Pseudo-entropy for descendant operators in two-dimensional conformal field theories

Song He[a,c,1], Jie Yang[b,2], Yu-Xuan Zhang[a,3], Zi-Xuan Zhao[a,b,4]

[a] *Center for Theoretical Physics and College of Physics, Jilin University, Changchun 130012, People's Republic of China*
[b] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, People's Republic of China*
[c] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, 14476 Golm, Germany*

Abstract

We study the late-time properties of pseudo-(Rényi) entropy of locally excited states in rational conformal field theories (RCFTs). The two non-orthogonal locally excited states used to construct the transition matrix are generated by acting different descendant operators on the vacuum. We prove that for the cases where two descendant operators are generated by a single Virasoro generator respectively acting on a primary operator, the late-time excess of pseudo-entropy and pseudo-Rényi entropy always coincides with the logarithmic of the quantum dimension of the corresponding primary operator. Furthermore, we consider two linear combination operators generated by the generic summation of Virasoro generators. We find their pseudo-Rényi entropy and pseudo-entropy may get additional contributions, as the mixing of holomorphic and anti-holomorphic parts of the correlation function enhances the entanglement. Finally, we assert the pseudo-Rényi entropy and pseudo-entropy are still the logarithmic quantum dimension of the primary operator when the correlation function of linear combination operators can be divided into the product of its holomorphic part and anti-holomorphic part. We offer some examples to illustrate the phenomenon.

[1] hesong@jlu.edu.cn
[2] yangjie@cnu.edu.cn
[3] yuxuanz18@jlu.edu.cn
[4] zhaozixuan@cnu.edu.cn

Contents

1 Introduction

The discovery of AdS/CFT correspondence [1–3] has motivated much research related to quantum information theory in the high-energy physics community in recent years. Among them, quantum entanglement, as a carrier of quantum information, play an increasingly significant role in probing the structure of quantum field theories (QFTs) [4–10], the emergence of geometry [11–13], black hole information paradox [14–18].

Recently, a new entanglement measure, called *pseudo-entropy*, was proposed in [19] as a generalization of entanglement entropy. Specifically, pseudo-entropy is a two-state vector version of entanglement entropy, defined as follows. Given two non-orthogonal states $|\psi\rangle$ and $|\varphi\rangle$ in the Hilbert space $\mathcal{H}_\mathcal{S}$ of a composed quantum system $\mathcal{S} = A \cup B$, we first constructs an operator called *transition matrix* acting on $\mathcal{H}_\mathcal{S}$,

$$\mathcal{T}^{\psi|\varphi} \equiv \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle} = \frac{\rho_\psi \rho_\varphi}{\mathrm{tr}[\rho_\psi \rho_\varphi]}. \tag{1}$$

The pseudo-entropy of subsystem A, then, is obtained by calculating the von Neumann entropy of the reduced transition matrix $\mathcal{T}_A^{\psi|\varphi} \equiv \mathrm{tr}_B[\mathcal{T}^{\psi|\varphi}]$,

$$S(\mathcal{T}_A^{\psi|\varphi}) = -\mathrm{tr}[\mathcal{T}_A^{\psi|\varphi} \log \mathcal{T}_A^{\psi|\varphi}]. \tag{2}$$

In general, the reduced transition matrix is non-Hermitian and pseudo-entropy can be complex-valued. When $|\varphi\rangle = |\psi\rangle$, pseudo-entropy reduces to entanglement entropy. Like entanglement entropy, in practice, especially in QFTs, one usually computes a quantity called *pseudo-Rényi entropy*,

$$S_A^{(n)} = \frac{1}{1-n}\log \mathrm{tr}\big[\big(\mathcal{T}_A^{\psi|\varphi}\big)^n\big], \tag{3}$$

instead of pseudo-entropy to avoid the computation of the logarithm of the matrix. The limit $n \to 1$ gives back the pseudo-entropy.

Pseudo-entropy is originally proposed from the study of the generalization of holography entanglement entropy [19]. In the AdS/CFT context, the pseudo-entropy of a boundary subsystem is proposed to be dual to the area of a minimal surface in the Euclidean time-dependent AdS space [19]. In addition, it is found that pseudo-entropy is closely related to the postselection experiments in quantum information (i.e., in addition to the initial state, the system's final state is also specified [20]). The first is that the input of the pseudo-entropy—transition matrix (1) plays the role of density matrix when one computes the weak values [21, 22] of observables in the post-selected system. Secondly, pseudo-entropy is considered to characterize the averaged number of EPR pairs that could have been distilled in the post-selected systems [19, 23]. There are also many research interests and prospects driving the study of pseudo-entropy in QFTs [24–30]. See [31–36] for other related developments of pseudo-entropy.

The present paper aims to study the properties of pseudo-entropy of locally descendant excited states in two-dimensional (2D) conformal field theories (CFTs). Our study can be traced back to the research on entanglement entropy in local quantum quenches in 2D CFTs [37–52].[5] It is found that the excess of Rényi entropy of the local primary or descendant excited states in rational conformal field theories (RCFTs) saturates to a constant equal to the logarithm of the quantum dimension [62] of the local operator's conformal family [39, 44, 45]. Such saturation is well explained by the picture of quasiparticle pairs propagation [38]. The related research is extended to the pseudo-entropy in parallel [30]. When considering the real-time evolution of the pseudo-Rényi entropy of locally primary excited states in RCFTs, the early-time behavior of the excess of pseudo-Rényi entropy depends on the respective spatial positions of two identical primary operators, which is not universal. Nevertheless, its late-time behavior is universal, which only depends on the quantum dimension of the primary operator, just like the entanglement entropy. The result suggests that the picture of quasiparticle pairs propagation is preserved in the pseudo-entropy. We generalize the previous study [30] on the pseudo-(Rényi) entropy to descendant operators in this paper. Specifically, we would like to explore the late-time behavior of pseudo-Rényi entropy of two descendant operators in RCFTs. We construct the transition matrix using two locally excited states created by the operator

$$V_\alpha(x) = \sum_{\{n_i\},\{\bar{n}_j\}} \alpha_{\{n_i\}\{\bar{n}_j\}} \cdot \prod_{i,j} L_{-n_i}\bar{L}_{-\bar{n}_j}\mathcal{O}(x) \tag{4}$$

[5] See [53–61] for studies on other information quantities (such as information metric, negativity, reflected entropy, etc) in local or global quantum quenches in CFTs.

and evaluate the pseudo-Rényi entropy using the replica method [5] and conformal mapping. In (4), $\mathcal{O}(x)$ is a primary operator in Schrödinger picture with chiral and anti-chiral conformal dimension Δ, L_{-n} ($\bar{L}_{-n}$) are holomorphic (anti-holomorphic) Virasoro generators, and $\alpha_{\{n_i\},\{\bar{n}_j\}} \in \mathbb{C}$ are superposition coefficients. Since the two-point function between descendant operators of different levels does not vanish, the transition matrices we are permitted to construct have more degrees of freedom than the cases of the primary operator. It is interesting to see whether the late-time behavior of the pseudo-(Rényi) entropy of subsystems corresponding to these transition matrices has contributions other than the quantum dimension.

The rest of this paper is organized as follows. In section 2, we briefly review the replica method for locally excited states in 2D CFTs and provide our convention and some useful formulae for the later calculations. In section 3, we mainly focus on the late-time behavior of the 2nd pseudo-Rényi entropy of locally descendant excited states. For simplicity, we study the cases that a single holomorphic Virasoro generator generates the descendants. More general and complicated situations are discussed in section 4, where we derive the late-time behavior of the k-th pseudo-Rényi entropy for the generic descendant states. We end with conclusions and prospects in section 5. Some calculation details are presented in the appendices.

2 Setup in 2D CFTs

2.1 Replica method with local operators

Our focus is the pseudo-Rényi entropy of locally excited states created by acting the operator V_α (4) on the ground state in RCFTs, which can be formulated in the path integral formalism using the replica method. W can consider a RCFT that lives on a plane and has a vacuum state $|\Omega\rangle$. We firstly prepare two locally excited states using V_α to construct a real-time evolved transition matrix $\mathcal{T}^{1|2}(t)$,

$$|\psi_1\rangle \equiv e^{-\epsilon H} V_\alpha(x_1)|\Omega\rangle, \quad |\psi_2\rangle \equiv e^{-\epsilon H} V_\beta(x_2)|\Omega\rangle, \quad \mathcal{T}^{1|2}(t) \equiv e^{-iHt} \frac{|\psi_1\rangle\langle\psi_2|}{\langle\psi_2|\psi_1\rangle} e^{iHt}. \tag{5}$$

Notice that an infinitesimally small parameter ϵ has been introduced to suppress the high energy modes [63]. We can obtain the reduced transition matrix of subsystem A at time t by tracing out the degrees of freedom of A^c (the complement of A), $\mathcal{T}_A^{1|2}(t) = \mathrm{tr}_{A^c}[\mathcal{T}^{1|2}(t)]$. It turns out that the excess of the n-th pseudo-Rényi entropy of A with respect to the ground state, defined as $\Delta S^{(n)}(T_A^{1|2}(t)) := S^{(n)}(T_A^{1|2}(t)) - S^{(n)}\big(\mathrm{tr}_{A^c}\big[|\Omega\rangle\langle\Omega|\big]\big)$, is of the form [30]

$$\begin{aligned}\Delta S^{(n)}(\mathcal{T}_A^{1|2}(t)) = \frac{1}{1-n}\big[&\log\langle \prod_{k=1}^{n} V_\alpha(w_{2k-1},\bar{w}_{2k-1}) V_\beta^\dagger(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_n} \\ &-n\log\langle V_\alpha(w_1,\bar{w}_1)V_\beta^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1}\big]\end{aligned} \tag{6}$$

using the replica method. In (6), Σ_n denotes a n-sheeted Riemann surface with cuts on each copy corresponding to A, and $(w_{2k-1},\bar{w}_{2k-1})$and $(w_{2k},\bar{w}_{2k})$ are coordinates on the kth-sheet surface. The term in the first line is given by a $2n$-point correlation function on Σ_n, while a two-point function gives the one in the second line on Σ_1. We can have

$$
\begin{aligned}
w_{2k-1} &= x_1 + t - i\epsilon, \quad w_{2k} = x_2 + t + i\epsilon, \\
\bar{w}_{2k-1} &= x_1 - t + i\epsilon, \quad \bar{w}_{2k} = x_2 - t - i\epsilon.
\end{aligned} \tag{7}
$$

2.2 Convention and useful formulae

The $2n$-point correlation function on Σ_n in Eq.(6) can be evaluated with the help of a conformal mapping of Σ_n to the complex plane Σ_1. The subsystem is $A = [0, \infty)$ hereafter for convenience. We can then map Σ_n to Σ_1 using the simple conformal mapping

$$
w = z^n. \tag{8}
$$

Let us first focus on the case of $n = 2$. The calculation of $\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))$ is related to the four-point function known pretty well for exactly solvable CFTs. In our convention, using Eq.(8), the 4 points z_1, z_2, z_3, z_4 in the complex plane are given by

$$
\begin{aligned}
z_1 = -z_3 = i\sqrt{-x_1 - t + i\epsilon}, \quad \bar{z}_1 = -\bar{z}_3 = -i\sqrt{-x_1 + t - i\epsilon}, \\
z_2 = -z_4 = i\sqrt{-x_2 - t - i\epsilon}, \quad z_2 = -z_4 = -i\sqrt{-x_2 + t + i\epsilon}.
\end{aligned} \tag{9}
$$

The key point is that one should treat $t \pm i\epsilon$ as a pure imaginary number in all algebraic calculations and take t to be real only in the final expression of the pseudo-Rényi entropy. To evaluate the four-point correlation function, it is useful to focus on the cross ratios

$$
\begin{aligned}
\eta &:= \frac{z_{12}z_{34}}{z_{13}z_{24}} = \frac{(x_1 + x_2 + 2t) + 2\sqrt{(x_1+t)(x_2+t) + \epsilon^2 + i\epsilon(x_1 - x_2)}}{4\sqrt{(x_1+t)(x_2+t) + \epsilon^2 + i\epsilon(x_1 - x_2)}}, \\
\bar{\eta} &:= \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} = \frac{(x_1 + x_2 - 2t) + 2\sqrt{(x_1-t)(x_2-t) + \epsilon^2 - i\epsilon(x_1 - x_2)}}{4\sqrt{(x_1-t)(x_2-t) + \epsilon^2 - i\epsilon(x_1 - x_2)}},
\end{aligned} \tag{10}
$$

where $z_{ij} = z_i - z_j$, and a useful relation is

$$
1 - \eta = \frac{z_{14}z_{23}}{z_{13}z_{24}}. \tag{11}
$$

Since we are mainly interested in the late-time ($t \to \infty$) behavior of pseudo-Rényi entropy, one can find some useful late-time formulae from (9)

$$
\begin{aligned}
&\lim_{t\to\infty} z_1 \sim \lim_{t\to\infty} z_4 \sim -\sqrt{t}, \qquad \lim_{t\to\infty} z_2 \sim \lim_{t\to\infty} z_3 \sim \sqrt{t}, \\
&\lim_{t\to\infty} z_{12} \sim \lim_{t\to\infty} z_{13} \sim -\sqrt{t}, \quad \lim_{t\to\infty} z_{24} \sim \lim_{t\to\infty} z_{34} \sim \sqrt{t}, \\
&\lim_{t\to\infty} z_{14} \sim \lim_{t\to\infty} z_{23} \sim \sqrt{\frac{1}{t}}.
\end{aligned} \tag{12}
$$

For the cross ratios $(\eta, \bar{\eta})$, as shown in [30], we can have

$$
\begin{aligned}
&\lim_{t\to\infty}(\eta, \bar{\eta}) = (1 + \frac{(x_2 - x_1 + 2i\epsilon)^2}{16t^2}, -\frac{(x_2 - x_1 - 2i\epsilon)^2}{16t^2}) \simeq (1, 0), \\
&\partial_i\eta \sim \frac{1}{t^{\frac{3}{2}}}, \quad \partial_i\partial_j\eta \sim \frac{1}{t}, \quad \partial_i\partial_j\partial_k\eta \sim \frac{1}{t^{\frac{5}{2}}}, \quad \partial_i\partial_j\partial_k\partial_l\eta \sim \frac{1}{t^2}(i \neq j \neq k \neq l).
\end{aligned} \tag{13}
$$

For general n-th pseudo-Rényi entropy, the $2n$ points $z_1, z_2, ..., z_{2n}$ in the z-coordinates are given by

$$
\begin{aligned}
z_{2k+1} =& e^{2\pi i \frac{k+1/2}{n}}(-x_1 - t + i\epsilon)^{\frac{1}{n}}, \quad \bar{z}_{2k+1} = e^{-2\pi i \frac{k+1/2}{n}}(-x_1 + t - i\epsilon)^{\frac{1}{n}}, \\
z_{2k+2} =& e^{2\pi i \frac{k+1/2}{n}}(-x_2 - t - i\epsilon)^{\frac{1}{n}}, \quad \bar{z}_{2k+2} = e^{-2\pi i \frac{k+1/2}{n}}(-x_2 + t + i\epsilon)^{\frac{1}{n}}, \quad (k = 0, ..., n-1). \qquad (14)
\end{aligned}
$$

3 Second pseudo-Rényi entropy $\Delta S_A^{(2)}$ for descendent operators

The pseudo-Rényi entropy for locally excited states can be regarded as a generalization of the Rényi entropy for locally excited states [19]. In RCFTs, it is known that the excess of the Rényi entropy saturates to a constant equal to the logarithm of the quantum dimension of the inserted primary operator [39]. A similar result for pseudo-Rényi entropy is found in [30], and the result also holds for Rényi entropy constructed by two descendent operators in [45]. However, [45] only considers the late-time behavior of Rényi entropy established by two descendent operators with the same Virasoro generators and at the same insertion spatial coordinates. The pseudo-Rényi entropy with two descendent operators at different levels is still unknown. This section will explore the 2nd pseudo-Rényi entropy for some specific descendent operators.

3.1 $\Delta S_A^{(2)}$ for $V_\alpha = L_{-1}\mathcal{O},\ V_\beta = \mathcal{O}$

We first consider the simplest case, which is different from the previous studies [30]: $V_\alpha(x_1) = L_{-1}\mathcal{O}(x_1)$, $V_\beta(x_2) = \mathcal{O}(x_2)$. The 2nd pseudo-Rényi entropy, which, according to (6), is related to a four-point function on Σ_2,

$$
\exp\{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))\} = \frac{\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)L_{-1}\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2}}{\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1}^2}. \qquad (15)
$$

For the first descendant operators, the transformation law of them under the conformal mapping $w = z^2$ is given by

$$
\partial\mathcal{O}(w_i,\bar{w}_i) = (w_i')^{-\Delta}(\bar{w}_i')^{-\Delta}\left((w_i')^{-1}\partial\mathcal{O}(z_i,\bar{z}_i) - \Delta\frac{w_i''}{(w_i')^2}\mathcal{O}(z_i,\bar{z}_i)\right), \qquad (16)
$$

where the prime denotes the derivative with respect to z or $\bar{z}$. Then the four-point function in (15) can be written in the light of correlators on the plane as

$$
\begin{aligned}
&\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)L_{-1}\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2} \\
=&\Big(\prod_{i=1}^4 |w_i'|^{-2\Delta}\Big)\cdot\Big(\frac{\partial z_1}{\partial w_1}\frac{\partial z_3}{\partial w_3}\langle\partial\mathcal{O}(1)\mathcal{O}^\dagger(2)\partial\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} + \Delta^2\Big(\frac{\partial z_1}{\partial w_1}\Big)^2\frac{\partial^2 w_1}{\partial z_1^2}\Big(\frac{\partial z_3}{\partial w_3}\Big)^2\frac{\partial^2 w_3}{\partial z_3^2}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} \\
&-\Delta\frac{\partial z_1}{\partial w_1}\Big(\frac{\partial z_3}{\partial w_3}\Big)^2\frac{\partial^2 w_3}{\partial z_3^2}\langle\partial\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} - \Delta\frac{\partial z_3}{\partial w_3}\Big(\frac{\partial z_1}{\partial w_1}\Big)^2\frac{\partial^2 w_1}{\partial z_1^2}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\partial\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\Big),
\end{aligned}
$$
(17)

where we use the notation $\mathcal{O}(i) \equiv \mathcal{O}(z_i, \bar{z}_i)$. Due to the conformal symmetry, we can express the four-point functions involved in (17) as follows

$$\begin{aligned}
\langle \mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} =& |z_{13}z_{24}|^{-4\Delta} G(\eta,\bar{\eta}), \\
\langle \partial\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} =& |z_{13}z_{24}|^{-4\Delta}\partial_{z_1} G(\eta,\bar{\eta}) - \frac{2\Delta}{z_{13}}|z_{13}z_{24}|^{-4\Delta} G(\eta,\bar{\eta}), \\
\langle \mathcal{O}(1)\mathcal{O}^\dagger(2)\partial\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} =& |z_{13}z_{24}|^{-4\Delta}\partial_{z_3} G(\eta,\bar{\eta}) + \frac{2\Delta}{z_{13}}|z_{13}z_{24}|^{-4\Delta} G(\eta,\bar{\eta}), \\
\langle \partial\mathcal{O}(1)\mathcal{O}^\dagger(2)\partial\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} =& |z_{13}z_{24}|^{-4\Delta}\partial_{z_1}\partial_{z_3} G(\eta,\bar{\eta}) + \frac{2\Delta}{z_{13}}|z_{13}z_{24}|^{-4\Delta}(\partial_{z_1}-\partial_{z_3}) G(\eta,\bar{\eta}) \\
&+ \frac{-2\Delta(2\Delta+1)}{z_{13}^2}|z_{13}z_{24}|^{-4\Delta} G(\eta,\bar{\eta}),
\end{aligned} \tag{18}$$

where

$$G(\eta,\bar{\eta}) := \lim_{z\to\infty} |z|^{4\Delta}\langle \mathcal{O}(z,\bar{z})\mathcal{O}(1,1)\mathcal{O}(\eta,\bar{\eta})\mathcal{O}(0,0)\rangle_{\Sigma_1}. \tag{19}$$

Under the conformal mapping between Σ_2 and Σ_1, we can have

$$\begin{aligned}
&\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)L_{-1}\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2} \\
=&2^{-8\Delta}|z_1z_2z_3z_4|^{-2\Delta}|z_{13}z_{24}|^{-4\Delta}\cdot\Big\{\frac{1}{4z_1z_3}\Big[\partial_{z_1}\partial_{z_3} + \frac{2\Delta}{z_{13}}(\partial_{z_1}-\partial_{z_3}) - \frac{2\Delta(2\Delta+1)}{z_{13}^2}\Big]G(\eta,\bar{\eta}) \\
&+\frac{\Delta^2}{4z_1^2z_3^2}G(\eta,\bar{\eta}) - \frac{\Delta}{4z_1z_3^2}\Big[\partial_{z_1} - \frac{2\Delta}{z_{13}}\Big]G(\eta,\bar{\eta}) - \frac{\Delta}{4z_1^2z_3}\Big[\partial_{z_3} + \frac{2\Delta}{z_{13}}\Big]G(\eta,\bar{\eta})\Big\}.
\end{aligned} \tag{20}$$

At the late times ($t \to \infty$), as shown in [30], η and $\bar{\eta}$ approach to 1 and 0, respectively, which leads to the following late time behavior of $G(\eta,\bar{\eta})$ for RCFTs

$$\lim_{t\to\infty} G(\eta,\bar{\eta}) \simeq d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}, \tag{21}$$

where $d_{\mathcal{O}}$ is the quantum dimension of the operator $\mathcal{O}$. Hence we can obtain

$$\begin{aligned}
\lim_{t\to\infty}\partial_{z_1}G(\eta,\bar{\eta}) \simeq& \frac{2\Delta\partial_{z_1}\eta}{1-\eta}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}, \quad \lim_{t\to\infty}\partial_{z_3}G(\eta,\bar{\eta}) \simeq \frac{2\Delta\partial_{z_3}\eta}{1-\eta}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}, \\
\lim_{t\to\infty}\partial_{z_1}\partial_{z_3}G(\eta,\bar{\eta}) \simeq& \frac{2\Delta\partial_{z_1}\partial_{z_3}\eta}{1-\eta}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta} + \frac{2\Delta(2\Delta+1)\partial_{z_1}\eta\partial_{z_3}\eta}{(1-\eta)^2}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}.
\end{aligned} \tag{22}$$

On the other hand, the two-point function in (15) is

$$\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1} = \partial_{w_1}\frac{1}{|w_{12}|^{4\Delta}} = \frac{-2\Delta}{w_{12}}\cdot\frac{1}{|w_{12}|^{4\Delta}}. \tag{23}$$

Substituting (20), (22) and (23) into (15) and setting $z_3 = -z_1$, $z_4 = -z_2$, we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\exp\{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))\} \\
\simeq&\frac{w_{12}^2}{4\Delta^2}\eta^{2\Delta}(1-\bar{\eta})^{2\Delta}\Big\{\frac{-1}{4z_1^2}\Big[\frac{2\Delta\frac{z_1^2+z_2^2}{8z_1^3z_2}}{(1-\eta)d_{\mathcal{O}}} - \frac{2\Delta(2\Delta+1)\frac{(z_1^2-z_2^2)^2}{64z_1^4z_2^2}}{(1-\eta)^2d_{\mathcal{O}}} + \frac{2\Delta^2\frac{z_2^2-z_1^2}{4z_1^2z_2}}{z_1(1-\eta)d_{\mathcal{O}}} - \frac{\Delta(2\Delta+1)}{2z_1^2d_{\mathcal{O}}}\Big] \\
&+\frac{\Delta^2}{4z_1^4d_{\mathcal{O}}} - \frac{\Delta}{4z_1^3}\Big[\frac{2\Delta\frac{z_2^2-z_1^2}{8z_1^2z_2}}{(1-\eta)d_{\mathcal{O}}} - \frac{\Delta}{z_1d_{\mathcal{O}}}\Big] + \frac{\Delta}{4z_1^3}\Big[\frac{2\Delta\frac{z_1^2-z_2^2}{8z_1^2z_2}}{(1-\eta)d_{\mathcal{O}}} + \frac{\Delta}{z_1d_{\mathcal{O}}}\Big]\Big\} \\
\simeq&d_{\mathcal{O}}^{-1}.
\end{aligned} \tag{24}$$

In going from the second to the third line, we use Eq.(9) and perform the Laurent expansion at infinity. The late-time limit of the 2nd pseudo-Rényi entropy is thus given by

$$\lim_{t\to\infty} \Delta S^{(2)}\big(\mathcal{T}^{1|2}(t)\big) = \log d_{\mathcal{O}}. \tag{25}$$

In this simplest case, the late-time behavior of the 2nd pseudo-Rényi entropy of $L_{-1}\mathcal{O}$ with $\mathcal{O}$ is the same as that of the primary operator $\mathcal{O}$.

3.2 $\Delta S_A^{(2)}$ for $V_\alpha = L_{-n}\mathcal{O},\ V_\beta = \mathcal{O}$

We next consider a more complicated case that V_α is a general n-level descendant associated with the Virasoro generator L_{-n}, and V_β is still a primary. The two-point function of V_α and V_β reads [64]

$$\langle L_{-n}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}(w_2,\bar{w}_2)\rangle = \frac{(n+1)\Delta}{w_{21}^n}|w_{12}|^{-4\Delta}. \tag{26}$$

We then compute the four-point function on Σ_2. Under the conformal transformation, the level n descendant transforms as

$$L_{-n}O(w_i,\bar{w}_i) = (w_i')^{-(\Delta+n)}(\bar{w}_i')^{-\Delta}L_{-n}\mathcal{O}(z_i,\bar{z}_i) + ... \tag{27}$$

The ellipsis stands for operators with lower conformal dimensions, contributing to lower-order singularity in the correlation functions. Then at a late time

$$\begin{aligned}&\langle \mathcal{O}^{(-n)}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}^{(-n)}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2}\\ \sim&\Big(\prod_{i=1}^4 |w_i'|^{-2\Delta}\Big)(w_1')^{-n}(w_3')^{-n}\langle \mathcal{O}^{(-n)}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}.\end{aligned} \tag{28}$$

We can pick out the most singular terms of the four-point function on the z-plane in (28). According to (12) and (60) in appendix A, the leading contribution at late time in $\langle \mathcal{O}^{(-n)}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}$ should be

$$\begin{aligned}&\frac{(n-1)\Delta}{z_{41}^n}\langle \mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} + \frac{-\partial_{z_4}}{z_{41}^{n-1}}\langle \mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\\ =&\Big(\frac{(n-1)\Delta}{z_{41}^n} - \frac{\partial_{z_4}}{z_{41}^{n-1}}\Big)\Big(\frac{(n-1)\Delta}{z_{23}^n} - \frac{\partial_{z_2}}{z_{23}^{n-1}}\Big)\langle \mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} + ...\\ =&|z_{13}z_{24}|^{-4\Delta}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}\\ &\times\Bigg(\frac{(n-1)^2\Delta^2}{z_{41}^n z_{23}^n} - \frac{(n-1)\Delta}{z_{41}^n z_{23}^{n-1}}\cdot\frac{2\Delta\partial_{z_2}\eta}{1-\eta} - \frac{(n-1)\Delta}{z_{41}^{n-1}z_{23}^n}\cdot\frac{2\Delta\partial_{z_4}\eta}{1-\eta}\\ &+\frac{1}{z_{41}^{n-1}z_{23}^{n-1}}\cdot\Big(\frac{2\Delta(2\Delta+1)\partial_{z_2}\eta\cdot\partial_{z_4}\eta}{(1-\eta)^2} + \frac{2\Delta\partial_{z_2}\partial_{z_4}\eta}{1-\eta}\Big)\Bigg) + ...\end{aligned} \tag{29}$$

Combine (26), (29) with $t\to\infty$, the leading-order behavior of $\exp\{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))\}$ is

$$\begin{aligned}&\lim_{t\to\infty} e^{-\Delta S^{(2)}\left(\mathcal{T}_A^{1|2}(t)\right)}\\ \sim&\frac{w_{12}^{2n}}{(n+1)^2\Delta^2}\times\frac{1}{4^n z_1^n z_3^n}d_{\mathcal{O}}^{-1}\Bigg(\frac{(n-1)^2\Delta^2}{z_{41}^n z_{23}^n} - \frac{(n-1)\Delta}{z_{41}^n z_{23}^{n-1}}\cdot\frac{2\Delta\partial_{z_2}\eta}{1-\eta} - \frac{(n-1)\Delta}{z_{41}^{n-1}z_{23}^n}\cdot\frac{2\Delta\partial_{z_4}\eta}{1-\eta}\\ &+\frac{1}{z_{41}^{n-1}z_{23}^{n-1}}\cdot\Big(\frac{2\Delta(2\Delta+1)\partial_{z_2}\eta\cdot\partial_{z_4}\eta}{(1-\eta)^2} + \frac{2\Delta\partial_{z_2}\partial_{z_4}\eta}{1-\eta}\Big)\Bigg) + ...\\ \sim&\frac{1}{d_{\mathcal{O}}} + ...\end{aligned} \tag{30}$$

Again, the ellipsis denotes terms with sub-leading contributions. Hence the late-time limit of the 2nd pseudo-Rényi entropy of the transition matrix constructed by a primary $\mathcal{O}$ and its n-level descendant $L_{-n}\mathcal{O}$ is still $\log d_{\mathcal{O}}$.

3.3 $\Delta S_A^{(2)}$ for $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = L_{-m}\mathcal{O}$

In this subsection, we use the conformal block and operator product expansion (OPE) to show the phenomenon discovered in previous subsections is true for a general case: $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = L_{-m}\mathcal{O}$.

In terms of [64], the two-point function of V_α and V_β reads[6]

$$
\begin{aligned}
&\langle L_{-n}\mathcal{O}(w_1,\bar{w}_1)L_{-m}\mathcal{O}(w_2,\bar{w}_2)\rangle_{\Sigma_1}\\
=&\frac{1}{12}(-1)^n(w_1-w_2)^{-m-n}\frac{1}{|w_{12}|^{4\Delta}}\\
&\left(\frac{\Gamma(m+n)\left(cm\left(m^2-1\right)n\left(n^2-1\right)+24\Delta(m+n)(m+n+1)(mn-1)\right)}{\Gamma(m+2)\Gamma(n+2)}+12\Delta(\Delta(m+1)(n+1)+2)\right).
\end{aligned}
\tag{31}
$$

The late-time behavior of the four point function on Σ_2 of (6) can be derived according to(27)

$$
\begin{aligned}
&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_2}\\
\sim&\Big(\prod_{i=1}^{4}|w_i'|^{-2\Delta}\Big)(w_1')^{-n}(w_2')^{-m}(w_3')^{-n}(w_4')^{-m}\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}.
\end{aligned}
\tag{32}
$$

We can next pick out the most singular terms of the four-point function on the z-plane in (32). According to (12) , the leading contribution at late time in $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}$ comes

[6] Here the following equation to simplify the result has been used

$$
\sum_{k=1}^{m-1}\frac{(k+m)(-k+m+1)(k+n-1)!}{(k+1)!(n-2)!}=\frac{2\left(m^2n+m\left(2n^2-1\right)-n(n+1)\right)\Gamma(m+n)}{\Gamma(m+1)\Gamma(n+2)}-m(m+1)n+2.
$$

from the OPE of $\mathcal{O}^{(-n)}(1)\mathcal{O}^{\dagger}(4)$ and $\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)$.

$$\begin{aligned}
&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_2}\\
\sim&\Big(\prod_{i=1}^{4}|w_i'|^{-2\Delta}\Big)(w_1-w_2)^{-m-n}(w_3-w_4)^{-m-n}G(\eta,\bar{\eta})(-1)^{m+n}\\
&\Big(-\frac{1}{12\Gamma(m+2)^2\Gamma(n+2)^2}\Delta e^{i\pi n}(-1)^{-m-n}((m(m+1)n-2)\Gamma(m+2)\Gamma(n+2)\\
&-2(m+1)(m^2n+m(2n^2-1)-n(n+1))\Gamma(m+n))((-(-1)^2)^{-m-n}\Gamma(m+n)\\
&(m(m^2-1)e^{i\pi n}n(-1)^{m+n}(c(n^2-1)+24\Delta)+24\Delta e^{i\pi m}(n+1)1^{m+n}(m(2mn-m+n^2-1)-n))\\
&+12\Delta(-1)^{-n}\Gamma(m+2)\Gamma(n+2)(1^{-m}(n+1)(\Delta+m(\Delta+n))+(-1)^{-m}e^{i\pi m}(2-mn(n+1))))\\
&+\frac{1}{12}\Delta(m+1)(\Delta+m)1^{-m-2n}(-1)^{-2m-n}(\frac{1}{n(n+1)\Gamma(m+2)\Gamma(n-1)}\Gamma(m+n)\\
&(m(m^2-1)(-1)^n n(-1)^{m+n}(c(n^2-1)+24\Delta)+24\Delta(-1)^m(n+1)1^{m+n}(m(m(2n-1)+n^2-1)-n))\\
&+12\Delta 1^n(n-1)((-1)^m(n+1)(\Delta+m(\Delta+n))+(-1)^{m+1}1^m(mn(n+1)-2)))\\
&\frac{1}{12}\Delta(2\Delta+m)1^{-m-2n}(-1)^{-2m-n}(\frac{1}{m\Gamma(m)\Gamma(n+2)}\Gamma(m+n)\\
&(m(m^2-1)(-1)^n n(-1)^{m+n}(c(n^2-1)+24\Delta)+24\Delta(-1)^m(n+1)1^{m+n}(m(m(2n-1)+n^2-1)-n))\\
&+12\Delta(m+1)1^n((-1)^m(n+1)(\Delta+m(\Delta+n))+(-1)^{m+1}1^m(mn(n+1)-2)))\\
&+\frac{1}{144(m+1)\Gamma(m-1)\Gamma(m)\Gamma(m+2)^2\Gamma(n-1)\Gamma(n+2)}\\
&((-1)^{m+1}m1^{-2(m+n)}(-1)^{-m-n}(c(m^2-1)+24\Delta)\Gamma(m+n)((m+1)(-1)^{n+1}n(-1)^{m+n}\Gamma(m)\Gamma(m+2)\\
&(c(n^2-1)+24\Delta)\Gamma(m+n)+12\Delta 1^n\Gamma(m-1)((m+1)\Gamma(m)((-1)^m(-n-1)\Gamma(m+2)\Gamma(n+2)\\
&(\Delta+m(\Delta+n))+(-1)^m 1^m((mn(n+1)-2)\Gamma(m+2)\Gamma(n+2)+2n(n+1)\Gamma(m+n)))\\
&-2(-1)^m1^m(n+1)(2mn-m+n^2-1)\Gamma(m+2)\Gamma(m+n))))\Big)+\dots
\end{aligned}\tag{33}$$

The complete derivation detail of Eq.(33) is shown in appendix B.

Combine (31) and (33) and take the limit $t\to\infty$, the leading-order behavior of $\exp\{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))\}$ is

$$\lim_{t\to\infty} e^{-\Delta S^{(2)}\left(\mathcal{T}_A^{1|2}(t)\right)}\sim\frac{1}{d_{\mathcal{O}}}+\dots\tag{34}$$

The ellipsis denotes terms with sub-leading contributions. The late-time limit of the 2nd pseudo-Rényi entropy of the transition matrix constructed by a m-level descendant operator $L_{-m}\mathcal{O}$ and a n-level descendant operator $L_{-n}\mathcal{O}$ is $\log d_{\mathcal{O}}$.

4 k-th pseudo-Rényi entropy for generic descendent states

In the previous section, the 2nd pseudo-Rényi entropy corresponding to $L_{-n}\mathcal{O}$ and $L_{-m}\mathcal{O}$ is the same as the 2nd pseudo-Rényi entropy of the corresponding primary operator $\mathcal{O}$ at a late time, and they all equal to the logarithm of the quantum dimension of the primary operator. However, to derive the pseudo-entropy, it is reasonable to consider the k-th pseudo-Rényi entropy and take k analytic continuation to 1. In this section, the late-time behavior of k-th pseudo-Rényi entropy with two linear combination descendent operators will be checked whether it is still $\log d_{\mathcal{O}}$.

4.1 $\Delta S_A^{(k)}$ for $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = L_{-m}\mathcal{O}$

We begin with calculating the general case discussed above: $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = L_{-m}\mathcal{O}$. Since the anti-holomorphic part of these operators is still primary, we only focus on the holomorphic part here.

The late-time behavior of the $2k$-point function on Σ_k can be derived according to (27)

$$\begin{aligned}&\langle \mathcal{O}^{(-n)}(w_1)\mathcal{O}^{(-m)\dagger}(w_2)\ldots\mathcal{O}^{(-n)}(w_{2k-1})\mathcal{O}^{(-m)\dagger}(w_{2k})\rangle_{\Sigma_k},\\ \sim&\mathcal{F}(w_1,w_2,\ldots,w_{2k},m,n,\Delta)\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\ldots\mathcal{O}^{(-n)}(2k-1)\mathcal{O}^{(-m)\dagger}(2k)\rangle_{\Sigma_1}+\ldots,\end{aligned} \tag{35}$$

where

$$\mathcal{F}(w_1,w_2,\ldots,w_{2k},m,n,\Delta)=\big(\prod_{i=1}^{2k}|w_i'|^{-2\Delta}\big)(w_1')^{-n}(w_2')^{-m}\ldots(w_{2k-1}')^{-n}(w_{2k}')^{-m} \tag{36}$$

is the leading factor coming from the conformal transformation between correlation functions on Σ_k and correlation functions on Σ_1.

According to (14), at the late-time limit, we can find the following relations

$$\begin{aligned}\lim_{t\to\infty}(z_{2i+4}-z_{2i+1})\simeq&\ \frac{w_2-w_1}{kt}e^{2\pi i\frac{i+1}{k}}t^{\frac{1}{k}}\simeq 0,\\ \lim_{t\to\infty}(\bar z_{2i+2}-\bar z_{2i+1})\simeq&\ \frac{w_1-w_2}{kt}e^{-2\pi i\frac{i+\frac12}{k}}t^{\frac{1}{k}}\simeq 0.\end{aligned} \tag{37}$$

Hence the leading term of $2k$-point correlation function on Σ_1 comes from the OPE of $\mathcal{O}^{(-n)}(2i+1)\mathcal{O}^{(-m)\dagger}(2i+4)$, i.e.,

$$\begin{aligned}&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\ldots\mathcal{O}^{(-n)}(2k-1)\mathcal{O}^{(-m)\dagger}(2k)\rangle_{\Sigma_1}\\ \sim&\mathcal{D}_{1,4}\mathcal{D}_{3,6}\ldots\mathcal{D}_{2k-3,2k}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\ldots\mathcal{O}(2k-1)\mathcal{O}^\dagger(2k)\rangle_{\Sigma_1},\end{aligned} \tag{38}$$

where $\mathcal{D}_{2i+1,2i+4}$ is a derivative operator that only contains constants related to the information of two descendant operators and derivatives coming from the most singular part of the OPE of $\mathcal{O}^{(-n)}(2i+1)\mathcal{O}^{(-m)\dagger}(2i+4)$,

$$\mathcal{D}_{2i+1,2i+4}=\mathcal{D}(\partial_{2i+1},\partial_{2i+4};m,n,c,\Delta). \tag{39}$$

See appendix B for a concrete example of the $\mathcal{D}$-operator. We need to pick up the proper channel to expand the $2k$-point function into the holomorphic and the anti-holomorphic part, as graphically shown in figure 1. In each channel, only the identity operator contributes to the final result. Hence, the $2k$-point function breaks up into k two-point functions for the holomorphic part(and k for the anti-holomorphic part).

$$\begin{aligned}&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\ldots\mathcal{O}^{(-n)}(2k-1)\mathcal{O}^{(-m)\dagger}(2k)\rangle_{\Sigma_1}\\ \sim&(F_{00}[\mathcal{O}])^{k-1}\mathcal{D}_{1,4}\mathcal{D}_{3,6}\ldots\mathcal{D}_{2k-3,2k}\langle\mathcal{O}(1)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\langle\mathcal{O}(3)\mathcal{O}^\dagger(6)\rangle_{\Sigma_1}\ldots\langle\mathcal{O}(2k-3)\mathcal{O}^\dagger(2k)\rangle_{\Sigma_1}\\ \sim&(F_{00}[\mathcal{O}])^{k-1}\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\langle\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(6)\rangle_{\Sigma_1}\ldots\langle\mathcal{O}^{(-n)}(2k-3)\mathcal{O}^{(-m)\dagger}(2k)\rangle_{\Sigma_1}.\end{aligned} \tag{40}$$

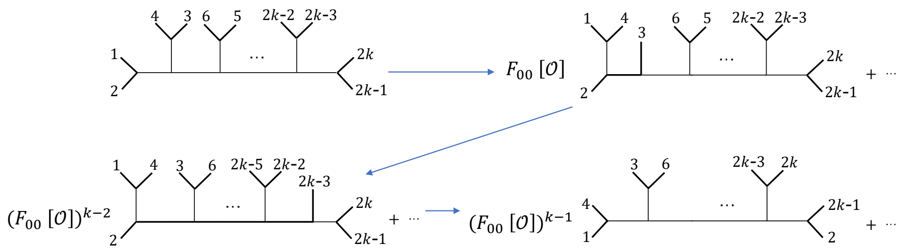

Figure 1: $k-1$ fusion transformations to obtain $\Delta S_A^{(k)}$

In the last line, the fact that $\mathcal{D}_{2i+1,2i+3}$ is a linear operator, and coordinates z_i and z_j are independent for $i \neq j$ has been applied.

Changing back into the w-coordinate, with the leading divergent term being transformed homogeneously and keeping the most divergent term, we can find

$$\begin{aligned}
&\langle \mathcal{O}^{(-n)}(w_1)\mathcal{O}^{(-m)\dagger}(w_2)\ldots\mathcal{O}^{(-n)}(w_{2k-1})\mathcal{O}^{(-m)\dagger}(w_{2k})\rangle_{\Sigma_k} \\
\sim&(F_{00}[\mathcal{O}])^{k-1}\mathcal{F}(w_1,w_2,\ldots,w_{2k},m,n,\Delta) \\
&\langle \mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\langle \mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(6)\rangle_{\Sigma_1}\ldots\langle \mathcal{O}^{(-n)}(2k-3)\mathcal{O}^{(-m)\dagger}(2k)\rangle_{\Sigma_1}+\ldots \\
\sim&(F_{00}[\mathcal{O}])^{k-1}\langle \mathcal{O}^{(-n)}(w_1)\mathcal{O}^{(-m)\dagger}(w_4)\rangle_{\Sigma_k}\langle \mathcal{O}^{(-n)}(w_3)\mathcal{O}^{(-m)\dagger}(w_6)\rangle_{\Sigma_k}\ldots\langle \mathcal{O}^{(-n)}(w_{2k-3})\mathcal{O}^{(-m)\dagger}(w_{2k})\rangle_{\Sigma_k} \\
&+\ldots
\end{aligned} \tag{41}$$

The two-point function of descendent operators on Σ_k and that on Σ_1 are similar at late time

$$\begin{aligned}
&\langle \mathcal{O}^{(-n)}(w_{2i+1})\mathcal{O}^{(-m)\dagger}(w_{2i+4})\rangle_{\Sigma_k} \\
\sim&(w'_{2i+1})^{-\Delta-n}(w'_{2i+4})^{-\Delta-m}\langle \mathcal{O}^{(-n)}(z_{2i+1})\mathcal{O}^{(-m)\dagger}(z_{2i+4})\rangle_{\Sigma_1} \\
\sim&(kz_{2i+1}^{k-1}e^{2\pi i\frac{i}{k}})^{-\Delta-n}(kz_{2i+4}^{k-1}e^{2\pi i\frac{i+1}{k}})^{-\Delta-m}\frac{C_0(m,n)}{((z_{2i+1}-z_{2i+4})e^{2\pi i\frac{i+1}{k}})^{2\Delta+m+n}} \\
\sim&e^{(-2\pi i\frac{1}{k})(-\Delta-n)}(kt^{\frac{k-1}{k}})^{-\Delta-n}(kt^{\frac{k-1}{k}})^{-\Delta-m}\frac{C_0(m,n)}{(w_{2i+1}-w_{2i+4})^{2\Delta+m+n}}(kt^{\frac{k-1}{k}})^{2\Delta+m+n} \\
\sim&e^{(-2\pi i\frac{1}{k})(-\Delta-n)}\langle \mathcal{O}^{(-n)}(w_1)\mathcal{O}^{(-m)\dagger}(w_2)\rangle_{\Sigma_1}.
\end{aligned} \tag{42}$$

Therefore, at a late time, for two descendent operators with a single Virasoro generator, we still have $\lim_{t\to\infty}\Delta S^{(k)} = \log d_{\mathcal{O}}$, and its pseudo-entropy is $\log d_{\mathcal{O}}$.

4.2 $\Delta S_A^{(k)}$ for Linear combination of descendent operators

Let us consider two linear combination operators constructed by operators in $\mathcal{O}$'s conformal family.

$$V_\alpha(w,\bar{w}) = \sum_i C_i V_i(w,\bar{w}), \qquad V_i(w,\bar{w}) = L_{-\{K_i\}}\bar{L}_{-\{\bar{K}_i\}}\mathcal{O}(w,\bar{w}), \tag{43}$$

$$V_\beta(w,\bar{w}) = \sum_i C'_i V'_i(w,\bar{w}), \qquad V'_i(w,\bar{w}) = L_{-\{K'_i\}}\bar{L}_{-\{\bar{K}'_i\}}\mathcal{O}^\dagger(w,\bar{w}), \tag{44}$$

where $L_{-\{K_i\}} \equiv L_{-k_{i1}}L_{-k_{i2}}...L_{-k_{in_i}}$, $(0 \le k_{i1} \le k_{i2} \le ... \le k_{in_i})$, and $L_{-\{\bar{K}_i\}} \equiv L_{-\bar{k}_{i1}}L_{-\bar{k}_{i2}}...L_{-\bar{k}_{i\bar{n}_i}}$, $(0 \le \bar{k}_{i1} \le \bar{k}_{i2} \le ... \le \bar{k}_{i\bar{n}_i})$. Likewise for $L_{-\{K'_i\}}$ and $L_{-\{\bar{K}'_i\}}$. If the combination coefficients C_i

(C_i') are required to be dimensionless, all $V_i(w,\bar{w})$ $\big(V_i'(w,\bar{w})\big)$ should have the same mass dimension, denoted as N (N'). This indicates that $\{K_i\}$ and $\{K_i'\}$ satisfy

$$|K_i|+|\bar{K}_i|=N,\quad |K_i'|+|\bar{K}_i'|=N',\quad \Big(|K_i|\equiv\sum_{j=1}^{n_i}k_{ij},\ |\bar{K}_i|\equiv\sum_{j=1}^{\bar{n}_i}\bar{k}_{ij}\Big). \tag{45}$$

The two point function of V_α and V_β is

$$\begin{aligned}
&\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)\rangle_{\Sigma_1}\\
=&\sum_i\sum_j C_iC_j'\langle L_{-\{K_i\}}\mathcal{O}(w_1)L_{-\{K_j'\}}\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}\langle \bar{L}_{-\{\bar{K}_i\}}\mathcal{O}(\bar{w}_1)\bar{L}_{-\{\bar{K}_j'\}}\mathcal{O}^\dagger(\bar{w}_2)\rangle_{\Sigma_1}\\
=&\sum_i\sum_j C_iC_j'\frac{c_0(\{K_i\},\{K_j'\})}{(w_1-w_2)^{2\Delta+|K_i|+|K_j'|}}\frac{c_0(\{\bar{K}_i\},\{\bar{K}_j'\})}{(\bar{w}_1-\bar{w}_2)^{2\Delta+|\bar{K}_i|+|\bar{K}_j'|}},
\end{aligned} \tag{46}$$

where the coefficient c_0 depends on the decomposition of generic Virasoro generators. At the late time, the $2k$-point function reads

$$\begin{aligned}
&\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)\dots V_\alpha(w_{2k-1},\bar{w}_{i_{2k-1}})V_\beta(w_{j_{2k}},\bar{w}_{2k})\rangle_{\Sigma_k}\\
=&\sum_{i_1}\sum_{j_2}\cdots\sum_{i_{2k-1}}\sum_{j_{2k}}C_{i_1}C_{j_2}'\dots C_{i_{2k-1}}C_{j_{2k}}'\\
&\langle L_{-\{K_{i_1}\}}\mathcal{O}(w_1)L_{-\{K_{j_2}'\}}\mathcal{O}^\dagger(w_2)\dots L_{-\{K_{i_{2k-1}}\}}\mathcal{O}(w_{2k-1})L_{-\{K_{j_{2k}}'\}}\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}\\
&\langle \bar{L}_{-\{\bar{K}_{i_1}\}}\mathcal{O}(\bar{w}_1)\bar{L}_{-\{\bar{K}_{j_2}'\}}\mathcal{O}^\dagger(\bar{w}_2)\dots \bar{L}_{-\{\bar{K}_{i_{2k-1}}\}}\mathcal{O}(\bar{w}_{2k-1})\bar{L}_{-\{\bar{K}_{j_{2k}}'\}}\mathcal{O}^\dagger(\bar{w}_{2k})\rangle_{\Sigma_k}\\
\sim& d_{\mathcal{O}}^{-(k-1)}\sum_{i_1}\sum_{j_2}\cdots\sum_{i_{2k-1}}\sum_{j_{2k}}C_{i_1}C_{j_2}'\dots C_{i_{2k-1}}C_{j_{2k}}'\\
&\langle L_{-\{K_{i_1}\}}\mathcal{O}(w_1)L_{-\{K_{j_4}'\}}\mathcal{O}^\dagger(w_4)\rangle_{\Sigma_k}\dots\langle L_{-\{K_{i_{2k-3}}\}}\mathcal{O}(w_{2k-3})L_{-\{K_{j_{2k}}'\}}\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}\\
&\langle \bar{L}_{-\{\bar{K}_{i_1}\}}\mathcal{O}(\bar{w}_1)\bar{L}_{-\{\bar{K}_{j_2}'\}}\mathcal{O}^\dagger(\bar{w}_2)\rangle_{\Sigma_k}\dots\langle \bar{L}_{-\{\bar{K}_{i_{2k-1}}\}}\mathcal{O}(\bar{w}_{2k-1})\bar{L}_{-\{\bar{K}_{j_{2k}}'\}}\mathcal{O}^\dagger(\bar{w}_{2k})\rangle_{\Sigma_k}\\
\sim& d_{\mathcal{O}}^{-(k-1)}\sum_{i_1}\sum_{j_2}\cdots\sum_{i_{2k-1}}\sum_{j_{2k}}C_{i_1}C_{j_2}'\dots C_{i_{2k-1}}C_{j_{2k}}'\\
&\frac{c_0(\{K_{i_1}\},\{K_{j_4}'\})}{(w_1-w_4)^{2\Delta+|K_{i_1}|+|K_{j_4}'|}}\cdots\frac{c_0(\{K_{i_{2k-3}}\},\{K_{j_{2k}}'\})}{(w_{2k-3}-w_{2k})^{2\Delta+|K_{i_{2k-3}}|+|K_{j_{2k}}'|}}\\
&\frac{c_0(\{\bar{K}_{i_1}\},\{\bar{K}_{j_2}'\})}{(\bar{w}_1-\bar{w}_2)^{2\Delta+|\bar{K}_{i_1}|+|\bar{K}_{j_2}'|}}\cdots\frac{c_0(\{\bar{K}_{i_{2k-1}}\},\{\bar{K}_{j_{2k}}'\})}{(\bar{w}_{2k-1}-\bar{w}_{2k})^{2\Delta+|\bar{K}_{2k-1}|+|\bar{K}_{2k}'|}}.
\end{aligned} \tag{47}$$

The first formula transforms the correlation function on Σ_k into combinations of its holomorphic and anti-holomorphic parts. In the second formula, we have extracted its leading term on Σ_1, separated it into k two-point functions using the fusion rule, and then changed the correlation function on Σ_1 back to Σ_k. The third formula has used the property of the two-point part on Σ_k (42).

Combine (46) and (47), the k-th pseudo-Rényi entropy for linear combination of descendent operators is

$$\Delta S^{(k)} = \frac{1}{1-k}\log\frac{\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)\dots V_\alpha(w_{2k-1},\bar{w}_{i_{2k-1}})V_\beta(w_{j_{2k}},\bar{w}_{2k})\rangle_{\Sigma_k}}{(\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)\rangle_{\Sigma_1})^k}$$

$$\sim\frac{1}{1-k}\log\left\{d_{\mathcal{O}}^{-(k-1)}\times\right.$$

$$\left.\frac{\sum\limits_{i_1}\cdots\sum\limits_{j_{2k}}C_{i_1}\dots C'_{j_{2k}}c_0(\{K_{i_1}\},\{K'_{j_4}\})\dots c_0(\{K_{i_{2k-3}}\},\{K'_{j_{2k}}\})c_0(\{\bar{K}_{i_1}\},\{\bar{K}'_{j_2}\})\dots c_0(\{\bar{K}_{i_{2k-1}}\},\{\bar{K}'_{j_{2k}}\})}{(\sum\limits_i\sum\limits_j C_iC'_jc_0(\{K_i\},\{K'_j\})c_0(\{\bar{K}_i\},\{\bar{K}'_j\}))^k}\right\}. \tag{48}$$

For the last formula, we have applied the restrictive condition (48) and the fact that when $\epsilon\to 0$, all z_i and $\bar{z}_i$ are real.

There are two types of contributions to the pseudo-entropy of two linear combination operators. The first one takes a universal form, depending on the quantum dimension of the corresponding primary operator. There may also have an extra contribution to the pseudo-entropy. To see this, consider the 2nd pseudo-Rényi entropy,

$$\Delta S^{(2)}\sim\log d_{\mathcal{O}}$$

$$-\log\left(\frac{\sum\limits_{i_1,i_3}\sum\limits_{j_2,j_4}C_{i_1}C_{i_3}C'_{j_2}C'_{j_4}c_0(\{K_{i_1}\},\{K'_{j_4}\})c_0(\{K_{i_3}\},\{K'_{j_2}\})c_0(\{\bar{K}_{i_1}\},\{\bar{K}'_{j_2}\})c_0(\{\bar{K}_{i_3}\},\{\bar{K}'_{j_4}\})}{(\sum\limits_i\sum\limits_j C_iC'_jc_0(\{K_i\},\{K'_j\})c_0(\{\bar{K}_i\},\{\bar{K}'_j\}))^2}\right). \tag{49}$$

In general, the numerator is not equal to the denominator in the last line of (49). So the k-th pseudo-Renyi entropy may acquire additional correction. It is zero when correlation functions containing V_α and V_β can be divided into the product of the holomorphic part and antiholomorphic part. However, the extra correction may be nonzero in general. The following two examples can illustrate the phenomenon.

- Example 1 with $V_\alpha(w_1,\bar{w}_1)=(L_{-1}+\bar{L}_{-1})\mathcal{O}(w_1,\bar{w}_1)$ $V_\beta(w_2,\bar{w}_2)=(L_{-1}+\bar{L}_{-1})\mathcal{O}(w_2,\bar{w}_2)$

  The two-point function is

$$\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)\rangle_{\Sigma_1}=\frac{-4\Delta(4\Delta+1)}{(x_1-x_2)^{4\Delta+2}}. \tag{50}$$

  Formula (50) is easy to check. Here, we replace x_1+t and x_2+t into w_1 and w_2 in the final result. The four-point function is

$$\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)V_\alpha(w_3,\bar{w}_3)V_\beta(w_4,\bar{w}_4)\rangle_{\Sigma_2}\sim\frac{8\Delta^2(16\Delta+1+8\Delta)}{(x_1-x_2)^{8\Delta+4}}. \tag{51}$$

  We explicitly show how to compute the four-point function on Σ_2 without showing the derivation of (51) in detail. Again we keep the final result replaced by x_1 and x_2.

  From (50) and (51), we can have

$$\Delta S^{(2)}\sim\log d_{\mathcal{O}}+\log 2. \tag{52}$$

In this case, the correlation function of V_α and V_β can not be divided into the product of the holomorphic part and antiholomorphic part, and $\Delta S^{(2)}$ contains an extra correction $\log 2$ besides $\log d_{\mathcal{O}}$.

- Example 2 with $V_\alpha(w_1,\bar{w}_1) = L_{-\{K_1\}}\bar{L}_{-\{\bar{K}_1\}}\mathcal{O}((w_1,\bar{w}_1)$, $V_\beta(w_2,\bar{w}_2) = L_{-\{K_2'\}}\bar{L}_{-\{\bar{K}_2'\}}\mathcal{O}(w_2,\bar{w}_2)$

  According to (46), the two-point function on Σ_1 reads

$$\begin{aligned}
&\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)\rangle_{\Sigma_1}\\
=&\langle L_{-\{K_1\}}\mathcal{O}(w_1)L_{-\{K_2'\}}\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}\langle \bar{L}_{-\{\bar{K}_1\}}\mathcal{O}(\bar{w}_1)\bar{L}_{-\{\bar{K}_2'\}}\mathcal{O}^\dagger(\bar{w}_2)\rangle_{\Sigma_1}\\
=&\frac{c_0(\{K_1\},\{K_2'\})}{(w_1-w_2)^{2\Delta+|K_1|+|K_2'|}}\frac{c_0(\{\bar{K}_1\},\{\bar{K}_2'\})}{(\bar{w}_1-\bar{w}_2)^{2\Delta+|\bar{K}_1|+|\bar{K}_2'|}}.
\end{aligned} \tag{53}$$

  The $2k$-point correlation function is

$$\begin{aligned}
&\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)\ldots V_\alpha(w_{2k-1},\bar{w}_{i_{2k-1}})V_\beta(w_{j_{2k}},\bar{w}_{2k})\rangle_{\Sigma_k}\\
=&\langle L_{-\{K_1\}}\mathcal{O}(w_1)L_{-\{K_2'\}}\mathcal{O}^\dagger(w_2)\ldots L_{-\{K_1\}}\mathcal{O}(w_{2k-1})L_{-\{K_2'\}}\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}\\
&\langle \bar{L}_{-\{\bar{K}_1\}}\mathcal{O}(\bar{w}_1)\bar{L}_{-\{\bar{K}_2'\}}\mathcal{O}^\dagger(\bar{w}_2)\ldots \bar{L}_{-\{\bar{K}_1\}}\mathcal{O}(\bar{w}_{2k-1})\bar{L}_{-\{\bar{K}_2'\}}\mathcal{O}^\dagger(\bar{w}_{2k})\rangle_{\Sigma_k}\\
\sim& d_{\mathcal{O}}^{-(k-1)}\langle L_{-\{K_1\}}\mathcal{O}(w_1)L_{-\{K_2'\}}\mathcal{O}^\dagger(w_4)\rangle_{\Sigma_k}\ldots\langle L_{-\{K_1\}}\mathcal{O}(w_{2k-3})L_{-\{K_2'\}}\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}\\
&\langle \bar{L}_{-\{\bar{K}_1\}}\mathcal{O}(\bar{w}_1)\bar{L}_{-\{\bar{K}_2'\}}\mathcal{O}^\dagger(\bar{w}_2)\rangle_{\Sigma_k}\ldots\langle \bar{L}_{-\{\bar{K}_1\}}\mathcal{O}(\bar{w}_{2k-1})\bar{L}_{-\{\bar{K}_2'\}}\mathcal{O}^\dagger(\bar{w}_{2k})\rangle_{\Sigma_k}\\
\sim& d_{\mathcal{O}}^{-(k-1)}\frac{c_0(\{K_1\},\{K_2'\})}{(w_1-w_4)^{2\Delta+|K_1|+|K_2'|}}\cdots\frac{c_0(\{K_1\},\{K_2'\})}{(w_{2k-3}-w_{2k})^{2\Delta+|K_1|+|K_2'|}}\\
&\frac{c_0(\{\bar{K}_1\},\{\bar{K}_2'\})}{(\bar{w}_1-\bar{w}_2)^{2\Delta+|\bar{K}_1|+|\bar{K}_2'|}}\cdots\frac{c_0(\{\bar{K}_1\},\{\bar{K}_2'\})}{(\bar{w}_{2k-1}-\bar{w}_{2k})^{2\Delta+|\bar{K}_1|+|\bar{K}_2'|}}.
\end{aligned} \tag{54}$$

  In this case the correlation function of V_α and V_β can be divided, and $\lim\limits_{t\to\infty}\Delta S^{(k)} = \log d_{\mathcal{O}}$, i.e. $\Delta S^{(k)}$ has no extra correction.

5 Conclusion and prospect

In this paper, we investigate the pseudo-Rényi entropy of local descendent operators in RCFTs, extending the previous studies in [30] [39] [45]. In [30] [45], it has been found that the late-time excess of the pseudo-Rényi entropy of two primary states and the Rényi entropy of a descendent state equal to the logarithmic quantum dimension of the primary operator in RCFTs. It is a natural question to consider the pseudo-Rényi entropy of the descendent states.

Firstly, we show that in some special cases: $V_\alpha = L_{-1}\mathcal{O}$, $V_\beta = \mathcal{O}$ and $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = \mathcal{O}$ with $\mathcal{O}$ being primary, the late-time excess of the 2nd pseudo-Rényi entropy (6) is still logarithmic of the quantum dimension of the primary operator. Using the conformal block and operator product expansion, we compute the 2nd pseudo-Rényi entropy constructed by two descendent operators with different Virasoro generators. We show that their 2nd pseudo-Rényi entropy is the same as their primaries for such states. Although the calculation looks quite complicated, the leading divergent terms in the late time limit are simple, behaving as the one for primary operators.

Further, we compute k-th pseudo-Rényi entropy with two descendent operators $L_{-n}\mathcal{O}$ and $L_{-m}\mathcal{O}$. We extract the most divergent term of the $2k$-point function on Σ_k with an overall factor $\mathcal{F}$ (35), and then associate the $2k$-point function of descendent operators with the $2k$-point function of primary operators (38) with some derivative operators of the form

$$\mathcal{D}_{2i+1,2i+4} = \mathcal{D}(\partial_{2i+1}, \partial_{2i+4}; m, n, c, \Delta). \tag{55}$$

We find the $2k$-point function breaks up into k two-point functions for the holomorphic part(and k for the anti-holomorphic part). The two-point function only depends on the conformal weight and some constant (42). As a result, in this case, the pseudo-entropy of the descendent operators is the same as primaries.

Finally, we discuss the most generic descendent operators, which are two linear combination operators constructed by operators in $\mathcal{O}$'s conformal family

$$V_\alpha(w_1, \bar{w}_1) = \sum_i C_i V_i(w_1, \bar{w}_1), \quad V_\beta(w_2, \bar{w}_2) = \sum_j C'_j V_j(w_2, \bar{w}_2). \tag{56}$$

Unsurprisingly, we find that the pseudo-Rényi entropy of these operators is generally different from that of the primary operator $\mathcal{O}$. The entropies are the same as the ones of the primary when the correlation function of V_α and V_β can be divided into the product of the holomorphic part and the anti-holomorphic part. A typical example is

$$V_\alpha(w_1, \bar{w}_1) = L_{-\{K_1\}} \bar{L}_{-\{\bar{K}_1\}} \mathcal{O}(w_1, \bar{w}_1), \quad V_\beta(w_2, \bar{w}_2) = L_{-\{K'_2\}} \bar{L}_{-\{\bar{K}'_2\}} \mathcal{O}(w_2, \bar{w}_2). \tag{57}$$

Otherwise, there is an extra contribution. A typical example of extra contribution is

$$V_\alpha(w_1, \bar{w}_1) = (L_{-1} + \bar{L}_{-1}) \mathcal{O}(w_1, \bar{w}_1), \quad V_\beta(w_2, \bar{w}_2) = (L_{-1} + \bar{L}_{-1}) \mathcal{O}(w_2, \bar{w}_2). \tag{58}$$

In general, the k-th pseudo-Rényi entropy for two linear combination operators only depends on the coefficients of the two-point functions and the combination coefficients,

$$\Delta S^{(k)} \sim \frac{1}{1-k} \log \Bigg\{ d_{\mathcal{O}}^{-(k-1)} \times \frac{\sum\limits_{i_1} \cdots \sum\limits_{j_{2k}} C_{i_1} \ldots C'_{j_{2k}} c_0(\{K_{i_1}\}, \{K'_{j_4}\}) \ldots c_0(\{K_{i_{2k-3}}\}, \{K'_{j_{2k}}\}) c_0(\{\bar{K}_{i_1}\}, \{\bar{K}'_{j_2}\}) \ldots c_0(\{\bar{K}_{i_{2k-1}}\}, \{\bar{K}'_{j_{2k}}\})}{(\sum\limits_i \sum\limits_j C_i C'_j c_0(\{K_i\}, \{K'_j\}) c_0(\{\bar{K}_i\}, \{\bar{K}'_j\}))^k} \Bigg\}. \tag{59}$$

Noticing the current results in RCFTs, one can directly calculate the pseudo-entropy of generic local operators in Liouville CFT, holographic CFTs, non-diagonal CFTs, etc. Since the spectra in such theories have different structures, the associated pseudo-entropy will be highly different from those in RCFTs. In particular, since holomorphic and antiholomorphic conformal blocks have different structures in non-diagonal CFTs, the late-time behavior of the entanglement entropy and pseudo-entropy associated with locally excited states will not be the same as the ones demonstrated in the current paper. We would like to leave them to future work.

Acknowledgements

We thank Wu-zhong Guo, Linlin Huang, Pak Hang Chris Lau, Yang Liu, Yuan Sun, and Long Zhao for the valuable discussions. S.H. would appreciate the financial support from Jilin University, Max Planck Partner group, and Natural Science Foundation of China Grants (No.12075101, No.12047569).

A Reduction of $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}$

With following the standard way [64], we can compute the four-point function $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}$ in this section.

$$
\begin{aligned}
&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\\
=&-\frac{1}{2\pi i}\sum_{i=2}^{4}\oint_{\mathcal{C}(z_i)}\mathrm{d}z(z-z_1)^{-n+1}\langle T(z)\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\\
=&\frac{-1}{2\pi i}\oint_{\mathcal{C}(z_2)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\left\{\frac{\Delta\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}}{(z-z_2)^2}+\frac{\partial_{z_2}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}}{z-z_2}+\mathrm{reg}(z-z_2)\right\}\\
&+\frac{-1}{2\pi i}\oint_{\mathcal{C}(z_3)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\left\{\frac{n(n^2-1)c/12+2n\Delta}{(z-z_3)^{n+2}}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\right.\\
&+\sum_{k=1}^{n-1}\frac{(n+k)\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-(n-k))}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}}{(z-z_3)^{k+2}}+\frac{(\Delta+n)\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}}{(z-z_3)^2}\\
&\left.+\frac{\partial_{z_3}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}}{z-z_3}+\mathrm{reg}(z-z_3)\right\}\\
&+\frac{-1}{2\pi i}\oint_{\mathcal{C}(z_4)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\left\{\frac{\Delta\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}}{(z-z_4)^2}+\frac{\partial_{z_4}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}}{z-z_4}+\mathrm{reg}(z-z_4)\right\}\\
=&\frac{(n-1)\Delta}{z_{21}^n}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}+\frac{-\partial_{z_2}}{z_{21}^{n-1}}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\\
&+\frac{(n-1)\Delta}{z_{41}^n}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}+\frac{-\partial_{z_4}}{z_{41}^{n-1}}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\\
&+(-1)^n\frac{\big(n(n^2-1)c/12+2n\Delta\big)(2n-1)!}{(n+1)!(n-2)!}\frac{\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}}{z_{13}^{2n}}\\
&+(-1)^n\sum_{k=1}^{n-1}\frac{(n+k)!}{(k+1)!(n-2)!}\frac{\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-(n-k))}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}}{z_{13}^{n+k}}\\
&+\frac{(n-1)(\Delta+n)}{z_{31}^n}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}+\frac{-\partial_{z_3}}{z_{31}^{n-1}}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}
\end{aligned}\tag{60}
$$

B Reduction of$\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}$

In terms of (12), the most divergent term of $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}$ should only contain z_{14} and z_{23}, as any terms containing z_{13},z_{24},z_{12} and z_{34} are subleading. So we can firstly expand $\mathcal{O}(1)$'s Virasoro generator,

$$\begin{aligned}
&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\\
\sim&-\frac{1}{2\pi i}\oint_{\mathcal{C}(z_4)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\langle T(z)\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\\
\sim&-\frac{1}{2\pi i}\oint_{\mathcal{C}(z_4)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\langle\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)(\frac{m(m^2-1)c/12+2m\Delta}{(z-z_4)^{m+2}}\mathcal{O}^{\dagger}(4)\\
&+\sum_{k=1}^{m-1}\frac{(m+k)}{(z-z_4)^{k+2}}\mathcal{O}^{-(m-k)\dagger}(4)+\frac{(\Delta+m)}{(z-z_4)^2}\mathcal{O}^{(-m)\dagger}+\frac{\partial_3\mathcal{O}^{(-m)\dagger}(4)}{z-z_4})\rangle_{\Sigma_1}\\
\sim&(-1)^m\frac{(n+m-1)!}{(m+1)!(n-2)!}\frac{m(m^2-1)c/12+2m\Delta}{z_{41}^{n+m}}\langle\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
&+(-1)^n\sum_{k=1}^{m-1}\frac{(n+k-1)!}{(k+1)!(n-2)!}\frac{(m+k)}{z_{14}^{n+k}}\langle\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-(m-k))\dagger}(4)\rangle_{\Sigma_1}\\
&+\frac{(n-1)(\Delta+m)}{z_{41}^{n}}\langle\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\\
&-\frac{\partial_4}{z_{41}^{n-1}}\langle\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}.
\end{aligned}\tag{61}$$

The correlation function with four Virasoro generators is deformed into correlation functions containing no more than 3 Virasoro generators. We can then expand $\mathcal{O}(4)$'s Virasoro generator,

$$\begin{aligned}
&\langle\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\\
\sim&-\frac{1}{2\pi i}\oint_{c(z_1)}\frac{\mathrm{d}z}{(z-z_4)^{m-1}}\langle T(z)\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
\sim&-\oint_{c(z_1)}\frac{\mathrm{d}z}{(z-z_4)^{m-1}}\langle(\frac{\Delta}{(z-z_1)^2}\mathcal{O}(1)+\frac{\partial_1 O(1)}{Z-Z_1})\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
\sim&\frac{(m-1)\Delta}{z_{14}^{m}}\langle(\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}-\frac{\partial_1}{z_{14}^{m-1}}\langle(\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}.
\end{aligned}\tag{62}$$

Form (61) and (62), we can read the exact form of $\mathcal{D}_{1,4}$ introduced in (38)

$$\begin{aligned}
\mathcal{D}_{1,4}=&(-1)^m\frac{(n+m-1)!}{(m+1)!(n-2)!}\frac{m(m^2-1)c/12+2m\Delta}{z_{41}^{n+m}}\\
&+(-1)^n\sum_{k=1}^{m-1}\frac{(n+k-1)!}{(k+1)!(n-2)!}\frac{(m+k)}{z_{14}^{n+k}}(\frac{(m-k-1)\Delta}{z_{14}^{m-k}}-\frac{\partial_1}{z_{14}^{m-k-1}})\\
&+\frac{(n-1)(\Delta+m)}{z_{41}^{n}}(\frac{(m-1)\Delta}{z_{14}^{m}}-\frac{\partial_1}{z_{14}^{m-1}})\\
&-\frac{\partial_4}{z_{41}^{n-1}}(\frac{(m-1)\Delta}{z_{14}^{m}}-\frac{\partial_1}{z_{14}^{m-1}}).
\end{aligned}\tag{63}$$

We can expand $\mathcal{O}(2)$'s Virasoro generator and $\mathcal{O}(3)$'s Virasoro generator in a similar way,

$$
\begin{aligned}
&\langle\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
\sim&-\oint_{c(z_3)}\frac{\mathrm{d}z}{(z-z_2)^{m-1}}\langle\mathcal{O}(1)T(z)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
\sim&-\oint_{c(z_3)}\frac{\mathrm{d}z}{(z-z_2)^{m-1}}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)(\frac{n(n^2-1)c/12+2n\Delta}{(z-z_3)^{n+2}}\mathcal{O}(3)\\
&+\sum_{l=1}^{n-1}\frac{(n+l)}{(z-z_3)^{l+2}}\mathcal{O}^{-(n-l)}(3)+\frac{(\Delta+n)}{(z-z_3)^2}\mathcal{O}^{-n}(3)+\frac{\partial_3}{z-z_3}\mathcal{O}^{(-n)}(3))\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
\sim&(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-(n-l)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
&+\frac{(m-1)(\Delta+n)}{z_{32}^{m}}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}-\frac{\partial_3}{z_{32}^{m-1}}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}.
\end{aligned}
\tag{64}
$$

Finally, we can have

$$
\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\sim\frac{(n-1)\Delta}{z_{23}^{n}}G-\frac{\partial_2}{z_{23}^{n-1}}G,
\tag{65}
$$

where G is $\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}$, and in late-time limit, it's $d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}$.

Combining (61),(62),(64) and (65), we have

$$
\begin{aligned}
&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\\
\sim&(-1)^m\frac{(n+m-1)!}{(m+1)!(n-2)!}\frac{m(m^2-1)c/12+2m\Delta}{z_{41}^{n+m}}\{(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}G-\frac{\partial_2}{z_{23}^{n-l-1}}G)\\
&+\frac{(m-1)(\Delta+n)}{z_{32}^m}(\frac{(n-1)\Delta}{z_{23}^n}G-\frac{\partial_2}{z_{23}^{n-1}}G)-\frac{1}{z_{32}^{m-1}}(\frac{n(n-1)\Delta}{z_{23}^{n+1}}G+\frac{(n-1)\Delta\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_2}{z_{23}^n}G-\frac{\partial_3\partial_2}{z_{23}^{n-1}}G)\}\\
&+(-1)^n\sum_{k=1}^{m-1}\frac{(n+k-1)!}{(k+1)!(n-2)!}\frac{(m+k)}{z_{14}^{n+k}}\{\frac{(m-k-1)\Delta}{z_{14}^{m-k}}[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}G-\frac{\partial_2}{z_{23}^{n-l-1}}G)\\
&+\frac{(m-1)(\Delta+n)}{z_{32}^m}(\frac{(n-1)\Delta}{z_{23}^n}G-\frac{\partial_2}{z_{23}^{n-1}}G)-\frac{1}{z_{32}^{m-1}}(\frac{n(n-1)\Delta}{z_{23}^{n+1}}G+\frac{(n-1)\Delta\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_2}{z_{23}^n}G-\frac{\partial_3\partial_2}{z_{23}^{n-1}}G)]\\
&-\frac{1}{z_{14}^{m-k-1}}[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_1G+\\
&(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-l-1}}G)+\frac{(m-1)(\Delta+n)}{z_{32}^m}(\frac{(n-1)\Delta}{z_{23}^n}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-1}}G)\\
&-\frac{1}{z_{32}^{m-1}}(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_1G+\frac{(n-1)\Delta\partial_1\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_1\partial_2}{z_{23}^n}G-\frac{\partial_1\partial_3\partial_2}{z_{23}^{n-1}}G)]\}\\
&+\frac{(n-1)(\Delta+m)}{z_{41}^n}\{\frac{(m-1)\Delta}{z_{14}^m}[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}G-\frac{\partial_2}{z_{23}^{n-l-1}}G)\\
&+\frac{(m-1)(\Delta+n)}{z_{32}^m}(\frac{(n-1)\Delta}{z_{23}^n}G-\frac{\partial_2}{z_{23}^{n-1}}G)-\frac{1}{z_{32}^{m-1}}(\frac{n(n-1)\Delta}{z_{23}^{n+1}}G+\frac{(n-1)\Delta\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_2}{z_{23}^n}G-\frac{\partial_3\partial_2}{z_{23}^{n-1}}G)]\\
&-\frac{1}{z_{14}^{m-1}}[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_1G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-l-1}}G)+\frac{(m-1)(\Delta+n)}{z_{32}^m}(\frac{(n-1)\Delta}{z_{23}^n}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-1}}G)\\
&-\frac{1}{z_{32}^{m-1}}(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_1G+\frac{(n-1)\Delta\partial_1\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_1\partial_2}{z_{23}^n}G-\frac{\partial_1\partial_3\partial_2}{z_{23}^{n-1}}G)]\}\\
&-\frac{1}{z_{41}^{n-1}}\{\frac{m(m-1)\Delta}{z_{14}^{m+1}}[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}G-\frac{\partial_2}{z_{23}^{n-l-1}}G)+\frac{(m-1)(\Delta+n)}{z_{32}^m}(\frac{(n-1)\Delta}{z_{23}^n}G-\frac{\partial_2}{z_{23}^{n-1}}G)\\
&-\frac{1}{z_{32}^{m-1}}(\frac{n(n-1)\Delta}{z_{23}^{n+1}}G+\frac{(n-1)\Delta\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_2}{z_{23}^n}G-\frac{\partial_3\partial_2}{z_{23}^{n-1}}G)]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(m-1)\Delta}{z_{14}^m}[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_4 G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_4 G-\frac{\partial_4\partial_2}{z_{23}^{n-l-1}}G)\\
&+\frac{(m-1)(\Delta+n)}{z_{32}^m}(\frac{(n-1)\Delta}{z_{23}^n}\partial_4 G-\frac{\partial_4\partial_2}{z_{23}^{n-1}}G)\\
&-\frac{1}{z_{32}^{m-1}}(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_4 G+\frac{(n-1)\Delta\partial_4\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_4\partial_2}{z_{23}^n}G-\frac{\partial_4\partial_3\partial_2}{z_{23}^{n-1}}G)]\\
&-\frac{(m-1)}{z_{14}^m}[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_1 G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_1 G-\frac{\partial_1\partial_2}{z_{23}^{n-l-1}}G)+\frac{(m-1)(\Delta+n)}{z_{32}^m}(\frac{(n-1)\Delta}{z_{23}^n}\partial_1 G-\frac{\partial_1\partial_2}{z_{23}^{n-1}}G)\\
&-\frac{1}{z_{32}^{m-1}}(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_1 G+\frac{(n-1)\Delta\partial_1\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_1\partial_2}{z_{23}^n}G-\frac{\partial_1\partial_3\partial_2}{z_{23}^{n-1}}G)]\\
&-\frac{1}{z_{14}^{m-1}}[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_4\partial_1 G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_4\partial_1 G-\frac{\partial_4\partial_1\partial_2}{z_{23}^{n-l-1}}G)\\
&+\frac{(m-1)(\Delta+n)}{z_{32}^m}(\frac{(n-1)\Delta}{z_{23}^n}\partial_4\partial_1 G-\frac{\partial_4\partial_1\partial_2}{z_{23}^{n-1}}G)\\
&-\frac{1}{z_{32}^{m-1}}(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_4\partial_1 G+\frac{(n-1)\Delta\partial_4\partial_1\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_4\partial_1\partial_2}{z_{23}^n}G-\frac{\partial_4\partial_1\partial_3\partial_2}{z_{23}^{n-1}}G)]\}. \qquad (66)
\end{aligned}
$$

The correlation function of four descendent operators becomes the correlation functions of their corresponding primary operators with some constants and derivatives.

For $i \neq j \neq k \neq l$, we can have

$$
\begin{aligned}
&\partial_i G=\frac{2\Delta\partial_i\eta}{1-\eta}G,\\
&\partial_j\partial_i G=\frac{2\Delta\partial_j\partial_i\eta}{1-\eta}G+\frac{2\Delta(2\Delta+1)\partial_j\eta\partial_i\eta}{(1-\eta)^2}G,\\
&\partial_k\partial_j\partial_i G=\frac{2\Delta\partial_k\partial_j\partial_i\eta}{1-\eta}G+2\Delta(2\Delta+1)[\frac{\partial_j\partial_i\eta\partial_k\eta+\partial_j\partial_k\eta\partial_i\eta+\partial_k\partial_i\eta\partial_j\eta}{(1-\eta)^2}G+\frac{(2\Delta+2)\partial_j\eta\partial_i\eta\partial_k\eta}{(1-\eta)^3}G]\\
&\sim 2\Delta(2\Delta+1)[\frac{\partial_j\partial_i\eta\partial_k\eta+\partial_j\partial_k\eta\partial_i\eta+\partial_k\partial_i\eta\partial_j\eta}{(1-\eta)^2}G+\frac{(2\Delta+2)\partial_j\eta\partial_i\eta\partial_k\eta}{(1-\eta)^3}G],\\
&\partial_l\partial_k\partial_j\partial_i G=\frac{2\Delta\partial_l\partial_k\partial_j\partial_i\eta}{1-\eta}G+\frac{(2\Delta(2\Delta+1))}{(1-\eta)^2}(\partial_k\partial_j\partial_i\eta\partial_l\eta+\partial_k\partial_j\partial_l\eta\partial_i\eta+\partial_l\partial_j\partial_i\eta\partial_k\eta+\partial_k\partial_l\partial_i\eta\partial_j\eta\\
&+\partial_i\partial_j\eta\partial_l\partial_k\eta+\partial_i\partial_k\eta\partial_j\partial_l\eta+\partial_i\partial_l\eta\partial_k\partial_j\eta)G+\frac{2\Delta(2\Delta+1)(2\Delta+2)}{(1-\eta)^3}(\partial_j\partial_i\eta\partial_k\eta\partial_l\eta+\partial_j\partial_k\eta\partial_i\eta\partial_l\eta\\
&+\partial_k\partial_i\eta\partial_j\eta\partial_l\eta+\partial_l\partial_i\eta\partial_k\eta\partial_j\eta+\partial_j\partial_l\eta\partial_k\eta\partial_i\eta+\partial_l\partial_k\eta\partial_i\eta\partial_j\eta)G\\
&+\frac{2\Delta(2\Delta+1)(2\Delta+1)(2\Delta+3)}{(1-\eta)^4}\partial_i\eta\partial_j\eta\partial_k\eta\partial_l\eta G\\
&\sim\frac{(2\Delta(2\Delta+1))}{(1-\eta)^2}(\partial_i\partial_j\eta\partial_l\partial_k\eta+\partial_i\partial_k\eta\partial_j\partial_l\eta+\partial_i\partial_l\eta\partial_k\partial_j\eta)G+\frac{2\Delta(2\Delta+1)(2\Delta+1)}{(1-\eta)^3}(\partial_j\partial_i\eta\partial_k\eta\partial_l\eta+\partial_j\partial_k\eta\partial_i\eta\partial_l\eta\\
&+\partial_k\partial_i\eta\partial_j\eta\partial_l\eta+\partial_l\partial_i\eta\partial_k\eta\partial_j\eta+\partial_j\partial_l\eta\partial_k\eta\partial_i\eta+\partial_l\partial_k\eta\partial_i\eta\partial_j\eta)G+\frac{2\Delta(2\Delta+1)(2\Delta+2)(2\Delta+3)}{(1-\eta)^4}\partial_i\eta\partial_j\eta\partial_k\eta\partial_l\eta G.
\end{aligned}
$$
(67)

References

[1] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105–114, [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[4] H. Casini and M. Huerta, *A Finite entanglement entropy and the c-theorem*, *Phys. Lett. B* **600** (2004) 142–150, [hep-th/0405111].

[5] P. Calabrese and J. L. Cardy, *Entanglement entropy and quantum field theory*, *J. Stat. Mech.* **0406** (2004) P06002, [hep-th/0405152].

[6] A. Kitaev and J. Preskill, *Topological entanglement entropy*, *Phys. Rev. Lett.* **96** (2006) 110404, [hep-th/0510092].

[7] H. Casini, I. Salazar Landea and G. Torroba, *The g-theorem and quantum information theory*, *JHEP* **10** (2016) 140, [1607.00390].

[8] T. Nishioka, *Entanglement entropy: holography and renormalization group*, *Rev. Mod. Phys.* **90** (2018) 035007, [1801.10352].

[9] E. Witten, *APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory*, *Rev. Mod. Phys.* **90** (2018) 045003, [1803.04993].

[10] H. Casini and M. Huerta, *Lectures on entanglement in quantum field theory*, 2201.13310.

[11] M. Van Raamsdonk, *Building up spacetime with quantum entanglement*, *Gen. Rel. Grav.* **42** (2010) 2323–2329, [1005.3035].

[12] J. Maldacena and L. Susskind, *Cool horizons for entangled black holes*, *Fortsch. Phys.* **61** (2013) 781–811, [1306.0533].

[13] M. Rangamani and T. Takayanagi, *Holographic Entanglement Entropy*, vol. 931. Springer, 2017, 10.1007/978-3-319-52573-0.

[14] S. W. Hawking, *Breakdown of Predictability in Gravitational Collapse*, *Phys. Rev. D* **14** (1976) 2460–2473.

[15] S. D. Mathur, *The Information paradox: A Pedagogical introduction*, *Class. Quant. Grav.* **26** (2009) 224001, [0909.1038].

[16] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, *Black Holes: Complementarity or Firewalls?*, *JHEP* **02** (2013) 062, [1207.3123].

[17] G. Penington, *Entanglement Wedge Reconstruction and the Information Paradox*, *JHEP* **09** (2020) 002, [1905.08255].

[18] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, *The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole*, *JHEP* **12** (2019) 063, [1905.08762].

[19] Y. Nakata, T. Takayanagi, Y. Taki, K. Tamaoka and Z. Wei, *New holographic generalization of entanglement entropy*, *Phys. Rev. D* **103** (2021) 026005, [2005.13801].

[20] Y. Aharonov and L. Vaidman, *The two-state vector formalism: an updated review*, *Time in quantum mechanics* (2008) 399–447.

[21] Y. Aharonov, P. G. Bergmann and J. L. Lebowitz, *Time symmetry in the quantum process of measurement*, *Phys. Rev.* **134** (Jun, 1964) B1410–B1416.

[22] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan and R. W. Boyd, *Colloquium: Understanding quantum weak values: Basics and applications*, *Rev. Mod. Phys.* **86** (Mar, 2014) 307–316.

[23] I. Akal, T. Kawamoto, S.-M. Ruan, T. Takayanagi and Z. Wei, *Page curve under final state projection*, *Phys. Rev. D* **105** (2022) 126026, [2112.08433].

[24] A. Mollabashi, N. Shiba, T. Takayanagi, K. Tamaoka and Z. Wei, *Pseudo Entropy in Free Quantum Field Theories*, *Phys. Rev. Lett.* **126** (2021) 081601, [2011.09648].

[25] G. Camilo and A. Prudenziati, *Twist operators and pseudo entropies in two-dimensional momentum space*, 2101.02093.

[26] A. Mollabashi, N. Shiba, T. Takayanagi, K. Tamaoka and Z. Wei, *Aspects of pseudoentropy in field theories*, *Phys. Rev. Res.* **3** (2021) 033254, [2106.03118].

[27] T. Nishioka, T. Takayanagi and Y. Taki, *Topological pseudo entropy*, *JHEP* **09** (2021) 015, [2107.01797].

[28] K. Goto, M. Nozaki and K. Tamaoka, *Subregion spectrum form factor via pseudoentropy*, *Phys. Rev. D* **104** (2021) L121902, [2109.00372].

[29] J. Mukherjee, *Pseudo Entropy in U(1) gauge theory*, *JHEP* **10** (2022) 016, [2205.08179].

[30] W.-z. Guo, S. He and Y.-X. Zhang, *On the real-time evolution of pseudo-entropy in 2d CFTs*, *JHEP* **09** (2022) 094, [2206.11818].

[31] M. Miyaji, *Island for gravitationally prepared state and pseudo entanglement wedge*, *JHEP* **12** (2021) 013, [2109.03830].

[32] Y. Ishiyama, R. Kojima, S. Matsui and K. Tamaoka, *Notes on pseudo entropy amplification*, *PTEP* **2022** (2022) 093B10, [2206.14551].

[33] A. Bhattacharya, A. Bhattacharyya and S. Maulik, *Pseudocomplexity of purification for free scalar field theories*, *Phys. Rev. D* **106** (2022) 086010, [2209.00049].

[34] W.-z. Guo, S. He and Y.-X. Zhang, *Constructible reality condition of pseudo entropy via pseudo-Hermiticity*, 2209.07308.

[35] K. Doi, J. Harper, A. Mollabashi, T. Takayanagi and Y. Taki, *Pseudo Entropy in dS/CFT and Time-like Entanglement Entropy*, 2210.09457.

[36] Z. Li, Z.-Q. Xiao and R.-Q. Yang, *On holographic time-like entanglement entropy*, 2211.14883.

[37] F. C. Alcaraz, M. I. Berganza and G. Sierra, *Entanglement of low-energy excitations in Conformal Field Theory*, *Phys. Rev. Lett.* **106** (2011) 201601, [1101.2881].

[38] M. Nozaki, T. Numasawa and T. Takayanagi, *Quantum Entanglement of Local Operators in Conformal Field Theories*, *Phys. Rev. Lett.* **112** (2014) 111602, [1401.0539].

[39] S. He, T. Numasawa, T. Takayanagi and K. Watanabe, *Quantum dimension as entanglement entropy in two dimensional conformal field theories*, *Phys. Rev. D* **90** (2014) 041701, [1403.0702].

[40] M. Nozaki, *Notes on Quantum Entanglement of Local Operators*, *JHEP* **10** (2014) 147, [1405.5875].

[41] P. Caputa, M. Nozaki and T. Takayanagi, *Entanglement of local operators in large-N conformal field theories*, *PTEP* **2014** (2014) 093B06, [1405.5946].

[42] P. Caputa, J. Simón, A. Štikonas and T. Takayanagi, *Quantum Entanglement of Localized Excited States at Finite Temperature*, *JHEP* **01** (2015) 102, [1410.2287].

[43] W.-Z. Guo and S. He, *Rényi entropy of locally excited states with thermal and boundary effect in 2D CFTs*, *JHEP* **04** (2015) 099, [1501.00757].

[44] P. Caputa and A. Veliz-Osorio, *Entanglement constant for conformal families*, *Phys. Rev. D* **92** (2015) 065010, [1507.00582].

[45] B. Chen, W.-Z. Guo, S. He and J.-q. Wu, *Entanglement Entropy for Descendent Local Operators in 2D CFTs*, *JHEP* **10** (2015) 173, [1507.01157].

[46] P. Caputa, T. Numasawa and A. Veliz-Osorio, *Out-of-time-ordered correlators and purity in rational conformal field theories*, *PTEP* **2016** (2016) 113B06, [1602.06542].

[47] T. Numasawa, *Scattering effect on entanglement propagation in RCFTs*, *JHEP* **12** (2016) 061, [1610.06181].

[48] S. He, *Conformal bootstrap to Rényi entropy in 2D Liouville and super-Liouville CFTs*, *Phys. Rev. D* **99** (2019) 026005, [1711.00624].

[49] W.-Z. Guo, S. He and Z.-X. Luo, *Entanglement entropy in (1+1)D CFTs with multiple local excitations*, *JHEP* **05** (2018) 154, [1802.08815].

[50] L. Apolo, S. He, W. Song, J. Xu and J. Zheng, *Entanglement and chaos in warped conformal field theories*, *JHEP* **04** (2019) 009, [1812.10456].

[51] P. Caputa, T. Numasawa, T. Shimaji, T. Takayanagi and Z. Wei, *Double Local Quenches in 2D CFTs and Gravitational Force*, *JHEP* **09** (2019) 018, [1905.08265].

[52] L. Bianchi, S. De Angelis and M. Meineri, *Radiation, entanglement and islands from a boundary local quench*, 2203.10103.

[53] M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi and K. Watanabe, *Distance between Quantum States and Gauge-Gravity Duality*, *Phys. Rev. Lett.* **115** (2015) 261602, [1507.07555].

[54] M. Miyaji, *Butterflies from Information Metric*, *JHEP* **09** (2016) 002, [1607.01467].

[55] J. Zhang and P. Calabrese, *Subsystem distance after a local operator quench*, *JHEP* **02** (2020) 056, [1911.04797].

[56] X. Wen, P.-Y. Chang and S. Ryu, *Entanglement negativity after a local quantum quench in conformal field theories*, *Phys. Rev. B* **92** (2015) 075109, [1501.00568].

[57] J. Kudler-Flam and S. Ryu, *Entanglement negativity and minimal entanglement wedge cross sections in holographic theories*, *Phys. Rev. D* **99** (2019) 106014, [1808.00446].

[58] J. Kudler-Flam, I. MacCormack and S. Ryu, *Holographic entanglement contour, bit threads, and the entanglement tsunami*, *J. Phys. A* **52** (2019) 325401, [1902.04654].

[59] J. Kudler-Flam, Y. Kusuki and S. Ryu, *Correlation measures and the entanglement wedge cross-section after quantum quenches in two-dimensional conformal field theories*, *JHEP* **04** (2020) 074, [2001.05501].

[60] J. Kudler-Flam, M. Nozaki, S. Ryu and M. T. Tan, *Entanglement of local operators and the butterfly effect*, *Phys. Rev. Res.* **3** (2021) 033182, [2005.14243].

[61] J. Kudler-Flam, Y. Kusuki and S. Ryu, *The quasi-particle picture and its breakdown after local quenches: mutual information, negativity, and reflected entropy*, *JHEP* **03** (2021) 146, [2008.11266].

[62] G. W. Moore and N. Seiberg, *Naturality in Conformal Field Theory*, *Nucl. Phys. B* **313** (1989) 16–40.

[63] P. Calabrese and J. L. Cardy, *Evolution of entanglement entropy in one-dimensional systems*, *J. Stat. Mech.* **0504** (2005) P04010, [`cond-mat/0503393`].

[64] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory.* Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997, 10.1007/978-1-4612-2256-9.