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DEFORMATIONS AND REPRESENTATIONS OF LIE ALGEBROIDS

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ABSTRACT. We study a class of derived representations of a Lie algebroid. The dg-category of these representations enhances the classical category of representations in the sense that the cohomology objects of a derived representation are classical representations. The adjoint complex is canonically an object of our category of representations. Our main contribution is the construction of an extension of the functor of de Rham- Lie cochains to this category, referred to as *Crainic-Moerdijk*-cochains. We show that, when applied to the adjoint representation, we obtain the *Deformation Complex* of Crainic and Moerdijk.

1. INTRODUCTION

The adjoint representation of a Lie algebra $\mathfrak{g}$ is of central importance in the deformation theory of $\mathfrak{g}$. Indeed, the differential graded Lie algebra controlling deformations of $\mathfrak{g}$ is equivalent to $C^*_{Lie}(\mathfrak{g}, \mathfrak{g}^{ad})$. Lie algebroids can be viewed as globalisations of Lie algebras, indeed a k-Lie algebroid over $spec(k)$ is nothing but a Lie algebra. It is desirable then to have a similar cohomological computation of the DGLA controlling deformations of a Lie algebroid over R, or more generally over a smooth k-scheme X. An elegant, if somewhat ad-hoc, solution to the computation of the deformations of $\mathcal{L}$ is given in work of Crainic and Moerdijk, [3], in terms of their *Deformation Complex*, $Def_k(\mathcal{L})$. We note however that it does not fit the general framework of computing deformations as a special case of Lie (algebroid) cohomology, applied to a suitable representation. Indeed, if one attempts this, a problem arises - the classical definition of representation of $\mathcal{L}$ does not provide us with a suitable candidate for an adjoint representation. In their work, [1], Abad and Crainic define the notion of *representation up to homotopy.* These objects generalise classical representations of $\mathcal{L}$, and their cohomology objects are representations of $\mathcal{L}$ in the classical sense.

Further, when $\mathcal{L}$ is equipped with the additional structure of a connection ∇, they can define an adjoint representation and prove that it has the desired properties. We view the additional choice of a connection ∇ as slightly unsatisfactory. In specifically algebro-geometric contexts it is very unsatisfactory as the requisite connection need not exist, indeed the Atiyah class of $\mathcal{L}$ provides an obstruction to existence. It is the goal of this note to propose studying instead a different definition (appearing in the paper [2] where it is also referred to as a *representation up to homotopy*), also weakening the classical definition of a representation of $\mathcal{L}$. Further, we define a suitable cohomological complex attached to such an object, and show that in the special case of the adjoint complex (which now exists canonically) we recover $Def_k(\mathcal{L})$. The basic idea, present already in [2] and [1], is to impose a certain relation only homotopically. This homotopy is part of the data of a derived representation in our sense. Crucially, this extra data allows us to *deform* the notion of R-linear de Rham-Lie cochains, providing us with a definition of cochains with values in a derived representation. We stress here that one of our goals is to ensure that our constuctions are suited to the algebro-geometric context.

2. Derived Representations

2.1. **Basic Properties and Definitions.** k will denote a field of characteristic 0 throughout and R will denote a finite type k-algebra, assumed smooth for simplicity. We let $\mathcal{L}$ denote a Lie algebroid for (R, k), with anchor map $\rho : \mathcal{L} \to \Theta_R := Der_k(R, R)$. We will assume throughout that the $\mathcal{L}$ is projective as an R-module. We will write $\mathcal{L}^{Lie}$ for the underlying Lie_k-algebra of $\mathcal{L}$. Let us recall the notion of *representation* of $\mathcal{L}$.

Definition 2.1. *We say that V is a* pre-representation *of $\mathcal{L}$ if it admits both an R-module structure (written multiplicatively), and an $\mathcal{L}^{Lie}$-module structure, written $[l, -]$, so that the relation*

$$[l, rv] = r[l, v] + \rho(l)(r)v$$

holds identically. If in addition it holds identically that $[rl, v] = r[l, v]$, then we say that V is a representation *of $\mathcal{L}$. There are evident categories of representations and pre-representations. We denote the category of representations $Rep_k(\mathcal{L})$.*

Remark. We note here that we can sheafify the above definition to make sense of Lie algebroids over a smooth k-scheme X, so that the above definition is the special case of $X = spec(R)$. All results proven in this note hold in this more general context, although we will often work locally and assume that X is affine.

Remark.
- $\bigotimes_R$ endows the category of pre-representations with a symmetric monoidal product.
- The R-linear dual of a pre-representation is naturally a pre-representation.
- The two term complex, ρ, naturally admits the structure of a (complex of) pre-representations. It will be called the *adjoint* pre-representation and denoted ρ^{ad}.

Definition 2.2. *Let (V, ∂) be a cohomologically graded finite length complex of pre-representations of $\mathcal{L}$. We note that the remarks above imply that $V[-1] \otimes_R \Theta_R \otimes_R \mathcal{L}^\vee$ is naturally a pre-representation. A morphism of pre-representations,*

$$\Gamma_V : V \to V[-1] \otimes_R \Theta_R \otimes_R \mathcal{L}^\vee,$$

is called a contraction map *if the following relation holds identically:*

$$[\partial, \Gamma_V](v)(r, l) = [rl, v] - r[l, v].$$

Finally, we define a derived representation *of $\mathcal{L}$ to be a tuple (V, ∂, Γ_V) as above. There is an evident notion of morphism of derived representations and will denote the category of derived representations as $dRep_k(\rho)$.*

Remark. It is perhaps more natural to encode the contraction map as a morphism

$$\Omega^1_R \otimes_R \mathcal{L} \longrightarrow End_R(V, V)[-1],$$

with the property that $[\partial, \Gamma(df \otimes l)] = [fl, -] - f[l, -]$. The formulation of Definition 3.2 is however more convenient from the point of view of Γ-symbols, which we introduce below.

Remark. We list here some properties of the category $dRep_k(\rho)$, all of these properties can be verified immediately from the definitions.

- There is an embedding $Ch(Rep_k(\mathcal{L})) \to dRep_k(\rho)$ defined by demanding the contraction map Γ vanish.

- There are natural functors $H^i(-) : dRep_k(\rho) \to Rep_k(\mathcal{L})$. Indeed, Γ witnesses the relation $[rl, -] = r[l, -]$ *homotopically.*
- Defining $\Gamma_{V \otimes_R W} := \Gamma_V \otimes_R id_W + id_V \otimes_R \Gamma_W$ endows $dRep_k(\rho)$ with the structure of a symmetric monoidal category.
- The adjoint complex, ρ, naturally admits the structure of a pre-representation. The contraction Γ_ρ, is defined to be the natural map

  $$\Gamma_\rho := id_{\Theta_R} \otimes coev_L : \Theta_R \to \Theta_R \otimes_R \mathcal{L} \otimes_R \mathcal{L}^\vee \cong \mathcal{L} \otimes_R \Theta_R \otimes_R \mathcal{L}^\vee,$$

  where $coev_{\mathcal{L}}$ is dual to the pairing between $\mathcal{L}$ and the dual $\mathcal{L}^\vee$.

We state now the main theorem of this note, the proof and relevant definitions will be given further below.

Theorem 2.1. *The functor of dR-Lie cochains admits a natural extension to a functor (*Crainic-Moerdijk-*cochains)*

$$C^*_{CM}(\rho, -) : dRep_k(\rho) \to Ch(k).$$

This functor preserves quasi-isomorphisms and thus induces a functor on the corresponding $(\infty, 1)$-localised categories. Moreover, there is an isomorphism

$$C^*_{CM}(\rho, \rho^{ad}) \cong Def_k(\mathcal{L}).$$

2.2. **De Rham-Lie Cochains for Derived Representations.** In this subsection, which forms the technical heart of this note, we will show how to extend the functor of de Rham-Lie cochains (see Definition 3.4 below),

$$C^*_{dR,Lie}(\mathcal{L}, -) : Ch(Rep_k(\mathcal{L})) \to Ch(k),$$

to a functor,

$$C^*_{CM}(\rho, -) : dRep_k(\rho) \to Ch(k).$$

The subtle point is that we can no longer naively take alternating R-multilinear maps from $\mathcal{L}$ into V as our cochains. This is because the Lie-cohomology differential will not preseve the R-multilinearity property, as the equation $[rl, -] = r[l, -]$ now only holds in the *homotopical* sense. The solution to this problem involves defining the notion of Γ-*symbol* for a k-multilinear map $D \in Hom_k(\wedge_k^n \mathcal{L}, V)$. This is heavily inspired by the *Deformation Complex* of Crainic-Moerdijk.

Definition 2.3. *Let V and W be two R-modules, equipped with a map*

$$\Gamma : W \to V \otimes_R \Theta_R \otimes_R \mathcal{L}^\vee,$$

*and let $D \in Hom_k(\wedge_k^n \mathcal{L}, V)$ and $E \in Hom_R(\wedge_k^{n-1} \mathcal{L}, W)$. We say that E is a Γ-*symbol *for D, and write $E \in Symb^\Gamma(D)$, if the following equation holds identically for all $r \in R, l_i \in \mathcal{L}$;*

$$D(rl_1 \wedge l_2 \wedge ... \wedge l_n) = rD(l_1 \wedge ... \wedge l_n) + \Gamma(E(l_2 \wedge ... \wedge l_n))(r, l_1).$$

Remark. We think of the E as measuring the difference, relative to Γ, between D and an R-multilinear function. In particular, note that for $\Gamma = 0$, the relation ensures that D is R-multilinear.

Of course, our choice of notation here is meant to be suggestive - the contraction map, Γ_V, attached to a an object $V \in dRep_k(\rho)$, gives us a means by which to require that $\mathcal{L}^{Lie}$-cochains valued in V be *almost* R-multilinear. For convenience we recall here the definition, for a Lie-algebra $\mathfrak{g}$ (resp. Lie algebroid $\mathcal{L}$), of the cohomology with coefficients in a cohomologically graded complex of representations. See [3] for a more detailed reference, and in particular one taking more care with signs.

Definition 2.4. *Let (V, ∂_V) be a complex of $\mathfrak{g}$-representations. Then the n-th space of $\mathfrak{g}$-cochains with coefficients in V is*

$$\bigoplus_{i+j=n} Hom_k(\wedge_k^i \mathfrak{g}, V^j).$$

The differential is $\delta_{tot} = \delta_{Lie} + \partial_V$, where we define

$$\delta_{Lie} D(x_1 \wedge ... \wedge x_{n+1}) = \sum_i (-1)^i [x_i, D(x_1 \wedge ... \wedge \widehat{x_i} \wedge ... \wedge x_{n+1})]$$

$$+ \sum_{i,j} (-1)^{i+j} D([x_i, x_j] \wedge ... \wedge \widehat{x_i} \wedge ... \wedge \widehat{x_j} \wedge ... \wedge x_{n+1}).$$

The resulting complex is denoted $C^*_{Lie}(\mathfrak{g}, V)$.

If $\mathcal{L}$ is a Lie algebroid over R and V a representation of $\mathcal{L}$, then the sub-space of $C^*_{Lie}(\mathcal{L}^{Lie}, V)$ consisting of R-linear cochains is preserved by the differential. We denote the resulting complex $C^*_{dR,Lie}(\mathcal{L}, V)$.

We now define the spaces of n-cochains for the functor $C^*_{CM}(\rho, -)$.

Definition 2.5. *Let $V \in dRep_k(\rho)$ be a derived representation with contraction map Γ_V. We define $C^n_{\Gamma,Lie}(\mathcal{L}^{Lie}, V)$ to be the subspace of $C^n_{Lie}(\mathcal{L}^{Lie}, V)$ consisting of those tuples*

$$\{(D_{i,j}) \in \bigoplus_{i+j=n} Hom_k(\wedge^i_k \mathcal{L}^{Lie}, V^j) | D_{i-1,j+1} \in Symb^{\Gamma_{V_{j+1}}}(D_{i,j})\}.$$

We come now to the main technical result of this note, namely that the Lie-cohomology differential preserves the Γ-symbol condition. In order to simplify notation we will now write simply $Symb^\Gamma$, where this is to be understood as $Symb^{\Gamma_{V_j}}$ for the appropriate index j.

Lemma 2.2. *The differential δ_{tot} maps $C^n_{\Gamma,Lie}(\mathcal{L}^{Lie}, V)$ to $C^{n+1}_{\Gamma,Lie}(\mathcal{L}^{Lie}, V)$. Explicitly, if $(D_{i,j}) \in C^n_{\Gamma,Lie}(\mathcal{L}^{Lie}, V)$, then*

$$\delta_{Lie}(D_{i-1,j+1}) + \partial_V(D_{i,j}) \in Symb^\Gamma(\delta_{Lie}(D_{i,j}) + \partial_V(D_{i+1,j-1})).$$

Proof. We ignore signs, and begin by grouping the terms arising in

$$(\delta_{Lie} D)(rl_1 \wedge l_2 \wedge ... \wedge l_{n+1})$$

into four different types.

(1) Terms of the form

$$D([l_i, l_j] \wedge rl_1 \wedge ... \wedge \widehat{l_i} \wedge ... \wedge \widehat{l_j} \wedge ... \wedge l_{n+1})$$

for 1 different from i and j,

(2) those of the form

$$[l_i, D(rl_1 \wedge ... \wedge \widehat{l_i} \wedge ... \wedge l_{n+1})]$$

for i not equal to 1,

(3) those of the form $D([rl_1, l_i] \wedge ... \wedge l_{n+1})$

(4) and finally the remaining term

$$[rl_1, D(l_2 \wedge ... \wedge l_{n+1})].$$

Recall that by assumption $D_{i-1,j+1}$ is a Γ-symbol for $D_{i,j}$. Terms of type (1) pose no problem, the defect of R-linearity is exactly given by the symbol in such cases. Already for types (2) and (3) we obtain extra terms not accounted for in the Γ-symbol relation. The point is that such cancel out. Indeed, simplifying the notation somewhat liberally as we go, we expand type (2) terms using the definition of a Γ-symbol as

$$[l_i, rD_{i,j}(l_1 \wedge ...) + \Gamma(D_{i-1,j+1})(l_2 \wedge ... \wedge l_{n+1})(r, l_1)] =$$

$$r[l_i, D] + \rho(l_i)(r)D + [l_i, \Gamma(D)].$$

We refer to these three subtypes of terms as of types (2.1), (2.2) and (2.3) respectively. Similarly we further expand the summands of type (3) to obtain

$$D(\rho(l_i)(r)l_1 \wedge ...) + D(r[l_1, l_i] \wedge ...) =$$

$$\rho(l_i)(r)D + \Gamma(D)(\rho(l_i(r), l_i) + rD([l_1, l_i] \wedge ...) + \Gamma(D)(r, [l_1, l_i]),$$

which subtypes of summands we refer to as of type (3.1),...(3.4). Summands of types (2.1) and (3.3) are easily disposed of. Summands of type (2.2) cancel with those of type (3.1). This leaves those of type (2.3), (3.2) and (3.4). At this point we pause to record a relation between Γ and ρ, which is an immediate consequence of the assumption that

$$\Gamma_V : V \to V[-1] \otimes_R \Theta_R \otimes_R \mathcal{L}^\vee$$

is a map of pre-representations. Explicitly, for all tuples (v, r, l_1, l_2), we have a relation

$$[l_1, \Gamma(v)(r, l_2)] = \Gamma([l_1, v])(r, l_2) + \Gamma(v)(\rho(l_1)(r), l_2) + \Gamma(v)(r, [l_1, l_2]).$$

Applying this relation to the sum of terms of types (2.3), (3.2) and (3.4) we see that they cancel leaving only terms of the form $\Gamma([l_i, D_{i-1,j+1}])(r, l_1)$ which accounts for the remaining summands of $\delta_{Lie}D_{i-1,j+1}$. To prove the lemma, we must now account for summand (4), as well as for $\partial D_{i+1,j-1}$ and $\partial D_{i,j}$. The contraction relation for Γ immediately implies the desired result and thus we are done. $\square$

Remark. We note now that we can functorially equip the functor $C^*_{CM}(\rho,-)$ with an increasing filtration (finite when restricted to derived representations of finite cohomological amplitude) called the *Hodge filtration*, and denoted $\mathcal{F}_{hdg}$. Namely, $\mathcal{F}^n_{hdg}C^*_{CM}$ consists of those cochains given by tuples $(D_{i,j})_{i,j}$ so that for all $i<n$ we have $D_{i,j}=0$.

We record now an obvious lemma, which makes precise the idea that our cochain complex consists of k-linear cycles which are *almost* R-linear.

Lemma 2.3. *We have an isomorphism of complexes of k-modules,*

$$Gr^j_{\mathcal{F}_{hdg}}C^*_{CM}(\rho,V)\cong Map_R(\wedge^j_R\mathcal{L},V).$$

Proof. The only thing to note is that the lowest, with respect to the Hodge filtration, non-zero term must be R-linear by the Γ-symbol condition. □

Theorem 2.4. *The functor of de Rham- Lie cochains admits a canonical extension to the category of derived representations,*

$$C^*_{CM}(\rho,-):dRep_k(\rho)\to Ch(k).$$

Moreover, this functor takes quasi-isomorphisms to quasi-isomorphims.

Proof. We set $C^*_{CM}(\rho,(V,\partial,\Gamma)):=C^*_{\Gamma,Lie}(\mathcal{L}^{Lie},V)$ with the differential constructed above. It remains to be checked that in the case of a complex of classical representations of $\mathcal{L}$ we recover the usual definition. Recall that in this case we have $\Gamma=0$, and so the Γ-symbol relation says that each $D_{i,j}$ is in fact R-multilinear, and so we recover the classic definition. Finally, one checks that exact triangles of complexes of R-modules are sent to exact triangles of complexes of vector spaces by noting that this is true upon taking the associated graded with respect to the filtation $\mathcal{F}_{hdg}$, which in turn is true because $\mathcal{L}$ is projective. This implies that quasi-isomorphisms map to quasi-isomorphisms immediately. □

Corollary 2.1. *Let V be a derived representation and recall that all the cohomology objects $H^i(V)$ are classical representations. Then we have a convergent spectral sequence*

$$H^i_{dR,Lie}(\rho,H^j(V))\implies H^{i+j}_{CM}(\rho,V).$$

Proof. One considers the canonical filtration on V, with associated graded given by $H^i(V)[-i]$ in degree i and takes the associated spectral sequence. □

We record now a simple lemma, which will be of use in what follows;

Lemma 2.5. *The functor $C^*_{CM}(\rho^{ad}, -)$ of cochains is symmetric monoidal, that is to say there is a natural morphism*

$$C^*_{CM}(\rho^{ad}, V_1) \otimes C^*_{CM}(\rho^{ad}, V_2) \to C^*_{CM}(\rho^{ad}, V_1 \otimes V_2).$$

Proof. The functor of cochains for the underlying Lie algebra $\mathcal{L}^{Lie}$ is symmetric monoidal and one can easily check that the Γ-symbol condition is preserved. □

Theorem 2.6. *We have an isomorphism of complexes (in fact of differential graded Lie algebras),*

$$C^*_{CM}(\rho, \rho^{ad}) \cong Def_k(\mathcal{L}),$$

between the dR-Lie cochains with coefficients in the adjoint derived representation and the Deformation Complex *of Crainic-Moerdijk (*[3]*).*

Proof. This can be confirmed very easily. We note that the DGLA structure exists as ρ^{ad} is a Lie algebra object in $dRep_k(\rho)$ and $C^*_{CM}(\rho^{ad}, -)$ is symmetric-monoidal. □

Remark. As mentioned in the introduction, the DGLA $Def_k(\mathcal{L})$ is known, by work of Crainic-Moerdijk ([3]), to control deformations of the Lie algebroid $\mathcal{L}$ and so the above is a generalisation of the well known cohomological description of deformations of a Lie algebra.

Corollary 2.2. *If V is a representation of $\mathcal{L}$, then the DGLA $Def_k(\mathcal{L})$ acts on cochains with values in $\mathcal{L}$*

Proof. We note that this result is proposition 6 in subsection 4.8 of [3]. In our language a strengthening holds, the Lie algebra object ρ^{ad} in $dRep_k(\rho)$ acts as a Lie algebra on any module. The result follows from taking cochains. □

3. Comparisons With Other Constructions In The Literature

We should say something about how the notion of module studied in this paper compares with some other homotopical weakenings of Lie algebroid representations. We will focus mostly on the case of Vaintrob's construction from [7], as it is the most suited to the algebraic context. Indeed, as mentioned in the introduction, the definition of [1] does not produce an adjoint module in the algebraic context. We stress again that this is not merely an issue of non-canonicity, a connection on $\mathcal{L}$ need not exist. For example, one cannot define the adjoint module for the Lie algebroid corresponding to a Poisson structure on a smooth projective surface of general type, which does not admit a connection on its canonical bundle (whence the sheaf of forms certainly cannot admit a connection.)

There is also a definition given in the paper [4], which again relies on some constructions which need not exist algebraically - for example see the notion of *horizontal lift* in definition 2.10 of loc. cit, again there are algebro-geometric obstructions to the existence of such.

3.1. **Vaintrob's Lie Algebroid Modules.**

In the paper [7] a notion of module, studied further in work of Mehta, [6], is given. In spirit it is essentially an instance of Koszul duality, and in the special case of the tangent Lie algebroid it goes back to Kapranov in [5]

Definition 3.1. *A* weak module *in the sense of [Vai] is defined to be a module over the differential graded algebra* $\mathcal{A} := C^*_{dR,Lie}(\mathcal{L}, R)$. *The category of such will simply be denoted* $\mathcal{A} - mod_k$.

Remark. In [6] these are referred to as *Lie algebroid modules.*

We note that

- a weak module M has an associated cohomology, which is just the cohomology of the underlying differential graded vector space M,
- there is a good notion of adjoint module, given by the differential graded Lie algebra, $Der(\mathcal{A})$, of derivations of the differential graded algebra (DGA) $\mathcal{A}$,

- the resulting cohomology is isomorphic to the deformation cohomology of $\mathcal{L}$,
- there is a functor $Rep_k(\mathcal{L}) \to \mathcal{A}$ obtained by taking cochains with values in M.

With an eye to comapring the two approaches, let us note that the construction of cochains in this note can be seen as a functor from $dRep_k(\rho)$ to the category $\mathcal{A}-mod_k$ of weak modules.

Lemma 3.1. *The functor $C^*_{CM}(\rho, -)$ is naturally valued in $\mathcal{A} - mod_k$. It preserves cohomology and takes the adjoint module in $dRep_k(\rho)$ to the adjoint module in $\mathcal{A} - mod_k$.*

Proof. Recall that $dRep_k(\rho)$ is symmetric monoidal with unit object the unit representation of $\mathcal{L}$. The lemma now follows from the fact that C^*_{CM} is symmetric-monoidal and that it returns $\mathcal{A}$ upon application to the unit $\mathcal{L}$-module. That cohomology is preserved is essentially tautological and that the respective adjoint modules match up can be found in [3] (subsection 2.5 corollary 1.) □

So we have seen that the obvious innovation associated to these two notions of module (i.e. the canonical construction of an adjoint module) match up functorially. A notable difference between the two notions arises when one considers the appropriate notion of quasi-isomorphism in the respective categories. In $dRep_k(\mathcal{L})$ we simply take as our quasi-isomorphisms those maps of derived representations which are quasi-isomorphisms of the underlying complexes of vector spaces. Now this is very simple, and one would perhaps like to take it as the definition of quasi-isomorphism in $\mathcal{A} - mod_k$. However, if one does this the natural map $Rep_k(\mathcal{L}) \to \mathcal{A} - mod_k$ will lose a lot of information, whence the appropriate notion of quasi-isomorphism must be more complicated than that in $dRep_k(\mathcal{L})$. Indeed the natural functor

$$C^*_{dR,Lie}(\mathcal{L}, -) : Rep_k(\mathcal{L}) \to \mathcal{A} - mod_k,$$

will send some non-zero objects to zero. For a simple example one could take the $T_{\mathbb{C}^*}$-module corresponding to the line bundle $\mathcal{O}$ with flat connection $\partial_z - \frac{\lambda}{z}$, where λ is non-integral.

4. Example computations

Let us compute deformation cohomology in some extreme cases, using the fact that is is cohomology with coefficients in the derived representation ρ^{ad}, in the sense described in this paper.

- Let us take $\mathcal{L} := T_X$, the tangent Lie algebroid to a smooth scheme X. Then the anchor map is the identity, ρ^{ad} is contractible, and thus cohomology vanishes by Theorem 3.4. Note that this is corollary 2 in subsection 4.2 of [3].

- Let $\mathcal{F} \subset T_X$ be a foliation, then ρ^{ad} is quasi-isomorphic to $\nu[-1]$, a shift of the *Bott representation*, whence we obtain a shift of cohomology with coefficients in ν. Note this is also proposition 4 in section 4.5 of [3].

- Let $\mathcal{L}$ be the Lie algebroid associated to a constant rank infinitesimal action of a Lie algebra $\mathfrak{g}$ on a scheme X. Then Theorem 3 of subsection 4.7 of [3] gives a long exact sequence relating deformation cohomology of $\mathcal{L}$ with cohomologies in the isotropy Lie algebra and the normal bundle to the induced foliation. This is immediate from the spectral sequence of corollary 3.1. Indeed the relevant complex has a length two filtration, so really we are dealing with a long exact sequence.

- Let L be a line bundle on a smooth scheme X, and consider the *Atiyah Lie algebroid* $At_X(L)$, i.e. the differential operators on L with order at most 1. Then the anchor map $At_X(L) \to T_X$ is the (surjective) symbol map. The adjoint complex is thus isomorphic to $\mathcal{O}_X[0]$, whence we obtain cohomology with coefficients in the structure sheaf, $H^*(X, \mathcal{O}_X)$. Recalling that $H^1(X, \mathcal{O}_X)$ controls first order deformations of any line bundle, we deduce that all first order deformations of $At_X(L)$ are induced from those of L. A similar result holds for the Atiyah algebroid of a higher rank vector bundle E, we obtain cohomology $H^*(X, End_X(E))$ as deformation cohomology and again we see that all deformations of the algebroid are induced from those of the bundle.

Let us further say something about CM-cochains in the case of a derived module of length greater 2, unforturnately we are not able to prove the following so we state it as a conjecture. In fact that there should be a theory of derived representations and their cochains resulting in an isomorphism as claimed was the initial inspiration for this note. We write H^*_{CM} for the cohomology of C^*_{CM} and refer to it as *CM-cohomology.*

Conjecture 4.1. *There is an isomorphism $H^*_{CM}(\rho, Sym(\rho^{ad})) \cong HH^*(\mathcal{U}_R(\mathcal{L}))$ between CM-cohomology with coefficients in the symmetric algebra of the adjoint module and Hochschild cohomology of the universal enveloping algebra of $\mathcal{L}$.*

Remark.

- If $R = k$, so that $\mathcal{L}$ is just a Lie algebra $\mathfrak{g}$, this is a well known corrollary of the PBW theorem and a computation of Chevalley-Eilenberg.
- More generally if the anchor map ρ vanishes, so that $\mathcal{L}$ is an R-Lie algebra, one can check that this amounts to a combination of the Hochschild-Kostant-Rosenberg isomorphism and the corresponding result for Lie algebras.
- If $\mathcal{L}$ is the tangent Lie algebroid then ρ^{ad} vanishes, so $Sym(\rho^{ad})$ is the unit $\mathcal{L}$ module and the universal enveloping algebra is the algebra of differential operators on $X = spec(R)$. In this case it is well known that one obtains de Rham cohomology of X for both spaces.

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