \) be the clone of \( x \) in \( C_2 \). Let \( z \) be the vertex after \( x \) on \( P \). Then, \( z \notin C_1 \) since \( P \) is a shortest path and \( C_1 \) is a clique. It follows that \( z \in X \); in particular, \( z \in N_X(x) = N_X(y) \). But, \( y, v \in C_2 \), and so, \( d(v, w) \leq 2 + d(z, w) = 2 + d(u, w) - 2 = d(u, w) \), a contradiction.

Now, consider a resolving set \( S \) of \( G \). Then, there exists \( w \in S \) such that \( d(u, w) \neq d(v, w) \). But, by the above paragraph, necessarily \( w \in V(C_1) \cup V(C_2) \), and so, \( S \cap (V(C_1) \cup V(C_2)) \neq \emptyset \). 

It follows from the above claim that, for any equivalence class \( C \) of \( \sim \) and any resolving set \( S \), \( S \) contains at least \( |C| - 1 \) vertices in \( V(C) := \bigcup_{C \in \mathcal{C}} V(C) \). We now present an upper bound on the size of \( S \cap V(C) \) when \( |C| \geq |X| + 2 \).

**Claim 5.3.** For every equivalence class \( C \) of \( \sim \), if \( |C| \geq |X| + 2 \) then, for any minimum resolving set \( S \) of \( G \), \( |S \cap V(C)| \leq |X| + |C| \cdot \max\{1, t(C)\} \).

**Proof.** Let \( \mathcal{C} \) be an equivalence class such that \( |C| \geq |X| + 2 \). Suppose, towards a contradiction, that there exists a minimum resolving set \( S \) of \( G \) such that \( |S \cap V(C)| > |X| + |C| \cdot \max\{1, t(C)\} \). Consider the subset \( S^0 \) of \( V(G) \) obtained from \( S \) as follows:

1. add every vertex of \( S \) to \( S^0 \);
2. delete all vertices of \( S \cap V(C) \) from \( S^0 \), and add all the vertices of \( X \) to \( S^0 \);
3. if \( t(C) = 0 \) then, for every clique \( C \in \mathcal{C} \), add an arbitrary vertex of \( C \) to \( S^0 \), and otherwise, for every pair of twin vertices in \( C \in \mathcal{C} \), add one of them to \( S^0 \).

Now, note that the second step removes at least \( |X| + |C| \cdot \max\{1, t(C)\} \) vertices from \( S^0 \) and adds exactly \( |X| \) vertices to \( S^0 \). Since the third step adds \( |C| \cdot \max\{1, t(C)\} \) vertices to \( S^0 \), it follows that \( |S^0| < |S| \). Let us next prove that \( S^0 \) is a resolving set of \( G \), which if true would contradict the minimality of \( S \), and thus, conclude the proof.

Consider any two distinct vertices \( u, v \in V(G) \). Since \( X \subseteq S^0 \), if \( \{u, v\} \cap X \neq \emptyset \), then any vertex in \( \{u, v\} \cap X \) resolves \( u \) and \( v \). Thus, we may assume that \( u, v \in V(G) \setminus X \). Suppose first that one of \( u \) and \( v \) belongs to \( V(C) \), say \( u \in V(C) \) without loss of generality. Let \( C \in \mathcal{C} \) be the clique such that \( u \in V(C) \). If \( v \notin V(C) \), then the vertex in \( S^0 \cap V(C) \) surely resolves \( u \) and \( v \). Thus, suppose that \( v \in V(C) \) as well. If \( N(u) \cap X \neq N(v) \cap X \), then any vertex in \( (N(u) \setminus N(v)) \cap X \) or \( (N(v) \setminus N(u)) \cap X \) resolves \( u \) and \( v \). Otherwise, \( u \) and \( v \) are true twins, and so, by construction, one of them belongs to \( S^0 \) which resolves the pair. Finally, suppose that \( u, v \notin X \cup V(C) \). Since \( S \) is a resolving set of \( G \), there exists \( w \in S \) such that \( d(w, u) \neq d(w, v) \), say \( d(w, u) < d(w, v) \) without loss of generality. If \( w \) belongs to \( S^0 \) as well, then we are done. Thus, suppose that \( w \notin S^0 \). Then, \( w \in S \cap V(C) \) by construction. Let \( P \) be a shortest path from \( u \) to \( w \). Since \( u \notin X \cup V(C) \), necessarily \( V(P) \cap X = \emptyset \): let \( x \in X \) be the closest internal vertex of \( P \) to \( w \). Then, \( \text{dist}(w, u) = \text{dist}(w, x) + \text{dist}(x, u) \), and so, \( x \in N(v) \). It follows that \( d(w, v) \leq 1 + d(x, w) = 1 + d(u, w) - 1 = d(u, w) \), a contradiction. Second, suppose that \( x \in C_1 \) and let \( y \) be the clone of \( x \) in \( C_2 \). Let \( z \) be the vertex after \( x \) on \( P \). Then, \( z \notin C_1 \) since \( P \) is a shortest path and \( C_1 \) is a clique. It follows that \( z \in X \); in particular, \( z \in N_X(x) = N_X(y) \). But, \( y, v \in C_2 \), and so, \( d(v, w) \leq 2 + d(z, w) = 2 + d(u, w) - 2 = d(u, w) \), a contradiction.
and since $d(w, u) < d(w, v) \leq d(w, x) + d(x, v)$, we conclude that $d(x, u) < d(x, v)$. Since $X \subseteq S^o$, it follows that there is a vertex in $S^o$ that resolves $u$ and $v$. Therefore, $S^o$ is a resolving set of $G$, which concludes the proof.

Let $C$ be an equivalence class of $\sim$, and let $S$ be a resolving set of $G$. For every $i \geq 0$, we denote by $C_i^S$ ($C_{\geq i}^S$, respectively) the set of cliques $C \in C$ such that $|S \cap V(C)| = i$ ($|S \cap V(C)| \geq i$, respectively).

**Claim 5.4.** Let $C$ be an equivalence class of $\sim$ such that $|C| \geq |X| + 2$. Then, for any minimum resolving set $S$ of $G$, the following hold:

1. if $t(C) = 0$, then $|C_{=0}^S| \leq 1$ and $|C_{\geq 2}^S| \leq |X| + 1$;
2. if $t(C) \neq 0$, then $|C_{\geq t(C)+1}^S| \leq |X| + 1$.

**Proof.** Suppose first that $t(C) = 0$. Then, by Claim 5.2, $|C_{=0}^S| \leq 1$. Since $|C_{=0}^S| + |C_{=1}^S| + |C_{\geq 2}^S| = |C|$, it follows that $|C_{=1}^S| \geq |C| - 1 - |C_{\geq 2}^S|$. Now, by definition, $|S \cap V(C)| \geq |C_{=1}^S| + 2 \cdot |C_{\geq 2}^S|$, and so, by the above,

$$|S \cap V(C)| \geq |C| - 1 - |C_{\geq 2}^S| + 2 \cdot |C_{\geq 2}^S| = |C| - 1 + |C_{\geq 2}^S|.$$ 

Thus, if $|C_{\geq 2}^S| \geq |X| + 2$, then $|S \cap V(C)| \geq |C| - 1 + |X| + 2$, a contradiction to Claim 5.3.

Second, suppose that $t(C) \neq 0$. Then, by Observation 2.1, any resolving set of $G$ contains at least $t(C)$ vertices from each clique in $C$, which implies, in particular, that $(C_{=t(C)}, C_{\geq t(C)+1}^S)$ is a partition of $C$. Now, by definition,

$$|S \cap V(C)| \geq t(C) \cdot |C_{=t(C)}^S| + (t(C) + 1) \cdot |C_{\geq t(C)+1}^S| = t(C) \cdot (|C_{=t(C)}^S| + |C_{\geq t(C)+1}^S|) + |C_{\geq t(C)+1}^S| = t(C) \cdot |C| + |C_{\geq t(C)+1}^S|.$$ 

Thus, if $|C_{\geq t(C)+1}^S| \geq |X| + 2$, then $|S \cap V(C)| \geq t(C) \cdot |C| + |X| + 2$, a contradiction to Claim 5.3.

The above claim states that if some equivalence class $C$ of $\sim$ contains at least $|X| + 3$ cliques, then, for any minimum resolving set $S$ of $G$, if $t(C) = 0$, then $C_{=1}^S \neq \emptyset$, and otherwise, $C_{\geq t(C)}^S \neq \emptyset$. The following reduction rule is based on this claim.

**Reduction Rule 5.3.** If there exists an equivalence class $C$ of $\sim$ such that $|C| \geq 2^{|X|+2} + |X| + 2$, then remove a clique $C \in C$ from $G$ and reduce $k$ by $\max\{1, t(C)\}$.

We next prove that Reduction Rule 5.3 is safe. To this end, we first prove the following.

**Claim 5.5.** Let $C_1$ and $C_2$ be two identical cliques. Then, for every $u_1 \in V(C_1)$ and $v_2 \in V(C_2)$, $d(u_1, v_2) = d(u_2, v_1)$, where $u_2 \in c(u_1, C_2)$ and $v_1 \in c(v_2, C_1)$.
Proof. Suppose first that \( w_2 \) belongs to \( X \). If \( w_{p-1} \) belongs to \( X \) as well, then \( u_2 P[w_2, w_{p-1}]v_1 \) is a path from \( u_2 \) to \( v_1 \) since \( N_X(u_2) = N_X(v_2) \) and \( N_X(v_1) = N_X(v_2) \). Suppose next that \( w_{p-1} \) belongs to \( C_2 \) and let \( t \in c(w_{p-1}, C_1) \). Since \( P \) is a shortest path and \( C_2 \) is a clique, \( w_{p-2} \) must belong to \( X \). Thus, the path \( u_2 P[w_2, w_{p-2}]v_1 \) is a path from \( u_2 \) to \( v_1 \) since \( N_X(w_{p-1}) = N_X(t) \). Second, suppose that \( w_2 \) belongs to \( C_1 \) and let \( t \in c(w_2, C_2) \). By symmetry, we may assume that \( w_{p-1} \) belongs to \( C_2 \) (we fall back into the previous case otherwise), and let \( r \in c(w_{p-1}, C_1) \). Since \( P \) is a shortest path and \( C_1, C_2 \) are cliques, it must be that \( w_3, w_{p-2} \notin X \), and thus, the path \( u_2 P[w_3, w_{p-2}]v_1 \) is a path from \( u_2 \) to \( v_1 \) since \( N_X(w_2) = N_X(t) \) and \( N_X(w_{p-1}) = N_X(r) \). In all cases, we obtain that \( d(u_2, v_1) \leq d(u_1, v_2) \), and by symmetry, we conclude that in fact equality holds.

Claim 5.6. Reduction Rule 5.3 is safe.

Proof. Assume that there exists an equivalence class \( C \) of \( \sim \) such that \( |C| \geq 2|X| + 2 \) and consider a clique \( C_1 \in C \). We first claim that there exists a minimum resolving set \( S \) of \( G \) such that if \( t(C) = 0 \), then \( C_1 \in \mathcal{C}_t(C)^S \), and otherwise, \( C_1 \in \mathcal{C}_t(C)^S \). Indeed, consider a minimum resolving set \( S \) of \( G \). Assume that \( C_1 \notin \mathcal{C}_t(C)^S \) (we are done otherwise). Since \( |C| \geq 2|X| + 2 \), then \( \mathcal{C}_t(C)^S \neq \emptyset \) by Claim 5.4, and so, let \( C_2 \in \mathcal{C}_t(C)^S \). We now construct a resolving set \( S^0 \) from \( S \) as follows. First, we add every vertex in \( S \setminus (V(C_1) \cup V(C_2)) \) to \( S^0 \). Then, for every \( x \in V(C_1) \cap S \), we add a clone of \( x \) in \( C_2 \) to \( S^0 \). Similarly, for every \( x \in V(C_2) \cap S \), we add a clone of \( x \) in \( C_1 \) to \( S^0 \). Let us show that \( S^0 \) is indeed a resolving set of \( G \). Suppose to the contrary that there exist \( u, v \in V(G) \) such that no vertex in \( S^0 \) resolves \( u \) and \( v \). Assume first that \( u, v \in V(C_1) \) or \( u, v \in V(C_2) \). Since \( S \) is a resolving set of \( G \), there exists \( w \in S \setminus S^0 \) such that \( d(w, u) \neq d(w, v) \). In particular, either \( w \in V(C_1) \) or \( w \in V(C_2) \). Now, if \( w \in V(C_1) \), then a clone \( w_2 \in V(C_2) \) of \( w \) belongs to \( S^0 \) by construction. But, by Claim 5.2, \( d(w_2, u) = d(w, u) \neq d(w, v) = d(w_2, v) \), a contradiction to our assumption. By symmetry, we conclude similarly if \( w \in V(C_2) \). Thus, at least one of \( u \) and \( v \) belongs to \( V(C_1) \cup V(C_2) \). Suppose first that exactly one of \( u \) and \( v \) belongs to \( V(C_1) \cup V(C_2) \), say \( u \in V(C_1) \cup V(C_2) \) without loss of generality. Let us suppose that \( u \in V(C_1) \) (the case where \( u \in V(C_2) \) is handled symmetrically). Since \( S \) is a resolving set of \( G \), there exists \( w \in S \) such that \( d(w, u_2) \neq d(w, v) \), where \( u_2 \in c(u, C_2) \). If \( w \in S^0 \), then \( w \notin V(C_1) \cup V(C_2) \) by construction, and so, by Claim 5.2, \( d(w, u) = d(w, u_2) \neq d(w, v) \), a contradiction to our assumption. Suppose therefore that \( w \notin S^0 \). If \( w \in V(C_2) \), then a clone \( w_1 \in c(w, C_1) \) belongs to \( S^0 \) by construction. But, by Claim 5.2, \( d(w_1, v) = d(w, v) \neq d(w, u_2) = d(w_1, u) \), a contradiction to our assumption. Similarly, if \( w \in V(C_1) \), then a clone \( w_2 \in c(w, C_2) \) belongs to \( S^0 \) by construction. But, by Claim 5.2 and Claim 5.5, \( d(w_2, v) = d(w, v) \neq d(w, u_2) = d(w_2, u) \), a contradiction to our assumption. Second, suppose that both \( u \) and \( v \) belong to \( V(C_1) \cup V(C_2) \).
and $v$ belong to $V(C_1) \cup V(C_2)$. Then, either (1) $u$ and $v$ belong to the same clique or (2) $u$ and $v$ belong to different cliques. Assume first that (1) holds, say $u, v \in V(C_1)$ without loss of generality. Since $S$ is a resolving set of $G$, there exists $w \in S$ such that $d(w, u_2) \neq d(w, v_2)$, where $u_2, v_2 \in V(C_2)$ are clones of $u, v$, respectively. If $w \in S^o$, then $w \notin V(C_1) \cup V(C_2)$ by construction, and so, by Claim 5.2, $d(w, u) = d(w, u_2) \neq d(w, v_2) = d(w, v)$, a contradiction to our assumption. Suppose therefore that $w \notin S^o$. Since $C_2$ is a clique, it must then be that $w \in V(C_1)$. But, a clone $u_2 \in c(w, C_2)$ belongs to $S^o$ by construction, and $d(w_2, u) = d(w, u_2) \neq d(w, v_2) = d(w_2, v)$ by Claim 5.5, a contradiction to our assumption. Assume finally that (2) holds, say $w \notin V(C_1)$ and $v \in V(C_2)$ without loss of generality. Since $S$ is a resolving set, there exists $w \in S$ such that $d(w, u_2) \neq d(w, v_1)$, where $u_2 \in V(C_2)$ and $v_1 \in V(C_1)$ are clones of $u$ and $v$, respectively. If $w \in S^o$, then $w \notin V(C_1) \cup V(C_2)$ by construction, and so, by Claim 5.2, $d(w, u) = d(w, u_2) \neq d(w, v_1) = d(w, v)$, a contradiction to our assumption. Suppose therefore that $w \notin S^o$. If $w \in V(C_1)$, then a clone $w_2 \in c(w, C_2)$ belongs to $S^o$ by construction. But, by Claim 5.5, $d(w_2, u) = d(w, u_2) \neq d(w, v_1) = d(w_2, v)$, a contradiction. By symmetry, we conclude similarly if $w \in V(C_2)$. Therefore, $S^o$ is a resolving set of $G$, and since $C_1 \in C_{\text{max}}\{t(C)\}$ by construction, our claim follows.

Let us now show that $G$ has a resolving set of size at most $k$ if and only if $G - V(C)$ has a resolving set of size at most $k - \max\{1, t(C)\}$. Assume first that $G$ has a resolving set $S$ of size at most $k$. By the above paragraph, we may assume that if $t(C) = 0$, then $C \in C_{=1}^S$, and otherwise $C \in C_{=t(C)}^S$. Now, suppose that $R = S \setminus V(C)$ is not a resolving set of $G - V(C)$ (we are done otherwise). Then, there exist $u, v \in V(G) \setminus V(C)$ such that no vertex in $R$ resolves $u$ and $v$. Since $S$ is a resolving set of $G$, there exists $w \in S \setminus R$ such that $d(w, u) \neq d(w, v)$, and in particular, $w \in V(C)$. It follows that no clone $w_B$ of $w$ in a clique $B \in C \setminus \{C\}$ belongs to $S$, as otherwise, by Claim 5.2, $d(w_B, u) = d(w, u) \neq d(w, v) = d(w_B, v)$, a contradiction to our assumption. Since $S$ contains at least one vertex from each pair of true twins, this implies in particular that $w$ has no true twin. It follows that $t(C) = 0$. Indeed, if $t(C) \neq 0$, then $|S \cap V(C)| \geq t(C) + 1$ since $S \cap V(C)$ contains $w$ and at least one vertex from each pair of true twins, a contradiction to the choice of $C$. Now, $|C| \geq 2^{|X|+2} + |X| + 2$, and so, by Claim 5.4, $|C_{=1}^S| \geq 2^{|X|+2}$. It follows that there exist four distinct cliques $C_1, C_2, C_3, C_4 \in C_{=1}^S$ such that the vertex in $S \cap V(C_4)$ is a clone of the vertex in $S \cap V(C_i)$ for every $i \in [3]$. Without loss of generality, we may assume that $C \neq C_1, C_2, C_3$. Let $z \in S \cap V(C_1)$ and let $w_1 \in V(C_1)$ be a clone of $w$ (recall that $w \in S \cap V(C_1)$). We claim that the set $R^*$ obtained from $R$ by replacing $z$ with $w_1$ is a resolving set of $G - V(C)$. Indeed, suppose to the contrary that there exist $x, y \in V(G) \setminus V(C)$ such that no vertex in $R^*$ resolves $x$ and $y$. Then, either (1) $w$ resolves $x$ and $y$ or (2) $z$ resolves $x$ and $y$.

Assume first that (1) holds. If $x, y \notin V(C_1)$, then by Claim 5.2, $d(w_1, x) = d(w, x) \neq d(w, y) = d(w_1, y)$, a contradiction to our assumption. It follows that at least one of $x$ and $y$ belongs to $C_1$. Suppose first that exactly one of $x$ and $y$ belongs to $C_1$, say $x \in V(C_1)$ without loss of generality. Since $S$ is a resolving
set of \( G \), there exists \( t \in S \) such that \( d(t, x^o) \neq d(t, y) \), where \( x^o \in c(x, C) \). If \( t = w \), then by Claim 5.2, \( d(w_1, y) = d(w, y) \neq d(w, x^o) = d(w_1, x) \), a contradiction to our assumption. Now, suppose that \( t = z \) and let \( z_2 \in c(z, C) \) (recall that \( z_2 \in S \cap R^* \)). If \( y \in V(C_2) \), then \( d(z_2, y) \leq 1 \neq d(z_2, x) \), and if \( y \notin V(C_2) \), then by Claim 5.2, \( d(z_2, y) = d(z, y) \neq d(z, x^o) = d(z_2, x) \), a contradiction in both cases to our assumption. Thus, \( t \notin V(C) \cup V(C_1) \). But then, by Claim 5.2, \( d(t, x) = d(t, x^o) \neq d(t, y) \), a contradiction to our assumption. Suppose therefore that both \( x \) and \( y \) belong to \( C_1 \). Since \( S \) is a resolving set of \( G \), there exists \( t \in S \) such that \( d(t, x^o) \neq d(t, y) \), where \( x^o, y^o \in V(C) \) are the clones of \( x, y \) respectively. If \( t = w \), then either \( x^o = w \) or \( y^o = w \), say the latter holds without loss of generality. But then \( d(w_1, y) = 0 < d(w_1, x) \), a contradiction to our assumption. If \( t = z \), then by Claim 5.2, \( d(z_2, x) = d(z, x^o) \neq d(z, y^o) = d(z_2, y) \), where \( z_2 \in V(C_2) \) is the clone of \( z \), a contradiction to our assumption. Thus, \( t \neq z, w \), and so, by Claim 5.2, \( d(t, x) = d(t, x^o) \neq d(t, y^o) = d(t, y) \), a contradiction to our assumption.

Assume now that (2) holds and, for all \( i \in \{2, 3\} \), let \( z_i \in V(C_i) \) be the clone of \( z \). Then, at least one of \( x \) and \( y \) belongs to \( V(C_1) \cup V(C_2) \), as otherwise, by Claim 5.2, \( d(z_2, x) = d(z, x^o) \neq d(z, y^o) = d(z_2, y) \). The same argument shows that at least one of \( x \) and \( y \) belongs to \( V(C_1) \cup V(C_3) \). It follows that either one of \( x \) and \( y \) belongs to \( C_2 \), while the other belongs to \( C_3 \), in which case \( z_2 \) resolves \( x \) and \( y \), or at least one of \( x \) and \( y \) belongs to \( C_1 \). Suppose first that \( x, y \in V(C_1) \). Then, either \( x = z \) or \( y = z \), say the latter holds without loss of generality. In particular, \( z \neq w_1 \). Since \( S \) is a resolving set of \( G \), there exists \( t \in S \) such that \( d(t, x^o) \neq d(t, y^o) \), where \( x^o, y^o \in V(C) \) are the clones of \( x, y \), respectively. Since \( z \neq w_1, t \neq w \) by construction, and in particular, \( t \notin V(C) \). Now, if \( t = z \), then by Claim 5.2, \( d(z_2, x) = d(z, x^o) \neq d(z, y^o) = d(z_2, y) \), and if \( t \neq z \), then \( t \in S \backslash (V(C_1) \cup V(C)) \subseteq R^* \), and so, \( t \) resolves \( x \) and \( y \) since by Claim 5.2, \( d(t, x) = d(t, x^o) \neq d(t, y^o) = d(t, y) \), a contradiction in both cases to our assumption. Thus, it must be that exactly one of \( x \) and \( y \) belongs to \( C_1 \), say \( x \in V(C_1) \) without loss of generality. Then, \( y \notin V(C_2) \), as otherwise, \( z_2 \) resolves \( x \) and \( y \), and similarly, \( y \notin V(C_3) \). Now, since \( S \) is a resolving set of \( G \), there exists \( t \in S \) such that \( d(t, x^o) \neq d(t, y) \), where \( x^o \in V(C) \) is the clone of \( x \). If \( t = w \), then by Claim 5.2, \( d(w_1, x) = d(w, x^o) \neq d(w, y) = d(w_1, y) \), a contradiction to our assumption. If \( t = z \), then by Claim 5.2, \( d(z_2, x) = d(z, x^o) \neq d(z, y^o) = d(z_2, y) \), a contradiction to our assumption. Otherwise, \( t \in S \backslash (V(C_1) \cup V(C)) \subseteq R^* \), and so, \( t \) resolves \( x \) and \( y \) since by Claim 5.2, \( d(t, x) = d(t, x^o) \neq d(t, y) \), a contradiction to our assumption. Thus, \( R^* \) is indeed a resolving set of \( G - V(C) \) of size at most \( k - 1 \).

Conversely, assume that \( G - V(C) \) has a resolving set \( S \) of size at most \( k - \max \{1, t(C)\} \). Consider the set \( R \) constructed from \( S \) as follows. First, add every vertex of \( S \) to \( R \). Then, letting \( B^o \) be a clique of \( S_{max \{1, t(C)\}} \) (note that since \( |C| \geq 2^{|X|+2} + |X| + 2 \), it follows from Claim 5.4 that \( S_{max \{1, t(C)\}} \neq \emptyset \)), for every \( x^o \in S \cap V(B^o) \), we add a clone \( x \in V(C) \) of \( x^o \) to \( R \). We contend that \( R \) is a resolving set of \( G \). Indeed, consider \( u, v \in V(G) \). We may assume that \( R \cap \{u, v\} = \emptyset \), as otherwise it is clear that a vertex in \( R \) resolves \( u \) and \( v \). Now, if \( \{u, v\} \cap V(C) = \emptyset \),
then there exists a vertex \( w \in S \subseteq R \) that resolves \( u \) and \( v \) since \( S \) is a resolving set of \( G \). Assume next that \( |\{u, v\} \cap V(C)| = 1 \), say \( u \in V(C) \) without loss of generality. Suppose that no vertex in \( R \cap V(C) \) resolves \( u \) and \( v \). Note that in this case, \( v \in X \) and, for every \( G \) of \( \mathcal{C} \), \( d(x, u) = 1 = d(x, v) \). Let \( u^o \in V(B^o) \) be the clone of \( u \), where \( B^o \in \mathcal{C}^S_{\max\{1,t(C)\}} \) is the clique considered in the construction of \( R \). Since \( S \) is a resolving set of \( G - V(C) \), there exists \( t \in S \) such that \( d(t, u^o) \neq d(t, v) \). Assume next that \( B \in \mathcal{C} \{ v \} \), such that \( d(t, u^o) \neq d(t, v) \), that is, \( t \in R \) resolves \( u \) and \( v \). Suppose therefore that \( t \in V(B^o) \). Then, it must be that \( t = u^o \). Indeed, if \( t \neq u^o \), then \( d(t, u^o) = 1 \), but since for all \( x \in R \cap V(C) \), \( d(x, u) = 1 = d(x, v) \), it follows by construction that \( d(t, v) = 1 \), a contradiction. Thus, \( t \) must have a true twin \( t' \). Indeed, if \( t \) has no true twin, then by construction, \( u \in R \), a contradiction to our assumption. Now, \( B^o \in \mathcal{C}^S_{\max\{1,t(C)\}} \), \( t' \notin S \). But, since, for every \( x \in R \cap V(C) \), \( d(x, v) = 1 \), it follows by construction that, for every \( y \in S \cap V(B^o) \), \( d(y, v) = 1 = d(y, t') \). Thus, there must exist \( w \in S \setminus V(B^o) \) such that \( d(w, t') \neq d(w, v) \). But then, by Claim 5.2, \( d(w, u) = d(w, t') \neq d(w, v) \), that is, \( w \in R \) resolves \( u \) and \( v \). Suppose finally that both \( u \) and \( v \) belong to \( C \). We claim that there exists a clique \( B \in \mathcal{C} \{ v \} \) such that \( u \) and \( v \) have a true twin, a contradiction to the choice of \( R \). Now, if neither \( u \) nor \( v \) have true twins, then since \( S \) contains at least one vertex from each pair of twins,

\[
|S \cap V(C \{ v \})| \geq |S \cap \bigcup_{B \in \mathcal{C} \{ v \}} c(uB) \cup c(vB)| + (|C| - 1) \cdot t(C) \\
\geq |C \{ v \}| + (|C| - 1) \cdot t(C) \\
\geq 2|X| + 1 + (|C| - 1) \cdot t(C),
\]

a contradiction to Claim 5.3. Similarly, if exactly one of \( u \) and \( v \) has a true twin, then since \( S \) contains at least one vertex from every other pair of twins,

\[
|S \cap V(C \{ v \})| \geq |S \cap \bigcup_{B \in \mathcal{C} \{ v \}} c(uB) \cup c(vB)| + (|C| - 1) \cdot (t(C) - 1) \\
\geq 2|C \{ v \}| + (|C| - 1) \cdot (t(C) - 1) \\
\geq 2(2|X| + |X| + 1) + (|C| - 1) \cdot t(C - 1),
\]

a contradiction to Claim 5.3. Finally, if both \( u \) and \( v \) have true twins, then since \( S \)}
contains at least one vertex from every other pair of twins,

\[ |S \cap V(C \setminus \{C\})| \geq |S \cap \bigcup_{B \in C \setminus \{C\}} c(u, B) \cup c(v, B)| + (|C| - 1) \cdot (t(C) - 2) \]

\[ \geq 3|C \setminus \{C\}| + (|C| - 1) \cdot (t(C) - 2) \]

\[ \geq 3(2^{2^{X+1}} + |X| + 1) + (|C| - 1) \cdot t(C - 2), \]

a contradiction to Claim 5.3. Thus, as claimed, there exists a clique \( B \in C \setminus \{C\} \) such that a clone \( u_B \) of \( u \) in \( B \) and a clone \( v_B \) of \( v \) in \( B \) are resolved by a vertex \( w \in S \setminus V(B) \). Also, since \( d(w, u) = d(w, u_B) \neq d(w, v_B) = d(w, v) \) by Claim 5.2, we conclude that \( w \) resolves \( u \) and \( v \). Since \( R \) has size at most \( k \), the claim follows.

Now observe that once Reduction Rule 5.3 has been exhaustively applied to \((G, k)\), each equivalence class of \( \sim \) contains at most \( 2^{2^{X+1}} + |X| + 1 \) cliques. Since there are at most \( 2^{2^{X+1}} \) equivalence classes and each clique of \( G - X \) has size at most \( 2^{2^{X+1}} \), we conclude that \( G \) contains at most \( 2^{2^{2^{X+1}} \cdot (2^{2^{X+1}} + |X| + 1) \cdot 2^{2^{X+1}} + |X|} \) vertices.

## 6 Conclusion

As the Metric Dimension problem is \( \text{W}[2] \)-hard when parameterized by the solution size [27], the next natural step is to understand its parameterized complexity under structural parameterizations. We continued this line of research, following in the steps of [28, 29, 30], and more recently [31, 32]. Our most technical result is a proof that the Metric Dimension problem is \( \text{W}[1] \)-hard when parameterized by the feedback vertex set number of the graph. We thereby improved the result by Bonnet and Purohit [31] that states the problem is \( \text{W}[1] \)-hard when parameterized by the treewidth, and answered an open question in [27]. It is easy to see that the problem admits an \( \text{FPT} \) algorithm when parameterized by the larger parameter, the vertex cover number of the graph. We proved that it does not admit a polynomial kernel under the standard complexity assumption. We also proved a similar result for an incomparable parameter, the distance to clique. On the positive side, we proved that the problem admits \( \text{FPT} \) algorithms when parameterized by the distance to cluster and the distance to co-cluster, which are smaller parameters than the vertex cover number.

Although this work advances the understanding of structural parameterizations of Metric Dimension, it falls short of completing the picture (see Figure 1). We find it hard to extend the positive results to the parameters like minimum clique cover, distance to disjoint paths, feedback edge set, and bandwidth. It would be interesting to find \( \text{FPT} \) algorithms or prove that such algorithms are highly unlikely to exist for these parameters. The \( \text{FPT} \) algorithm parameterized by treedepth in [30] relies on the meta-result. Is it possible to get an \( \text{FPT} \) algorithm whose running time is a single or double exponent in the treedepth? It would also be interesting to investigate
the parameterized complexity of the problem when the parameter is the distance to
cograph. Recall that the problem is polynomial-time solvable in cographs [23].

Bonnet and Purohit [31] conjectured that the problem is W[1]-hard even when
parameterized by the treewidth plus the solution size. Towards resolving this con-
jecture, an interesting question would be to investigate whether the problem admits
an FPT algorithm when parameterized by the feedback vertex set number plus the
solution size. Note that even an XP algorithm parameterized by the feedback vertex
set number is not apparent.

Acknowledgement

The authors would like to thank Florent Foucaud for pointing us to Gutin et al. [34].
The article contains a result that subsumes our result conditionally refuting the poly-
nomial kernel for METRIC DIMENSION parameterized by the vertex cover number.

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