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Holographic torus correlators in AdS_3 gravity coupled to scalar field

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Abstract: This paper investigates holographic torus correlators of generic operators at conformal infinity and a finite cutoff within AdS_3 gravity coupled with a free scalar field. Using a near-boundary analysis and solving the gravitational boundary value problem, we solve Einstein's equation and calculate mixed correlators for massless and massive coupled scalar fields. The conformal Ward identity on the torus has been reproduced holographically, which can be regarded as a consistency check. Further, recurrence relations for a specific class of higher-point correlators are derived, validating AdS_3/CFT_2 with non-trivial boundary topology. While the two-point scalar correlator is accurately computed on the thermal AdS_3 saddle, the higher-point correlators associated with scalar and stress tensor operators are explored.

Keywords: AdS-CFT Correspondence, Field Theories in Lower Dimensions, Gauge-Gravity Correspondence, Conformal and W Symmetry

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Contents

1 Introduction

Understanding the nonperturbative effects in strongly coupled physics has been a longstanding challenge. As a concrete realization of the holographic principle [1, 2], the Anti-de Sitter gravity/conformal field theory (AdS/CFT) correspondence [3–5] provides a potent tool for analytically studying strongly coupled field theories. A crucial application of this correspondence lies in obtaining the correlators of local operators in the boundary CFT through gravitational perturbative calculations performed in the bulk. The stress tensor correlators, which contain information about a system's energy, momentum, and stress distribution, have garnered substantial research attention. The holographic computations of the stress tensor correlators have been done in many remarkable works [6–10] when considering a boundary CFT with a trivial topology. Simultaneously, it is imperative to investigate holographic correlators on nontrivial topological manifolds as they offer valuable insights into the holographic principle and the behavior of CFTs in curved spacetimes.

Exploring correlators in CFTs with nontrivial topologies through the variational principle involves solving Einstein's equations in a general bulk geometry. While solutions near the boundary are well-established [11–15], the absence of the maximal symmetry in pure AdS poses a substantial challenge for addressing the global boundary value problem. This difficulty persists even when dealing with linearized equations, as discussed in previous literature [16–19]. Moreover, the research community's understanding of higher-point stress tensor correlators in strongly coupled CFTs with nontrivial topology remains limited. There is a pressing need for explicit results obtained through holographic computations to shed light on these

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complicated correlation functions. In our previous research [20], we delved into the study of holographic stress tensor correlators within the framework of AdS_3/CFT_2 when considering a CFT defined on a torus. We expanded upon this approach in [21], extending our analysis to AdS_5/CFT_4. We computed the Euclidean thermal two-point correlators for the stress tensor and the U(1) current in the latter case. Recent advancements in this field have also led to investigations of correlation functions within holographic CFTs featuring nontrivial topologies, as reported in [22, 23]. These studies contribute to the ongoing quest for a deeper understanding of holography in diverse CFT settings.

When calculating the holographic stress tensor correlators, one may consider a pure gravitational system in the bulk, where the boundary metric is the only source in the corresponding CFT. Extending the computation of correlators to holographic CFTs that contain multiple sources is a natural progression. A crucial research direction involves investigating Einstein's gravity minimally coupled to the matter field. This study focuses on calculating holographic torus correlators in a minimally coupled Einstein-scalar system within the framework of AdS_3/CFT_2. Before this, holographic correlators of Einstein-scalar theories have been extensively investigated on manifolds with trivial topology [14, 24–28]. By considering multiple sources, we can compute holographic correlators that involve a mixed insertion of both the dual scalar operator and the stress tensor. The contribution of the inserted stress tensor is obtained by perturbatively solving Einstein's equation, leading to the derivation of the holographic torus Ward identities. This result serves as a concrete verification of AdS_3/CFT_2 with non-trivial topology. The derivation of recurrence relations is presented herein to compute higher-point correlators. To obtain the precise form of holographic correlators, we subsequently calculate the two-point scalar correlator on the thermal AdS_3 saddle. The procedure for computing holographic correlators is extended to an Einstein-scalar system featuring a boundary at a finite cutoff. From the perspective of field theory, such a bulk theory is dual to T^2-deformed CFT [29–32].

The remainder of this paper is organized as follows. In section 2, we review fundamental techniques for computing holographic torus correlators. In section 3, we calculate the holographic correlators on the conformal boundary. We begin by considering the scenario where the coupled scalar field is massless in subsection 3.1. The recurrence relations on a general saddle are derived in subsubsection 3.1.3, and the precise two-point scalar correlator on the thermal AdS_3 saddle is computed in subsubsection 3.1.1. In subsection 3.2, we investigate the scenario where the coupled scalar field is massive and derive the corresponding recurrence relations. In section 4, we present a method for calculating holographic torus correlators at a finite cutoff. We analyze the two-point correlators $\langle T_{ij}O\rangle$ and $\langle T_{ij}T_{kl}\rangle$ on a general saddle in subsection 4.1. The exact three-point correlators $\langle T_{ij}OO\rangle$ on the thermal AdS_3 saddle is computed in subsection 4.2. In section 5 we provide a comprehensive summary and future perspectives.

2 Holographic setup and torus correlators

Let us briefly review the basics of calculating holographic correlators. Our calculations are within the framework of AdS_3/CFT_2 correspondence. The three-dimensional bulk $\mathcal{M}$ is a solid torus that encompasses Einstein gravity (with a negative cosmological constant)

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and a free scalar field,[1]

$$I_{\text{bulk}} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{\mathcal{G}}\Big(\mathcal{R}-2\Big) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{\gamma}K + \frac{1}{2}\int_{\mathcal{M}} d^3x\sqrt{\mathcal{G}}\Big((\partial\Phi)^2+m^2\Phi^2\Big) + I_{\text{ct}}. \tag{2.1}$$

Here, we have set the AdS radius $l=1$. The second term is the Gibbons-Hawking term [33], where $K=\gamma^{ij}K_{ij}$ with the induced metric γ_{ij} and the extrinsic curvature K_{ij}. Einstein's equations and scalar equation of motion (EOM) are obtained from the variations of (2.1) w.r.t. $\mathcal{G}^{\mu\nu}$ and Φ,

$$\mathcal{R}_{\mu\nu} - 2\mathcal{G}_{\mu\nu} + 8\pi G[\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2\mathcal{G}_{\mu\nu}] = 0, \tag{2.2}$$

$$\Box_{(\mathcal{G})}\Phi - m^2\Phi = 0. \tag{2.3}$$

The Laplacian $\Box_{(\mathcal{G})}\Phi = \frac{1}{\sqrt{\mathcal{G}}}\partial_\mu(\sqrt{\mathcal{G}}\mathcal{G}^{\mu\nu}\partial_\nu\Phi)$. The on-shell action diverges when evaluated at the boundary, requiring the inclusion of the counterterm I_{ct} in (2.1) to eliminate the divergence. This prescription is called holographic renormalization [14] (see also [12, 25, 26, 34–36]) and we will review it in appendix C.

It is convenient to work in the Fefferman-Graham (FG) coordinate system [11, 15], in which the bulk metric $\mathcal{G}_{\mu\nu}$ takes the form

$$ds^2 = \mathcal{G}_{\mu\nu}dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(\rho,x)dx^i dx^j. \tag{2.4}$$

The radial coordinate $\rho\in[0,\pi^{-2}]$ and $\rho=0$ is the conformal boundary. One can introduce the complex tangential coordinates[2] $(x^1,x^2)=(z,\bar{z})$ with identifications $(z,\bar{z})\cong(z+1,\bar{z}+1)\cong(z+\tau,\bar{z}+\bar{\tau})$, where τ is the modular parameter of the conformal torus boundary. In the FG coordinate system, bulk solutions $g_{ij}(\rho,x)$ and $\Phi(\rho,x)$ can be constructed near the conformal boundary. As discussed in [37] (see also [5, 14]), there are two homogeneous solutions g^-_{ij} and ρg^+_{ij} for the metric, and two homogeneous solutions $\rho^{1-\frac{\Delta}{2}}\phi^-$ and $\rho^{\frac{\Delta}{2}}\phi^+$ for the scalar field, where $\Delta=1+\sqrt{1+m^2}$ is the scaling dimension of the dual operator. The bulk solutions can be formally written as

$$\begin{aligned} g_{ij}(\rho,x) &= g^-_{ij}(\rho,x) + \rho g^+_{ij}(\rho,x),\\ \phi(\rho,x) &= \phi^-(\rho,x) + \rho^{\Delta-1}\phi^+(\rho,x), \end{aligned} \tag{2.5}$$

where $\phi(\rho,x)=\rho^{\frac{\Delta}{2}-1}\Phi(\rho,x)$. The leading order of each solution has a specific interpretation. $g^-_{ij}(0,x)$ and $\phi^-(0,x)$ are the sources couple to the boundary stress tensor T^{ij} and the dual scalar operator O respectively. $g^+_{ij}(0,x)$ and $\phi^+(0,x)$ are related to one-point correlators $\langle T_{ij}\rangle$ and $\langle O\rangle$ in the boundary CFT.

By appropriately choosing the scaling dimension, each solution in (2.5) can be written as a series expansion of ρ, called the FG expansion. The cases for $\Delta=1,2,\cdots$ have been considered in [14], where the series expansion contains the integer powers of ρ and the

[1] We follow the convention of [14] for the bulk action.

[2] Throughout this paper, we work with Euclidean signature.

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logarithmic terms. In particular, $\Delta = 2$ corresponds to massless scalar field coupling whose FG expansions take the forms

$$g_{ij}(\rho, x) = \sum_{n=0}^{\infty} \rho^n \Big[g_{(2n)ij} + \rho \ln \rho h_{(2n+2)ij} \Big](x),$$
$$\phi(\rho, x) = \sum_{n=0}^{\infty} \rho^n \Big[\phi_{(2n)} + \rho \ln \rho \psi_{(2n+2)} \Big](x). \tag{2.6}$$

The expansions of two homogeneous solutions g^-_{ij} and ρg^+_{ij} overlap on the positive integer orders of ρ, which requires the presence of the logarithmic terms $\rho^n \ln \rho h_{(2n)ij}(x)$ for $n \geq 1$. Similarly, the logarithmic terms $\rho^n \ln \rho \psi_{(2n)}(x)$ for $n \geq 1$ are added to the scalar field expansion. The zeroth-order coefficients $g_{(0)ij}(x)$ and $\phi_{(0)}(x)$ (corresponding to $g^-_{ij}(0, x)$ and $\phi^-(0, x)$ above) are the boundary values of the metric and scalar field. The first-order coefficients $g_{(2)ij}(x)$ and $\phi_{(2)}(x)$ (corresponding to $g^+_{ij}(0, x)$ and $\phi^+(0, x)$ above) are determined by the state of boundary CFT, which are independent of the boundary values. Plugging (2.4)(2.6) into (2.2)(2.3) and solving the EOMs order by order, one obtains each coefficient of (2.6) in terms of $g_{(0)ij}(x)$, $\phi_{(0)}(x)$, $g_{(2)ij}(x)$ and $\phi_{(2)}(x)$. Especially, $h_{(2)ij}(x)$ and $\psi_{(2)}(x)$ correspond to the conformal anomalies of boundary CFT, which contribute to one-point correlators $\langle T_{ij} \rangle$ and $\langle O \rangle$ respectively [14, 28, 38]. More generally, for the case where Δ is a rational number, the fractional powers of ρ appear in the FG expansions [37], which will be discussed in section 3.2.

In the case of massless scalar field coupling, one-point correlators are obtained through holographic renormalization [14],

$$\langle T_{ij} \rangle = \frac{1}{8\pi G} \left(g_{(2)ij} - 4\pi G \partial_i \phi_{(0)} \partial_j \phi_{(0)} - 2\pi G \partial^k \phi_{(0)} \partial_k \phi_{(0)} g_{(0)ij} - \frac{1}{2} R_{(0)} g_{(0)ij} \right),$$
$$\langle O \rangle = 2 \left(\phi_{(2)} - \frac{1}{4} \Box_{(0)} \phi_{(0)} \right), \tag{2.7}$$

where the indices i, j, k, l are raised by $g^{ij}_{(0)}$. $\langle T_{ij} \rangle$ has the following trace relation and conservation equation,

$$\langle T^i_i \rangle = -\frac{1}{16\pi G} R_{(0)} - \frac{1}{2} \partial^i \phi_{(0)} \partial_i \phi_{(0)}, \tag{2.8}$$
$$\nabla^j_{(0)} \langle T_{ij} \rangle = \langle O \rangle \partial_i \phi_{(0)}. \tag{2.9}$$

Detailed derivation is reviewed in appendix C.

2.1 Prescription of holographic torus correlators

Throughout this paper, we primarily calculate holographic correlators within the framework of AdS_3/CFT_2 correspondence. This correspondence is characterized by the Gubser-Klebanov-Polyakov-Witten (GKPW) relation [4, 5], which establishes the equivalence between the bulk gravitational partition function and the boundary generating functional,

$$Z_G[\phi_{(0)}, g_{(0)ij}] = \left\langle \exp \int_{\partial \mathcal{M}} d^2x \sqrt{g_{(0)}} \left(\phi_{(0)} O - \frac{1}{2} g^{ij}_{(0)} T_{ij} \right) \right\rangle_{\text{CFT}}. \tag{2.10}$$

In the semiclassical limit, the gravitational partition function can be approximated as a sum over all saddles, $Z_G[\phi_{(0)}, g_{(0)ij}] \approx \sum_\alpha e^{-I^{(\alpha)}_{\text{on-shell}}[\phi_{(0)}, g_{(0)ij}]}$. Assuming that the partition function

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$Z_G[\phi_{(0)}, g_{(0)ij}]$ is dominated by only one saddle, holographic correlators can be obtained by taking functional derivatives of its on-shell action [26, 27, 39, 40],

$$\begin{aligned}&\Big\langle \prod_{k=1}^{m} T_{i_k j_k}(x_k) \prod_{l=1}^{n} O(y_l) \Big\rangle_c \\ &= -\left(\prod_{k=1}^{m} \frac{-2}{\sqrt{g_{(0)}(x_k)}}\right)\left(\prod_{l=1}^{n} \frac{1}{\sqrt{g_{(0)}(y_l)}}\right)\left(\prod_{k=1}^{m} \frac{\delta}{\delta g_{(0)}^{i_k j_k}(x_k)} \prod_{l=1}^{n} \frac{\delta}{\delta \phi_{(0)}(y_l)}\right) I_{\text{on-shell}}\Big|_{\substack{\phi_{(0)}=0 \\ g_{(0)ij}=\eta_{ij}}},\end{aligned} \tag{2.11}$$

where the subscript c on the left-hand side implies the connected part of the correlator. The functional derivative is evaluated at the fixed point $(\phi_{(0)}, g_{(0)ij}) = (0, \eta_{ij})$. We are considering three types of holographic correlators. For the case where $m \neq 0$ and $n = 0$, we obtain m-point stress tensor correlators, which have been computed in a prior study [20]. When $m = 0$ and $n \neq 0$, we obtain n-point scalar correlators, as discussed in section 3.1.1. However, our primary focus lies on the case where $m \neq 0$ and $n \neq 0$, resulting in $(m+n)$-point mixed correlators. Throughout this paper, we will calculate mixed correlators in various scenarios and compare them with results in torus CFT.

To calculate holographic correlators, we introduce perturbations $\delta\phi_{(0)}$ and $\delta g_{(0)ij}$ to the boundary values and then solve the bulk equations of motion (2.2) and (2.3) with the perturbed Dirichlet boundary conditions. However, the bulk fields are constructed by both the boundary values ($\phi_{(0)}$ and $g_{(0)ij}$) and the state-dependent data ($\phi_{(2)}$ and $g_{(2)ij}$). When perturbing the boundary values, the changes $\delta\phi_{(2)}$ and $\delta g_{(2)ij}$ in the state-dependent data are not uniquely determined. In previous work [20], we employed the global regularity condition to constrain the remaining degrees of freedom. The crucial requirement is that the perturbed bulk solutions described by the FG expansions near the boundary can be smoothly extended to global solutions. As mentioned earlier, the bulk $\mathcal{M}$ is a solid torus, and the cutoff surface at $\rho = \rho_c$ is a torus with the induced metric $\gamma_{ij}(\rho_c, x)$. As ρ_c approaches π^{-2}, the cutoff torus shrinks into a circle and the bulk fields must be regular here. For the perturbed scalar field, we have

$$\lim_{\rho \to \pi^{-2}} \delta\Phi(\rho, x) = \delta\Phi^*(x) \quad \text{finite}, \tag{2.12}$$

which plays a crucial role in section 3.1.1. The global regularity condition for the perturbed bulk metric $\delta g_{ij}(\rho, x)$ is more complicated. After changing the bulk metric, the new FG coordinates differ from the original FG coordinates by a boundary preserving diffeomorphism.[3] Then, the perturbed bulk metric can be divided into two parts: the change by the boundary preserving diffeomorphism and the variation in the new FG coordinate system. Transforming them into a smoothly compatible chart covering the circle at $\rho = \pi^{-2}$ and requiring all its components to be globally regular, one can obtain some constraints on $\delta g_{ij}(\pi^{-2}, x)$. In previous work [20], we have used these constraints to fix the undetermined constants in two-point stress tensor correlators. For higher-point correlators, the calculation of the constraints becomes increasingly tedious.

[3] For a specific form of this boundary preserving diffeomorphism, please refer to appendix B of [20].

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There is an alternative method to fix the remaining degrees of freedom in holographic correlators [41, 42] (see also section 7.2 of [43]). For the correlator with at least one stress tensor insertion, the undetermined part can always be written as a one-point-averaged correlator,[4]

$$\int_{\partial\mathcal{M}} d^2z \langle T^{ij}(z)X\rangle = \int_{\partial\mathcal{M}} d^2z \left(\frac{\delta\langle X\rangle}{\delta g_{(0)ij}(z)} + \text{contact terms}\right). \tag{2.13}$$

Let us calculate the first term on the right-hand side. Consider the global variations $\delta g_{(0)\bar{z}\bar{z}} = \alpha$ and $\delta g_{(0)zz} = \bar{\alpha}$ on the boundary metric, we have

$$\begin{aligned} ds^2 &= dzd\bar{z} + \alpha d\bar{z}^2 + \bar{\alpha}dz^2 \\ &= (1+\alpha+\bar{\alpha})d[z+\alpha(\bar{z}-z)]d[\bar{z}+\bar{\alpha}(z-\bar{z})] + O(\alpha^2). \end{aligned} \tag{2.14}$$

After performing the global Weyl transformation $g_{(0)ij} \mapsto (1-\alpha-\bar{\alpha})g_{(0)ij}$ plus the coordinate transformation $(z', \bar{z}') = (z+\alpha(\bar{z}-z), \bar{z}+\bar{\alpha}(z-\bar{z}))$, the torus metric (at leading order) changes back to the Euclidean one $ds^2 = dz'd\bar{z}' + O(\alpha^2)$ with the new modular parameter $\tau' = \tau + \alpha(\bar{\tau}-\tau)$. For a general correlator, the changes under the global metric variation and the global Weyl transformation are

$$\begin{aligned} (\delta\langle X\rangle)_{\text{metric}} &= \alpha\int_{\partial\mathcal{M}} d^2z \frac{\delta\langle X\rangle}{\delta g_{(0)\bar{z}\bar{z}}(z)} + \bar{\alpha}\int_{\partial\mathcal{M}} d^2z \frac{\delta\langle X\rangle}{\delta g_{(0)zz}(z)}, \\ (\delta\langle X\rangle)_{\text{Weyl}} &= -(\alpha+\bar{\alpha})\int_{\partial\mathcal{M}} d^2z \frac{\delta\langle X\rangle}{\delta g_{(0)z\bar{z}}(z)}. \end{aligned} \tag{2.15}$$

The change of $\langle X\rangle$ under the coordinate transformation $(z, \bar{z}) \mapsto (z', \bar{z}')$ is

$$(\delta\langle X\rangle)_{\text{coordinate}} = \alpha\mathcal{L}_{(z-\bar{z})\partial_z}\langle X\rangle + \bar{\alpha}\mathcal{L}_{(\bar{z}-z)\partial_{\bar{z}}}\langle X\rangle, \tag{2.16}$$

where $\mathcal{L}$ denotes the Lie derivative. The sum of the above three transformations is equivalent to varying the modular parameter $\delta\tau = \alpha(\bar{\tau}-\tau)$, which gives the following two relations [20],

$$\begin{aligned} (\bar{\tau}-\tau)\partial_\tau\langle X\rangle &= \mathcal{L}_{(z-\bar{z})\partial_z}\langle X\rangle + \int_{\partial\mathcal{M}} d^2z \left(\frac{\delta\langle X\rangle}{\delta g_{(0)\bar{z}\bar{z}}(z)} - \frac{\delta\langle X\rangle}{\delta g_{(0)z\bar{z}}(z)}\right), \\ (\tau-\bar{\tau})\partial_{\bar{\tau}}\langle X\rangle &= \mathcal{L}_{(\bar{z}-z)\partial_{\bar{z}}}\langle X\rangle + \int_{\partial\mathcal{M}} d^2z \left(\frac{\delta\langle X\rangle}{\delta g_{(0)zz}(z)} - \frac{\delta\langle X\rangle}{\delta g_{(0)z\bar{z}}(z)}\right). \end{aligned} \tag{2.17}$$

Later, we will utilize these relations to compute the two-point and three-point mixed correlators and derive recurrence relations for higher-point mixed correlators.

3 Correlators in holographic CFT

In this section, holographic correlators on the conformal boundary are calculated in two cases: the coupled scalar field is massless ($\Delta = 2$) and massive ($1 \leq \Delta < 2$). We mainly focus on obtaining mixed correlators. The basic idea is to perturb the boundary values $\phi_{(0)}$ and $g_{(0)ij}$ and solve the trace relation and the conservation equation order by order. Furthermore, we compute the two-point scalar correlator and give a preliminary discussion of the higher-point scalar correlators.

[4] In this paper, the complex variable z collectively represents both z and its complex conjugate $\bar{z}$.

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3.1 Massless scalar field

3.1.1 Perturbative solution of massless scalar EOM

As we perturb the boundary value of the scalar field $\Phi_{(0)}(x) = \epsilon f(x)$, the bulk metric $\mathcal{G}^{[\epsilon]}_{\mu\nu}$ also varies due to the back-reaction. Hence, the scalar EOM should be considered in a changing background,

$$\frac{1}{\sqrt{\mathcal{G}^{[\epsilon]}}}\partial_\mu\left(\sqrt{\mathcal{G}^{[\epsilon]}}\mathcal{G}^{[\epsilon]\mu\nu}\partial_\nu\Phi^{[\epsilon]}\right) = 0. \tag{3.1}$$

The perturbed bulk fields have Taylor expansions $\Phi^{[\epsilon]} = \sum_{k=0}^{\infty}\epsilon^k\Phi^{[k]}$ and $\mathcal{G}^{[\epsilon]}_{\mu\nu} = \sum_{k=0}^{\infty}\epsilon^k\mathcal{G}^{[k]}_{\mu\nu}$. Plugging them into (3.1), we read off the differential equation for $\Phi^{[k]}$. Meanwhile, $\Phi^{[k]}$ has the FG expansion in the coordinate system (2.4), and the subleading term $\Phi^{[k]}_{(2)}$ corresponds to the one-point scalar correlator $\langle O\rangle^{[k]}$. The k-th order variation of $\langle O\rangle^{[k]}$ gives the $(n+1)$-point scalar correlator in the boundary CFT.

The zeroth-order coefficients $\Phi^{[0]}$ and $\mathcal{G}^{[0]}_{\mu\nu}$ are given by the saddle before perturbation. In the FG coordinate system (2.4), they satisfy the boundary conditions[5] $\Phi^{[0]}|_{\rho\to 0} = 0$ and $\rho\mathcal{G}^{[0]}_{ij}|_{\rho\to 0} = \eta_{ij}$. The simplest type of saddle is characterized by the disappearance of the bulk scalar field. For instance, the thermal AdS_3 solution

$$ds^2 = \mathcal{G}^{[0]}_{\mu\nu}dx^\mu dx^\nu = \frac{1}{1+r^2}dr^2 + r^2d\varphi^2 + (1+r^2)dt^2, \quad \Phi^{[0]}(r,\varphi,t) = 0, \tag{3.2}$$

where (φ, t) are the real tangential coordinates with $\varphi \sim \varphi + 2\pi$ and $t \sim t + 2\pi T$. The radial coordinate $r \in [0,\infty)$ and $r \to \infty$ is the conformal infinity. Such kind of pure gravitational saddle is considered trivial since it does not contribute to the one-point scalar correlator of the dual CFT.

We further compute the two-point scalar correlator of the saddle (3.2). The perturbed scalar field $\Phi^{[\epsilon]}$ satisfies the Dirichlet boundary condition at conformal infinity and regularity condition at $r = 0$,

$$\Phi^{[\epsilon]}(r,\varphi,t)\Big|_{r\to\infty} = \epsilon f(\varphi,t), \tag{3.3}$$

$$\Phi^{[\epsilon]}(r,\varphi,t)\Big|_{r\to 0} \quad \text{regular}. \tag{3.4}$$

Since $\Phi^{[0]}$ is turned off, the first-order EOM is a homogeneous differential equation,

$$\frac{1}{\sqrt{\mathcal{G}^{[0]}}}\partial_\mu\left(\sqrt{\mathcal{G}^{[0]}}\mathcal{G}^{[0]\mu\nu}\partial_\nu\Phi^{[1]}\right) = 0. \tag{3.5}$$

The investigation of scalar field solutions within a fixed gravitational background has been extensively explored in numerous works [22, 44–53]. The majority of these studies utilize the

[5] In fact, the boundary condition of the massless scalar field can be extended to $\Phi^{[0]}|_{\rho\to 0} = C$, where C is a constant. The boundary field theory can be regarded as a marginal deformation of the original CFT,

$$S' = S_{\text{CFT}} + C\int_{\partial\mathcal{M}} d^2z O.$$

This class of boundary conditions will lead to the same holographic Ward identities.

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Minkowski signature and implement the ingoing boundary condition at the horizon. The derivation in the Euclidean signature space is analogous. The double periodicity of $\Phi^{[1]}$ allows us to represent it formally as a Fourier expansion,

$$\Phi^{[1]}(r,\varphi,t)=\sum_{m,n=-\infty}^{+\infty}\Phi^{[1]}_{m,n}(r)e^{\frac{-imt}{T}}e^{in\varphi}. \tag{3.6}$$

Plugging (3.2)(3.6) into (3.5), we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r(1+r^2)\frac{d}{dr}\Phi^{[1]}_{m,n}\right)+\left(\frac{-m^2}{(1+r^2)T^2}-\frac{n^2}{r^2}\right)\Phi^{[1]}_{m,n}=0. \tag{3.7}$$

Applying the following coordinate transformation and field redefinition,

$$x=1-\frac{1}{1+r^2},\qquad \Phi^{[1]}_{m,n}(r)\to x^{\frac{n}{2}}(1-x)y^{[1]}_{m,n}(x), \tag{3.8}$$

we rewrite (3.7) in the form of hypergeometric equation,

$$x(1-x)\frac{d^2}{dx^2}y^{[1]}_{m,n}+[\gamma-(\alpha+\beta+1)x]\frac{d}{dx}y^{[1]}_{m,n}-\alpha\beta y^{[1]}_{m,n}=0, \tag{3.9}$$

where the parameters $\alpha=1+\frac{im}{2T}+\frac{n}{2}$, $\beta=1-\frac{im}{2T}+\frac{n}{2}$, and $\gamma=1+n$. The new radial coordinate $x\in[0,1]$ with the conformal boundary at $x=1$ and the circle at $x=0$. The boundary conditions for $y^{[1]}_{m,n}$ are read off from (3.3)(3.4),

$$(1-x)y^{[1]}_{m,n}(x)\Big|_{x\to1}=f_{m,n}, \tag{3.10}$$

$$x^{\frac{n}{2}}y^{[1]}_{m,n}(x)\Big|_{x\to0}\quad\text{regular}, \tag{3.11}$$

where $f_{m,n}=\frac{1}{4\pi^2T}\int_{-\pi}^{\pi}d\varphi\int_{-\pi T}^{\pi T}dtf(\varphi,t)e^{\frac{imt}{T}}e^{-in\varphi}$ is the Fourier coefficient of $f(\varphi,t)$. Equation (3.9) has two independent solutions in the neighborhood of $x=0$, and only one of them satisfies the regularity condition (3.11). The form of the allowed solution varies depending on the sign of n,

$$y^{[1]}_{m,n}(x)=\begin{cases}A^{[1]}_{m,n}F(\alpha,\beta,\gamma,x), & n\geq0,\\ A^{[1]}_{m,n}x^{-n}F(\alpha-n,\beta-n,1-n,x), & n<0.\end{cases} \tag{3.12}$$

$F(\alpha,\beta,\gamma,x)$ is the hypergeometric function [54] (and note that $\gamma-\alpha-\beta=-1$), which has the following expansion in the neighborhood of $x=1$,

$$\begin{aligned}F(\alpha,\beta,\gamma,x)=&\frac{\Gamma(\gamma)(1-x)^{-1}}{\Gamma(\alpha)\Gamma(\beta)}+\frac{\Gamma(\gamma)}{\Gamma(\alpha-1)\Gamma(\beta-1)}\Big[\ln(1-x)\\&+[\psi(\alpha)+\psi(\beta)+2\gamma_E-1]\Big]+O(1-x),\end{aligned} \tag{3.13}$$

where $\psi(\lambda)=\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$ is the logarithmic differentiation of the Gamma function and γ_E is the Euler constant. Plugging (3.12)(3.13) into (3.10), we find

$$A^{[1]}_{m,n}=\begin{cases}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}f_{m,n}, & n\geq0,\\ \frac{\Gamma(\alpha-n)\Gamma(\beta-n)}{\Gamma(1-n)}f_{m,n}, & n<0.\end{cases} \tag{3.14}$$

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Putting everything together, we obtain the global solution $\Phi^{[1]}$, then we expand it around the conformal boundary,

$$\begin{aligned}
&\Phi^{[1]}(x,\varphi,t)\\
&=\sum_{m,n=-\infty}^{+\infty}\left\{1+\left[\left(\frac{m^2}{4T^2}+\frac{n^2}{4}\right)\left[\psi\left(1+\frac{im}{2T}+\frac{|n|}{2}\right)+\psi\left(1-\frac{im}{2T}+\frac{|n|}{2}\right)+2\gamma_E-1\right]-\frac{|n|}{2}\right](1-x)\right.\\
&\left.+\left(\frac{m^2}{4T^2}+\frac{n^2}{4}\right)(1-x)\ln(1-x)\right\}f_{m,n}e^{\frac{-imt}{T}}e^{in\varphi}+O((1-x)^2).
\end{aligned}\tag{3.15}$$

Now, let us return to the FG coordinate system (2.4). Applying the following coordinate transformation

$$x=\left(\frac{1-\pi^2\rho}{1+\pi^2\rho}\right)^2,\qquad \varphi=\pi(z+\bar{z}),\qquad t=-i\pi(z-\bar{z}),\tag{3.16}$$

the bulk metric of the thermal AdS_3 can be rewritten as

$$ds^2=\frac{d\rho^2}{4\rho^2}+\frac{1}{\rho}\Big[dzd\bar{z}-\pi^2\rho(dz^2+d\bar{z}^2)+\pi^4\rho^2dzd\bar{z}\Big],\tag{3.17}$$

and the FG coefficients of $\Phi^{[1]}$ take the forms

$$\begin{aligned}
\Phi^{[1]}_{(0)}(z)&=f(z),\\
\Phi^{[1]}_{(2)}(z)&=4\pi^2\sum_{m,n=-\infty}^{+\infty}\left[\left(\frac{m^2}{4T^2}+\frac{n^2}{4}\right)\left[\psi\left(1+\frac{im}{2T}+\frac{|n|}{2}\right)+\psi\left(1-\frac{im}{2T}+\frac{|n|}{2}\right)\right.\right.\\
&\qquad\left.\left.+2\gamma_E-1+\ln(4\pi^2)\right]-\frac{|n|}{2}\right]f_{m,n}e^{i\pi\left(n+\frac{im}{T}\right)z}e^{i\pi\left(n-\frac{im}{T}\right)\bar{z}},\\
\Psi^{[1]}_{(2)}(z)&=4\pi^2\sum_{m,n=-\infty}^{+\infty}\left(\frac{m^2}{4T^2}+\frac{n^2}{4}\right)f_{m,n}e^{i\pi\left(n+\frac{im}{T}\right)z}e^{i\pi\left(n-\frac{im}{T}\right)\bar{z}}.
\end{aligned}\tag{3.18}$$

By definition, the perturbed one-point correlator $\langle O\rangle^{[1]}=2(\Phi^{[1]}_{(2)}+\Psi^{[1]}_{(2)})$. Taking the functional derivative w.r.t. $f(z')$, we obtain the two-point correlator,

$$\begin{aligned}
\langle O(z)O(z')\rangle=\frac{8\pi^2}{T}\sum_{m,n=-\infty}^{+\infty}&\left[\left(\frac{m^2}{4T^2}+\frac{n^2}{4}\right)\left[\psi\left(1+\frac{im}{2T}+\frac{|n|}{2}\right)+\psi\left(1-\frac{im}{2T}+\frac{|n|}{2}\right)\right.\right.\\
&\left.\left.+2\gamma_E+\ln(4\pi^2)\right]-\frac{|n|}{2}\right]e^{i\pi\left(n+\frac{im}{T}\right)(z-z')}e^{i\pi\left(n-\frac{im}{T}\right)(\bar{z}-\bar{z}')}.
\end{aligned}\tag{3.19}$$

An alternative method for obtaining the scalar two-point correlator is presented herein. We start from the Poincare AdS_3 with the metric $ds^2=\frac{dy^2}{4y^2}+\frac{dwd\bar{w}}{y}$. Applying the following coordinate transformation

$$y=\frac{4\pi^2\rho}{(1+\pi^2\rho)^2}e^{-i2\pi(z-\bar{z})},\quad w=\frac{1-\pi^2\rho}{1+\pi^2\rho}e^{-i2\pi z},\quad \bar{w}=\frac{1-\pi^2\rho}{1+\pi^2\rho}e^{i2\pi\bar{z}},\tag{3.20}$$

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we recover the bulk metric (3.17). The calculation of the scalar two-point correlator involves perturbing the boundary value of the scalar field and subsequently solving for the subleading order $\Phi^{[1]}(\rho, z, \bar{z})$ from (3.5). Consider the following Fourier mode of the scalar field,

$$\Phi^{[1]}_{\omega,k}(x)e^{-i\omega t}e^{ik\varphi}, \tag{3.21}$$

where the coordinates (x, t, φ) are defined in (3.16), and (ω, k) are real parameters. Plugging (3.21) into (3.5) and using the boundary conditions

$$\begin{aligned} \Phi^{[1]}_{\omega,k}(x)\Big|_{x\to 1} &= f_{\omega,k}, \\ \Phi^{[1]}_{\omega,k}(x)\Big|_{x\to 0} &\quad \text{regular}, \end{aligned} \tag{3.22}$$

we obtain

$$\begin{aligned} \Phi^{[1]}_{\omega,k}(x) = &\frac{\Gamma\left(1+\frac{i\omega}{2}+\frac{|k|}{2}\right)\Gamma\left(1-\frac{i\omega}{2}+\frac{|k|}{2}\right)}{\Gamma(1+|k|)} f_{\omega,k}x^{\frac{|k|}{2}}(1-x) \\ &\times F(1+\frac{i\omega}{2}+\frac{|k|}{2}, 1-\frac{i\omega}{2}+\frac{|k|}{2}, 1+|k|, x). \end{aligned} \tag{3.23}$$

Integrating (3.21) over the momentum space, we find

$$\begin{aligned} \Phi^{[1]}(x,t,\varphi) &= \int_{\mathbb{R}^2} d\omega dk \Phi^{[1]}_{\omega,k}(x)e^{-i\omega t}e^{ik\varphi} \\ &= \int_{\mathbb{R}^2} dt' d\varphi' K_{\mathrm{g}}(x, t-t', \varphi-\varphi')f(t', \varphi'), \end{aligned} \tag{3.24}$$

where the bulk-boundary propagator takes the form

$$\begin{aligned} K_{\mathrm{g}}(x,t-t',\varphi-\varphi') = &\frac{1}{4\pi^2}\int_{\mathbb{R}^2} d\omega dk \Big[\frac{\Gamma\left(1+\frac{i\omega}{2}+\frac{|k|}{2}\right)\Gamma\left(1-\frac{i\omega}{2}+\frac{|k|}{2}\right)}{\Gamma(1+|k|)} x^{\frac{|k|}{2}}(1-x) \\ &\times F(1+\frac{i\omega}{2}+\frac{|k|}{2}, 1-\frac{i\omega}{2}+\frac{|k|}{2}, 1+|k|, x)e^{-i\omega(t-t')}e^{ik(\varphi-\varphi')}\Big]. \end{aligned} \tag{3.25}$$

Now let us return to the thermal AdS_3 spacetime, which can be regarded as a quotient of the global AdS_3 spacetime [55, 56]. The thermal AdS_3 propagator, denoted as K_{t}, satisfies the following periodicity conditions:

$$K_{\mathrm{t}}(x, t-t', \varphi-\varphi') = K_{\mathrm{t}}(x, t-t'+2\pi T, \varphi-\varphi') = K_{\mathrm{t}}(x, t-t', \varphi-\varphi'+2\pi). \tag{3.26}$$

This propagator can be constructed from the global AdS_3 propagator K_{g} via the method of images [44, 57–59]. We shift the Euclidean time t and the angle coordinate φ in (3.25) by integer multiples of $2\pi T$ and 2π, respectively, followed by summation over all such propagators to obtain

$$\begin{aligned} K_{\mathrm{t}}(x, t-t', \varphi-\varphi') = &\sum_{m,n=-\infty}^{\infty} K_{\mathrm{g}}(x, t-t'+2\pi m T, \varphi-\varphi'+2\pi n) \\ = &\frac{1}{4\pi^2}\int_{\mathbb{R}^2} d\omega dk \Big[\frac{\Gamma\left(1+\frac{i\omega}{2}+\frac{|k|}{2}\right)\Gamma\left(1-\frac{i\omega}{2}+\frac{|k|}{2}\right)}{\Gamma(1+|k|)} x^{\frac{|k|}{2}}(1-x) \\ &\times F(1+\frac{i\omega}{2}+\frac{|k|}{2}, 1-\frac{i\omega}{2}+\frac{|k|}{2}, 1+|k|, x)e^{-i\omega(t-t')}e^{ik(\varphi-\varphi')} \\ &\times \sum_{m,n=-\infty}^{\infty} e^{-i2\pi m T\omega}e^{i2\pi nk}\Big]. \end{aligned} \tag{3.27}$$

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Consider the Dirac comb function defined as $\text{Ш}_L(u) = \sum_{n=-\infty}^{\infty} \delta(u - nL)$, which has the Fourier series expansion

$$\text{Ш}_L(u) = \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{i2\pi n \frac{u}{L}}. \tag{3.28}$$

Plugging (3.28) into (3.27) we obtain

$$\begin{aligned} K_{\text{t}}(x, t-t', \varphi-\varphi') = \frac{1}{4\pi^2 T} \sum_{m,n=-\infty}^{\infty} \Big[& \frac{\Gamma\left(1+\frac{im}{2T}+\frac{|n|}{2}\right)\Gamma\left(1-\frac{im}{2T}+\frac{|n|}{2}\right)}{\Gamma(1+|n|)} x^{\frac{|n|}{2}}(1-x) \\ & \times F(1+\frac{im}{2T}+\frac{|n|}{2}, 1-\frac{im}{2T}+\frac{|n|}{2}, 1+|n|, x) e^{-i\frac{m}{T}(t-t')} e^{in(\varphi-\varphi')} \Big]. \end{aligned} \tag{3.29}$$

It follows that

$$\begin{aligned} \Phi^{[1]}(x,t,\varphi) &= \int_{-\pi T}^{\pi T} dt' \int_{-\pi}^{\pi} d\varphi' K_{\text{t}}(x, t-t', \varphi-\varphi') f(t', \varphi') \\ &= \sum_{m,n=-\infty}^{\infty} \Bigg[\frac{\Gamma\left(1+\frac{im}{2T}+\frac{|n|}{2}\right)\Gamma\left(1-\frac{im}{2T}+\frac{|n|}{2}\right)}{\Gamma(1+|n|)} x^{\frac{|n|}{2}}(1-x) \\ &\quad \times F(1+\frac{im}{2T}+\frac{|n|}{2}, 1-\frac{im}{2T}+\frac{|n|}{2}, 1+|n|, x) e^{-i\frac{m}{T}t} e^{in\varphi} f_{m,n} \Bigg]. \end{aligned} \tag{3.30}$$

This solution is consistent with the one derived from the previous approach. Once again, we expand $\Phi^{[1]}$ around the boundary and extract the FG coefficients (3.18). By computing the variation of the perturbed one-point correlator $\langle O \rangle^{[1]} = 2(\Phi^{[1]}_{(2)} + \Psi^{[1]}_{(2)})$, we recover the two-point correlator (3.19).

In computing higher-order coefficients, the back-reaction of the scalar field to the bulk geometry needs to be considered. The k-th order EOM (for $k \geq 2$) is a nonhomogeneous differential equation given by

$$\frac{1}{\sqrt{\mathcal{G}^{[0]}}} \partial_\mu \left(\sqrt{\mathcal{G}^{[0]}} \mathcal{G}^{[0]\mu\nu} \partial_\nu \Phi^{[k]} \right) = M^{[k]}. \tag{3.31}$$

Here $M^{[k]}$ is a functional of $f(\varphi, t)$, whose explicit form is written in terms of the lower-order coefficients of the metric and the scalar field,

$$M^{[k]} = -\sum_{l=1}^{k-1} \left[\frac{1}{\sqrt{\mathcal{G}}} \partial_\mu \left(\sqrt{\mathcal{G}} \mathcal{G}^{\mu\nu} \partial_\nu \right) \right]^{[k-l]} \Phi^{[l]}. \tag{3.32}$$

Plugging (3.2)(3.6)(3.8) into (3.31), we obtain

$$x(1-x)\frac{d^2}{dx^2} y^{[k]}_{m,n} + [\gamma - (\alpha+\beta+1)x] \frac{d}{dx} y^{[k]}_{m,n} - \alpha\beta y^{[k]}_{m,n} = \tilde{M}^{[k]}_{m,n}, \tag{3.33}$$

where $\tilde{M}^{[k]}_{m,n} = 4(1-x)^2 x^{\frac{n}{2}} M^{[k]}_{m,n}$ and $M^{[k]}_{m,n}$ is the Fourier coefficient of $M^{[k]}$. Consider a Green's function with the following Dirichlet boundary problem,

$$\begin{aligned} & \left(x(1-x)\partial_x^2 + [\gamma - (\alpha+\beta+1)x]\partial_x - \alpha\beta \right) \mathscr{G}_{m,n}(x, x_0) = \delta(x - x_0), \\ & (1-x)\mathscr{G}_{m,n}(x, x_0)\Big|_{x\to 1} = 0, \qquad x^{\frac{n}{2}} \mathscr{G}_{m,n}(x, x_0)\Big|_{x\to 0} \quad \text{regular}. \end{aligned} \tag{3.34}$$

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The explicit form of $\mathscr{G}_{m,n}$ is discussed in appendix B. $y^{[k]}_{m,n}$ can be formally written as

$$y^{[k]}_{m,n}(x) = 4\int_0^1 dx_0(1-x_0)^2 x_0^{\frac{n}{2}} M^{[k]}_{m,n}(x_0)\mathscr{G}_{m,n}(x,x_0). \tag{3.35}$$

To obtain the exact solution, we need to calculate $M^{[k]}_{m,n}(x_0)$, which contains the metric coefficients $\mathcal{G}^{[l]}_{\mu\nu}$ for $l \leq k-1$. The first-order perturbation $\mathcal{G}^{[1]}_{\mu\nu} = 0$ due to the absence of matter terms in the first-order of (2.2). Consequently, $M^{[2]} = 0$, resulting in the vanishing of the connected three-point correlator. In fact, since the bulk scalar theory is free, any higher-point scalar correlator[6] at order $\mathcal{O}(G^0)$ can be factorized into a product of lower-point scalar correlators. Therefore, the connected part of the correlator can only be non-zero at higher orders of G. To see this, we consider the connected four-point scalar correlator, which corresponds to $M^{[3]}$ in (3.32). Expanding the Einstein's equation (2.2) to the second-order in ϵ, we obtain

$$\frac{1}{2}[\nabla_\mu\nabla^\sigma\mathcal{G}^{[2]}_{\sigma\nu}+\nabla_\nu\nabla^\sigma\mathcal{G}^{[2]}_{\sigma\mu}-\nabla_\mu\nabla_\nu\mathcal{G}^{[2]\sigma}_{\sigma}-\nabla^2\mathcal{G}^{[2]}_{\mu\nu}]-2\mathcal{G}^{[2]}_{\mu\nu} = -8\pi G\partial_\mu\Phi^{[1]}\partial_\nu\Phi^{[1]}, \tag{3.36}$$

where ∇_μ is the covariant derivative operator of the metric $\mathcal{G}^{[0]}_{\mu\nu}$, and Greek indices are raised by $\mathcal{G}^{[0]\mu\nu}$. The second-order perturbation $\mathcal{G}^{[2]}_{\mu\nu}$ is proportional to Newton's constant G, yielding a non-zero connected four-point scalar correlator at order $\mathcal{O}(G^1)$, which is suppressed in the semiclassical limit. Unfortunately, due to the lack of a viable method for solving the differential equation (3.36), we are unable to provide an explicit form of the four-point correlator in this context. Similarly, one can expand Einstein's equation to higher orders in ϵ and obtain differential equations for higher order perturbations of the metric. One can verify that these perturbations are always proportional to G, implying that the connected higher-point scalar correlators are non-zero at order $\mathcal{O}(G^1)$.

As a direct generalization, we can activate the scalar field source $\phi_{(0)}(z) = \epsilon(z)$. The boundary field theory should then be considered as the CFT deformed by the marginal operator O,

$$S' = S_{\text{CFT}} + \int_{\partial\mathcal{M}} d^2z\epsilon(z)O(z). \tag{3.37}$$

The deformed one-point correlators can be perturbatively constructed from the correlators on the pure gravitational saddle,

$$\begin{aligned}\langle T_{ij}(z)\rangle_\epsilon &= \langle T_{ij}\rangle + \int_{\partial\mathcal{M}} d^2z_1\int_{\partial\mathcal{M}} d^2z_2\epsilon(z_1)\epsilon(z_2)\langle T_{ij}(z)O(z_1)O(z_2)\rangle + O(\epsilon^3),\\ \langle O(z)\rangle_\epsilon &= \int_{\partial\mathcal{M}} d^2z_1\epsilon(z_1)\langle O(z)O(z_1)\rangle + O(\epsilon^2).\end{aligned} \tag{3.38}$$

3.1.2 Two-point and three-point mixed correlators

In terms of holographic correlators (2.11), we specify the boundary values of bulk fields as

$$\begin{aligned}\phi_{(0)}(z) &= \epsilon_1 f(z),\\ g_{(0)ij}(z) &= \eta_{ij} + \epsilon_2\chi_{ij}(z),\end{aligned} \tag{3.39}$$

[6]Here we are referring to a correlator that has more than two scalar operator insertions.

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where ϵ_1 and ϵ_2 are infinitesimal parameters. The perturbed one-point correlators can be expanded in powers of ϵ_1 and ϵ_2,

$$\langle T_{ij}\rangle^{[\epsilon_1,\epsilon_2]} = \sum_{k,l=1}^{\infty} \epsilon_1^k \epsilon_2^l \langle T_{ij}\rangle^{[k,l]},$$
$$\langle O\rangle^{[\epsilon_1,\epsilon_2]} = \sum_{k,l=1}^{\infty} \epsilon_1^k \epsilon_2^l \langle O\rangle^{[k,l]}. \tag{3.40}$$

Plugging (3.39)(3.40) into (2.8)(2.9), we obtain

$$\sum_{m,k,l=0}^{\infty} \epsilon_1^k \epsilon_2^{l+m} g_{(0)}^{[0,m]ij} \langle T_{ij}\rangle^{[k,l]} = -\sum_{m=0}^{\infty} \epsilon_2^m \left[\frac{1}{16\pi G} R_{(0)}^{[0,m]} + \frac{1}{2}\epsilon_1^2 g_{(0)}^{[0,m]ij} \partial_i f \partial_j f\right],$$
$$\sum_{m,k,l=0}^{\infty} \epsilon_1^k \epsilon_2^{l+m} \nabla_{(0)}^{[0,m]j} \langle T_{ij}\rangle^{[k,l]} = \sum_{k,l=0}^{\infty} \epsilon_1^{k+1} \epsilon_2^l \langle O(z)\rangle^{[k,l]} \partial_i f. \tag{3.41}$$

Here $g_{(0)}^{ij}$, $R_{(0)}$, and $\nabla_{(0)}^j$ depend only on the boundary metric $g_{(0)ij}$. We derive the differential equations for stress tensor correlators and mixed correlators from the coefficients of each order in (3.41), while the scalar correlators remain unspecified.

For instance, the first-order coefficients satisfy

$$\eta^{ij}\langle T_{ij}\rangle^{[1,0]} = 0, \tag{3.42}$$
$$\partial^j \langle T_{ij}\rangle^{[1,0]} = \langle O\rangle^{[0,0]} \partial_i f, \tag{3.43}$$
$$\eta^{ij}\langle T_{ij}\rangle^{[0,1]} = \frac{1}{16\pi G}[\partial^i\partial^j \chi_{ij} - \partial^i \partial_i \chi^j_j] + \chi^{ij}\langle T_{ij}\rangle^{[0,0]}, \tag{3.44}$$
$$\partial^j \langle T_{ij}\rangle^{[0,1]} = \frac{1}{2}[(2\partial^k \chi^j_k - \partial^j \chi^k_k)\langle T_{ij}\rangle^{[0,0]} + \partial_i \chi^{jk}\langle T_{jk}\rangle^{[0,0]}]. \tag{3.45}$$

In deriving (3.45), we have utilized the translation invariance of $\langle T_{ij}\rangle^{[0,0]}$, with indices raised by η^{ij}. It is convenient to work in the complex coordinates $(z, \bar{z})$ with the Euclidean metric $\eta_{z\bar{z}} = \eta_{\bar{z}z} = \frac{1}{2}$, $\eta_{zz} = \eta_{\bar{z}\bar{z}} = 0$. Taking functional derivatives on both sides of (3.42)(3.43)(3.44)(3.45), we obtain

$$\frac{\delta\langle T_{z\bar{z}}(z)\rangle}{\delta\phi_{(0)}(z_1)} = 0, \tag{3.46}$$
$$\partial_{\bar{z}} \frac{\delta\langle T_{zz}(z)\rangle}{\delta\phi_{(0)}(z_1)} = \frac{1}{2}\langle O\rangle \partial_z \delta(z - z_1), \tag{3.47}$$
$$\partial_z \frac{\delta\langle T_{\bar{z}\bar{z}}(z)\rangle}{\delta\phi_{(0)}(z_1)} = \text{ c.c. of } \partial_{\bar{z}} \frac{\delta\langle T_{zz}(z)\rangle}{\delta\phi_{(0)}(z_1)}, \tag{3.48}$$
$$\frac{\delta\langle T_{z\bar{z}}(z)\rangle}{\delta g_{(0)ij}(z_1)} = \frac{1}{16\pi G}\left(\partial_z^2 \frac{\delta\chi_{\bar{z}\bar{z}}(z)}{\delta\chi_{ij}(z_1)} - \partial_z\partial_{\bar{z}} \frac{\delta\chi_{z\bar{z}}(z)}{\delta\chi_{ij}(z_1)} + \partial_{\bar{z}}^2 \frac{\delta\chi_{zz}(z)}{\delta\chi_{ij}(z_1)}\right)$$
$$+ \frac{\delta\chi_{zz}(z)}{\delta\chi_{ij}(z_1)}\langle T_{\bar{z}\bar{z}}\rangle + \frac{\delta\chi_{z\bar{z}}(z)}{\delta\chi_{ij}(z_1)}\langle T_{z\bar{z}}\rangle + \frac{\delta\chi_{\bar{z}\bar{z}}(z)}{\delta\chi_{ij}(z_1)}\langle T_{zz}\rangle, \tag{3.49}$$

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$$\partial_{\bar{z}}\frac{\delta\langle T_{zz}(z)\rangle}{\delta g_{(0)ij}(z_1)}=\frac{1}{16\pi G}\left(-\partial_z\partial_{\bar{z}}^2\frac{\delta\chi_{zz}(z)}{\delta\chi_{ij}(z_1)}+\partial_z^2\partial_{\bar{z}}\frac{\delta\chi_{z\bar{z}}(z)}{\delta\chi_{ij}(z_1)}-\partial_z^3\frac{\delta\chi_{\bar{z}\bar{z}}(z)}{\delta\chi_{ij}(z_1)}\right)$$
$$+2\partial_z\frac{\delta\chi_{\bar{z}\bar{z}}(z)}{\delta\chi_{ij}(z_1)}\langle T_{zz}\rangle+2\partial_{\bar{z}}\frac{\delta\chi_{zz}(z)}{\delta\chi_{ij}(z_1)}\langle T_{z\bar{z}}\rangle, \tag{3.50}$$

$$\partial_z\frac{\delta\langle T_{\bar{z}\bar{z}}(z)\rangle}{\delta g_{(0)ij}(z_1)}=\text{ c.c. of }\partial_{\bar{z}}\frac{\delta\langle T_{zz}(z)\rangle}{\delta g_{(0)\bar{i}\bar{j}}(z_1)}. \tag{3.51}$$

The first three equations determine the two-point mixed correlators. (3.46) gives $\langle T_{z\bar{z}}(z)O(z_1)\rangle=0$. (3.47) and (3.48) are first-order differential equations, which can be solved as

$$\frac{\delta\langle T_{zz}(z)\rangle}{\delta\phi_{(0)}(z_1)}=\frac{1}{\text{Im}\tau}\int_{\partial\mathcal{M}}d^2w\frac{\delta\langle T_{zz}(w)\rangle}{\delta\phi_{(0)}(z_1)}+\frac{1}{2\pi}\langle O\rangle\partial_z G_\tau(z-z_1),$$
$$\frac{\delta\langle T_{\bar{z}\bar{z}}(z)\rangle}{\delta\phi_{(0)}(z_1)}=\frac{1}{\text{Im}\tau}\int_{\partial\mathcal{M}}d^2w\frac{\delta\langle T_{\bar{z}\bar{z}}(w)\rangle}{\delta\phi_{(0)}(z_1)}+\frac{1}{2\pi}\langle O\rangle\partial_{\bar{z}}\overline{G_\tau(z-z_1)}, \tag{3.52}$$

where $G_\tau(z-z_1)$ is the torus Green's function with modular parameter τ (see appendix A for the definition), and $\text{Im}\tau$ is the imaginary part of τ. As we discussed above, (3.52) leaves two one-point-averaged correlators undetermined. By definition (2.11), we can rewrite them as the global metric variations of $\langle O\rangle$, and then apply (2.17) to obtain

$$\int_{\partial\mathcal{M}}d^2w\frac{\delta\langle T_{zz}(w)\rangle}{\delta\phi_{(0)}(z_1)}=\frac{1}{2}\int_{\partial\mathcal{M}}d^2w\frac{\delta\langle O(z_1)\rangle}{\delta g_{(0)\bar{z}\bar{z}}(w)}=-i\text{Im}\tau\partial_\tau\langle O\rangle-\frac{1}{2}\langle O\rangle,$$
$$\int_{\partial\mathcal{M}}d^2w\frac{\delta\langle T_{\bar{z}\bar{z}}(w)\rangle}{\delta\phi_{(0)}(z_1)}=\frac{1}{2}\int_{\partial\mathcal{M}}d^2w\frac{\delta\langle O(z_1)\rangle}{\delta g_{(0)zz}(w)}=i\text{Im}\tau\partial_{\bar{\tau}}\langle O\rangle-\frac{1}{2}\langle O\rangle. \tag{3.53}$$

Putting everything together, we obtain

$$\langle T_{zz}(z)O(z_1)\rangle=-i\partial_\tau\langle O\rangle-\frac{1}{2\pi}\left[\wp_\tau(z-z_1)+2\zeta_\tau\left(\frac{1}{2}\right)\right]\langle O\rangle,$$
$$\langle T_{\bar{z}\bar{z}}(z)O(z_1)\rangle=\text{ c.c. of }\langle T_{zz}(z)O(z_1)\rangle,$$
$$\langle T_{z\bar{z}}(z)O(z_1)\rangle=0, \tag{3.54}$$

where $\wp_\tau(z)$ and $\zeta_\tau(z)$ are the Weierstrass $\wp$-function and ζ-function respectively. Note that the two-point mixed correlators are expressed as functions of the scalar one-point correlator. The latter is determined by the FG coefficients of the saddle (2.7). In our context, the scalar one-point correlator vanishes as a consequence of the Z_2 symmetry $\Phi\to-\Phi$. More generally, since our calculations are based on the semiclassical approximation, the precise one-point correlator could be nonzero at higher-orders in $1/c$, but it is suppressed in the large-c limit. The last three equations (3.49)(3.50)(3.51) determine the two-point stress tensor correlators. Note that these equations are derived with the scalar field turned off, and stress tensor correlators obtained from them agree with the results in pure gravity [20].

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Let us further consider the second-order coefficients of (3.41),

$$\eta^{ij}\langle T_{ij}\rangle^{[2,0]} = -\frac{1}{2}\partial^i f\partial_i f, \tag{3.55}$$

$$\partial^j\langle T_{ij}\rangle^{[2,0]} = \langle O\rangle^{[1,0]}\partial_i f, \tag{3.56}$$

$$\eta^{ij}\langle T_{ij}\rangle^{[1,1]} = \chi^{ij}\langle T_{ij}\rangle^{[1,0]}, \tag{3.57}$$

$$\begin{aligned}\partial^j\langle T_{ij}\rangle^{[1,1]} = {}& \chi^{jk}\partial_k\langle T_{ij}\rangle^{[1,0]} + \frac{1}{2}[(2\partial^k\chi^j_k - \partial^j\chi^k_k)\langle T_{ij}\rangle^{[1,0]} + \partial_i\chi^{jk}\langle T_{jk}\rangle^{[1,0]}] \\ & + \langle O\rangle^{[0,1]}\partial_i f,\end{aligned} \tag{3.58}$$

$$\begin{aligned}\eta^{ij}\langle T_{ij}\rangle^{[0,2]} = {}& \chi^{ij}\langle T_{ij}\rangle^{[0,1]} - \chi^{ik}\chi^j_k\langle T_{ij}\rangle^{[0,0]} - \frac{1}{64\pi G}[4\partial^i\chi^j_i\partial^k\chi_{jk} - 4\partial^i\chi^j_i\partial_j\chi^k_k \\ & + \partial^i\chi^j_j\partial_i\chi^k_k - 3\partial^i\chi^j_k\partial_i\chi^k_j + 2\partial^i\chi^j_k\partial_j\chi^k_i - 4\chi^{ij}(\partial_i\partial_j\chi^k_k \\ & + \partial^k\partial_k\chi_{ij} - 2\partial_i\partial^k\chi_{jk})],\end{aligned} \tag{3.59}$$

$$\begin{aligned}\partial^j\langle T_{ij}\rangle^{[0,2]} = {}& \chi^{jk}\partial_k\langle T_{ij}\rangle^{[0,1]} + \frac{1}{2}[(2\partial^k\chi^j_k - \partial^j\chi^k_k)\langle T_{ij}\rangle^{[0,1]} + \partial_i\chi^{jk}\langle T_{jk}\rangle^{[0,1]}] \\ & + \frac{1}{2}[\chi^{kl}(\partial^j\chi_{ik} - \partial_i\chi^j_k - \partial_k\chi^j_i) + \chi^{jk}(\partial_k\chi^l_i - \partial_i\chi^l_k - \partial^l\chi_{ik})]\langle T_{jl}\rangle^{[0,0]} \\ & + \frac{1}{2}[\chi^{kl}(\partial^j\chi_{kl} - 2\partial_k\chi^k_l) + \chi^{jk}(\partial_k\chi^l_l - 2\partial^l\chi_{kl})]\langle T_{ij}\rangle^{[0,0]}.\end{aligned} \tag{3.60}$$

The last two equations (3.59)(3.60) give the three-point stress tensor correlators, which have been considered in our previous work. For the remaining four equations, (3.55)(3.56) give the correlators of type $\langle T_{ij}(z)O(z_1)O(z_2)\rangle$,

$$\begin{aligned}\langle T_{zz}(z)O(z_1)O(z_2)\rangle = {}& -i\partial_\tau\langle O(z_1)O(z_2)\rangle - \frac{1}{2\pi}\sum_{n=1}^{2}\Bigg[\Bigg(\left[\wp_\tau(z-z_n) + 2\zeta_\tau\left(\frac{1}{2}\right)\right] \\ & + \left[\zeta_\tau(z-z_n) - 2\zeta_\tau\left(\frac{1}{2}\right)(z-z_n)\right]\partial_{z_n}\Bigg)\langle O(z_1)O(z_2)\rangle \\ & - \left(\pi\partial_{z_n}\delta(z-z_n)\partial_{z_n} + \left[\wp_\tau(z-z_n) + 2\zeta_\tau\left(\frac{1}{2}\right)\right]\partial_{z_n}\partial_{\bar{z}_n}\right)\delta(z_1-z_2)\Bigg], \\ \langle T_{\bar{z}\bar{z}}(z)O(z_1)O(z_2)\rangle = {}& \text{c.c. of } \langle T_{zz}(z)O(z_1)O(z_2)\rangle, \\ \langle T_{z\bar{z}}(z)O(z_1)O(z_2)\rangle = {}& -\frac{1}{2}\partial_z\delta(z-z_1)\partial_{\bar{z}}\delta(z-z_2) + (z_1\leftrightarrow z_2),\end{aligned} \tag{3.61}$$

and (3.57)(3.58) give the correlators of type $\langle T_{ij}(z)T_{kl}(z_1)O(z_2)\rangle$,

$$\begin{aligned}\langle T_{zz}(z)T_{zz}(z_1)O(z_2)\rangle = {}& -i\partial_\tau\langle T_{zz}(z_1)O(z_2)\rangle - \frac{1}{2\pi}\Bigg[2\left(\wp_\tau(z-z_1) + 2\zeta_\tau\left(\frac{1}{2}\right)\right) \\ & + \left(\wp_\tau(z-z_2) + 2\zeta_\tau\left(\frac{1}{2}\right)\right) + \left(\zeta_\tau(z-z_1) - 2\zeta_\tau\left(\frac{1}{2}\right)(z-z_1)\right)\partial_{z_1} \\ & + \left(\zeta_\tau(z-z_2) - 2\zeta_\tau\left(\frac{1}{2}\right)(z-z_1)\right)\partial_{z_2}\Bigg]\langle T_{zz}(z_1)O(z_2)\rangle, \\ \langle T_{\bar{z}\bar{z}}(z)T_{\bar{z}\bar{z}}(z_1)O(z_2)\rangle = {}& \text{c.c. of } \langle T_{zz}(z)T_{zz}(z_1)O(z_2)\rangle, \\ \langle T_{zz}(z)T_{\bar{z}\bar{z}}(z_1)O(z_2)\rangle = {}& -i\partial_\tau\langle T_{\bar{z}\bar{z}}(z_1)O(z_2)\rangle - \frac{1}{2\pi}\Bigg[\left(\wp_\tau(z-z_2) + 2\zeta_\tau\left(\frac{1}{2}\right)\right) \\ & + \left(\zeta_\tau(z-z_2) - 2\zeta_\tau\left(\frac{1}{2}\right)(z-z_1)\right)\partial_{z_2}\Bigg]\langle T_{\bar{z}\bar{z}}(z_1)O(z_2)\rangle,\end{aligned}$$

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$$\langle T_{z\bar{z}}(z)T_{zz}(z_1)O(z_2)\rangle = \frac{1}{2}\delta(z-z_1)\langle T_{zz}(z)O(z_2)\rangle,$$
$$\langle T_{z\bar{z}}(z)T_{\bar{z}\bar{z}}(z_1)O(z_2)\rangle = \text{ c.c. of } \langle T_{z\bar{z}}(z)T_{zz}(z_1)O(z_2)\rangle,$$
$$\langle T_{z\bar{z}}(z)T_{z\bar{z}}(z_1)O(z_2)\rangle = 0. \tag{3.62}$$

In this context, we have utilized the principle of translation invariance, represented as $\sum_{i=1}^{N}\partial_{z_i}\langle X(z_1,\ldots,z_N)\rangle = 0$. While our calculations effectively reduce the stress tensors in the mixed correlators, they leave the scalar correlators undetermined. Unfortunately, the application of FG expansions to the scalar equations of motion (EOM) results in the breakdown of the constraint on $\phi_{(2)}$, rendering it impossible to establish a differential equation for $\langle O\rangle$. Determining scalar correlators requires a global solution to the scalar EOM, a topic we have explored in section 3.1.1.

3.1.3 Recurrence relations of mixed correlators

In principle, with increasingly tedious calculations, one can obtain the arbitrary higher-point mixed correlator by the above method. Some special perturbations simplify (3.41) and lead to recurrence relations. For instance, if we only turn on the variations of the scalar field $\delta\phi_{(0)}(z) = \epsilon_1 f(z)$ and $\bar{z}\bar{z}$ component of the metric $\delta g_{(0)\bar{z}\bar{z}}(z) = \epsilon_2 F(z)$, geometric quantities can be represented as polynomials in ϵ_2,

$$\begin{aligned} g^{ij}_{(0)} &= \eta^{ij} - 4\epsilon_2 F\delta^i_z\delta^j_z, \\ \Gamma^k_{(0)ij} &= \epsilon_2\left[\delta^k_z\delta^{\bar{z}}_i\partial_j F + \delta^k_z\delta^{\bar{z}}_j\partial_i F - \frac{1}{2}\eta^{kl}\delta^{\bar{z}}_i\delta^{\bar{z}}_j\partial_l F\right] + 2\epsilon_2^2\delta^k_z\delta^{\bar{z}}_i\delta^{\bar{z}}_j F\partial_z F, \\ R_{(0)} &= -4\epsilon_2\partial_z^2 F. \end{aligned} \tag{3.63}$$

Plugging (3.63) into (3.41) we obtain

$$\begin{aligned} \langle T_{z\bar{z}}\rangle^{[k,l]} &= F\langle T_{zz}\rangle^{[k,l-1]} + \frac{\delta_{k,0}\delta_{l,1}}{16\pi G}\partial_z^2 F - \frac{\delta_{k,2}}{2}[\delta_{l,0}\partial_z f\partial_{\bar{z}} f - \delta_{l,1}(\partial_z f)^2 F], \\ \partial_{\bar{z}}\langle T_{zz}\rangle^{[k,l]} &= \frac{1}{2}\langle O\rangle^{[k-1,l]}\partial_z f + 2F\partial_z\langle T_{zz}\rangle^{[k,l-1]} + 3\partial_z F\langle T_{zz}\rangle^{[k,l-1]} - \partial_z\langle T_{z\bar{z}}\rangle^{[k,l]}, \end{aligned} \tag{3.64}$$

where $\delta_{m,n}$ is the Kronecker symbol defined as $\delta_{m,n} = 1$ for $m = n$ and $\delta_{m,n} = 0$ for $m \neq n$. The terms containing the Kronecker symbols come from the conformal anomaly and contribute to some specific correlators (as contact terms most of the time). Taking functional derivatives on both sides of (3.64) and solving the differential equation, we obtain

$$\begin{aligned} &\frac{\delta^{k+l}\langle T_{zz}(z)\rangle}{\prod_{i=1}^{k}\delta\phi_{(0)}(x_i)\prod_{j=1}^{l}\delta g_{(0)\bar{z}\bar{z}}(y_j)} \\ &= C(x,y) + \frac{1}{2\pi}\sum_{i=1}^{k}\left[\left(\partial_z G_\tau(z-x_i) - G_\tau(z-x_i)\partial_{x_i}\right)\frac{\delta^{k+l-1}\langle O(x_i)\rangle}{\prod_{i'\neq i}\delta\phi_{(0)}(x_{i'})\prod_{j=1}^{l}\delta g_{(0)\bar{z}\bar{z}}(y_j)}\right] \\ &\quad + \frac{1}{\pi}\sum_{j=1}^{l}\left[\left(2\partial_z G_\tau(z-y_j) - G_\tau(z-y_j)\partial_{y_j}\right)\frac{\delta^{k+l-1}\langle T_{zz}(y_j)\rangle}{\prod_{i=1}^{k}\delta\phi_{(0)}(x_i)\prod_{j'\neq j}\delta g_{(0)\bar{z}\bar{z}}(y_{j'})}\right] \\ &\quad - \frac{\delta_{k,0}\delta_{l,1}}{16\pi^2 G}\partial_z^3 G_\tau(z-y) + \frac{\delta_{k,2}}{2\pi}\Big[\delta_{l,0}\partial_{\bar{x}_1}[\partial_{x_1}G_\tau(z-x_1)\partial_{x_1}\delta(x_1-x_2)] \\ &\quad + \delta_{l,1}\partial_y G_\tau(z-y)\partial_y\delta(y-x_1)\partial_y\delta(y-x_2) + (x_1\leftrightarrow x_2)\Big]. \end{aligned} \tag{3.65}$$

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The undetermined function $C(x,y)=\frac{1}{\mathrm{Im}\tau}\int d^2w\frac{\delta^{k+l}\langle T_{zz}(w)\rangle}{\prod_{i=1}^{k}\delta\phi_{(0)}(x_i)\prod_{j=1}^{l}\delta g_{(0)\bar{z}\bar{z}}(y_j)}$ is calculated using (2.11)(2.17)(3.64),

$$
\begin{aligned}
&C(x,y)\\
&=-\frac{2^l}{2\mathrm{Im}\tau}\left[2i\mathrm{Im}\tau\partial_\tau+(k+2l)+\sum_{i=1}^{k}(x_i-\bar{x}_i)\partial_{x_i}+\sum_{j=1}^{l}(y_j-\bar{y}_j)\partial_{y_j}\right]\langle\prod_{i=1}^{k}O(x_i)\prod_{j=1}^{l}T_{zz}(y_j)\rangle\\
&\quad+\frac{\delta_{k,2}}{2\mathrm{Im}\tau}\Big[\delta_{l,0}\partial_{x_1}\partial_{\bar{x}_1}\delta(x_1-x_2)+\delta_{l,1}\partial_y\delta(y-x_1)\partial_y\delta(y-x_2)+(x_1\leftrightarrow x_2)\Big].
\end{aligned}
\tag{3.66}
$$

Combining (3.65) and (3.66), we find the recurrence relation

$$
\begin{aligned}
&\langle T_{zz}(z)\prod_{i=1}^{k}O(x_i)\prod_{j=1}^{l}T_{zz}(y_j)\rangle\\
&=-\frac{1}{2\pi}\left[2\pi i\partial_\tau+\sum_{i=1}^{k}\left(\wp_\tau(z-x_i)+2\zeta_\tau\left(\frac{1}{2}\right)+\left[\zeta_\tau(z-x_i)-2\zeta_\tau\left(\frac{1}{2}\right)(z-x_i)\right]\partial_{x_i}\right)\right.\\
&\quad\left.+\sum_{j=1}^{l}\left(2\left[\wp_\tau(z-y_j)+2\zeta_\tau\left(\frac{1}{2}\right)\right]+\left[\zeta_\tau(z-y_j)-2\zeta_\tau\left(\frac{1}{2}\right)(z-y_j)\right]\partial_{y_j}\right)\right]\langle\prod_{i=1}^{k}O(x_i)\prod_{j=1}^{l}T_{zz}(y_j)\rangle\\
&\quad+\frac{\delta_{k,0}\delta_{l,1}}{32\pi^2G}\wp''_\tau(z-y)+\frac{\delta_{k,2}}{2\pi}\left[\delta_{l,0}\left(\pi\partial_{x_1}\delta(z-x_1)\partial_{x_1}+\left[\wp_\tau(z-x_1)+2\zeta_\tau\left(\frac{1}{2}\right)\right]\partial_{x_1}\partial_{\bar{x}_1}\right)\delta(x_1-x_2)\right.\\
&\quad\left.+\frac{\delta_{l,1}}{2}\left[\wp_\tau(z-y)+2\zeta_\tau\left(\frac{1}{2}\right)\right]\partial_y\delta(y-x_1)\partial_y\delta(y-x_2)+(x_1\leftrightarrow x_2)\right].
\end{aligned}
\tag{3.67}
$$

Note that (3.64) should also contain a differential equation for $\langle T_{\bar{z}\bar{z}}\rangle^{[k,l]}$,

$$
\partial_z\langle T_{\bar{z}\bar{z}}\rangle^{[k,l]}=\frac{1}{2}\langle O\rangle^{[k-1,l]}\partial_{\bar{z}}f+\partial_{\bar{z}}F\langle T_{zz}\rangle^{[k,l-1]}+2\partial_z[F\langle T_{z\bar{z}}\rangle^{[k,l-1]}]-\partial_{\bar{z}}\langle T_{z\bar{z}}\rangle^{[k,l]}.
\tag{3.68}
$$

Through a similar procedure, we obtain

$$
\begin{aligned}
&\langle T_{\bar{z}\bar{z}}(z)\prod_{i=1}^{k}O(x_i)\prod_{j=1}^{l}T_{zz}(y_j)\rangle\\
&=-\frac{1}{2\pi}\left[-2\pi i\partial_{\bar{\tau}}+\sum_{i=1}^{k}\left(\overline{\wp_\tau(z-x_i)}+2\overline{\zeta_\tau\left(\frac{1}{2}\right)}+\left[\overline{\zeta_\tau(z-x_i)}-2\overline{\zeta_\tau\left(\frac{1}{2}\right)}(\bar{z}-\bar{x}_i)\right]\partial_{\bar{x}_i}\right)\right.\\
&\quad\left.+\sum_{j=1}^{l}\left(\overline{\zeta_\tau(z-y_j)}-2\overline{\zeta_\tau\left(\frac{1}{2}\right)}(\bar{z}-\bar{y}_j)\right)\partial_{\bar{y}_j}\right]\langle\prod_{i=1}^{k}O(x_i)\prod_{j=1}^{l}T_{zz}(y_j)\rangle\\
&\quad+\sum_{j=1}^{l}\delta(z-y_j)\langle T_{z\bar{z}}(z)\prod_{i=1}^{k}O(x_i)\prod_{j'\neq j}T_{zz}(y_{j'})\rangle-\frac{\delta_{k,0}\delta_{l,1}}{32\pi G}\partial_z\partial_{\bar{z}}\delta(z-y)\\
&\quad+\frac{\delta_{k,2}}{2\pi}\left[\delta_{l,0}\left(\pi\partial_{\bar{x}_1}\delta(z-x_1)\partial_{\bar{x}_1}+\left[\overline{\wp_\tau(z-x_1)}+2\overline{\zeta_\tau\left(\frac{1}{2}\right)}\right]\partial_{x_1}\partial_{\bar{x}_1}\right)\delta(x_1-x_2)\right.\\
&\quad\left.+\frac{\delta_{l,1}}{2}\left[\overline{\wp_\tau(z-y)}+2\overline{\zeta_\tau\left(\frac{1}{2}\right)}\right]\partial_y\delta(y-x_1)\partial_y\delta(y-x_2)+(x_1\leftrightarrow x_2)\right].
\end{aligned}
\tag{3.69}
$$

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With the omission of the contact terms, (3.67) and (3.69) agree with the Ward identity in the torus CFT [60–63] by the Brown-Henneaux central charge relation $c = \frac{3}{2G}$ [64]. The dual operator O is a primary operator with conformal weights $h = \bar{h} = 1$.

3.2 Massive scalar field

We further compute mixed correlators in the case of gravity coupled to a massive scalar field. Dual operators are classified according to their scaling dimensions. The operator is irrelevant for $\Delta > 2$, in which case, we can only consider the infinitesimal source; otherwise, it would cause a self-consistent problem [5]. For $\Delta = 2$, the operator is marginal and has been discussed in section 3.1. In this section, we are interested in the relevant operator where $\Delta < 2$. The scaling dimension has a lower bound $\Delta \geq 1$ called the BF bound [65, 66], which guarantees the stability of the AdS background.

We continue to adopt the FG coordinate system (2.4). Plugging (2.4) into (2.2)(2.3), we obtain the EOMs of the bulk fields,

$$\begin{aligned}
&\rho(2\partial_\rho^2 g_{ij} - 2\partial_\rho g_{ik} g^{kl} \partial_\rho g_{lj} + g^{kl}\partial_\rho g_{kl}\partial_\rho g_{ij}) + R_{(g)ij} - g^{kl}\partial_\rho g_{kl} g_{ij} \\
&+ 8\pi G \rho^{1-\Delta}[\Delta(\Delta-2) g_{ij}\phi^2 + \rho \partial_i\phi\partial_j\phi] = 0, \\
&\nabla_{(g)i}(g^{kl}\partial_\rho g_{kl}) - \nabla^k_{(g)}(\partial_\rho g_{ki}) + 16\pi G\rho^{1-\Delta}\left[\frac{2-\Delta}{2}\phi\partial_i\phi + \rho\partial_\rho\phi\partial_i\phi\right] = 0, \\
&g^{kl}\partial_\rho^2 g_{kl} - \frac{1}{2} g^{ij}\partial_\rho g_{jk} g^{kl}\partial_\rho g_{li} + 16\pi G\rho^{-\Delta}\left[(2-\Delta)\left(\frac{(1-\Delta)}{2}\phi^2 + \rho\phi\partial_\rho\phi\right) + \rho^2\partial_\rho\phi\partial_\rho\phi\right] = 0, \\
&\Box_{(g)}\phi + 2\rho(\partial_\rho \ln g \partial_\rho\phi + 2\partial_\rho^2\phi) + (2-\Delta)(\partial_\rho \ln g\phi + 4\partial_\rho\phi) = 0,
\end{aligned} \tag{3.70}$$

where $\phi(\rho, x) = \rho^{\frac{\Delta}{2}-1}\Phi(\rho, x)$. It is natural to expand g_{ij} and ϕ in powers of ρ and then solve the above equations order by order. The bulk EOMs contain $\rho^{-\Delta}$, and Δ is not necessarily an integer. We thus cannot simply assume that g_{ij} and ϕ can be expanded as formal Taylor series in ρ. The authors of [37] prove that for rational scaling dimensions, the FG expansions include fractional orders of ρ. Suppose that $\frac{2-\Delta}{2} = \frac{N}{M}$, where M and N are relatively prime and $0 < \frac{N}{M} \leq \frac{1}{2}$. As we mentioned in section 2, each bulk field has two homogeneous solutions, and their expansions in ρ are given by [37],

$$\begin{aligned}
g^{\pm}_{ij}(\rho, x) &= \sum_{n=0}^{N-1} \rho^n \left[g^{\pm}_{(2n)ij} + \sum_{m=2}^{\infty} \rho^{m\frac{N}{M}} g^{\pm}_{(2n+2m\frac{N}{M})ij} \right](x), \\
\phi^{\pm}(\rho, x) &= \sum_{n=0}^{N-1} \rho^n \left[\phi^{\pm}_{(2n)} + \sum_{m=1}^{\infty} \rho^{m\frac{N}{M}} \phi^{\pm}_{(2n+2m\frac{N}{M})} \right](x).
\end{aligned} \tag{3.71}$$

Plugging (3.71) into (2.5) and adding logarithmic terms to the overlap of two solutions, we obtain the FG expansions for g_{ij} and ϕ, which will be shown later.

Let us first assume that $N \geq 2$. Since M and N are relatively prime, the smallest integer that $m\frac{N}{M}$ passes through is N. Thus, the first logarithmic term that appears in the scalar

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field expansion[7] is $\rho\ln\rho\psi_{(2)}$ instead of $\rho^{1-2\frac{N}{M}}\ln\rho\psi_{(2-4\frac{N}{M})}$. The FG expansions take the forms

$$\begin{aligned}
g_{ij}(\rho,x) &= g_{(0)ij}(x) + \sum_{m=2}^{\lfloor\frac{M}{N}\rfloor} \rho^{m\frac{N}{M}} g_{(2m\frac{N}{M})ij}(x) + \rho[g_{(2)ij} + \ln\rho h_{(2)ij}](x) + \cdots, \\
\phi(\rho,x) &= \phi_{(0)}(x) + \sum_{m=1}^{\lfloor\frac{M}{N}\rfloor} \rho^{m\frac{N}{M}} \phi_{(2m\frac{N}{M})}(x) + \rho^{1-2\frac{N}{M}}\phi_{(2-4\frac{N}{M})}(x) + \rho^{1-\frac{N}{M}}\phi_{(2-2\frac{N}{M})}(x) \\
&\quad + \rho[\phi_{(2)} + \ln\rho\psi_{(2)}](x) + \cdots .
\end{aligned} \tag{3.72}$$

Plugging (3.72) into (3.70), we obtain

$$\begin{aligned}
h_{(2)ij} &= 0, \\
g^{ij}_{(0)}g_{(2)ij} &= \frac{1}{2}R_{(0)} - 64\pi G\frac{N}{M}\left(1-\frac{N}{M}\right)\phi_{(0)}\phi_{(2-4\frac{N}{M})}, \\
\nabla^j g_{(2)ij} &= \partial_i(g^{kl}_{(0)}g_{(2)kl}) + 16\pi G\left[\frac{N}{M}\phi_{(0)}\partial_i\phi_{(2-4\frac{N}{M})} + \left(1-\frac{N}{M}\right)\phi_{(2-4\frac{N}{M})}\partial_i\phi_{(0)}\right].
\end{aligned} \tag{3.73}$$

The next step is to calculate the one-point correlators $\langle T_{ij}\rangle$ and $\langle O\rangle$ by taking variations of the renormalized on-shell action. Following the spirit of [14], we construct the counterterm as (see appendix D for a detailed derivation)

$$I_{\rm ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}_\varepsilon} d^2x\sqrt{\gamma}\left[1 + \frac{N}{M}\frac{8\pi G\phi^2}{\varepsilon^{-2\frac{N}{M}}} + \sum_{m=3}^{\lfloor\frac{M}{N}\rfloor}\frac{(\pi G)^{\frac{m}{2}}\tilde{C}_{(2m\frac{N}{M})}\phi^m}{\varepsilon^{-m\frac{N}{M}}} + \frac{1}{4}R_{(\gamma)}\ln\varepsilon\right], \tag{3.74}$$

where $\{\tilde{C}_{(2m\frac{N}{M})}\}$ are some constants that do not show up in the finite part of the renormalized on-shell action. The renormalized one-point correlators are given by

$$\begin{aligned}
\langle T_{ij}\rangle &= \frac{1}{8\pi G}\left(g_{(2)ij} - {\rm Tr}g_{(2)}g_{(0)ij}\right) - 2\frac{N}{M}\phi_{(0)}\phi_{(2-4\frac{N}{M})}g_{(0)ij}, \\
\langle O\rangle &= 2\left(1-2\frac{N}{M}\right)\phi_{(2-4\frac{N}{M})}.
\end{aligned} \tag{3.75}$$

Combining (3.73) and (3.75), we obtain the trace relation and the conservation equation,

$$\begin{aligned}
\langle T^i_i\rangle &= -\frac{1}{16\pi G}R_{(0)} + 2\frac{N}{M}\phi_{(0)}\langle O\rangle, \\
\nabla^j_{(0)}\langle T_{ij}\rangle &= \langle O\rangle\partial_i\phi_{(0)}.
\end{aligned} \tag{3.76}$$

One can compute holographic mixed correlators from (3.76) by utilizing the same techniques in section 3.1. In particular, we are concerned with the holographic recurrence relations and wish to compare them with the results in CFT$_2$. Plugging the Taylor expansions (3.40)

[7] The discussion here is slightly different from [37]. In the latter, the FG expansion of ϕ contains $\rho^{1-2\frac{N}{M}}\ln\rho\psi_{(2-4\frac{N}{M})}$ and $\rho^{1-\frac{N}{M}}\ln\rho\psi_{(2-2\frac{N}{M})}$. We can keep these two terms, plug the expansions of g_{ij} and ϕ into (3.70), and solve the equations order by order. Eventually we will find that $\psi_{(2-4\frac{N}{M})} = \psi_{(2-2\frac{N}{M})} = 0$ for $N \geq 2$.

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into (3.76), we read off

$$\langle T_{z\bar{z}}\rangle^{[k,l]} = F\langle T_{zz}\rangle^{[k,l-1]} + \frac{N}{2M}\langle O\rangle^{[k-1,l]} f - \frac{\delta_{k,0}\delta_{l,1}}{16\pi G}\partial_z^2 F, \tag{3.77}$$

$$\partial_{\bar{z}}\langle T_{zz}\rangle^{[k,l]} = \frac{1}{2}\langle O\rangle^{[k-1,l]}\partial_z f + 2F\partial_z\langle T_{zz}\rangle^{[k,l-1]} + 3\partial_z F\langle T_{zz}\rangle^{[k,l-1]} - \partial_z\langle T_{z\bar{z}}\rangle^{[k,l]}, \tag{3.78}$$

$$\partial_z\langle T_{\bar{z}\bar{z}}\rangle^{[k,l]} = \frac{1}{2}\langle O\rangle^{[k-1,l]}\partial_{\bar{z}} f + \partial_{\bar{z}} F\langle T_{zz}\rangle^{[k,l-1]} + 2\partial_z[F\langle T_{z\bar{z}}\rangle^{[k,l-1]}] - \partial_{\bar{z}}\langle T_{z\bar{z}}\rangle^{[k,l]}. \tag{3.79}$$

After taking the functional derivatives, (3.77) gives $\langle T_{z\bar{z}}O\dots OT_{zz}\dots T_{zz}\rangle$, while (3.78) and (3.79) provide the differential equations of $\langle T_{zz}O\dots OT_{zz}\dots T_{zz}\rangle$ and $\langle T_{\bar{z}\bar{z}}O\dots OT_{zz}\dots T_{zz}\rangle$, respectively. We solve these differential equations by the torus Green's function and compute the one-point-averaged correlators using (2.17). The final results are listed as follows:

$$\begin{aligned}
&\langle T_{zz}(z)\prod_{i=1}^{k}O(x_i)\prod_{j=1}^{l}T_{zz}(y_j)\rangle \\
&= -\frac{1}{2\pi}\Bigg[2\pi i\partial_\tau + \sum_{i=1}^{k}\Bigg(\left(1-\frac{N}{M}\right)\left[\wp_\tau(z-x_i)+2\zeta_\tau\left(\frac{1}{2}\right)\right] + \left[\zeta_\tau(z-x_i) - 2\zeta_\tau\left(\frac{1}{2}\right)(z-x_i)\right]\partial_{x_i}\Bigg) \\
&\quad + \sum_{j=1}^{l}\Bigg(2\left[\wp_\tau(z-y_j)+2\zeta_\tau\left(\frac{1}{2}\right)\right] + \left[\zeta_\tau(z-y_j) - 2\zeta_\tau\left(\frac{1}{2}\right)(z-y_j)\right]\partial_{y_j}\Bigg)\Bigg]\langle\prod_{i=1}^{k}O(x_i)\prod_{j=1}^{l}T_{zz}(y_j)\rangle \\
&\quad + \frac{\delta_{k,0}\delta_{l,1}}{32\pi^2 G}\wp_\tau''(z-y),
\end{aligned} \tag{3.80}$$

$$\begin{aligned}
&\langle T_{\bar{z}\bar{z}}(z)\prod_{i=1}^{k}O(x_i)\prod_{j=1}^{l}T_{zz}(y_j)\rangle \\
&= -\frac{1}{2\pi}\Bigg[-2\pi i\partial_{\bar{\tau}} + \sum_{i=1}^{k}\Bigg(\left(1-\frac{N}{M}\right)\left[\overline{\wp_\tau(z-x_i)}+2\overline{\zeta_\tau\left(\frac{1}{2}\right)}\right] + \left[\overline{\zeta_\tau(z-x_i)} - 2\overline{\zeta_\tau\left(\frac{1}{2}\right)}(\bar{z}-\bar{x}_i)\right]\partial_{\bar{x}_i}\Bigg) \\
&\quad + \sum_{j=1}^{l}\left(\overline{\zeta_\tau(z-y_j)} - 2\overline{\zeta_\tau\left(\frac{1}{2}\right)}(\bar{z}-\bar{y}_j)\right)\partial_{\bar{y}_j}\Bigg]\langle\prod_{i=1}^{k}O(x_i)\prod_{j=1}^{l}T_{zz}(y_j)\rangle \\
&\quad + \sum_{j=1}^{l}\delta(z-y_j)\langle T_{z\bar{z}}(z)\prod_{i=1}^{k}O(x_i)\prod_{j'\neq j}T_{zz}(y_{j'})\rangle - \frac{\delta_{k,0}\delta_{l,1}}{32\pi G}\partial_z\partial_{\bar{z}}\delta(z-y).
\end{aligned} \tag{3.81}$$

Except for the contact terms, (3.80) and (3.81) agree with the Ward identity in the torus CFT, and O is a primary operator with conformal weights $h = \bar{h} = 1 - \frac{N}{M} = \frac{\Delta}{2}$.

For the case of $N = 1$, the fractional order $\frac{m}{M}$ passes through 1 and the scalar conformal anomaly $\psi_{(2-\frac{4}{M})}$ emerges. The FG expansions take the forms

$$\begin{aligned}
g_{ij}(\rho,x) &= g_{(0)ij}(x) + \sum_{m=2}^{M-1}\rho^{\frac{m}{M}}g_{(\frac{2m}{M})ij}(x) + \rho[g_{(2)ij} + \ln\rho h_{(2)ij}](x) + \cdots, \\
\phi(\rho,x) &= \phi_{(0)}(x) + \sum_{m=1}^{M-3}\rho^{\frac{m}{M}}\phi_{(\frac{2m}{M})}(x) + \rho^{1-\frac{2}{M}}\left[\phi_{(2-\frac{4}{M})} + \ln\rho\psi_{(2-\frac{4}{M})}\right](x) + \cdots.
\end{aligned} \tag{3.82}$$

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Plugging (3.82) into the bulk on-shell action, we can read off its divergent part as $\varepsilon \to 0$, which is canceled by the following counterterm,

$$I_{\rm ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}_\varepsilon} d^2x\sqrt{\gamma}\Bigg[1+\frac{1}{M}\frac{8\pi G\phi^2}{\varepsilon^{-\frac{2}{M}}}+\sum_{m=3}^{M-1}\frac{(\pi G)^{\frac{m}{2}}\tilde{D}_{\left(\frac{2m}{M}\right)}\phi^m}{\varepsilon^{-\frac{m}{M}}} + \frac{1}{4}\left(R_{(\gamma)}+\varepsilon(\pi G)^{\frac{M}{2}}\tilde{D}_{(2)}\phi^M\right)\ln\varepsilon\Bigg]. \tag{3.83}$$

Unlike the counterterm (3.74), the subleading terms here have a finite contribution to the renormalized on-shell action. More precisely, the finite part of $\phi^m\varepsilon^{\frac{m}{M}-1}$ is proportional to $\phi_{(0)}^M$ (using the FG coefficients in appendix D). Thus, the renormalized one-point correlators take the following forms

$$\begin{aligned}\langle T_{ij}\rangle &= \frac{1}{8\pi G}\Big(g_{(2)ij}-\mathrm{Tr}g_{(2)}g_{(0)ij}\Big)-\frac{2}{M}\phi_{(0)}\phi_{\left(2-\frac{4}{M}\right)}g_{(0)ij}+(\pi G)^{\frac{M}{2}-1}\mathcal{A}_M\phi_{(0)}^M g_{(0)ij},\\ \langle O\rangle &= 2\left(1-\frac{2}{M}\right)\phi_{\left(2-\frac{4}{M}\right)}+(\pi G)^{\frac{M}{2}-1}\mathcal{B}_M\phi_{(0)}^{M-1},\end{aligned} \tag{3.84}$$

where $\mathcal{A}_M$ and $\mathcal{B}_M$ are some constants. Since the bulk action (2.1) is invariant under $\Phi \to -\Phi$, additional terms in (3.84) occur only when M is even. For example, when $N = 1$ and $M = 4$, the counterterm is constructed as

$$I_{\rm ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}_c} d^2x\sqrt{\gamma}\left[1+2\pi G\varepsilon^{\frac{1}{2}}\phi^2+\frac{1}{4}R_{(\gamma)}\ln\varepsilon+2\pi^2G^2\phi^4\varepsilon\ln\varepsilon\right], \tag{3.85}$$

and the renormalized one-point correlators are

$$\begin{aligned}\langle T_{ij}\rangle &= \frac{1}{8\pi G}\Big(g_{(2)ij}-\mathrm{Tr}g_{(2)}g_{(0)ij}\Big)-\frac{1}{2}\phi_{(0)}\phi_{(1)}g_{(0)ij}+\frac{7\pi G}{4}\phi_{(0)}^4g_{(0)ij},\\ \langle O\rangle &= \phi_{(1)}+2\pi G\phi_{(0)}^3.\end{aligned} \tag{3.86}$$

These terms contribute an additional trace anomaly,

$$\langle T_i^i\rangle = -\frac{1}{16\pi G}R_{(0)}+\frac{2}{M}\phi_{(0)}\langle O\rangle+(\pi G)^{\frac{M}{2}-1}\mathcal{C}_M\phi_{(0)}^M, \tag{3.87}$$

which is also proportional to $\phi_{(0)}^M$. Moreover, one can easily prove that the conservation equation takes the same form as (2.9). To calculate the higher-point mixed correlators, we perturbed the boundary values as (3.39) and expanded the one-point correlators as (3.40). The differential equations for $\langle T_{ij}\rangle^{[k,l]}$ are obtained from the trace relation and the conservation equation,

$$\begin{aligned}\langle T_{z\bar{z}}\rangle^{[k,l]} &= -\frac{1}{4}\sum_{m=1}^{l}g_{(0)}^{[0,m]ij}\langle T_{ij}\rangle^{[k,l-m]}+\frac{1}{2M}\langle O\rangle^{[k-1,l]}f-\frac{1}{64\pi G}\delta_{k,0}R_{(0)}^{[0,l]}\\ &\quad+\delta_{k,M}\delta_{l,0}(\pi G)^{\frac{M}{2}-1}\mathcal{D}_M^{T_{z\bar{z}}}f^M,\\ \partial_{\bar{z}}\langle T_{zz}\rangle^{[k,l]} &= \frac{1}{4}\sum_{m=1}^{l}[\partial_z g_{(0)}^{[0,m]ij}+g_{(0)}^{[0,m]ij}\partial_z-2\delta_z^i\nabla_{(0)}^{[0,m]j}]\langle T_{ij}\rangle^{[k,l-m]}+\frac{1}{2}\left(1-\frac{1}{M}\right)\langle O\rangle^{[k-1,l]}\partial_z f\\ &\quad-\frac{1}{2M}\partial_z\langle O\rangle^{[k-1,l]}f+\frac{1}{64\pi G}\delta_{k,0}\partial_z R_{(0)}^{[0,l]}+\delta_{k,M}\delta_{l,0}(\pi G)^{\frac{M}{2}-1}\mathcal{D}_M^{T_{zz}}f^{M-1}\partial_z f,\\ \partial_z\langle T_{\bar{z}\bar{z}}\rangle^{[k,l]} &= \text{ c.c. of }\partial_{\bar{z}}\langle T_{zz}\rangle^{[k,l]}.\end{aligned} \tag{3.88}$$

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Additional terms proportional to $\delta_{k,M}\delta_{l,0}$ in (3.88) contribute contact terms to the mixed correlators when the number of scalar operators $k \geq M$.

4 Correlators in holographic CFT with a finite cutoff

In the preceding sections, we have computed the mixed correlators within the framework of AdS_3/CFT_2 and successfully derived the appropriate Ward identities. In what follows, we will be interested in the holographic aspects of the AdS_3 with Dirichlet boundary at finite cutoff. $(y, z, \bar{z})$ are the FG coordinates in bulk, and the boundary $\partial\mathcal{M}_c$ is a hard radial cutoff at $y = y_c$. A natural holographic dictionary is given by the generalized GKPW relation,

$$Z_G[\phi_{(c)}, g_{(c)ij}] = \left\langle \exp \int_{\partial\mathcal{M}_c} d^2x \sqrt{g_{(c)}} \left(\phi_{(c)} O - \frac{1}{2} g^{ij}_{(c)} T_{ij} \right) \right\rangle_{\text{EFT}}, \tag{4.1}$$

where $\phi_{(c)}(z) = \phi(y_c, z)$ and $g_{(c)ij}(z) = g_{ij}(y_c, z)$. We consider the semiclassical limit and assume that the gravitational partition function is dominated by only one saddle. Then the left-hand side $Z_G[\phi_{(c)}, g_{(c)ij}] \approx e^{-I_{\text{on-shell}}[\phi_{(c)}, g_{(c)ij}]}$. The bulk action takes the same form as (2.1) and one can introduce the counterterm on $\partial\mathcal{M}_c$,

$$I_{\text{ct}} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}_c} d^2z \sqrt{\gamma} \left[1 + \frac{1}{4} R_{(\gamma)} \ln y_c + \Psi \right]. \tag{4.2}$$

Here Ψ is some local function of $(y_c, \phi, \partial_i\phi, g_{ij})$, whose specific form is obtained by holographic renormalization. For example, in massless scalar field coupling $\Psi = 2\pi G y_c \ln y_c g^{kl} \partial_k\phi\partial_l\phi$. The renormalized one-point correlators take the forms

$$\begin{aligned} \langle T_{ij} \rangle &= \frac{1}{8\pi G} \left[\partial_y g_{ij} - g^{kl} \partial_y g_{kl} g_{ij} \right]_{y=y_c} - X_{ij}, \\ \langle O \rangle &= \left[2y^{\frac{2-\Delta}{2}} \partial_y \left(y^{\frac{2-\Delta}{2}} \phi \right) \right]_{y=y_c} - Y, \end{aligned} \tag{4.3}$$

where we have used the notation

$$\begin{aligned} X_{ij} &= \frac{1}{8\pi G y_c} \left[\Psi g_{ij} - 2 \frac{\partial\Psi}{\partial g^{ij}} \right]_{y=y_c}, \\ Y &= \frac{1}{8\pi G y_c} \left[\frac{\partial\Psi}{\partial\phi} - \frac{1}{\sqrt{g}} \partial_k \left(\sqrt{g} \frac{\partial\Psi}{\partial(\partial_k\phi)} \right) \right]_{y=y_c}. \end{aligned} \tag{4.4}$$

Plugging (4.3) into (3.70), we obtain the trace relation

$$\begin{aligned} \langle T^i_i \rangle = 4\pi G y_c \Big[(\langle T_{ij} \rangle + X_{ij})^2 - (\langle T^i_i \rangle + X^i_i)^2 \Big] + \frac{1}{2} y_c^{\Delta-1} (\langle O \rangle + Y)^2 \\ - \frac{1}{2} y_c^{2-\Delta} \partial^i \phi_{(c)} \partial_i \phi_{(c)} + \frac{\Delta(2-\Delta)}{2} y_c^{1-\Delta} \phi^2_{(c)} - \frac{R_{(c)}}{16\pi G} - X^i_i, \end{aligned} \tag{4.5}$$

and the conservation equation

$$\nabla^k_{(c)} \langle T_{ki} \rangle = \langle O \rangle \partial_i \phi_{(c)}. \tag{4.6}$$

The indices i, j, k, l are raised by $g^{ij}_{(c)}$.

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From the field theory perspective, the dual EFT in (4.1) is obtained by T^2 deformation of the original CFT [29]. The flow equation of the EFT action is

$$\frac{\partial S}{\partial \lambda}=\int d^2z\sqrt{g_{(c)}}\left[(T_{ij}+X_{ij})^2-(T^i_i+X^i_i)^2+\frac{1}{4\pi G y_c}\left(t^y_y-\frac{R_{(c)}}{16\pi G}-X^i_i\right)\right]. \tag{4.7}$$

Here $\lambda=2\pi G y_c$ is the deformation parameter. t^y_y is the radial component of the matter stress tensor. For example, when the coupled scalar field is massless, we have

$$\begin{aligned} t^y_y&=-\frac{1}{2}\partial^i\phi_{(c)}\partial_i\phi_{(c)}+\frac{y_c}{2}\left(O-\frac{\ln y_c}{2}\Box_{(c)}\phi_{(c)}\right)^2,\\ X_{ij}&=-\frac{\ln y_c}{2}\left(\partial_i\phi_{(c)}\partial_j\phi_{(c)}-\frac{1}{2}\partial^k\phi_{(c)}\partial_k\phi_{(c)}g_{(c)ij}\right). \end{aligned} \tag{4.8}$$

By turning off the sources, the flow equation can be simplified as follows,

$$\frac{\partial S}{\partial\lambda}\Big|_{\substack{\phi_{(c)}=0\\ g_{(c)ij}=\eta_{ij}}}=\int d^2z\sqrt{\eta}\left[T_{ij}T^{ij}-(T^i_i)^2+\frac{1}{8\pi G}O^2\right]. \tag{4.9}$$

It can be observed that, in the presence of a matter field in the bulk, the dual field theory at a finite cutoff deviates from being a standard $T\bar{T}$-deformed CFT. The deformation operator encompasses both the $T\bar{T}$ operator and the dual scalar operator.

According to the generalized GKPW relation, holographic correlators are computed by (2.11), by replacing $\phi_{(0)}$ and $g_{(0)ij}$ with $\phi_{(c)}$ and $g_{(c)ij}$ respectively. We change the boundary values like (3.39) and solve (4.5) and (4.6) perturbatively. The terms of order $\epsilon_1^k\epsilon_2^l$ read

$$\langle T_{z\bar{z}}\rangle^{[k,l]}=\bar{A}\langle T_{zz}\rangle^{[k,l]}+A\langle T_{\bar{z}\bar{z}}\rangle^{[k,l]}+B\langle O\rangle^{[k,l]}+\mathcal{F}^{[k,l]}_{T_{z\bar{z}}}, \tag{4.10}$$

$$\partial_{\bar{z}}\langle T_{zz}\rangle^{[k,l]}=-\partial_z\langle T_{z\bar{z}}\rangle^{[k,l]}+\mathcal{F}^{[k,l]}_{T_{zz}}, \tag{4.11}$$

$$\partial_z\langle T_{\bar{z}\bar{z}}\rangle^{[k,l]}=-\partial_{\bar{z}}\langle T_{z\bar{z}}\rangle^{[k,l]}+\mathcal{F}^{[k,l]}_{T_{\bar{z}\bar{z}}}. \tag{4.12}$$

Here $\mathcal{F}^{[k,l]}_{T_{z\bar{z}}}$, $\mathcal{F}^{[k,l]}_{T_{zz}}$ and $\mathcal{F}^{[k,l]}_{T_{\bar{z}\bar{z}}}$ consist of lower-order coefficients and local functions of f and χ_{ij}. A, $\bar{A}$, and B are constants determined by the one-point correlators at the saddle,

$$\begin{aligned} A&=\frac{8\pi G y_c\langle T_{zz}\rangle^{[0,0]}}{1+16\pi G y_c\langle T_{z\bar{z}}\rangle^{[0,0]}},\quad \bar{A}=\frac{8\pi G y_c\langle T_{\bar{z}\bar{z}}\rangle^{[0,0]}}{1+16\pi G y_c\langle T_{z\bar{z}}\rangle^{[0,0]}},\\ B&=\frac{y_c^{\Delta-1}\langle O\rangle^{[0,0]}}{4(1+16\pi G y_c\langle T_{z\bar{z}}\rangle^{[0,0]})}. \end{aligned} \tag{4.13}$$

Plugging (4.10) into (4.11) and (4.12) we obtain

$$(\partial_{\bar{z}}+\bar{A}\partial_z)\langle T_{zz}\rangle^{[k,l]}+A\partial_z\langle T_{\bar{z}\bar{z}}\rangle^{[k,l]}=-B\partial_z\langle O\rangle^{[k,l]}-\partial_z\mathcal{F}^{[k,l]}_{T_{z\bar{z}}}+\mathcal{F}^{[k,l]}_{T_{zz}}, \tag{4.14}$$

$$\bar{A}\partial_{\bar{z}}\langle T_{zz}\rangle^{[k,l]}+(\partial_z+A\partial_{\bar{z}})\langle T_{\bar{z}\bar{z}}\rangle^{[k,l]}=-B\partial_{\bar{z}}\langle O\rangle^{[k,l]}-\partial_{\bar{z}}\mathcal{F}^{[k,l]}_{T_{z\bar{z}}}+\mathcal{F}^{[k,l]}_{T_{\bar{z}\bar{z}}}. \tag{4.15}$$

Adding $(\partial_z+A\partial_{\bar{z}})$(4.14) and $-A\partial_z$(4.15) we find

$$\begin{aligned} &(\partial_z\partial_{\bar{z}}+\bar{A}\partial_z^2+A\partial_{\bar{z}}^2)\langle T_{zz}\rangle^{[k,l]}\\ &\quad=-B\partial_z^2\langle O\rangle^{[k,l]}-\partial_z^2\mathcal{F}^{[k,l]}_{T_{z\bar{z}}}+(\partial_z+A\partial_{\bar{z}})\mathcal{F}^{[k,l]}_{T_{zz}}-A\partial_z\mathcal{F}^{[k,l]}_{T_{\bar{z}\bar{z}}}. \end{aligned} \tag{4.16}$$

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The matrix M defined by $M^{ij}\partial_i\partial_j = \partial_z\partial_{\bar{z}} + \bar{A}\partial_z^2 + A\partial_{\bar{z}}^2$ is positive definite[8] as long as $|A| < \frac{1}{2}$. We introduce the following coordinate transformation,

$$
\begin{aligned}
z &= \frac{\bar{A}+\tilde{A}}{4\tilde{A}}\left[\left(1+\Delta_A+\frac{i(1+\Delta_A)\mathrm{Im}A}{\Delta_A(\mathrm{Re}A+\tilde{A})}\right)Z+\left(1-\Delta_A-\frac{i(1-\Delta_A)\mathrm{Im}A}{\Delta_A(\mathrm{Re}A+\tilde{A})}\right)\bar{Z}\right],\\
\bar{z} &= \frac{A+\tilde{A}}{4\tilde{A}}\left[\left(1-\Delta_A+\frac{i(1-\Delta_A)\mathrm{Im}A}{\Delta_A(\mathrm{Re}A+\tilde{A})}\right)Z+\left(1+\Delta_A-\frac{i(1+\Delta_A)\mathrm{Im}A}{\Delta_A(\mathrm{Re}A+\tilde{A})}\right)\bar{Z}\right],
\end{aligned}
\tag{4.17}
$$

where $\tilde{A} = \mathrm{sgn}(\mathrm{Re}A)|A|$ and $\Delta_A = \sqrt{\frac{1-2\tilde{A}}{1+2\tilde{A}}}$. The periods in Z are 1 and Ω with the new modular parameter

$$
\Omega = \frac{(\Delta_A+1)(A+\tilde{A})\tau + (\Delta_A-1)(\bar{A}+\tilde{A})\bar{\tau}}{2\Delta_A(A_1+\tilde{A})+2i\mathrm{Im}A}. \tag{4.18}
$$

Plugging (4.17) into the left-hand side of (4.16) we obtain

$$
(\partial_z\partial_{\bar{z}} + \bar{A}\partial_z^2 + A\partial_{\bar{z}}^2)\langle T_{zz}\rangle^{[k,l]} = \frac{2\Delta_A^2\tilde{A}(1+2\tilde{A})(\mathrm{Re}A+\tilde{A})}{\Delta_A^2(\mathrm{Re}A+\tilde{A})^2+\mathrm{Im}A^2}\partial_Z\partial_{\bar{Z}}\langle T_{zz}\rangle^{[k,l]}. \tag{4.19}
$$

Then equation (4.16) can be solved by the torus Green's function (which is defined in appendix A),

$$
\begin{aligned}
&\frac{\delta^{k+l}\langle T_{zz}(z)\rangle^{[k,l]}}{\prod_{i=1}^k \delta f(x_i)\prod_{j=1}^l \delta\chi_{\alpha_j\beta_j}(y_j)}\\
&\quad= \int d^2W\left(\frac{1}{\mathrm{Im}\Omega}+\frac{1}{\pi}\tilde{G}_\Omega(W-Z)\partial_W\partial_{\bar{W}}\right)\frac{\delta^{k+l}\langle T_{zz}(w)\rangle^{k,l}}{\prod_{i=1}^k \delta f(x_i)\prod_{j=1}^l \delta\chi_{\alpha_j\beta_j}(y_j)}.
\end{aligned}
\tag{4.20}
$$

Later, we will perform this procedure to compute the specific holographic correlators. Note that (4.17) is available in the domain $\{A, |A| < \frac{1}{2}\}\backslash\{A, \mathrm{Re}A = 0 \text{ and } 0 < |\mathrm{Im}A| < \frac{1}{2}\}$. When A is a pure imaginary number, we can apply the following coordinate transformation,

$$
\begin{aligned}
z &= \frac{(i+\Delta'_A)^2|A| + [1+(\Delta'_A)^2]\bar{A}}{4i\Delta'_A|A|}Z' - \frac{(-i+\Delta'_A)^2|A| + [1+(\Delta'_A)^2]\bar{A}}{4i\Delta'_A|A|}\bar{Z}',\\
\bar{z} &= \frac{(i+\Delta'_A)^2|A| + [1+(\Delta'_A)^2]A}{4i\Delta'_A|A|}Z' - \frac{(-i+\Delta'_A)^2|A| + [1+(\Delta'_A)^2]A}{4i\Delta'_A|A|}\bar{Z}',
\end{aligned}
\tag{4.21}
$$

where $\Delta'_A = \sqrt{\frac{(1+2|A|)(|A|-\mathrm{Re}A)}{(1-2|A|)(|A|+\mathrm{Re}A)}}$. The periods in Z' are 1 and $\Omega' = \frac{1}{\mathrm{Im}A}[\mathrm{Im}(A\tau) + \frac{-i+\Delta'_A}{i+\Delta'_A}|A|\mathrm{Im}\tau]$. The coordinate transformation (4.21) fails when A is a real number.

To end this section, let us discuss the ambiguity of the counterterm, as also mentioned in [29]. It should be noted that the counterterm (4.2) lacks covariance due to its explicit

[8] M is always positive definite in our consideration. Using the trace relation (4.5) we find

$$
|A|^2 = \frac{1}{4+\frac{1+4\pi G y_c^{\Delta}(\langle O\rangle^{[0,0]})^2}{|8\pi G y_c\langle T_{zz}\rangle^{[0,0]}|^2}} < \frac{1}{4}.
$$

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dependence on y_c. Consequently, counterterms of the same form constructed in alternative FG coordinate systems exhibit differences from (4.2), which tend to zero as $y_c \to 0$. More generally, any counterterm that effectively cancels the divergence of the on-shell action in the CFT limit is permissible. Diverse choices of counterterms yield distinct holographic trace relations, resulting in varied flow equations for the boundary field theory.

4.1 Two-point correlators on general saddles

For simplicity, we consider the massless scalar field coupling in the remainder of this paper. From the counterterm (4.2) with $\Psi = 2\pi G y_c \ln y_c g^{kl}\partial_k\phi\partial_l\phi$, it follows that

$$\begin{aligned} X_{ij} &= -\frac{\ln y_c}{2}\left(\partial_i\phi_{(c)}\partial_j\phi_{(c)} - \frac{1}{2}\partial^k\phi_{(c)}\partial_k\phi_{(c)}g_{(c)ij}\right), \\ Y &= -\frac{\ln y_c}{2}\Box_{(c)}\phi_{(c)}. \end{aligned} \tag{4.22}$$

The two-point mixed correlators on a general saddle are computed by implementing the procedure illustrated in (4.10)–(4.20). The results are listed as follows:

$$\begin{aligned} \langle T_{zz}(z)O(z_1)\rangle =& \frac{-i[(1-A)\partial_\tau - A\partial_{\bar\tau}]\langle O\rangle}{1-2\mathrm{Re}A} + \frac{(\Delta_A^2+\tilde A-\mathrm{Re}A)\mathrm{Im}(A\tau)}{2\pi\Delta_A|A|^2(1-2\mathrm{Re}A)\mathrm{Im}\tau} \\ &\times\Big\{\int d^2w\Bigg[\left(\partial_w^2\tilde G_\Omega(W-Z) - \frac{2\pi\Delta_A|A|^2}{(\Delta_A^2+\tilde A-\mathrm{Re}A)\mathrm{Im}(A\tau)}\right) \\ &\times\left(\frac{\langle O\rangle}{2}\delta(w-z_1) - B\langle O(w)O(z_1)\rangle\right)\Bigg] - 2B\ln y_c\partial_{z_1}^2\partial_{\bar z_1}\tilde G_\Omega(Z_1-Z)\Big\}, \\ \langle T_{\bar z\bar z}(z)O(z_1)\rangle =& \text{ c.c. of } \langle T_{zz}(z)O(z_1)\rangle, \\ \langle T_{z\bar z}(z)O(z_1)\rangle =& \bar A\langle T_{zz}(z)O(z_1)\rangle + A\langle T_{\bar z\bar z}(z)O(z_1)\rangle + B\langle O(z)O(z_1)\rangle \\ &- 2B\ln y_c\partial_z\partial_{\bar z}\delta(z-z_1)]. \end{aligned} \tag{4.23}$$

We also compute the two-point stress tensor correlators,

$$\begin{aligned} &\langle T_{zz}(z)T_{zz}(z_1)\rangle \\ &= \frac{-i[(1-A)\partial_\tau - A\partial_{\bar\tau}]\langle T_{zz}\rangle}{1-2\mathrm{Re}A} - \frac{(1-A)\langle T_{zz}\rangle}{1-2\mathrm{Re}A} \\ &\quad + \frac{(\Delta_A^2+\tilde A-\mathrm{Re}A)\mathrm{Im}(A\tau)}{4\pi\Delta_A|A|^2(1-2\mathrm{Re}A)\mathrm{Im}\tau}\Bigg\{\frac{1}{1+16\pi G y_c\langle T_{z\bar z}\rangle}\Bigg[2\langle T_{zz}\rangle\partial_{z_1} \\ &\quad + 16\pi G y_c(2\langle T_{z\bar z}\rangle\partial_{z_1} + \langle T_{zz}\rangle\partial_{\bar z_1}) - \frac{1}{16\pi G}\partial_{z_1}^3\Bigg]\partial_{z_1}\tilde G_\Omega(Z_1-Z) \\ &\quad - 2B\int d^2w\left[\left(\partial_w^2\tilde G_\Omega(W-Z) - \frac{2\pi\Delta_A|A|^2}{(\Delta_A^2+\tilde A-\mathrm{Re}A)\mathrm{Im}(A\tau)}\right)\langle O(w)T_{zz}(z_1)\rangle\right]\Bigg\}, \end{aligned}$$

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$$
\begin{aligned}
&\langle T_{zz}(z)T_{\bar{z}\bar{z}}(z_1)\rangle \\
&= \frac{-i[(1-A)\partial_\tau - A\partial_{\bar{\tau}}]\langle T_{\bar{z}\bar{z}}\rangle}{1-2\mathrm{Re}A} + \frac{(1-2\mathrm{Re}A)\langle T_{z\bar{z}}\rangle - A\langle T_{\bar{z}\bar{z}}\rangle}{\mathrm{Im}\tau(1-2\mathrm{Re}A)} \\
&\quad + \frac{(\Delta_A^2+\tilde{A}-\mathrm{Re}A)\mathrm{Im}(A\tau)}{4\pi\Delta_A|A|^2(1-2\mathrm{Re}A)\mathrm{Im}\tau}\Big\{\frac{1}{1+16\pi G y_c\langle T_{z\bar{z}}\rangle}\Big[16\pi G y_c\langle T_{z\bar{z}}\rangle(\langle T_{\bar{z}\bar{z}}\rangle\partial_{z_1}^2 \\
&\quad + \langle T_{z\bar{z}}\rangle\partial_{z_1}\partial_{\bar{z}_1} + \langle T_{zz}\rangle\partial_{\bar{z}_1}^2) + \frac{y_c}{4}\langle O\rangle^2\partial_{z_1}\partial_{\bar{z}_1} - \frac{1}{16\pi G}\partial_{z_1}^2\partial_{\bar{z}_1}^2\Big]\tilde{G}_\Omega(Z_1-Z) \\
&\quad - 2B\int d^2w\left[\left(\partial_w^2\tilde{G}_\Omega(W-Z) - \frac{2\pi\Delta_A|A|^2}{(\Delta_A^2+\tilde{A}-\mathrm{Re}A)\mathrm{Im}(A\tau)}\right)\langle O(w)T_{\bar{z}\bar{z}}(z_1)\rangle\right]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\langle T_{z\bar{z}}(z)T_{zz}(z_1)\rangle \\
&= \bar{A}\langle T_{zz}(z)T_{zz}(z_1)\rangle + A\langle T_{\bar{z}\bar{z}}(z)T_{zz}(z_1)\rangle + B\langle O(z)T_{zz}(z_1)\rangle \\
&\quad + \frac{1}{2(1+16\pi G y_c\langle T_{z\bar{z}}\rangle)}\left(\langle T_{zz}\rangle + \frac{1}{16\pi G}\partial_z^2\right)\delta(z-z_1),
\end{aligned}
$$

$$
\begin{aligned}
&\langle T_{z\bar{z}}(z)T_{z\bar{z}}(z_1)\rangle \\
&= \bar{A}\langle T_{zz}(z)T_{z\bar{z}}(z_1)\rangle + A\langle T_{\bar{z}\bar{z}}(z)T_{z\bar{z}}(z_1)\rangle + B\langle O(z)T_{z\bar{z}}(z_1)\rangle \\
&\quad - \frac{1}{2(1+16\pi G y_c\langle T_{z\bar{z}}\rangle)}\left(2\langle T_{z\bar{z}}\rangle - \frac{y_c}{4}\langle O\rangle^2 + \frac{1}{16\pi G}\partial_z\partial_{\bar{z}}\right)\delta(z-z_1),
\end{aligned}
$$

$$
\begin{aligned}
\langle T_{\bar{z}\bar{z}}(z)T_{\bar{z}\bar{z}}(z_1)\rangle &= \text{ c.c. of } \langle T_{zz}(z)T_{zz}(z_1)\rangle, \\
\langle T_{z\bar{z}}(z)T_{\bar{z}\bar{z}}(z_1)\rangle &= \text{ c.c. of } \langle T_{z\bar{z}}(z)T_{zz}(z_1).
\end{aligned} \tag{4.24}
$$

As a special case, we plug $A = -\frac{\pi^2 y_c}{1+2\pi^2 y_c}$ and $B = 0$ into (4.24) to derive the two-point stress tensor correlators in the cutoff thermal AdS_3 [20].

4.2 Three-point mixed correlators on thermal AdS_3 saddle

Let us now specify the saddle as thermal AdS_3 with the scalar field turned off,

$$
ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\Big[dZd\bar{Z} - \pi^2\rho(dZ^2+d\bar{Z}^2) + \pi^4\rho^2 dZd\bar{Z}\Big], \quad \phi(\rho,Z) = 0. \tag{4.25}
$$

The periods in Z are 1 and Ω. We introduce the following coordinate transformation,

$$
y = \frac{\rho}{(1-\pi^2\rho_c)^2}, \quad z = \frac{Z-\pi^2\rho_c\bar{Z}}{1-\pi^2\rho_c}, \quad \bar{z} = \frac{-\pi^2\rho_c Z + \bar{Z}}{1-\pi^2\rho_c}. \tag{4.26}
$$

The new FG coordinates $(y, z, \bar{z})$ are defined in a manner that ensures the boundary metric on $\partial\mathcal{M}_c$ takes on an Euclidean form. The periods in z are 1 and $\tau = \frac{\Omega-\pi^2\rho_c\bar{\Omega}}{1-\pi^2\rho_c}$. Combining (4.3)(4.22) we obtain the one-point correlators

$$
\begin{aligned}
\langle T_{zz}\rangle &= \frac{-\pi}{8G\sqrt{1+4\pi^2 y_c}}, \quad \langle T_{\bar{z}\bar{z}}\rangle = \frac{-\pi}{8G\sqrt{1+4\pi^2 y_c}}, \\
\langle T_{z\bar{z}}\rangle &= \frac{4\pi^3 y_c}{8G\sqrt{1+4\pi^2 y_c}\left(1+\sqrt{1+4\pi^2 y_c}\right)^2}, \quad \langle O\rangle = 0.
\end{aligned} \tag{4.27}
$$

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Plugging (4.27) into (4.23) and (4.24), we find that the two-point mixed correlators $\langle T_{ij}O\rangle$ vanish, and the stress tensor correlators $\langle T_{ij}T_{kl}\rangle$ consist with the results in pure gravity. We further compute the three-point mixed correlators $\langle T_{ij}OO\rangle$,

$$
\begin{aligned}
&\langle T_{zz}(z)O(z_1)O(z_2)\rangle \\
&= -\frac{i}{2}\partial_\tau\langle O(z_1)O(z_2)\rangle + \partial_{z_2}\left[\left(\frac{1+2\pi^2 y_c}{2\pi\sqrt{1+4\pi^2 y_c}}\partial_{z_2}\tilde{G}_\Omega(Z_2-Z) - i\frac{\mathrm{Im}z_2}{\mathrm{Im}\tau}\right)\langle O(z_1)O(z_2)\rangle\right] \\
&\quad + \frac{\pi y_c}{2\sqrt{1+4\pi^2 y_c}}\left[\frac{2\pi}{\sqrt{1+4\pi^2 y_c}}\left(\frac{i}{2}(\partial_\tau-\partial_{\bar\tau}) + i\frac{\mathrm{Im}z_2}{\mathrm{Im}\tau}(\partial_{z_2}-\partial_{\bar z_2}) + \frac{1}{\mathrm{Im}\tau}\right)\right. \\
&\quad \left. + \partial_{z_2}\tilde{G}_\Omega(Z_2-Z)\partial_{\bar z_2} - \partial_{\bar z_2}\tilde{G}_\Omega(Z_2-Z)\partial_{z_2}\right]\langle O(z_1)O(z_2)\rangle \\
&\quad + \frac{y_c \ln y_c}{2}\partial_{z_2}\partial_{\bar z_2}\left[\left(\frac{1}{\pi}\partial_{z_2}^2\tilde{G}_\Omega(Z_2-Z) - \frac{1}{\mathrm{Im}\tau\sqrt{1+4\pi^2 y_c}}\right)\langle O(z_1)O(z_2)\rangle\right] \\
&\quad - \frac{y_c}{8}\int d^2w\left[\left(\frac{1}{\pi}\partial_w^2\tilde{G}_\Omega(W-Z) - \frac{1}{\mathrm{Im}\tau\sqrt{1+4\pi^2 y_c}}\right)\langle O(w)O(z_1)\rangle\langle O(w)O(z_2)\rangle\right] \\
&\quad - \left(\partial_{\bar z_2} - \frac{\pi^2 y_c \ln y_c}{2\sqrt{1+4\pi^2 y_c}}\partial_{z_2}\right)\left[\left(\frac{1}{\pi}\partial_w^2\tilde{G}_\Omega(W-Z) - \frac{1}{\mathrm{Im}\tau\sqrt{1+4\pi^2 y_c}}\right)\partial_{z_2}\delta(z_1-z_2)\right] \\
&\quad + \frac{\pi^2 y_c \ln y_c}{2\sqrt{1+4\pi^2}}\partial_{\bar z_2}\left[\left(\frac{1}{\pi}\partial_w^2\tilde{G}_\Omega(W-Z) - \frac{1}{\mathrm{Im}\tau\sqrt{1+4\pi^2 y_c}}\right)\partial_{\bar z_2}\delta(z_1-z_2)\right] \\
&\quad - \frac{y_c(\ln y_c)^2}{2}\partial_{z_2}\partial_{\bar z_2}\left[\left(\frac{1}{\pi}\partial_w^2\tilde{G}_\Omega(W-Z) - \frac{1}{\mathrm{Im}\tau\sqrt{1+4\pi^2 y_c}}\right)\partial_{z_2}\partial_{\bar z_2}\delta(z_1-z_2)\right] + (z_1\leftrightarrow z_2),
\end{aligned}
$$

$$
\begin{aligned}
&\langle T_{z\bar z}(z)O(z_1)O(z_2)\rangle \\
&= -\frac{\pi^2 y_c}{2(1+2\pi^2 y_c)}\Big[\langle T_{zz}(z)O(z_1)O(z_2)\rangle + \langle T_{\bar z\bar z}(z)O(z_1)O(z_2)\rangle\Big] \\
&\quad + \frac{1}{2(1+2\pi^2 y_c)}\left[\sqrt{1+4\pi^2 y_c}\left[\frac{y_c}{4}\langle O(z)O(z_1)\rangle\langle O(z)O(z_2)\rangle - \partial_z\delta(z-x_1)\partial_{\bar z}\delta(z-z_2)\right.\right. \\
&\quad \left. - y_c\ln y_c\langle O(z)O(z_1)\rangle\partial_z\partial_{\bar z}\delta(z-z_2) + y_c(\ln y_c)^2\partial_z\partial_{\bar z}\delta(z-z_1)\partial_z\partial_{\bar z}\delta(z-z_2)\right] \\
&\quad \left. + \pi^2 y_c\ln y_c[\partial_z\delta(z-z_1)\partial_z\delta(z-z_2) + \partial_{\bar z}\delta(z-z_1)\partial_{\bar z}\delta(z-z_2)]\right] + (z_1\leftrightarrow z_2),
\end{aligned}
$$

$$
\langle T_{\bar z\bar z}(z)O(z_1)O(z_2)\rangle = \text{ c.c. of } \langle T_{zz}(z)O(z_1)O(z_2)\rangle. \tag{4.28}
$$

The calculation of $\langle O(z)O(z')\rangle$ can be accomplished by following the procedure in section 3.1.1. The radial coordinate x and the field redefinition $y^{[1]}_{m,n}$, introduced in (3.8), continue to be employed. The boundary conditions are modified to

$$
\begin{aligned}
y^{[1]}_{m,n}(x)\Big|_{x\to x_c} &= \frac{f_{m,n}}{x_c^{\frac{n}{2}}(1-x_c)}, \\
x^{\frac{n}{2}}y^{[1]}_{m,n}(x)\Big|_{x\to 0} &\quad \text{regular.}
\end{aligned} \tag{4.29}
$$

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The exact solution takes the form

$$y^{[1]}_{m,n}(x)=\begin{cases}\frac{f_{m,n}F(\alpha,\beta,\gamma,x)}{x_c^{\frac{n}{2}}(1-x_c)F(\alpha,\beta,\gamma,x_c)}, & n\geq 0,\\ \frac{f_{m,n}x^{-n}F(\alpha-n,\beta-n,1-n,x)}{x_c^{-\frac{n}{2}}(1-x_c)F(\alpha-n,\beta-n,1-n,x_c)}, & n<0.\end{cases} \tag{4.30}$$

It follows that

$$\Phi^{[1]}(x,\varphi,t)=\sum_{m,n=-\infty}^{+\infty}\left[\frac{x^{\frac{|n|}{2}}(1-x)F(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},x)}{x_c^{\frac{|n|}{2}}(1-x_c)F(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},x_c)}f_{m,n}e^{\frac{-imt}{T}}e^{in\varphi}\right], \tag{4.31}$$

where $\tilde{\alpha}=1+\frac{im}{2T}+\frac{|n|}{2}$, $\tilde{\beta}=1-\frac{im}{2T}+\frac{|n|}{2}$ and $\tilde{\gamma}=1+|n|$. Now, let us go back to the FG coordinate system $(y,z,\bar{z})$ via the following transformation,

$$x=\left(\frac{1-4\pi^2\left(1+\sqrt{1+4\pi^2y_c}\right)^{-2}y}{1+4\pi^2\left(1+\sqrt{1+4\pi^2y_c}\right)^{-2}y}\right)^2,\quad \varphi=\pi(z+\bar{z}),\quad t=-i\pi\frac{z-\bar{z}}{\sqrt{1+4\pi^2y_c}}. \tag{4.32}$$

The modular parameter of the cutoff torus $\tau=iT\sqrt{1+4\pi^2y_c}$. The first-order perturbed one-point correlator $\langle O\rangle^{[1]}$ is calculated using (4.3), and then the functional derivative w.r.t. $f(z')$ is taken to obtain the two-point scalar correlator,

$$\begin{aligned}&\langle O(z)O(z')\rangle\\ &=\frac{8\pi^2}{|\tau|}\sum_{m,n=-\infty}^{+\infty}\Big\{\frac{1}{1+4\pi^2y_c}\left[\frac{1}{4\pi^2y_c}-\frac{\tilde{\alpha}\tilde{\beta}F(\tilde{\alpha}+1,\tilde{\beta}+1,\tilde{\gamma}+1,\frac{1}{1+4\pi^2y_c})}{\tilde{\gamma}(1+4\pi^2y_c)F(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\frac{1}{1+4\pi^2y_c})}-\frac{|n|}{2}\right]\\ &\quad-\frac{1}{\sqrt{1+4\pi^2y_c}}\left(\frac{n^2}{4}+\frac{m^2}{4|\tau|^2}\right)\ln y_c\Big\}e^{i\pi\left(n-\frac{m}{\tau}\right)(z-z')}e^{i\pi\left(n+\frac{m}{\tau}\right)(\bar{z}-\bar{z}')}.\end{aligned} \tag{4.33}$$

To end this section, let us discuss the non-locality of holographic correlators. We consider the thermal AdS_3 saddle, whose scalar one-point correlator $\langle O\rangle=0$ and scalar two-point correlator is explicitly obtained in (4.33). As previously mentioned, the two-point mixed correlators $\langle T_{ij}O\rangle=0$, and the stress tensor correlators $\langle T_{ij}T_{kl}\rangle$ are consistent with the results obtained in the absence of a matter field [20]. Hence, no indication of non-locality can be discerned in these two-point correlators. Non-locality may potentially become evident in the higher-order correlators. For example, the three-point mixed correlator $\langle T_{zz}(z)O(z_1)O(z_2)\rangle$ involves an integral of the form

$$-\frac{y_c}{8}\int d^2w\left(\frac{1}{\pi}\partial_w^2\tilde{G}_\Omega(W-Z)-\frac{1}{\mathrm{Im}\tau\sqrt{1+4\pi^2y_c}}\right)\langle O(w)O(z_1)\rangle\langle O(w)O(z_2)\rangle, \tag{4.34}$$

which could suggest the non-locality of the correlator. Similar integrals can also be observed in the three-point stress tensor correlators. One can verify that, due to the vanishing of $\langle O\rangle$, the three-point stress tensor correlators are consistent with the results in pure gravity. For example, the three-point correlator $\langle T_{zz}(z)T_{zz}(z_1)T_{zz}(z_2)\rangle$ in a cutoff thermal AdS_3 is given by equation (C.22) in the previous work [20], which contains an integral

$$\begin{aligned}\pi Gy_c\int d^2w\left(\frac{1}{\pi}\partial_w^2\tilde{G}_\Omega(W-Z)-\frac{1}{\mathrm{Im}\tau\sqrt{1+4\pi^2y_c}}\right)\Big[&\langle T_{w\bar{w}}(w)T_{zz}(z_1)\rangle\langle T_{w\bar{w}}(w)T_{zz}(z_2)\rangle\\ &-\langle T_{ww}(w)T_{zz}(z_1)\rangle\langle T_{\bar{w}\bar{w}}(w)T_{zz}(z_2)\rangle+(z_1\leftrightarrow z_2)\Big].\end{aligned} \tag{4.35}$$

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It is noteworthy to mention that the techniques utilized in this section to compute the thermal AdS_3 correlators can also be applied to other gravitational saddles, such as the Euclidean BTZ black hole [67, 68]. Analogous terms as presented in (4.34) and (4.35) can also be found in the three-point correlators of these saddles.

5 Conclusions and perspectives

In this paper, we investigate the holographic torus correlators of the scalar operators both at conformal infinity and at a finite cutoff. The three-dimensional bulk in our study contains Einstein gravity and a free scalar field. Firstly, we employ the near-boundary analysis to solve Einstein's equation and calculate the mixed correlators in two cases: when the coupled scalar field is massless and massive. Additionally, we derive recurrence relations for a specific class of higher-point correlators. Our results are consistent with the Ward identity in torus CFT, providing concrete validation of AdS_3/CFT_2 with non-trivial topology. Secondly, to fully determine the holographic correlators, we explore the global solution of the massless scalar EOM and accurately compute the scalar two-point correlator on the thermal AdS_3 saddle. However, determining the higher-point scalar correlators remains unresolved. A robust algorithm for computing scalar correlators on a general saddle is still lacking.

Utilizing our results to compute the precise correlators on a non-trivial saddle would be interesting. A well-known AdS-sliced domain wall solution is the Janus solution [69, 70], and its holographic correlators are calculated in [28, 71]. The authors of [72, 73] investigate the AdS_3 Janus solution. We wonder whether there are analogous solutions in the bulk with a torus boundary. Another important direction is to introduce the scalar potential in the bulk. Exact bulk solutions in the presence of an exponential potential have been explored in numerous works [74–79]. Extending our study to the model that includes a general self-interacting potential of the scalar field is necessary.

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A Green's functions on torus

In this appendix, we give the definitions of Green's functions $G_\tau(z-w)$ and $\tilde{G}_\tau(z-w)$ on a torus with modular parameter τ. They satisfy the following differential equations,

$$\begin{aligned}\frac{1}{\pi}\partial_{\bar{z}}G_\tau(z-w) &= \delta(z-w) - \frac{1}{\mathrm{Im}\tau},\\ \frac{1}{\pi}\partial_z\partial_{\bar{z}}\tilde{G}_\tau(z-w) &= \delta(z-w) - \frac{1}{\mathrm{Im}\tau},\end{aligned} \tag{A.1}$$

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where $\delta(z-w)$ is the delta function with respect to the measure $d^2z = \frac{i}{2}dz \wedge d\bar{z}$. $G_\tau(z-w)$ can be represented by the Weierstrass ζ-function,

$$G_\tau(z-w) = \zeta_\tau(z-w) - 2\zeta_\tau\left(\frac{1}{2}\right)(z-w) + \frac{2\pi i}{\mathrm{Im}\tau}\mathrm{Im}(z-w). \tag{A.2}$$

$\zeta_\tau(z)$ is defined by a series expansion [80]

$$\zeta_\tau(z) = \frac{1}{z} + \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \left(\frac{1}{z-(m+n\tau)} + \frac{1}{(m+n\tau)} + \frac{z}{(m+n\tau)^2}\right). \tag{A.3}$$

The inclusion of the last two terms in (A.2) guarantees the double periodicity of $G_\tau(z-w)$. $\tilde{G}_\tau(z-w)$ can be represented by the Weierstrass σ-function,

$$\tilde{G}_\tau(z-w) = \ln(|\sigma_\tau(z-w)|^2) - \zeta_\tau\left(\frac{1}{2}\right)(z-w)^2 - \overline{\zeta_\tau\left(\frac{1}{2}\right)}(\bar{z}-\bar{w})^2 - \frac{2\pi}{\mathrm{Im}\tau}[\mathrm{Im}(z-w)]^2. \tag{A.4}$$

$\sigma_\tau(z)$ is defined as

$$\sigma_\tau(z) = z \prod_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \left(1 - \frac{z}{m+n\tau}\right) e^{\frac{z}{m+n\tau} + \frac{z^2}{2(m+n\tau)^2}}. \tag{A.5}$$

B Green's function $\mathscr{G}_{m,n}(x, x_0)$

In this section, we attempt to construct the exact Green's function for the hypergeometric equation, which is used to calculate the higher-order scalar coefficients in section 3.1.1. The Green's function $\mathscr{G}_{m,n}(x, x_0)$ satisfies the following differential equation

$$\Big(x(1-x)\partial_x^2 + [\gamma - (\alpha+\beta+1)x]\partial_x - \alpha\beta\Big)\mathscr{G}_{m,n}(x, x_0) = \delta(x-x_0), \tag{B.1}$$

where the parameters $\alpha = 1 + \frac{im}{2T} + \frac{n}{2}$, $\beta = 1 - \frac{im}{2T} + \frac{n}{2}$ and $\gamma = 1+n$ for $m, n \in \mathbb{Z}$. A particular solution to this differential equation can be written as

$$y^*_{m,n}(x, x_0) = \frac{y^{(3)}_{m,n}(x_0)y^{(4)}_{m,n}(x_0)}{\Delta(x_0)x_0(1-x_0)} H(x-x_0)\left(\frac{y^{(4)}_{m,n}(x)}{y^{(4)}_{m,n}(x_0)} - \frac{y^{(3)}_{m,n}(x)}{y^{(3)}_{m,n}(x_0)}\right), \tag{B.2}$$

where $\Delta = y^{(3)}_{m,n}(y^{(4)}_{m,n})' - y^{(4)}_{m,n}(y^{(3)}_{m,n})'$ is the Wronskian. $H(x-x_0)$ is the Heaviside step function defined as $H(x-x_0) = 1$ for $x > x_0$ and $H(x-x_0) = 0$ for $x \le x_0$. $y^{(3)}_{m,n}$ and $y^{(4)}_{m,n}$ are two independent solutions of the hypergeometric equation in the neighborhood of $x = 1$. Note that $\gamma - \alpha - \beta \equiv -1$, these two solutions take the forms

$$\begin{aligned} y^{(3)}_{m,n}(x) &= F(\alpha, \beta, 2, 1-x), \\ y^{(4)}_{m,n}(x) &= F(\alpha, \beta, 2, 1-x)\ln(1-x) + \frac{1}{(\alpha-1)(\beta-1)(1-x)} + \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{k!(2)_k} \\ &\quad \times [\psi(\alpha+k) + \psi(\beta+k) - \psi(2+k) - \psi(1+k)](1-x)^k, \end{aligned} \tag{B.3}$$

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where $F(\alpha,\beta,2,1-x)$ is the hypergeometric function, $(\lambda)_k=1$ for $k=0$ and $(\lambda)_k=\lambda(\lambda+1)\cdots(\lambda+k-1)$ for $k\geq 1$, and $\psi(\lambda)=\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$. If α and β are negative integers, which means that $m=0$ and $n=-4,-6,-8,\cdots$, then $\psi(1+\frac{n}{2}+k)$ diverges for $k\leq -1-\frac{n}{2}$. The second solution should be modified to

$$y^{(4)}_{0,n}(x)=F(1+\frac{n}{2},1+\frac{n}{2},2,1-x)\ln(1-x)+\frac{4}{n^2(1-x)}+\sum_{k=0}^{-1-\frac{n}{2}}\frac{\left[(-1-\frac{n}{2})_k\right]^2}{k!(2)_k}\times\left[2\psi\left(-k-\frac{n}{2}\right)-\psi(2+k)-\psi(1+k)\right](1-x)^k. \tag{B.4}$$

Obviously, the particular solution $y^*_{m,n}$ satisfies

$$\begin{aligned}(1-x)y^*_{m,n}(x,x_0)\Big|_{x\to 1}&=\frac{F(\alpha,\beta,2,1-x_0)}{\Delta(x_0)x_0(1-x_0)(\alpha-1)(\beta-1)},\\ x^{\frac{n}{2}}y^*_{m,n}(x,x_0)\Big|_{x\to 0}&=0.\end{aligned} \tag{B.5}$$

Let $y^{(1)}_{m,n}$ and $y^{(2)}_{m,n}$ be two independent solutions of the hypergeometric equation in the neighborhood of $x=0$. The general solution to (B.1) takes the form

$$\mathscr{G}_{m,n}(x,x_0)=A_{m,n}y^{(1)}_{m,n}(x)+B_{m,n}y^{(2)}_{m,n}(x)+y^*_{m,n}(x,x_0), \tag{B.6}$$

where $A_{m,n}$ and $B_{m,n}$ are undetermined constants. As we discussed above, the Green's function $\mathscr{G}_{m,n}$ satisfies regularity condition at $x=0$

$$x^{\frac{n}{2}}\mathscr{G}_{m,n}(x,x_0)\Big|_{x\to 0}\quad\text{regular}. \tag{B.7}$$

The asymptotic behaviors of the two solutions are $y^{(1)}_{m,n}\sim 1$, $y^{(2)}_{m,n}\sim x^{-n}$ for $n\geq 1$, $y^{(1)}_{m,n}\sim 1$, $y^{(2)}_{m,n}\sim \ln x$ for $n=0$, and $y^{(1)}_{m,n}\sim x^{-n}$, $y^{(2)}_{m,n}\sim 1$ for $n\leq -1$. Thus the regularity condition (B.7) requires that $B_{m,n}=0$. $A_{m,n}$ can be determined by the boundary condition of $\mathscr{G}_{m,n}$ at $x=1$

$$(1-x)\mathscr{G}_{m,n}(x,x_0)\Big|_{x\to 1}=0, \tag{B.8}$$

which is discussed as follows:

- For $n\geq 0$, the homogeneous solution $y^{(1)}_{m,n}$ takes the form

$$y^{(1)}_{m,n}(x)=F(\alpha,\beta,\gamma,x). \tag{B.9}$$

  The hypergeometric function can be expanded in the neighborhood of $x=1$

$$F(\alpha,\beta,\gamma,x)=\frac{\Gamma(\gamma)(1-x)^{-1}}{\Gamma(\alpha)\Gamma(\beta)}+\frac{\Gamma(\gamma)\ln(1-x)}{\Gamma(\alpha-1)\Gamma(\beta-1)}+O(1), \tag{B.10}$$

  where $O(1)$ contains the regular terms as $x\to 1$. Plugging (B.5)(B.6)(B.10) into (B.8) we obtain

$$A_{m,n}=-\frac{\Gamma(\alpha)\Gamma(\beta)F(\alpha,\beta,2,1-x_0)}{\Delta(x_0)x_0(1-x_0)(\alpha-1)(\beta-1)\Gamma(\gamma)}. \tag{B.11}$$

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- For $n<0$, the homogeneous solution $y^{(1)}_{m,n}$ takes the form

$$y^{(1)}_{m,n}(x)=x^{-n}F(\alpha-n,\beta-n,1-n,x). \tag{B.12}$$

The expansion of $y^{(1)}_{m,n}$ in the neighborhood of $x=1$ is

$$y^{(1)}_{m,n}(x)=\frac{\Gamma(1-n)(1-x)^{-1}}{\Gamma(\alpha-n)\Gamma(\beta-n)}+\frac{\Gamma(1-n)\ln(1-x)}{\Gamma(\alpha-n-1)\Gamma(\beta-n-1)}+O(1). \tag{B.13}$$

Plugging (B.5)(B.6)(B.13) into (B.8) we obtain

$$A_{m,n}=-\frac{\Gamma(\alpha-n)\Gamma(\beta-n)F(\alpha,\beta,2,1-x_0)}{\Delta(x_0)x_0(1-x_0)(\alpha-1)(\beta-1)\Gamma(1-n)}. \tag{B.14}$$

Finally, we can put everything together to obtain the explicit form of $\mathscr{G}_{m,n}(x,x_0)$. Note that (B.5) does not apply to the zero mode $y^*_{0,0}$, which satisfies $\lim_{x\to1}(1-x)y^*_{0,0}(x,x_0)=0$. The zero mode of Green's function can be written as

$$\mathscr{G}_{0,0}(x,x_0)=-H(x-x_0)\frac{(1-x_0)\ln x\ln\left(\frac{1-x}{1-x_0}\right)}{x_0\ln x_0(1-x)}. \tag{B.15}$$

C Review of holographic renormalization

To obtain the finite holographic correlators of boundary CFT, one needs to introduce additional boundary terms to cancel the divergence in the on-shell action. In what follows we will review this prescription in the context of AdS_3 coupled to a massless scalar field. The first step is to determine the coefficients in the FG expansions (2.6). In the FG coordinate system (2.4), Einstein's equations and scalar EOM can be rewritten as [12, 14]

$$\begin{aligned}
&\rho(2\partial_\rho^2 g_{ij}-2\partial_\rho g_{ik}g^{kl}\partial_\rho g_{lj}+g^{kl}\partial_\rho g_{kl}\partial_\rho g_{ij})+R_{(g)ij}-g^{kl}\partial_\rho g_{kl}g_{ij}+8\pi G\partial_i\phi\partial_j\phi=0,\\
&\nabla_{(g)i}(g^{kl}\partial_\rho g_{kl})-\nabla^k_{(g)}(\partial_\rho g_{ki})+16\pi G\partial_\rho\phi\partial_i\phi=0,\\
&g^{kl}\partial_\rho^2 g_{kl}-\frac{1}{2}g^{ij}\partial_\rho g_{jk}g^{kl}\partial_\rho g_{li}+16\pi G\partial_\rho\phi\partial_\rho\phi=0,\\
&\Box_{(g)}\phi+2\rho\partial_\rho\ln g\partial_\rho\phi+4\rho\partial_\rho^2\phi=0,
\end{aligned} \tag{C.1}$$

where we have used $m=0$ and $\phi(\rho,x)=\Phi(\rho,x)$. $R_{(g)ij}$ and $\nabla_{(g)i}$ indicate the Ricci tensor and the covariant derivative operator of the metric g_{ij} respectively. Plugging (2.6) into (C.1) we obtain

$$\begin{aligned}
h_{(2)ij}&=-4\pi G\left(\partial_i\phi_{(0)}\partial_j\phi_{(0)}-\frac{1}{2}g^{kl}_{(0)}\partial_k\phi_{(0)}\partial_l\phi_{(0)}g_{(0)ij}\right),\\
\psi_{(2)}&=-\frac{1}{4}\Box_{(0)}\phi_{(0)},\\
g^{ij}_{(0)}g_{(2)ij}&=\frac{1}{2}R_{(0)}+4\pi G g^{kl}_{(0)}\partial_k\phi_{(0)}\partial_l\phi_{(0)},\\
\nabla^j_{(0)}g_{(2)ij}&=\partial_i(g^{kl}_{(0)}g_{(2)kl})+16\pi G\phi_{(2)}\partial_i\phi_{(0)}.
\end{aligned} \tag{C.2}$$

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Here $R_{(0)}$ represents the Ricci scalar of the boundary metric $g_{(0)ij}$. The second step is to extract the divergent part of the on-shell action. We follow the regularization procedure in [12]. The bulk integral is restricted to the domain $\rho \in [\varepsilon, \pi^{-2}]$, and the boundary integral is evaluated at the finite cutoff $\rho = \varepsilon$. Plugging (2.2)(2.3) into (2.1) and using the FG coordinate system (2.4), we obtain the regularized on-shell action

$$I_{\text{reg}} = \frac{1}{8\pi G}\int_{\mathcal{M}_\varepsilon} d^2x d\rho \frac{1}{\rho^2}\sqrt{g} + \frac{1}{4\pi G}\int_{\partial\mathcal{M}_\varepsilon} d^2x \left[\partial_\varepsilon\sqrt{g} - \frac{1}{\varepsilon}\sqrt{g}\right]. \tag{C.3}$$

Plugging (2.6)(C.2) into (C.3) we have

$$I_{\text{reg}} = \frac{-1}{8\pi G}\int_{\partial\mathcal{M}_\varepsilon} d^2x\sqrt{g_{(0)}}\left[\frac{1}{\varepsilon} + \left(\frac{1}{4}R_{(0)} + 2\pi G g^{kl}_{(0)}\partial_k\phi_{(0)}\partial_l\phi_{(0)}\right)\ln\varepsilon\right] + O(1). \tag{C.4}$$

The last step is to construct the renormalized on-shell action. The counterterm I_{ct} is to cancel the divergent part of (C.4),

$$I_{\text{ct}} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}_\varepsilon} d^2x\sqrt{g_{(0)}}\left[\frac{1}{\varepsilon} + \left(\frac{1}{4}R_{(0)} + 2\pi G g^{kl}_{(0)}\partial_k\phi_{(0)}\partial_l\phi_{(0)}\right)\ln\varepsilon\right] + \text{regular part}. \tag{C.5}$$

The selection of counterterm is not unique, since the regular parts of two allowed counterterms can be different. The authors of [14] (see also [25]) provide a method to obtain an explicit counterterm. This method requires us to invert the FG expansions (2.6). Plugging the inverted expansions and the coefficients (C.2) into (C.5), we can rewrite $g_{(0)ij}$, $R_{(0)}$ and $\phi_{(0)}$ in terms of g_{ij}, $R_{(g)}$ and ϕ. We only keep the terms that diverge as $\varepsilon \to 0$, and finally, the counterterm takes the form

$$I_{\text{ct}} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}_\varepsilon} d^2x\sqrt{\gamma}\left[1 + \left(\frac{1}{4}R_{(\gamma)} + 2\pi G\gamma^{kl}\partial_k\phi\partial_l\phi\right)\ln\varepsilon\right], \tag{C.6}$$

where $\gamma_{ij}(\varepsilon, x) = \frac{1}{\varepsilon}g_{ij}(\varepsilon, x)$ is the induced metric on the cutoff surface $\partial\mathcal{M}_\varepsilon$. The renormalized on-shell action $I_{\text{ren}} = I_{\text{reg}} + I_{\text{ct}}$, which is finite as $\varepsilon \to 0$.

The holographic one-point correlators are defined as the functional derivatives of renormalized on-shell action,

$$\begin{aligned}
\langle T_{ij}\rangle &= \lim_{\varepsilon\to 0}\left(\frac{2}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma^{ij}}(I_{\text{reg}} + I_{\text{ct}})\right) \\
&= \frac{1}{8\pi G}\left(g_{(2)ij} - 4\pi G\partial_i\phi_{(0)}\partial_j\phi_{(0)} - 2\pi G\partial^k\phi_{(0)}\partial_k\phi_{(0)}g_{(0)ij} - \frac{1}{2}R_{(0)}g_{(0)ij}\right), \\
\langle O\rangle &= \lim_{\varepsilon\to 0}\left(\frac{-1}{\varepsilon\sqrt{\gamma}}\frac{\delta}{\delta\phi}(I_{\text{reg}} + I_{\text{ct}})\right) \\
&= 2\left(\phi_{(2)} - \frac{1}{4}\Box_{(0)}\phi_{(0)}\right).
\end{aligned} \tag{C.7}$$

The indices i, j, k, l are raised by $g^{ij}_{(0)}$. In (C.7), we use the fact that $\int_{\partial\mathcal{M}_\varepsilon} d^2x\sqrt{g}R_{(g)}$ is a topological invariant. Plugging (C.7) into (C.2), we obtain the trace relation (2.8) and the conservation equation (2.9).

In the preceding discussion, we assumed the scalar field to be massless. However, for the massive bulk scalar, the FG expansion may differ from the form given in eq. (2.6). Nevertheless, if an exact FG expansion exists, the counterterm can still be constructed using the procedure outlined above. In appendix D, we will demonstrate holographic renormalization for a general rational number Δ.

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D Holographic renormalization for massive scalar field coupling

In this appendix, we will construct the counterterm in the case of gravity coupled to a massive scalar field. To begin with, let us assume that $N \geq 2$ (where $\frac{2-\Delta}{2} = \frac{N}{M}$ for a rational scaling dimension Δ). The FG expansions of the bulk fields are shown in (3.72). We first compute the coefficients $g_{(2m\frac{N}{M})ij}$ and $\phi_{(2m\frac{N}{M})}$ for $1 \leq m \leq \lfloor \frac{M}{N} \rfloor$. Plugging (3.72) into (3.70) and extracting the coefficients of order $\rho^{m\frac{N}{M}-1}$, we obtain

$$\begin{aligned}
&2m\frac{N}{M}\left(m\frac{N}{M}-1\right)g_{(2m\frac{N}{M})ij} + F_{(2m\frac{N}{M})ij} + \frac{1}{2\left(m\frac{N}{M}-2\right)}g^{kl}_{(0)}F_{(2m\frac{N}{M})kl}g_{(0)ij} = 0,\\
&2m\frac{N}{M}\left[(m+2)\frac{N}{M}-1\right]\phi_{(2m\frac{N}{M})} + m\frac{N^2}{M^2}\mathrm{ln}g_{(2m\frac{N}{M})}\phi_{(0)} + G_{(2m\frac{N}{M})} = 0,
\end{aligned} \tag{D.1}$$

where $F_{(2m\frac{N}{M})ij}$ and $G_{(2m\frac{N}{M})}$ can be expressed by the lower-order coefficients,

$$\begin{aligned}
F_{(2m\frac{N}{M})ij} = &\sum_{\substack{m_1+m_2+m_3=m\\ m_i\in\{0,1,\cdots,m-1\}}}\Bigg[m_1m_3\frac{N^2}{M^2}\Big(g_{(2m_1\frac{N}{M})kl}g^{kl}_{(2m_2\frac{N}{M})}g_{(2m_3\frac{N}{M})ij}\\
&-2g_{(2m_1\frac{N}{M})ik}g^{kl}_{(2m_2\frac{N}{M})}g_{(2m_3\frac{N}{M})lj}\Big) - m_2\frac{N}{M}g_{(2m_1\frac{N}{M})kl}g^{kl}_{(2m_2\frac{N}{M})}g_{(2m_3\frac{N}{M})ij}\Bigg]\\
&-32\pi G\frac{N}{M}\left(1-\frac{N}{M}\right)\sum_{\substack{m_1+m_2+m_3=m-2\\ m_i\in\{0,1,\cdots,m-2\}}}\phi_{(2m_1\frac{N}{M})}\phi_{(2m_2\frac{N}{M})}g_{(2m_3\frac{N}{M})ij},\\
G_{(2m\frac{N}{M})} = &\sum_{\substack{m_1+m_2=m\\ m_i\in\{1,2,\cdots,m-1\}}} m_1(1+m_2)\frac{N^2}{M^2}\mathrm{ln}g_{(2m_1\frac{N}{M})}\phi_{(2m_2\frac{N}{M})}.
\end{aligned} \tag{D.2}$$

For $m = 1$ we have $g_{(2\frac{N}{M})ij} = 0$ and $\phi_{(2\frac{N}{M})} = 0$. For $m = 2$ we have

$$\begin{aligned}
g_{(4\frac{N}{M})ij} &= -4\pi G\phi^2_{(0)}g_{(0)ij},\\
\phi_{(4\frac{N}{M})} &= \frac{4\pi G}{4-\frac{M}{N}}\phi^3_{(0)}.
\end{aligned} \tag{D.3}$$

By applying (D.1) and (D.2) multiple times, we find that for $3 \leq m \leq \lfloor \frac{M}{N} \rfloor$, the coefficients of the metric and the scalar field take the forms

$$g_{(2m\frac{N}{M})ij} = (\pi G)^{\frac{m}{2}}A_{(2m\frac{N}{M})}\phi^m_{(0)}g_{(0)ij}, \tag{D.4}$$

$$\phi_{(2m\frac{N}{M})} = (\pi G)^{\frac{m}{2}}B_{(2m\frac{N}{M})}\phi^{m+1}_{(0)}, \tag{D.5}$$

where $A_{(2m\frac{N}{M})}$ and $B_{(2m\frac{N}{M})}$ are some constants. Since we consider the gravity coupled to a free scalar field, the bulk action (2.1) is invariant under $\Phi \to -\Phi$, and thus $A_{(2m\frac{N}{M})} = B_{(2m\frac{N}{M})} = 0$ for m is an odd number [37]. Plugging (3.72) into (3.70) and extracting the coefficients of order $\mathrm{ln}\rho$, we obtain

$$g^{ij}_{(0)}h_{(2)ij} = 0, \qquad \psi_{(2)} = 0. \tag{D.6}$$

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Extracting the coefficients of order ρ^0, we obtain

$$\begin{aligned}
h_{(2)ij} &= 0, \\
g^{ij}_{(0)} g_{(2)ij} &= \frac{1}{2} R_{(0)} - 64\pi G \frac{N}{M}\left(1-\frac{N}{M}\right)\phi_{(0)}\phi_{\left(2-4\frac{N}{M}\right)}, \\
\nabla^k_{(0)} g_{(2)ki} &= \partial_i (g^{kl}_{(0)} g_{(2)kl}) + 16\pi G\left[\frac{N}{M}\phi_{(0)}\partial_i\phi_{\left(2-4\frac{N}{M}\right)} + \left(1-\frac{N}{M}\right)\phi_{\left(2-4\frac{N}{M}\right)}\partial_i\phi_{(0)}\right].
\end{aligned} \tag{D.7}$$

Following the spirit of [14], we plug the FG expansions into the bulk on-shell action, and evaluate the boundary integral at a cutoff surface $\partial\mathcal{M}_\varepsilon$,

$$\begin{aligned}
I_{\text{reg}} &= \frac{1}{8\pi G}\int_{\mathcal{M}_\varepsilon} d^2x d\rho\sqrt{g}\left[\frac{1}{\rho^2} + 2\frac{N}{M}\left(1-\frac{N}{M}\right)\frac{8\pi G\phi^2}{\rho^{2-2\frac{N}{M}}}\right] + \frac{1}{16\pi G}\int_{\partial\mathcal{M}_\varepsilon} d^2x \frac{4}{\rho}\left[\rho\partial_\rho\sqrt{g} - \sqrt{g}\right] \\
&= -\frac{1}{8\pi G}\int_{\partial\mathcal{M}_\varepsilon} d^2x\sqrt{g_{(0)}}\left[\frac{1}{\varepsilon} - \frac{1}{2}\left(1-2\frac{N}{M}\right)\frac{8\pi G\phi^2_{(0)}}{\varepsilon^{1-2\frac{N}{M}}} + \sum_{m=3}^{\lfloor\frac{M}{N}\rfloor}\frac{(\pi G)^{\frac{m}{2}} C_{\left(2m\frac{N}{M}\right)}\phi^m_{(0)}}{\varepsilon^{1-m\frac{N}{M}}}\right. \\
&\quad \left. + \frac{1}{4}R_{(0)}\ln\varepsilon\right] + O(1).
\end{aligned} \tag{D.8}$$

Plugging the inverted FG expansions into (D.8) and keeping the terms that diverge as $\varepsilon \to 0$, we obtain the allowed counterterm (3.74) for $N \geq 2$.

For the case $N = 1$, the two homogeneous solutions of the scalar field overlap on the order $\rho^{1-\frac{2}{M}}$. Plugging the FG expansions (3.82) into (3.70), we find that (D.4) holds for $2 \leq m \leq M-1$, and (D.5) holds for $1 \leq m \leq M-3$. The order $\rho^{-\frac{2}{M}}$ of the scalar EOM gives the scalar conformal anomaly,

$$\begin{aligned}
\psi_{\left(2-\frac{4}{M}\right)} &= -\frac{M}{2(M-2)}\left[\frac{M-2}{M^2}\ln g_{\left(2-\frac{4}{M}\right)}\phi_{(0)} + G_{\left(2-\frac{4}{M}\right)}\right] \\
&\propto (\pi G)^{\frac{M-2}{2}}\phi^{M-1}_{(0)}.
\end{aligned} \tag{D.9}$$

The order $\ln\rho$ of Einstein's equations gives the trace part of the metric conformal anomaly,

$$\begin{aligned}
g^{ij}_{(0)} h_{(2)ij} &= -64\pi G\frac{M-1}{M^2}\psi_{\left(2-\frac{4}{M}\right)}\phi_{(0)} \\
&\propto (\pi G)^{\frac{M}{2}}\phi^M_{(0)}.
\end{aligned} \tag{D.10}$$

The order ρ^0 of Einstein's equations gives

$$\begin{aligned}
h_{(2)ij} &= -32\pi G\frac{M-1}{M^2}\psi_{\left(2-\frac{4}{M}\right)}\phi_{(0)} g_{(0)ij}, \\
g^{ij}_{(0)} g_{(2)ij} &= \frac{1}{2}R_{(0)} + \frac{1}{2}g^{ij}_{(0)}F_{(2)ij}, \\
\nabla^k_{(0)} g_{(2)ki} &= \partial_i(g^{kl}_{(0)} g_{(2)kl}) + 16\pi G\left[\frac{2-2M}{M^2}\phi_{(0)}\partial_i\psi_{\left(2-\frac{4}{M}\right)} + \frac{M^2-2M+2}{M^2}\psi_{\left(2-\frac{4}{M}\right)}\partial_i\phi_{(0)}\right] \\
&\quad + H_{(2)i} + 16\pi G\left[\frac{1}{M}\phi_{(0)}\partial_i\phi_{\left(2-\frac{4}{M}\right)} + \frac{M-1}{M}\phi_{\left(2-\frac{4}{M}\right)}\partial_i\phi_{(0)}\right],
\end{aligned} \tag{D.11}$$

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where

$$H_{(2)i} = \sum_{\substack{m_1+m_2=M \\ m_i\in\{1,2,\cdots,M-1\}}} \frac{m_2}{M}\left[\partial_i\left(g^{kl}_{(\frac{2m_1}{M})}g_{(\frac{2m_2}{M})kl}\right) - \nabla^k_{(\frac{2m_1}{M})}g_{(\frac{2m_2}{M})ki}\right]$$
$$+ 16\pi G \sum_{\substack{m_1+m_2=M-2 \\ m_i\in\{1,2,\cdots,M-3\}}} \frac{m_1+1}{M}\phi_{(\frac{2m_1}{M})}\partial_i\phi_{(\frac{2m_2}{M})}$$
$$\propto (\pi G)^{\frac{M}{2}}\phi_{(0)}^{M-1}\partial_i\phi_{(0)}. \tag{D.12}$$

Plugging the FG expansions (3.82) into the regularized on-shell action, we obtain

$$I_{\text{reg}} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}_\varepsilon} d^2x\sqrt{g_{(0)}}\left[\frac{1}{\varepsilon} - \frac{1}{2}\left(1-\frac{2}{M}\right)\frac{8\pi G\phi_{(0)}^2}{\varepsilon^{1-\frac{2}{M}}} + \sum_{m=3}^{M-1}\frac{(\pi G)^{\frac{m}{2}}D_{(\frac{2m}{M})}\phi_{(0)}^m}{\varepsilon^{1-\frac{m}{M}}}\right.$$
$$\left. + \frac{1}{4}\left[R_{(0)} + (\pi G)^{\frac{M}{2}}D_{(2)}\phi_{(0)}^M\right]\ln\varepsilon\right] + O(1). \tag{D.13}$$

Using the invert FG expansion, we can rewrite $g_{(0)ij}$ and $\phi_{(0)}$ in terms of g_{ij} and ϕ, and finally obtain the allowed counterterm (3.83) for $N = 1$.

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