Geometric Pricing: How Low Dimensionality Helps in Approximability

Parinya Chalermsook∗ Khaled Elbassioni† Danupon Nanongkai‡ He Sun∗

Abstract

Consider the following toy problem. There are $m$ rectangles and $n$ points on the plane. Each rectangle $R$ is a consumer with budget $B_R$, who is interested in purchasing the cheapest item (point) inside $R$, given that she has enough budget. Our job is to price the items to maximize the revenue. This problem can also be defined on higher dimensions. We call this problem the geometric pricing problem.

In high dimensions, the above problem is equivalent to the unlimited-supply profit-maximizing pricing problem, which has been studied extensively in approximation algorithms and algorithmic game theory communities. Previous studies suggest that the latter problem is too general to obtain a sub-linear approximation ratio (in terms of the number of items) even when the consumers are restricted to have very simple purchase strategies.

In this paper, we study a new class of problems arising from a geometric aspect of the pricing problem. It intuitively captures typical real-world assumptions that have been widely studied in marketing research, healthcare economics, etc. It also helps classify other well-known pricing problems, such as the highway pricing problem and the graph vertex pricing problem on planar and bipartite graphs. Moreover, this problem turns out to have close connections to other natural geometric problems such as the geometric versions of the unique coverage and maximum feasible subsystem problems.

We show that the low dimensionality arising in this pricing problem does lead to improved approximation ratios, by presenting sublinear-approximation algorithms for two central versions of the problem: unit-demand uniform-budget min-buying and single-minded pricing problems. Our algorithm is obtained by combining algorithmic pricing and geometric techniques. These results suggest that considering geometric aspect might be a promising research direction in obtaining improved approximation algorithms for such pricing problems. To the best of our knowledge, this is one of very few problems in the intersection between geometry and algorithmic pricing areas. Thus its study may lead to new algorithmic techniques that could benefit both areas.

∗IDSIA, Lugano, Switzerland, Email: parinya@uchicago.edu. Work partially done while the author was at the University of Chicago, Chicago, IL, USA, and Max-Planck-Institut für Informatik, Saarbrücken, Germany.
†Max-Planck-Institut für Informatik, Saarbrücken, Germany. Email: {elbassio,hsun}@mpi-inf.mpg.de
‡Theory and Applications of Algorithms Research Group, University of Vienna, Vienna, Austria. Email: danupon@gmail.com. Work partially done while the author was at Georgia Institute of Technology, Atlanta, GA, USA and Max-Planck-Institut für Informatik, Saarbrücken, Germany.
1 Introduction

This paper studies a geometric version of two central unlimited-supply pricing problems. We are given a set \( \mathcal{I} \) of \( n \) consumers and a set \( \mathcal{C} \) of \( m \) items. Every item \( I \in \mathcal{I} \) is represented by a point \( I = (I[1], \ldots, I[d]) \in \mathbb{R}^d_+ \), where \( \mathbb{R}^d_+ \) denotes the set of non-negative reals and \( I[j] \) expresses the quality of item \( I \) in the \( j \)-th attribute. Every consumer \( C \in \mathcal{C} \) is represented by a point \( C = (C[1], \ldots, C[d]) \in \mathbb{R}^d_+ \), where \( C[j] \) is the criterion of consumer \( C \in \mathcal{C} \) in the \( j \)-th attribute. Each consumer \( C \) is additionally equipped with budget \( B_C \in \mathbb{R}^d_+ \) and a consideration set

\[
S_C = \{ I : I[j] \geq C[j], \text{ for all } 1 \leq j \leq d \}.
\]

In the \( d \)-dimensional uniform-budget unit-demand min-buying pricing problem (\( d \)-UUDP-MIN), once we assign prices to items, each consumer \( C \) will buy the cheapest item \( I \) in \( S_C \) if the price of item \( I \) is at most \( B_C \). In the \( d \)-dimensional single-minded pricing problem (\( d \)-SMP), consumer \( C \) will buy the all items in \( S_C \) if the total price of those items is at most \( B_C \). The objective is to set the price of items in \( \mathcal{I} \) in order to maximize the revenue. That is, we want to find \( p : \mathcal{I} \to \mathbb{R} \geq 0 \) that maximizes \( \sum_{C \in \mathcal{C}, \min_{I \in S_C} p(I) \leq B_C} \min_{I \in S_C} p(I) \) in the case of \( d \)-UUDP-MIN and \( \sum_{C \in \mathcal{C}, \sum_{I \in S_C} p(I) \leq B_C} \sum_{I \in S_C} p(I) \) in the case of \( d \)-SMP. Fig. 1 illustrates the problem: Each item corresponds to a point in the plane. The consideration set of each consumer \( C \) is represented by an (unbounded) axis-parallel rectangle with point \( C \) as a lower-left corner.

The above problems when \( d \) is unbounded (called UUDP-MIN and SMP) have been widely studied recently (e.g., [44, 45, 46, 14, 45, 6]) and are known to be \( O(\log m) \)-approximable [1]; so we have a reasonable approximation guarantee when there are not many consumers. However, in many cases, one would expect the number of consumers to be much larger than the number of items \( n \). In this case, we are still at the trivial \( O(n) \) approximation ratio, and there are evidences that suggest that getting a sub-linear approximation ratios might be impossible: Unless \( \text{NP} \subseteq \text{DTIME}(n^{\text{polylog} n}) \), these problem are hard to approximate within a \( 2^{\log^{1-\epsilon} n} \) for any constant \( \epsilon > 0 [15] \). Moreover, assuming a stronger (but still plausible) assumption, these problems are hard to approximate to within a factor of \( n^\epsilon \) for some \( \epsilon > 0 [44] \).

Motivated by various types of assumptions, the pricing problems with special structures have been studied (e.g., when there is a price-ladder constraint [44, 14, 45, 46, 14], consideration sets are small [6, 14] or consideration sets correspond to paths on graphs [6, 22, 31, 22, 27]). In these cases, better approximation ratios are usually possible.

In this paper we consider the geometric structure of pricing problems arising naturally from real-world scenarios, which turns out to be quite general. Our motivation is two-fold: We hope that the geometric structures will lead to better approximation algorithms, and we found these problems interesting on their own as they have connections to other pricing and geometric problems.

Our problems are motivated by the following simple observation on the consumers’ behavior. Consider a setting where we sell cars. If a consumer has car \( A \) with horse power 130HP in her consideration set, she would not mind buying car \( B \) with horse power 150HP. Maybe she does not want \( B \) because it is less energy-efficient or has lower reputation. But, if we list all attributes of the cars that people care about and it happens that \( B \) is not worse than \( A \) in all other aspects, then \( B \) should also be in the list.
In particular, instead of looking at a full generality where each consumer $C$ considers any set of items $S_C$, it is reasonable to assume that each consumer has some criterion in mind for each attribute of the cars, and her consideration set consists of any car that passes all her criteria, i.e. consumers judge items according to their attributes. This natural assumption has been a model of study in other fields such as marketing research, healthcare economics and urban planning. It is referred to as the \textit{attribute-based screening process}. In particular, using criteria to define consideration sets as in Eq. \eqref{consider} is called \textit{conjunctive screening rule}. Besides being natural, this assumption has been supported by a number of studies where it is concluded that consumers typically use a conjunctive screening rule in obtaining their consideration sets (see further detail in Section \ref{related_work}).

It is also interesting that $d$-SMP captures many previously studied problems as special cases. For example, 2-SMP generalizes the highway pricing problem \cite{32, 6, 23, 31} and thus our algorithmic results on 2-SMP can immediately be applied to this problem. Moreover, 3-SMP generalizes the upward case of the tollbooth pricing problem \cite{22, 10} as well as the graph vertex pricing problem on planar graphs \cite{6, 16}. 4-SMP generalizes the unlimited-supply version of the \textit{exhibition} problem \cite{19}, the graph vertex pricing problem on bipartite graphs \cite{6, 39}, and the “rectangle version” of the \textit{unique coverage problem} (UC) \cite{20}, which are the geometric variants of UC studied recently in \cite{24, 37}.

Moreover, SMP is a special case of the \textit{maximum feasible subsystem with 0/1 coefficients} problem (MRFS) \cite{22}. Elbassioni et al. \cite{22} showed that a very special geometric version of MRFS (the “interval version”) admits much better approximation ratios than the general one. A geometric MRFS can be seen as a special case of “2-MRFS” in our terminologies, and it is thus interesting whether “d-MRFS” is easier than general MRFS for other values of $d$. Our geometric SMP is a special case of $d$-MRFS. Thus, solving $d$-SMP serves as the first step towards solving $d$-MRFS.

\subsection{Our Results and Techniques}

We show that geometric structures lead to breaking the linear-approximation barrier: While the pricing problems are likely to be hard to approximate within a factor of $n^{1-\epsilon}$ in the general cases, we obtain an $o(n)$-approximation algorithms in the geometric setting, as follows.

\begin{theorem}
For any $d > 0$, there is a $\tilde{O}_d(n^{1-\epsilon(d)})$-approximation algorithm for $d$-UUDP-MIN and $d$-SMP where function $\epsilon(d) := \frac{1}{4d-1}$ and $\tilde{O}_d$ treats $d$ as a constant and hides a polylog($n$) factor.
\end{theorem}

The essential idea behind our algorithm is to partition the problem instance into sub-instances without decreasing the optimal revenue (we call this \textit{consideration-preserving decomposition}). This is done by using Dilworth’s Theorem (partitioning items into chains and anti-chains) and epsilon-nets to find subsets of items satisfying certain structural properties. Subsequently, we show that the dimensions of these sub-instances can be reduced through the notion of \textit{consideration-preserving embedding}. In the end of our algorithm, we are left with a sub-linear number of sub-instances, each of which can be solved almost optimally in polynomial time. Returning the best solution among the solutions of these sub-instances guarantees a sub-linear approximation ratio.

The spirit of our technique is in some sense in a similar flavor to Chan’s algorithm \cite{17} which computes a \textit{conflict-free} coloring of $d$-dimensional points (w.r.t. rectangle ranges) using $O(n^{1-0.632/(2d-3-0.368)})$ colors. In particular, in the 2-dimensional cases of both our geometric pricing and Chan’s conflict-free coloring problems, the upper bounds of $O(\sqrt{n})$ can be obtained by a simple application of Dilworth’s theorem (Ajwani et al. \cite{3} obtained a better bound in this case for the latter problem). However, the techniques of the two results are different in higher dimensions.
We also obtain QPTASs for 2-UUDP-MIN and 2-SMP. We present this in Appendix B and C. These results, together with a widely-believed assumption that the existence of a QPTAS for any problem implies that PTAS exists for the same problem (e.g., [9, 23]), imply that the value of $\epsilon(d)$ in Theorem 1.1 could be improved slightly to $1/4^{d-2}$. As a by-product of these results, we show a QPTAS for 2-SMP which subsumes the previous QPTAS for highway pricing [23].

**Hardness** We also study the hardness of approximation of our problems. We show that 3-UUDP-MIN and 2-SMP are NP-hard, and 4-UUDP-MIN and 4-SMP are APX-hard. Hence, our problem is already non-trivial for small $d$. Our hardness proofs establish a cute connection between our problem and the vertex cover problem on graphs of low order dimensions [37, 48]. Moreover, we show that the hardness of our problem tends to increase as we increase $d$, and the whole generality is captured when $d = n$. In particular, we show that when the dimension is sufficiently high (i.e. $d \geq \log^2 n$), the problems are hard to approximate to within a factor of $d^{1/4-\epsilon}$ for any $\epsilon > 0$.

Table 1 concludes our results for $d$-UUDP-MIN and $d$-SMP.

<table>
<thead>
<tr>
<th>Problem</th>
<th>$d = 1$</th>
<th>$d = 2$</th>
<th>$d = 3$</th>
<th>$d = 4$</th>
<th>large $d$ {range}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$-UUDP-MIN</td>
<td>Upper bound</td>
<td>Polytim</td>
<td>QPTAS</td>
<td>NP-hard</td>
<td>$n^{1-\frac{\epsilon}{\log d}}$ {constant $d$}</td>
</tr>
<tr>
<td></td>
<td>Lower bound</td>
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<td>$d^{1/4-\epsilon}$ {constant $d$}</td>
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<tr>
<td></td>
<td>Lower bound</td>
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<td>$d^{1/4-\epsilon}$ {constant $d$}</td>
</tr>
</tbody>
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Table 1: Results of $d$-UUDP-MIN and $d$-SMP for small values of $d$.

**1.2 Related Work**

Rusmevichientong et al. [44, 45, 46] defined the non-parametric multi-product pricing problem, motivated by the possibility that the data about consumers’ preferences and budgets can be predicted based on previous data, which can be gathered and mined by web sites designed for this purpose, e.g., [33, 46]. This problem is what we call uniform-budget unit-demand pricing problem (UUDP). Rusmevichientong et al. proposed many decision rules such as min-buying, max-buying and rank-buying and showed that UUDP-MIN allows a polynomial-time algorithm, assuming the price-ladder constraint, i.e., a predefined total order on the prices of all products. Aggarwal et al. [1] later showed that the price ladder constraint also leads to a 4-approximation algorithm for the max-buying case, even in the case of limited supply.

We note that the price ladder constraint is closely related to our notion of attributes in the following sense. It can be shown that 1-UUDP-MIN satisfies the price ladder constraint (this is the reason we can solve it in polynomial time). Moreover, although 2-UUDP-MIN does not satisfy this constraint, it partially satisfies the constraint in the sense that if one item is better than another item in all attributes then we can assume that it has a higher price. This property plays an important role in obtaining QPTAS for 2-UUDP-MIN and also holds for general $d$.

Other variants defined later include non-uniform and utility-maximizing unit-demand, single-minded (SMP), tollbooth and highway models [1, 32]. These problems were later found to have important connections to algorithmic mechanism design [2, 7, 32] and online pricing problems [6, 13]. As we mentioned in the introduction, many problems can be approximated within the factor of $O(\log m + \log n)$ and $O(n)$, and these seem to be tight.

The observation that consumers make decisions based on attributes has been used in other areas outside computer science. For example, most pricing models are captured by the two-stage consider-then-choose model (e.g., [29, 42, 43, 30, 33, 38, 34, 31]) in marketing research: Each consumer first
tended to a price function for the original problem $P$ then there is an $UUDP-MIN$ and $1-$

For any $\text{Theorem 2.1.}$

and obtains a solution of $d$ holds that $\text{Definition 2.2.}$

We call a collection $USM$ (see Appendix B). Our algorithm partitions the input instance into many subinstances and tries to collect the profit and consideration-preserving decomposition, defined below, allows us to do so without losing revenue.

Our algorithm partitions the input instance into many subinstances and tries to collect the profit from some of them. The notion of consideration-preserving decomposition, defined below, allows us to do so without losing revenue.

**Definition 2.2.** We call a collection $\{(C'_1, I'_1), \ldots, (C'_k, I'_k)\}$ a consideration-preserving decomposition of the problem $(C, I)$ if and only if for any $C \in C$ and $I \in SC$, there exists (not necessarily unique) $i$ such that $C \in C'_i$ and $I \in I'_i$.

By definition, for any consumer $C$ and item $I$ the fact that consumer $C$ considers item $I$ is preserved by at least one instance $(C'_i, I'_i)$. The following lemma says that this decomposition preserves the total revenue.

**Lemma 2.3.** For any consideration-preserving decomposition $\{(C'_1, I'_1), \ldots, (C'_k, I'_k)\}$ of $(C, I)$, it holds that $\sum_{i=1}^{k} OPT(C'_i, I'_i) \geq OPT(C, I)$. Moreover, any price function for $P(C'_i, I'_i)$ can be extended to a price function for the original problem $P(C, I)$ that gives revenue at least $OPT(C'_i, I'_i)$.

This is simply by applying the optimal price function of one problem to the other (see Appendix $A.1$ for the full proof). In the rest of our discussion, we mainly use two different types of consideration-preservation decomposition, as explained in the following observation.
Observation 2.4. Given an input instance \((C', \mathcal{I}')\), let \(C' = \bigcup_{i=1}^{k} C_i'\). Then \(\left\{ (C_1', \mathcal{I}_1'), \ldots, (C_k', \mathcal{I}_k') \right\}\) is a consideration-preserving decomposition of \((C', \mathcal{I}')\). Similarly, if \(\mathcal{I}' = \bigcup_{i=1}^{k} \mathcal{I}_i'\), then we have that \(\left\{ (C', \mathcal{I}_1'), \ldots, (C', \mathcal{I}_k') \right\}\) is a consideration-preserving decomposition of \((C', \mathcal{I}')\).

2.2 Algorithm

At a high level, the algorithm proceeds in four steps where each step involves consideration-preserving decomposition (see Fig. 2 for an overview). In Step 1, we partition \(\mathcal{I}\) into different subsets where every subset satisfies certain properties, i.e. the elements in each subset either form a chain or an antichain. The problem on those subsets in which elements form a chain can be solved easily, and we deal with the antichains in later steps. In Step 2, we partition consumers in \(C\) into two types, those with large and small consideration sets. We use the algorithm of [6, 14] to deal with consumers with small consideration sets and handle the rest consumers in later steps. In Step 3, we find a subset of items, i.e. a “hitting set”, and partition consumers further into several sets. Each set of consumers has the following property: There is some item desired by all consumers in the set. Using this property, we show in Step 4 that the problem can be further partitioned into a few problems where each of them can be viewed as a \((d−1)\)-UUDP-MIN problem. (We call this a “consideration-preserving embedding”.)

Step 1: Partitioning items into chains and antichains Let \((C, \mathcal{I})\) be an input of \(d\)-UUDP-MIN. First we define a partially ordered set \((\mathcal{I}, \leq)\) on the item set as follows. We say that \(I_1 \leq I_2\) if and only if \(I_1\) has a lower quality than \(I_2\) in every attribute, i.e. \(I_1[\cdotj] \leq I_2[\cdotj]\) for all \(\cdotj \in [d]\). We say that a subset \(\mathcal{I}' \subseteq \mathcal{I}\) is a chain if \(\mathcal{I}'\) can be written as \(\mathcal{I}' = \{I_1, \ldots, I_z\}\) such that \(I_j \leq I_{j+1}\) for all \(j \in [z−1]\). We say that \(\mathcal{I}' \subseteq \mathcal{I}\) is an antichain if and only if for any pair of items \(I, I' \in \mathcal{I}'\), neither \(I \leq I'\) nor \(I' \leq I\).

Lemma 2.5. For any \(\epsilon > 0\) and any \(s = n^{\epsilon/4}, t = n^{1−\epsilon/4}\), we can partition \(\mathcal{I}\) into \(A_1, \ldots, A_s\) and \(B_1, \ldots, B_t\) in polynomial-time. Moreover, each \(A_i\) is an antichain and each \(B_j\) is a chain.

Proof Idea. (See Section A.2 for detailed definitions and proofs.) By Dilworth’s theorem [21, 26], the minimum chain decomposition equals to the maximum antichain size. We will use the fact that both minimum chain decomposition and maximum-size antichain can be computed in polynomial time as follows: As long as the maximum-size antichain is bigger than \(n^{\epsilon/4}\), we repeatedly extract such an antichain out of the input; otherwise, we would have the decomposition into at most \(n^{\epsilon/4}\) chains, so we stop. 

By Observation 2.4, the collection \(\left\{ (C, A_1), \ldots, (C, A_s), (C, B_1), \ldots, (C, B_t) \right\}\) is a consideration-preserving decomposition of \((C, \mathcal{I})\). It follows by Lemma 2.3 that \(\sum_{i=1}^{s} \text{OPT}(C, A_i) + \sum_{j=1}^{t} \text{OPT}(C, B_j) \geq \text{OPT}(C, \mathcal{I})\). Further, observe that if there exists \(j\) such that \(\text{OPT}(C, B_j) \geq \text{OPT}(C, \mathcal{I})/(2n^{1−\epsilon/4})\), then we would be done: the \(d\)-UUDP-MIN problem \(P(C, B_j)\) can be seen as a \(1\)-UUDP-MIN problem (since \(B_j\) is a chain) and hence can be solved optimally! (See Lemma 2.11 for detailed analysis) Otherwise \(\text{OPT}(C, B_j) \leq \text{OPT}(C, \mathcal{I})/(2n^{1−\epsilon/4})\) for every \(j\). Therefore \(\sum_{j=1}^{t} \text{OPT}(C, B_j) \leq \text{OPT}(C, \mathcal{I})/(2n^{1−\epsilon/4})\) for every \(j\). Therefore \(\sum_{j=1}^{t} \text{OPT}(C, B_j) \leq \text{OPT}(C, \mathcal{I})/2n^{1−\epsilon/4}\).
\[ n^{1-\varepsilon/4} \cdot \frac{\text{OPT}(C, \mathcal{I})}{(2n^{1-\varepsilon/4})} < \frac{\text{OPT}(C, \mathcal{I})}{2}. \] If this is not the case then we know that there must be an antichain \( A_1 \) such that \( \text{OPT}(C, A_1) \geq \frac{\text{OPT}(C, \mathcal{I})}{2n^{\varepsilon/4}} \).

**Step 2: Dealing with small consideration sets**

For simplicity, let us assume that we know \( i \) such that \( \text{OPT}(C, A_i) \geq \frac{\text{OPT}(C, \mathcal{I})}{2n^{\varepsilon/4}} \). Now we focus on collecting revenue from the subproblem \( \mathcal{P}(C, A_i) \). Let \( \mathcal{C}_1 \subseteq \mathcal{C} \) be the set of consumers who are interested in at most \( n^{1-2\varepsilon/4} \) items in \( A_i \), i.e. \( \mathcal{C}_1 = \{ C \in \mathcal{C} : |S_C \cap A_i| \leq n^{1-2\varepsilon/4} \} \), and define \( \mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1 \). Since \( \{(\mathcal{C}_1, A_i), (\mathcal{C}_2, A_i)\} \) is a consideration-preserving decomposition of \( (C, A_i) \), we have \( \text{OPT}(\mathcal{C}_1, A_i) + \text{OPT}(\mathcal{C}_2, A_i) \geq \text{OPT}(C, A_i) \geq \frac{\text{OPT}(C, \mathcal{I})}{2n^{\varepsilon/4}} \).

Now we need an algorithm of [6, 14]. Balcan and Blum give an approximation algorithm for \( \text{SMP} \) whose approximation guarantee depends on the sizes of consideration sets. Briest and Krysta, by using a slight modification of this algorithm, give an approximation algorithm with the same guarantee for \( \text{UDP-MIN} \). Their result, stated in terms of \( \text{UUDP-MIN} \), is summarized in the following theorem. (For completeness, we provide the proof in Appendix A.3.)

**Theorem 2.6.** [17] Given a \( \text{UUDP-MIN} \) instance \( (C, \mathcal{I}, \{S_C\}_{C \subseteq C}) \), there is a deterministic \( O(k) \)-approximation algorithm of \( \text{UUDP-MIN} \), where \( k := \max_{C \subseteq C} |S_C| \).

We remark that we extend this technique to deal with any pricing problem with subadditive revenue in the full version of this paper.

If \( \text{OPT}(\mathcal{C}_1, A_i) \geq \frac{\text{OPT}(C, \mathcal{I})}{(4n^{\varepsilon/4})} \), then we could invoke the algorithm in Theorem 2.6 on \( (\mathcal{C}_1, A_i) \) to get a solution with approximation ratio \( O(\max_{C \subseteq \mathcal{C}_1} |S_C \cap A_i|) = O(n^{1-2\varepsilon/4}) \). This yields a solution that gives a desired revenue of \( \Omega \left( \frac{\text{OPT}(C, A_i)}{n^{1-2\varepsilon/4}} \right) = \Omega \left( \frac{\text{OPT}(C, \mathcal{I})}{n^{1-\varepsilon/4}} \right) \). Otherwise we have \( \text{OPT}(\mathcal{C}_1, A_i) < \frac{\text{OPT}(C, \mathcal{I})}{4n^{\varepsilon/4}} \). Then \( \text{OPT}(\mathcal{C}_2, A_i) = \Omega \left( \frac{\text{OPT}(C, \mathcal{I})}{n^{\varepsilon/4}} \right) \). We will deal with this case in the next steps.

**Step 3: Partitioning consumers using a small hitting set**

First, we apply the epsilon net theorem [36] to derive the following lemma.

**Lemma 2.7.** We can find a set \( H \subseteq A_i \) of size \( \tilde{O}(n^{2\varepsilon}) \) in randomized polynomial time such that for any \( C \subseteq \mathcal{C}_2 \), there exists \( I \in H \) such that \( I \supseteq C \).

**Proof.** The instance \( (C_2, A_i) \) defines a set system \( \{S_C\}_{C \subseteq C_2} \) over \( A_i \), where \( S_C = \{ I \in A_i : I \supseteq C \} \). We note that each set \( S_C \) has descriptive complexity at most \( d \), i.e. set \( S_C \) can be described by \( d \) linear inequalities of the form \( S_C = \bigcap_{d'=1}^{d} \{ I \in \mathcal{I} : |d'| \geq C[d'] \} \). In this case, the set system has VC dimension \( O(d) \), c.f. [39]. More specifically, it is well known (e.g., [5]) that any collection of \( d \)-dimensional axis-parallel boxes has VC dimension \( O(d) \). We will not formally define VC-dimension here. The following theorem is all we need.

**Theorem 2.8.** ([18, 36]; Epsilon net theorem) Let \( \mathcal{X} \) be a set system of VC-dimension at most \( d' \) over \( N \). Then for any \( \delta \in (0, 1) \), we can find a set \( H \subseteq N \) with \( |H| = O\left( \frac{d'}{\delta} \log \frac{N}{\delta} \right) \) in randomized polynomial time such that, for all \( X_i \in \mathcal{X} \) with \( |X_i| \geq \delta |N| \), it holds that \( H \cap X_i \neq \emptyset \).

Using the theorem with \( \delta = n^{-2\varepsilon/4} \), we can find a set \( H \subseteq A_i \) of size at most \( \tilde{O}(n^{2\varepsilon/4}) \), and since we have \( |S_C \cap A_i| \geq \delta n \) for all \( C \in \mathcal{C}_2 \), we are guaranteed that \( H \cap S_C \neq \emptyset \) for all \( C \in \mathcal{C}_2 \). \( \square \)
Reducing the dimension embedding. For any Fig. 4(b) for an idea). We defer the formal statement and proof of this claim to Section 2.3. Finally, of all points.) In other words, the problem can be viewed as a (\(d-1\))-UUDP-MIN problem.

For each \(I \in H\), let \(C_I = \{C \in C_2 \mid I \in S_C\}\), i.e., \(C_I\) consists of all consumers in \(C_2\) that consider item \(I\). Observe that \(\bigcup_{I \in H} C_I = C_2\), and therefore by Lemma 2.3 we have \(\sum_{I \in H} \text{OPT}(C_I, A_i) \geq \text{OPT}(C_2, A_i) \geq \Omega(\text{OPT}(C, I)/n^{\epsilon/4})\). Since \(|H| = O(n^{2\epsilon/3})\), there exists \(I^* \in H\) such that

\[
\text{OPT}(C_{I^*}, A_i) = \tilde{\Omega}(\text{OPT}(C, I) \cdot n^{-\epsilon/4}/|H|) = \tilde{\Omega}(\text{OPT}(C, I)/n^{3\epsilon/4}).
\]

Now we, again, assume that we know \(I^*\) and turn our focus to the subproblem \(\mathcal{P}(C_{I^*}, A_i)\).

**Step 4: Reducing the dimension** We have now reached the most crucial step. We will (crucially) rely on the fact that all consumers in \(C_{I^*}\) consider item \(I^*\), and that \(A_i\) is an antichain. For each \(j \leq d\), define \(A_i^j\) as the set of items in \(A_i\) that are at least as good as \(I^*\) in the \(j\)-th coordinate, i.e., \(A_i^j = \{i \in A_i \mid I[j] \geq I^*[j]\}\). See Fig. 4(a) for an example in the case of 2-UUDP-MIN.

**Lemma 2.9.** \(A_i = \bigcup_{j=1}^{d} A_i^j\).

This lemma holds simply because \(A_i\) is an antichain (in any antichain, no item can completely dominate the other, so at least one coordinate of any \(I \in I^*_I\) has to be at least as good as \(I^*\); see detailed proof in Appendix A.4). Then \(\{(C_{I^*}, A_i^1), \ldots, (C_{I^*}, A_i^d)\}\) is a consideration-preserving decomposition of \((C_{I^*}, A_i)\) and thus there exists \(j\) such that \(\text{OPT}(C_{I^*}, A_i^j) \geq \text{OPT}(C_{I^*}, A_i)/d = \tilde{\Omega}(\text{OPT}(C, I)/n^{3\epsilon/4})\). Observe that, for all \(C \in C_{I^*}\) and \(I \in A_i^j\), \(C[j] \leq I^*[j] \leq I[j]\). This implies that we can ignore the \(j\)-th coordinate when we solve \(\mathcal{P}(C_{I^*}, A_i^j)\). (In particular, for any \(C \in C_{I^*}\), the consideration set \(S_C = \{I \geq C \mid I \in A_i^j\}\) remains the same even when we drop the \(j\)-th coordinate of all points.) In other words, the problem can be viewed as a \((d-1)\)-UUDP-MIN problem (see Fig. 4(b) for an idea). We defer the formal statement and proof of this claim to Section 2.3. Finally, we can invoke the \(\tilde{\Omega}(d(n^{1-\epsilon}))\)-approximation algorithm for \((d-1)\)-UUDP-MIN to collect the revenue of \(\tilde{\Omega}(\text{OPT}(C, I)n^{-3\epsilon/4}/n^{1-\epsilon}) = \tilde{\Omega}(\text{OPT}(C, I)/n^{1-\epsilon/4})\). Therefore we obtain an approximation ratio of \(\tilde{\Omega}(d(n^{1-\epsilon/4}))\) in all cases. Algorithm 1 summarizes our algorithm for solving \(d\)-UUDP-MIN.

### 2.3 Consideration-preserving Embedding

To formally discuss the reduction of dimensions, we introduce the notion of consideration-preserving embedding. For any \(d\), let \((C, I)\) be any instance of \(d\)-UUDP-MIN. For any \(d'\), consider one-to-one functions \(f\) and \(g\) that map points in \(\mathbb{R}^d\) to the ones in \(\mathbb{R}^{d'}\). We say that \((f, g)\) is a consideration-preserving embedding if, for any item \(I \in I\) and consumer \(C \in C\), we have that \(I \geq C\) if and only
Algorithm 1 UUDP-MIN-APPROX(d)

1: if \( d = 1 \) then
2: \( \) Solve the problem \( \mathcal{P}(\mathcal{C}, \mathcal{I}) \) optimally using an algorithm for 1-UUDP-MIN (cf. Appendix A.5)
3: else
4: Partition \( \mathcal{I} \) into antichains \( A_1, \ldots, A_s \) and chains \( B_1, \ldots, B_t \) where \( s \leq n^{d/4} \) and \( t \leq n^{1-d/4} \) as in Step 1.
5: We claim that the problems \( \mathcal{P}(\mathcal{C}, B_1), \ldots, \mathcal{P}(\mathcal{C}, B_t) \) are equivalent to 1-UUDP-MIN problems (cf. Section A.2).
6: Solve them optimally using an algorithm for 1-UUDP-MIN (cf. Appendix A.5).
7: for \( i = 1, \ldots, s \) do
8: Partition \( \mathcal{C} \) into \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) as in Step 2. Find an \( O(\max_{C \in C_1} |C \cap A_i|) = O(n^{1-2d/4}) \) approximate solution of problem \( \mathcal{P}(\mathcal{C}_1, A_i) \).
9: Find a hitting set \( H \) of \( (\mathcal{C}_2, A_i) \) as in Step 3
10: Define \( \mathcal{C}_1 \) as in Step 3
11: Define \( A_i^1, \ldots, A_i^d \) as in Step 4
12: Solve problem \( \mathcal{P}(\mathcal{C}_1, A_i^1), \ldots, \mathcal{P}(\mathcal{C}_1, A_i^d) \) using an \( O(n^{1-\epsilon}) \)-approximation algorithm for \((d-1)\)-UUDP-MIN
13: end for
14: end for
15: end if
16: return the solution with highest revenue among the solutions of all solved problems

if \( g(I) \geq f(C) \). That is, the fact that consumer \( C \) is considering or not considering item \( I \) must be preserved in \( f(C) \) and \( g(I) \).

Given a consideration-preserving embedding \((f, g)\), we can naturally define a \( d'\)-UUDP-MIN problem \( \mathcal{P}(f(C), g(I)) \) where \( f(C) = \{f(C) \mid C \in \mathcal{C}\} \), \( g(I) = \{g(I) \mid I \in \mathcal{I}\} \) and the budget \( B_{f(C)} \) is \( B_C \) for any \( C \in \mathcal{C} \).

Observe that, although \((C, I)\) and \((f(C), g(I))\) correspond to points on different spaces, they represent the same pricing problem (i.e., the consumers’ consideration sets and budgets are exactly the same). Thus, we sometimes say that \((C, I)\) and \((f(C), g(I))\) are equivalent. The following observation follows trivially.

Observation 2.10. For any instance \((C, I)\), let \((f, g)\) be a consideration-preserving embedding of \((C, I)\) into \( \mathbb{R}^{d'} \). Then we have that \( \text{OPT}(C, I) = \text{OPT}(f(C), g(I)) \). Moreover, if \( f \) and \( g \) are polynomial-time computable then a solution for \( \mathcal{P}(f(C), g(I)) \) can be efficiently transformed into one for \( \mathcal{P}(C, I) \) that gives the same revenue.

The transformation in the above lemma is trivial: For any price function \( p \) for \((f(C), g(I))\), we simply price item \( I \in \mathcal{I} \) to \( p(g(I)) \). Observe that we will receive the same revenue from both problems using this pricing strategy.

In Step 1, we claimed that when the items form a chain, our instance would be equivalent to 1-UUDP-MIN. Now we prove this fact formally below.

Lemma 2.11. Let \((C, I)\) be a \( d\)-UUDP-MIN instance where \((I, \leq)\) is a chain. Then \((C, I)\) is equivalent to a 1-UUDP-MIN instance. Moreover, the corresponding consideration-preserving embedding \((f, g)\) can be computed in polynomial time.

Proof. Order items in \( I \) by \( I_1 \leq I_2 \leq \ldots \). Now map each item into a one-dimensional point: \( g(I_i) = (i) \). Moreover, map each consumer according to \( f(C) = g(I_i) \), where \( i \) is the minimum number such that \( I_i \geq C \). Observe that \((f, g)\) is a consideration-preserving embedding since \( S_C = \{I_i, I_{i+1}, \ldots \} \) while \( S_{f(C)} = \{g(I_i), g(I_{i+1}), \ldots \} \) for any \( C \in \mathcal{C} \). (Note that this embedding
might create redundancy since it is possible that \( f(C) = f(C') \) for some \( C \neq C' \). This can be fixed easily by slightly perturbing the points.)

In Step 4, we also claimed the dimension reduction of sub-instances \((C_I, A_I^j)\), and we now prove the claim formally. Recall that the item \( I^* \in A_I^j \) has the property that \( I^* \geq C \) for all \( C \in C_I \) and \( I^*[j] \leq I[j] \) for all \( I \in A_I^j \).

**Lemma 2.12.** The instance \((C_I, A_I^j)\) is equivalent to a \((d - 1)\)-UUDP-MIN instance. Moreover, the corresponding consideration-preserving embedding \((f, g)\) can be computed in polynomial time.

**Proof.** Consider “ignoring” the \( j \)-th coordinate as follows. For any \( C \in C_I \) and \( I \in A_I^j \), let \( f(C) = (C[1], C[2], \ldots, C[j - 1], C[j + 1], \ldots, C[d]) \) and \( g(I) = (I[1], I[2], \ldots, I[j - 1], I[j + 1], \ldots, I[d]) \). Observe that for any \( C \in C_I \) and \( I \in A_I^j \), \( I \geq C \) trivially implies that \( g(I) \geq f(C) \). Conversely, if \( g(I) \geq f(C) \) then \( I \geq C \) since \( I[j] \geq I^*[j] \geq C[j] \). Thus, \((f, g)\) is a consideration-preserving embedding. \( \square \)

## 3 Hardness

We provide hardness results in both scenarios when the number of attributes \( d \) is small and when \( d \) is large. We sketch our results here. More details can be found in Appendix D.

**Few attributes** First we discuss the \( \text{NP} \)-hardness of 3-UUDP-MIN and APX-hardness of 4-UUDP-MIN. These hardness results hold even when the consumer budgets are either 1 or 2. We perform a reduction from Vertex Cover \([28, 4]\), essentially using the same ideas as in \([32]\), except for the fact that we use Schnyder’s result \([47, 48]\) to “embed” the instance into posets of low order dimensions.

First, let us recall the reduction in \([32]\). We start from a graph \( G = (V, E) \), which is an input instance of Vertex Cover. We create two types of consumers: (i) poor consumer \( C_e \) for each edge \( e \) with budget 1 and (ii) rich consumer \( C_v \) for each vertex \( v \) with budget 2. The items are \( I = \{I_v : v \in V\} \). Each poor consumer \( C_e \) has a consideration set containing two items \( I_u \) and \( I_v \) where \( e = (u, v) \) and each rich consumer \( C_v \) considers only one item \( I_v \). Using the analysis essentially the same as \([32]\), one can show that the problem is \( \text{NP} \)-hard if we start from Vertex Cover on planar graphs and APX-hard if we start from Vertex Cover on cubic graphs.

Therefore, it only remains to map consumers and items to points in \( \mathbb{R}^d \) (where \( d = 3, 4 \)) such that for each consumer \( C_i \), the set of items that pass her criteria (i.e., \( \{I \in I_{C_i} \mid I[i] \geq C[i]\} \)) is exactly her consideration set. The main idea is to first embed the problem into an adjacency poset of the input graph. Then, we invoke Schnyder’s theorem \([47, 48]\) to again embed this poset into a Euclidean space.

An adjacency poset of a graph can be constructed as follows. First we construct a 2-layer poset with minimal elements in the first layer and maximal elements in the second layer. For each edge \( e \in E \), we have a minimal element in the poset corresponding to \( e \) (for convenience, we also denote the poset element by \( e \)). For each vertex \( v \in V \), we have a maximal poset element corresponding to \( v \). There is a relation \( e \preceq v \) if and only if vertex \( v \) is an endpoint of \( e \).

The last task is to “embed” poset elements into points in the Euclidean space in such a way that, for any poset elements \( e_1 \) and \( e_2 \), \( e_1 \preceq e_2 \) if and only if \( q_{e_1}[i] \geq q_{e_2}[i] \) for all \( i \) where \( q_{e_1} \) and \( q_{e_2} \) are points that \( e_1 \) and \( e_2 \) are mapped to, respectively. If we can do this, we would be done, simply by defining the coordinates of each consumer \( C_e \) to be \( q_e \), and the coordinates of each consumer \( C_v \) to be \( q_v \). Similarly, we define the coordinates of each item \( I_v \) as \( q_v \). In order to obtain such an embedding, we use part of Schnyder’s theorem \([47]\) which states that any planar graph has an
adjacency poset of dimension three, and any 4-colorable graph (including cubic graphs) has an
adjacency poset of dimension four. Moreover, embedding these graphs into Euclidean spaces can
be done in polynomial time [48].

Finally we note that \(2\text{-SMP}\) is strongly \(NP\)-hard and \(4\text{-SMP}\) is \(APX\)-hard. The proof follows
from the fact that these problems generalize Highway pricing and graph vertex pricing on bipartite
graphs, respectively, and can be found in the full version.

**Many attributes** We establish a connection between the UUDP-MIN with bounded-size consider-
ation sets and our problem. This connection immediately implies hardness results for \(d\text{-UUDP-MIN}\)
when \(d\) is at least poly-logarithmic in \(n\). Our main result in this section is the following:

**Theorem 3.1.** (Informal) Let \(A = (C, I, \{S_C\}_{C \in C})\) be an instance of UUDP-MIN where \(B = \max_{C \in C} |S_C|\). We can (with high probability of success) create an instance \(A' = (C', I')\) of \(d\text{-UUDP-MIN}\), where \(d = O(B^2 \log n)\), that is “equivalent” to \(A\).

In other words, the above theorem shows that any UUDP-MIN instance with consideration sets
of size bounded by \(B\), can be realized by a \(d\text{-UUDP-MIN}\) instance for \(d = O(B^2 \log n)\). Combining
this with the result in [15], we have a hardness of \(\Omega(d^{1/4-\epsilon})\) for any \(\epsilon > 0\).

We remark that our reduction here in fact works independently of the decision model, so this
result works for SMP and UDP-Util as well.

### 4 Open Problems

Several interesting problems are open. The most important problem is whether we can obtain
better approximation factors for \(d\text{-UUDP-MIN}\) and \(d\text{-SMP}\). We tend to believe that there is an
\(f(d)\)-approximation algorithm for \(d\text{-UUDP-MIN}\) and \(d\text{-SMP}\) where \(f(d)\) is a function that depends
on \(d\) only. However, it seems to be a very challenging task to obtain approximation ratio like
\(\log^{O(d)} n\) or \(O_d(\log^{1-\epsilon(d)} m)\), for some constant \(\epsilon(d) > 0\) depending on \(d\).

One promising direction in attacking the above problems is to improve Theorem 2.1, e.g., getting
\(O_d(\rho \cdot \text{polylog}(n))\) for \(d\text{-UUDP-MIN}\) using a \(\rho\)-approximation algorithm of \((d - 1)\text{-UUDP-MIN}\) as a blackbox. A positive resolution to this problem would imply \((\log^{O(d)} n)\)-approximation algorithm
for \(d\text{-UUDP-MIN}\). We believe that, even resolving this problem would require some new insights on
geometric and poset structures.

There are two special cases that can be thought of as barriers in dealing with standard versions
of SMP and UUDP-MIN, and we believe that these two special cases serve as good starting points
in attacking our problems. The first problem is the geometric version of the Maximum Expanding
Subsequence (Mes) problem which is the key problem to show the hardness of UUDP-MIN [14]. The
second problem is the Unique Coverage problem [20] when the sets have constant VC-dimension.
Another interesting problem is to obtain \(PTAS\)s for 2-UUDP-MIN and 2-SMP (e.g., by extending
the techniques in [31]).
References


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Appendix

A Proof Omitted from Section 2

A.1 Proof of Lemma 2.3

Let $p^*$ be the optimal price function for $\mathcal{P}(\mathcal{C}, \mathcal{I})$. For each $i = 1, \ldots, k$, we define $p^*_i : \mathcal{I}'_i \to \mathbb{R}$ by

$$p^*_i(I) = p^*(I) \text{ if } I \in \mathcal{I}'_i, \text{ and } p^*_i(I) = \infty \text{ otherwise.}$$

Let $r_i$ be the total revenue made by $p^*_i$ in $\mathcal{P}(\mathcal{C}'_i, \mathcal{I}'_i)$. We argue below that

$$\sum_{i=1}^k r_i \geq \text{OPT}(\mathcal{C}, \mathcal{I}).$$

Let $\mathcal{C}^* \subseteq \mathcal{C}$ be the set of consumers who make a positive payment with respect to $p^*$. For each consumer $C \in \mathcal{C}^*$, denote by $\varphi(C) \in \mathcal{I}$ the item that consumer $C$ buys with respect to the price $p^*$. So we can write $\text{OPT}(\mathcal{C}, \mathcal{I})$ as

$$\text{OPT}(\mathcal{C}, \mathcal{I}) = \sum_{C \in \mathcal{C}^*} p^*(\varphi(C)).$$

For each $i = 1, \ldots, k$, let $\mathcal{C}^*_i \subseteq \mathcal{C}'_i$ be the set of consumers $C \in \mathcal{C}'_i$ such that $\varphi(C) \in \mathcal{I}'_i$. That is, $\mathcal{C}^*_i$ is a set of consumers whose item she bought in $\text{OPT}(\mathcal{C}, \mathcal{I})$ is in $\mathcal{I}'_i$. Notice that

$$r_i \geq \sum_{C \in \mathcal{C}^*_i} p^*(\varphi(C)).$$

Since $\{(\mathcal{C}'_i, \mathcal{I}'_i)\}_{i=1}^k$ is a consideration-preserving decomposition, we have that

$$\bigcup_{i=1}^k \mathcal{C}^*_i \supseteq \mathcal{C}^*,$$}

since for any $C \in \mathcal{C}^*$, we must have $\varphi(C) \in \mathcal{I}_i$ for some $i$. By summing Eq.(4) over all $i = 1, \ldots, k$, we have

$$\sum_{i=1}^k r_i \geq \sum_{i=1}^k \sum_{C \in \mathcal{C}^*_i} p^*(\varphi(C)) \quad (\text{by Eq.(4)})$$

$$\geq \sum_{C \in \mathcal{C}^*} p^*(\varphi(C)) \quad (\text{by Eq.(5)})$$

$$= \text{OPT}(\mathcal{C}, \mathcal{I}) \quad (\text{by Eq.(3)})$$

This proves Eq.(2) and thus the first claim.

Now suppose we have a price $p' : \mathcal{I} \to \mathbb{R}$ that collects revenue $r'$ in $\mathcal{P}(\mathcal{C}'_i, \mathcal{I}'_i)$. We define a function $p : \mathcal{I} \to \mathbb{R}$ by $p(I) = p'(I)$ for $I \in \mathcal{I}'_i$ and $p(I) = \infty$ otherwise. We can use $p'$ to obtain a revenue of $r'$ from $\mathcal{P}(\mathcal{C}, \mathcal{I})$. This proves the second claim.
A.2 Decomposing items into small number of chains and antichains

We will use the following theorem, first proved by Dilworth [21], and its polynomial computability follows from the equivalence between Dilworth’s theorem and König’s theorem [26].

**Theorem A.1.** Let \((S, \leq)\) be a partially ordered set, and \(Z\) be the maximum number of elements in any antichain of \(S\). Then there is a polynomial-time algorithm that produces a partition of \(S\) into \(Z\) chains \(S_1, \ldots, S_Z\).

We now use the theorem to prove Lemma 2.5

of Lemma 2.3. Initially, let \(i = 1\). In iteration \(i\), we check if the size of maximum antichain in \(\mathcal{I}\) is at least \(t = n^{1-\epsilon/4}\). If so, we find the maximum antichain \(A_i\), update \(\mathcal{I} = \mathcal{I} \setminus A_i\), and proceed to the next iteration; otherwise, we stop the iterations. Notice that the number of iterations is at most \(s = n^{\epsilon/4}\), and when the iteration stops, the size of maximum-size antichain is at most \(t \leq n^{1-\epsilon/4}\). We apply the above theorem to compute a decomposition of \(\mathcal{I}\) into \(t\) chains, denoted by \(B_1,\ldots, B_t\).

This concludes the proof of Lemma 2.5. \(\square\)

A.3 Proof of Balcan-Blum Theorem for UUDP-MIN (cf. Theorem 2.6)

We first explain a randomized algorithm, and then we discuss how to derandomize it. This part is essentially the same as [6] [14]. First, we randomly construct a set \(\mathcal{I}^* \subseteq \mathcal{I}\) where each item \(i\) is independently added to \(\mathcal{I}^*\) with probability \(1/k\) (recall that \(k = \max_{C \in \mathcal{C}} |S_C|\)). Then let \(\mathcal{C}^*\) be a set of consumer \(C\) such that \(|S_C \cap \mathcal{I}^*| = 1\) (i.e. consumers who care about exactly one item in \(\mathcal{I}^*\)).

We show that the problem \(\mathcal{P}(\mathcal{C}^*, \mathcal{I}^*)\) has expected revenue at least \(\Omega(\text{OPT}(\mathcal{C}, \mathcal{I})/k)\).

Let \(p\) be the optimal price function for \((\mathcal{C}, \mathcal{I})\) and \(\varphi : \mathcal{C} \to \mathcal{I} \cup \{\bot\}\) be a function that maps each consumer to the item she buys with respect to \(p\) (let \(\varphi(C) = \bot\) if consumer \(C\) buys nothing and \(p(\bot) = 0\)). Therefore, we have that \(\text{OPT}(\mathcal{C}, \mathcal{I}) = \sum_{C} p(\varphi(C))\). We denote by \(p^*\) the price function \(p\) restricted to \(\mathcal{I}^*\). For each \(C\), if \(C \in \mathcal{C}^*\) and \(\varphi(C) \in \mathcal{I}^*\), the revenue created by \(p^*\) in \((\mathcal{C}^*, \mathcal{I}^*)\) would be at least \(p(\varphi(C))\). Therefore,

\[
E[\text{OPT}(\mathcal{C}^*, \mathcal{I}^*)] \geq \sum_{C \in \mathcal{C}} \Pr[\varphi(C) \in \mathcal{I}^* \text{ and } C \in \mathcal{C}^*] \times p(\varphi(C)).
\]

Notice that, for any \(C \in \mathcal{C}\) and \(I \in S_C\),

\[
\Pr[I \in \mathcal{I}^* \text{ and } C \in \mathcal{C}^*] \geq \frac{1}{k} \left(1 - \frac{1}{k}\right)^{k-1} \geq \frac{1}{ke},
\]

which implies that \(E[\text{OPT}(\mathcal{C}^*, \mathcal{I}^*)] \geq \frac{1}{ke} \cdot \text{OPT}(\mathcal{C}, \mathcal{I})\).

**Derandomization:** First, note that we can assume that \(k = O(\log m + \log n)\). Otherwise, we can use the result of [11] [32] [8] (see [8] Section 4 for the result in a general setting) to obtain \(O(\log m + \log n)\) approximation algorithm for UUDP-MIN, which will also be \(O(k)\)-approximation.

Now, assuming that \(k = O(\log m + \log n)\), we follow the argument of Balcan and Blum [6]. In particular, we observe that we need only \(k\)-wise independence among the events of the form \(I \in \mathcal{I}^*\) and \(C \in \mathcal{C}^*\), for any \(I\) and \(C\), in order to get the above expectation result. In this case, we can use the tools from Even et al [25] to derandomize the above algorithm while blowing up the running time by a factor of \(2^{O(k)} = \text{poly}(m, n)\). For more details, we refer the readers to [6].
A.4 Proof of Lemma 2.9
Recall that each \(A_i\) is an antichain, i.e., for any distinct \(I_1, I_2 \in A_i\), there exists \(1 \leq d_1, d_2 \leq d\) such that \(I_1[d_1] < I_2[d_1]\) and \(I_1[d_2] > I_2[d_2]\). In particular, if \(I_1 = I^*\), then we have that for any \(I \in A_i\), there exists coordinate \(j\) such that \(I[j] \geq I^*\). This means that \(I \in A^*_i\). The lemma follows.

A.5 Polynomial-Time Algorithm for 1-UUDP-MIN
We provide a polynomial-time algorithm for solving 1-UUDP-MIN. Let \(I_1, \ldots, I_n\) be a sequence of items ordered non-increasingly by their coordinates. We can assume without loss of generality that their coordinates are different (by slightly perturbing their values), and we say that consumer \(C\) is at level \(j\) if her coordinate lies between \(I_{j-1}\) and \(I_j\). Notice that, for any consumer \(C\) at level \(j\), we have \(S_C = \{I_1, \ldots, I_j\}\).

Claim A.2. Let \(p^*\) be an optimal price. Then we can assume that \(p^*(I_1) \geq p^*(I_2) \geq \ldots \geq p^*(I_n)\).

Proof. Suppose that \(p^*(I_i) < p^*(I_j)\) for some \(i < j\). Recall that \(I_i \geq I_j\), so for each consumer \(C\) such that \(C \leq I_j\), we know that \(C\) does not buy item \(I_j\) with respect to this solution. Thus, we can reduce \(p^*(I_i)\) slightly, while maintaining the same revenue.

The claim will ensure that consumers at level \(j\) only buy item \(I_j\) but not any other items in \(\{I_1, \ldots, I_{j-1}\}\), and this allows us to solve the problem by dynamic programming. For each \(j = 1, \ldots, n\), for each price \(P \in \mathbb{R}\) we have a table entry \(T[j, P]\) that keeps the maximum revenue achievable from consumers at levels 1, \ldots, \(j\) and items \(\{I_1, \ldots, I_j\}\) where the price of \(I_j\) is set to \(P\). Notice that it is easy to compute the profit from consumers at level \(j\) if we know \(p(I_j) = P\). Denote such value by \(\gamma\). Then we have that \(T[j, P] = \gamma + \max_{P' \geq P} T[j-1, P']\). Finally, we note that there are at most \(|C|\) possibilities of prices \(P\) because one can assume without loss of generality that, for UUDP-MIN, the prices always belong to \(\{B_C\}_{C \in C}\).

B QPTAS for 2-UUDP-MIN
We note that we will write \(O(\log m)\) instead of \(O(\log n + \log m)\) since we assume that \(n \leq m\) in this paper. (Otherwise, we already have approximation ratio of \(O(\log m) = O(\log n).)\)

We explain the main idea first. The intuition can be realized by solving the following simple case: Assume for now that we have \(\Theta(n^2)\) items, which form a set \(\{(2i - 1, 2j - 1) : 1 \leq i, j \leq n\}\). In this case it is possible to have two different consumers at the same coordinate, i.e. \(C = C'\), while there is exactly one item at each point \((2i - 1, 2j - 1)\). Assume further that each consumer has budget either 1 or 2. We show below how to solve this case in polynomial time.

Note that there is an optimal solution such that each item is priced either 1 or 2: otherwise we could increase the price by small amount to collect more revenue. Now, for any item point \((2i - 1, 2j - 1)\) and any price assignment \(p\), define

\[
r_p(i, j) := \min_{|I| \geq 2i-1, |J| \geq 2j-1} \min_{I \subseteq J} \{p(I)\}
\]

to be the minimum price among the items dominating \((2i - 1, 2j - 1)\). This quantity immediately tells us how much revenue we will get from consumers at point \((2i - 2, 2j - 2)\): each consumer will buy an item at price \(r_p(i, j)\) if and only if she has budget at least \(r_p(i, j)\).

By the definition of \(r_p\), we know that for any fixed value \(j\), \(r_p(i, j)\) is non-decreasing in terms of \(i\). In other words, for any pricing \(p\) and integer \(j\), there exists a “threshold” \(\gamma(p, j)\) such that
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\( r_p(i', j) = 1 \) for all \( i' \leq \gamma(p, j) \) and \( r_p(i', j) = 2 \) for all \( i' > \gamma(p, x) \). Additionally, for any \( j \), \( \gamma(p, j) \geq \gamma(p, j + 1) \). Using these observations, we are ready to define the dynamic programming table. The table entry \( T[i, j] \) is defined to be the maximum revenue we can get among the price assignment \( p \) such that \( r_p(i', j) = 1 \) for all \( i' \leq i \) and \( r_p(i', j) = 2 \) for all \( i' > i \). The table \( T \) can be computed as follows.

\[
T[i, j] = \max_{i' \leq i} \{ T[i', j + 1] + m_1(i', j) + 2m_2(i', j) \}
\]

where \( m_1(i', j) \) is the number of consumers of the form \((2i'' - 2, 2j - 2)\) for \( i'' \leq i' \) with budget 1 and \( m_2(i', j) \) is the number of consumers of the form \((2i'' - 2, 2j - 2)\) for \( i'' > i' \) with budget 2. Moreover, let \( T[i, n + 1] = 0 \) for all \( i \). The optimal solution is then \( \max_i T[i, 1] \).

The above discussion captures almost all the key ideas for solving the general 2-UUDP-MIN problem. To get a QPTAS in the general case, we extend these ideas in the following way.

- Consider a slight generalization when there is only one item in each column and row of grid cells (cf. Lemma B.1) while each budget is still 1 and 2. In this case, we cannot pick arbitrary value of \( i' \) when we compute \( T[i, j] \) as in Eq.(6) since it might not correspond to any pricing. Through some additional observations, table \( T \) can be computed as follows: Let \( I_j \) be the item whose \( y \)-coordinate is \( j \). If \( i = I_j[1] \) then we can use Eq.(6); otherwise, \( T[i, j] = T[i, j + 1] + m_1(i, j) + 2m_2(i, j) \). This algorithm runs in \( O(n^3) \) time.

- When there are \( q \) different budgets, say \( B_1, B_2, \ldots, B_q \), we can solve the problem in \( n^{O(q)} \) time. This is done by defining \( T[i_1, \ldots, i_q, j] \) to be the maximum revenue we can get among the price assignment \( p \) such that, for all \( q' : 1 \leq q' \leq q \), \( r_p(i', j) = B_{q'} \) for all \( i_{q' - 1} < i' \leq i_{q'} \) (where we let \( i_0 = -1 \) and \( i_q = n \)).

- Finally, we obtain a QPTAS by “discretizing” the prices so that there are not many choices of item prices (cf. Lemma B.2). This enables us to assume that the prices are in \( \Gamma = \{0, (1 + \epsilon)^0, (1 + \epsilon)^1, \ldots, (1 + \epsilon)^9\} \) where \( q = O(\log_{1+\epsilon} m) \), and we can get the algorithm running in time \( n^{O(\Gamma)} = n^{O(\log mn)} \).

### B.1 Preprocessing

The following lemma says that we can assume the input lies on the grid where each row and column of the grid contains exactly one item.

**Lemma B.1.** We are given an instance \((C, I)\) of 2-UUDP-MIN. Then we can, in polynomial time, transform \((C, I)\) into an “equivalent” instance \((C', I')\) such that

- Each consumer \( C' \in C' \) has even coordinates \((2i, 2j)\) for \( 0 \leq i, j \leq n \).

- Each item \( I' \in I' \) has odd coordinate \((2i - 1, 2j - 1)\) for \( 1 \leq i, j \leq n \).

- For each odd number \( 2i - 1 \), \( 1 \leq i \leq n \), there is exactly one item \( I' \in I' \) with \( I'[1] = 2i - 1 \) and exactly one item \( I' \) with \( I'[2] = 2i - 1 \).

**Proof.** We sweep the horizontal line from top to bottom, and whenever the line meets the items \( I_1', \ldots, I_z' \) such that \( I_1'[1] < I_2'[1] < \ldots < I_z'[1] \) with the same \( y \)-coordinate \( y' \), we break ties as follows. Let \( \delta \) be the vertical distance from the line to the next item point below the line. We set the new \( y \)-coordinates of these items to \( I_j'[2] = y' - (z - j)\delta/2z \). Notice that some consumers whose \( y \)-coordinates lie in \([y', y' - \delta)\) get affected by this move. We also change the \( y \)-coordinates
of those consumers to $y’ – \delta/2$. Then we add the horizontal grid lines between the space of every consecutive items, while making sure that consumer points are on the line passing $y – \delta/2$. It is easy to see that this process preserves the consideration set of every consumer. We repeat the above steps until the sweeping line passes the bottommost item.

We do a similar sweep of vertical line from right to left, inserting the grid lines along the way. In the end, each consumer lies on the intersection of the grid lines and each item in its cell, which guarantees that no two items appear in the same row or column of the grid.

\[\blacksquare\]

**B.2 Details of QPTAS for UUDP-MIN**

First, we can make the following simple assumption.

**Lemma B.2.** We can assume that the prices are in the form $(1 + \epsilon)^{0}, (1 + \epsilon)^{1}, ..., (1 + \epsilon)^{q}$ or zero where $q = O(\log_{1+\epsilon} m)$ by sacrificing $(1 + \epsilon)$ in the approximation factor.

**Proof.** We use the following standard arguments. Consider an optimal price $p^*$. For each item $I_j$, if the price is non-zero, we round down the price $p^*(I_j)$ to the nearest scale of $(1 + \epsilon)^{i}$, so the price of each item gets decreased by at most a factor of $(1 + \epsilon)$. Consider a consumer $C$ who bought $I_j$ with price $p^*$. After the rounding, she can still afford $I_j$, so we can still collect at least $(1 – \epsilon)p^*(I_j)$ from $C$.

Now, assuming that the optimal price $p^*$ has the above structure, we show how to solve the problem in quasi-polynomial time. First, we reorder the items based on their $y$-coordinates in descending order, so we have $I_1[2] > I_2[2] > ... > I_n[2]$. A consumer $C$ is said to belong to level $j$ if it lies between the rows of $I_j$ and that of $I_{j+1}$, so each consumer belongs to exactly one level. Moreover, observe that a consumer $C$ at level $j$ is only interested in (a subset of) items in $\{I_1, ..., I_j\}$ (since $I_{j’}[2] < C[2]$ for any $j’ > j$). We define a subproblem $P_j$ as the pricing problem with items $\{I_1, ..., I_j\}$ and consumers at levels $1, ..., j$. We use the dynamic programming technique to solve this problem.

**Profiles** We will remember the profile for each subproblem $P_j$. A profile $\Pi$ of $P_j$ consists of $O(\log m)$ item indices $\pi_1, ..., \pi_q \in \{1, ..., j\}$. Each value $\pi_i$ is supposed to tell us the index of the item $I$ of price $(1 + \epsilon)^i$ with maximum value $I[1]$. That is, we say that a price $p$ for $P_j$ is consistent with profile $\Pi = (\pi_1, ..., \pi_q)$ if, for each $i$, the item $I_{\pi_i}$ has the highest value in the first coordinate among the items with price at most $(1 + \epsilon)^i$, i.e., for all $i$,

$$\pi_i = \arg\max_{j’} \{I_{j’}[1] \mid p(I_{j’}) \leq (1 + \epsilon)^i\}.$$

Since $\{I_{j’} \mid p(I_{j’}) \leq (1 + \epsilon)^i\} \subseteq \{I_{j’} \mid p(I_{j’}) \leq (1 + \epsilon)^{i+1}\}$ for any $i$,

$$I_{\pi_1}[1] \leq I_{\pi_2}[1] \leq ... \leq I_{\pi_q}[1].$$

Observe that if two prices $p’$ and $p''$ have the same $P_j$ profile, then consumers at level $j$ see no difference between these two prices, as shown formally by the following lemma. We say that an item $I_k$ is a profile item for profile $\Pi = (\pi_1, ..., \pi_q)$ if and only if $k = \pi_{q’}$ for some $q’ \in [q]$.

**Lemma B.3.** Let $\Pi$ be a profile for subproblem $P_j$, and let $p$ be any price function for $P_j$ that is consistent with profile $\Pi$. Then we can assume without loss of generality that every consumer at level $j$ only purchases profile items.
Proof. Suppose that a consumer $C$ buys an item $I$ in $I$ with $p(I) = (1 + \epsilon)^q$ which is not a profile item. Then consider the profile item $I_{\pi_q}$, which satisfies $\Pi[1] \geq I[1]$, so we must have $I_{\pi_q} \in S_C$. We can therefore assume that consumer $C$ buys $I_{\pi_q}$ instead of $I$.

Let $\Pi = (\pi_1, \ldots, \pi_q)$ be a profile for $P_j$ and $\Pi' = (\pi_1', \ldots, \pi_q')$ be a profile for $P_{j-1}$. We say that $\Pi$ is consistent with $\Pi'$ if for any price $p' : \{I_1, \ldots, I_j \} \rightarrow \mathbb{R}$ that is consistent with $\Pi'$, we can extend $p'$ to $p$ by assigning value $p(I_j)$ such that $p$ is consistent with $\Pi$. Notice that consistency between any two profiles can be checked in polynomial time by trying all $q$ possibilities of prices.

We recall that we use $p^*$ to denote the optimal price.

Lemma B.4. There are profiles $\Pi^1, \ldots, \Pi^n$ for $P_1, \ldots, P_n$ respectively such that for each $j \in \{1, \ldots, n-1\}$, $\Pi^j$ is consistent with $\Pi^{j+1}$. Moreover, all such profiles are consistent with price $p^*$.

Proof. For each subproblem $P_j$, we define the profile $\Pi^j = (\pi^j_1, \ldots, \pi^j_q)$ based on the price $p^*$ (there is only one possible profile consistent with $p^*$). It is clear that $\Pi^j$ is always consistent with $\Pi^{j+1}$. □

Dynamic Programming Table For each $j = 1, \ldots, n$ and for each profile $\Pi$ of $P_j$, we use a table entry $T(j, \Pi)$ to store the maximum revenue achievable among the price function for $P_j$ that is consistent with the profile $\Pi$. Since there are $n^{O(\log m)}$ possibilities for the profile $\Pi$, the table size is $n^{O(\log m)}$. We now show the computation of the table. To compute $T(j, \Pi)$, we recall that given the profile $\Pi$, the revenue from consumers at level $j$ can be computed efficiently. Denote such revenue by $r_j(\Pi)$. The following equation holds:

$$T(j, \Pi) = r_j(\Pi) + \max_{\Pi'} \text{consistent with } \Pi T(j-1, \Pi')$$

Computing the Solution For each table entry $T(j, \Pi)$, we can keep track of the profile $\Pi'$ such that $T(j-1, \Pi')$ is the entry that is used to compute $T(j, \Pi)$. Let $T(n, \Pi)$ be the entry that contains the maximum value over all $\Pi$. The value in this entry represents the revenue we can get from the optimal pricing $p^*$, so it is enough to reconstruct the price function $p^*$. We first obtain a sequence of profiles $\Pi^1, \ldots, \Pi^n = \Pi$ such that $\Pi^j$ is a profile for $P_j$ and that $\Pi^j$ is consistent with $\Pi^{j-1}$ for any $j = 1, \ldots, n$. This sequence allows us to reconstruct a price function that is consistent with all the profiles in polynomial time.

C QPTAS for 2-SMP

In this section, we show that QPTAS for 2-SMP.

C.1 Overview

We sketch the key ideas here and leave the details in next sections. First, consider the special case where each consumer has budget 1 or 2 and each item must be priced either 0 or 1. The exact optimal solution of this case can be found in $n^{O(\log^2 mn)}$ time. We later show how to extend the idea to the general cases, which turns out to be easy for the case of highway problem but need a few more ideas for the case of 2-SMP.

Algorithm for highway pricing problem reviewed: Let us first start with the highway pricing problem which can be casted as a special case of 2-SMP where items are in the form $(1, n), (2, n-1), \ldots, (n, 1)$. The main idea used in [23], casted in our language of “partition tree” (for convenience in explaining our 2-SMP algorithm later) is the following.

We first construct a balanced binary tree called a partition tree and denoted by $T$. We define the vertical gridline in the middle to be a level-0 line, denoted by $l_r$, dividing the items equally to left
and right sides. This line corresponds to the root node \( r \) of the tree. We also assign the consumers whose consideration set contains items on both sides to the root node. Then we recursively define the subtrees on the subproblems on the two sides of line \( \ell_r \) as shown in Figure 5 until we reach the subproblem containing only one item. For any node \( v \in T \), let \( C_v \) be the set of consumers assigned to \( v \), and \( \ell_v \) be the line associated with node \( v \).

Now we show a top-down recursive algorithm to solve this problem. This algorithm can be converted to a dynamic program by working bottom-up instead. At the root node \( r \) of \( T \), we would like to compute \( f_r(I_{L,1}, I_{L,2}, I_{L,3}, I_{R,1}, I_{R,2}, I_{R,3}) \) which is defined to be the optimal revenue that we can collect from consumers in \( C \setminus C_r \) when we price the items in such a way that \( I_{L,1}, I_{L,2} \) and \( I_{L,3} \) (\( I_{R,1} \), \( I_{R,2} \), and \( I_{R,3} \), respectively) are the first, second, and third closest items on the left (respectively, right) of \( \ell_r \) that have price 1. To avoid long notation, let us denote \( \{I_{L,1}, I_{L,2}, I_{L,3}, I_{R,1}, I_{R,2}, I_{R,3}\} \) by \( \Gamma_r \) and \( f_r(I_{L,1}, I_{L,2}, I_{L,3}, I_{R,1}, I_{R,2}, I_{R,3}) \) by \( f_r(\Gamma_r) \). If we can compute \( f_r(\Gamma_r) \) for all \( \Gamma_r \), then the optimal revenue can be obtained via the following formula.

\[
\text{Optimal revenue} = \max_{\Gamma_r} f_r(\Gamma_r) + m_1(\Gamma_r) + 2m_2(\Gamma_r) \tag{7}
\]

where, for any node \( v \), \( m_1(\Gamma_v) \) is the number of consumers in \( C_v \) whose consideration sets contain exactly one item in \( \Gamma_v \), and \( m_2(\Gamma_v) \) is the number of consumers in \( C_v \) with budget 2 whose consideration sets contain exactly two items in \( \Gamma_v \). The point is that we can calculate the revenue from consumers in \( C_r \) as \( m_1(\Gamma_r) + 2m_2(\Gamma_r) \) and use \( f_r(\Gamma_r) \) to compute the revenue obtained from the rest of the consumers.

It is left to compute \( f_r(\Gamma_r) \). Let \( u \) and \( v \) be the left and right children of \( r \), respectively. In order to compute \( f_r(\Gamma_r) \), we will compute \( f_u(\Gamma_r, \Gamma_u) \) which is the maximum revenue we can collect from consumers assigned to the descendants of \( u \) (excluding \( u \)) where \( \Gamma_r \) is the set of six items of price 1 that are closest to \( \ell_r \) as defined earlier. And, similarly, \( \Gamma_u = \{I'_{L,1}, I'_{L,2}, I'_{L,3}, I'_{R,1}, I'_{R,2}, I'_{R,3}\} \) is the set of six items of price 1 that are closest to \( \ell_u \). Moreover, we require that \( \Gamma_u \) must be consistent with \( \Gamma_r \) in the sense that there is some price assignment such that items in \( \Gamma_u \) are the items closest to \( \ell_u \) of price 1 and items in \( \Gamma_r \) are the items closest to \( \ell_u \) of price 1 as well. (For example, if we let \( \Gamma_r = \{I_{L,1}, I_{L,2}, I_{L,3}, I_{R,1}, I_{R,2}, I_{R,3}\} \) then an item \( I \) with property \( I_{L,3}[1] < I[1] < I_{L,2}[1] \) cannot

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{partition_tree.png}
\caption{A partition tree}
\end{figure}
be in \( \Gamma_u \) since this item must have price 0.) We use \( \Gamma_u \bowtie \Gamma_r \) to denote “\( \Gamma_u \) is consistent with \( \Gamma_r \)”.

We define \( f_v(\Gamma_r, \Gamma_u) \) in a similar way.

Once we have \( f_u(\Gamma_r, \Gamma_u) \) and \( f_v(\Gamma_r, \Gamma_v) \) for all \( \Gamma_u \bowtie \Gamma_r \) and \( \Gamma_v \bowtie \Gamma_r \), we can compute \( f_r(\Gamma_r) \):

\[
 f_r(\Gamma_r) = \max_{\Gamma_u \bowtie \Gamma_r} \{ f_u(\Gamma_r, \Gamma_u) + m_1(\Gamma_u) + 2m_2(\Gamma_u) \} + \max_{\Gamma_v \bowtie \Gamma_r} \{ f_v(\Gamma_r, \Gamma_v) + m_1(\Gamma_v) + 2m_2(\Gamma_v) \}. \tag{8}
\]

The main point here is that there is no consistency requirement between \( \Gamma_u \) and \( \Gamma_v \) so we have two independent subproblems. We define the function \( f_z \), for all nodes \( z \) in \( T \) similarly: Let \( r = v_0, v_1, v_2, \ldots, v_q = z \). We have to compute \( f_z(\Gamma_{v_0}, \Gamma_{v_1}, \ldots, \Gamma_{v_q}) \) for all \( \Gamma_{v_0}, \Gamma_{v_1}, \ldots, \Gamma_{v_q} \) such that \( \Gamma_{v_i} \bowtie \Gamma_{v_j} \) for all \( i \neq j \).

The computation of \( f_z(\Gamma_{v_0}, \Gamma_{v_1}, \ldots, \Gamma_{v_q}) \) is done in the same way as Eq.\( \tag{8} \) for every non-leaf node \( z \). At leaf node \( z \), \( f_z(\Gamma_{v_0}, \Gamma_{v_1}, \ldots, \Gamma_{v_q}) \) can also be easily computed: \( f_z(\Gamma_{v_0}, \Gamma_{v_1}, \ldots, \Gamma_{v_q}) = m_1(\Gamma_z) + 2m_2(\Gamma_z) \).

Observe that \( q = O(\log m + \log n) \) and there are \( n^6 \) choices for each \( \Gamma_{v_i} \). Therefore, we can precompute \( f_z(\Gamma_{v_0}, \Gamma_{v_1}, \ldots, \Gamma_{v_q}) \) for all \( n^{O(\log m + \log n)} \) combinations of \( \Gamma_{v_0}, \Gamma_{v_1}, \ldots, \Gamma_{v_q} \). By working bottom-up from the leaf nodes, the running time becomes \( \text{poly}(m)n^{O(\log m + \log n)} \).

**Algorithm for 2-SMP (special case):** To solve the special case of 2-SMP defined above, we need to modify a few definitions in a right way. Let us again consider the top-down algorithm and start at the root node \( r \) of the partition tree \( T \). (Recall that we can assume that there is at most one item in each row and column so we can still define the partition tree by drawing the vertical line through the point in the middle when sorted by the first dimension.)

One problem immediately appears: \( f_r(\Gamma_r) \) cannot be used to compute the optimal revenue as we did in Eq.\( \tag{7} \). The reason is that we cannot compute the revenue from \( C_r \) using \( m_1(\Gamma_r) + 2m_2(\Gamma_r) \) anymore. To fix this, we have to redefine \( C_r \) in the following way: We assign all consumers lying on the left (respectively, right) of \( I_r \) to the left (respective, right) child and keep only those consumers lying exactly on the vertical line going through \( I_r \) in \( C_r \).

Now we can compute the revenue from the newly defined \( C_r \) and a function that computes the total revenue. To do this, we define \( f_r(I_1, I_2, I_3) \) to be the total revenue we can get from consumers in \( C \setminus C_r \) by pricing the items in such a way that, among the items on the right side of \( I_r \), items \( I_1, I_2, \) and \( I_3 \) are the items with price 1 that have the highest, second highest, and third highest values in the second dimension, respectively. Again, let \( \Gamma_r \) denote a possible choice of \( \{I_1, I_2, I_3\} \) and write \( f_r(\Gamma_r) \) instead of \( f_r(I_1, I_2, I_3) \). If we can compute \( f_r(\Gamma_r) \) then we can get the optimal revenue by Eq.\( \tag{7} \), where \( m_1(\Gamma_r) \) and \( m_2(\Gamma_r) \) is as defined earlier (with the new definition of \( C_r \)).

Some more complications lie in computing \( f_r(\Gamma_r) \), for any \( \Gamma_r \). As before, we will compute \( f_u(\Gamma_r, \Gamma_u) \) and \( f_v(\Gamma_r, \Gamma_v) \) where \( u \) and \( v \) are the left and right children of \( r \), respectively. However, we have to carefully define \( f_u(\Gamma_r, \Gamma_u) \) and \( f_v(\Gamma_r, \Gamma_v) \), in a different way.

We define \( f_u(\Gamma_r, \Gamma_u) \), for any \( \Gamma_u = \{I_1, I_2, I_3\} \), to be the maximum revenue from the consumers assigned to the descendants of \( u \) when we price the items in such a way that, among the items lying on the right side of \( I_u \) and left side of \( I_r \), items \( I_1, I_2, \) and \( I_3 \) are the items with price 1 that have the highest, second highest, and third highest values in the second dimension, respectively. Note that we do not need to check any consistency between \( \Gamma_r \) and \( \Gamma_u \). For any choice of \( \Gamma_r \) and \( \Gamma_u \), there is always a price assignment such that items in \( \Gamma_r \) and \( \Gamma_u \) are the items of price 1 that have the highest values in the second dimension in their respective regions. In this case, we say that \( \Gamma_r \bowtie \Gamma_u \) is always true for any \( \Gamma_r \) and \( \Gamma_u \).

On the other hand, we define \( f_v(\Gamma_r, \Gamma_v) \), for any \( \Gamma_v = \{I_1, I_2, I_3\} \), to be the maximum revenue from the consumers assigned to the descendants of \( v \) when we price the items in such a way that,
among the items lying on the right side of $I_v$, items $I_1$, $I_2$, and $I_3$ are the items with price 1 that have the highest, second highest, and third highest values in the second dimension, respectively. In this case, we have to make sure that $\Gamma_v$ is consistent with $\Gamma_r$, i.e., there is some price assignment such that items in $\Gamma_r$ and $\Gamma_u$ are the items of price 1 that have the highest values in the second dimension in their respective regions.

Now we have defined \( f_u(\Gamma_r, \Gamma_u) \) and \( f_v(\Gamma_r, \Gamma_v) \), we compute \( f_r(\Gamma_r) \) using Eq. (8). As in the case of the highway pricing problem, we can extend the definition to other nodes. In particular, at a leaf node $z$ we have to compute \( f_z(\Gamma_v^0, \Gamma_v^1, \ldots, \Gamma_v^q) \) where $q = O(\log m + \log n)$. Hence, this case can be solved in \( \text{poly}(|C|) \cdot |I|^{\text{poly}(\log |I|)} \) time.

**Algorithm for general 2-SMP:** We now remove the restrictions that each item must be priced 0 or 1 and each budget must be 1 or 2. The removal of the restriction on item price does not affect the case of highway pricing problem since this can be easily assumed (see, e.g., [31]). Moreover, we can still assume that the maximum budget is $O(mn)$. Now we can deal with the general highway problem by redefining \( f_i(\Gamma_r) \): Let \( \Gamma_r = \{I_{L_0, 0}, I_{L_1, 1}, \ldots, I_{L_q, q}, I_{R_0, 0}, I_{R_1, 1}, \ldots, I_{R_q, q}\} \) where $q = O(\log mn)$. For any $i \leq q$, we want to price in such a way that $I_{L,i}$ is the item closest to $I_r$ on the left such that the sum of the price of all items between $I_r$ and $I_{L,i}$ is at least \( (1 + \epsilon)^i \). Computing \( f_r(\Gamma_r) \) can be done in the same manner as before and consistency checking is easy to deal with. Function \( f_v(\Gamma_v^0, \Gamma_v^1, \ldots, \Gamma_v^q) \), for any node $v_q$ at level $q$ in $T$, can be defined in a similar manner.

For 2-SMP, we may not in general assume the item prices to be 0/1. Instead, we show that it can be assumed that each item must have price 0, or \( (1 + \epsilon)^j \), for any $j = 0, 1, \ldots, O(\log m)$. A natural extension of the above idea is to define the notion of “volume of regions”: For each item $I$, let $H_I$ and $V_I$ denote the horizontal and vertical line cutting through item $I$, respectively. Any rectangle resulting from drawing some horizontal and vertical lines through some items are called *regions* and the regions that do not contain other regions are called *minimal regions*. For any price assignment, we define the *volume* of a region to be the sum of the price of all items within the region.

Now, similar to the highway problem, we define $\Gamma_r = \{I_0, I_1, \ldots, I_k\}$ (note that $k = O(\log m)$) as the “region guess”: We define \( f_r(\Gamma_r) \) to be the maximum revenue from $C \setminus C_r$ when we price in such
a way that, for any \( i \), item \( \mathbf{I}_i \) is the highest item (in the second dimension) such that the volume of the region on the right of the vertical line \( V_{\mathbf{I}_i} \) and above the horizontal line \( H_{\mathbf{I}_i} \) (including \( \mathbf{I}_i \)) is at least \((1 + \epsilon)^i\). Using these volume guesses, we can approximate the upper and lower bounds of the revenue from each consumer \( \mathbf{C} \) at node \( z \) by looking at \( \Gamma_v \) for all ancestors \( v \) of \( z \). This is because each consumer’s consideration set will contain some set of regions \( B_1, B_2, \ldots \) with volume guesses \((1 + \epsilon)^{i_1}, (1 + \epsilon)^{i_2}, \ldots \) respectively (such as the blue regions in Figure 6). Also, this consideration set will also be contained in some set of regions \( R_1, R_2, \ldots \) with volume guesses \((1 + \epsilon)^{j_1+1}, (1 + \epsilon)^{j_2+1}, \ldots \) (such as the blue and red regions together in Figure 6).

However, in contrast to the case of highway problem, the consistency between the guesses (e.g., between \( \Gamma_r \) and its children \( \Gamma_u \) and \( \Gamma_v \)) is harder to guarantee. In order to guarantee the consistency, we add another parameter, denoted by \( \Delta_r \) (\( \epsilon_\mathbf{B} \)), whose budgets are less than \( \epsilon B \). Notice that we only lose the revenue of at most \( \epsilon B_{\max} / mn \). Notice that we only lose the revenue of at most \( \epsilon B_{\max} / mn \). Notice that we only lose the revenue of at most \( \epsilon B_{\max} / mn \). Notice that we only lose the revenue of at most \( \epsilon B_{\max} / mn \).

C.2 Preprocessing

Fix some \( \epsilon > 0 \). Given an instance \((\mathcal{I}, \mathcal{C})\), our goal is to compute a price that collects a revenue of at least \( (1 - O(\epsilon)) \text{OPT} \). Recall that we can assume that the consumers are on the intersection of grid lines, and the items are in the grid cells (cf. Lemma B.1). First we process the input so that the budgets and prices are polynomially bounded. Moreover, the optimal solution only assigns prices of the form \((1 + \epsilon)^\ell\) for some \( \ell \leq O(\log m) \). The proof of this fact only uses standard arguments (along the same line as in [3]).

**Lemma C.1.** Let \( M = O(mn / \epsilon) \). The input instance \( \mathcal{P} \) can be reduced to \( \mathcal{P}' \) with the following properties.

- For each consumer \( \mathbf{C} \), the budget of \( \mathbf{C} \) in \( \mathcal{P}' \) is between 1 and \( M \).
- Any price \( p' \) that \( \alpha \)-approximates the optimal pricing of \( \mathcal{P}' \) can be transformed in polynomial time into another price \( p \) that gives \((1 + 3\epsilon)\alpha\)-approximation for \( \mathcal{P} \).
- There is a \((1 + \epsilon)\)-approximate solution \( \bar{p} \) satisfying the following: For all \( \mathbf{I} \in \mathcal{I} \), \( 1 \leq \bar{p}(\mathbf{I}) \leq M \), and \( \bar{p}(\mathbf{I}) \) is in the form \((1 + \epsilon)^j\) for some \( j \leq O(\log m) \).

**Proof.** Let \( B_{\max} \) be the maximum budget among all consumers. We first remove all consumers whose budgets are less than \( \epsilon B_{\max} / mn \). Notice that we only lose the revenue of at most \( \epsilon B_{\max} \leq \epsilon \text{OPT} \) by this removal. We denote the new set of consumers by \( \mathcal{C}' \). Now look at the optimal price \( p^* \) for the resulting instance. If for some \( \mathbf{I} \in \mathcal{I} \), the price \( p^*(\mathbf{I}) \) is less than \( \epsilon B_{\max} / mn \), we change its price to \( p'(\mathbf{I}) = 0 \) and remove item \( \mathbf{I} \) completely from the instance. Again, since each such item
can only be sold to at most \( m \) consumers, discarding it only decreases the revenue by \( \epsilon B_{\text{max}}/n \).

There are at most \( n \) such items, so we lose a revenue of at most \( \epsilon \text{OPT} \) in total. Let \( \mathcal{I}' \) denote the resulting set of items.

Next we scale each consumer budget by \( M' = mn/\epsilon B_{\text{max}} \) to get a new budget, i.e. \( B_C' = M'B_C \).

Now we have a complete description of the instance \( \mathcal{P}' \) in which consumer budgets are between 1 and \( M' \). Let \( \text{OPT}' \) be the optimal value of the new instance. First we try to lower bound the value of \( \text{OPT}' \). Consider the same price \( p^* : \mathcal{I}' \rightarrow \mathbb{R} \) scaled up by a factor of \( M' \). The revenue from this price is at least \( (1-2\epsilon)M'\text{OPT} \), so we have that \( \text{OPT}' \geq (1-2\epsilon)M'\text{OPT} \).

We are now ready to prove the second part. Assume that we have a price \( p' \) that gives \( \alpha \)-approximation for \( \mathcal{P}' \), so the revenue collected by \( p' \) is at least \( \text{OPT}'/\alpha \). We construct the price \( p \) by scaling down the price of \( p' \) by \( M' \). Notice that for each consumer \( C \) who can afford his consideration set in \( \mathcal{P}' \) with price \( p'(S_C) \), he can also afford his set in \( \mathcal{P} \) with price \( p'(S_C)/M' \). Therefore, the revenue collected by \( p \) is at least \( \text{OPT}'/\alpha M' \geq (1-2\epsilon)\text{OPT}/\alpha \). This argument also implies that \( \text{OPT} \geq \text{OPT}'/M' \).

Finally we show that there is a good solution \( \hat{p} \) that only assigns prices in the form \((1+\epsilon)^j\), as follows. We round down the price of \( p^* \) to the nearest scale of \((1+\epsilon)^j\) for some \( j \). For each consumer \( C \) who purchases item \( I \) w.r.t. price \( p^* \), by scaling down every item price, she can still afford her consideration set \( S_C \), whose new price is at least \( p^*(S_C)/(1+\epsilon) \geq (1-\epsilon)p^*(S_C) \).

From now on, we assume that our input instance and its optimal price are in such format. Our goal is to devise a QPTAS for this instance. We note here that in some special cases of single-minded pricing problems, especially the Highway problem, an even stronger statement can be assumed; namely, that the optimal price is integral \([32]\). It seems that such a nice property may not hold in our case, and we anyway do not need it.

### C.3 Partition tree

We first construct a (almost balanced) binary tree \( T \) where each node in \( T \) is associated with a rectangular region in the plane (from now on, whenever we talk about region, we always mean a rectangular one). We call this tree the partition tree. It can be constructed recursively as follows. In the beginning, we have \( T = \{r\} \) where \( r \) is the root of the tree whose region \( A_r \) is the whole grid. We repeat the following process: For each leaf \( v \in T \), if the region \( A_v \) of \( v \) contains at least two items, we choose a vertical grid line \( \ell_v \) dividing the items in a balanced manner to the left and right side. We then add the left child \( v' \) of \( v \) with the region \( A_{v'} \) being the region of \( A_v \) on the left side of \( \ell_v \). We also add the right child \( v'' \) of \( v \) associated with the region \( A_{v''} \) on the right side of \( \ell_v \). We repeat the process until every leaf is associated with a region containing only one item; see Figure 7(a).

For each node \( v \in T \), we define the item set \( \mathcal{I}_v \) to be the set of all items in the region \( A_v \). Fix a price \( p : \mathcal{I} \rightarrow \mathbb{R} \). For any region \( A \), we define the “volume” \( \text{vol}_p(A) \) to be the total price among all items in the region, i.e. \( \text{vol}_p(A) = \sum_{I \in A} p(I) \). The following simple claim is crucial in designing our algorithm.

**Claim C.2.** Let \( p^* \) be an optimal price. Then for any region \( A \), there are only \( n^{O(\log m)} \) possible values of \( \text{vol}_{p^*}(A) \).

**Proof.** Let \( x_j \) denote the number of items \( I \) in \( A \) with price \( p^*(I) = (1+\epsilon)^j \). Notice that we can write the volume of \( A \) as \( \sum_{j=1}^{\ell} x_j (1+\epsilon)^j \) where \( x_j \) only takes non-negative integer values at most \( n \). So we have at most \( n^{O(\log m)} \) possibilities for the volume. \( \square \)
C.4 Horizontal partition and local profile

From the construction, each node \( v \) of the partition tree, is associated with a vertical line \( \ell_v \) which divides the plane into two regions. We further partition the right region using vertical line, as follows.

Consider a non-leaf node \( v \in \mathcal{T} \) with left child \( v' \) and right child \( v'' \). A horizontal partition for node \( v \), denoted by \( H_v \), is a collection of (not-necessarily distinct) horizontal lines \( \ell_{v,1}^v, \ldots, \ell_{v,n}^v \), partitioning the region of \( A_{v''} \) into many pieces; note that the left endpoints of these lines are on \( \ell_v \). The line \( \ell_{v,j}^v \) is supposed to mark the highest \( y \)-coordinate such that the volume inside \( A_{v''} \) above \( \ell_{v,j}^v \) is at least \((1 + \epsilon)^j\). Notice that each node \( v \) has at most \( n^{O(\log m)} \) feasible partitions since there are at most \( n \) possibilities for the choice of each \( \ell_{v,j}^v \).

Now if we fix a horizontal partition of every non-leaf node in the partition tree, we can define minimal regions for each non-leaf node \( v \) as follows. For each node \( v \), we consider all vertical and horizontal lines associated with \( v \) and all its ancestors (i.e., all lines in \( \ell_u \) and \( H_u \) where \( u = v \) or \( v \) is an ancestor of \( v \)). Let \( \mathcal{L}_v \) denote the set of these lines. \( \mathcal{L}_v \) naturally defines minimal regions: We say that a region \( A \) is minimal with respect to \( \mathcal{L}_v \) if and only if \( A \) is a rectangle whose four boundaries are the lines in \( \mathcal{L}_v \), and there is no line in \( \mathcal{L}_v \) that intersects with the interior of \( A \).

Now, we define a local profile of a node \( v \). It consists of (i) horizontal partitions for \( v \) and for all its ancestors, and (ii) numbers on every minimal region resulting from vertical and horizontal lines. The numbers are supposedly the “volume guesses” of every minimal region of \( v \).

Now we try to guess the “right” local profile of every node in the partition tree. We show that if this guess is right, then we get a good approximation of the optimal solution. Moreover, we can use dynamic programming to make the right guess.

C.5 Dynamic Programming Solution

A global profile (or just profile in short) of a node \( v \) consists of the local profile of \( v \) and all its ancestors in such a way that the volumes of minimal regions of \( v \) is consistent with its ancestors. More formally, fix a node \( v \). A profile \( \Pi_v \) for \( v \) consists of, for any ancestor \( v' \) of \( v \), \( \Pi_{v,v'} \) which is the local profile that node \( v \) wants its ancestor \( v' \) to have (we also think of \( v \) has an ancestor of itself for convenience). As a reminder, for each ancestor \( v' \) of \( v \), local profile \( \Pi_{v,v'} \) some horizontal partition \( H_{v'} \) and the “volume guess” of each minimal region of \( v \). In addition, we restrict that these local profiles \( \Pi_{v,v'} \) have to be consistent in themselves in the following sense. For each vertex \( v' \), for any minimal region \( A' \) of \( \Pi_{v,v'} \) that is further partitioned into minimal regions \( A'_1, A'_2, \ldots, A'_\gamma \) of \( \Pi_{v,v'} \) for some descendant \( v'' \) of \( v' \), the number \( z_{A'} \) at \( \Pi_{v,v''} \) is equal to the sum of the numbers \( z_{A'_j} \) of \( \Pi_{v,v''} \).

We argue that the number of global profiles for each node is not too large, i.e. only \( n^{\text{poly log} m} \). There are \( n^{O(\log m)} \) horizontal partitions for each ancestor \( v' \) of \( v \), making a total of \( n^{O(\log m \log n)} \) possibilities of the lines \( \ell_{v,j}' \). Now fix a choice of such horizontal partitions. If we draw all lines \( \ell_{v,j}' \) involved in the global profiles, we will see a number of regions formed by intersections between these lines and the vertical lines \( \ell_{v''} \). Since there are \( O(log m \log n) \) such horizontal lines and \( O(log n) \) vertical lines involved, we have at most \( O(log m \log^2 n) \) minimal rectangular regions. Each region has at most \( n^{O(\log m)} \) possible volumes, so there are at most \( n^{O(\log^2 m \log^2 n)} \) global profiles for each node \( v \).

Now we define a valid tree profile \( \Pi \) for \( \mathcal{T} \) as the set of global profiles \( \{\Pi_v\}_{v \in \mathcal{T}} \) such that \( \Pi_v \) is a global profile for node \( v \). Moreover, for every parent-child pair \( v, v' \) where \( v \) is a parent of \( v' \) in \( \mathcal{T} \), the profile \( \Pi_{v'} \) agrees with \( \Pi_v \). That is, all profiles about ancestors of \( v \) in \( \Pi_v \) and \( \Pi_{v'} \) are exactly
the same.

Given a valid tree profile Π, we have the notion of cost of the profile Π (denoted by \(\text{Cost}(\Pi)\)) which is supposed to approximate the total revenue we can collect by a price function consistent with Π. The cost of a profile can be computed as follows. For each node \(v \in \mathcal{T}\), let \(\mathcal{C}_v\) be the set of all consumers on line \(\ell_v\). For each consumer \(\mathcal{C} \in \mathcal{C}_v\), the rectangular region enclosed by horizontal line \(\mathcal{C}[2]\) and vertical line \(\ell_v\) is the actual amount the consumer needs to pay. This is the amount we do not know, but we can approximate: We let \(v_0, v_1, \ldots, v_\alpha\) be a sequence of ancestors of \(v\) such that \(v\) is on the left subtree of \(v_i\) (in the order from \(v\) to the root), where \(v_0 = v\). And we let for each \(i, j_i\) be the maximum number such that \(\ell_{v_i}^{j_i}\) does not lie below \(\mathcal{C}[2]\). The cost of consumer \(\mathcal{C}\) is just the sum \(\sum_{i=0}^{\alpha} (1 + \epsilon)^{j_i}\) if \(B_{\mathcal{C}} \leq \sum_{i=0}^{\alpha} (1 + \epsilon)^{j_i}\) and zero otherwise. The cost at node \(v\) is just the total cost of all consumers in \(\mathcal{C}_v\), and the cost of the profile is the sum of the cost over all nodes \(v \in \mathcal{T}\).

**Lemma C.3.** There is a valid tree profile \(\Pi^*\) such that the cost is at least \((1 - \epsilon)\text{OPT}\).

**Proof.** We start from the optimal price \(p^*\) and construct the valid profile as follows. For each node \(v\), we define a feasible partition of \(v\) by choosing the line \(\ell_v^{j_i}\) to be at the highest y-coordinate such that the total volume enclosed is at least \((1 + \epsilon)^{j_i}\). Then we create a profile \(\Pi^*_v\) for each node \(v\) according to the actual volume of each minimal region. Notice that this gives a valid tree profile.

Our goal now is to compute the valid profile \(\Pi\) of maximum cost by dynamic programming, and the profile will automatically suggest a near-optimal pricing.

**Computing the Solution:** Let \(v \in \mathcal{T}\). We say that a price \(p : I_v \to \mathbb{R}\) is consistent with global profile \(\Pi_v\) if and only if for every minimal region \(A\) of \(\Pi_v\) that is completely contained in \(A_v\), we have \(\text{vol}_p(A) = z_A\). The minimum cost profile can be computed in a bottom-up fashion, as follows. For a leaf node \(v\), a global profile for \(v\) automatically determines the price of the only item in \(A_v\); discard a profile which does not have consistent price.

The following lemma shows that a price \(p\) consistent with a valid tree profile \(\Pi\) can be computed from \(\Pi\).
Lemma C.4. For each node $v$ with left child $v'$ and right child $v''$, let $p' : I_{v'} \rightarrow \mathbb{R}$ and $p'' : I_{v''} \rightarrow \mathbb{R}$ be the prices that are consistent with the profile $\Pi_{v'}$ and $\Pi_{v''}$ respectively. Then the price $p : I_v \rightarrow \mathbb{R}$ defined to agree with $p'$ on $I_{v'}$ and with $p''$ on $I_{v''}$, is consistent with $\Pi_v$.

Proof. Consider a minimal region $A \subseteq A_v$ and a volume guess $z_A$ in $\Pi_v$. If $A \subseteq A_v$ where $A$ is the union of minimal regions $A'_1, \ldots, A'_r$ of $\Pi_v$ (similar argument can be made in case $A \subseteq A_v$), then by assumption that $\Pi_v$ is consistent with $\Pi_v$, we know that the total value $z_A = \sum_{j=1}^r z'_{A'_j}$. Since $p'$ is consistent with the profile $\Pi_v$, we have that $\text{vol}_{p'}(A) = \sum_{j} \text{vol}_{p'}(A'_j) = \sum_{j} z'_{A'_j} = z_A$ as desired. □

We have shown that a valid tree profile $\Pi$ always has a price $p$ consistent with it. The following lemma basically says that this price $p$ gives a revenue close to the cost of the profile, which in turn imply that the maximum cost profile gives the revenue of at least $(1 - O(\epsilon))\text{OPT}$.

Lemma C.5. For any valid tree profile $\Pi$, let $p$ be a price consistent with $\Pi$ and let $p' = p/(1 + \epsilon)$. Then $p'$ collects revenue at least $(1 - \epsilon)$ fraction of the profile cost.

D Omitted hardness results

D.1 Hardness of 3-UUDP-MIN and 4-UUDP-MIN

In this section we show that 3-UUDP-MIN is NP-hard, and 4-UUDP-MIN is APX-hard by a reduction from Vertex Cover. Our reduction relies on the concepts of adjacency poset and its embedding into Euclidean space. We describe basic terminologies here. Given a graph $G = (V, E)$, an adjacency poset $(V \cup E, \preceq_G)$ of graph $G$ can be constructed as follows: First we define a poset with its maximal elements corresponding to vertices in $V$ and its minimal elements corresponding to edges $E$. For each vertex $v$ and each edge $e$, we have the relation $e \preceq_G v$ if and only if vertex $v$ is an endpoint of $e$. We say that a map $\varphi : V \cup E \rightarrow \mathbb{R}^d$ is an embedding of adjacency poset $(V \cup E, \preceq_G)$ into $\mathbb{R}^d$ if and only if it preserves the relations $\preceq_G$, i.e., for any two elements $a, b \in V \cup E$, we have that $a \preceq_G b$ iff $\varphi(a)[i] \leq \varphi(b)[i]$ for all coordinates $i \in [d]$.

Now we describe our reductions. Since two reductions are essentially the same, we give a general procedure which will imply both results. Given an instance $G = (V, E)$ of Vertex Cover, we first construct an adjacency poset $(V \cup E, \preceq_G)$ for $G$, and then we compute the embedding $\varphi$ of this poset into Euclidean space $\mathbb{R}^d$. We will use the graph $G$, as well as the embedding $\varphi$, to define the instance of $d$-UUDP-MIN as follows:

- **Consumers**: We have two types of consumers, i.e. the rich consumers and the poor ones.
  For each vertex $v \in V$, we create a rich consumer $C_v$ with budget 2 at coordinates $\varphi(v)$. For each edge $e \in E$, we create a poor consumer $C_e$ with budget 1 at coordinates $\varphi(e)$.

- **Items**: For each vertex $v \in V$, we create item $I_v$ at coordinates $\varphi(v)$.

Note that for each $e = (u, v)$, each poor consumer $C_e$ has $S_{C_e} = \{I_v, I_u\}$, while each rich consumer $C_v$ has $S_{C_v} = \{I_v\}$. We denote the resulting instance by $(C, I)$.

The following lemma gives a characterization of the optimal solution for $(C, I)$. It says that we may assume without loss of generality that every poor consumer gets some item.

Lemma D.1. For any price $p$ that is a solution for $(C, I)$ constructed above, we can transform $p$ to $p'$ such that every poor consumer buys some item with respect to $p'$, and the revenue of $p'$ is at least as much as the revenue of $p$. 

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Proof. Consider edge \( e = (u,v) \). Suppose poor consumer \( C_e \) does not get any item, so it implies that both items \( I_u \) and \( I_v \) have price \( p(I_u) = p(I_v) = 2 \) (recall that, since budgets are 1 or 2, the optimal prices would never set prices that are not in \( \{1,2\} \)). We define the price function \( p' \) by setting \( p'(I_u) = 1 \) while \( p'(I_w) = p(I_w) \) for all other vertices \( w \in V \setminus \{u\} \). The only rich consumer that gets affected is \( C_u \), whose payment may decrease by one. However, we earn the revenue of one back from poor consumer \( C_e \). For \( e' \in E : e' \neq e \), poor consumer \( C_{e'} \) is never affected because his budget is one. Overall, changing the price from \( p \) to \( p' \) never decreases revenue.

Let \( p^* \) be the optimal price for \((C,I)\) and \( VC(G) \) denote the size of minimum vertex cover of \( G \). We show the following connection between the size of minimum vertex cover and the optimal revenue collected by \( p^* \).

**Theorem D.2.** The optimal revenue collected by \( p^* \) is exactly \( 2n - VC(G) + m \).

Proof. From the previous lemma, we can assume that the pricing \( p^* \) sells items to every poor consumer. In other words, if \( V' = \{v : p^*(I_v) = 1\} \), it must be the case that \( V' \) is a vertex cover: otherwise, let \( e = (u, w) \) be an edge which is not covered by any vertex in \( V' \), so \( C_e \) is only interested in items with price 2, which he cannot afford. This contradicts the assumption that \( p^* \) sells items to every poor consumer.

The revenue collected from poor consumers is exactly \( m \). Each rich consumer \( C_v \) in the vertex cover gets the item with price 1 while others get the items with price 2, so the total revenue is \( m + VC(G) + 2(n - VC(G)) \).

This theorem immediately implies the gap between Yes-Instance and No-Instance for \( d \)-UUDP-MIN. The only detail we left out is the computation of the embedding \( \varphi \), and this is where the hardness proofs of 3-UUDP-MIN and 4-UUDP-MIN depart (other steps are exactly the same). For 3-dimensional case, we start from planar graphs whose adjacency poset can be embedded into \( \mathbb{R}^3 \). Since planar vertex cover has a polynomial-time approximation scheme, we only get \( \text{NP} \)-hardness here. For 4-dimensional case, we start from vertex cover in cubic graphs, which is known to be \( \text{APX} \)-hard, but unfortunately we can only embed its adjacency poset into \( \mathbb{R}^4 \), thus obtaining the hardness of 4-UUDP-MIN.

**NP-Hardness of 3-UUDP-MIN** To show the \( \text{NP} \)-hardness, we start from Vertex Cover in planar graphs, which is known to be \( \text{NP} \)-complete [28]. We will use the following theorem, due to Schnyder [47].

**Theorem D.3.** Let \((V \cup E, \leq_G)\) be an incident poset of a planar graph \( G \). Then there exists an embedding \( \varphi \) from the poset into \( \mathbb{R}^3 \).

Schnyder shows later that the crucial step in the theorem can be computed in polynomial time [48], which immediately implies the following theorem.

**Theorem D.4.** 3-UUDP-MIN is \( \text{NP} \)-hard even when the consumer budgets are either 1 or 2.

**APX-Hardness of 4-UUDP-MIN** We will be using the fact that Vertex Cover in cubic graphs is \( \text{APX} \)-hard [4], stated in the language convenient for our use below.

**Theorem D.5.** For some \( 0 < \alpha < \beta < 1 \), it is \( \text{NP} \)-hard to distinguish between (i) the graph that has a vertex cover of size at most \( \alpha n \), and (ii) the graph whose minimum vertex cover is at least \( \beta n \).
Now we assume that our input graph $G$ is a cubic graph and use the following theorem to embed the adjacency poset of $G$ into $\mathbb{R}^4$.

**Theorem D.6** (Schnyder). An adjacency poset of any 4-colorable graph can be embedded into $\mathbb{R}^4$. Moreover, the embedding is computable in polynomial time.

It only requires a straightforward computation to prove the following theorem.

**Theorem D.7.** 4-UUDP-MIN is APX-hard even when the consumer budgets are either 1 or 2.

**Proof.** In the Yes-Instance, we can collect the revenue of $(2 - \alpha) n + m$. However, in the No-Instance, the revenue is at most $(2 - \beta) n + m$. Since the graph is cubic, we may assume that $m = \gamma n$ for some $1 \leq \gamma < 2$. Hence we have a gap of $(2 - \alpha + \gamma)/(2 - \beta + \gamma)$. \hfill \square

### D.2 NP-hardness of 2-SMP

Highway problem can be defined as follows: We are given a line $P = (v_0, \ldots, v_n)$ consisting of $n$ edges and $n + 1$ vertices and a set of consumers $C$ where each consumer $C$ corresponds to a subpath $P_C$ of $P$ and is equipped with budget $B_C$. Our goal is to set price to edges so as to maximize the revenue, where each consumer $C$ buys a path $P_C$ if she can afford the whole path; otherwise she buys nothing.

**Lemma D.8.** There is a polynomial-time algorithm that transforms an instance of Highway problem to an instance of 2-SMP.

**Proof.** For each $i = 1, \ldots, n$, each edge $(v_{i-1}, v_i)$, we create an item $I_i$ at coordinates $(i, n + 1 - i)$. Then for each consumer $C$ whose path is $P_C = (v_{s}, \ldots, v_t)$, we create a consumer point at $(s + 1, n + 1 - t)$. It is easy to see that the consideration set remains unchanged. \hfill \square

### D.3 APX-hardness of 4-SMP

We perform a reduction from Graph Vertex Pricing on bipartite graphs. In this problem, we are given a graph $G = (V, E)$, where each vertex corresponds to item and each edge $e \in E$ corresponds to a consumer, additionally equipped with budget $B_e$. Each consumer edge is interested in items that correspond to her incident vertices. Our goal is to set a price $p : V \to \mathbb{R}$ so as to maximize our revenue.

Given an instance $(G, \{B_e\}_{e \in E})$ of Graph Vertex Pricing where graph $G$ is a bipartite graph $(U \cup W, E)$, we create an instance of 4-SMP as follows. Suppose we have $U = \{u_1, \ldots, u_{|U|}\}$ and $W = \{w_1, \ldots, w_{|W|}\}$. For each vertex $u_i \in U$, we have a corresponding item $I^u_i$ with coordinates $(i, |U| + 1 - i, \infty, \infty)$. Similarly, for each vertex $w_j \in W$, we have a corresponding item $I^w_j$ with coordinates $(\infty, \infty, j, |W| + 1 - j)$. Finally, for each edge $(u_i, w_j) \in E$, we have a consumer $C_{ij} = (i, |U| + 1 - i, j, |W| + 1 - j)$, whose budget is the same as the budget of edge $(u_i, w_j)$. The following claim is almost immediate.

**Claim D.9.** For each consumer $C_{ij}$, we have that $S_{C_{ij}} = \{I^u_i, I^w_j\}$

**Proof.** It is easy to see that $\{I^u_i, I^w_j\} \subseteq S_{C_{ij}}$. Notice that for $i' < i$, we have $C_{ij}[1] > I^u_{i'}[1]$, so any such item $I^u_{i'}$ cannot belong to $S_{C_{ij}}$. Similarly for $i' > i$, we have $C_{ij}[2] > I^w_{i'}[2]$, so such an item cannot belong to $S_{C_{ij}}$. By using similar arguments for items of the form $I^w_{j'}$ for $j' \neq j$, we reach the conclusion that $S_{C_{ij}} = \{I^u_i, I^w_j\}$. \hfill \square
Since the 4-SMP instance is equivalent to the instance of Graph Vertex Pricing, the maximum revenue is preserved. Using the APX-hardness result of Graph Vertex Pricing on bipartite graphs [39], we conclude that 4-SMP is APX-hard.

D.4 Hardness Results in Higher Dimensions

In this section, we present the proof of Theorem 3.1. Let $A = (\mathcal{I}, \mathcal{C})$ be an instance of UUDP-MIN where every consumer $\mathcal{C}$ has its consideration set $S_C$ of size at most $B$. Let $\mathcal{I} = \{I_1, \ldots, I_n\}$. For each $i \in [d]$, we pick a random permutation $\pi_i : [n] \to [n]$, so we have $d$ permutations $\pi_1, \ldots, \pi_d$. The function $\varphi$ on items $\mathcal{I}$ can be defined as $\varphi(I_j)[i] = \pi_i(j)$, and we extend the function to the set of consumers as follows: $\varphi(\mathcal{C})[i] = \min_{j \in S_C} \pi_i(j)$. Now we have a well-defined function $\varphi$.

Lemma D.10. With probability at least $1 - \frac{1}{n}$, for all consumer $\mathcal{C} \in \mathcal{C}$, the consideration set $S'_C$ defined by $S'_C = \{I_j : \varphi(I_j) \text{ dominates } \varphi(\mathcal{C})\}$ is exactly $S_C$.

Proof. Since we define $\varphi(\mathcal{C})$ to be the minimum of $\varphi(I_j)$ over all items in $S_C$, we have $S_C \subseteq S'_C$. Let $k$ be the index of an item that does not belong to $S_C$. We show the following claim.

Claim D.11. The probability that $\varphi(I_k)$ dominates $\varphi(\mathcal{C})$ is at most $\frac{1}{n B^2 + 1}$.

Proof. Fix some $i \in [d]$. The bad event that $\pi_i(k) \geq \min_{j \in S_C} \pi_i(j)$ happens only if $\pi_i$ does not put $k$ in the last position among $S_C \cup \{k\}$. This probability is exactly $1 - (1/(B + 1))$. Therefore, the bad event happens for all values of $i$ with probability at most $d = O(B^2 \log n)$. This claim immediately implies the lemma: By the union bound, the probability that $\varphi(I_k)$ dominates $\varphi(\mathcal{C})$ is at most $\frac{1}{n B^2 + 1}$. So we have that $\Pr[S_C \neq S'_C] \leq \frac{1}{n B^2 + 1}$. There are at most $n^B$ possible consideration sets of size at most $B$, so by union bounds, the probability that a bad event $S_C \neq S'_C$ happens for some consumer $\mathcal{C}$ is at most $1/n$.

$n$ attributes capture general problem Finally, we end this section with the proof that $n$-UUDP-MIN captures the whole generality of UUDP-MIN: Consider an instance $(\mathcal{C}, \mathcal{I}, \{S_C\}_{\mathcal{C} \in \mathcal{C}})$ of UUDP-MIN. Denote the set of items by $\mathcal{I} = \{I_1, \ldots, I_n\}$. Notice that we can define the coordinates of each consumer by $C[i] = 0$ if $I_i \in S_C$, and $C[i] = 1$ otherwise. We define the coordinates of each item as $I_i[i] = 0$ and $I_i[j] = 1$ for all $j \neq i$. It is easy to check that the consideration sets are preserved by this reduction.