Twisted ambidexterity in equivariant homotopy theory: Two approaches

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Bastiaan Cnossen

aus

Leiden, Niederlande

Bonn, September 2023

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

Gutachter und Betreuer: Prof. Dr. Stefan Schwede

Gutachter: Prof. Dr. Marc Hoyois

Tag der Promotion: September 20, 2023

Erscheinungsjahr: 2024

Contents

Abstract

This dissertation explores the phenomenon of twisted ambidexterity in equivariant stable homotopy theory for compact Lie groups, which encompasses, and sheds a new light on, equivariant Atiyah duality and the Wirthmüller isomorphism.

In Part I, we take a homotopy-theoretic approach, defining twisted ambidexterity in a general parametrized setup via a form of assembly map $f_!(- \otimes D_f) \to f_*$. When applied to equivariant homotopy theory for a compact Lie group G, we show that parametrized genuine G-spectra form the universal theory of stable G-equivariant objects which satisfy twisted ambidexterity for the orbits G/H. In simple terms, this says that genuine equivariant spectra differ from naive equivariant spectra only by the existence of Wirthmüller isomorphisms.

In Part II, we take a differential-geometric approach, following ideas from motivic homotopy theory. We introduce for every separated differentiable stack $\mathcal{X}$ an ∞-category $\mathrm{SH}(\mathcal{X})$ of genuine sheaves of spectra on $\mathcal{X}$, which for a smooth manifold returns ordinary sheaves of spectra and for the classifying stack of a compact Lie group returns genuine equivariant spectra. We prove a form of relative Poincaré duality in this setting: for a proper representable submersion f of separated differentiable stacks, there is an equivalence $f_\sharp \simeq f_*(- \otimes S^{T_f})$ between its relative homology and a twist of its relative cohomology by the relative tangent sphere bundle. When specialized to quotient stacks of equivariant smooth manifolds, this recovers both equivariant Atiyah duality and the Wirthmüller isomorphism in stable equivariant homotopy theory.

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my advisor, Stefan Schwede, for taking me under his wing and for providing invaluable guidance and support throughout my doctoral research.

I am further deeply indebted to Marc Hoyois, who provided me with several crucial insights without which Part II of this dissertation would not have been written.

It is a pleasure to thank Shachar Carmeli for countless fruitful conversations, which have taught me a great deal of mathematics and have shaped me as a mathematician. I am further grateful to Marco Volpe for answering a variety of questions concerning sheaves and six-functor formalisms. Thanks should also go to Louis Martini and Sebastian Wolf for their explanations regarding parametrized category theory.

Special thanks go to Sil Linskens for the three pleasant years we shared an office in Bonn, during which we had innumerable conversations and discussions, and for providing helpful feedback on earlier versions of this dissertation.

Further thanks go to my collaborators: Shachar Carmeli, Tobias Lenz, Sil Linskens, Maxime Ramzi and Lior Yanovski. The collaborative process has been joyful and stimulating, and has taught me various valuable skills concerning writing and communicating mathematics.

I would like to acknowledge the following people with whom I have had useful interactions related to the contents of this dissertation: Shay Ben-Moshe, Robert Burklund, Tim Campion, Dustin Clausen, Adrian Clough, Peter Haine, Fabian Hebestreit, Kaif Hilman, Branko Juran, Shai Keidar, Adeel Khan, Dominik Kirstein, Christian Kremer, Jonas McCandless, Maarten Mol, Florian Naef, Irakli Patchkoria, Dmitri Pavlov and Michael Stahlhauer.

In the typesetting of commutative diagrams, I have frequently made use of the quiver web editor developed by Nathanael Arkor.

Last but not least, I would like to thank my family and friends for their moral support during these four years of research.

Introduction

The goal of this dissertation is to study the phenomenon of twisted ambidexterity in equivariant stable homotopy theory, both from a homotopical perspective and as well as from a differential-geometric perspective.

Twisted ambidexterity

We use the term 'twisted ambidexterity' to refer to a general phenomenon in which a certain homology/colimit functor agrees 'up to a twist' with the corresponding cohomology/limit functor. Several well-known results in algebraic topology and homotopy theory can be regarded as instances of this phenomenon:

(1) *Twisted Poincaré duality*: for a compact smooth manifold X of dimension n, there is an isomorphism between the cohomology groups $H^k(X;\mathbb{Z})$ of X and the (shifted) homology groups $H_{n-k}(X;O_X)$ of X with local coefficients in the orientation sheaf O_X of X;

(2) *Atiyah duality*: for a compact smooth manifold X, the Spanier-Whitehead dual of the suspension spectrum $\Sigma^\infty_+ X$ is equivalent to the Thom spectrum X^{-TX} of the stable normal bundle of X, see [Ati61];

(3) *Klein's dualizing spectrum*: if X is a compact space and $E \in \mathrm{Sp}^X$ is a spectrum parametrized over X, there is an equivalence $\mathrm{Nm}_X \colon \mathrm{colim}_X(E \otimes D_X) \xrightarrow{\sim} \lim_X E$ of spectra, where $D_X \in \mathrm{Sp}^X$ is the dualizing spectrum of X due to Klein [Kle01];

(4) *Tate vanishing*: for a finite group G and a $K(n)$-local spectrum E with a G-action, there is an equivalence $\mathrm{Nm}\colon E_{hG} \xrightarrow{\sim} E^{hG}$ in $\mathrm{Sp}_{K(n)}$, see [GS96; HS96];

(5) *Ambidexterity in chromatic homotopy theory:* for a π-finite space X and an X-indexed family of $K(n)$-local spectra $E \in \mathrm{Sp}^X_{K(n)}$, there is an equivalence $\mathrm{Nm}_X\colon \mathrm{colim}_X E \xrightarrow{\sim} \lim_X E$, see [HL13];

(6) An *equivariant* version of Atiyah duality: for a compact Lie group G and a compact smooth G-manifold X, the dual of the suspension spectrum $\Sigma^\infty_+ X$ in Sp_G is the Thom spectrum X^{-TX};

(7) The *Wirthmüller isomorphism* in stable equivariant homotopy theory: for a compact Lie group G, a closed subgroup H and a genuine H-spectrum E, there is an equivalence of genuine G-spectra $\mathrm{ind}_H^G(E \otimes S^{-L}) \xrightarrow{\sim} \mathrm{coind}_H^G(E)$, see [Wir74];

(8) *Ambidexterity in motivic homotopy theory*: for a proper smooth morphism $f\colon X \to Y$ of schemes over some base scheme S, there is an equivalence $f_\sharp \simeq f_*\Sigma^{T_f}$ of functors $\mathrm{SH}(X) \to \mathrm{SH}(Y)$, see [Ayo07, Théorème 1.7.17].

Although seemingly unrelated at first, there is a common pattern to these examples. In each case, one is given a functor $X \mapsto C(X)$ which assigns to a certain type of geometric object X a stable ∞-category $C(X)$, thought of as a choice of 'parametrized spectra over X'. The map from X to the point induces a symmetric monoidal functor $C(\mathrm{pt}) \to C(X)$ which admits both a left adjoint L as well as a right adjoint R; we want to think of L and R as the 'homology' and 'cohomology' of X in the context of C. Each of the above equivalences can be interpreted as an identification $L(- \otimes D_X) \xrightarrow{\sim} R(-)$ of these two functors up to a 'twist' by some dualizing object $D_X \in C(X)$. We regard these equivalences as twisted analogues of the ambidexterity equivalences from examples (4) and (5), which explains our choice of terminology.

The aforementioned examples of twisted ambidexterity can be divided into two classes, which one could think of as 'homotopical twisted ambidexterity' and 'geometric twisted ambidexterity':

(I) Examples (3)-(5) are of homotopy-theoretic nature: the objects X only carry homotopical information, and the twisted norm maps are produced using higher categorical manipulations. This provides these maps with good formal properties, making it easy to relate them to various categorical duality phenomena. A disadvantage of this approach is that it does not provide much control over the resulting dualizing objects.

(II) Examples (1)-(2) and (6)-(8) are of differential-geometric nature: the objects X come equipped with geometric structure, and the equivalences are implemented geometrically via a form of Pontryagin-Thom construction. As a result, one obtains an explicit description of the dualizing object in terms of the tangent bundle of X.

The goal of this dissertation is to provide a detailed study of these two approaches to twisted ambidexterity in the context of stable equivariant homotopy theory. Part I is devoted to a study of approach (I), while Part II deals with approach (II). These two parts can be read independently, up to a few sporadic cross-references: they are written as individual articles and come with their own abstracts and introductions. Part III provides an outlook for future research and explores the relationship between approaches (I) and (II).

We shall now provide a broad overview of the contents of these two parts, referring to the technical summary for more precise statements of the results.

Part I: Twisted ambidexterity in equivariant homotopy theory

In Part I of this dissertation, we introduce a generic formulation of twisted ambidexterity in terms of parametrized category theory which covers the homotopical examples of twisted ambidexterity. We will be primarily interested in the context of equivariant stable homotopy theory for a compact Lie group G, where one assigns to a G-space X the ∞-category Sp_G^X of *genuine G-spectra parametrized over X*, in the spirit of May and Sigurdsson [MS06]. The general framework then provides for every morphism $f\colon X \to Y$ of G-spaces a *twisted norm map*

$$\mathrm{Nm}_f\colon f_!(- \otimes D_f) \xrightarrow{\sim} f_*(-)$$

of functors $\mathrm{Sp}_G^X \to \mathrm{Sp}_G^Y$, where $D_f \in \mathrm{Sp}_G^X$ is the *relative dualizing spectrum* of f, and $f_!$ and f_* denote the left and right adjoint, respectively, to the pullback functor $f^*\colon \mathrm{Sp}_G^Y \to \mathrm{Sp}_G^X$. It follows from results of [MS06] that the map Nm_f is an equivalence whenever f has compact fibers. For example, when Y is a point and $X = G/H$ is an orbit associated to some closed subgroup $H \leqslant G$, the resulting equivalence is the well-known *Wirthmüller isomorphism* $\mathrm{ind}_H^G(- \otimes S^{-L}) \simeq \mathrm{coind}_H^G$, due to [Wir74]. It is known since the work of Blumberg [Blu06] that the Wirthmüller isomorphisms are in some sense the only difference between genuine G-spectra and the so-called 'naive G-spectra', which do not support Wirthmüller isomorphisms. The main result of Part I of this dissertation is another manifestation of this philosophy:

> The ∞-categories Sp_G^X of parametrized genuine G-spectra over G-spaces X constitute the universal theory of stable G-equivariant objects which supports formal Wirthmüller isomorphisms.

A precise formulation of this result requires the language of parametrized category theory and will be given in the technical summary below, see Theorem 2.

Part II: Relative Poincaré duality for differentiable stacks

In Part II of this dissertation, we study a differential-geometric form of twisted ambidexterity in equivariant homotopy theory, which we refer to as 'relative Poincaré duality'.

Given a compact Lie group G, we will introduce for every smooth G-manifold an ∞-category $\mathrm{SH}_G(M)$ of *genuine sheaves of G-spectra on M*. When $M = G/H$ is an orbit for some closed subgroup $H \leqslant G$, this ∞-category is equivalent to the ∞-category Sp_H of genuine H-spectra. When G is the trivial group, it is equivalent to the ∞-category $\mathrm{Shv}(M;\mathrm{Sp})$ of ordinary sheaves of spectra on M.

The map from M to the point induces a symmetric monoidal functor $\mathrm{Sp}_G = \mathrm{SH}_G(\mathrm{pt}) \to \mathrm{SH}_G(M)$. The left and right adjoints of this functor assign to a genuine sheaf of G-spectra on M its *genuine equivariant sheaf homology/cohomology*. We will prove a version of equivariant Poincaré duality for these functors: when M is compact, these two adjoints agree up to tensoring with the tangent sphere bundle S^{T_M} of M. More generally, we will prove a *relative* version of this result, where the compact G-manifold M gets replaced by a G-equivariant proper smooth submersion $f\colon M \to N$. The pullback functor $f^*\colon \mathrm{SH}_G(N) \to \mathrm{SH}_G(M)$ at the level of genuine sheaves admits both a left adjoint $f_\sharp$ as well as a right adjoint f_*, thought of as the *relative* homology and cohomology of f. Relative Poincaré duality is the statement that there is an equivalence of the form $f_\sharp(-) \simeq f_*(- \otimes S^{T_f})$, where $S^{T_f} \in \mathrm{SH}_G(M)$ is the one-point compactification of the relative tangent bundle of f.

By construction, the ∞-category $\mathrm{SH}_G(M)$ of genuine sheaves on M will only depend on the quotient stack $M/\!/G$ of M: we will introduce for every separated differentiable stack $\mathcal{X}$ an ∞-category $\mathrm{SH}(\mathcal{X})$ of *genuine sheaves of spectra on $\mathcal{X}$*, which specializes to $\mathrm{SH}_G(M)$ when $\mathcal{X}$ is the quotient stack $M/\!/G$. Our main result of Part II is the following version of relative Poincaré duality in this context:

> For a proper representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of separated differentiable stacks, there is an equivalence $\mathfrak{p}_f\colon f_\sharp(-) \xrightarrow{\sim} f_*(- \otimes S^{T_f})$ of functors $\mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$.

The proof of this result is an adaptation of the usual proof of Atiyah duality for compact smooth manifolds in the context of differentiable stacks. The construction of the ∞-category $\mathrm{SH}(\mathcal{X})$ and many of its formal properties are direct analogues of similar constructions in motivic homotopy theory, and the proof strategy of our main result closely follows that of Hoyois [Hoy17] in the motivic setting.

Technical summary

We will now provide a detailed overview of the results contained in this dissertation.

Part I: Twisted ambidexterity in equivariant homotopy theory

Chapter I.2: Parametrized category theory

In Chapter I.2, we recall the setup of parametrized (higher) category theory over an ∞-topos $\mathcal{B}$ and establish some analogues of well-known results in non-parametrized category theory.

We start in Section I.2.1 with recollections, following Martini and Wolf [Mar21; MW21; MW22]. The central notion is that of a *presentably symmetric monoidal $\mathcal{B}$-category*, Definition I.2.15, which is a limit-preserving functor $\mathcal{C}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ satisfying the condition that for all morphisms $f : A \to B$ in $\mathcal{B}$ the restriction functors $f^* := \mathcal{C}(f)$: $\mathcal{C}(B) \to \mathcal{C}(A)$ admit left adjoints $f_! : \mathcal{C}(A) \to \mathcal{C}(B)$ which satisfy *base change* and the *projection formula*. For an object $A \in \mathcal{B}$, we may form the cotensor $\mathcal{C}^A$ of $\mathcal{C}$ by A, given by $\mathcal{C}^A(B) := \mathcal{C}(A \times B)$. The main result of Section I.2.2 is the following classification of $\mathcal{C}$-linear $\mathcal{B}$-functors $\mathcal{C}^A \to \mathcal{C}^B$:

Theorem 1 (Theorem I.2.32)**.** *For objects $A, B \in \mathcal{B}$, there is an equivalence of ∞-categories $\mathcal{C}(A \times B) \xrightarrow{\sim} \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{C}^B)$ which sends an object $D \in \mathcal{C}(A \times B)$ to the $\mathcal{C}$-linear $\mathcal{B}$-functor $(\mathrm{pr}_B)_!(\mathrm{pr}_A^*(-) \otimes D)\colon \mathcal{C}^A \to \mathcal{C}^B$, where $\mathrm{pr}_A\colon A \times B \to A$ and $\mathrm{pr}_B\colon A \times B \to B$ are the projections.*

In Section I.2.3 we give two instances in which we can prove the existence of the *formal inversion* $\mathcal{C}[S^{-1}]$ of a small collection of objects S in a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$, that is, the presentably symmetric monoidal $\mathcal{B}$-category obtained from $\mathcal{C}$ by universally inverting the objects in S with respect to the monoidal structure.

Chapter I.3: Twisted ambidexterity

In Chapter I.3 we introduce the notion of *twisted ambidexterity* for a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$. Given an object $A \in \mathcal{B}$, the diagonal $\mathcal{B}$-functor $A^*: \mathcal{C} \to \mathcal{C}^A$ admits both a left adjoint $A_!: \mathcal{C}^A \to \mathcal{C}$ as well as a right adjoint $A_*: \mathcal{C}^A \to \mathcal{C}$. In Section I.3.1, we construct the *dualizing object* $D_A \in \mathcal{C}(A)$ and the *twisted norm map* $\mathrm{Nm}_A: A_!(- \otimes D_A) \to A_*(-)$. An object $A \in \mathcal{B}$ is called *twisted $\mathcal{C}$-ambidextrous* if this map is an equivalence. We show that Nm_A exhibits $A_!(- \otimes D_A)$ in a precise sense as the universal $\mathcal{C}$-linear colimit preserving approximation of A_*, see Proposition I.3.7 for a precise statement.

In Section I.3.2, we show that our notion of twisted ambidexterity is closely related to the notion of *ambidexterity* by Hopkins and Lurie [HL13]: an n-truncated object $A \in \mathcal{B}$ is $\mathcal{C}$-ambidextrous in the sense of [HL13] if and only if not only A but also all its iterated diagonals $A \to A^{S^n}$ are twisted $\mathcal{C}$-ambidextrous. It then follows inductively that the dualizing object D_A is equivalent to the monoidal unit of $\mathcal{C}(A)$ and the the twisted norm map reduces to the norm map $\mathrm{Nm}_A: A_! \to A_*$ of Hopkins and Lurie.

In Section I.3.3, we explain the close connection between twisted ambidexterity and Costenoble-Waner duality, a form of duality theory in parametrized homotopy theory introduced [CW16] and [MS06]. Most importantly, we show that an object $A \in \mathcal{B}$ is twisted $\mathcal{C}$-ambidextrous if and only if the monoidal unit $\mathbb{1}_A \in \mathcal{C}(A) = \mathcal{C}(1 \times A)$ is left Costenoble-Waner dualizable, see Proposition I.3.28. In this case, the left dual of $\mathbb{1}_A$ is given by the dualizing object $D_A \in \mathcal{C}(A) = \mathcal{C}(A \times 1)$.

Chapter I.4: Equivariant homotopy theory

Chapter I.4 contains the main result of Part I of this dissertation: a universal property of the G-category $\underline{\mathrm{Sp}}^G$ of genuine G-spectra for a compact Lie group G in terms of twisted ambidexterity. A *G-category* is by definition an ∞-category parametrized over the ∞-topos $\mathcal{B} = \mathrm{Spc}^G$ of G-spaces, or equivalently a functor $\mathrm{Orb}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$. The G-category $\underline{\mathrm{Sp}}^G$ is informally given by sending an orbit G/H to the ∞-category Sp^H of genuine H-spectra, see Section I.4.1 for a precise construction. In Section I.4.2 we prove the main result of Part I:

Theorem 2 (Theorem I.4.8)**.** *For a compact Lie group G, the G-category $\underline{\mathrm{Sp}}^G$ is initial among fiberwise stable presentably symmetric monoidal G-categories $\mathcal{C}$ such that all compact G-spaces are twisted $\mathcal{C}$-ambidextrous.*

In fact, we show it suffices to have twisted ambidexterity for the orbits G/H, in which case the resulting twisted ambidexterity isomorphism specializes to a formal Wirthmüller isomorphism of the form $\mathrm{Nm}_{G/H}: \mathrm{ind}_H^G(- \otimes D_{G/H}) \xrightarrow{\sim} \mathrm{coind}_H^G(-)$. In particular, Theorem 2 may be interpreted as saying that parametrized genuine G-spectra form the universal theory of stable G-equivariant objects which admit formal Wirthmüller isomorphisms.

In Section I.4.3 and Section I.4.4, we replace the ∞-topos of G-spaces by that of *orbispaces* and that of *proper G-spaces*, respectively, and extend the above universal property to the contexts of *orbispectra* and *proper equivariant homotopy theory*, respectively.

Part II: Relative Poincaré duality for differentiable stacks

Chapter II.2: Foundations on differentiable stacks

Chapter II.2 consists of preliminary material concerning differentiable stacks: sheaves of spaces $\mathcal{X}$ on the site Diff of smooth manifolds and open covers which admit a *representable atlas* $M \twoheadrightarrow \mathcal{X}$. In Section II.2.3, we discuss how every Lie groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ gives rise to a *classifying stack* $\mathbb{B}\mathcal{G}$, and show how $\mathbb{B}\mathcal{G}$ can be used to classify both left actions of $\mathcal{G}$ on smooth manifolds as well as principal $\mathcal{G}$-bundles over smooth manifolds. In Section II.2.4 we discuss local properties of maps of stacks and in Section II.2.5 we recall the notion of a vector bundle over a stack $\mathcal{X}$.

Chapter II.3: Geometry of differentiable stacks

In Chapter II.3 we discuss various geometrical aspects of differentiable stacks. In Section II.3.1, we introduce the *coarse moduli space* $|\mathcal{X}|_{\mathrm{mod}}$ of a stack $\mathcal{X}$ on Diff and show that its open subspaces are in one-to-one correspondence with the open substacks of $\mathcal{X}$. This allows us to construct the *open complement* $\mathcal{X} \setminus \mathcal{Z}$ of a closed embedding $\mathcal{Z} \hookrightarrow \mathcal{X}$ in Section II.3.2. A geometrically well-behaved class of differentiable stacks, discussed in Section II.3.3, are the *separated* differentiable stacks: those whose diagonal is proper. In Section II.3.4 we introduce the *isotropy groups* of a differentiable stack, and show that a morphism between separated differentiable stacks is representable if and only if it induces injections on isotropy groups. We define the *relative tangent bundle* T_f and *normal bundle* $\mathcal{N}_f$ of a representable morphism $f\colon \mathcal{Y} \to \mathcal{X}$ in Section II.3.5 and show in Section II.3.6 that every embedding $i\colon \mathcal{S} \hookrightarrow \mathcal{X}$ of separated differentiable stacks admits a *tubular neighborhood*: an open neighborhood of $\mathcal{S}$ in $\mathcal{X}$ which is equivalent to an open neighborhood of $\mathcal{S}$ inside $\mathcal{N}_i$. In Section II.3.7 we prove that every separated differentiable stack is locally isomorphic to a quotient stack $\mathbb{R}^n//G$ for some smooth linear action of a compact Lie group G on a Euclidean space $\mathbb{R}^n$, which lets us reduce various statements about differentiable stacks to their analogues for equivariant smooth manifolds.

Chapter II.4: Genuine sheaves on differentiable stacks

In Chapter II.4 we introduce for every separated differentiable stack $\mathcal{X}$ the ∞-categories $\mathrm{H}(\mathcal{X})$ and $\mathrm{SH}(\mathcal{X})$ of *genuine sheaves of spaces/spectra on* $\mathcal{X}$. In Section II.4.1, we introduce the site $\mathrm{Sub}_{/\mathcal{X}}$ of representable submersions over $\mathcal{X}$, equipped with the *open cover topology*. We define the ∞-category $\mathrm{H}(\mathcal{X})$ in Section II.4.2 as the ∞-category of *homotopy invariant sheaves* on $\mathrm{Sub}_{/\mathcal{X}}$: those sheaves $\mathcal{F}$ for which the map $\mathcal{F}(\mathcal{Y}) \to \mathcal{F}(\mathcal{Y} \times \mathbb{R})$ induced by the projection $\mathcal{Y} \times \mathbb{R} \to \mathcal{Y}$ is an equivalence for every $\mathcal{Y} \in \mathrm{Sub}_{/\mathcal{X}}$. There is a forgetful functor $\gamma^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X})$ to the ∞-category of ordinary sheaves on $\mathcal{X}$ which admits fully faithful left and right adjoints, see Subsection 4.2.5. The ∞-category $\mathrm{SH}(\mathcal{X})$ is defined in Section II.4.3: in case $\mathcal{X} = M//G$ is a global quotient stack we obtain $\mathrm{SH}(\mathcal{X})$ from $\mathrm{H}(\mathcal{X})_*$ by formally inverting the sphere bundle $S^{\mathcal{E}}$ of every vector bundle $\mathcal{E} \in \mathrm{Vect}(\mathcal{X})$, and this determines the general case by imposing descent along open covers. In Section II.4.4, we show that genuine sheaves over classifying stacks of compact Lie groups give back classical equivariant homotopy theory:

Theorem 3 (Theorem II.4.4.16, Proposition II.4.4.17)**.** *For a compact Lie group G, there are equivalences of ∞-categories* $\mathrm{H}(\mathbb{B}G) \simeq \mathrm{Spc}^G$ *and* $\mathrm{SH}(\mathbb{B}G) \simeq \mathrm{Sp}^G$.

In Section II.4.5, following [DG22], we study universal characterizations of the assignments $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ and $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ in terms of the notion of a *pullback formalism*.

Chapter II.5: Localization sequences

In Chapter II.5, we prove the localization theorem for pointed genuine sheaves, closely following the proof strategy of Khan [Kha19]:

Theorem 4 (Theorem II.5.2.16)**.** *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks and let $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ be its open complement. Then the functor $i_*\colon \mathrm{H}(\mathcal{Z})_* \to \mathrm{H}(\mathcal{X})_*$ is fully faithful, and there is a preferred cofiber sequence $j_\sharp j^* \xrightarrow{\text{counit}} \mathrm{id} \xrightarrow{\text{unit}} i_* i^*$.*

Here $j^*\colon \mathrm{H}(\mathcal{X})_* \to \mathcal{H}(\mathcal{U})_*$ and $i^*\colon \mathrm{H}(\mathcal{X})_* \to \mathcal{H}(\mathcal{Z})_*$ are the pullback functors along i and j, and $j_\sharp$ and i_* are their left and right adjoint, respectively. The analogous statement for the functor $i_*\colon \mathrm{SH}(\mathcal{Z}) \to \mathrm{SH}(\mathcal{X})$ also follows, and as a corollary the ∞-category $\mathrm{SH}(\mathcal{X})$ can be exhibited as a *recollement* of the ∞-categories $\mathrm{SH}(\mathcal{Z})$ and $\mathrm{SH}(\mathcal{U})$.

Chapter II.6: Relative Poincaré duality for differentiable stacks

Chapter II.6 contains the main result of Part II of this dissertation: a relative version of Poincaré duality for separated differentiable stacks in the context of genuine sheaves of spectra. For a representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of separated differentiable stacks, we define a *dualizing sheaf* $\omega_f \in \mathrm{SH}(\mathcal{Y})$ as $\omega_f := \mathrm{pr}_{1\sharp}\Delta_* \mathbb{1}_{\mathcal{Y}}$, where $\mathrm{pr}_1\colon \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is the projection to the first factor and $\Delta\colon \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is the diagonal of f. A choice of tubular neighborhood of Δ provides an equivalence $\omega_f \simeq S^{T_f}$ between the dualizing sheaf and the suspension spectrum of the one-point compactification of the relative tangent bundle of f. We further define a *Poincaré duality map* $\mathfrak{p}_f\colon f_\sharp \to f_*(- \otimes \omega_f)$, a natural transformation of functors $\mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$.

Theorem 5 (Relative Poincaré duality, Theorem II.6.1.7)**.** *If $f\colon \mathcal{Y} \to \mathcal{X}$ be a proper representable submersion between separated differentiable stacks, the Poincaré duality map $\mathfrak{p}_f\colon f_\sharp(-) \to f_*(- \otimes \omega_f)$ is an equivalence.*

The proof is close in spirit to the proof of Atiyah duality for a compact smooth manifold M. The main ingredient of the proof is the construction of a Pontryagin-Thom collapse map associated to a closed embedding, which we will introduce in Section II.6.2. Using the auxiliary notion of a *kernel operator*, introduced in Section II.6.3, we give a proof of relative Poincaré duality in Section II.6.4. Section II.6.5 discusses various important consequences of relative Poincaré duality, like relative Atiyah duality and proper base change.

Part III: Outlook

In Part III, we indicate potential directions for future research. In Section III.2, we explain in some detail the expected close relation between twisted ambidexterity from Part I and relative Poincaré duality from Part II. In Section III.3, we discuss six-functor formalisms on the site SepStk of separated differentiable stacks and formulate a conjecture concerning a six-functor formalism of genuine sheaves of spectra. In Section III.4, we propose a notion of *proper genuine sheaves* on an arbitrary differentiable stack and deduce relative Poincaré duality in this context.

Appendices

This dissertation comes with a large amount of appendices. In Appendix A, we recall unstraightening techniques from Lurie [Lur17] and Drew and Gallauer [DG22, Appendix A] involving symmetric monoidal structures. In Appendix B, we provide a condensed proof of a theorem by Campion [Cam23] which shows that the suspension spectrum functor $\Sigma^\infty \colon \mathrm{Spc}_*^G \to \mathrm{Sp}^G$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is initial among morphisms $F \colon \mathrm{Spc}_*^G \to \mathcal{D}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ whose target is stable and which send all compact pointed G-spaces to dualizable objects. In Appendix C we collect some basic results on smooth manifolds. Appendix D recalls background material on Lie groupoids, including a variety of examples, the definition of Lie groupoid actions and principal bundles, and the notion of Morita equivalence.

In Appendix E we provide an extensive account of foundations on ∞-topoi. Their definition and some of their most relevant properties are recalled in Appendix E.1, and the correspondence between groupoid objects and effective epimorphisms is discussed in Appendix E.2. In Appendix E.3 we define groupoid actions and principal bundles in ∞-topoi and recall how they can be classified using the classifying stack of the groupoid. We finish with a discussion of sheaf topoi in Appendix E.4 and hypercompleteness in Appendix E.5.

In Appendix F, we provide a detailed treatment of Beck-Chevalley transformations, double Beck-Chevalley transformations and projection formula maps.

Conventions

We will adopt the standard notational conventions from higher category theory, following [Lur09] and [Lur17]. One notable exception is our terminology for the notion of an *∞-groupoid*: In Part I we use the classical word ‘space’, but in Part II we use the word ‘anima’, introduced by [CS23, Section 5.1.4], to make a clearer distinction between the geometry and the homotopy theory in that part. Accordinly we denote the ∞-topos of ∞-groupoids by Spc in Part I and by An in Part II. Similarly, the ∞-topos of G-spaces for a compact Lie group G will be denoted by Spc^G in Part I and by An_G in Part II, where in the latter case we refer to its objects as *genuine G-animae*.

Part I

Twisted ambidexterity in equivariant homotopy theory

Abstract

We develop the concept of twisted ambidexterity in a parametrized presentably symmetric monoidal ∞-category, which generalizes the notion of ambidexterity by Hopkins and Lurie and the Wirthmüller isomorphisms in equivariant stable homotopy theory, and is closely related to Costenoble-Waner duality. Our main result establishes the parametrized ∞-category of genuine G-spectra for a compact Lie group G as the universal example of a presentably symmetric monoidal ∞-category parametrized over G-spaces which is both stable and satisfies twisted ambidexterity for compact G-spaces. We further extend this result to the settings of orbispectra and proper genuine G-spectra for a Lie group G which is not necessarily compact.

I.1 Introduction

A well-known phenomenon in representation theory, sometimes referred to as *ambidexterity*, is that if H is a subgroup of a finite group G, the restriction functor from G-representations to H-representations admits a left adjoint ind_H^G and a right adjoint coind_H^G which are naturally equivalent to each other. An analogue of this result in stable equivariant homotopy theory was established by Wirthmüller [Wir74]: there is an equivalence $\mathrm{ind}_H^G \simeq \mathrm{coind}_H^G$ between the induction and coinduction functors from genuine H-spectra to genuine G-spectra.

The situation becomes more interesting when G is a non-discrete compact Lie group. For a closed subgroup $H \leqslant G$, the induction and coinduction functors are only equivalent up to a 'twist': for every genuine H-spectrum X there is a natural equivalence of genuine G-spectra

$$\mathrm{ind}_H^G(X \otimes S^{-L}) \xrightarrow{\sim} \mathrm{coind}_H^G(X),$$

called the *Wirthmüller isomorphism*, where S^{-L} is the inverse of the representation sphere of the tangent H-representation $L = T_{eH}(G/H)$. The construction of the comparison map in this case is geometric in nature and therefore substantially more involved than in the case of finite groups, see for example Schwede [Sch18, Section 3.2]. As a result, it is not immediately clear how to extend it to more general settings, like that of proper equivariant homotopy theory [Deg+19] or that of orbispectra [Par20], where some of the required geometric constructions are not available.

In [MS06], May and Sigurdsson approach the Wirthmüller isomorphism using the language of *parametrized homotopy theory*. They split up the construction of the isomorphism into two steps:

(1) The first step is formal: inspired by work of Costenoble and Waner [CW16], May and Sigurdsson set up a notion of parametrized duality theory called *Costenoble-Waner duality*. This theory provides an entirely categorical construction of a genuine H-spectrum $D_{G/H}$ equipped with a natural transformation

$$\mathrm{ind}_H^G(- \otimes D_{G/H}) \implies \mathrm{coind}_H^G(-).$$

This transformation is an equivalence if and only if $\mathbb{S}_H$ is *Costenoble-Waner dualizable*, in which case its Costenoble-Waner dual is given by $D_{G/H}$.

(2) The second step is geometric: May and Sigurdsson use a parametrized form of the Pontryagin-Thom construction to produce explicit duality data that exhibit the genuine H-spectrum S^{-L} as Costenoble-Waner dual to $\mathbb{S}_H$. As a consequence, one obtains an equivalence $D_{G/H} \simeq S^{-L}$ and the resulting map $\mathrm{ind}_H^G(X \otimes S^{-L}) \to \mathrm{coind}_H^G(X)$ is an equivalence.

A key advantage of separating the construction into these two steps is that the first step does not need any geometric input and works in much greater generality: it is an instance of a notion we call *twisted ambidexterity*. This allows for the construction of similar transformations in a wider range of contexts, including proper equivariant homotopy theory. It further allows one to formulate the Wirthmüller isomorphisms as a *property* of a parametrized homotopy theory, making it possible to talk about the universal example of a parametrized homotopy theory satisfying this property.

Twisted ambidexterity in parametrized homotopy theory

The concept of twisted ambidexterity comes up in homotopy theory in the setting of local systems on spaces. For a space A, let Sp^A denote the functor category $\mathrm{Fun}(A, \mathrm{Sp})$, also known as the ∞-category of local systems of spectra on A. The constant local system functor $A^*\colon \mathrm{Sp} \to \mathrm{Sp}^A$ admits both a left adjoint $A_! = \mathrm{colim}_A\colon \mathrm{Sp}^A \to \mathrm{Sp}$ as well as a right adjoint $A_* = \lim_A\colon \mathrm{Sp}^A \to \mathrm{Sp}$. Although A_* does not preserve colimits in general, it can be universally approximated from the left by a colimit-preserving functor via a *twisted norm map*

$$\mathrm{Nm}_A\colon A_!(- \otimes D_A) \implies A_*(-),$$

as shown by Nikolaus and Scholze [NS18, Theorem I.4.1(v)]. The parametrized spectrum $D_A \in \mathrm{Sp}^A$ is the *dualizing spectrum of* A, introduced and studied by John Klein [Kle01].[1] In case A is a compact space, the functor A_* already preserves colimits, and the twisted norm map Nm_A is an equivalence. When $A = M$ is a compact smooth manifold, $D_M = S^{-T_M}$ is the inverse of the one-point-compactification of the tangent bundle of M, and the resulting equivalence between the cohomology of M and a shift of the homology of M recovers twisted Poincaré duality.

In this article, we introduce a framework for twisted ambidexterity which generalizes the above story for local systems of spectra in two ways. As a first generalization, we replace

[1] Klein considered connected spaces $A = BG$ for topological groups G, and wrote D_G rather than D_{BG}.

the ∞-category of spectra by an arbitrary presentably symmetric monoidal ∞-category C, in which case the transformation $A_!(-\otimes D_A)\colon C^A \to C$ is the universal C-linear colimit-preserving approximation of $A_*\colon C^A \to C$. As a second generalization, following Ando, Blumberg and Gepner [ABG18], we consider homotopy theories parametrized over an arbitrary ∞-topos $\mathcal{B}$ in place of the ∞-category of spaces, allowing for applications in equivariant homotopy theory. The role of C is now played by certain limit-preserving functors $C\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ known as *presentably symmetric monoidal $\mathcal{B}$-categories*, see Definition 2.15. Given an object $A \in \mathcal{B}$, one can enhance $A_!$ and A_* to *parametrized* functors $C^A \to C$, and one can again construct an object $D_A \in C(A)$ and a twisted norm map $\mathrm{Nm}_A\colon A_!(-\otimes D_A) \to A_*$ universally approximating A_* by a C-linear colimit-preserving parametrized functor; see Proposition 3.7 for a precise statement. If Nm_A is an equivalence, we will say that A is *twisted C-ambidextrous*.

The notion of twisted ambidexterity can be regarded as a generalization of the notion of *ambidexterity* by Hopkins and Lurie [HL13]: if A is an n-truncated space, then the twisted norm map reduces to the norm map of [HL13] whenever the latter is defined, and A is C-ambidextrous in the sense of [HL13, Definition 4.3.4] if and only if each of the spaces $A, \Omega A, \Omega^2 A, \ldots, \Omega^{n+1} A$ is twisted C-ambidextrous, see Proposition 3.15.

A *relative* version of twisted ambidexterity for morphisms $f\colon A \to B$ in $\mathcal{B}$ is obtained by replacing the ∞-topos $\mathcal{B}$ by its slice $\mathcal{B}_{/B}$, producing a twisted norm map $\mathrm{Nm}_f\colon f_!(-\otimes D_f) \to f_*(-)$.

Twisted ambidexterity in equivariant homotopy theory

The Wirthmüller isomorphism in equivariant homotopy theory discussed before may be understood as a special case of twisted ambidexterity. Given a compact Lie group G, we work in the setting of *G-categories*, defined as ∞-categories parametrized over the ∞-topos of G-spaces. There is a G-category $\underline{\mathrm{Sp}}^G$ of genuine G-spectra, which assigns to the orbit space G/H of a closed subgroup $H \leqslant G$ the ∞-category of genuine H-spectra. The G-category $\underline{\mathrm{Sp}}^G$ is presentably symmetric monoidal and is fiberwise stable, meaning that the ∞-category $\underline{\mathrm{Sp}}^G(A)$ is stable for every G-space A. Furthermore, it follows from results of [MS06] that every compact G-space is twisted $\underline{\mathrm{Sp}}^G$-ambidextrous in the sense discussed above. The main result of this article is that $\underline{\mathrm{Sp}}^G$ is in fact *universal* among G-categories satisfying the above properties:

Theorem A (Theorem 4.8)**.** *For a compact Lie group G, the G-category $\underline{\mathrm{Sp}}^G$ is initial among fiberwise stable presentably symmetric monoidal G-categories C such that all* *compact G-spaces are twisted $\mathcal{C}$-ambidextrous.*

In fact, it suffices to require that the orbit G/H is twisted ambidextrous for every closed subgroup $H \leqslant G$. In this case, the resulting twisted ambidexterity isomorphism specializes to a formal Wirthmüller isomorphism of the form

$$\mathrm{Nm}_{G/H}\colon \mathrm{ind}_H^G(-\otimes D_{G/H}) \xrightarrow{\sim} \mathrm{coind}_H^G(-).$$

Therefore, Theorem A may be interpreted as saying that genuine equivariant spectra form the universal theory of stable G-equivariant objects which admit formal Wirthmüller isomorphisms.

For the proof of Theorem A, we relate twisted ambidexterity for compact G-spaces to invertibility of representation spheres. If $\mathcal{C}$ is a pointed presentably symmetric monoidal G-category, one can show that the underlying ∞-category $\mathcal{C}(1)$ of $\mathcal{C}$ comes equipped with a canonical tensoring by pointed G-spaces, and thus the representation spheres S^V act on $\mathcal{C}(1)$.

Theorem B (Theorem 4.7)**.** *Let G be a compact Lie group and let $\mathcal{C}$ be a fiberwise stable presentably symmetric monoidal G-category. Then the following conditions are equivalent:*

(1) For any G-representation V, the representation sphere S^V acts invertibly on $\mathcal{C}(1)$;

(2) Every compact G-space is twisted $\mathcal{C}$-ambidextrous;

(3) For every closed subgroup $H \leqslant G$, the orbit G/H is twisted $\mathcal{C}$-ambidextrous.

Since $\underline{\mathrm{Sp}}^G$ is initial with respect to condition (1), Theorem A is as a direct consequence of Theorem B. For the proof of Theorem B, we show in Section 3.3 that twisted ambidexterity can be formulated in terms of *Costenoble-Waner duality*, a parametrized form of duality theory. The main ingredient for the implication (1) $\implies$ (2) is then a result of May and Sigurdsson [MS06] about Costenoble-Waner duality in equivariant stable homotopy theory. The main ingredient for the implication (2) $\implies$ (1) is a result of Campion [Cam23], recalled in Appendix B, which roughly says that dualizability of compact G-spaces implies invertibility of the representation spheres.

Our methods directly extend to the contexts of *orbispectra* and *proper genuine G-spectra*. We refer to Section 4.3 and Section 4.4 for precise definitions of the words appearing in the following two theorems:

Theorem C (Theorem 4.20)**.** *The orbicategory $\underline{\mathrm{OrbSp}}$ of orbispectra is initial among fiberwise stable presentably symmetric monoidal orbicategories $\mathcal{C}$ such that every relatively compact morphism of orbispaces is twisted $\mathcal{C}$-ambidextrous.*

Theorem D (Theorem 4.29)**.** *For a Lie group G which is not necessarily compact, the proper G-category $\underline{\mathrm{Sp}}^G$ of proper G-spectra is initial among fiberwise stable presentably symmetric monoidal proper G-categories C such that every relatively compact morphisms of proper G-spaces is twisted C-ambidextrous.*

In particular, Theorem D shows that for every closed subgroup H of a Lie group G with compact orbit space G/H there is a formal Wirthmüller isomorphism

$$\mathrm{Nm}_{G/H} \colon \mathrm{ind}_H^G(- \otimes D_{G/H}) \implies \mathrm{coind}_H^G(-).$$

The methods developed in this article also allow us to give a formal description of the ∞-category of proper genuine G-spectra of Degrijse et al. [Deg+19] in case the Lie group G has *enough bundle representations* in the sense of Definition 4.30: it is obtained from the ∞-category of pointed proper G-spaces by inverting the sphere bundles S^ξ associated to finite-dimensional vector bundles ξ over the classifying orbispace $\mathbb{B}G$, see Corollary 4.34.

Organization

In Chapter I.2, we introduce the setting of parametrized higher category theory we will use in this article. Several foundational definitions and results from [Mar21; MW21; MW22] are recalled in Section 2.1. In Section 2.2, we provide for every parametrized presentably symmetric monoidal ∞-category C a classification of C-linear functors $C^A \to C^B$ in terms of objects of $C(A \times B)$, where A and B are objects of the base ∞-topos. Section 2.3 contains a discussion of formally inverting objects in parametrized symmetric monoidal ∞-categories.

In Chapter I.3, we introduce the notion of twisted ambidexterity in a parametrized presentably symmetric monoidal ∞-category C. The twisted norm maps $\mathrm{Nm}_f \colon f_!(- \otimes D_f) \Rightarrow f_*(-)$ for a morphism $f \colon A \to B$ in the base ∞-topos are constructed in Section 3.1, where also their universal property is established. In Section 3.2, we relate twisted ambidexterity to the notion of ambidexterity by [HL13] and to the notion of parametrized semiadditivity by [Nar16; CLL23]. In Section 3.3 we discuss the relation between twisted ambidexterity and Costenoble-Waner duality.

In Chapter I.4, we apply our methods in the context of equivariant homotopy theory. Section 4.1 introduces the parametrized ∞-category of genuine G-spectra for a compact Lie group G, and Section 4.2 establishes its universal property in terms of twisted ambidexterity. In Section 4.3 and Section 4.4 we extend this to the contexts of orbispectra and proper equivariant spectra, respectively.

Relation to other work

Our treatment of twisted ambidexterity draws on ideas from a wide range of prior work. The main inspiration is the concept of Costenoble-Waner duality in parametrized homotopy theory, introduced by Costenoble and Waner [CW16] under the name 'homological duality' and further developed by May and Sigurdsson [MS06]. The untwisted notion of ambidexterity appeared in the works of Hopkins and Lurie [HL13] and has been further studied by Harpaz [Har20] and Carmeli, Schlank and Yanovski [CSY22; CSY21]. A treatment in the context of representation theory was given by Balmer and Dell'Ambrogio [BD20]. In the case of local systems on spaces, the universal property of the twisted norm map resembles classical assembly maps; see, for example, the discussion following Theorem I.4.1 of Nikolaus and Scholze [NS18]. The dualizing object in this case is the dualizing spectrum introduced by Klein [Kle01] and studied by Bauer [Bau04] and Rognes [Rog08]. Our twisted norm maps are similar to those constructed by Hoyois [Hoy17] and Bachmann and Hoyois [BH21], which in turn are closely related to the 'purity equivalences' of Cisinski and Déglise [CD19]. Although similar in appearance, our approach is different from the formal Wirthmüller isomorphisms of Fausk, Hu and May [FHM03] and Balmer, Dell'Ambrogio and Sanders [BDS16], where no parametrized homotopy theory is involved.

Our characterization of genuine equivariant spectra in terms of fiberwise stability and formal Wirthmüller isomorphisms is heavily inspired by work of Blumberg [Blu06], who described the category of genuine G-spectra for a compact Lie group G in terms of continuous functors that satisfy excision and have (geometrically defined) Wirthmüller isomorphisms. For finite groups, a parametrized universal property for genuine G-spectra in terms of stability and Wirthmüller equivalences $\mathrm{ind}_H^G(-) \simeq \mathrm{coind}_H^G(-)$ was outlined by Nardin [Nar16], and a similar result in the context of global homotopy theory was proved by Lenz, Linskens and the author in [CLL23].

Acknowledgements

I am truly grateful to the following individuals for their contributions to this article. Stefan Schwede sparked the idea behind this project and has been a source of support and encouragement throughout the process. Louis Martini and Sebastian Wolf were invaluable resources for discussing the intricacies of parametrized category theory. Sil Linskens was always available for helpful discussions, especially regarding formal inversions and proper equivariant homotopy theory. Shachar Carmeli provided the idea for the universal property of the twisted norm map. Tim Campion graciously allowed me to include a result from his PhD dissertation in the appendix. I further had useful conversations with Tobias Lenz, Maxime Ramzi, and Lior Yanovski about some of the contents of this article. Finally, I would like to thank the Max Planck Institute for Mathematics for its financial support.

I.2 Parametrized category theory

We start by recalling the setup of parametrized category theory we are working in, and establishing some analogues of well-known results in non-parametrized category theory. Our main references are the articles [Mar21], [MW21] and [MW22] of Martini and Wolf; see in particular [MW22, Section 2.6] for a short overview of the theory. An earlier framework for parametrized category theory was given by Barwick, Dotto, Glasman, Nardin and Shah [Bar+16a; Bar+16b; Sha21; Nar16].

Convention 2.1. Since most of the ∞-categories we will be working with are presentable, we will by convention take all ∞-categories to be large unless explicitly specified that they are small. In particular, Cat_∞ denotes the (very large) ∞-category of large ∞-categories; in [MW22], this is denoted by $\widehat{\mathrm{Cat}}_\infty$ instead.

2.1 Recollections on parametrized category theory

Throughout this section, we fix an ∞-topos $\mathcal{B}$.

Definition 2.2. A *$\mathcal{B}$-category* is a sheaf of ∞-categories on $\mathcal{B}$, i.e., a limit-preserving functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. Its *underlying ∞-category* $\Gamma(\mathcal{C})$ is the ∞-category $\mathcal{C}(1)$, where $1 \in \mathcal{B}$ is the terminal object. Given two $\mathcal{B}$-categories $\mathcal{C}$ and $\mathcal{D}$, a *$\mathcal{B}$-functor* is a natural transformation $\mathcal{C} \to \mathcal{D}$. We let $\mathrm{Cat}(\mathcal{B}) \subseteq \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ denote the (very large) ∞-category of $\mathcal{B}$-categories and $\mathcal{B}$-functors.

By [Mar21, Proposition 3.5.1], a $\mathcal{B}$-category may equivalently be encoded as a *(large) category internal to* $\mathcal{B}$, which is the perspective used in [Mar21; MW21; MW22].

When $\mathcal{B}$ is the ∞-topos Spc of spaces (a.k.a. ∞-groupoids or animae), the underlying category functor provides an equivalence

$$\Gamma\colon \mathrm{Cat}(\mathrm{Spc}) \xrightarrow{\simeq} \mathrm{Cat}_\infty;$$

its inverse sends an ∞-category C to the functor $\mathrm{Spc}^{\mathrm{op}} \to \mathrm{Cat}_\infty\colon A \mapsto \mathrm{Fun}(A, C)$. More generally, when $\mathcal{B} = \mathrm{PSh}(T)$ is the ∞-topos of presheaves on some small ∞-category T, restriction to the representable objects induces an equivalence $\mathrm{Cat}(\mathrm{PSh}(T)) \xrightarrow{\sim} \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat}_\infty)$, and the theory of $\mathcal{B}$-categories reduces to that of T-∞-categories studied by Barwick et al. [Bar+16b]. Since all the examples of ∞-topoi considered in this article will be presheaf topoi, the reader uncomfortable with the language of ∞-topoi may replace $\mathcal{B}$ by $\mathrm{PSh}(T)$ throughout.

Example 2.3 ($\mathcal{B}$-groupoids)**.** Every object $A \in \mathcal{B}$ can naturally be regarded as a $\mathcal{B}$-category via the Yoneda embedding $\mathcal{B} \hookrightarrow \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Spc}) \subseteq \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)$. The $\mathcal{B}$-categories of this form are called *$\mathcal{B}$-groupoids*.

Example 2.4 (Base change)**.** Any geometric morphism $f^*\colon \mathcal{A} \rightleftarrows \mathcal{B} : f_*$ induces an adjunction $f^*\colon \mathrm{Cat}(\mathcal{A}) \rightleftarrows \mathrm{Cat}(\mathcal{B}) : f_*$, see [Mar21, Section 3.3], [MW21, Section 2.6]. The right adjoint f_* is explicitly given by precomposing a $\mathcal{B}$-category $C\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ with $f^*\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$.

Example 2.5 (Locally constant $\mathcal{B}$-categories)**.** Applying Example 2.4 to the geometric morphism $\mathrm{LConst}\colon \mathrm{Spc} \rightleftarrows \mathcal{B} : \Gamma$, we obtain an adjunction $\mathrm{LConst}\colon \mathrm{Cat}_\infty \rightleftarrows \mathrm{Cat}(\mathcal{B}) : \Gamma$, where Γ is the underlying ∞-category functor. The $\mathcal{B}$-categories in the image of LConst are called *locally constant*.[1]

Example 2.6 (Passing to slice topoi)**.** For an object $B \in \mathcal{B}$, applying Example 2.4 to the étale geometric morphism $- \times B = \pi_B^*\colon \mathcal{B} \rightleftarrows \mathcal{B}_{/B} : (\pi_B)_*$, we get an adjunction

$$\pi_B^*\colon \mathrm{Cat}(\mathcal{B}) \rightleftarrows \mathrm{Cat}(\mathcal{B}_{/B}) : (\pi_B)_*.$$

For $C \in \mathrm{Cat}(\mathcal{B})$, the $\mathcal{B}_{/B}$-category $\pi_B^* C$ is given by precomposing C with the forgetful functor $\mathcal{B}_{/B} \to \mathcal{B}$.

Example 2.7 (Parametrized functor categories)**.** By [Mar21, Proposition 3.2.11], the ∞-category $\mathrm{Cat}(\mathcal{B})$ of $\mathcal{B}$-categories is cartesian closed: for all $C, \mathcal{D} \in \mathrm{Cat}(\mathcal{B})$ there is an internal hom-object[2] $\underline{\mathrm{Fun}}_{\mathcal{B}}(C, \mathcal{D})$, called the *$\mathcal{B}$-category of $\mathcal{B}$-functors from C to $\mathcal{D}$*. We let $\mathrm{Fun}_{\mathcal{B}}(C, \mathcal{D})$ denote the underlying ∞-category of $\underline{\mathrm{Fun}}_{\mathcal{B}}(C, \mathcal{D})$. Its 2-morphisms are called *$\mathcal{B}$-transformations*.

Definition 2.8. A *symmetric monoidal $\mathcal{B}$-category* is a commutative monoid in the ∞-category $\mathrm{Cat}(\mathcal{B})$, or equivalently a limit-preserving functor $C\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{CMon}(\mathrm{Cat}_\infty)$. For

[1] In [MW21] these are simply called *constant*.

[2] Martini [Mar21] denotes $\underline{\mathrm{Fun}}_{\mathcal{B}}(C, \mathcal{D})$ by $[C, \mathcal{D}]$.

$B \in \mathcal{B}$, we denote the tensor product and monoidal unit of $C(B)$ by $- \otimes_B -$ and $\mathbb{1}_B$, respectively.

Presentable $\mathcal{B}$-categories

We give a brief overview of the theory of presentable $\mathcal{B}$-categories, developed by Martini and Wolf [MW22].

Definition 2.9. A $\mathcal{B}$-category $C\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is called *fiberwise presentable* if it factors (necessarily uniquely) through the subcategory $\mathrm{Pr}^{\mathrm{L}} \subseteq \mathrm{Cat}_\infty$ of presentable ∞-categories and colimit preserving functors. We say that C is *presentable* if it is fiberwise presentable and additionally satisfies the following two conditions:

(1) (Left adjoints) For every morphism $f\colon A \to B$ in $\mathcal{B}$, the restriction functor $f^*\colon C(B) \to C(A)$ has a left adjoint $f_!\colon C(A) \to C(B)$;

(2) (Left base change) For every pullback square

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B' \arrow[r, "\beta"'] & B \end{tikzcd}$$

in $\mathcal{B}$, the Beck-Chevalley transformation $f'_!\alpha^* \Rightarrow \beta^* f_!$ is an equivalence.

If C and $\mathcal{D}$ are presentable $\mathcal{B}$-categories, we say that a $\mathcal{B}$-functor $F\colon C \to \mathcal{D}$ *preserves (parametrized) colimits* if the following two properties are satisfied:

(1) For every object $B \in \mathcal{B}$, the functor $F(B)\colon C(B) \to \mathcal{D}(B)$ preserves small colimits;

(2) For every morphism $f\colon A \to B$ in $\mathcal{B}$, the Beck-Chevalley transformation $f_! \circ F(A) \Longrightarrow F(B) \circ f_!$ is an equivalence.

By [MW22, Theorem A] and [MW21, Proposition 4.2.3], these definitions agree with the definitions of Martini and Wolf [MW22].

Remark 2.10. For a presentable $\mathcal{B}$-category, the restriction functor $f^*\colon C(B) \to C(A)$ is assumed to admit a right adjoint $f_*\colon C(A) \to C(B)$ for every morphism $f\colon A \to B$. By passing to right adjoints in condition (2) in Definition 2.9, we see that also the other Beck-Chevalley transformation $\beta_* f^* \Rightarrow f'^*\alpha_*$ is an equivalence. It follows that any presentable $\mathcal{B}$-category admits all (parametrized) limits and colimits.

The next definition introduces the presentable $\mathcal{B}$-category $\Omega_{\mathcal{B}}$, the $\mathcal{B}$-parametrized analogue of the ∞-category of spaces.

Definition 2.11. The target functor $d_0\colon \mathcal{B}^{\Delta^1} \to \mathcal{B}$ is a cartesian fibration, and by [Lur09, Theorem 6.1.3.9] is classified by a limit-preserving functor

$$\Omega_{\mathcal{B}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}, \qquad B \mapsto \mathcal{B}_{/B}, \qquad (f\colon A \to B) \mapsto (f^*\colon \mathcal{B}_{/B} \to \mathcal{B}_{/A}).$$

The pullback functors $f^*\colon \mathcal{B}_{/B} \to \mathcal{B}_{/A}$ have left adjoints $f \circ -\colon \mathcal{B}_{/A} \to \mathcal{B}_{/B}$ which satisfy the Beck-Chevalley condition, so $\Omega_{\mathcal{B}}$ is a presentable $\mathcal{B}$-category. We call $\Omega_{\mathcal{B}}$ the *$\mathcal{B}$-category of $\mathcal{B}$-groupoids*.[3]

Definition 2.12. Let $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ denote the (non-full) subcategory of $\mathrm{Cat}(\mathcal{B})$ spanned by the presentable $\mathcal{B}$-categories and the colimit preserving $\mathcal{B}$-functors. For presentable $\mathcal{B}$-categories $\mathcal{C}$ and $\mathcal{D}$, we let $\mathrm{Fun}^{\mathrm{L}}_{\mathcal{B}}(\mathcal{C},\mathcal{D})$ denote the subcategory of $\mathrm{Fun}_{\mathcal{B}}(\mathcal{C},\mathcal{D})$ spanned by those $\mathcal{B}$-functors $\mathcal{C} \to \mathcal{D}$ that preserve colimits.

Recall from [MW21, Definition 2.7.2] that the ∞-categories $\mathrm{Cat}(\mathcal{B}_{/B})$ assemble into a (very large) $\mathcal{B}$-category $\mathrm{Cat}_{\mathcal{B}}$. The subcategories $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}) \subseteq \mathrm{Cat}(\mathcal{B}_{/B})$ assemble into a parametrized subcategory $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}} \subseteq \mathrm{Cat}_{\mathcal{B}}$, see [MW22, Definition 6.4.1].

Proposition 2.13 ([MW21, Proposition 4.5.1], [MW22, Definition 6.4.1, Remark 6.4.2, Corollary 6.4.11])**.** *The $\mathcal{B}$-categories $\mathrm{Cat}_{\mathcal{B}}$ and $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}$ are complete and cocomplete, and the inclusion $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathrm{Cat}_{\mathcal{B}}$ preserves limits.* □

Corollary 2.14. *For each $B \in \mathcal{B}$ the adjunction $\pi_B^*\colon \mathrm{Cat}(\mathcal{B}) \rightleftarrows \mathrm{Cat}(\mathcal{B}_{/B}) :(\pi_B)_*$ from Example 2.6 restricts to an adjunction $\pi_B^*\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \rightleftarrows \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}) :(\pi_B)_*$.* □

Tensor products of presentable $\mathcal{B}$-categories

Given two presentable $\mathcal{B}$-categories $\mathcal{C}$ and $\mathcal{D}$, there exists a presentable $\mathcal{B}$-category $\mathcal{C} \otimes \mathcal{D}$ called the *tensor product* of $\mathcal{C}$ and $\mathcal{D}$, which is characterized by the property that colimit-preserving $\mathcal{B}$-functors into a third presentable $\mathcal{B}$-category $\mathcal{E}$ correspond to $\mathcal{B}$-functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve colimits in both variables, see [MW22, Section 8.2]. The tensor product equips $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ with the structure of a symmetric monoidal ∞-category $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^{\otimes}$ whose monoidal unit is $\Omega_{\mathcal{B}}$. In fact, in [MW22, Proposition 8.2.9] it is shown that a parametrized version of this construction equips the (very large) $\mathcal{B}$-category $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}$ of presentable $\mathcal{B}$-categories with the structure of a symmetric monoidal $\mathcal{B}$-category.

[3]It is called the *universe for groupoids* by [Mar21].

By [MW22, Proposition 8.2.11], a formula for the tensor product of two presentable $\mathcal{B}$-categories $\mathcal{C}$ and $\mathcal{D}$ is given by $\mathcal{C} \otimes \mathcal{D} \simeq \underline{\mathrm{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$, the full sub-$\mathcal{B}$-category of $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ spanned by the limit-preserving $\mathcal{B}$-functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. For a third presentable $\mathcal{B}$-category $\mathcal{E}$, there is an equivalence

$$\mathrm{Fun}^{\mathrm{L}}_{\mathcal{B}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathcal{B}}(\mathcal{C}, \underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathcal{D}, \mathcal{E})),$$

and it follows in particular that the $\mathcal{B}$-functor $\mathcal{C} \otimes -\colon \mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}$ preserves colimits.

Definition 2.15 ([MW22, Definition 8.2.10])**.** A *presentably symmetric monoidal $\mathcal{B}$-category* is a commutative algebra object in the symmetric monoidal ∞-category $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^{\otimes}$.

As $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^{\otimes}$ is a subcategory of $\mathrm{Cat}(\mathcal{B})^{\times}$, a symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$ is presentably symmetric monoidal if and only if $\mathcal{C}$ is presentable and the tensor product $\mathcal{B}$-functor $- \otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits in both variables. Unwinding definitions, this boils down to the following two non-parametrized conditions:

(1) (Fiberwise presentably symmetric monoidal) For every $B \in \mathcal{B}$, the tensor product functor $- \otimes_B -\colon \mathcal{C}(B) \times \mathcal{C}(B) \to \mathcal{C}(B)$ of $\mathcal{C}(B)$ preserves small colimits in both variables.

(2) (Left projection formula) For every morphism $f\colon A \to B$ in $\mathcal{B}$ and all objects $X \in \mathcal{C}(B)$ and $Y \in \mathcal{C}(A)$, the exchange morphism $f_!(f^*(X) \otimes_A Y) \to X \otimes_B f_!(Y)$ is an equivalence.

In particular, the data of a presentably symmetric monoidal $\mathcal{B}$-category is the same as that of a limit-preserving functor $\mathcal{C}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ such that all the symmetric monoidal restriction functors $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admit left adjoints that satisfy base change and satisfy the left projection formula. Such structure has also been called a *Wirthmüller context*[4] [ABG18, Definition 6.7, Proposition 6.8] or a *presentable pullback formalism* [DG22, Definition 4.5].

Embedding $\mathcal{B}$-modules into presentable $\mathcal{B}$-categories

The ∞-topos $\mathcal{B}$ comes equipped with a presentably symmetric monoidal structure given by cartesian product, and thus we may consider left modules over it in Pr^{L}, which we will refer to as *$\mathcal{B}$-modules*. It was shown in [MW22, Section 8.3] that the $\mathcal{B}$-modules embed fully faithfully into presentable $\mathcal{B}$-categories:

[4]In [ABG18], the Beck-Chevalley conditions and the left projection formula were not taken as part of the axioms.

Proposition 2.16 ([MW22, Proposition 8.3.2, Lemma 8.3.3, Proposition 8.3.5, Proposition 8.3.6]). *There is a symmetric monoidal fully faithful functor*

$$- \otimes_{\mathcal{B}} \Omega_{\mathcal{B}} \colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}) \hookrightarrow \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$$

which admits a right adjoint $\Gamma^{\mathrm{lin}} \colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}})$ *whose composition with the forgetful functor* $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ *is the global section functor* $\Gamma \colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}$. □

Given a $\mathcal{B}$-module $\mathcal{D}$, the presentable $\mathcal{B}$-category $\mathcal{D} \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$ is given at an object $B \in \mathcal{B}$ by the relative tensor product $\mathcal{D} \otimes_{\mathcal{B}} \mathcal{B}_{/B}$ in Pr^{L}. The functoriality of this expression in B is informally described as follows: given a morphism $f \colon A \to B$ in $\mathcal{B}$, one considers the composition functor $f_! \colon \mathcal{B}_{/A} \to \mathcal{B}_{/B}$ as a $\mathcal{B}$-linear functor in Pr^{L}, tensors it with $\mathcal{D}$ over $\mathcal{B}$ to get a map $\mathcal{D} \otimes_{\mathcal{B}} \mathcal{B}_{/A} \to \mathcal{D} \otimes_{\mathcal{B}} \mathcal{B}_{/B}$, and then passes to its right adjoint. We refer to [MW22, Section 8.3] for a precise construction.

It follows directly from Proposition 2.16 that every commutative $\mathcal{B}$-algebra $\mathcal{D}$ in Pr^{L} gives rise to a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{D} \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$:

Corollary 2.17. *The adjunction from Proposition 2.16 induces an adjunction*

$$- \otimes_{\mathcal{B}} \Omega_{\mathcal{B}} \colon \mathrm{CAlg}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}) \underset{}{\overset{}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})) : \Gamma^{\mathrm{lin}}. \qquad \square$$

We will also need a $\mathcal{C}$-linear version of this result:

Lemma 2.18. *For every presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$, the adjunction from Proposition 2.16 induces an adjunction*

$$- \otimes_{\Gamma(\mathcal{C})} \mathcal{C} \colon \mathrm{Mod}_{\Gamma(\mathcal{C})}(\mathrm{Pr}^{\mathrm{L}}) \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})) : \Gamma^{\mathcal{C}}.$$

Furthermore:

(1) the left adjoint $- \otimes_{\Gamma(\mathcal{C})} \mathcal{C}$ is fully faithful and symmetric monoidal;

(2) the right adjoint $\Gamma^{\mathcal{C}}$ preserves colimits and satisfies the projection formula: the canonical map

$$\mathcal{D} \otimes_{\Gamma(\mathcal{C})} \Gamma^{\mathcal{C}}(\mathcal{E}) \to \Gamma^{\mathcal{C}}((\mathcal{D} \otimes_{\Gamma(\mathcal{C})} \mathcal{C}) \otimes_{\mathcal{C}} \mathcal{E})$$

is an equivalence for every $\Gamma(\mathcal{C})$-module $\mathcal{D}$ and $\mathcal{C}$-module $\mathcal{E}$.

Proof. The adjunction is obtained as a concatenation of the following two adjunctions:

$$\mathrm{Mod}_{\Gamma(\mathcal{C})}(\mathrm{Pr}^{\mathrm{L}}) \underset{\Gamma^{\mathrm{lin}}}{\overset{- \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}}{\rightleftarrows}} \mathrm{Mod}_{\Gamma(\mathcal{C}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})) \underset{\mathrm{fgt}}{\overset{- \otimes_{\Gamma(\mathcal{C}) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}} \mathcal{C}}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})).$$

The first adjunction is an instance of [Lur17, Example 7.3.2.8] while the second adjunction is the base change adjunction of the counit map $\Gamma(C) \otimes_{\mathcal{B}} \Omega_{\mathcal{B}} \to C$. As both left adjoints are symmetric monoidal, so is their composite $- \otimes_C \Gamma(C)$.

To show that the functor $\Gamma^C \colon \mathrm{Mod}_C(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})) \to \mathrm{Mod}_{\Gamma(C)}(\mathrm{Pr}^{\mathrm{L}})$ preserves colimits, it suffices to show that the functor $\Gamma \colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}$ preserves colimits, as colimits in module categories are computed underlying. As $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \simeq (\mathrm{Pr}^{\mathrm{R}}(\mathcal{B}))^{\mathrm{op}}$ by [MW22, Proposition 6.4.7], we may pass to opposite categories and instead show that $\Gamma^R \colon \mathrm{Pr}^{\mathrm{R}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{R}}$ preserves limits. By [MW22, Proposition 6.4.10] the inclusion $\mathrm{Pr}^{\mathrm{R}}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ preserves limits, hence it suffices to show that the evaluation functor $\Gamma = \mathrm{ev}_1 \colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$ preserves limits. This is clear, as limits in $\mathrm{Cat}(\mathcal{B}) = \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ are computed pointwise.

We next show that the functor Γ^C satisfies the projection formula. Recall that this is automatic whenever $\mathcal{D}$ is dualizable in $\mathrm{Mod}_{\Gamma(C)}(\mathrm{Pr}^{\mathrm{L}})$, see for example [FHM03, Proposition 3.12] or [Car+22, Lemma 4.15]. Since we already showed that Γ^C commutes with colimits, it will suffice to show that the ∞-category $\mathrm{Mod}_{\Gamma(C)}(\mathrm{Pr}^{\mathrm{L}})$ is generated under colimits by dualizable objects. Since the free $\Gamma(C)$-module functor $\Gamma(C) \otimes - \colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Mod}_{\Gamma(C)}(\mathrm{Pr}^{\mathrm{L}})$ is symmetric monoidal and its image generates $\mathrm{Mod}_{\Gamma(C)}(\mathrm{Pr}^{\mathrm{L}})$ under colimits, it suffices to show that Pr^{L} is generated under colimits by dualizable objects. This holds by [RS22, Lemma 7.14].

Finally, we show that the left adjoint $- \otimes_{\Gamma(C)} C$ is fully faithful, or equivalently that the unit $\mathcal{D} \to \Gamma^C(\mathcal{D} \otimes_{\Gamma(C)} C)$ of the adjunction is an equivalence. This is a special case of the projection formula applied to $\mathcal{E} = C$. □

For general C-modules $\mathcal{D}$ and $\mathcal{E}$ in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$, it is not clear in general how the parametrized relative tensor product $\mathcal{D} \otimes_C \mathcal{E}$ looks like. The situation improves when $\mathcal{D}$ comes from a $\Gamma(C)$-module in Pr^{L}:

Corollary 2.19. *Let $C \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$, let $\mathcal{D}, \mathcal{E} \in \mathrm{Mod}_C(\mathrm{Pr}^{\mathrm{L}})$, and assume that $\mathcal{D} = \mathcal{D}_0 \otimes_{\Gamma(C)} C$ for some $\mathcal{D}_0 \in \mathrm{Mod}_{\Gamma(C)}(\mathrm{Pr}^{\mathrm{L}})$. Then there is for every $A \in \mathcal{B}$ an equivalence of $C(A)$-linear ∞-categories*

$$\mathcal{D}(A) \otimes_{C(A)} \mathcal{E}(A) \xrightarrow{\sim} (\mathcal{D} \otimes_C \mathcal{E})(A).$$

Proof. By passing to the slice category $\mathcal{B}_{/A}$, we may assume that A is terminal in $\mathcal{B}$. In this case, Lemma 2.18 provides equivalences of $\Gamma(C)$-linear ∞-categories

$$\mathcal{D}(1) \otimes_{C(1)} \mathcal{E}(1) = \Gamma^C(\mathcal{D}) \otimes_{\Gamma(C)} \Gamma^C(\mathcal{E}) \xleftarrow{\sim} \mathcal{D}_0 \otimes_{\Gamma(C)} \Gamma^C(\mathcal{E}) \xrightarrow{\sim} \Gamma^C(\mathcal{D} \otimes_C \mathcal{E}) = (\mathcal{D} \otimes_C \mathcal{E})(1).$$

□

2.2 Classification of $\mathcal{C}$-linear functors

Given a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$, one may define for every object $A \in \mathcal{B}$ a presentable $\mathcal{B}$-category $\mathcal{C}^A$ given by $\mathcal{C}^A(B) = \mathcal{C}(A \times B)$. It comes equipped with a natural tensoring over $\mathcal{C}$ in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. The goal of this section is to give a classification of $\mathcal{C}$-linear $\mathcal{B}$-functors $F\colon \mathcal{C}^A \to \mathcal{C}^B$ for objects $A, B \in \mathcal{B}$: we will show in Theorem 2.32 below that evaluation at the object $\Delta_! \mathbb{1}_A \in \mathcal{C}(A \times A) = \mathcal{C}^A(A)$ determines an equivalence

$$\mathrm{ev}_{\Delta_! \mathbb{1}}\colon \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{C}^B) \xrightarrow{\sim} \mathcal{C}(A \times B),$$

whose inverse sends an object $D \in \mathcal{C}(A \times B)$ to the $\mathcal{B}$-functor $(\mathrm{pr}_B)_!(\mathrm{pr}_A^*(-) \otimes D)\colon \mathcal{C}^A \to \mathcal{C}^B$.

The $\mathcal{B}$-category of $\mathcal{C}$-linear $\mathcal{B}$-categories

Fix a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$, and consider the ∞-category $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ of left $\mathcal{C}$-modules in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. We refer to the objects of $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ as *$\mathcal{C}$-linear $\mathcal{B}$-categories* and to the morphisms as *$\mathcal{C}$-linear $\mathcal{B}$-functors*. In particular, $\mathcal{C}$-linear $\mathcal{B}$-functors will always preserve colimits by convention.

For any object $B \in \mathcal{B}$, we obtain an object $\pi_B^* \mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}))$. By [MW22, Definition 7.2.2], the ∞-categories $\mathrm{Mod}_{\pi_B^* \mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}))$ assemble into a $\mathcal{B}$-category $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}})$. By [MW22, Proposition 7.2.7], we have:

(1) The $\mathcal{B}$-category $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}})$ is complete and cocomplete;

(2) The $\mathcal{B}$-functor $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}) \to \mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}$ preserves limits and colimits;

(3) The relative tensor product $- \otimes_{\mathcal{C}} -\colon \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}) \times \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}) \to \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}})$ preserves colimits in both variables.

Since $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ is closed symmetric monoidal, so is $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$: for every pair of $\mathcal{C}$-linear $\mathcal{B}$-categories $\mathcal{D}$ and $\mathcal{E}$, there exists an internal hom object $\underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$. We let $\mathrm{Fun}_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$ denote its underlying ∞-category, whose objects are the $\mathcal{C}$-linear $\mathcal{B}$-functors $F\colon \mathcal{D} \to \mathcal{E}$, and whose morphisms will be referred to as *$\mathcal{C}$-linear natural transformations*. Given two $\mathcal{C}$-linear $\mathcal{B}$-functors $F, F'\colon \mathcal{D} \to \mathcal{E}$, we let $\mathrm{Nat}_{\mathcal{C}}(F, F')$ denote the mapping space from F to F' in $\mathrm{Fun}_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$.

Definition 2.20. A $\mathcal{C}$-linear $\mathcal{B}$-functor $F\colon \mathcal{D} \to \mathcal{E}$ is called an *internal left adjoint in* $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ if there is a $\mathcal{C}$-linear $\mathcal{B}$-functor $G\colon \mathcal{E} \to \mathcal{D}$ equipped with $\mathcal{C}$-linear transformations $\varepsilon\colon FG \Rightarrow \mathrm{id}$ and $\eta\colon \mathrm{id} \Rightarrow GF$ satisfying the triangle identities.

We have the following characterization of internal left adjoints:

Lemma 2.21. *A $\mathcal{C}$-linear $\mathcal{B}$-functor $F\colon \mathcal{D} \to \mathcal{E}$ is an internal left adjoint in* $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ *if and only if its right adjoint $G\colon \mathcal{D} \to \mathcal{E}$ in* $\mathrm{Cat}(\mathcal{B})$ *preserves colimits and satisfies the projection formula: for every $B \in \mathcal{B}$, every $C \in \mathcal{C}(B)$ and every $E \in \mathcal{E}(B)$, the canonical map $C \otimes_B G(E) \to G(C \otimes_B E)$ is an equivalence in $\mathcal{D}(B)$.*

Proof. By Proposition A.7, the right adjoint G of G is a right adjoint internal to $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}(\mathcal{B}))$ if and only if G satisfies the projection formula. The adjunction lifts further to $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ if and only if G preserves colimits. □

The property for a $\mathcal{C}$-linear functor F to be an internal left adjoint can be checked locally in $\mathcal{B}$:

Proposition 2.22. *Let $F\colon \mathcal{D} \to \mathcal{E}$ be a $\mathcal{C}$-linear $\mathcal{B}$-functor and assume that $B \twoheadrightarrow 1$ is an an effective epimorphism in $\mathcal{B}$. Then F is an internal left adjoint in* $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ *if and only if $\pi_B^* F$ is an internal left adjoint in* $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}))$.

Proof. Let $G\colon \mathcal{E} \to \mathcal{D}$ be the right adjoint of F. Since $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is a parametrized subcategory of $\mathrm{Cat}_{\mathcal{B}}$, G preserves colimits if and only if $\pi_B^* G$ does. By Lemma 2.21, it thus remains to show that G satisfies the projection formula if and only if $\pi_B^* G$ does. This is true because checking that for objects $A \in \mathcal{B}$, $C \in \mathcal{C}(A)$ and $E \in \mathcal{E}(A)$ the map $C \otimes_B G(E) \to G(C \otimes_B E)$ is an equivalence in $\mathcal{D}(A)$ can be done after pulling back along the effective epimorphism $A \times B \twoheadrightarrow A$. □

Free and cofree $\mathcal{C}$-linear $\mathcal{B}$-categories

Continue to fix a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$. We will next associate to every object $A \in \mathcal{B}$ two $\mathcal{C}$-linear $\mathcal{B}$-categories $\mathcal{C}[A]$ and $\mathcal{C}^A$, the *free* and *cofree $\mathcal{C}$-linear $\mathcal{B}$-categories on A*, and prove that they are in fact equivalent, see Corollary 2.29 below. The arguments are entirely analogous to those of [Car+22, Section 4.1].

Definition 2.23. Let $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ be a $\mathcal{C}$-linear $\mathcal{B}$-category. For an object $A \in \mathcal{B}$, we define the $\mathcal{C}$-linear $\mathcal{B}$-categories $\mathcal{D}[A]$ and $\mathcal{D}^A$ by

$$\mathcal{D}[A] := \mathrm{colim}_A\, \mathcal{D} \qquad \text{and} \qquad \mathcal{D}^A := \lim_A \mathcal{D},$$

the A-indexed colimit and limit of the constant diagram on $\mathcal{D}$ in the $\mathcal{B}$-category $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}})$. For a morphism $f\colon A \to B$ in $\mathcal{B}$, we denote by

$$f_!\colon \mathcal{D}[A] \to \mathcal{D}[B] \qquad\qquad \text{and} \qquad\qquad f^*\colon \mathcal{D}^B \to \mathcal{D}^A$$

the induced $\mathcal{C}$-linear $\mathcal{B}$-functors. Their right adjoints in $\mathrm{Cat}(\mathcal{B})$ are denoted by

$$f^*\colon \mathcal{D}[B] \to \mathcal{D}[A] \qquad\qquad \text{and} \qquad\qquad f_*\colon \mathcal{D}^A \to \mathcal{D}^B.$$

Since the forgetful functors $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}) \to \mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}}$ preserve limits by [MW22, Proposition 7.2.7] and [MW22, Proposition 6.4.9], the underlying $\mathcal{B}$-category of $\mathcal{D}^A$ is the $\mathcal{B}$-category $(\pi_A)_*\pi_A^*\mathcal{D} = \mathcal{D}(A \times -)\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

Corollary 2.24. *There is a natural equivalence of $\mathcal{C}$-linear $\mathcal{B}$-categories*

$$\underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}[A], \mathcal{D}) \xrightarrow{\sim} \mathcal{D}^A.$$

Proof. The internal hom $\mathcal{B}$-functor $\underline{\mathrm{Fun}}_{\mathcal{C}}(-, \mathcal{D})\colon \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \to \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}})$ turns colimits in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}})$ into limits and returns $\mathcal{D}$ when evaluated at the monoidal unit $\mathcal{C} \in \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$, hence we have

$$\underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}[A], \mathcal{D}) = \underline{\mathrm{Fun}}_{\mathcal{C}}(\mathrm{colim}_A\, \mathcal{C}, \mathcal{D}) \simeq \lim_A \underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}, \mathcal{D}) \simeq \lim_A \mathcal{D} = \mathcal{D}^A. \qquad \square$$

Lemma 2.25. *For objects $A, B \in \mathcal{B}$, there is an equivalence*

$$\mathcal{C}[A] \otimes_{\mathcal{C}} \mathcal{D}[B] \simeq \mathcal{D}[A \times B].$$

Proof. Since $- \otimes_{\mathcal{C}} -$ preserves colimits in both variables and $\mathcal{C} \otimes_{\mathcal{C}} \mathcal{D} \simeq \mathcal{D}$, this follows from the observation that $\mathrm{colim}_A\, \mathrm{colim}_B \simeq \mathrm{colim}_{A\times B}$: both sides are left adjoint to the diagonal $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}) \to \underline{\mathrm{Fun}}_{\mathcal{B}}(A \times B, \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}))$. $\square$

Proposition 2.26. *Let $\mathcal{D}$ be a $\mathcal{C}$-linear $\mathcal{B}$-category.*

(1) For every morphism $f\colon A \to B$ in $\mathcal{B}$, the $\mathcal{C}$-linear $\mathcal{B}$-functor $f_!\colon \mathcal{C}[A] \to \mathcal{C}[B]$ admits a $\mathcal{C}$-linear right adjoint $f^*\colon \mathcal{C}[B] \to \mathcal{C}[A]$;

(2) For every pullback square in $\mathcal{B}$ as on the left, the induced square of $\mathcal{C}$-linear $\mathcal{B}$-functors as on the right is (vertically) right adjointable in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$*:*

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\ B' \arrow[r, "\beta"] & B \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{C}[A'] \arrow[r, "\alpha_!"] \arrow[d, "f'_!"'] & \mathcal{C}[A] \arrow[d, "f_!"] \\ \mathcal{C}[B'] \arrow[r, "\beta_!"] & \mathcal{C}[B]; \end{tikzcd}$$

(3) *For every morphism $f\colon A \to B$ in $\mathcal{B}$, the C-linear $\mathcal{B}$-functor $f^*\colon \mathcal{D}^B \to \mathcal{D}^A$ admits a C-linear left adjoint $f_!\colon \mathcal{D}^A \to \mathcal{D}^B$;*

(4) *For every pullback square in $\mathcal{B}$ as on the left, the induced square of C-linear $\mathcal{B}$-functors as on the right is (vertically) left adjointable in* $\mathrm{Mod}_C(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$*:*

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\ B' \arrow[r, "\beta"] & B \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}^{A'} & \mathcal{D}^{A} \arrow[l, "\alpha^*"'] \\ \mathcal{D}^{B'} \arrow[u, "f'^*"] & \mathcal{D}^{B}; \arrow[l, "\beta^*"'] \arrow[u, "f^*"'] \end{tikzcd}$$

(5) *For every morphism $f\colon A \to B$ in $\mathcal{B}$, there are naturally commutative squares of C-linear $\mathcal{B}$-functors*

$$\begin{tikzcd} \underline{\mathrm{Fun}}_C(C[B], \mathcal{D}) \arrow[r, "\simeq"] \arrow[d, "-\circ f_!"'] & \mathcal{D}^B \arrow[d, "f^*"] \\ \underline{\mathrm{Fun}}_C(C[A], \mathcal{D}) \arrow[r, "\simeq"] & \mathcal{D}^A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \underline{\mathrm{Fun}}_C(C[A], \mathcal{D}) \arrow[r, "\simeq"] \arrow[d, "-\circ f^*"'] & \mathcal{D}^A \arrow[d, "f_!"] \\ \underline{\mathrm{Fun}}_C(C[B], \mathcal{D}) \arrow[r, "\simeq"] & \mathcal{D}^B. \end{tikzcd}$$

Proof. Parts (3) and (4) follow immediately from (1) and (2) because of the natural equivalence $\underline{\mathrm{Fun}}_C(C[A], \mathcal{D}) \simeq \mathcal{D}^A$. For (5), the left-hand square is simply naturality in A of the equivalence $\underline{\mathrm{Fun}}_C(C[A], \mathcal{D}) \simeq \mathcal{D}^A$, and the right-hand square is obtained from this by passing to internal left adjoints in $\mathrm{Mod}_C(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$. For parts (1) and (2), consider the base change functor $C \otimes_{\Omega_{\mathcal{B}}} -\colon \mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}} \simeq \mathrm{Mod}_{\Omega_{\mathcal{B}}}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}) \to \mathrm{Mod}_C(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}})$. It is a colimit-preserving symmetric monoidal 2-functor, and thus we have $C[A] \simeq C \otimes_{\Omega_{\mathcal{B}}} \Omega_{\mathcal{B}}[A]$. It thus suffices to show the claim when $C = \Omega_{\mathcal{B}}$. In this case we have an equivalence $\Omega_{\mathcal{B}}[A] \simeq \mathrm{PSh}_{\Omega}(A)$ by [MW21, Theorem 6.1.1], and by [MW21, Lemma 6.1.3] the functor $f_!\colon \mathrm{PSh}_{\Omega}(A) \to \mathrm{PSh}_{\Omega}(B)$ is given by left Kan extension. The right adjoint $f^*\colon \mathrm{PSh}_{\Omega}(B) \to \mathrm{PSh}_{\Omega}(A)$ of $f_!$ is therefore just the restriction functor, which is a morphism in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$, proving (1). For part (2), it suffices to check that for every object $C \in \mathcal{B}$, evaluating the square at C gives a vertically right adjointable square of ∞-categories. In the case at hand, this square looks like

$$\begin{tikzcd} \mathcal{B}_{/A' \times C} \arrow[r, "(\alpha\times 1)\circ -"] \arrow[d, "(f'\times 1)\circ -"'] & \mathcal{B}_{/A\times C} \arrow[d, "(f\times 1)\circ -"] \\ \mathcal{B}_{/B'\times C} \arrow[r, "(\beta\times 1)\circ -"] & \mathcal{B}_{/B\times C}, \end{tikzcd}$$

which is adjointable as $\mathcal{B}$ satisfies base change. $\square$

Corollary 2.27. *For every object $A \in \mathcal{B}$, the C-linear pairing*

$$C[A] \otimes_C C[A] \simeq C[A \times A] \xrightarrow{\Delta^*} C[A] \xrightarrow{A_!} C$$

is part of a duality datum in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$*, exhibiting the* $\mathcal{C}$*-linear* $\mathcal{B}$*-category* $\mathcal{C}[A]$ *as self-dual.*

Proof. The coevaluation is given by $\mathcal{C} \xrightarrow{A^*} \mathcal{C}[A] \xrightarrow{\Delta_!} \mathcal{C}[A \times A] \simeq \mathcal{C}[A] \otimes_{\mathcal{C}} \mathcal{C}[A]$. The first triangle identity follows from the following commutative diagram:

$$\begin{tikzcd}
 & & \mathcal{C}[A\times A\times A] \arrow[dr, "(\Delta\times 1)^*"] & & \\
 & \mathcal{C}[A\times A] \arrow[ur, "(1\times\Delta)_!"] \arrow[dr, "\Delta^*"] & & \mathcal{C}[A\times A] \arrow[dr, "(\mathrm{pr}_2)_!"] & \\
\mathcal{C}[A] \arrow[ur, "\mathrm{pr}_1^*"] \arrow[rr, equal] & & \mathcal{C}[A] \arrow[ur, "\Delta_!"] \arrow[rr, equal] & & \mathcal{C}[A],
\end{tikzcd}$$

where the square commutes via the Beck-Chevalley equivalence. The other triangle identity is analogous. □

Lemma 2.28. *Let* $f\colon A \to B$ *be a morphism in* $\mathcal{B}$*. Then the following diagrams commute:*

$$\begin{tikzcd}
\mathcal{C}[B] \arrow[r, "f^*"] \arrow[d, "\simeq"'] & \mathcal{C}[A] \arrow[d, "\simeq"] \\
\mathcal{C}[B]^\vee \arrow[r, "(f_!)^\vee"] & \mathcal{C}[A]^\vee
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathcal{C}[A] \arrow[r, "f_!"] \arrow[d, "\simeq"'] & \mathcal{C}[B] \arrow[d, "\simeq"] \\
\mathcal{C}[A]^\vee \arrow[r, "(f^*)^\vee"] & \mathcal{C}[B]^\vee.
\end{tikzcd}$$

Proof. We will prove the left diagram. The proof for the right diagram is analogous and is left to the reader. Expanding the definition of $(f_!)^\vee$ by plugging in the explicit evaluation and coevaluation maps from Corollary 2.27, we see it is given by the composite

$$\mathcal{C}[B] \xrightarrow{(\mathrm{pr}_B)^*} \mathcal{C}[B\times A] \xrightarrow{(1\times(f,1))_!} \mathcal{C}[B\times B\times A] \xrightarrow{(\Delta\times 1)^*} \mathcal{C}[B\times A] \xrightarrow{(\mathrm{pr}_A)_!} \mathcal{C}[A].$$

Observe that the maps $1\times(f,1)$ and $\Delta\times 1$ fit into a pullback square

$$\begin{tikzcd}
A \arrow[r, "(f,1)"] \arrow[d, "(f,1)"'] \arrow[dr, phantom, "\lrcorner"] & B\times A \arrow[d, "1\times(f,1)"] \\
B\times A \arrow[r, "\Delta\times 1"] & B\times B\times A,
\end{tikzcd}$$

and thus it follows from base change that the above composite is homotopic to

$$\mathcal{C}[B] \xrightarrow{(\mathrm{pr}_B)^*} \mathcal{C}[B\times A] \xrightarrow{(f,1)^*} \mathcal{C}[A] \xrightarrow{(f,1)_!} \mathcal{C}[B\times A] \xrightarrow{(\mathrm{pr}_A)_!} \mathcal{C}[A].$$

But this composite is the functor $f^*\colon \mathcal{C}[B] \to \mathcal{C}[A]$, as desired. □

Under the equivalence $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}[A], \mathcal{C}^A) \simeq \mathcal{C}^A(A) = \mathcal{C}(A\times A)$, the object $\Delta_!\mathbb{1}_A$ corresponds to a $\mathcal{C}$-linear $\mathcal{B}$-functor $\mathcal{C}[A] \to \mathcal{C}^A$.

Corollary 2.29. *For every object $A \in \mathcal{B}$, the $\mathcal{B}$-functor $\mathcal{C}[A] \to \mathcal{C}^A$ is an equivalence of $\mathcal{C}$-linear $\mathcal{B}$-categories. Furthermore, for every map $f\colon A \to B$ in $\mathcal{B}$, the following diagrams commute:*

$$\begin{tikzcd} \mathcal{C}[B] \arrow[r, "\simeq"] \arrow[d, "f^*"'] & \mathcal{C}^B \arrow[d, "f^*"] \\ \mathcal{C}[A] \arrow[r, "\simeq"] & \mathcal{C}^A \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{C}[A] \arrow[r, "\simeq"] \arrow[d, "f_!"'] & \mathcal{C}^A \arrow[d, "f_!"] \\ \mathcal{C}[B] \arrow[r, "\simeq"] & \mathcal{C}^B. \end{tikzcd}$$

Proof. By Corollary 2.27, the pairing $\mathcal{C}[A] \otimes_{\mathcal{C}} \mathcal{C}[A] \to \mathcal{C}$ adjoints over to an equivalence $\mathcal{C}[A] \xrightarrow{\sim} \underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}[A], \mathcal{C})$. Composing this with the equivalence $\underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}[A], \mathcal{C}) \simeq \mathcal{C}^A$ from Corollary 2.24 gives an equivalence $\mathcal{C}[A] \xrightarrow{\sim} \mathcal{C}^A$. It remains to show that this is the $\mathcal{B}$-functor $\mathcal{C}[A] \to \mathcal{C}^A$ classified by $\Delta_! \mathbb{1}_A \in \mathcal{C}^A(A)$. Adjoining over once more, it suffices to show that the dual $\mathcal{C} \to \mathcal{C}^{A \times A}$ of the pairing $\mathcal{C}[A \times A] \to \mathcal{C}$ is classified by the object $\Delta_! \mathbb{1}_A \in \mathcal{C}(A \times A)$. But by Proposition 2.26(5) and the construction of the pairing, this is the composite $\mathcal{C} \xrightarrow{A^*} \mathcal{C}^A \xrightarrow{\Delta_!} \mathcal{C}^{A \times A}$. This proves the first claim.

The two commutative diagrams follow from a combination of Lemma 2.28 and Proposition 2.26(5). $\square$

Classification of $\mathcal{C}$-linear functors

As a result of Corollary 2.29, the $\mathcal{C}$-linear $\mathcal{B}$-category $\mathcal{C}^A$ is, in a precise sense, the free $\mathcal{C}$-linear $\mathcal{B}$-category on A: any $\mathcal{C}$-linear $\mathcal{B}$-functor $\mathcal{C}^A \to \mathcal{D}$ to another $\mathcal{C}$-linear $\mathcal{B}$-category $\mathcal{D}$ is fully determined by where it sends the object $\Delta_! \mathbb{1}_A \in \mathcal{C}(A \times A) = \mathcal{C}^A(A)$.

Corollary 2.30. *For an object $A \in \mathcal{B}$ and a $\mathcal{C}$-linear $\mathcal{B}$-category $\mathcal{D}$, the composite*

$$\underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{D}) \xrightarrow{(-)^A} \underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}^{A \times A}, \mathcal{D}^A) \xrightarrow{\mathrm{ev}_{\Delta_! \mathbb{1}_A}} \mathcal{D}^A$$

is an equivalence of $\mathcal{C}$-linear $\mathcal{B}$-categories, where the last map denotes evaluation at $\Delta_! \mathbb{1}_A \in \mathcal{C}(A \times A)$.

Proof. Let $U \in \mathcal{C}[A](A)$ denote the object classified by the identity $\mathcal{C}[A] \to \mathcal{C}[A]$ under the equivalence $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}[A], \mathcal{C}[A]) \simeq \mathcal{C}[A](A)$. By Corollary 2.29, it suffices to show the statement for $\mathcal{C}[A]$ instead of $\mathcal{C}^A$: the composite

$$\underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}[A], \mathcal{D}) \xrightarrow{(-)^A} \underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}[A]^A, \mathcal{D}^A) \xrightarrow{\mathrm{ev}_U} \mathcal{D}^A$$

is an equivalence of $\mathcal{C}$-linear $\mathcal{B}$-categories. But it is clear from functoriality in $\mathcal{D}$ that this composite is simply the equivalence $\underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}[A], \mathcal{D}) \xrightarrow{\sim} \mathcal{D}^A$ from Corollary 2.24. $\square$

Observation 2.31. Since the functor $(-)^A\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ is lax symmetric monoidal (as it is the composite of the symmetric monoidal functor $A^*\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B})$ and its right adjoint A_*), the $\mathcal{B}$-category $\mathcal{D}^A$ is canonically tensored over $\mathcal{C}^A$. We claim that the inverse of the equivalence of Corollary 2.30 is adjoint to the composite $\mathcal{D}^A \otimes_{\mathcal{C}} \mathcal{C}^A \xrightarrow{-\otimes_A -} \mathcal{D}^A \xrightarrow{A_!} \mathcal{D}$. To see this, it suffices to show that the composite $\mathcal{D}^A \to \underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}^A$ is equivalent to the identity. Expanding definitions, this composite is given by

$$\mathcal{D}^A \xrightarrow{\mathrm{pr}_1^*} \mathcal{D}^{A\times A} \simeq \mathcal{D}^{A\times A} \otimes \mathcal{C} \xrightarrow{1\otimes\Delta_!\mathbb{1}_A} \mathcal{D}^{A\times A} \otimes_{\mathcal{C}} \mathcal{C}^{A\times A} \xrightarrow{-\otimes_{A\times A}-} \mathcal{D}^{A\times A} \xrightarrow{\mathrm{pr}_{2!}} \mathcal{D}^A.$$

By the projection formula, this is equivalent to the composite

$$\mathrm{pr}_{2!}\Delta_!(\Delta^*\mathrm{pr}_1^*(-)\otimes_A \mathbb{1}_A)\colon \mathcal{D}^A \to \mathcal{D}^A,$$

which is equivalent to the identity since $\Delta\colon A \to A\times A$ is a section of both projections $\mathrm{pr}_1, \mathrm{pr}_2\colon A\times A \to A$.

Specializing to $\mathcal{D} = \mathcal{C}^B$, we arrive at the main result of this subsection:

Theorem 2.32. *For objects $A, B \in \mathcal{B}$, evaluation at $\Delta_!\mathbb{1}_A \in \mathcal{C}^A(A)$ induces an equivalence of $\mathcal{C}$-linear $\mathcal{B}$-categories*

$$\underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{C}^B) \xrightarrow{\sim} \mathcal{C}^{A\times B}.$$

The inverse is adjoint to the composite

$$\mathcal{C}^{A\times B}\otimes\mathcal{C}^A \xrightarrow{1\otimes\mathrm{pr}_A^*} \mathcal{C}^{A\times B}\otimes\mathcal{C}^{A\times B} \xrightarrow{-\otimes_{A\times B}-} \mathcal{C}^{A\times B} \xrightarrow{\mathrm{pr}_{B!}} \mathcal{C}^B. \qquad \square$$

Definition 2.33. Let $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ be a $\mathcal{C}$-linear $\mathcal{B}$-category and let $F\colon \mathcal{C}^A \to \mathcal{D}$ be a $\mathcal{C}$-linear $\mathcal{B}$-functor. We define $D_F := F_A(\Delta_!\mathbb{1}_A) \in \mathcal{D}(A)$. Conversely, for an object $D \in \mathcal{D}(A)$, we let $F_D\colon \mathcal{C}^A \to \mathcal{D}$ denote the $\mathcal{C}$-linear $\mathcal{B}$-functor $F_D = A_!(-\otimes_A D)\colon \mathcal{C}^A \to \mathcal{D}$.

More informally, Corollary 2.30 says that every $\mathcal{C}$-linear $\mathcal{B}$-functor $F\colon \mathcal{C}^A \to \mathcal{D}$ is naturally equivalent to the $\mathcal{B}$-functor $A_!(-\otimes_A D_F)$. One should be a bit careful with the interpretation of these symbols "$A_!$" and "$\otimes_A$", as they depend on $\mathcal{D}$. For example, in the case of Theorem 2.32, where $\mathcal{D} = \mathcal{C}^B$ for some $B \in \mathcal{B}$, the statement is that every $\mathcal{C}$-linear functor $F\colon \mathcal{C}^A \to \mathcal{C}^B$ is equivalent to $\mathrm{pr}_{B!}(\mathrm{pr}_A^*(-)\otimes_{A\times B} D_F)$ for a unique $D_F \in \mathcal{C}^B(A) = \mathcal{C}(A\times B)$.

Corollary 2.34. *Let $A \in \mathcal{B}$, let $\mathcal{D}$ be a $\mathcal{C}$-linear $\mathcal{B}$-category and let $F, G\colon \mathcal{C}^A \to \mathcal{D}$ be two $\mathcal{C}$-linear functors. Evaluation at $\Delta_!\mathbb{1}_A$ induces an equivalence of spaces*

$$\mathrm{Nat}_{\mathcal{C}}(F, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(A)}(D_F, D_G).$$

Proof. This follows immediately from the fact that evaluation at $\Delta_!\mathbb{1}_A$ is an equivalence from $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{D})$ to $\mathcal{D}(A)$ by Corollary 2.30, so that it in particular induces equivalences on mapping spaces. $\square$

2.3 Formal inversions

If $\mathcal{C}$ is a presentably symmetric monoidal ∞-category and S is a small subcategory of $\mathcal{C}$, there is another presentably symmetric monoidal ∞-category $\mathcal{C}[S^{-1}]$ equipped with the universal morphism $\mathcal{C} \to \mathcal{C}[S^{-1}]$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ out of $\mathcal{C}$ which sends all objects in S to invertible objects, see Robalo [Rob15, Section 2.1] and Hoyois [Hoy17, Section 6.1]. The goal of this subsection is to discuss a parametrized variant of this construction.

Definition 2.35 (Formal inversion)**.** Let $\mathcal{C}$ is a presentably symmetric monoidal $\mathcal{B}$-category, and let $S \subseteq \mathcal{C}$ be a full subcategory. A morphism $p\colon \mathcal{C} \to \mathcal{C}'$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ is said to *exhibit $\mathcal{C}'$ as a formal inversion of S in $\mathcal{C}$* if for every other $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ the map of spaces

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))}(\mathcal{C}', \mathcal{D}) \to \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))}(\mathcal{C}, \mathcal{D})$$

given by precomposition with p is a monomorphism hitting those path components corresponding to $\mathcal{B}$-functors $F\colon \mathcal{C} \to \mathcal{D}$ which invert the objects of S: for every $A \in \mathcal{B}$, the functor $F_A\colon \mathcal{C}(A) \to \mathcal{D}(A)$ sends every object $X \in S(A)$ to an invertible object in $\mathcal{D}(A)$.

If such formal inversion $p\colon \mathcal{C} \to \mathcal{C}'$ exists, it is uniquely determined by this universal property and we will denote it by $p\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$. It is not clear to the author in what generality parametrized formal inversions can be expected to exist. In this article, we will contend ourselves with some specific situations in which the formal inversion can be shown to exist.

A first such situation is when the subcategory S is generated by a (non-parametrized) full subcategory $S_0 \subseteq \mathcal{C}(1)$, in the following sense:

Definition 2.36. Let $\mathcal{C}$ be a $\mathcal{B}$-category and let $S \subseteq \mathcal{C}$ be a subcategory. Given a small subcategory $S_0 \subseteq \Gamma(S) \subseteq \Gamma(\mathcal{C})$, we will say that *$S$ is generated by S_0* if the following condition is satisfied: for every $X \in S(B)$, there exists an effective epimorphism $f\colon A \to B$ in $\mathcal{B}$ and an object $Y \in S_0 \subseteq \mathcal{C}(1) = \Gamma(\mathcal{C})$ such that $f^*X \simeq A^*Y$.

Note that in this case, a morphism $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ inverts the objects of S if and only if the underlying functor $\Gamma(F)\colon \Gamma(\mathcal{C}) \to \Gamma(\mathcal{D})$ inverts the objects of S_0.

Definition 2.37. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ and let $S \subseteq \mathcal{C}$ be a symmetric monoidal subcategory which is generated by a small subcategory $S_0 \subseteq \Gamma(\mathcal{C})$. Let $\Gamma(\mathcal{C})[S_0^{-1}]$ denote the (non-parametrized) formal inversion of S_0 in $\Gamma(\mathcal{C})$. We define the commutative $\mathcal{C}$-algebra $\mathcal{C}[S_0^{-1}]$ in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ as the image of $\Gamma(\mathcal{C})[S_0^{-1}]$ under the adjunction

$$- \otimes_{\Gamma(\mathcal{C})} \mathcal{C}\colon\ \mathrm{CAlg}_{\Gamma(\mathcal{C})}(\mathrm{Pr}^{\mathrm{L}}) \rightleftarrows \mathrm{CAlg}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))\ : \Gamma^{\mathcal{C}}$$

from Lemma 2.18.

Proposition 2.38. *Let $C \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ and let $S \subseteq C$ be a symmetric monoidal subcategory which is generated by $S_0 \subseteq \Gamma(C)$. Then the symmetric monoidal left adjoint $C \to C[S_0^{-1}]$ is a formal inversion of S in C.*

Proof. Let $F\colon C \to \mathcal{D}$ be morphism in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$. By the adjunction between $- \otimes_{\Gamma(C)} C$ and Γ^C, there is a one-to-one correspondence between morphisms $C[S_0^{-1}] \to \mathcal{D}$ of commutative C-algebras and morphisms $\Gamma(C)[S_0^{-1}] \to \Gamma(\mathcal{D})$ of commutative $\Gamma(C)$-algebras. The claim thus follows from the universal property of the formal inversion $\Gamma(C) \to \Gamma(C)[S_0^{-1}]$, combined with the fact that the $\mathcal{B}$-functor $F\colon C \to \mathcal{D}$ inverts the objects of S if and only if the underlying functor $\Gamma(F)\colon \Gamma(C) \to \Gamma(\mathcal{D})$ inverts the objects of S_0. □

Observation 2.39. In the situation of Definition 2.37, it follows directly from fully faithfulness of the functor $C \otimes_{\Gamma(C)}\colon \mathrm{Mod}_{\Gamma(C)}(\mathrm{Pr}^{\mathrm{L}}) \hookrightarrow \mathrm{Mod}_C(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ that the underlying ∞-category of $C[S_0^{-1}]$ is given by the non-parametrized formal inversion $\Gamma(C)[S_0^{-1}]$ of S_0 in $\Gamma(C)$. In particular, Proposition 2.38 not only shows that the formal inversion of S in C exists, but also that its underlying morphism in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is a non-parametrized formal inversion.

More generally, evaluating the map $C \to C[S_0^{-1}]$ at an object $A \in \mathcal{B}$ gives a symmetric monoidal left adjoint

$$C(A) \to C[S_0^{-1}](A)$$

which exhibits $C[S_0^{-1}](A)$ as a non-parametrized formal inversion of $S(A)$ in $C(A)$. Indeed, this follows immediately by applying the above observation to the slice category $\mathcal{B}_{/A}$, where we use that passing to slices preserves formal inversions by Proposition 2.47 below.

The upshot of Observation 2.39 is that the formal inversion in the above setting can be obtained as a *pointwise* formal inversion, in the sense that $C[S^{-1}]$ is given at $B \in \mathcal{B}$ as the non-parametrized formal inversion $C(B)[S(B)^{-1}]$. This observation will lead to the construction of parametrized formal inversions in more general situations. We start by recalling the functoriality of the formal inversion construction in the non-parametrized setting.

Definition 2.40. We define $\mathrm{Cat}_{\infty,\mathrm{aug}}$ as the full subcategory of $\mathrm{Ar}(\mathrm{Cat}_\infty) = \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ spanned by the morphisms in Cat_∞ corresponding to a fully faithful functor $\iota_S\colon S \hookrightarrow C$, where S is a small ∞-category. The functor $\iota_S\colon S \hookrightarrow C$ is called an *augmentation* of the

∞-category $\mathcal{C}$ and the pair $(\mathcal{C}, \iota_S)$ is called an *augmented ∞-category*. We will often abuse notation and write $(\mathcal{C}, S)$ for $(\mathcal{C}, \iota_S)$, identifying S with its image in $\mathcal{C}$.

Note that the forgetful functor $\mathrm{Cat}_{\infty,\mathrm{aug}} \to \mathrm{Cat}_\infty, (\mathcal{C}, S) \mapsto \mathcal{C}$ is faithful: for another augmented ∞-category $(\mathcal{D}, T)$, a morphism $(\mathcal{C}, S) \to (\mathcal{D}, T)$ in $\mathrm{Cat}_{\infty,\mathrm{aug}}$ may be identified with a functor $\mathcal{C} \to \mathcal{D}$ which sends the full subcategory $S \subseteq \mathcal{C}$ into $T \subseteq \mathcal{D}$.

Definition 2.41. We define the ∞-category $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ of *augmented presentably symmetric monoidal ∞-categories* as the pullback

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Cat}_{\infty,\mathrm{aug}} \arrow[d] \\ \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \arrow[r] & \mathrm{Cat}_\infty. \end{tikzcd}$$

Its objects are pairs $(\mathcal{C}, S)$ of a presentably symmetric monoidal ∞-category $\mathcal{C}$ equipped with an augmentation $\iota_S \colon S \hookrightarrow \mathcal{C}$.
We define a section $(-)_{\mathrm{inv}} \colon \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ of the forgetful functor by equipping a symmetric monoidal $\mathcal{C}$ with its collection of invertible objects.

Lemma 2.42. *The functor $(-)_{\mathrm{inv}} \colon \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ admits a left adjoint*

$$\mathcal{L} \colon \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

given on objects by sending a pair $(\mathcal{C}, S)$ to the formal inversion $\mathcal{C}[S^{-1}]$.

Proof. It suffices to show that for every pair $(\mathcal{C}, S)$, the formal inversion $\mathcal{C}[S^{-1}]$ is a left adjoint object to $(\mathcal{C}, S)$ under the functor $(-)_{\mathrm{inv}} \colon \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}}$. This follows from the universal property of $\mathcal{C}[S^{-1}]$: for every $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, precomposition with the functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ induces an inclusion of path components

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})}(\mathcal{C}[S^{-1}], \mathcal{D}) \hookrightarrow \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})}(\mathcal{C}, \mathcal{D})$$

whose image precisely consists of those functors $\mathcal{C} \to \mathcal{D}$ which send all objects of S to invertible objects in $\mathcal{D}$, i.e. the space of maps in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ from $(\mathcal{C}, S)$ to $\mathcal{D}_{\mathrm{inv}}$. □

We will now construct the pointwise formal inversion $\mathcal{L}(\mathcal{C}, S)$ in case $\mathcal{B} = \mathrm{PSh}(T)$ is a presheaf topos.

Construction 2.43. Let T be a small ∞-category, let $\mathcal{B} = \mathrm{PSh}(T)$ and let $\mathcal{C}$ be a presentably symmetric monoidal $\mathcal{B}$-category equipped with a small full subcategory $S \subseteq \mathcal{C}$. We let

$(\mathcal{C}, S)\colon T^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ denote the lift of the functor $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ which equips the ∞-category $\mathcal{C}(B)$ with the augmentation $S(B)$ for every $B \in T$. Composing with the functor $\mathcal{L}$ gives rise to a new functor

$$\mathcal{L}(\mathcal{C}, S)\colon T^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}),$$

given on objects by $B \mapsto \mathcal{C}(B)[S(B)^{-1}]$. This uniquely extends to a limit-preserving functor $\mathcal{L}(\mathcal{C}, S)\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

We claim that under suitable conditions, $\mathcal{L}(\mathcal{C}, S)$ is a presentably symmetric monoidal $\mathcal{B}$-category modeling the parametrized formal inversion $\mathcal{C}[S^{-1}]$. To this end, we need some basic properties of (non-parametrized) formal inversions.

Lemma 2.44. *Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a presentably symmetric monoidal ∞-category and let $S \subseteq S'$ be two small subcategories of $\mathcal{C}$. Assume that for every $A \in \mathcal{B}$ and $X' \in S'(A)$ there exists objects $X \in S(A)$ and $Y \in \mathcal{C}(A)$ such that $X \simeq X' \otimes Y$. Then any formal inversion $p\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ of S in $\mathcal{C}$ also exhibits $\mathcal{C}[S^{-1}]$ as a formal inversion of S' in $\mathcal{C}$.*

Proof. This is immediate from the observation that any symmetric monoidal $\mathcal{B}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ which inverts S must also invert S': if $X \simeq X' \otimes Y$, we get $F(X) \simeq F(X') \otimes F(Y)$ and thus if $F(X)$ is invertible, so must be $F(X')$ and $F(Y)$. □

Lemma 2.45. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ and let $S \subseteq \mathcal{C}$ be a small subcategory. Then the canonical map*

$$\mathcal{D} \otimes_{\mathcal{C}} \mathcal{C}[S^{-1}] \to \mathcal{D}[F(S)^{-1}]$$

obtained by adjunction from the $\mathcal{C}$-linear functor $\mathcal{C}[S^{-1}] \to \mathcal{D}[F(S)^{-1}]$ is an equivalence.

Proof. It follows from spelling out the adjunctions and universal properties that both sides are $\mathcal{D}$-algebras in Pr^{L} which admit a (necessarily unique) symmetric monoidal $\mathcal{D}$-linear map into another $\mathcal{D}$-algebra $\mathcal{E}$ if and only if the unit map $\mathcal{D} \to \mathcal{E}$ carries the objects of $F(S)$ to invertible objects in $\mathcal{E}$. The claim thus follows from the Yoneda lemma. □

We are now ready to prove that in certain cases the pointwise formal inversion $\mathcal{L}(\mathcal{C}, S)$ represents the parametrized formal inversion $\mathcal{C}[S^{-1}]$.

Proposition 2.46. *Let $\mathcal{B} = \mathrm{PSh}(T)$ be the ∞-topos of presheaves on some small ∞-category T. Let $\mathcal{C}$ be a presentably symmetric monoidal $\mathcal{B}$-category equipped with a full subcategory $S \subseteq \mathcal{C}$. Assume that the following property is satisfied:*

($*$) *For every morphism $f\colon A \to B$ in $T \subseteq \mathcal{B}$ and every object $X \in S(A)$, there exists objects $Y \in S(B)$ and $Z \in C(A)$ such that $f^*Y \simeq X \otimes Z \in C(A)$.*

Then the functor $\mathcal{L}(C,S)\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is a presentably symmetric monoidal $\mathcal{B}$-category and the map $C = \mathcal{L}(C,\emptyset) \to \mathcal{L}(C,S)$ exhibits $\mathcal{L}(C,S)$ as a formal inversion of S in C.

Proof. We first show that $\mathcal{L}(C,S)$ is a presentably symmetric monoidal $\mathcal{B}$-category and that the map $C \to \mathcal{L}(C,S)$ preserves colimits. This may be proved by pulling back C to the slice topos $\mathrm{PSh}(T)_{/B} \simeq \mathrm{PSh}(T_{/B})$ for every $B \in T$, hence we may assume that T admits a terminal object B. In this case, consider the full subcategory $S' \subseteq C$ given at $A \in T$ by

$$S'(A) := \{f^*X \in C(A) \mid X \in S(B)\},$$

where $f\colon A \to B$ is the unique map to the terminal object. Note that $S' \subseteq S$, as S is a parametrized subcategory of C. It is immediate from the definition that S' is generated by $S_0 := S(B)$, in the sense of Definition 2.36, and thus Proposition 2.38 provides a formal inversion $F\colon C \to C[S_0^{-1}]$ of S' in C. As in the proof of Lemma 2.44, the assumption ($*$) guarantees that F even inverts all the objects in S, and it follows that the map $C \to C[S_0^{-1}]$ uniquely extends to a map

$$\mathcal{L}(C,S) \to C[S_0^{-1}]$$

in $\mathrm{Fun}(T^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))$. It follows from Observation 2.39 and Lemma 2.44 that this map is pointwise an equivalence and hence that it is an equivalence of symmetric monoidal $\mathcal{B}$-categories. Since $C[S_0^{-1}]$ is presentably symmetric monoidal, it follows that so is $\mathcal{L}(C,S)$. Similarly we deduce that the map $C \to \mathcal{L}(C,S)$ preserves colimits.

We will now verify that $C \to \mathcal{L}(C,S)$ is indeed a formal inversion. Assume that $\mathcal{D}$ is a presentably symmetric monoidal $\mathcal{B}$-category. Because of the adjunction $(-)_{\mathrm{inv}} \dashv \mathcal{L}$, fiberwise-colimit-preserving symmetric monoidal $\mathcal{B}$-functors $G\colon \mathcal{L}(C,S) \to \mathcal{D}$ correspond to fiberwise-colimit-preserving symmetric monoidal $\mathcal{B}$-functors $G'\colon C \to \mathcal{D}$ inverting the objects in S. It thus remains to show that if G' preserves all $\mathcal{B}$-colimits, then so does its extension G. This may be tested after passing to the slice topoi $\mathrm{PSh}(T)_{/B} \simeq \mathrm{PSh}(T_{/B})$ for $B \in T$, and since the construction of $\mathcal{L}(C,S)$ commutes with passage to slice topoi we may assume that T admits a terminal object. In this case, there is a symmetric monoidal equivalence $\mathcal{L}(C,S) \simeq C[S_0^{-1}]$, and since $C[S_0^{-1}]$ is a formal inversion of S in C it follows that G' admits a unique colimit-preserving symmetric monoidal extension to $C[S_0^{-1}] \to \mathcal{D}$. This in particular preserves fiberwise colimits, so must agree with G, finishing the proof. $\square$

We finish the section by proving that parametrized formal inversions are preserved under passing to slice topoi.

Proposition 2.47. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a morphism in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ *which exhibits $\mathcal{C}'$ as a formal inversion of a small subcategory S in $\mathcal{C}$. Then for every object $B \in \mathcal{B}$, the induced $\mathcal{B}_{/B}$-functor $\pi_B^* F\colon \pi_B^*\mathcal{C} \to \pi_B^*\mathcal{C}'$ exhibits $\pi_B^*\mathcal{C}'$ as a formal inversion of $\pi_B^* S$ in $\pi_B^*\mathcal{C}$.*

Proof. Consider $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}))$. Because of the symmetric monoidal adjunction

$$\pi_B^*\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \rightleftarrows \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}) : (\pi_B)_*$$

from Corollary 2.14, a morphism $F\colon \pi_B^*\mathcal{C} \to \mathcal{D}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}))$ is the same as a morphism $F'\colon \mathcal{C} \to (\pi_B)_*\mathcal{D}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$. Since the unit $\mathcal{C} \to (\pi_B)_*\pi_B^*\mathcal{C}$ and the counit $\pi_B^*(\pi_B)_*\mathcal{D} \to \mathcal{D}$ are symmetric monoidal, it follows that F inverts the objects of $\pi_B^* S$ if and only if F' inverts the objects of S. Since the map

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}))}(\pi_B^*\mathcal{C}', \mathcal{D}) \to \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}))}(\pi_B^*\mathcal{C}, \mathcal{D})$$

given by precomposition with $\pi_B^* F$ corresponds under the adjunction to the map

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))}(\mathcal{C}', (\pi_B)_*\mathcal{D}) \to \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))}(\mathcal{C}, (\pi_B)_*\mathcal{D})$$

given by precomposition with F, we conclude that $\pi_B^* F$ exhibits $\pi_B^*\mathcal{C}'$ as a formal inversion of $\pi_B^* S$ in $\pi_B^*\mathcal{C}$. □

I.3 Twisted ambidexterity

Fix an ∞-topos $\mathcal{B}$ and a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$. In Section 3.1, we will associate to every morphism $f\colon A \to B$ in $\mathcal{B}$ a *relative dualizing object* $D_f \in \mathcal{C}(A)$ together with a *twisted norm map* $\mathrm{Nm}_f\colon f_!(- \otimes D_f) \Rightarrow f_*(-)$, which informally speaking exhibits $f_!(- \otimes D_f)$ as the universal parametrized $\mathcal{C}$-linear approximation to f_*. We will show in Section 3.2 that when f is n-truncated for some n, the twisted norm map reduces to the (untwisted) norm map $\mathrm{Nm}_f\colon f_! \Rightarrow f_*$ from Hopkins and Lurie [HL13] whenever the latter is defined, and use this to express ambidexterity in terms of twisted ambidexterity. In Section 3.3 we will explain the close relation between twisted ambidexterity and *Costenoble-Waner duality*, a parametrized form of monoidal duality due to [CW16; MS06].

3.1 The twisted norm map

To define the twisted norm map Nm_f, we will first treat the case where the target B of f is the terminal object of $\mathcal{B}$. The case for arbitrary B will be obtained by passing to the slice topos $\mathcal{B}_{/B}$.

Definition 3.1. For an object $A \in \mathcal{B}$ we define the *dualizing object* $D_A \in \mathcal{C}(A)$ of A as

$$D_A := \mathrm{pr}_{1*}\Delta_!\,\mathbb{1}_A \in \mathcal{C}(A),$$

where $\mathrm{pr}_1\colon A \times A \to A$ is the first projection and $\Delta\colon A \to A \times A$ is the diagonal of A. We let $c_A\colon \mathrm{pr}_1^* D_A \to \Delta_!\,\mathbb{1}_A$ denote the counit.

Under the equivalence $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{C}) \simeq \mathcal{C}(A)$ of Theorem 2.32, the object $D_A \in \mathcal{C}(A)$ corresponds to a $\mathcal{C}$-linear $\mathcal{B}$-functor $A_!(- \otimes D_A)\colon \mathcal{C}^A \to \mathcal{C}$. As a special case of Corollary 2.34 we immediately obtain the following corollary:

Corollary 3.2. *For an object $A \in \mathcal{B}$, evaluation at $\Delta_!\,\mathbb{1}_A$ induces an equivalence of spaces*

$$\mathrm{Nat}_{\mathcal{C}}(A^*A_!(- \otimes_A D_A), \mathrm{id}_{\mathcal{C}^A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}(A\times A)}(\mathrm{pr}_1^* D_A, \Delta_!\,\mathbb{1}_A). \qquad \text{(I.3.1)}$$

Proof. By naturality in $\mathcal{D}$ of the equivalence $\underline{\mathrm{Fun}}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}^A$ from Corollary 2.30, evaluating $A^*A_!(- \otimes_A D_A)\colon \mathcal{C}^A \to \mathcal{C}^A$ at $\Delta_! \mathbb{1}_A$ gives $\mathrm{pr}_1^* D_A$. The statement then follows from Corollary 2.34, since all three functors A^*, $A_!$ and $- \otimes_A D_A$ are canonically $\mathcal{C}$-linear. □

Definition 3.3. For $A \in \mathcal{B}$, we define a $\mathcal{C}$-linear $\mathcal{B}$-transformation

$$\widetilde{\mathrm{Nm}}_A\colon A^*A_!(- \otimes_A D_A) \implies \mathrm{id}_{\mathcal{C}^A}$$

as the $\mathcal{C}$-linear $\mathcal{B}$-transformation corresponding under the equivalence (I.3.1) to the counit map $\mathrm{pr}_1^* D_A = \mathrm{pr}_1^* \mathrm{pr}_{1*} \Delta_! \mathbb{1}_A \to \Delta_! \mathbb{1}_A$. We will refer to $\widetilde{\mathrm{Nm}}_A$ as the *adjoint twisted norm map* for A. We define the *twisted norm map*

$$\mathrm{Nm}_A\colon A_!(- \otimes_A D_A) \implies A_*(-)$$

as the $\mathcal{B}$-transformation between $\mathcal{B}$-functors $\mathcal{C}^A \to \mathcal{C}$ adjoint to $\widetilde{\mathrm{Nm}}_A$.

Definition 3.4. An object $A \in \mathcal{B}$ is called *twisted $\mathcal{C}$-ambidextrous* if the twisted norm map Nm_A is an equivalence.

A relative version of twisted ambidexterity is obtained by passing to slices of $\mathcal{B}$:

Definition 3.5. For a morphism $f\colon A \to B$ in $\mathcal{B}$, define the *relative dualizing object* $D_f \in \mathcal{C}(A)$ of f as $D_f := \mathrm{pr}_{1*}(\Delta_f)_! \mathbb{1}_A \in \mathcal{C}(A)$, where $\mathrm{pr}_1\colon A \times_B A \to A$ is the first projection and $\Delta_f\colon A \to A \times_B A$ is the diagonal of f. We define the *twisted norm map*

$$\mathrm{Nm}_f\colon f_!(- \otimes_A D_f) \implies f_*(-)$$

as the $\mathcal{B}_{/B}$-transformation between $\mathcal{B}_{/B}$-functors $(\pi_B^* \mathcal{C})^A \to \pi_B^* \mathcal{C}$ as in Definition 3.3. We will say that f is *twisted $\mathcal{C}$-ambidextrous* if the transformation Nm_f is an equivalence.

We may obtain an explicit formula for the transformation $\widetilde{\mathrm{Nm}}_A$ as follows.

Lemma 3.6. *For $A \in \mathcal{B}$, the transformation $\widetilde{\mathrm{Nm}}_A\colon A^*A_!(- \otimes_A D_A) \Rightarrow \mathrm{id}_{\mathcal{C}^A}$ is equivalent to the composite*

$$\begin{aligned}
A^*A_!(- \otimes_A D_A) &\overset{l.b.c.}{\simeq} \mathrm{pr}_{2!} \mathrm{pr}_1^*(- \otimes_A D_A) \\
&\simeq \mathrm{pr}_{2!}(\mathrm{pr}_1^*(-) \otimes_{A \times A} \mathrm{pr}_1^* D_A) \\
&\Rightarrow \mathrm{pr}_{2!}(\mathrm{pr}_1^*(-) \otimes_{A \times A} \Delta_! \mathbb{1}_A) \\
&\overset{l.p.f.}{\simeq} \mathrm{pr}_{2!} \Delta_!(\Delta^* \mathrm{pr}_1^*(-) \otimes_A \mathbb{1}_A) \\
&\simeq \mathrm{id}_{\mathcal{C}^A}.
\end{aligned}$$

Here l.b.c. denotes the left base change equivalence, l.p.f. denotes the left projection formula equivalence, and the non-invertible arrow in the middle is induced by the counit $\mathrm{pr}_1^* D_A = \mathrm{pr}_1^* \mathrm{pr}_{1*} \Delta_! \mathbb{1}_A \to \Delta_! \mathbb{1}_A$.

Proof. By definition, the adjoint twisted norm map $\widetilde{\mathrm{Nm}}_A$ is defined to be the map whose image under the equivalence $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{C}^A) \xrightarrow{\sim} \mathcal{C}(A \times A)$ from Theorem 2.32 is the counit map $\mathrm{pr}_1^* D_A \to \Delta_! \mathbb{1}_A$. Recall that an inverse to this equivalence is given by sending $D \in \mathcal{C}(A \times A)$ to the $\mathcal{C}$-linear $\mathcal{B}$-functor $\mathrm{pr}_{2!}(\mathrm{pr}_1^*(-) \otimes_{A \times A} D) \colon \mathcal{C}^A \to \mathcal{C}^A$. It follows that morphism $\widetilde{\mathrm{Nm}}_A$ in $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{C}^A)$ is equivalent to the map $\mathrm{pr}_{2!}(\mathrm{pr}_1^*(-) \otimes_{A \times A} \mathrm{pr}_1^* D_A) \to \mathrm{pr}_{2!}(\mathrm{pr}_1^*(-) \otimes_{A \times A} \Delta_! \mathbb{1})A)$ induced by the counit. Unwinding definitions, the identifications of the source and target of these maps happens through the left base change equivalence and the left projection formula, giving the claim. ◻

The twisted norm map Nm_A exhibits the $\mathcal{B}$-functor $A_!(- \otimes D_A) \colon \mathcal{C}^A \to \mathcal{C}$ in a suitable sense as the universal $\mathcal{C}$-linear approximation of the $\mathcal{B}$-functor $A_* \colon \mathcal{C}^A \to \mathcal{C}$. More precisely, the dual adjoint norm map $\widetilde{\mathrm{Nm}}_A$ exhibits $A_!(- \otimes D_A)$ as terminal among $\mathcal{C}$-linear $\mathcal{B}$-functor $F \colon \mathcal{C}^A \to \mathcal{C}$ equipped with a $\mathcal{C}$-linear transformation $A^* F \to \mathrm{id}$:

Proposition 3.7 (Universal property twisted norm map)**.** *For a $\mathcal{C}$-linear functor $F \colon \mathcal{C}^A \to \mathcal{C}$, the composite*

$$\mathrm{Nat}_{\mathcal{C}}(F, A_!(- \otimes D_A)) \xrightarrow{A^* \circ -} \mathrm{Nat}_{\mathcal{C}}(A^* F, A^* A_!(- \otimes D_A)) \xrightarrow{\widetilde{\mathrm{Nm}}_A \circ -} \mathrm{Nat}_{\mathcal{C}}(A^* F, \mathrm{id}_{\mathcal{C}^A})$$

is an equivalence of spaces.

Proof. Observe that the three instances of the equivalence of Corollary 2.34 fit in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Nat}_{\mathcal{C}}(F, A_!(- \otimes D_A)) \arrow[r, "A^* \circ -"] \arrow[d, "\simeq"', "\mathrm{ev}_{\Delta_! \mathbb{1}_A}"] & \mathrm{Nat}_{\mathcal{C}}(A^* \circ F, A^* \circ A_!(- \otimes D_A)) \arrow[r, "\widetilde{\mathrm{Nm}}_A \circ -"] \arrow[d, "\simeq"', "\mathrm{ev}_{\Delta_! \mathbb{1}_A}"] & \mathrm{Nat}_{\mathcal{C}}(A^* \circ F, \mathrm{id}_{\mathcal{C}^A}) \arrow[d, "\simeq"', "\mathrm{ev}_{\Delta_! \mathbb{1}_A}"] \\
\mathrm{Hom}_{\mathcal{C}(A)}(D_F, D_A) \arrow[r, "\mathrm{pr}_1^*"] & \mathrm{Hom}_{\mathcal{C}(A \times_B A)}(\mathrm{pr}_1^* D_F, \mathrm{pr}_1^* D_A) \arrow[r, "c^*_{\mathrm{pr}_1} \circ -"] & \mathrm{Hom}_{\mathcal{C}(A \times_B A)}(\mathrm{pr}_1^* D_F, \Delta_! \mathbb{1}_A).
\end{tikzcd}$$

The left square commutes by naturality of the equivalence $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{D}) \simeq \mathcal{D}(A)$ in $\mathcal{D}$ and the right square commutes because an equivalence of categories preserves composition. It thus remains to show that the bottom composite in the diagram is an equivalence. But this is clear since it is given by the adjunction equivalence on hom-spaces for the adjunction $\mathrm{pr}_1^* \dashv \mathrm{pr}_{1*}$. ◻

As a consequence of the universal property of Nm_A, we may express twisted ambidexterity in terms of *internal left adjoints* in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$, in the sense of Definition 2.20:

Proposition 3.8. *The object A is twisted $\mathcal{C}$-ambidextrous if and only if the $\mathcal{C}$-linear $\mathcal{B}$-functor $A^*\colon \mathcal{C} \to \mathcal{C}^A$ is an internal left adjoint in* $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$.

Proof. Note that A is twisted $\mathcal{C}$-ambidextrous if and only if the adjoint twisted norm map

$$\widetilde{\mathrm{Nm}}_A\colon A^*A_!(- \otimes_A D_A) \implies \mathrm{id}_{\mathcal{C}^A}$$

exhibits $A_!(- \otimes_A D_A)$ as a $\mathcal{C}$-linear right adjoint of A^*, so that one implication is clear. Conversely, if A^* is an internal left adjoint with $\mathcal{C}$-linear right adjoint A_*, the $\mathcal{C}$-linear counit $A^*A_* \to \mathrm{id}$ equips A_* with the same universal property of $A_!(- \otimes D_A)$ of Proposition 3.7: for every $\mathcal{C}$-linear $\mathcal{B}$-functor $F\colon \mathcal{C}^A \to \mathcal{C}$, the composite

$$\mathrm{Nat}_{\mathcal{C}}(F, A_*) \xrightarrow{A^* \circ -} \mathrm{Nat}_{\mathcal{C}}(A^*F, A^*A_*) \xrightarrow{\widetilde{\mathrm{Nm}}_A \circ -} \mathrm{Nat}_{\mathcal{C}}(A^*F, \mathrm{id}_{\mathcal{C}^A})$$

is an equivalence of spaces. It follows that $A_!(- \otimes_A D_A)$ and A_* are equivalent as $\mathcal{C}$-linear $\mathcal{B}$-functors $\mathcal{C}^A \to \mathcal{C}$, necessarily via the twisted norm map. □

Remark 3.9. Applying Proposition 3.8 to the slice topos $\mathcal{B}_{/B}$, it follows that a morphism $f\colon A \to B$ in $\mathcal{B}$ is twisted $\mathcal{C}$-ambidextrous if and only if the $\mathcal{C}$-linear $\mathcal{B}_{/B}$-functor $f^*\colon \pi_B^*\mathcal{C} \to (\pi_B^*\mathcal{C})^A$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}))$, i.e. the right adjoint f_* preserves $\mathcal{B}_{/B}$-parametrized colimits and satisfies the projection formula. Applying the criterion Lemma 2.21 to the slice topos $\mathcal{B}_{/B}$, this can be made explicit in terms of non-parametrized criteria: f is twisted $\mathcal{C}$-ambidextrous if and only if for every pullback diagram

$$\begin{tikzcd} A'' \arrow[r, "\alpha"] \arrow[d, "f''"'] \arrow[dr, phantom, "\lrcorner"] & A' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\ B'' \arrow[r, "\beta"] & B' \arrow[r] & B \end{tikzcd}$$

in $\mathcal{B}$ the following three conditions hold:

(1) (Preserving fiberwise colimits) The functor $f'_*\colon \mathcal{C}(A') \to \mathcal{C}(B')$ preserves colimits;

(2) (Preserving groupoid-indexed colimits) The Beck-Chevalley square

$$\begin{tikzcd} \mathcal{C}(A') \arrow[r, "\alpha^*"] \arrow[d, "f'_*"'] & \mathcal{C}(A'') \arrow[d, "f''_*"] \\ \mathcal{C}(B') \arrow[r, "\beta^*"] & \mathcal{C}(B'') \end{tikzcd}$$

is horizontally left adjointable;

(3) (Right projection formula) For objects $X \in \mathcal{C}(A')$ and $Y \in \mathcal{C}(B')$, the canonical morphism

$$f'_*(X) \otimes Y \to f'_*(X \otimes f'^*(Y))$$

in $\mathcal{C}(B')$ is an equivalence.

Conditions (1) and (2) correspond to the assumption that the $\mathcal{B}_{/B}$-functor f_* preserves $\mathcal{B}_{/B}$-parametrized colimits, while condition (3) is equivalent to the condition f_* is $\mathcal{C}$-linear.

Example 3.10. When $\mathcal{B} = \mathrm{Spc}$ is the ∞-topos Spc of spaces and $\mathcal{C} = \mathrm{Sp}$ is the ∞-category of spectra, the universal property of the twisted norm map appears as [NS18, Theorem I.4.1(v)]. Since every colimit-preserving functor between stable presentable ∞-categories is Sp-linear, the universal property simplifies to the statement that the twisted norm map $\mathrm{colim}_A(- \otimes D_A) \Rightarrow \lim_A(-)$ exhibits its source as the universal colimit-preserving aproximation of its target. The object $D_A \in \mathrm{Sp}^A$ is called the *dualizing spectrum* of A, and was studied by Klein [Kle01]. As a functor $A \to \mathrm{Sp}$, it may be identified with the composite

$$A \xrightarrow{a \mapsto \mathrm{Map}_A(a,-)} \mathrm{Spc}^A \xrightarrow{\Sigma^\infty_+} \mathrm{Sp}^A \xrightarrow{\lim_A} \mathrm{Sp}.$$

Example 3.11. Let $\mathcal{B} = \mathrm{Spc}$ be the ∞-topos of spaces and let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. By Proposition 3.8, a space A is twisted $\mathcal{C}$-ambidextrous if and only if it is *$\mathcal{C}$-adjointable* in the terminology of [Car+22, Definition 4.17]. In particular, the following are examples of twisted $\mathcal{C}$-ambidextrous spaces:

(1) If $\mathcal{C}$ is stable, then every compact space is twisted $\mathcal{C}$-ambidextrous, [Car+22, Example 4.24].

(2) If $\mathcal{C}$ is m-semiadditive for some $m \geq -2$, then every m-finite space is twisted $\mathcal{C}$-ambidextrous, [Car+22, Example 4.22].

(3) It $\mathcal{C}$ is an n-category (i.e., its mapping spaces are $(n-1)$-truncated), then every n-connected space is twisted $\mathcal{C}$-ambidextrous, [Car+22, Example 4.23].

(4) Let $\mathcal{C} = \mathrm{Pr}^{\mathrm{L}}_\kappa$ be the ∞-category of κ-presentable ∞-categories for a regular cardinal κ. Then *every* space is twisted $\mathcal{C}$-ambidextrous, [Car+22, Example 4.26].

The twisted $\mathcal{C}$-ambidextrous morphisms form a well-behaved class of morphisms in $\mathcal{B}$:

Proposition 3.12. *The collection of twisted $\mathcal{C}$-ambidextrous morphisms is*

(1) *closed under composition;*

(2) *closed under arbitrary disjoint unions;*

(3) *closed under base change;*

(4) *closed under cartesian products;*

(5) *a local class of morphisms in $\mathcal{B}$, in the sense of [Lur09, Definition 6.1.3.8].*

Proof. (1) For closure under composition, let $f\colon A \to B$ and $g\colon B \to C$ be twisted $\mathcal{C}$-ambidextrous morphisms. We may write the $\mathcal{B}_{/C}$-functor $(gf)^*\colon [C,\mathcal{C}]_C \to [A,\mathcal{C}]_C$ as a composite of $g^*\colon [C,\mathcal{C}]_C \to [B,\mathcal{C}]_C$ and $f^*\colon [B,\mathcal{C}]_C \to [A,\mathcal{C}]_C$. These are both left adjoints internal to $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/C}))$ by Remark 3.9, and thus so is $(gf)^*$. Another application of Remark 3.9 then shows that gf is twisted $\mathcal{C}$-ambidextrous as well.

(2) Closure under disjoint unions follows directly from Remark 3.9, using the equivalence $\mathcal{C}(\bigsqcup_i A_i) \simeq \prod_i \mathcal{C}(A_i)$. (3) Closure under base change is immediate from Remark 3.9. (4) Closure under cartesian products follows from closure under base change and closure under composition.

(5) We check that the twisted $\mathcal{C}$-ambidextrous morphisms form a local class in $\mathcal{B}$. Given closure under disjoint unions, it remains to show that for any pullback square

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\ B' \arrow[r, two heads, "\beta"] & B \end{tikzcd}$$

in $\mathcal{B}$ such that f' is twisted $\mathcal{C}$-ambidextrous $\beta\colon B' \twoheadrightarrow B$ is an effective epimorphism, then also f is twisted $\mathcal{C}$-ambidextrous, i.e. the $\mathcal{B}_{/B}$-functor $f^*\colon \pi_B^*\mathcal{C} \to (\pi_B^*\mathcal{C})^A$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B}))$. By Proposition 2.22, it suffices to check this after pulling f^* back along β to the slice $\mathcal{B}_{/B'}$. But there it becomes the condition that the $\mathcal{B}_{/B'}$-functor $f'^*\colon \pi_{B'}^*\mathcal{C} \to (\pi_{B'}^*\mathcal{C})^{A'}$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/B'}))$, which holds by assumption on f'. □

Corollary 3.13. *For $\mathcal{B} = \mathrm{Spc}$, the ∞-topos of spaces, a map $f\colon A \to B$ is twisted $\mathcal{C}$-ambidextrous if and only if each of its fibers are twisted $\mathcal{C}$-ambidextrous.* □

3.2 Relation to ambidexterity and parametrized semiadditivity

In [HL13, Construction 4.1.8, Remark 4.1.12], Hopkins and Lurie introduce for every 'Beck-Chevalley fibration' $q\colon \mathcal{C} \to \mathcal{B}$ a collection of *$\mathcal{C}$-ambidextrous morphisms $f\colon A \to B$ in $\mathcal{B}$*, each of which come equipped with a norm equivalence $\mathrm{Nm}_f\colon f_! \xrightarrow{\sim} f_*$. In this subsection, we will compare this notion of ambidexterity with our notion of twisted ambidexterity. As a consequence, we relate twisted ambidexterity with the notion of parametrized semiadditivity introduced by Nardin [Nar16] and Lenz, Linskens and the author [CLL23].

Definition 3.14 (Iterated diagonals)**.** Let $f\colon A \to B$ be a morphism in $\mathcal{B}$. The *diagonal* $\Delta(f)$ of f is the map $(1,1)\colon A \to A \times_B A$. The *iterated diagonals* $\Delta^k(f)$ of f are defined inductively by letting $\Delta^0(f) := f$ and $\Delta^{k+1}(f) := \Delta(\Delta^k(f))$.

The functor $\mathcal{C}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ can be unstraightened to a cartesian fibration $\widetilde{\mathcal{C}} \to \mathcal{B}$. Since $\mathcal{C}$ has parametrized colimits, this is a Beck-Chevalley fibration, in the sense of [HL13, Definition 4.1.3].

Proposition 3.15. *Let $f\colon A \to B$ be a morphism in $\mathcal{B}$ and assume that f is n-truncated for some natural number n.*

(1) The morphism f is $\mathcal{C}$-ambidextrous (in the sense of [HL13, Construction 4.1.8] applied to the Beck-Chevalley fibration $\widetilde{\mathcal{C}} \to \mathcal{B}$) if and only if each of the iterated diagonals $\Delta^k(f)$ for $k = 0, 1, \dots, n+1$ is twisted $\mathcal{C}$-ambidextrous (in the sense of Definition 3.5).

(2) Similarly, f is weakly $\mathcal{C}$-ambidextrous (in the sense of [HL13, Construction 4.1.8] applied to the Beck-Chevalley fibration $\widetilde{\mathcal{C}} \to \mathcal{B}$) if and only if each of the iterated diagonals $\Delta^k(f)$ for $k = 1, \dots, n+1$ is twisted $\mathcal{C}$-ambidextrous.

(3) If f is weakly $\mathcal{C}$-ambidextrous, then there is an equivalence $D_f \simeq \mathbb{1}_A$ between the dualizing object D_f and the monoidal unit $\mathbb{1}_A \in \mathcal{C}(A)$, and the composite

$$f_!(-) \simeq f_!(- \otimes_A D_f) \xrightarrow{\mathrm{Nm}_f} f_*(-)$$

is equivalent to the norm map $f_! \to f_*$ of [HL13, Remark 4.1.12].

Proof. We prove the three claims by simultaneous induction on n. For $n = -2$, f is an equivalence and parts (1) and (2) are vacuous as every equivalence is both $\mathcal{C}$-ambidextrous as well as twisted $\mathcal{C}$-ambidextrous. In this case, $f_!$ and f_* are both inverse to f^* and therefore admit a canonical equivalence $\mathrm{Nm}_f \colon f_! \simeq f_*$ of parametrized functors $C^A \to C$, which agrees with the norm map of Hopkins and Lurie. Evaluating this equivalence at $\Delta_! \mathbb{1}_A \in C(A \times A) = C^A(A)$ gives an equivalence $\mathbb{1}_A \simeq D_f$ in $C(A)$, and the last statement of (3) then holds by construction.

Now assume that $n \geq -1$. The diagonal $\Delta(f)$ of f is $(n-1)$-truncated, so by part (1) of the induction hypothesis it is C-ambidextrous (that is, f is weakly C-ambidextrous) if and only if all the iterated diagonals $\Delta^k(f)$ for $k = 1, \ldots, n+1$ are twisted C-ambidextrous, proving part (2). In this case, there is an equivalence

$$D_f = \mathrm{pr}_{1*}\Delta_! \mathbb{1}_A \xrightarrow[\simeq]{\mathrm{Nm}_\Delta} \mathrm{pr}_{1*}\Delta_* \mathbb{1}_A = \mathbb{1}_A \quad \in \quad C(A).$$

Plugging in (the inverse of) this equivalence in the twisted norm map Nm_f and using the description of Nm_f given in Lemma 3.6 (applied to the slice topos $\mathcal{B}_{/B}$), one sees that the composite in (3) is adjoint to the following composite:

$$f^* f_!(-) \simeq \mathrm{pr}_{2!}\mathrm{pr}_1^*(-) \xrightarrow{u_\Delta^*} \mathrm{pr}_{2!}\Delta_*\Delta^*\mathrm{pr}_1^*(-) \xleftarrow[\simeq]{\mathrm{Nm}_\Delta} \mathrm{pr}_{2!}\Delta_!\Delta^*\mathrm{pr}_1^*(-) \simeq \mathrm{id} \circ \mathrm{id} = \mathrm{id}\,.$$

But this composite is precisely (a parametrized version of) the map $\nu_f^{(n+1)} \colon f^* f_! \to \mathrm{id}$ of [HL13, Construction 4.1.8], and its adjoint $f_! \to f_*$ is the norm map of [HL13, Remark 4.1.12], finishing the proof of (3).

Finally we deduce part (1) from (2) and (3). Given (2), we may assume that f is weakly C-ambidextrous, and we need to show it is twisted C-ambidextrous if and only if it is C-ambidextrous. In other words, we need to show that the twisted norm map is an equivalence if and only if the norm map $f_! \to f_*$ of Hopkins and Lurie is an equivalence. This is immediate from part (3). $\square$

Specializing the result of Proposition 3.15 to the case where $\mathcal{B}$ is the ∞-topos of spaces, we obtain the following corollary:

Corollary 3.16. *Let C be a presentably symmetric monoidal ∞-category and let A be a connected n-truncated space. Then the following conditions are equivalent:*

(1) The space A is C-ambidextrous in the sense of [HL13, Definition 4.3.4];

(2) Each of the objects $A, \Omega A, \ldots, \Omega^{n+1} A$ is twisted C-ambidextrous in the sense of Definition 3.4.

Proof. Letting $f \colon A \to \mathrm{pt}$ denote the map from A to the point, we observe that each of the fibers of the iterated diagonal $\Delta^k f$ is given by the k-fold loop space $\Omega^k A$. It follows from

Corollary 3.13 that $\Omega^k A$ is twisted $\mathcal{C}$-ambidextrous if and only if the iterated diagonal $\Delta^k A$ of A is twisted $\mathcal{C}$-ambidextrous. The claim thus follows from Proposition 3.15. ◻

As a consequence, we obtain a characterization of higher semiadditivity in terms of twisted ambidexterity. An advantage of this characterization over the usual definition of ambidexterity is that all the twisted norm maps are a priori defined rather than through an inductive process.

Corollary 3.17. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category and let $-2 \leq m \leq \infty$. Then $\mathcal{C}$ is m-semiadditive if and only if each m-finite space is twisted $\mathcal{C}$-ambidextrous.*

Proof. This is immediate from Corollary 3.16, as the iterated loop spaces of an m-finite space are again m-finite. ◻

Corollary 3.18. *Let $\mathcal{C}$ be a presentably symmetric monoidal $\mathcal{B}$-category, let $A \in \mathcal{B}$ and let $-2 \leq m \leq \infty$. Then the following two conditions are equivalent:*

(1) the ∞-category $\mathcal{C}(A)$ is m-semiadditive;

(2) the fold map $\operatorname{colim}_X A \to A$ *is twisted $\mathcal{C}$-ambidextrous for every m-finite space X.*

Proof. For an object $A \in \mathcal{B}$, consider the (unique) colimit-preserving functor $L_A \colon \mathrm{Spc} \to \mathcal{B}$ sending the point to A, given on objects by $X \mapsto \operatorname{colim}_X A$. The ∞-category $\mathcal{C}(A)$ is encoded as a Spc-category by the composite

$$\mathrm{Spc}^{\mathrm{op}} \xrightarrow{L_A} \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Pr}^{\mathrm{L}}.$$

Since L_A preserves pullbacks, a morphism $f \colon X \to Y$ of spaces is twisted $\mathcal{C}(A)$-ambidextrous if and only if the map $L_A(f) \colon L_A(X) \to L_A(Y)$ is twisted $\mathcal{C}$-ambidextrous. Applying this to $Y = \mathrm{pt}$ shows that $X \to *$ is twisted $\mathcal{C}(A)$-ambidextrous if and only if the map $\operatorname{colim}_X A = L_A(X) \to A$ is twisted $\mathcal{C}$-ambidextrous. The claim now follows from Corollary 3.17. ◻

In [Nar16] and [CLL23], parametrized notions of semiadditivity were introduced. By Proposition 3.15, these may be expressed in terms of twisted ambidexterity:

Corollary 3.19. *Let T be a small ∞-category and let $P \subseteq T$ be an atomic orbital subcategory, in the sense of [CLL23, Definition 4.3.1]. Let $\mathcal{C}$ be a presentably symmetric monoidal* $\mathrm{PSh}(T)$*-category. Then $\mathcal{C}$ is P-semiadditive in the sense of [CLL23, Definition 4.5.1] if and only if $\mathcal{C}$ is fiberwise semiadditive and every morphism $p \colon A \to B$ in P is twisted $\mathcal{C}$-ambidextrous.*

Proof. By [CLL23, Corollary 4.5.5], C is P-semiadditive if and only if it is fiberwise semiadditive and the norm map $\mathrm{Nm}_p \colon p_! \to p_*$ from [CLL23, Construction 4.3.6] is an equivalence for every morphism $p \colon A \to B$ in P. By [CLL23, Remark 4.3.7], this norm map agrees with the norm map Nm_p defined by Hopkins and Lurie, which in turn agrees with the twisted norm map Nm_p by Proposition 3.15. This finishes the proof. $\square$

Corollary 3.20. *Let T be an atomic orbital ∞-category, in the sense of [Nar16, Definition 4.1]. Let C be a presentably symmetric monoidal* $\mathrm{PSh}(T)$*-category. Then C is T-semiadditive in the sense of [Nar16, Definition 5.3] if and only if C is fiberwise semiadditive and every morphism $f \colon A \to B$ in T is twisted C-ambidextrous.*

Proof. This follows immediately from the previous corollary for $P = T$, since by [CLL23, Proposition 4.6.3] the norm map from [CLL23, Construction 4.3.6] is equivalent to the norm map constructed in [Nar16, Construction 5.2]. $\square$

3.3 Costenoble-Waner duality

There is a close link between twisted ambidexterity and Costenoble-Waner duality, a form of duality theory in parametrized homotopy theory introduced in the early 2000's by Costenoble and Waner [CW16] and subsequently developed in more detail by May and Sigurdsson [MS06]. The goal of this subsection is to introduce a general form of Costenoble-Waner duality in an arbitrary presentably symmetric monoidal $\mathcal{B}$-category C and explain its relationship with twisted ambidexterity.

Recall that an object X of a symmetric monoidal ∞-category $\mathcal{D}$ is called *dualizable* if there exists another object $Y \in \mathcal{D}$, called the *dual* of X, together with an evaluation map $\varepsilon \colon X \otimes Y \to \mathbb{1}$ and a coevaluation map $\eta \colon \mathbb{1} \to Y \otimes X$ satisfying the triangle identities:

$$\begin{tikzcd}[column sep=small] & X \otimes Y \otimes X \arrow[dr, "\varepsilon\otimes X"] & \\ X \otimes \mathbb{1} \arrow[ur, "X\otimes\eta"] \arrow[r, equal] & X \arrow[r, equal] & \mathbb{1} \otimes X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & Y \otimes X \otimes Y \arrow[dr, "Y\otimes\varepsilon"] & \\ \mathbb{1} \otimes Y \arrow[ur, "\eta\otimes Y"] \arrow[r, equal] & Y \arrow[r, equal] & Y \otimes \mathbb{1}. \end{tikzcd}$$

It is not difficult to see that this is equivalent to the $\mathcal{D}$-linear functor $X \otimes - \colon \mathcal{D} \to \mathcal{D}$ admitting a $\mathcal{D}$-linear right adjoint, necessarily of the form $Y \otimes - \colon \mathcal{D} \to \mathcal{D}$, with $\mathcal{D}$-linear unit and counit.

In our approach to Costenoble-Waner duality, we will generalize the above perspective to the parametrized setting. In place of the correspondence between objects $X \in \mathcal{D}$ and $\mathcal{D}$-linear functors $\mathcal{D} \to \mathcal{D}$ in the non-parametrized setting, we will use the equivalence $\mathrm{Fun}_C(C^A, C^B) \simeq C(A \times B)$ from Theorem 2.32 in the parametrized setting, where C is a presentably symmetric monoidal $\mathcal{B}$-category. In particular, any object $X \in \mathcal{C}(A \times B)$ determines a $\mathcal{C}$-linear $\mathcal{B}$-functor $F_X \colon \mathcal{C}^A \to \mathcal{C}^B$ given by the composite

$$F_X \colon \mathcal{C}^A \xrightarrow{\mathrm{pr}_A^*} \mathcal{C}^{A\times B} \xrightarrow{-\otimes_{A\times B} X} \mathcal{C}^{A\times B} \xrightarrow{\mathrm{pr}_{B!}} \mathcal{C}^B,$$

and every $\mathcal{C}$-linear $\mathcal{B}$-functor $F \colon \mathcal{C}^A \to \mathcal{C}^B$ is of this form for a unique object $D_F \in \mathcal{C}(A\times B)$.

Convention 3.21. Whenever we write $X \in \mathcal{C}(A \times B)$, we think of X as being directed from A towards B. If we wish to think of X as being directed from B towards A, we will write $X \in \mathcal{C}(B \times A)$ instead. This also applies when $B = 1$ is the terminal object of $\mathcal{B}$, meaning that we distinguish between $X \in \mathcal{C}(A) = \mathcal{C}(A \times 1)$ and $X \in \mathcal{C}(A) = \mathcal{C}(1 \times A)$.

Definition 3.22 (cf. [MS06, Construction 17.1.3, Proposition 17.1.4])**.** For an object $A \in \mathcal{B}$, we define $U_A := \Delta_! \mathbb{1}_A \in \mathcal{C}(A \times A)$. For objects $A, B, C \in \mathcal{B}$ and objects $X \in \mathcal{C}(A \times B)$ and $Y \in \mathcal{C}(B \times C)$, their *composition product* $Y \odot X \in \mathcal{C}(A \times C)$ is defined as

$$Y \odot X := (\mathrm{pr}_{AC})_!(\mathrm{pr}_{AB}^* X \otimes \mathrm{pr}_{BC}^* Y),$$

where $\mathrm{pr}_{AB} \colon A \times B \times C \to A \times B$ denotes the projection and similarly for pr_{BC} and pr_{AC}. This gives rise to a functor $- \odot - \colon \mathcal{C}(B \times C) \times \mathcal{C}(A \times B) \to \mathcal{C}(A \times C)$.

The following lemma may be regarded as a justification for the definition of the composition product:

Lemma 3.23. *In the above situation, there are natural equivalences*

$$\begin{aligned} F_{U_A} &\simeq \mathrm{id}_{\mathcal{C}^A} &&\in \quad \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{C}^A), \\ F_{Y\odot X} &\simeq F_Y \circ F_X &&\in \quad \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{C}^C). \end{aligned}$$

Proof. The first equivalence is immediate, as the equivalence $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}^A, \mathcal{C}^A) \simeq \mathcal{C}(A \times A)$ of Theorem 2.32 is given by evaluation at $\Delta_! \mathbb{1}_A$. For the second equivalence, plugging in the definition of $Y \odot X$ and using the projection formula for $\mathrm{pr}_{AC!}$ shows that $F_{Y\odot X}$ is given by the composite

$$\mathcal{C}^A \xrightarrow{\mathrm{pr}_A^*} \mathcal{C}^{A\times C} \xrightarrow{\mathrm{pr}_{AC}^*} \mathcal{C}^{A\times B\times C} \xrightarrow{-\otimes \mathrm{pr}_{AB}^* X \otimes \mathrm{pr}_{BC}^* Y} \mathcal{C}^{A\times C\times B} \xrightarrow{\mathrm{pr}_{AC!}} \mathcal{C}^{A\times C} \xrightarrow{\mathrm{pr}_{C!}} \mathcal{C}^C.$$

Using symmetric monoidality of the functors pr_{AB}^* and pr_{BC}^* and using the base change equivalence $\mathrm{pr}_B^* \mathrm{pr}_{B!} \simeq \mathrm{pr}_{BC!} \mathrm{pr}_{AB}^*$, this is equivalent to the composite

$$\mathcal{C}^A \xrightarrow{\mathrm{pr}_A^*} \mathcal{C}^{A\times B} \xrightarrow{-\otimes X} \mathcal{C}^{A\times B} \xrightarrow{\mathrm{pr}_{B!}} \mathcal{C}^B \xrightarrow{\mathrm{pr}_B^*} \mathcal{C}^{B\times C} \xrightarrow{-\otimes Y} \mathcal{C}^{B\times C} \xrightarrow{\mathrm{pr}_{C!}} \mathcal{C}^C.$$

But this is simply $F_Y \circ F_X$, finishing the proof. $\square$

We will frequently use Lemma 3.23 to deduce properties of the composition product which can be somewhat tedious to prove by hand. For example, it follows directly from Lemma 3.23 that the composition product is associative and unital up to homotopy. The objects $U_A = \Delta_! \mathbb{1}_A \in \mathcal{C}(A \times A)$ serve as identities with respect to the composition product: for an object $X \in \mathcal{C}(A \times B)$ there are equivalences

$$X \odot U_A \simeq X \qquad \text{and} \qquad U_B \odot X \simeq X.$$

For brevity, we will mostly suppress the associativity and unitaliy equivalences from the notation and treat them as identities, just like we do for functors.

We may now introduce a parametrized analogue of monoidal duality, first discovered by Costenoble and Waner [CW16] in the context of equivariant homotopy theory.

Definition 3.24 (cf. [MS06, Definition 16.4.1, Chapter 18])**.** An object $X \in \mathcal{C}(A \times B)$ is called *left Costenoble-Waner dualizable* if there is another object $Y \in \mathcal{C}(B \times A)$, called the *left Costenoble-Waner dual of X*, together with morphisms

$$\varepsilon \colon X \odot Y \to U_B \qquad \text{and} \qquad \eta \colon U_A \to Y \odot X$$

in $\mathcal{C}(B \times B)$ resp. $\mathcal{C}(A \times A)$ satisfying the triangle identities

$$\begin{tikzcd} & X \odot Y \odot X \arrow[dr, "\varepsilon \odot X"] & \\ X \odot U_A \arrow[ur, "X \odot \eta"] \arrow[r, equal] & X \arrow[r, equal] & U_B \odot X \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & Y \odot X \odot Y \arrow[dr, "Y \odot \varepsilon"] & \\ U_A \odot Y \arrow[ur, "\eta \odot Y"] \arrow[r, equal] & Y \arrow[r, equal] & Y \odot U_B. \end{tikzcd}$$

Conversely, we call X the *right Costenoble-Waner dual* of Y. Sometimes we say *left dual* and *right dual* for brevity. Note that $X \in \mathcal{C}(A \times B)$ is left Costenoble-Waner dualizable if and only if it is right Costenoble-Waner dualizable when treated as an object $X \in \mathcal{C}(B \times A)$.

Warning 3.25. In [MS06], the phrase 'Costenoble-Waner duality' is only used when A is the terminal object of $\mathcal{B}$. When B is the terminal object, they use the phrase 'fiberwise duality', compare also Lemma 3.33 below.

Due to the translation from Lemma 3.23 between the composition product and and composition of $\mathcal{C}$-linear $\mathcal{B}$-functors, we can express Costenoble-Waner duality in terms of internal adjunctions in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$, in the sense of Definition 2.20.

Lemma 3.26. *An object $X \in \mathcal{C}(A \times B)$ is left Costenoble-Waner dualizable if and only if the $\mathcal{C}$-linear $\mathcal{B}$-functor $F_X \colon \mathcal{C}^A \to \mathcal{C}^B$ associated to X is an internal left adjoint in* $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$*. In this case, the $\mathcal{C}$-linear right adjoint of F_X is given by $F_Y \colon \mathcal{C}^B \to \mathcal{C}^A$, where $Y \in \mathcal{C}(B \times A)$ is a left*[1] *Costenoble-Waner dual of X.*

[1] The fact that *left* duals correspond to *right* adjoints is unfortunate but seems to be the standard convention.

Proof. If X is left Costenoble-Waner dualizable with right dual Y, we may use Lemma 3.23 to turn the evaluation and coevaluation ε and ν into $\mathcal{C}$-linear counit and unit maps $F_X \circ F_Y \to \mathrm{id}_{\mathcal{C}^B}$ and $\mathrm{id}_{\mathcal{C}^A} \to F_Y \circ F_X$, respectively. The triangle identities for the Costenoble-Waner duality between X and Y translate into the triangle identities of an internal adjunction $F_X \dashv F_Y$ in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$. Conversely, if F_X is an internal left adjoint internal to $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ with $\mathcal{C}$-linear right adjoint $G\colon \mathcal{C}^B \to \mathcal{C}^A$, then it follows from Theorem 2.32 that G is of the form F_Y for some object $Y \in \mathcal{C}(B \times A)$, and by Lemma 3.23 the unit and counit of the adjunction give rise to the coevaluation and evaluation satisfying the triangle identities, and thus providing duality data between X and Y. $\square$

Remark 3.27. May and Sigurdsson [MS06, Section 16.4] define Costenoble-Waner duality as a special case of a notion they call *duality in a closed symmetric bicategory*, applied to a certain bicategory $\mathcal{E}x$ of parametrized equivariant spectra [MS06, Construction 17.1.3]. From our perspective, one might think of their bicategory $\mathcal{E}x$ as the full subcategory of the homotopy 2-category of $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ spanned by objects of the form $\mathcal{C}^A$ for $A \in \mathcal{B}$. Indeed, due to Theorem 2.32 and Lemma 3.23, we obtain the following more explicit description of this bicategory:

- The objects $\mathcal{C}^A$ correspond to objects A of $\mathcal{B}$;
- Given $A, B \in \mathcal{B}$, the category of morphisms $\mathcal{C}^A \to \mathcal{C}^B$ can be identified with the homotopy category $\mathrm{Ho}(\mathcal{C}(A \times B))$;
- The identity morphisms are given by $U_A \in \mathcal{C}(A \times A)$;
- The composition is given by the composition product $- \odot -\colon \mathcal{C}(B \times C) \times \mathcal{C}(A \times B) \to \mathcal{C}(A \times C)$.

When applied to the ∞-topos Spc^G of G-spaces for a compact Lie group G and to the G-category $\underline{\mathrm{Sp}}^G$ of genuine G-spectra, to be defined in Section 4.1 below, this is essentially the bicategory $\mathcal{E}x$ of May and Sigurdsson.

We get the following reformulation of twisted ambidexterity in terms of Costenoble-Waner duality:

Proposition 3.28. *An object $A \in \mathcal{B}$ is twisted $\mathcal{C}$-ambidextrous if and only if the monoidal unit $\mathbb{1}_A \in \mathcal{C}(A) = \mathcal{C}(1 \times A)$ is left Costenoble-Waner dualizable. In this case, the left dual of $\mathbb{1}_A$ is given by the dualizing object $D_A \in \mathcal{C}(A) = \mathcal{C}(A \times 1)$.*

Proof. The $\mathcal{C}$-linear $\mathcal{B}$-functor $F_{\mathbb{1}_A}\colon \mathcal{C} \to \mathcal{C}^A$ associated to $\mathbb{1}_A$ is $A^*\colon \mathcal{C} \to \mathcal{C}^A$. It follows from Lemma 3.26 that $\mathbb{1}_A$ is Costenoble-Waner dualizable if and only if $A^*\colon \mathcal{C} \to \mathcal{C}^A$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$, which by Proposition 3.8 is true if and only if A is twisted $\mathcal{C}$-ambidextrous. In this case, the right adjoint of A^* is classified by the object $D_A \in \mathcal{C}(A) = \mathcal{C}(A \times 1)$. □

Characterizations of Costenoble-Waner duality

For later use, we recall various alternative characterizations of Costenoble-Waner duality from [MS06] and [CW16].

Lemma 3.29 (cf. [MS06, Proposition 16.4.6])**.** *Consider objects $X \in \mathcal{C}(A \times B)$ and $Y \in \mathcal{C}(B \times A)$ and let $\varepsilon\colon X \odot Y \to U_B$ be a morphism in $\mathcal{C}(B \times B)$. Then the following conditions are equivalent:*

(1) The object Y is a left Costenoble-Waner dual of X with evaluation map ε.

(2) For every $C \in \mathcal{B}$ and objects $W \in \mathcal{C}(A \times C)$ and $Z \in \mathcal{C}(B \times C)$, the map

$$\mathrm{Hom}_{\mathcal{C}(A\times C)}(W, Z \odot X) \xrightarrow{-\odot Y} \mathrm{Hom}_{\mathcal{C}(B\times C)}(W \odot Y, Z \odot X \odot Y) \xrightarrow{\varepsilon} \mathrm{Hom}_{\mathcal{C}(B\times C)}(W \odot Y, Z)$$

is an equivalence;

(3) Condition (2) holds for $C = A$, $W = U_A$, $Z = X$ and for $C = B$, $W = Y$, $Z = U_B$.

Proof. Given (1), an inverse to the map in (2) is given by

$$\mathrm{Hom}_{\mathcal{C}(B\times C)}(W \odot Y, Z) \xrightarrow{-\odot X} \mathrm{Hom}_{\mathcal{C}(A\times C)}(W \odot Y \odot X, Z \odot X) \xrightarrow{\eta} \mathrm{Hom}_{\mathcal{C}(A\times C)}(W, Z \odot X).$$

It is clear that (2) implies (3). Given (3), we may take $C = A$, $W = U_A$, $Z = Y$ and define the coevaluation $\eta\colon U_A \to Y \odot X$ to be the inverse image of the identity on X under the equivalence from (2). One of the triangle identities holds by construction. For the other one, we take $C = B$, $W = Y$ and $Z = U_B$, and observe that both $\mathrm{id}_Y\colon Y \to Y$ as well as $(Y \odot \varepsilon) \circ (\eta \odot Y)\colon Y \to Y$ are sent to the map $\varepsilon\colon Y \odot X \to U_A$ by the equivalence from (2), implying that they are homotopic. □

Definition 3.30. If the equivalent conditions of Lemma 3.29 hold, we say that ε *exhibits Y as a left Costenoble-Waner dual of X.*

Any object $X \in \mathcal{C}(A \times B)$ admits a *weak* dual. For simplicity, we will only introduce this when A is the terminal object of $\mathcal{B}$.

Definition 3.31 (cf. [CW16, Definition 2.9.1])**.** Let $B \in \mathcal{B}$ and consider an object $X \in \mathcal{C}(B) = \mathcal{C}(1 \times B)$. We define the *weak Costenoble-Waner dual* of X to be

$$D_B^{CW}(X) := \mathrm{pr}_{2*}\,\underline{\mathrm{Hom}}_{B\times B}(\mathrm{pr}_1^* X, \Delta_! \mathbb{1}_B) \quad \in \quad \mathcal{C}(B),$$

where $\Delta\colon B \to B \times B$ is the diagonal of B, and $\mathrm{pr}_1, \mathrm{pr}_2\colon B \times B \to B$ are the two projections.

Note that for any object $Y \in \mathcal{C}(B) = \mathcal{C}(B \times 1)$, there is a one-to-one correspondence between morphisms $Y \to D_B^{CW}(X)$ in $\mathcal{C}(B)$ and morphisms $X \odot Y = \mathrm{pr}_1^* X \otimes_{B\times B} \mathrm{pr}_2^* Y \to \Delta_! \mathbb{1}_B = U_B$ in $\mathcal{C}(B \times B)$. In particular, the identity on $D_B^{CW}(X)$ gives rise to a map $\varepsilon_0\colon X \odot D_B^{CW}(X) \to U_B$.

Lemma 3.32 (cf. [CW16, Theorem 2.9.5])**.** *An object $X \in \mathcal{C}(B) = \mathcal{C}(1 \times B)$ is left Costenoble-Waner dualizable if and only if the map $\varepsilon_0\colon X \odot D_B^{CW}(X) \to U_B$ exhibits $D_B^{CW}(X)$ as a left Costenoble-Waner dual of X.*

Proof. The "if" direction is obvious. For the "only if", assume that X is left Costenoble-Waner dualizable with left dual $Y \in \mathcal{C}(B \times 1)$ and let $\varepsilon\colon X \odot Y \to U_B$ and $\eta\colon U_A \to Y \odot X$ be evaluation and coevaluation maps exhibiting this duality. The map ε adjoints over to a map $Y \to D_B^{CW}(X)$ and it will suffice to show that this is an equivalence. Indeed, a reasonably straightforward diagram chase shows that an inverse is given by the composite

$$D_B^{CW}(X) = U_A \odot D_B^{CW}(X) \xrightarrow{\eta \odot 1} Y \odot X \odot D_B^{CW}(X) \xrightarrow{1 \odot \varepsilon_0} Y \odot U_B = Y. \qquad \square$$

Lemma 3.33 (cf. [MS06, Proposition 18.1.1])**.** *An object $X \in \mathcal{C}(A) = \mathcal{C}(A \times 1)$ is left Costenoble-Waner dualizable if and only if it is dualizable in the symmetric monoidal ∞-category $\mathcal{C}(A)$. Its left Costenoble-Waner dual in $\mathcal{C}(A) = \mathcal{C}(1 \times A)$ is given by the monoidal dual in $\mathcal{C}(A)$.*

Proof. Consider objects $X \in \mathcal{C}(A) = \mathcal{C}(A \times 1)$ and $Y \in \mathcal{C}(A) = \mathcal{C}(1 \times A)$. Unwinding definitions, we see that $X \odot Y \simeq A_!(X \otimes_A Y) \in \mathcal{C}(1)$. It follows that a morphism $\varepsilon\colon X \odot Y \xrightarrow{\varepsilon} U_1 = \mathbb{1}$ in $\mathcal{C}(1)$ is the same data as a morphism $\varepsilon'\colon X \otimes_A Y \to A^* \mathbb{1} = \mathbb{1}_A$ in $\mathcal{C}(A)$. We claim that ε exhibits Y as a left Costenoble-Waner dual to X if and only if ε' exhibits Y as a dual of X in $\mathcal{C}(A)$. Using some diagram chasing, this follows from Lemma 3.29. As the proof is entirely analogous to the proof of [MS06, Proposition 18.1.1], we will omit it. $\square$

Preservation properties of Costenoble-Waner duality

Costenoble-Waner dualizable objects are preserved under a variety of constructions.

Lemma 3.34. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal left adjoint between presentably symmetric monoidal $\mathcal{B}$-categories. Then F preserves left/right Costenoble-Waner dualizable objects.*

Proof. It suffices to observe that F commutes with the composition products on $\mathcal{C}$ and $\mathcal{D}$, which is immediate from the definition. □

Corollary 3.35. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal left adjoint between presentably symmetric monoidal $\mathcal{B}$-categories. Then any twisted $\mathcal{C}$-ambidextrous object in $\mathcal{B}$ is also twisted $\mathcal{D}$-ambidextrous. Consequently, any twisted $\mathcal{C}$-ambidextrous morphism in $\mathcal{B}$ is also twisted $\mathcal{D}$-ambidextrous.*

Proof. Since F preserves monoidal units, the first statement is a consequence of Lemma 3.34 and Proposition 3.28. The second statement follows by passing to slice topoi $\mathcal{B}_{/B}$. □

Lemma 3.36. *If $X \in \mathcal{C}(A \times B)$ and $X' \in \mathcal{C}(B \times C)$ are left Costenoble-Waner dualizable with left duals Y and Y', then so is their composition product $X' \odot X \in \mathcal{C}(A \times C)$, with left dual $Y \odot Y'$.*

Proof. This is immediate from Lemma 3.26, since internal adjunctions compose. □

Lemma 3.37. *Let $X \in \mathcal{C}(A \times B)$ be left Costenoble-Waner dualizable with left dual $Y \in \mathcal{C}(B \times A)$*

(1) For a map $f\colon A' \to A$, the object $(f \times 1)^*X \in \mathcal{C}(A' \times B)$ is left Costenoble-Waner dualizable, with left dual given by $(1 \times f)^*Y$.

(2) For a map $g\colon B \to B'$, the object $(1 \times g)_!X \in \mathcal{C}(A \times B')$ is left Costenoble-Waner dualizable, with left dual given by $(g \times 1)_!Y$.

(3) For a twisted $\mathcal{C}$-ambidextrous map $f\colon A \to A'$, the object $(f \times 1)_!X \in \mathcal{C}(A' \times B)$ is left Costenoble-Waner dualizable, with left dual given by $(1 \times f)_!Y$.

(4) For a twisted $\mathcal{C}$-ambidextrous map $g\colon B' \to B$, the object $(1 \times g)^*X \in \mathcal{C}(A \times B')$ is left Costenoble-Waner dualizable, with left dual given by $(g \times 1)^*Y$.

Proof. In each case, the claim follows from Proposition 2.26(5) and the fact that adjunctions compose, using that for a morphism $f\colon A \to B$, the $\mathcal{B}$-functor $f_!\colon \mathcal{C}^A \to \mathcal{C}^B$ is always an internal left adjoint, while $f^*\colon \mathcal{C}^B \to \mathcal{C}^A$ is an internal left adjoint whenever f is twisted $\mathcal{C}$-ambidextrous. □

Corollary 3.38. *Assume that $X \in \mathcal{C}(1 \times A)$ is left Costenoble-Waner dualizable with left dual $Y \in \mathcal{C}(A \times 1)$. Then $A_! X \in \mathcal{C}(1)$ is a dualizable object with dual $A_! Y$.* □

Corollary 3.39. *Assume $A \in \mathcal{B}$ is twisted $\mathcal{C}$-ambidextrous. Then the object $A_! \mathbb{1}_A \in \mathcal{C}(1)$ is dualizable, with dual $A_! D_A$.*

Proof. Combine the previous corollary with Proposition 3.28. □

Proposition 3.40 (cf. [MS06, Proposition 16.8.1])**.** *For every object every $B \in \mathcal{B}$, the collection of left Costenoble-Waner dualizable objects in $\mathcal{C}(1 \times B)$ is closed under retracts. If $\mathcal{C}$ is fiberwise stable, then these objects form a stable (and hence thick) subcategory of $\mathcal{C}(1 \times B)$.*

Proof. By Lemma 3.32 and Lemma 3.29, an object $X \in \mathcal{C}(1 \times B)$ is left Costenoble-Waner dualizable if and only if for all $C \in \mathcal{B}$, $W \in \mathcal{C}(1 \times C)$ and $Z \in \mathcal{C}(B \times C)$, a certain map

$$\mathrm{Hom}_{\mathcal{C}(1\times C)}(W, Z \odot X) \to \mathrm{Hom}_{\mathcal{C}(B\times C)}(W \odot D_B^{CW}(X), Z)$$

is an equivalence. Since this map is natural in X, it follows that the collection of objects for which it is an equivalence is closed under retracts.

If $\mathcal{C}$ is fiberwise stable, we can lift the above map to a map at the level of mapping spectra. In that case both sides are exact in X, and it follows that the collection of objects for which it is an equivalence forms a stable subcategory. □

Corollary 3.41. *The collection of twisted $\mathcal{C}$-ambidextrous objects of $\mathcal{B}$ is closed under retracts. If $\mathcal{C}$ is fiberwise stable, then the collection of twisted $\mathcal{C}$-ambidextrous objects of $\mathcal{B}$ is also closed under finite colimits.*

Proof. Consider a retract diagram $A \xrightarrow{s} B \xrightarrow{r} A$ in $\mathcal{B}$, i.e. we have $rs = \mathrm{id}_A$. Assume that B is twisted $\mathcal{C}$-ambidextrous. It follows from Proposition 3.28 that $\mathbb{1}_B \in \mathcal{C}(B)$ is left Costenoble-Waner dualizable, so by Lemma 3.37 also $r_! \mathbb{1}_B$ is left Costenoble-Waner dualizable. Since $\mathbb{1}_A \in \mathcal{C}(A)$ is a retract of $r_! \mathbb{1}_B$, it is left Costenoble-Waner dualizable by Proposition 3.40, and thus A is twisted $\mathcal{C}$-ambidextrous by Proposition 3.28.

Now assume that $\mathcal{C}$ is fiberwise stable. Since $\mathcal{C}(\emptyset) = *$, it follows from pointedness of $\mathcal{C}$ that the initial object $\emptyset$ is twisted $\mathcal{C}$-ambidextrous. Consider a pushout diagram in $\mathcal{B}$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] \arrow[ur, phantom, "\ulcorner"] & D, \end{tikzcd}$$

and assume that A, B and C are twisted $\mathcal{C}$-ambidextrous. We need to show that D is twisted $\mathcal{C}$-ambidextrous. By descent, the functors h^* and k^* induce an equivalence $(h^*, k^*)\colon \mathcal{C}(D) \xrightarrow{\sim} \mathcal{C}(B) \times_{\mathcal{C}(A)} \mathcal{C}(C)$. It follows that the monoidal unit $\mathbb{1}_D \in \mathcal{C}(D)$ sits in a cofiber sequence

$$(hg)_! \mathbb{1}_A \to h_! \mathbb{1}_B \oplus k_! \mathbb{1}_C \to \mathbb{1}_D .$$

By Proposition 3.28, the objects $\mathbb{1}_A$, $\mathbb{1}_B$ and $\mathbb{1}_C$ are left Costenoble-Waner dualizable, and thus by Lemma 3.37 so are $(hg)_! \mathbb{1}_A$, $h_! \mathbb{1}_B$ and $k_! \mathbb{1}_C$. It thus follows from Proposition 3.40 that $\mathbb{1}_D$ is left Costenoble-Waner dualizable, hence D is twisted $\mathcal{C}$-ambidextrous by Proposition 3.28. This finishes the proof. $\square$

I.4 Equivariant homotopy theory

The goal of this section is to investigate the notion of twisted ambidexterity in stable equivariant homotopy theory. Throughout the section, we will use the terminology 'G-category' to refer to a $\mathcal{B}$-category when $\mathcal{B}$ is the ∞-topos Spc^G of G-spaces. Since Spc^G is equivalent to the presheaf category of the orbit ∞-category Orb_G of G, a G-category may equivalently be encoded as a functor $C\colon \mathrm{Orb}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$. We let $\mathrm{Pr}^{\mathrm{L}}_G := \mathrm{Pr}^{\mathrm{L}}(\mathrm{Spc}^G)$ denote the (very large) ∞-category of presentable G-categories.

In Section 4.1, we introduce the G-category $\underline{\mathrm{Sp}}^G$ of genuine G-spectra, informally given by sending an orbit G/H to the ∞-category Sp^H of genuine H-spectra. In Section 4.2 we prove that $\underline{\mathrm{Sp}}^G$ is the initial stable presentably symmetric monoidal G-category for which all compact G-spaces are twisted ambidextrous. In Section 4.3 and Section 4.4, we extend this result to the contexts of *orbispectra* and *proper equivariant homotopy theory*, respectively. In particular, we will obtain for every (not necessarily compact) Lie group G and cocompact subgroup H a Wirthmüller isomorphism

$$\mathrm{ind}_H^G(- \otimes D_{G/H}) \simeq \mathrm{coind}_H^G(-)$$

in the ∞-category of proper genuine G-spectra.

4.1 Parametrized genuine G-spectra

Let G be a compact Lie group, fixed throughout this subsection. The goal of this subsection is to introduce the G-category $\underline{\mathrm{Sp}}^G$ of genuine G-spectra and discuss its universal property in terms of inverting representation spheres.

Definition 4.1. Let $\{S^V\}$ be a set of representation spheres, where V runs over a set of representatives for the isomorphism classes of finite-dimensional irreducible G-representations.[1]

[1] Running over *all* finite-dimensional G-representations gives an equivalent ∞-category.

We define the presentably symmetric monoidal ∞-category Sp^G of *genuine G-spectra* as the formal inversion

$$\mathrm{Sp}^G := (\mathrm{Spc}^G_*)[\{S^V\}^{-1}]$$

of the representation spheres S^V in the ∞-category Spc^G_* of pointed G-spaces, see [Rob15, Definition 2.6] or Section 2.3.

By Gepner and Meier [GM20, Corollary C.7], the ∞-category Sp^G is equivalent to the ∞-category underlying the model category of orthogonal G-spectra with the stable model structure constructed in [MM02, Section III.4]. References on equivariant orthogonal spectra include [HHR16], [Sch20], [Sch18].

The ∞-categories Spc^G_* and Sp^G are presentably symmetric monoidal and come equipped with a symmetric monoidal left adjoint from Spc^G, making them into commutative Spc^G-algebras in Pr^{L}. We may now use the fully faithful functor

$$- \otimes_{\mathrm{Spc}^G} \Omega_{\mathrm{Spc}^G} \colon \mathrm{CAlg}_{\mathrm{Spc}^G}(\mathrm{Pr}^{\mathrm{L}}) \hookrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_G)$$

from Corollary 2.17 to regard them as presentably symmetric monoidal G-categories:

Definition 4.2. We define presentably symmetric monoidal G-categories $\underline{\mathrm{Spc}}^G$, $\underline{\mathrm{Spc}}^G_*$ and $\underline{\mathrm{Sp}}^G$ as

$$\underline{\mathrm{Spc}}^G := \Omega_{\mathrm{Spc}^G}, \qquad \underline{\mathrm{Spc}}^G_* := \mathrm{Spc}^G_* \otimes_{\mathrm{Spc}^G} \Omega_{\mathrm{Spc}^G}, \qquad \underline{\mathrm{Sp}}^G := \mathrm{Sp}^G \otimes_{\mathrm{Spc}^G} \Omega_{\mathrm{Spc}^G},$$

called the G-categories of *G-spaces*, *pointed G-spaces* and *genuine G-spectra*, respectively. We let

$$(-)_+ \colon \underline{\mathrm{Spc}}^G \to \underline{\mathrm{Spc}}^G_* \qquad \text{and} \qquad \Sigma^\infty_+ \colon \underline{\mathrm{Spc}}^G \to \underline{\mathrm{Sp}}^G$$

denote the induced maps, which are the unit maps for the algebra structures of $\underline{\mathrm{Spc}}^G_*$ and $\underline{\mathrm{Sp}}^G$ in $\mathrm{Pr}^{\mathrm{L}}_G$. Unwinding definitions, we see that these G-categories are given at a G-space B as follows:

- The ∞-category $\underline{\mathrm{Spc}}^G(B)$ is the slice $\mathrm{Spc}^G_{/B}$ of G-spaces over B.
- The ∞-category $\underline{\mathrm{Spc}}^G_*(B)$ is the relative tensor product $\mathrm{Spc}^G_{/B} \otimes_{\mathrm{Spc}^G} \mathrm{Spc}^G_*$, which by [Lur17, Example 4.8.1.21] is equivalent to the ∞-category $(\mathrm{Spc}^G_{/B})_*$ of *retractive G-spaces* over B.
- The ∞-category $\underline{\mathrm{Sp}}^G(B)$ is the relative tensor product $\mathrm{Spc}^G_{/B} \otimes_{\mathrm{Spc}^G} \mathrm{Sp}^G$. As Sp^G is pointed, this is equivalent to the relative tensor product $(\mathrm{Spc}^G_{/B})_* \otimes_{\mathrm{Spc}^G_*} \mathrm{Sp}^G$, which by Lemma 2.45 is in turn equivalent to $(\mathrm{Spc}^G_{/B})_*[\{S^V_B\}^{-1}]$, the formal inversion of the trivial sphere bundles $S^V_B := S^V \times B \to B$ in $(\mathrm{Spc}^G_{/B})_*$.

When $B = G/H$ is an orbit for some subgroup $H \leqslant G$, there are equivalences

$$\underline{\mathrm{Spc}}^G(G/H) \simeq \mathrm{Spc}^H, \qquad \underline{\mathrm{Spc}}^G_*(G/H) \simeq \mathrm{Spc}^H_*, \qquad \text{and} \qquad \underline{\mathrm{Sp}}^G(G/H) \simeq \mathrm{Sp}^H.$$

Indeed, the slice of Spc^G over G/H is equivalent to Spc^H by taking the fiber over eH, giving the first two equivalences. For the third equivalence, we observe that by Lemma 2.44 the ∞-category of genuine H-spectra can be obtained from the ∞-category of pointed H-spaces by just inverting the restricted representation spheres $\mathrm{res}^G_H(S^V)$ for irreducible G-representations V, as every irreducible H-representation is a direct summand of the restriction to H of an irreducible G-representation, see Bröcker and tom Dieck [BD95, Theorem 4.5].

The G-categories $\underline{\mathrm{Spc}}^G$, $\underline{\mathrm{Spc}}^G_*$ and $\underline{\mathrm{Sp}}^G$ admit the following universal properties:

Proposition 4.3. *Let C be a presentably symmetric monoidal G-category.*

(1) There exists a unique symmetric monoidal left adjoint G-functor $F\colon \underline{\mathrm{Spc}}^G \to C$;

(2) The G-functor F from (1) extends to a symmetric monoidal left adjoint $F'\colon \underline{\mathrm{Spc}}^G_* \to C$ if and only if $C(1)$ is pointed, in which case the extension F' is unique.

(3) If $C(1)$ is pointed, the G-functor F' from (2) extends to a symmetric monoidal left adjoint $F''\colon \underline{\mathrm{Sp}}^G \to C$ if and only if the functor $F'(1)\colon \mathrm{Spc}^G_* \to C(1)$ inverts the representation spheres S^V, in which case the extension F'' is unique.

Proof. Part (1) is immediate as $\underline{\mathrm{Spc}}^G$ is the monoidal unit of $\mathrm{Pr}^{\mathrm{L}}_G$. Parts (2) and (3) follow by combining the adjunction from Corollary 2.17 with the analogous universal properties of $\mathrm{Spc}^G_* \simeq \mathrm{Spc}^G \otimes \mathrm{Spc}_*$ and $\mathrm{Sp}^G = \mathrm{Spc}^G_*[\{S^V\}^{-1}]$ in Pr^{L}. □

For a G-space B, the functor $F_B\colon \mathrm{Spc}^G_{/B} \to C(B)$ from part (1) sends a morphism $f\colon A \to B$ to the object $f_! \mathbb{1}_A \in C(B)$. The functor $F'_B\colon (\mathrm{Spc}^G_{/B})_* \to C(B)$ sends a morphism $f\colon A \to B$ with section $s\colon B \to A$ to the cofiber of $\mathbb{1}_B \simeq f_! s_! s^* \mathbb{1}_A \to f_! \mathbb{1}_A$ in $C(B)$.

4.2 Twisted ambidexterity for genuine G-spectra

We continue to fix a compact Lie group G. The presentably symmetric monoidal G-category $\underline{\mathrm{Sp}}^G$ of genuine G-spectra is *fiberwise stable*, meaning that it takes values in the subcategory $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \subseteq \mathrm{Pr}^{\mathrm{L}}$ of stable presentable ∞-categories. Moreover, one can show that $\underline{\mathrm{Sp}}^G$ satisfies twisted ambidexterity for all compact G-spaces. The goal of this section is to show that $\underline{\mathrm{Sp}}^G$ is in a precise sense *universal* with these two properties, see Theorem 4.8 below.

We start by recalling a result of May and Sigurdsson [MS06] on Costenoble-Waner duality for G-spaces, based on ideas of Costenoble and Waner [CW16].

Construction 4.4 (cf. [Sch18, Construction 3.2.7])**.** Let $H \leqslant G$ be a closed subgroup of the compact Lie group G, and let $L = T_{eH}(G/H)$ denote the tangent H-representation of G/H. Choose an embedding $i\colon G/H \hookrightarrow V$ of G/H into a finite-dimensional orthogonal G-representation V, and let $W := V - (di)_{eH}(L)$ denote the orthogonal complement of the image of L in V. By scaling, we may assume that the map $j\colon G \times_H D(W) \to V$ given by $[g, w] \mapsto g \cdot (v_0 + w)$ is an embedding, where $D(W)$ is the unit disc of W; see [Bre72, Ch.0, Thm. 5.2, Ch. II, Cor. 5.2] for proofs that these choices are possible. The map j gives rise to a G-equivariant collapse map $c\colon S^V \to G_+ \wedge_H S^W$. Passing to genuine G-spectra and smashing with S^{-V} then gives a map of genuine G-spectra

$$\eta\colon \mathbb{S}_G \to G_+ \wedge_H (S^{-V} \wedge S^W) \simeq G_+ \wedge_H S^{-L} \simeq S^{-L} \odot \mathbb{S}_H .$$

Theorem 4.5 ([MS06, Theorem 18.6.5])**.** *Let $H \leqslant G$ be a closed subgroup of a compact Lie group G. Then the map $\eta\colon \mathbb{S}_G \to S^{-L} \odot \mathbb{S}_H$ from Construction 4.4 exhibits $S^{-L} \in \mathrm{Sp}^H$ as left Costenoble-Waner dual to $\mathbb{S}_H \in \mathrm{Sp}^H$.* □

Warning 4.6. May and Sigurdsson use different foundations on parametrized stable homotopy theory than we do, based on orthogonal spectrum objects in retractive topological spaces over B. For this reason, we need to be careful when citing results from [MS06]. Although one cannot strictly speaking cite [MS06, Theorem 18.6.5] in the case of Theorem 4.5, one observes that their proof carries through verbatim in our setting: May and Sigurdsson construct the relevant duality data already at the level of topological G-spaces using a notion of *V-duality*, see [MS06, Section 18.6], and the same commutative diagrams prove Theorem 4.5.

We can now prove our main result.

Theorem 4.7. *Let G be a compact Lie group and let $\mathcal{C}$ be a fiberwise stable presentably symmetric monoidal G-category. Let $F'\colon \underline{\mathrm{Spc}}^G_* \to \mathcal{C}$ be the unique symmetric monoidal left adjoint provided by Proposition 4.3(2). Then the following conditions are equivalent:*

(1) The functor $F'(1)\colon \mathrm{Spc}^G_* \to \mathcal{C}(1)$ inverts th representation sphere S^V for every G-representation V;

(2) The G-functor F' extends[2] *to a symmetric monoidal left adjoint $F'' \colon \underline{\mathrm{Sp}}^G \to \mathcal{C}$;*

(3) For every pair of closed subgroups $K \leqslant H \leqslant G$, the map of G-spaces $G/K \to G/H$ is twisted $\mathcal{C}$-ambidextrous;

(4) For every closed subgroup $H \leqslant G$, the orbit G/H is twisted $\mathcal{C}$-ambidextrous;

(5) Every compact G-space is twisted $\mathcal{C}$-ambidextrous;

(6) The functor $F'(1) \colon \mathrm{Spc}_*^G \to \mathcal{C}(1)$ sends compact pointed G-spaces to dualizable objects.

If the group G is finite, these conditions are moreover equivalent to:

(7) The G-category $\mathcal{C}$ is G-semiadditive, in the sense of [Nar16, Definition 5.3];

(8) The G-category $\mathcal{C}$ is G-stable, in the sense of [Nar16, Definition 7.1].

Proof. The implication (1) $\implies$ (2) was treated in Proposition 4.3. For the implication (2) $\implies$ (3), it suffices by Corollary 3.35 to show that the morphism $G/K \to G/H$ is twisted $\underline{\mathrm{Sp}}^G$-ambidextrous. Identifying the slice $\mathrm{Spc}^G_{/(G/H)}$ with the ∞-category of H-spaces, the morphism $G/K \to G/H$ corresponds to the H-space H/K, and thus we need to show that H/K is twisted ambidextrous for the H-category $\pi_H^* \underline{\mathrm{Sp}}^G \simeq \underline{\mathrm{Sp}}^H$. This is an instance of Theorem 4.5, applied to $K \leqslant H$. The fact that (3) implies (4) is clear. The implication (4) $\implies$ (5) follows from Corollary 3.41, since every compact G-space is a retract of a finite G-CW-complex, and finite G-CW-complexes are built from the orbits G/H using finite colimits. The implication (5) $\implies$ (6) holds by Corollary 3.39, as $F'(1)(B)$ is the cofiber of the map $\mathbb{1} \to B_! \mathbb{1}_B$ and dualizable objects in $\mathcal{C}(1)$ are closed under cofibers. The implication (6) $\implies$ (1) holds by Theorem B.1. This shows that conditions (1)-(6) are equivalent.

If G is a finite group, conditions (7) and (8) are equivalent since $\mathcal{C}$ is already assumed to be fiberwise stable. Since $\mathcal{C}$ is in particular fiberwise semiadditive, it follows from Corollary 3.20 that conditions (3) and (7) are equivalent. This finishes the proof. □

Theorem 4.8. *Let G be a compact Lie group. Then the G-category $\underline{\mathrm{Sp}}^G$ is initial among fiberwise stable presentably symmetric monoidal G-categories $\mathcal{C}$ such that all compact G-spaces are twisted $\mathcal{C}$-ambidextrous.*

Proof. This is immediate from the equivalence between (2) and (5) in Theorem 4.7. □

[2] If an extension exists, it is necessarily unique.

Theorem 4.9. *Let G be a finite group. Then the G-category $\underline{\mathrm{Sp}}^G$ is the initial presentably symmetric monoidal G-category which is G-stable in the sense of [Nar16].*

Proof. This is immediate from the equivalence between (2) and (8) in Theorem 4.7. □

4.3 Orbispectra

In this subsection, we will study a global analogue of the results established in the previous subsection: instead of working only with subgroups of a fixed compact Lie group G, we work with an indexing category Orb containing all compact Lie groups and *injective* group homomorphisms.

The global orbit category

We start by recalling the definition of the global orbit category Orb.

Definition 4.10. Let $\mathrm{TopGrpd}_1$ denote the ordinary category of topological groupoids. It is naturally enriched over itself via the internal mapping objects. Using the finite product preserving geometric realization functor

$$|-|\colon \mathrm{TopGrpd}_1 \hookrightarrow \mathrm{sTop} \xrightarrow{|-|} \mathrm{Top},$$

we obtain a topological enrichment on $\mathrm{TopGrpd}_1$. Explicitly, the geometric realization $|\mathcal{G}|$ of a topological groupoid $\mathcal{G}$ is given by the coend

$$|\mathcal{G}| = \int^{[n]\in\Delta} \Delta^n \times \mathcal{G}_n \quad \in \quad \mathrm{Top},$$

where Δ^n denotes the topological n-simplex and where $\mathcal{G}_n = \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} \cdots \times_{\mathcal{G}_0} \mathcal{G}_1$. We let TopGrpd denote the homotopy coherent nerve of the topologically enriched category $\mathrm{TopGrpd}_1$.

Definition 4.11 (Global indexing category)**.** Given a topological group G, we let $\mathbb{B}G$ denote the associated one-object topological groupoid. We define the ∞-category Glo as the full subcategory of TopGrpd spanned by the topological groupoids $\mathbb{B}G$ for compact Lie groups G. We call Glo the *global indexing category*.

Given compact Lie groups H and G, the mapping space $\mathrm{Hom}_{\mathrm{Glo}}(\mathbb{B}H, \mathbb{B}G)$ is the homotopy type of the geometric realization of the topological groupoid $\mathbb{H}\mathrm{om}(\mathbb{B}H, \mathbb{B}G)$. This topological groupoid is equivalent to the action groupoid of the topological G-space $\mathrm{Hom}_{\mathrm{Lie}}(H, G)$

of continuous group homomorphisms from H to G, with G-action given by conjugation. As a consequence, we get an identification

$$\mathrm{Hom}_{\mathrm{Glo}}(\mathbb{B}H,\mathbb{B}G) \simeq \mathrm{Hom}_{\mathrm{Lie}}(H,G)_{hG} \quad \in \quad \mathrm{Spc}.$$

Definition 4.12 (Global orbit category)**.** We let Orb $\subseteq$ Glo denote the wide subcategory, whose morphism spaces

$$\mathrm{Hom}_{\mathrm{Orb}}(\mathbb{B}H,\mathbb{B}G) \subseteq \mathrm{Hom}_{\mathrm{Glo}}(\mathbb{B}H,\mathbb{B}G)$$

consist of those components corresponding to *injective* group homomorphisms $H \hookrightarrow G$. We call Orb the *global orbit category*. We denote the presheaf category PSh(Orb) by OrbSpc and refer to this as the ∞-category of *orbispaces*. We refer to OrbSpc-categories as *orbicategories*.

Remark 4.13. Our definitions of Glo and Orb agree with those of [Rez14, p. 2.2] and [LNP22, Definition 6.1]. A close analogue of the definition was originally given by Gepner and Henriques [GH07, Section 4.1] for arbitrary topological groups, where both Glo and Orb were denoted as Orb. A slight difference with the definition of [GH07] is that they use the *fat realization* $||-||\colon$ TopGrpd $\to$ Top as opposed to the usual ('thin') geometric realization. As $||-||$ does not preserve finite products on the nose, defining composition is slightly subtle, but when taking sufficient care the resulting ∞-categories will be equivalent; see [Kör18, Remark 3.10] for a more detailed discussion.

Crucial for the comparision with equivariant homotopy theory for a compact Lie group G is the statement that the slice of Orb over $\mathbb{B}G$ is equivalent to the orbit category Orb_G of G. This was proved at the level of simplicially enriched categories by [GM20, Proposition 2.15], and at the level of ∞-categories by [LNP22, Lemma 6.13]. It follows that there is an equivalence of ∞-categories

$$\mathrm{Spc}^G \simeq \mathrm{OrbSpc}_{/\mathbb{B}G}$$

between the ∞-category of G-spaces and the ∞-category of orbispaces over $\mathbb{B}G$. Given a G-space A, we denote its associated orbispace by $A//G$, so that $*//G = \mathbb{B}G$. By restricting along the functor $-//G\colon \mathrm{Spc}_G \to \mathrm{OrbSpc}$, any orbicategory C has an underlying G-category which we will denote by π_G^*C.

The orbicategory of orbispectra

We will now define the orbicategory of orbispectra, informally given by the assignment $\mathbb{B}G \mapsto \mathrm{Sp}^G$, and prove its universal property in terms of twisted ambidexterity.

Definition 4.14. Define the presentably symmetric monoidal orbicategories $\underline{\mathrm{OrbSpc}}$ and $\underline{\mathrm{OrbSpc}_*}$ of *orbispaces* resp. *pointed orbispaces* as

$$\underline{\mathrm{OrbSpc}} := \Omega_{\mathrm{OrbSpc}}, \qquad \underline{\mathrm{OrbSpc}_*} := \mathrm{OrbSpc}_* \otimes_{\mathrm{OrbSpc}} \Omega_{\mathrm{OrbSpc}},$$

using the fully faithful embedding $- \otimes_{\mathrm{OrbSpc}} \Omega_{\mathrm{OrbSpc}} \colon \mathrm{CAlg}_{\mathrm{OrbSpc}}(\mathrm{Pr}^{\mathrm{L}}) \hookrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Orb}})$ from Proposition 2.16. Explicitly, they are given at an orbispace B by

$$\underline{\mathrm{OrbSpc}}(B) = \mathrm{OrbSpc}_{/B}, \qquad \underline{\mathrm{OrbSpc}_*}(B) = (\mathrm{OrbSpc}_{/B})_*.$$

Definition 4.15. We let $S \subseteq \underline{\mathrm{OrbSpc}_*}$ denote the subcategory spanned by those objects $X \in (\mathrm{OrbSpc}_{/B})_*$ whose restriction along any map $\mathbb{B}G \to B$ corresponds to a G-representation sphere in $(\mathrm{OrbSpc}_{/\mathbb{B}G})_* \simeq \mathrm{Spc}^G_*$. We define the presentably symmetric monoidal orbicategory $\underline{\mathrm{OrbSp}}$ of *orbispectra* as

$$\underline{\mathrm{OrbSp}} := \mathcal{L}(\underline{\mathrm{OrbSpc}_*}, S) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Orb}}),$$

using Construction 2.43. We define the orbifunctor $\Sigma^\infty \colon \underline{\mathrm{OrbSpc}_*} \to \underline{\mathrm{OrbSp}}$ as

$$\Sigma^\infty \colon \underline{\mathrm{OrbSpc}_*} = \mathcal{L}(\underline{\mathrm{OrbSpc}_*}, \emptyset) \to \mathcal{L}(\underline{\mathrm{OrbSpc}_*}, S) = \underline{\mathrm{OrbSp}}.$$

We let OrbSp denote the underlying ∞-category of $\underline{\mathrm{OrbSp}}$, referred to as the *$\infty$-category of orbispectra*. For an orbispace B, we also write $\mathrm{OrbSp}(B)$ for $\underline{\mathrm{OrbSp}}(B)$ and call it the *∞-category of orbispectra parametrized over B*.

Pardon [Par20] has previously defined a notion of orbispectra in the setting of topological stacks. Although his definition seems close in spirit to our definition, the precise mathematical connection is not known to the author.

Proposition 4.16. *The orbicategory $\underline{\mathrm{OrbSp}}$ is presentably symmetric monoidal and the orbifunctor $\Sigma^\infty \colon \underline{\mathrm{OrbSpc}_*} \to \underline{\mathrm{OrbSp}}$ exhibits it as the formal inversion of the representation spheres in $\underline{\mathrm{OrbSpc}_*}$.*

Proof. This is an instance of Proposition 2.46. The condition $(*)$ of that proposition is satisfied: for a closed subgroup $H \leqslant G$, every irreducible H-representation is a direct summand of the restriction to H of an irreducible G-representation, see Bröcker and tom Dieck [BD95, Theorem 4.5]. □

The orbicategory $\underline{\mathrm{OrbSp}}$ recovers the G-category $\underline{\mathrm{Sp}}^G$ of genuine G-spectra for every compact Lie group G:

Lemma 4.17. *For every compact Lie group G, the G-category $\pi_G^*\underline{\mathrm{OrbSp}}$ underlying the orbicategory $\underline{\mathrm{OrbSp}}$ of orbispectra is equivalent to the G-category $\underline{\mathrm{Sp}}^G$ of genuine G-spectra.*

Proof. Both come equipped with a symmetric monoidal left adjoint from $\underline{\mathrm{Spc}}_*^G$ exhibiting them as formal inversions of the G-representation spheres. For $\underline{\mathrm{Sp}}^G$ this is by Proposition 2.38, while for $\pi_G^*\underline{\mathrm{OrbSp}}$ this is by Proposition 2.47. $\square$

Remark 4.18. The functor $\underline{\mathrm{OrbSp}}\colon \mathrm{Orb}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is the restriction along $\mathrm{Orb} \hookrightarrow \mathrm{Glo}$ of the functor $\mathrm{Sp}_\bullet\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ constructed by [LNP22, Section 10]. Indeed, they construct a natural transformation $\Sigma^\infty_\bullet\colon \mathcal{S}_{\bullet,*} \to \mathrm{Sp}_\bullet$, where the functor $\mathcal{S}_{\bullet,*}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ constructed in [LNP22, Construction 6.16] restricts to the functor $\underline{\mathrm{OrbSpc}}_*$ using the natural equivalence $\mathrm{PSh}(\mathrm{Orb}_{/-}) \simeq \mathrm{PSh}(\mathrm{Orb})_{/-}$. Furthermore, it is shown in [LNP22, Proposition 10.5] that this transformation $\Sigma^\infty_\bullet$ is pointwise given by the standard suspension spectrum functor $\mathrm{Spc}_*^G \to \mathrm{Sp}^G$, so that it exhibits its target as a pointwise formal inversion of its source.

From the results of Section 4.2 we may deduce a universal property of $\underline{\mathrm{OrbSp}}$ in terms of twisted ambidexterity. To this end, recall that a morphism $f\colon A \to B$ in an ∞-topos is called *relatively compact* if for every compact object K in $\mathcal{B}$ and every morphism $K \to B$, the pullback $A \times_B K \to K$ is a compact object in the slice $\mathcal{B}_{/K}$ (or equivalently in $\mathcal{B}$ by [GHK22, Lemma 3.1.5]). If $\mathcal{B} = \mathrm{PSh}(T)$ is a presheaf topos, it suffices to check this when K is a representable object. In particular, a morphism of orbispaces $f\colon A \to B$ is relatively compact if and only if for every compact Lie group G and every map $\mathbb{B}G \to B$ of orbispaces, the pullback $A \times_B \mathbb{B}G$ corresponds to a compact G-space.

Proposition 4.19. *Let $\mathcal{C}$ be a fiberwise stable presentably symmetric monoidal orbicategory. Then the following conditions are equivalent:*

(1) The unique symmetric monoidal left adjoint $F'\colon \underline{\mathrm{OrbSpc}}_* \to \mathcal{C}$ inverts the representation spheres;

(2) The functor $F'\colon \underline{\mathrm{OrbSpc}}_* \to \mathcal{C}$ extends (necessarily uniquely) to a symmetric monoidal left adjoint $F'\colon \underline{\mathrm{OrbSp}} \to \mathcal{C}$;

(3) For every compact Lie group G and every compact G-space A, the map $A//G \to *//G = \mathbb{B}G$ of orbispaces is twisted $\mathcal{C}$-ambidextrous;

(4) Every relatively compact morphism $f\colon A \to B$ of orbispaces is twisted $\mathcal{C}$-ambidextrous.

Proof. The equivalence between (1) and (2) is immediate from the universal property of $\underline{\mathrm{OrbSp}}$. Note that condition (1) is satisfied if and only if each of the G-functors $\pi_G^* F' \colon \pi_G^* \underline{\mathrm{OrbSpc}}_* \to \pi_G^* C$ satisfies condition (1) of Theorem 4.7, while (3) is satisfied if and only if the G-category π_G^* satisfies condition (5) of Theorem 4.7, so the equivalence between (1) and (3) holds by applying Theorem 4.7 to the G-category $\pi_G^* C$ for every G. The equivalence between (3) and (4) holds by Proposition 3.12(5) and the above characterization of relatively compact morphisms in OrbSpc. □

Theorem 4.20. *The orbicategory* $\underline{\mathrm{OrbSp}}$ *is initial among fiberwise stable presentably symmetric monoidal orbicategories* C *such that every relatively compact morphism of orbispaces is twisted* C*-ambidextrous.*

Proof. This is immediate from the equivalence between (2) and (4) in the previous proposition. □

4.4 Proper equivariant stable homotopy theory

For a Lie group G, not assumed to be compact, Degrijse et al. [Deg+19] introduced an ∞-category Sp^G of *proper genuine* G*-spectra*. In this subsection, we will see that this ∞-category can be identified with the ∞-category of orbispectra parametrized over a certain orbispace $\mathbb{B}G$. As an application, we show that when G has *enough bundle representations*, the ∞-category Sp^G may be obtained from the ∞-category of pointed proper G-spaces by inverting the sphere bundles S^ξ associated to finite-dimensional vector bundles ξ over $\mathbb{B}G$, see Proposition 4.33.

Definition 4.21. For a Lie group G, we define its *proper orbit category* as the full subcategory $\mathrm{Orb}_G^{\mathrm{pr}} \subseteq \mathrm{Orb}_G$ spanned by the orbits G/K for compact subgroups $K \leqslant G$. We define the ∞-category $\mathrm{Spc}^G_{\mathrm{pr}}$ of *proper* G*-spaces* as the presheaf category $\mathrm{PSh}(\mathrm{Orb}_G^{\mathrm{pr}})$. A *proper* G*-category* is a $\mathrm{Spc}^G_{\mathrm{pr}}$-category, equivalently encoded by a functor $(\mathrm{Orb}_G^{\mathrm{pr}})^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

We start by identifying the ∞-category of proper G-spaces with a slice of the ∞-category of orbispaces.

Definition 4.22 (Classifying orbispace of a Lie group)**.** In analogy with $\mathrm{Orb} \subseteq \mathrm{TopGrpd}$, we define the ∞-category $\mathrm{Orb}' \subseteq \mathrm{TopGrpd}$ as the (non-full) subcategory whose objects are the one-point topological groupoids $\mathbb{B}G$ for (not necessarily compact) Lie groups G, and whose mapping spaces

$$\mathrm{Hom}_{\mathrm{Orb}'}(\mathbb{B}H, \mathbb{B}G) \subseteq \mathrm{Hom}_{\mathrm{TopGrpd}}(\mathbb{B}H, \mathbb{B}G) \simeq \mathrm{Hom}_{\mathrm{Lie}}(H, G)_{hG}$$

consist of those path components corresponding to *injective* continuous group homomorphisms $H \to G$. It is immediate that Orb' contains Orb. Given a Lie group G, we define its *classifying orbispace* $\mathbb{B}G$ as the composite

$$\mathbb{B}G \colon \mathrm{Orb}^{\mathrm{op}} \hookrightarrow (\mathrm{Orb}')^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Orb}'}(-,\mathbb{B}G)} \mathrm{Spc}.$$

Note that when G is compact, this is just the representable presheaf on $\mathbb{B}G \in \mathrm{Orb}$.

Construction 4.23. For a Lie group G, let $\mathrm{Orb}_{/\mathbb{B}G}$ denote the full subcategory of $\mathrm{Orb}'_{/\mathbb{B}G}$ spanned by the morphisms of the form $\mathbb{B}K \to \mathbb{B}G$ for compact subgroups $K \leqslant G$. At the level of topological categories, one can define a topologically enriched functor from Orb_G to Orb' by sending an orbit G/H to $\mathbb{B}H$, which induces a functor of ∞-categories between their homotopy coherent nerves. As Orb_G admits a terminal object G/G, this canonically gives rise to a functor $\mathrm{Orb}_G \to \mathrm{Orb}'/\mathbb{B}G$, which is easily seen to restrict to a functor $\mathrm{Orb}_G^{\mathrm{pr}} \to \mathrm{Orb}_{/\mathbb{B}G}$.

Lemma 4.24. *For a Lie group G, the above functor $\mathrm{Orb}_G^{\mathrm{pr}} \to \mathrm{Orb}_{/\mathbb{B}G}$ is an equivalence.*

Proof. We will argue just like [LNP22, Lemma 6.13]. The functor is essentially surjective by definition, so we must show it is fully faithful. Consider two objects G/H and G/K in $\mathrm{Orb}_G^{\mathrm{pr}}$, where we (non-canonically) choose representatives $H, K \subseteq G$ of the conjugacy classes $[H]$ and $[K]$ of subgroups of G. We have to show that the square

$$\begin{tikzcd}
(G/H)^K \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{TopGrpd}}(\mathbb{B}K, \mathbb{B}H) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}(H,K)_{hK} \arrow[d] \\
* \arrow[r] & \mathrm{Hom}_{\mathrm{TopGrpd}}(\mathbb{B}K, \mathbb{B}G) \arrow[r, "\simeq"] & \mathrm{Hom}_{\mathrm{Lie}}(H,G)_{hG}
\end{tikzcd}$$

is homotopy cartesian. The argument for this is identical to that of [LNP22, Lemma 6.13], so we will not repeat it here. Note that the quotient map $G \to G/C(H)$ used in that proof is still a fibration, as it is a locally trivial fiber bundle by [Pal61, Corollary 4.1]. $\square$

Corollary 4.25. *For every Lie group G, there is an equivalence $\mathrm{OrbSpc}_{/\mathbb{B}G} \simeq \mathrm{Spc}^G_{\mathrm{pr}}$ between the ∞-category of orbispaces over $\mathbb{B}G$ and the ∞-category of proper G-spaces.*

Proof. By Lemma 4.24, there is an equivalence $\mathrm{Spc}^G_{\mathrm{pr}} = \mathrm{PSh}(\mathrm{Orb}_G^{\mathrm{pr}}) \simeq \mathrm{PSh}(\mathrm{Orb}_{/\mathbb{B}G})$. The ∞-category $\mathrm{Orb}_{/\mathbb{B}G}$ is equivalent to the subcategory of $\mathrm{OrbSpc}_{/\mathbb{B}G}$ spanned by the maps $\mathbb{B}K \to \mathbb{B}G$ for compact Lie groups K, since both embed fully faithfully into $\mathrm{PSh}(\mathrm{Orb}')_{/\mathbb{B}G}$ with the same image. It follows that there is an equivalence $\mathrm{PSh}(\mathrm{Orb}_{/\mathbb{B}G}) \simeq \mathrm{OrbSpc}_{/\mathbb{B}G}$, finishing the proof. $\square$

Definition 4.26. Let G be a Lie group. By restricting along the forgetful functor $\mathrm{Spc}^G_{\mathrm{pr}} \simeq \mathrm{OrbSpc}_{/\mathbb{B}G} \xrightarrow{\mathrm{fgt}} \mathrm{OrbSpc}$, any orbicategory C gives rise to a proper G-category π_G^*. We define the proper G-category of *proper genuine G-spectra* $\underline{\mathrm{Sp}}^G$ as

$$\underline{\mathrm{Sp}}^G := \pi_G^* \underline{\mathrm{OrbSp}},$$

the underlying proper G-category of the orbicategory of orbispectra. When G is compact this agrees with the G-∞-category of genuine G-spectra $\underline{\mathrm{Sp}}^G$ by Lemma 4.17.

Proposition 4.27 (Linskens-Nardin-Pol [LNP22, Theorem 12.11])**.** *For every Lie group G, the underlying ∞-category of the proper G-category $\underline{\mathrm{Sp}}^G$ is equivalent to the ∞-category Sp^G of proper genuine G-spectra defiend by [Deg+19].*

Proof. By definition, the underlying ∞-category of $\underline{\mathrm{Sp}}^G$ is given as the limit of the functor $\underline{\mathrm{Sp}}^G \colon (\mathrm{Orb}^{\mathrm{pr}}_G)^{\mathrm{op}} \to \mathrm{Cat}_\infty$. By Remark 4.18, this functor is equivalent to the composite

$$(\mathrm{Orb}^{\mathrm{pr}}_G)^{\mathrm{op}} \simeq \mathrm{Orb}^{\mathrm{op}}_{/\mathbb{B}G} \to \mathrm{Orb}^{\mathrm{op}} \hookrightarrow \mathrm{Glo}^{\mathrm{op}} \xrightarrow{\mathrm{Sp}_\bullet} \mathrm{Cat}_\infty.$$

The limit of this diagram was shown by [LNP22, Theorem 12.11] to be equivalent to Sp^G, finishing the proof. □

Corollary 4.28. *The proper G-category $\underline{\mathrm{Sp}}^G$ satisfies twisted ambidexterity for all relatively compact morphisms of proper G-spaces. In particular, if $H \leqslant G$ is a cocompact subgroup, meaning that G/H is a compact topological space, then there is a formal Wirthmüller isomorphism*

$$\mathrm{ind}_H^G(- \otimes D_{G/H}) \simeq \mathrm{coind}_H^G(-) \colon \mathrm{Sp}^H \to \mathrm{Sp}^G.$$

Proof. The first statement is immediate as $\underline{\mathrm{Sp}}^G$ is the restriction of the orbicategory $\underline{\mathrm{OrbSp}}$ to the slice $\mathrm{Orb}_{/\mathbb{B}G}$ and $\underline{\mathrm{OrbSp}}$ satisfies twisted ambidexterity for all relatively compact morphisms of orbispaces. The second statement follows from the observation that the map of orbispaces $\mathbb{B}H \to \mathbb{B}G$ is relatively compact, by compactness of G/H. □

Theorem 4.29. *The proper G-category $\underline{\mathrm{Sp}}^G$ is the initial fiberwise stable presentably symmetric monoidal proper G-category satisfying twisted ambidexterity for all relatively compact morphisms of proper G-spaces.*

Proof. Just like in Proposition 4.19, one deduces from Theorem 4.7 that twisted ambidexterity for relatively compact morphisms of proper G-spaces is equivalent to the invertibility of representation spheres. The claim follows, since $\underline{\mathrm{Sp}}^G$ is a formal inversion of representation spheres in $\underline{\mathrm{Spc}}^G_*$ by Proposition 2.47. □

Enough bundle representations

Our results on formal inversions allow us to prove that in certain cases the ∞-category of proper genuine G-spectra may be obtained from the ∞-category of pointed proper G-spaces by inverting the sphere bundles of vector bundles over $\mathbb{B}G$.

Definition 4.30. Let $\mathrm{Rep}\colon \mathrm{Orb}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ denote the functor which sends $\mathbb{B}G$ to the ordinary category of finite-dimensional G-representations. We may limit-extend this to a functor

$$\mathrm{Vect}\colon \mathrm{OrbSpc}^{\mathrm{op}} \to \mathrm{Cat}_\infty,$$

and we refer to $\mathrm{Vect}(B)$ as the category of *vector bundles over B*. We say that an orbispace B *has enough bundle representations* if for any compact Lie group G, any map of orbispaces $\mathbb{B}G \to B$ and any G-representation V, there exists a vector bundle $\xi \in \mathrm{Vect}(B)$ such that the restriction $\xi|_{\mathbb{B}G} \in \mathrm{Vect}(\mathbb{B}G) = \mathrm{Rep}(G)$ contains V as a direct summand.

Example 4.31. For every compact Lie group G, $\mathbb{B}G$ has enough bundle representations by Bröcker and tom Dieck [BD95, Theorem 4.5].

Example 4.32. Let G be a discrete group and assume that $\mathbb{B}G$ is a finite orbispace, that is, it lies in the subcategory of OrbSpc generated under finite colimits by the $\mathbb{B}K$ for compact Lie groups K. Then $\mathbb{B}G$ has enough bundle representations. Indeed, under the identification $\mathrm{OrbSpc}_{/\mathbb{B}G} \simeq \mathrm{Spc}^G_{\mathrm{pr}}$, the orbispace $\mathbb{B}G$ corresponds to the universal proper G-space $\underline{EG}$, and by the assumption this is a finite proper G-CW-complex. Given a finite subgroup $K \leqslant G$, the unique map $\varphi\colon G/K \to \underline{EG}$ is a map of finite proper G-CW-complexes, hence by Lück and Oliver [LO01, Lemma 3.7] any K-representation is a direct summand of the restriction along φ of a G-vector bundle V over $\underline{EG}$. Since V in particular gives rise to a vector bundle over the orbispace $\mathbb{B}G$, this finishes the proof.

Proposition 4.33. *Assume the orbispace B has enough bundle representations. Then the ∞-category* $\mathrm{OrbSp}(B)$ *of orbispectra parametrized over B is equivalent to the formal inversion of sphere bundles* $\{S^\xi \mid \xi \in \mathrm{Vect}(B)\}$ *in the ∞-category* $(\mathrm{OrbSpc}_{/B})_*$ *of retractive orbispces over B:*

$$\mathrm{OrbSp}(B) \simeq (\mathrm{OrbSpc}_{/B})_*[\{S^\xi\}^{-1}]. \qquad \square$$

Proof. Let $\mathcal{B} = \mathrm{OrbSpc}_{/B}$ be the ∞-topos of orbispaces over B. By Proposition 2.47, the $\mathcal{B}$-functor $\pi_B^*\underline{\mathrm{OrbSpc}_*} \to \pi_B^*\underline{\mathrm{OrbSp}}$ is a formal inversion of the representation spheres $S^V \in \mathrm{Spc}^G_* \simeq \pi_B^*\underline{\mathrm{OrbSpc}_*}(\mathbb{B}G)$ for every compact Lie group G and a map of orbispaces $\mathbb{B}G \to B$. By the assumption that B has enough bundle representations, the parametrized subcategory of representation spheres is generated (in the sense of Definition 2.36) by the objects $S^\xi \in \mathrm{Spc}_*^B$ for all vector bundles $\xi \in \mathrm{Vect}(B)$. It thus follows from Observation 2.39 that the underlying functor $(\mathrm{OrbSpc}_{/B})_* \to \mathrm{OrbSp}(B)$ of this $\mathcal{B}$-functor is a formal inversion of the objects S^ξ, finishing the proof. □

Corollary 4.34. *Assume that G is a Lie group which has enough bundle representations. Then the ∞-category of proper genuine G-spectra is the formal inversion of the ∞-category proper pointed G-spaces by inverting the sphere bundles S^ξ associated to all finite-dimensional vector bundles ξ over $\mathbb{B}G$:*

$$\mathrm{Sp}^G \simeq (\mathrm{Spc}_*^G)[\{S^\xi\}^{-1}].$$

Part II

Relative Poincaré duality for differentiable stacks

Abstract

Following ideas from motivic homotopy theory, we introduce for every separated differentiable stack $\mathcal{X}$ an ∞-category $\mathrm{SH}(\mathcal{X})$ of genuine sheaves of spectra on $\mathcal{X}$, which for manifolds is given by ordinary sheaves of spectra while on the classifying stack of a compact Lie group is given by genuine G-spectra. We prove a form of relative Poincaré duality in this setting: for a proper representable submersion f of separated differentiable stacks, there is an equivalence $f_\sharp \simeq f_*(- \otimes S^{T_f})$ between its relative homology and a twist of its relative cohomology by the relative tangent sphere bundle of f. When specialized to quotient stacks of equivariant smooth manifolds, this recovers both equivariant Atiyah duality and the Wirthmüller isomorphism in stable equivariant homotopy theory.

II.1 Introduction

The goal of this article is to establish a version of relative Poincaré duality for separated differentiable stacks, following ideas from Hoyois [Hoy17] in the setting of equivariant motivic homotopy theory.

Relative Poincaré duality for smooth manifolds

Poincaré duality is an important result relating the homology and cohomology of manifolds: if M is a compact orientable n-dimensional manifold, then forming the cap product with the fundamental class $[M] \in H_n(M)$ induces an isomorphism of graded abelian groups

$$- \cap [M] \colon H^{n-*}(M;\mathbb{Z}) \xrightarrow{\cong} H_*(M;\mathbb{Z}).$$

The orientability condition on M can be dropped if one replaces the left-hand side by the cohomology of the *orientation sheaf* $\mathcal{O}_M \in \mathrm{Shv}(M;\mathrm{Ab})$; the resulting isomorphism $H^{n-*}(M;\mathcal{O}_M) \xrightarrow{\cong} H_*(M)$ is known as *twisted Poincaré duality*.

There is a more general version of Poincaré duality for *families* of manifolds, known as *relative Poincaré duality*. Consider a proper smooth submersion $f \colon M \to N$, thought of as an N-indexed family of compact smooth manifolds $M_x = f^{-1}(x)$ for $x \in N$. The pullback functor on sheaves of spectra $f^* \colon \mathrm{Shv}(N;\mathrm{Sp}) \to \mathrm{Shv}(M;\mathrm{Sp})$ admits both a left adjoint $f_\sharp$ and a right adjoint f_*, which we think of as the *relative sheaf homology/cohomology* of f, respectively. Indeed, given a sheaf of spectra $\mathcal{F}$ on M, one can show that the stalks of the sheaves $f_\sharp(\mathcal{F})$ and $f_*(\mathcal{F})$ on N are given by the sheaf homology/cohomology of the restriction of $\mathcal{F}$ to each of the fibers of f. Relative Poincaré duality states that these two functors $f_\sharp$ and f_* agree up to a 'twist' by the tangent sphere bundle $S^{T_f} \in \mathrm{Shv}(M;\mathrm{Sp})$, the suspension spectrum of the one-point compactification of the relative tangent bundle of f:

Proposition (Relative Poincaré duality, Volpe [Vol21, Proposition 6.18, Theorem 7.11])**.** *Let $f \colon M \to N$ be a proper smooth submersion between smooth manifolds. Then there is for every sheaf $\mathcal{F} \in \mathrm{Shv}(M;\mathrm{Sp})$ a natural equivalence $f_\sharp(\mathcal{F}) \simeq f_*(\mathcal{F} \otimes S^{T_f})$ of sheaves of spectra on N.*

Relative Poincaré duality for differentiable stacks

The goal of this article is to generalize the above result to the setting of (separated[1]) differentiable stacks. We may think of a differentiable stack as a generalization of a smooth manifold which is allowed to have *singularities*: instead of being locally isomorphic to a Euclidean space $\mathbb{R}^n$, it will locally be isomorphic to the quotient of $\mathbb{R}^n$ by a linear action of a compact Lie group G. While the geometric behavior of differentiable stacks closely parallels that of smooth manifolds, the allowed singularities make it possible to capture equivariant phenomena.

For every differentiable stack $\mathcal{X}$, we will define a stable ∞-category $\mathrm{SH}(\mathcal{X})$ of *genuine sheaves of spectra on* $\mathcal{X}$. When $\mathcal{X} = M$ is a smooth manifold, this reduces to the ∞-category $\mathrm{Shv}(M;\mathrm{Sp})$ of ordinary sheaves of spectra on M, while for a classifying stack $\mathbb{B}G$ of a compact Lie group G it recovers the ∞-category Sp_G of genuine G-spectra. The construction of $\mathrm{SH}(\mathcal{X})$ follows the definition of the stable motivic homotopy category in motivic homotopy theory: First one forms the ∞-category $\mathrm{H}(\mathcal{X}) = \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}})$ of homotopy invariant sheaves on the site of representable submersions over $\mathcal{X}$ (equipped with the open cover topology). Then, at least locally in $\mathcal{X}$, one defines $\mathrm{SH}(\mathcal{X})$ by inverting all sphere bundles of vector bundles over $\mathcal{X}$, which determines $\mathrm{SH}(\mathcal{X})$ for an arbitrary differentiable stack by asking the assignment $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ to satisfy descent with respect to open covers.

Consider a proper representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of differentiable stacks, which we want to think of as an $\mathcal{X}$-indexed family of compact smooth manifolds. This map admits a relative tangent bundle T_f over $\mathcal{Y}$, producing a tangent sphere bundle $S^{T_f} \in \mathrm{SH}(\mathcal{Y})$. The map f defines a symmetric monoidal pullback functor $f^*\colon \mathrm{SH}(\mathcal{X}) \to \mathrm{SH}(\mathcal{Y})$, which admits both a left adjoint $f_\sharp$ as well as a right adjoint f_*, thought of as the relative homology and cohomology of f. The following is our main result:

Theorem A (Relative Poincaré duality for genuine sheaves, Theorem 6.1.7)**.** *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a proper representable submersion between separated differentiable stacks. Then there is for every $\mathcal{F} \in \mathrm{SH}(\mathcal{Y})$ a natural equivalence $f_\sharp(\mathcal{F}) \simeq f_*(\mathcal{F} \otimes S^{T_f})$ in $\mathrm{SH}(\mathcal{X})$.*

As a consequence we will deduce relative Atiyah duality for differentiable stacks, and we will prove proper base change, smooth-proper base change and the proper projection formula for the pushforward functors $p_*\colon \mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$ of proper maps $p\colon \mathcal{Y} \to \mathcal{X}$.

[1] In this introduction, all differentiable stacks are assumed separated.

Organization

In Chapter II.2 we recall standard material on differentiable stacks, including their relation to Lie groupoids, local properties of maps of stacks and the notion of vector bundles over stacks. In Chapter II.3 we discuss various geometrical aspects of differentiable stacks, including the coarse moduli space of a stack (Section 3.1), open complements of closed substacks (Section 3.2), the isotropy groups of a stack (Section 3.4) and the notions of relative tangent bundles and normal bundles (Section 3.5). A geometrically well-behaved class of differentiable stacks, discussed in Section 3.3, are the *separated* differentiable stacks: those whose diagonal is proper. In Section 3.6 we show that every embedding of separated differentiable stacks admits a *tubular neighborhood* and in Section 3.7 we prove that every separated differentiable stack is locally isomorphic to a quotient stack $\mathbb{R}^n//G$ for some smooth linear action of a compact Lie group G on a Euclidean space $\mathbb{R}^n$.

In Chapter II.4 we introduce for every separated differentiable stack $\mathcal{X}$ the ∞-categories $\mathrm{H}(\mathcal{X})$ and $\mathrm{SH}(\mathcal{X})$ of *genuine sheaves of animae/spectra on* $\mathcal{X}$, show that ordinary sheaves are both a localization and a colocalization of genuine sheaves, and prove basic properties for genuine sheaves like smooth base change and the smooth projection formula. In Section 4.4, we show that genuine sheaves over classifying stacks of compact Lie groups give back classical equivariant homotopy theory:

Theorem B (Theorem 4.4.16, Proposition 4.4.17)**.** *For a compact Lie group G, there are equivalences of ∞-categories* $\mathrm{H}(\mathbb{B}G) \simeq \mathrm{An}_G$ *and* $\mathrm{SH}(\mathbb{B}G) \simeq \mathrm{Sp}_G$.

In Section 4.5 we show that the assignments $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ and $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ admit universal characterizations, phrased in the language of *pullback formalisms* introduced by [DG22].

In Chapter II.5, we prove the localization theorem for pointed genuine sheaves:

Theorem C (Theorem II.5.2.16)**.** *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks and let $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ be its open complement. Then the functor $i_*\colon \mathrm{H}(\mathcal{Z})_* \to \mathrm{H}(\mathcal{X})_*$ is fully faithful, and there is a preferred cofiber sequence $j_\sharp j^* \xrightarrow{\text{counit}} \mathrm{id} \xrightarrow{\text{unit}} i_* i^*$.*

In Chapter II.6 we give a proof of our main result, Theorem A stated above. We give a precise formulation in Section 6.1. The proof, which will be given in Section 6.4, is close in spirit to the proof of Atiyah duality for a compact smooth manifold M, using a version of *Pontryagin-Thom collapse map* constructed in Section 6.2. In Section 6.3 we introduce the auxiliary notion of a *kernel operator*. In Section 6.5 we discuss various important consequences of relative Poincaré duality, like relative Atiyah duality, proper base change and smooth-proper base change.

Conventions

In contrast to most sources on differentiable stacks, we will fully work in the homotopy-theoretic setting of ∞-categories, and adopt the standard notations and terminology from this setting. One notable exception, following [CS23, Section 5.1.4], is that we use the word 'anima' rather than 'space' to refer to the notion of an ∞-groupoid, and accordingly write An for the ∞-topos of animae/∞-groupoids. Similarly we write An_G for the ∞-category of G-spaces for a compact Lie group G and refer to its objects as 'genuine G-animae'.

Acknowledgements

I am deeply indebted to Marc Hoyois for his invaluable feedback and continuous encouragement throughout this project: his insightful contributions were instrumental in establishing several key results. I am also grateful to Shachar Carmeli and Marco Volpe for enlightening discussions on geometric aspects of sheaves and stacks. Furthermore, I am indebted to Dustin Clausen, Adrian Clough, Peter Haine, Adeel Khan, Sil Linskens, Florian Naef and Sebastian Wolf for useful conversations and for mathematical insights that have further enhanced this article.

A major source of inspiration for this work was the article [Hoy17], whose proof strategy we will follow closely. Additionally, important references that provided significant insights include [Kha19], [CD19], [Vol21], and [KR21].

II.2 Foundations on differentiable stacks

In this chapter, we will recall some of the foundations on differentiable stacks: sheaves of groupoids on the site of differentiable manifolds and open coverings which admit a representable atlas. All of the material in this chapter is well-known and sources on this topic are plentiful; a selection is [Pro96; BX11; Met03; Ler10; Hoy13]. Our treatment differs somewhat from these sources in that we will work in the homotopy-theoretic setting of sheaves of animae/∞-groupoids, sometimes called *∞-sheaves*, rather than sheaves of groupoids; a recollection of the theory of sheaf ∞-topoi is provided in Appendix E.4. We will see in Corollary 2.3.24 below that the objects of interest, the differentiable stacks, are in fact sheaves of groupoids, making our approach equivalent to the classical approach.

2.1 Representable morphisms of stacks

We start by introducing stacks on the site of smooth manifolds and discussing the notion of representable morphisms between stacks.

Definition 2.1.1. Let Diff denote the ordinary category of smooth manifolds[1] and smooth maps. We turn it into a site by equipping it with the open cover topology and let $\mathrm{Shv}(\mathrm{Diff})$ denote the associated ∞-topos of sheaves of animae on Diff. We refer to an object $\mathcal{X} \in \mathrm{Shv}(\mathrm{Diff})$ as a *stack on* Diff, or as a *stack* for short.

Note that the site $(\mathrm{Diff}, \mathrm{open})$ is subcanonical: for every smooth manifold $M \in \mathrm{Diff}$, the representable functor $\mathrm{Hom}_{\mathrm{Diff}}(-, M)\colon \mathrm{Diff} \to \mathrm{Set} \hookrightarrow \mathrm{An}$ is a sheaf. In other words, the Yoneda embedding $\mathrm{Diff} \hookrightarrow \mathrm{PSh}(\mathrm{Diff})$ factors through the subcategory of sheaves:

$$y\colon \mathrm{Diff} \hookrightarrow \mathrm{Shv}(\mathrm{Diff}).$$

[1] All smooth manifolds are assumed to be Hausdorff. They are allowed to be *impure*: different path components may have different dimensions.

We will often abuse notation by identifying smooth manifolds M with their associated sheaves $y(M)$; accordingly, we say that a stack $\mathcal{X}$ on Diff *is a smooth manifold* if it lies in the essential image of the Yoneda embedding. The Yoneda embedding y preserves all limits, and furthermore preserves arbitrary coproducts due to the sheaf condition.

Recall from Appendix E.2 the notion of an *effective epimorphism* in an ∞-topos. In the case of Shv(Diff), the effective epimorphisms can be characterized as those maps admitting *local sections*:

Definition 2.1.2. Let M be a smooth manifold and let $f\colon \mathcal{X} \to M$ be a map of sheaves. We say that f *admits local sections* if there exists an open cover $\{U_i\}_{i\in I}$ of M such that the map $U_i \hookrightarrow M$ factors through f for each $i \in I$.

Proposition 2.1.3. *If M is a smooth manifold, then a map $f\colon \mathcal{X} \to M$ in* Shv(Diff) *is an effective epimorphism if and only if it admits local sections.*

Proof. This is a special case of Lemma E.38. □

Since the site Diff does not admit all pullbacks, there is a subtlety in the definition of representable morphisms between stacks on Diff. However, pullbacks along *smooth submersions* always exist in Diff, see for example Proposition C.9. This leads to the following two-step definition of representability, which guarantees that every smooth map between smooth manifolds is representable when considered as a morphism of stacks:

Definition 2.1.4 (Representable morphisms, [EG11, Definition 2.1])**.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks on Diff.

(1) We call f a *representable submersion* if for any smooth manifold M and any morphism $M \to \mathcal{Y}$, the fiber product $M \times_{\mathcal{Y}} \mathcal{X}$ is a smooth manifold and the induced map $M \times_{\mathcal{Y}} \mathcal{X} \to M$ is a submersion. If the map $M \times_{\mathcal{Y}} \mathcal{X} \to M$ is surjective for every M, we say that f is a *representable surjective submersion*.

(2) We say f *representable* if for any representable submersion $M \to \mathcal{Y}$ from a smooth manifold M, the pullback $M \times_{\mathcal{Y}} \mathcal{X}$ is a smooth manifold.

It is clear from the definition that every representable submersion is a representable morphism. Furthermore, every smooth map $f\colon M \to N$ of smooth manifolds is representable when considered as a morphism of stacks, and it is a representable submersion of stacks if and only if it is a smooth submersion of smooth manifolds.

Warning 2.1.5. We warn the reader that representability behaves reasonably only when there is a sufficiently large supply of representable submersions into $\mathcal{Y}$. For instance, if there are no representable submersions from smooth manifolds to $\mathcal{Y}$, then every morphism $\mathcal{X} \to \mathcal{Y}$ is representable according to the above definition.

Remark 2.1.6. A smooth submersion between smooth manifolds has local sections if and only if it is surjective, see e.g. Lemma C.10. It follows from Proposition 2.1.3 that a representable submersion $f: \mathcal{X} \to \mathcal{Y}$ of stacks is in fact a representable *surjective* submersion if and only if it is an effective epimorphism.

Definition 2.1.7 (Open substack)**.** Let $j: \mathcal{U} \to \mathcal{X}$ be a representable submersion of stacks on Diff. We say that j exhibits $\mathcal{U}$ as an *open substack* of $\mathcal{X}$ if for every smooth manifold M and any morphism $M \to \mathcal{X}$, the induced map of smooth manifolds $M \times_{\mathcal{X}} \mathcal{U} \to M$ is an open embedding.

As we will establish next, representability of a stack is a local condition:

Lemma 2.1.8. *Let $\mathcal{X}$ be a stack on* Diff *and assume that there exists a collection of open substacks $\{j_\alpha : U_\alpha \hookrightarrow \mathcal{X}\}_{\alpha \in I}$ of $\mathcal{X}$ such that the map $\coprod_{\alpha \in I} U_\alpha \to \mathcal{X}$ is an effective epimorphism.*[2] *If each U_α is a smooth manifold, then so is $\mathcal{X}$.*

Proof. By the definition of effective epimorphisms, recalled in Definition E.12, $\mathcal{X}$ is the colimit of the Čech nerve of the map $\coprod_{\alpha \in I} U_\alpha \to \mathcal{X}$. Since each of the maps $f_\alpha : U_\alpha \hookrightarrow \mathcal{X}$ is assumed to be a representable submersion, the iterated pullbacks $U_{\alpha_1} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{\alpha_n}$ are all smooth manifolds again, hence the Čech nerve defines a simplicial diagram in Diff. We define a topological space M as the colimit of this simplicial diagram, regarded as a diagram in the ordinary category Top of topological spaces. More explicitly:

$$M = \operatorname{coeq}\left(\coprod_{\alpha \neq \beta} U_\alpha \times_{\mathcal{X}} U_\beta \rightrightarrows \coprod_{\alpha} U_\alpha \right).$$

Each of the maps $U_\alpha \to M$ is an open embedding, so M admits an open cover by the smooth manifolds U_α. As the smooth structures are compatible on their intersections, M itself obtains the structure of a smooth manifold. It follows that M is the colimit in Diff of the simplicial diagram, and consequently also the colimit in Shv(Diff) due to descent with respect to open covers. It follows that M is equivalent to $\mathcal{X}$, proving the claim. □

[2]Such a collection of open substacks is called an ‘open cover’, see Definition 4.1.2.

Corollary 2.1.9. *Let $\mathcal{X} \to M$ be a morphism of stacks on* Diff *and assume that M admits an open cover $\{V_\alpha\}$ such that each of the pullback stacks $V_\alpha \times_M \mathcal{X}$ is a smooth manifold. Then $\mathcal{X}$ is a smooth manifold.*

Proof. The maps $V_\alpha \times_M \mathcal{X} \to \mathcal{X}$ are pullbacks of the open embeddings $V_\alpha \hookrightarrow \mathcal{X}$ and thus are open substacks of $\mathcal{X}$. As the map $\bigsqcup_{\alpha \in I} V_\alpha \to M$ is an effective epimorphism, so is their base chane $\bigsqcup_{\alpha \in I} V_\alpha \times_M \mathcal{X} \to \mathcal{X}$ along the map $\mathcal{X} \to M$. The claim thus follows from Lemma 2.1.8. □

Proposition 2.1.10. *Consider a pullback diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

of stacks on Diff.

(1) If f is a representable submersion, then f' is a representable submersion.

(2) If g is a representable submersion and f is representable, then f' is representable.

(3) If g is an effective epimorphism and f' is a representable submersion, then f is a representable submersion.

(4) If g is a representable surjective submersion and f' is representable, then f is representable.

Proof. Parts (1) and (2) are immediate from the definitions. For part (3), assume that g is an effective epimorphism and let $M \to \mathcal{Y}$ be a morphism from a smooth manifold. It follows from Proposition 2.1.3 that the map $M \to \mathcal{Y}$ locally factors through $g\colon \mathcal{Y}' \twoheadrightarrow \mathcal{Y}$. Since the condition we are proving may be checked locally in M by Corollary 2.1.9, we may assume that already the map $M \to \mathcal{Y}$ factors through $\mathcal{Y}'$ via some map $s\colon M \to \mathcal{Y}'$. Now consider the following pullback diagram:

$$\begin{tikzcd} N \arrow[r] \arrow[d, "f''"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X}' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "f"] \\ M \arrow[r, "s"] & \mathcal{Y}' \arrow[r, two heads, "g"] & \mathcal{Y}. \end{tikzcd}$$

By assumption, f' is a representable submersion, and it follows that the pullback $N = M \times_{\mathcal{Y}'} \mathcal{X}' \cong M \times_{\mathcal{Y}} \mathcal{X}$ is a smooth manifold and that the map $f''\colon N \to M$ is a submersion. This proves that f is a representable submersion, finishing the proof of (3).

For part (4), let $M \to \mathcal{Y}$ be a representable submersion from some smooth manifold M. We have to show that the pullback $M \times_{\mathcal{Y}} \mathcal{X}$ is again a smooth manifold. By pulling back the whole situation along $M \to \mathcal{Y}$, we may assume that $\mathcal{Y} = M$ is a smooth manifold. We are then in the situation of a pullback square

$$\begin{tikzcd} N \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "f"] \\ M' \arrow[r, two heads, "g"] & M \end{tikzcd}$$

in $\mathrm{Shv}(\mathrm{Diff})$, where the stacks M, M' and N are all smooth manifolds and g is a surjective submersion. We have to show that $\mathcal{X}$ is a smooth manifold as well.

As g is a surjective smooth submersion, it follows from Lemma C.10 that g admits local sections: there exists an open cover $\{U_i\}_{i \in I}$ of M such that for every $i \in I$ there is a smooth map $s_i \colon U_i \to M'$ such that the composite $g \circ s_i \colon U_i \to M$ is equal to the inclusion of U_i into M. By Corollary 2.1.9, it suffices to show that each of the pullback stacks $\mathcal{X}_i := \mathcal{X} \times_M U_i$ is a smooth manifold. By base changing the whole situation along the map $U_i \hookrightarrow M$, we may thus assume that the map $g \colon M' \twoheadrightarrow M$ admits a section $s \colon M \to M'$.

We may now define a map $t = (s \circ f, \mathrm{id}) \colon \mathcal{X} \to N = M' \times_M \mathcal{X}$. Since it is a section of the map $g' \colon N \to \mathcal{X}$, it follows that the stack $\mathcal{X}$ is a retract in $\mathrm{Shv}(\mathrm{Diff})$ of the smooth manifold N. It follows from idempotent completeness of Diff, Corollary C.5, that $\mathcal{X}$ is itself a smooth manifold, finishing the proof. □

Corollary 2.1.11. *Representable submersions and representable morphisms of stacks are closed under compositions.*

Proof. The statement about representable submersions is clear from the definition, since smooth submersions of smooth manifolds are closed under composition. The statement about representable morphisms follows the definition, using part (1) of Proposition 2.1.10. □

Lemma 2.1.12. *Let $f_1 \colon \mathcal{X}_1 \to \mathcal{Y}_1$ and $f_2 \colon \mathcal{X}_2 \to \mathcal{Y}_2$ be two representable submersions of stacks on* Diff. *Then their cartesian product $f_1 \times f_2 \colon \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$ is again a representable submersion.*

Proof. Let $g = (g_1, g_2) \colon M \to \mathcal{Y}_1 \times \mathcal{Y}_2$ be a map from a smooth manifold. By assumption, the pullbacks $N_1 := M \times_{\mathcal{Y}_1} \mathcal{X}_1$ and $N_2 := M \times_{\mathcal{Y}_2} \mathcal{X}_2$ are smooth manifolds, and the maps $N_1 \to M$ and $N_2 \to M$ are smooth submersions. Now consider the following pullback diagram:

$$\begin{tikzcd}
N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & N_1 \times N_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X}_1 \times \mathcal{X}_2 \arrow[d, "f_1 \times f_2"] \\
M \arrow[r, "\Delta"] & M \times M \arrow[r, "g_1 \times g_2"] & \mathcal{Y}_1 \times \mathcal{Y}_2.
\end{tikzcd}$$

Since the vertical map $N_1 \times N_2 \to M \times M$ is a submersion, the left pullback exists in the category Diff, and it follows that N is a smooth manifold. Since the bottom composite is the map $g = (g_1, g_2)$, this finishes the proof. $\square$

Lemma 2.1.13. *Let $\mathcal{X}$ and $\mathcal{Y}$ be stacks on* Diff. *Then any representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is 0-truncated.*

Proof. A map f in a sheaf topos is 0-truncated if and only if its base change along any map $M \to \mathcal{Y}$ from a representable object M is 0-truncated. As f is representable, the pullback $N := M \times_{\mathcal{Y}} \mathcal{X}$ is a smooth manifold. It follows that the base change $N \to M$ of f is a smooth map between smooth manifolds, and in particular 0-truncated, finishing the proof. $\square$

2.2 Differentiable stacks

In this section, we introduce a special class of stacks on Diff called *differentiable stacks*, whose geometric behavior closely parallels that of smooth manifolds. These objects will play a central role throughout the entire remainder of this article.

Definition 2.2.1 (Differentiable stack)**.** Let $\mathcal{X}$ be a stack on Diff. A *representable atlas* for $\mathcal{X}$ is a smooth manifold M together with a representable surjective submersion $p\colon M \twoheadrightarrow \mathcal{X}$. A stack which admits a representable atlas is called a *differentiable stack*. We let

$$\mathrm{DiffStk} \subseteq \mathrm{Shv}(\mathrm{Diff})$$

denote the full subcategory spanned by the differentiable stacks.

Differentiable stacks are closed under a variety of constructions.

Lemma 2.2.2. *If $\mathcal{X}$ and $\mathcal{Y}$ are differentiable stacks with atlases $M \twoheadrightarrow \mathcal{X}$ and $N \twoheadrightarrow \mathcal{Y}$, then the cartesian product $\mathcal{X} \times \mathcal{Y}$ is a differentiable stack with atlas $M \times N \twoheadrightarrow \mathcal{X} \times \mathcal{Y}$.*

Proof. The product map is an effective epimorphism whose source is a smooth manifold. By Lemma 2.1.12, it is also a representable submersion, finishing the proof. $\square$

Lemma 2.2.3. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism, and assume that $M \twoheadrightarrow \mathcal{Y}$ is a representable atlas for $\mathcal{Y}$. Then the pullback $N := M \times_Y \mathcal{X} \to \mathcal{X}$ is a representable atlas for $\mathcal{X}$.*

Proof. The map $N \to \mathcal{X}$ is a representable submersion and an effective epimorphism since it is a base change of the map $M \twoheadrightarrow \mathcal{Y}$. Furthermore, the assumption that f is a representable submersion guarantees that N is a smooth manifold. $\square$

Lemma 2.2.4. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of stacks.*

(1) If $\mathcal{Y}$ is a differentiable stack, then also $\mathcal{X}$ is a differentiable stack;

(2) If f is a representable surjective submersion and $\mathcal{X}$ is a differentiable stack, then also $\mathcal{Y}$ is a differentiable stack.

Proof. Part (1) is immediate from Lemma 2.2.3. For part (2), note that any representable atlas $M \twoheadrightarrow \mathcal{X}$ of $\mathcal{X}$ gives a representable atlas for $\mathcal{Y}$ by postcomposing with f. $\square$

As a consequence, we see that differentiable stacks are closed under pullbacks along representable submersions on Shv(Diff):

Corollary 2.2.5. *Consider a pullback square*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

of stacks on Diff. *Assume that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Y}'$ are differentiable stacks and that f is a representable submersion. Then $\mathcal{X}'$ is also a differentiable stack.*

Proof. By Proposition 2.1.10, the morphism f' is a representable submersion, hence the claim follows from Lemma 2.2.4. $\square$

Lemma 2.2.6. *Let $\mathcal{X}$ be a differentiable stack with representable atlas $M \twoheadrightarrow \mathcal{X}$. Then a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ is representable if and only if the pullback $M \times_{\mathcal{X}} \mathcal{Y}$ is a smooth manifold. Furthermore, f is a representable submersion if and only if its base change $M \times_{\mathcal{X}} \mathcal{Y} \to M$ of f is a smooth submersion of smooth manifolds.*

Proof. This is immediate from Proposition 2.1.10, using that any morphism between smooth manifolds is representable. $\square$

Corollary 2.2.7. *Let $\mathcal{Y}$ be a differentiable stack. Then the diagonal $\Delta\colon \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable.*

Proof. Let $M \twoheadrightarrow \mathcal{Y}$ be a representable atlas for $\mathcal{Y}$. By Lemma 2.2.2, the product map $M \times M \twoheadrightarrow \mathcal{Y} \times \mathcal{Y}$ is again a representable atlas. By Lemma 2.2.6, it thus suffices to show that the differentiable stack $(M \times M) \times_{\mathcal{Y}\times\mathcal{Y}} \mathcal{Y} \cong M \times_{\mathcal{Y}} M$ is a smooth manifold. This is immediate since the map $M \twoheadrightarrow \mathcal{Y}$ is a representable submersion. □

Corollary 2.2.8. *Let $\mathcal{X}$ be a differentiable stack. Then any morphism $N \to \mathcal{X}$ from a smooth manifold N is representable.*

Proof. Let $M \twoheadrightarrow \mathcal{Y}$ be a representable atlas for $\mathcal{X}$. By Lemma 2.2.6 it suffices to show that the pullback $M \times_{\mathcal{X}} N$ is a smooth manifold. This is immediate from the fact that the map $M \twoheadrightarrow \mathcal{X}$ is a representable submersion. □

2.3 Classifying stacks of Lie groupoids

A rich supply of differentiable stacks is provided by the *classifying stacks* of Lie groupoids. In fact, we will show in Corollary 2.3.8 below that every differentiable stack is (non-canonically) equivalent to the classifying stack $\mathbb{B}\mathcal{G}$ of some Lie groupoid $\mathcal{G}$.

We refer to Appendix D for background material on Lie groupoids. We will also make use of some foundational results on the relation between groupoid objects and effective epimorphisms in an ∞-topos, recalled in Appendix E.2.

Definition 2.3.1 (Nerve of a Lie groupoid)**.** The *nerve* $N\mathcal{G}$ of a Lie groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ is the simplicial manifold $N\mathcal{G}\colon \Delta^{\mathrm{op}} \to \mathrm{Diff}$ whose n-simplices are given by the manifold

$$\mathcal{G}_n := \mathcal{G}_1 \times_{s,\mathcal{G}_0,t} \cdots \times_{s,\mathcal{G}_0,t} \mathcal{G}_1,$$

given as the n-fold iterated fiber product of $\mathcal{G}_1$ over $\mathcal{G}_0$ in Diff. The face maps of $N\mathcal{G}$ are obtained from source, target and multiplication maps of $\mathcal{G}$, while the degeneracy maps of $N\mathcal{G}$ are obtained from the unit map of $\mathcal{G}$.

Definition 2.3.2 (Classifying stack)**.** Let $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ be a Lie groupoid. We define its *classifying stack* $\mathbb{B}\mathcal{G} \in \mathrm{Shv}(\mathrm{Diff})$ as the geometric realization of its nerve in Shv(Diff):

$$\mathbb{B}\mathcal{G} := |N\mathcal{G}| = \operatorname{colim}\left(\cdots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \mathcal{G}_2 \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \mathcal{G}_1 \rightrightarrows \mathcal{G}_0 \right) \quad \in \quad \mathrm{Shv}(\mathrm{Diff}).$$

The classifying stack $\mathbb{B}\mathcal{G}$ of a Lie groupoid is indeed a differentiable stack:

Lemma 2.3.3. *Let $\mathcal{G}$ be a Lie groupoid. Then the canonical map $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$ is a representable atlas for $\mathbb{B}\mathcal{G}$. In particular, $\mathbb{B}\mathcal{G}$ is a differentiable stack.*

Proof. As groupoid objects in ∞-topoi are effective, the map $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$ is an effective epimorphism, see e.g. Corollary E.15. Since $\mathcal{G}_0$ is a smooth manifold, it remains to show that the map $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$ is a representable submersion. By Proposition 2.1.10, it suffices to show that this is the case after pulling it back along an effective epimorphism, for example along $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$ itself. But this base change $\mathrm{pr}_1 \colon \mathcal{G}_0 \times_{\mathbb{B}\mathcal{G}} \mathcal{G}_0 \to \mathcal{G}_0$ is equivalent to the map source map $s \colon \mathcal{G}_1 \to \mathcal{G}_0$, which by assumption on $\mathcal{G}$ is a submersion of smooth manifolds and hence a representable submersion in $\mathrm{Shv}(\mathrm{Diff})$. □

Example 2.3.4 (Classifying stack of Lie group)**.** Every Lie group G can be regarded as a one-object Lie groupoid $G \rightrightarrows \mathrm{pt}$ and thus gives rise to a differentiable stack $\mathbb{B}G$ called the *classifying stack of G*.

Example 2.3.5 (Global quotient stack)**.** Let M be a smooth manifold and let G be a Lie group acting smoothly on M. We obtain a Lie groupoid $G \ltimes M = (G \times M \rightrightarrows M)$, called the *action groupoid*. The associated differentiable stack $\mathbb{B}(G \ltimes M)$ is denoted by $M/\!/G$ and is called the *quotient stack* of M by G. Since the action groupoid $G \ltimes M$ is functorial in M, we obtain a functor

$$-/\!/G \colon \mathrm{Diff}_G \to \mathrm{DiffStk},$$

where Diff_G denotes the category of smooth G-manifolds. Note that the quotient stack $\mathrm{pt}/\!/G$ of the point is precisely the classifying stack $\mathbb{B}G$ of G.

A differentiable stack $\mathcal{X}$ is called a *global quotient stack* if it is equivalent to a quotient stack $M/\!/G$ for some *compact* Lie group G and some smooth G-manifold. We let $\mathrm{QtStk} \subseteq \mathrm{DiffStk}$ denote the full subcategory of global quotient stacks.

By Lemma 2.3.3, every Lie groupoid gives rise to a differentiable stack. We will now show that, conversely, every differentiable stack $\mathcal{X}$ can be presented by a Lie groupoid $\mathcal{G}$, in the sense that there exists an equivalence $\mathcal{X} \simeq \mathbb{B}\mathcal{G}$ of stacks. In fact, we will see that there is a one-to-one correspondence between choices of an atlas for $\mathcal{X}$ and choices of a presentation of $\mathcal{X}$ by a Lie groupoid.

Definition 2.3.6. We denote by $\mathrm{Atl}^{\mathrm{rep}}(\mathrm{Shv}(\mathrm{Diff})) \subseteq \mathrm{Fun}([1], \mathrm{Shv}(\mathrm{Diff}))$ the full subcategory of the arrow category of $\mathrm{Shv}(\mathrm{Diff})$ spanned by the representable atlases $M \twoheadrightarrow \mathcal{X}$ in the sense of Definition 2.2.1.

Proposition 2.3.7. *Sending a Lie groupoid $\mathcal{G}$ to the representable atlas $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$ defines an equivalence of ∞-categories* $\mathrm{LieGrpd} \xrightarrow{\sim} \mathrm{Atl}^{\mathrm{rep}}(\mathrm{Shv}(\mathrm{Diff}))$.

Proof. Recall from Proposition E.13 the equivalence

$$\mathrm{Grpd}(\mathrm{Shv}(\mathrm{Diff})) \xrightarrow{\sim} \mathrm{Atl}(\mathrm{Shv}(\mathrm{Diff}))$$

between the ∞-category of groupoid objects in $\mathrm{Shv}(\mathrm{Diff})$ and the ∞-category of atlases (i.e. effective epimorphisms) in $\mathrm{Shv}(\mathrm{Diff})$. This equivalence is given by sending a groupoid object $\mathcal{G}$ to the effective epimorphism $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$, and its inverse sends an effective epimorphism to its Čech nerve. We claim that this equivalence restricts to the desired equivalence $\mathrm{LieGrpd} \xrightarrow{\sim} \mathrm{Atl}^{\mathrm{rep}}(\mathrm{Shv}(\mathrm{Diff}))$ of full subcategories. By Lemma 2.3.3, the map $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$ is indeed a representable atlas if $\mathcal{G}$ is a Lie groupoid. Conversely, if $p\colon M \twoheadrightarrow \mathcal{X}$ is a representable atlas, its Čech nerve $\check{C}(p)$ is given at level $[n] \in \Delta$ by the n-fold fiber product

$$\check{C}(p)_n \simeq M \times_{\mathcal{X}} M \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} M,$$

which is a smooth manifold by the assumptions that M is a manifold and that the map p is a representable submersion. It follows that $\check{C}(p)$ is the nerve of a Lie groupoid $\mathcal{G} = (M \times_{\mathcal{X}} M \rightrightarrows M)$, finishing the proof. □

Corollary 2.3.8. *Every differentiable stack $\mathcal{X}$ is equivalent to the classifying stack $\mathbb{B}\mathcal{G}$ of some Lie groupoid $\mathcal{G}$.*

Proof. Let $M \twoheadrightarrow \mathcal{X}$ be a representable atlas for $\mathcal{X}$. By Proposition 2.3.7, there exists a Lie groupoid $\mathcal{G}$ with $M \cong \mathcal{G}_0$ and $\mathcal{X} \simeq \mathbb{B}\mathcal{G}$. This finishes the proof. □

Warning 2.3.9. Although we may think of Lie groupoids as *presentations* of differentiable stacks, it is not true that a morphism of Lie groupoids $\mathcal{H} \to \mathcal{G}$ contains the same data as a morphism $\mathbb{B}\mathcal{H} \to \mathbb{B}\mathcal{G}$ of stacks. Indeed, as made precise by Proposition 2.3.7, morphisms of stacks $\mathbb{B}\mathcal{H} \to \mathbb{B}\mathcal{G}$ comes from a morphism of Lie groupoids $\mathcal{H} \to \mathcal{G}$ precisely if it is compatible with the preferred atlases $\mathcal{H}_0 \twoheadrightarrow \mathbb{B}\mathcal{H}$ and $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$.

2.3.1 Differentiable stacks are Lie groupoids up to Morita equivalence

We will now recall the well-known fact that two Lie groupoids $\mathcal{H}$ and $\mathcal{G}$ have equivalent classifying stacks if and only if they are Morita equivalent, in the sense of Definition D.18. We start by characterizing when a morphism of Lie groupoids $f\colon \mathcal{H} \to \mathcal{G}$ induces an equivalence of differentiable stacks $\mathbb{B}f\colon \mathbb{B}\mathcal{H} \xrightarrow{\sim} \mathbb{B}\mathcal{G}$. For this, we need to recall the notions of fully faithfulness and essential surjectivity of morphisms of Lie groupoids.

Definition 2.3.10 ([HF19, Section 6.1])**.** Let $f\colon \mathcal{H} \to \mathcal{G}$ be a morphism of Lie groupoids.

(1) The morphism f is called *fully faithful* if the square

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r, "f_1"] \arrow[d, "{(s,t)}"'] & \mathcal{G}_1 \arrow[d, "{(s,t)}"] \\ \mathcal{H}_0\times\mathcal{H}_0 \arrow[r, "f_0\times f_0"] & \mathcal{G}_0\times\mathcal{G}_0 \end{tikzcd}$$

is a pullback square.

(2) The morphism f is called *essentially surjective* if the composite map

$$\mathcal{G}_1\times_{\mathcal{G}_0}\mathcal{H}_0 \xrightarrow{\mathrm{pr}_1} \mathcal{G}_1 \xrightarrow{t} \mathcal{G}_0$$

is a surjective submersion;

(3) The morphism f is called a *Morita map* if it is both fully faithful and essentially surjective.

(4) The morphism f is called *strongly surjective* if the smooth map $f_0\colon \mathcal{H}_0 \to \mathcal{G}_0$ is a surjective submersion.

(5) The morphism f is called a *Morita fibration* if it is both fully faithful and strongly surjective.

Warning 2.3.11. The conventions on the terminology of Morita maps are not very consistent in the literature: the notion we call 'Morita map' appears in the literature under the names 'equivalence' [Moe02, Definition 2.4], 'weak equivalence' [Pro96, Definition 12], 'essential equivalence' [Met03, Definition 58] and 'Morita equivalence' [Car11, Definition I.2.20]. Morita fibrations also appear under the names 'Morita morphisms' [BX11, Definition 2.24] and 'elementary Morita equivalences' [Noo05, Section 8]. Our convention follows [HF19, Section 6.1].

Remark 2.3.12 (Morita fibrations are Morita maps)**.** Every strongly surjective morphism f of Lie groupoids is essentially surjective, and thus every Morita fibration is a Morita map. Indeed, the condition in (4) that the map f_0 is a surjective submersion implies that the projection map $\mathrm{pr}_1\colon \mathcal{G}_1\times_{\mathcal{G}_0}\mathcal{H}_0 \to \mathcal{G}_1$ is a surjective submersion, as it is a base change of f_0. Since the target map $t\colon \mathcal{G}_1 \to \mathcal{G}_0$ is always a surjective submersion, it follows that $t\circ\mathrm{pr}_1$ is also a surjective submersion, which is condition (2).

The following result characterizes the properties of fully faithfulness and essential surjectivity of a morphism of Lie groupoids in terms of its underlying map of differentiable stacks.

Lemma 2.3.13 ([Met03, Proposition 60], [Car11, Proposition I.2.5]). *Let $f\colon \mathcal{H} \to \mathcal{G}$ be a morphism of Lie groupoids, and consider the induced map $\mathbb{B}f\colon \mathbb{B}\mathcal{H} \to \mathbb{B}\mathcal{G}$ on differentiable stacks.*

(1) The map $\mathbb{B}f$ is a monomorphism if and only if f is fully faithful;

(2) The map $\mathbb{B}f$ is an representable surjective submersion if and only if f is essentially surjective;

(3) The map $\mathbb{B}f$ is an equivalence if and only if f is a Morita map.

Proof. For part (1), consider the following diagram:

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[rr, "f_1"] \arrow[dr] \arrow[dd, "(s,t)"'] & & \mathcal{G}_1 \arrow[dr] \arrow[dd, "(s,t)", near start] & \\
& \mathbb{B}\mathcal{H} \arrow[rr, crossing over, "\mathbb{B}f", near start] & & \mathbb{B}\mathcal{G} \arrow[dd, "\Delta"] \\
\mathcal{H}_0 \times \mathcal{H}_0 \arrow[rr, "f_0 \times f_0", near end] \arrow[dr, two heads] & & \mathcal{G}_0 \times \mathcal{G}_0 \arrow[dr, two heads] & \\
& \mathbb{B}\mathcal{H} \times \mathbb{B}\mathcal{H} \arrow[rr, "\mathbb{B}f \times \mathbb{B}f"] \arrow[from=uu, crossing over, "\Delta", near start] & & \mathbb{B}\mathcal{G} \times \mathbb{B}\mathcal{G}.
\end{tikzcd}$$

The map $\mathbb{B}f\colon \mathbb{B}\mathcal{H} \to \mathbb{B}\mathcal{G}$ is a monomorphism if and only if the front square of the diagram is a pullback square. Since the map $\mathcal{H}_0 \twoheadrightarrow \mathbb{B}\mathcal{H}$ is an effective epimorphism and the left and right faces of the cube are pullback squares, it follows from the pasting lemma for pullback squares that the front square is a pullback square if and only if the back square is. As the back square defines fully faithfulness of f, this finishes the proof of (1)

For part (2), consider the following pullback diagram:

$$\begin{tikzcd}
\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_1 \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_0 \arrow[d, two heads] \\
\mathcal{H}_0 \arrow[r, "f_0"] & \mathcal{G}_0 \arrow[r] & \mathbb{B}\mathcal{G}.
\end{tikzcd}$$

The bottom composite $\mathcal{H}_0 \to \mathcal{G}_0 \to \mathbb{B}\mathcal{G}$ is equivalent to the composite $\mathcal{H}_0 \twoheadrightarrow \mathbb{B}\mathcal{H} \xrightarrow{\mathbb{B}f} \mathbb{B}\mathcal{G}$. As the map $\mathcal{H}_0 \twoheadrightarrow \mathbb{B}\mathcal{H}$ is a representable surjective submersion, it follows from part (4) of Lemma C.7 that the map $\mathbb{B}f$ is a representable surjective submersion if and only if the bottom composite $\mathcal{H}_0 \to \mathbb{B}\mathcal{G}$ is. Since this may be checked after pulling back along the atlas $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$, this is equivalent to the condition that the top composite $t \circ \mathrm{pr}_1\colon \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0 \to \mathcal{G}_0$

is a surjective submersion. Since this is the definition of fully faithfulness of f, this finishes the proof of (2).

Part (3) is a direct consequence of (1) and (2) since a morphism of stacks is an equivalence if and only if it is both a monomorphism and a representable surjective submersion. $\square$

We deduce that two Lie groupoids have equivalent classifying stacks if and only if they are Morita equivalent:

Proposition 2.3.14 (cf. [Pro96], [BX11, Theorem 2.26], [Met03])**.** *Given two Lie groupoids $\mathcal{H}$ and $\mathcal{G}$, the following conditions are equivalent:*

(1) The classifying stacks $\mathbb{B}\mathcal{H}$ and $\mathbb{B}\mathcal{G}$ are equivalent;

(2) The Lie groupoids $\mathcal{H}$ and $\mathcal{G}$ are Morita equivalent, in the sense of Definition D.18;

(3) There exists a third Lie groupoid $\mathcal{K}$ along with Morita fibrations $\mathcal{K} \to \mathcal{H}$ and $\mathcal{K} \to \mathcal{G}$;

(4) There exists a third Lie groupoid $\mathcal{K}$ along with Morita maps $\mathcal{K} \to \mathcal{H}$ and $\mathcal{K} \to \mathcal{G}$.

Proof. To prove that (1) implies (2), assume that an equivalence of stacks $\mathbb{B}\mathcal{H} \simeq \mathbb{B}\mathcal{G}$ has been given. It follows that the composite $\mathcal{H}_0 \twoheadrightarrow \mathbb{B}\mathcal{H} \simeq \mathbb{B}\mathcal{G}$ is a representable atlas for $\mathbb{B}\mathcal{G}$. We define a smooth manifold P as the following pullback:

$$\begin{tikzcd} P \arrow[r, two heads, "\beta"] \arrow[d, two heads, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{H}_0 \arrow[d, two heads] \\ \mathcal{G}_0 \arrow[r, two heads] & \mathbb{B}\mathcal{G}. \end{tikzcd}$$

Observe that the map $\alpha\colon P \twoheadrightarrow \mathcal{G}_0$ is a principal $\mathcal{H}$-bundle, in the sense of Definition D.15, as it is a base change of the principal $\mathcal{H}$-bundle $\mathcal{H}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$. Similarly the map $\beta\colon P \twoheadrightarrow \mathcal{H}_0$ is a principal $\mathcal{G}$-bundle. It is immediate that the actions of $\mathcal{G}$ and $\mathcal{H}$ commute as they act on two separate components of $P = \mathcal{G}_0 \times_{\mathbb{B}\mathcal{G}} \mathcal{H}_0$, and thus the triple (P, α, β) forms a Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$.

To prove that (2) implies (3), assume given a Morita equivalence (P, α, β) between $\mathcal{G}$ and $\mathcal{H}$, in the sense of Definition D.18. We start by defining a Lie groupoid $\mathcal{K}$. Set $\mathcal{K}_0 = P$ and define $\mathcal{K}_1$ via the following pullback diagram:

$$\begin{tikzcd} \mathcal{K}_1 \arrow[r, "(\alpha_1,\beta_1)"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_1 \times \mathcal{H}_1 \arrow[d, "s\times s"] \\ P \arrow[r, "(\alpha,\beta)"] & \mathcal{G}_0 \times \mathcal{H}_0. \end{tikzcd}$$

The commuting actions of $\mathcal{G}$ and $\mathcal{H}$ on P define an action map $t\colon \mathcal{K}_1 \to P$ and one checks that the maps $s,t\colon \mathcal{K}_1 \to P = \mathcal{K}_0$ are part of a Lie groupoid structure $\mathcal{K} = (\mathcal{K}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{K}_0)$. The Lie groupoid $\mathcal{K}$ comes equipped with morphisms of Lie groupoids $(\alpha_1,\alpha)\colon \mathcal{K} \to \mathcal{G}$ and $(\beta_1,\beta)\colon \mathcal{K} \to \mathcal{H}$. We claim that these are Morita fibrations, thus finishing the proof of (3). By symmetry, it suffices to check this for (α_1,α). Strong surjectivity is clear, as the map $\alpha\colon \mathcal{K}_0 = P \to \mathcal{G}_0$ is a surjective submersion by assumption. For fully faithfulness, we must show that the map

$$\mathcal{K}_1 \to \mathcal{G}_1 \times_{\mathcal{G}_0\times\mathcal{G}_0} (P\times P)$$

is a diffeomorphism. To provide an inverse, consider the shear map $\mathrm{shear}\colon \mathcal{H}_1 \times_{\mathcal{H}_0} P \to P\times_{\mathcal{G}_0} P$, which is a diffeomorphism as $\alpha\colon P \to \mathcal{G}_0$ is a principal $\mathcal{H}$-bundle. An inverse to the above map is then given by

$$\mathcal{G}_1 \times_{\mathcal{G}_0\times\mathcal{G}_0} (P\times P) \to \mathcal{K}_1 = (\mathcal{G}_1\times\mathcal{H}_1)\times_{\mathcal{G}_0\times\mathcal{H}_0} P, \quad (g,p,p') \mapsto (g,\mathrm{shear}^{-1}(p,p')),$$

giving the desired claim.

It is immediate from Remark 2.3.12 that (3) implies (4).

Finally, to see that (4) implies (1), observe that the Morita maps $\mathcal{K} \to \mathcal{G}$ and $\mathcal{K} \to \mathcal{H}$ induce equivalences of stacks

$$\mathbb{B}\mathcal{H} \xleftarrow{\sim} \mathbb{B}\mathcal{K} \xrightarrow{\sim} \mathbb{B}\mathcal{G}$$

by Lemma 2.3.13. It follows that $\mathbb{B}\mathcal{H}$ and $\mathbb{B}\mathcal{G}$ are equivalent, as desired. □

As shown above, the classifying stack functor $\mathbb{B}\colon \mathrm{LieGrpd} \to \mathrm{DiffStk}$ sends Morita maps to equivalences. A well-known result in the theory of differentiable stacks says that this functor is in fact *universal* with this property. For completeness we will state the precise result, although we shall not make use of it.

Theorem 2.3.15 ([Pro96, Section 3.4, Section 4])**.** *The functor* $\mathbb{B}\colon \mathrm{LieGrpd} \to \mathrm{DiffStk}$ *exhibits the* $(2,1)$*-category*[3] DiffStk *as the* $(2,1)$*-categorical Dwyer-Kan localization of the category* LieGrpd *at the Morita maps: for any other* $(2,1)$*-category* $\mathcal{D}$*, precomposition with* $\mathbb{B}$ *induces a fully faithful functor*

$$\mathrm{Fun}(\mathrm{DiffStk},\mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathrm{LieGrpd},\mathcal{D})$$

whose essential image consists of those functors $\mathrm{LieGrpd} \to \mathcal{D}$ *which send Morita maps to equivalences.*

Remark 2.3.16. One can prove that the functor $\mathbb{B}\colon \mathrm{LieGrpd} \to \mathrm{DiffStk}$ is in fact an ∞-categorical Dwyer-Kan localization, see [Nui16].

[3] See Corollary 2.3.24 below.

2.3.2 Classification of Lie groupoid actions on smooth manifolds

Given a Lie groupoid $\mathcal{G}$, recall from Definition D.14 the notion of a *left action* of $\mathcal{G}$ on a smooth manifold: given a smooth submersion $M \to \mathcal{G}_0$, a left $\mathcal{G}$-action on M consists of an associative and unital action $a\colon \mathcal{G}_1 \times_{s,\mathcal{G}_0} M \to M$ over $\mathcal{G}_0$. A smooth manifold equipped with a left $\mathcal{G}$-action is called a *smooth $\mathcal{G}$-manifold*. In this subsection, we show that such left actions are classified by representable maps of stacks to the classifying stack $\mathbb{B}\mathcal{G}$. More precisely, we show that the category $\mathrm{Diff}_{\mathcal{G}}$ of smooth $\mathcal{G}$-manifolds is equivalent to the subcategory of $\mathrm{DiffStk}_{/\mathbb{B}\mathcal{G}}$ spanned by the representable submersions into $\mathbb{B}\mathcal{G}$, where the equivalence sends M to the map $M//\mathcal{G} \to \mathrm{pt}//\mathcal{G} = \mathbb{B}\mathcal{G}$.

Definition 2.3.17. Given a smooth $\mathcal{G}$-manifold M, we obtain a Lie groupoid

$$\mathcal{G} \ltimes M = \left(\mathcal{G}_1 \times_{\mathcal{G}_0} M \overset{\mathrm{pr}_2}{\underset{a}{\rightrightarrows}} M \right)$$

called the *action groupoid* of M. We define the *quotient stack* $M//\mathcal{G} \in \mathrm{Shv}(\mathrm{Diff})$ as the classifying stack $\mathbb{B}(\mathcal{G} \ltimes M)$. The Lie groupoid $\mathcal{G} \ltimes M$ comes equipped with a morphism of Lie groupoids to $\mathcal{G}$:

$$\begin{tikzcd}
\mathcal{G}_1 \times_{\mathcal{G}_0} M \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[r, shift right, "a"'] \arrow[d, "\mathrm{pr}_1"'] & M \arrow[d] \\
\mathcal{G}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathcal{G}_0.
\end{tikzcd}$$

The notion of left actions of Lie groupoids on smooth manifolds may be phrased in terms of groupoid actions in the ∞-topos $\mathrm{Shv}(\mathrm{Diff})$ of stacks on Diff. Recall from Definition E.17 that a left action of a groupoid object $\mathcal{G}$ in an ∞-topos $\mathcal{B}$ consists of another groupoid object $\mathcal{H}$ in $\mathcal{B}$ equipped with a *cartesian* morphism $c\colon \mathcal{H} \to \mathcal{G}$ of simplicial objects in $\mathcal{B}$, meaning that all the naturality squares of c are pullback squares. In the case at hand, observe that the map $\mathcal{G} \ltimes M \to \mathcal{G}$ of Lie groupoids from Definition 2.3.17 is a cartesian morphism of groupoid objects in $\mathrm{Shv}(\mathrm{Diff})$, thus producing a functor

$$\mathcal{G} \ltimes -\colon\ \mathrm{Diff}_{\mathcal{G}} \to \mathrm{Act}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff}))$$

from the category of smooth $\mathcal{G}$-manifolds to the ∞-category of stacks on Diff equipped with a $\mathcal{G}$-action.

Lemma 2.3.18. *For a Lie groupoid $\mathcal{G}$, the functor $\mathcal{G} \ltimes -\colon \mathrm{Diff}_{\mathcal{G}} \to \mathrm{Act}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff}))$ is fully faithful. An object $c\colon \mathcal{H} \to \mathcal{G}$ of $\mathrm{Act}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff}))$ lies inside the essential image of this functor if and only if $\mathcal{H}_0 \in \mathrm{Shv}(\mathrm{Diff})$ is a smooth manifold and the map $\mathcal{H}_0 \to \mathcal{G}_0$ is a smooth submersion.*

Proof. Notice that a morphism $\varphi\colon M \to M'$ of smooth $\mathcal{G}$-manifolds determines and is determined by the induced morphism $\mathcal{G} \ltimes M \to \mathcal{G} \ltimes M'$ of Lie groupoids over $\mathcal{G}$, so that this functor is fully faithful. Furthermore, if $c\colon \mathcal{H} \to \mathcal{G}$ is a cartesian morphism of groupoid objects of $\mathrm{Shv}(\mathrm{Diff})$ such that $\mathcal{H}_0$ is a smooth manifold, then it follows that $\mathcal{H}_n \simeq \mathcal{H}_0 \times_{\mathcal{G}_0} \mathcal{G}_n$ is a smooth manifold for all $n \geq 0$, so that $\mathcal{H}$ is in fact a Lie groupoid. Defining $M := \mathcal{H}_0$, the map $d_1\colon \mathcal{G}_1 \times_{\mathcal{G}_0} M \simeq \mathcal{H}_1 \to \mathcal{H}_0 = M$ determines an action of $\mathcal{G}$ on M and one observes that $\mathcal{H}$ is equivalent to the action groupoid $\mathcal{G} \ltimes M$, showing that $\mathcal{H}$ lies in the essential image. □

Corollary 2.3.19. *Let $\mathcal{G}$ be a Lie groupoid. Sending a smooth $\mathcal{G}$-manifold M to the map $M//\mathcal{G} \to \mathbb{B}\mathcal{G}$ defines a fully faithful functor*

$$\mathrm{Diff}_{\mathcal{G}} \hookrightarrow \mathrm{DiffStk}_{/\mathbb{B}\mathcal{G}}$$

whose essential image consists of the representable submersions $\mathcal{X} \to \mathbb{B}\mathcal{G}$.

Proof. By Proposition E.20, there is an equivalence of ∞-categories

$$\mathrm{Act}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff})) \xrightarrow{\sim} \mathrm{Shv}(\mathrm{Diff})_{/\mathbb{B}\mathcal{G}}$$

given by sending a cartesian morphism $\varphi\colon \mathcal{H} \to \mathcal{G}$ of groupoid objects in $\mathrm{Shv}(\mathrm{Diff})$ to the map $\mathbb{B}\varphi\colon \mathbb{B}\mathcal{H} \to \mathbb{B}\mathcal{G}$. Since there is a cartesian square

$$\begin{tikzcd}
\mathcal{H}_0 \arrow[r, two heads] \arrow[d, "\varphi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{B}\mathcal{G} \arrow[d, "\mathbb{B}\varphi"] \\
\mathcal{G}_0 \arrow[r, two heads] & \mathbb{B}\mathcal{G},
\end{tikzcd}$$

it follows from Proposition 2.1.10 that the map φ_0 is a representable submersion if and only if $\mathbb{B}\varphi$ is a representable submersion. It follows from Lemma 2.3.18 that the subcategory $\mathrm{Diff}_{\mathcal{G}}$ of $\mathrm{Act}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff}))$ corresponds to the full subcategory of $\mathrm{Shv}(\mathrm{Diff})_{/\mathbb{B}\mathcal{G}}$ spanned by the representable submersions. As the source of every representable submersion $\mathcal{X} \to \mathbb{B}\mathcal{G}$ is a differentiable stack by Lemma 2.2.4, the claim follows. □

Corollary 2.3.20. *Let G be a Lie group. The assignment $M \mapsto (M//G \to \mathbb{B}G)$ defines a fully faithful functor*

$$-//G\colon \mathrm{Diff}_G \hookrightarrow \mathrm{DiffStk}_{/\mathbb{B}G},$$

whose essential image consists of the representable morphisms $\mathcal{X} \to \mathbb{B}G$.

Proof. Given Corollary 2.3.19, it remains to show that a morphism $f\colon \mathcal{X} \to \mathbb{B}G$ of differentiable stacks is representable if and only if it is a representable submersion. One direction is immediate, so assume that $\mathcal{X} \to \mathbb{B}G$ is representable. Consider the following pullback square:

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{pt} \arrow[r, two heads] & \mathbb{B}G. \end{tikzcd}$$

As the map $\mathrm{pt} \twoheadrightarrow \mathbb{B}G$ is a representable submersion, it follows that the pullback M is a smooth manifold. In particular, the map $M \to \mathrm{pt}$ is a smooth submersion. By Lemma 2.2.6, it follows that also $f\colon \mathcal{X} \to \mathbb{B}G$ is a representable submersion, finishing the proof. □

Corollary 2.3.21. *A differentiable stack $\mathcal{X}$ is a global quotient stack in the sense of Example 2.3.5 if and only if it admits a representable submersion $\mathcal{X} \to \mathbb{B}G$ for some compact Lie group G.* □

2.3.3 Classification of principal bundles on smooth manifolds

Let $\mathcal{G}$ be a Lie groupoid and let N be a smooth manifold. Recall from Definition D.15 the notion of a *smooth principal $\mathcal{G}$-bundle on N*: a surjective smooth submersion $p\colon P \twoheadrightarrow N$ equipped with a left $\mathcal{G}$-action $a\colon \mathcal{G}_1 \times_{\mathcal{G}_0} P \to P$ living over N for which the shear map $(a, \mathrm{pr}_2)\colon \mathcal{G}_1 \times_{\mathcal{G}_0} P \to P \times_N P$ is a diffeomorphism. The goal of this subsection is to give a classification of smooth principal $\mathcal{G}$-bundles on N in terms of morphisms $N \to \mathbb{B}\mathcal{G}$ of differentiable stacks: there is an equivalence of ∞-groupoids

$$\mathrm{Hom}_{\mathrm{DiffStk}}(N, \mathbb{B}G) \simeq \{\text{Smooth principal } \mathcal{G}\text{-bundles on } N\}.$$

Since the right-hand side is simply a 1-groupoid, it follows in particular that the stack $\mathbb{B}\mathcal{G}$ is a 1-stack, making our definition of differentiable stacks compatible with the standard literature.

We start by comparing the explicit definition of smooth principal bundles for Lie groupoids to the general topos-theoretic notion of principal bundles, recalled in Appendix E.3.

Proposition 2.3.22. *Let $\mathcal{G}$ be a Lie groupoid en let N be a smooth manifold. Then there is an equivalence*

$$\mathrm{PrnBdl}_{\mathcal{G}}(N) \xrightarrow{\sim} \mathrm{PrnBdl}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff}))_N$$

between the category of smooth principal $\mathcal{G}$-bundles over N, as defined in Definition D.15, and the ∞-category of principal $\mathcal{G}$-bundles over N internal to the ∞-topos Shv(Diff)*, as defined in Definition E.24.*

Proof. For a smooth principal $\mathcal{G}$-bundle $p\colon P \to N$, the action groupoid $\mathcal{G} \ltimes P$ is by assumption (2) of Definition D.15 a groupoid object in the slice $\mathrm{Diff}_{/N}$. The groupoid map $\mathcal{G} \ltimes P \to \mathcal{G}$ lifts by adjunction to a groupoid map $\mathcal{G} \ltimes P \to \mathcal{G} \times N$ in the slice $\mathrm{Diff}_{/N}$. This map is still a cartesian natural transformation, so forms an object of $\mathrm{Act}_{\mathcal{G}}(\mathrm{Diff}_{/N}) \subseteq \mathrm{Act}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff})_{/N})$. This produces a functor

$$\mathrm{PrnBdl}_{\mathcal{G}}(N) \to \mathrm{Act}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff})_{/N}).$$

Since a map $P \to P'$ of smooth principal $\mathcal{G}$-bundles over N is equivalently specified by a map $\mathcal{G} \ltimes P \to \mathcal{G} \ltimes P'$ commuting with the maps to N and to $\mathcal{G}$, this is a fully faithful functor. We claim that this inclusion factors through the full subcategory $\mathrm{PrnBdl}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff}))_N$ of principal $\mathcal{G}$-bundles over N internal to $\mathrm{Shv}(\mathrm{Diff})$. Notice that the map $(a, \mathrm{pr}_2)\colon \mathcal{G}_1 \times_{\mathcal{G}_0} P \to P \times_N P$ in condition (3) of Definition D.15 is precisely the shear map (internal to $\mathrm{Diff}_{/N}$) from Definition E.21, so that the map $P \to N$ is a formally principal $\mathcal{G}$-bundle in the sense of Definition E.24. Furthermore, the condition (1) from Definition D.15 that the map $P \to N$ is a surjective smooth submersion is by Remark 2.1.6 equivalent to the condition that the map $P \to N$ is an effective epimorphism in $\mathrm{Shv}(\mathrm{Diff})$, so that $P \to N$ is in fact a principal $\mathcal{G}$-bundle internal to $\mathrm{Shv}(\mathrm{Diff})$. We have therefore produced a fully faithful embedding

$$\mathrm{PrnBdl}_{\mathcal{G}}(N) \hookrightarrow \mathrm{PrnBdl}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff}))_N.$$

It remains to prove that this inclusion is essentially surjective. Let $\mathcal{P} \twoheadrightarrow N$ be a map in $\mathrm{Shv}(\mathrm{Diff})$ equipped with the structure of a principal $\mathcal{G}$-bundle. By Theorem E.28, there is a classifying map $c\colon N \to \mathbb{B}\mathcal{G}$ and a pullback square

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_0 \arrow[d, two heads] \\ N \arrow[r, "c"] & \mathbb{B}\mathcal{G}. \end{tikzcd}$$

Since the map $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$ is a surjective representable submersion by Lemma 2.3.3, it follows that also the map $\mathcal{P} \twoheadrightarrow N$ is a surjective smooth submersion. In particular the stack $\mathcal{P}$ is a smooth manifold, which we will henceforth denote as P. Just as in Lemma 2.3.18 it follows that P is equipped with a left $\mathcal{G}$-action, and since the action groupoid $\mathcal{G} \ltimes P \to P$ lives in $\mathrm{Diff}_{/N}$ it satisfies condition (2) of Definition D.15. As before, condition (3) is equivalent to the condition that the map $P \to N$ is a formally principal $\mathcal{G}$-bundle and condition (1) is equivalent to the condition that $P \to N$ is an effective epimorphism in $\mathrm{Shv}(\mathrm{Diff})$. It follows that the map $P \to N$ is indeed a smooth principal $\mathcal{G}$-bundle as desired. $\square$

As a corollary, we obtain that the classifying stack $\mathbb{B}\mathcal{G}$ of a Lie groupoid $\mathcal{G}$ is a 1-stack:

Proposition 2.3.23. *Let $\mathcal{G}$ be a Lie groupoid. For every smooth manifold N, there is a natural equivalence of ∞-groupoids*

$$\mathbb{B}\mathcal{G}(N) \simeq \mathrm{PrnBdl}_{\mathcal{G}}(N).$$

In particular, $\mathbb{B}\mathcal{G}(N)$ is an ordinary groupoid for every N.

Proof. By Theorem E.28, there is an equivalence of ∞-groupoids

$$\mathbb{B}\mathcal{G}(N) = \mathrm{Hom}_{\mathrm{Shv}(\mathrm{Diff})}(N, \mathbb{B}\mathcal{G}) \simeq \mathrm{PrnBdl}_{\mathcal{G}}(\mathrm{Shv}(\mathrm{Diff}))_N.$$

By Proposition 2.3.22, the right-hand side is equivalent to the ordinary groupoid $\mathrm{PrnBdl}_{\mathcal{G}}(N)$ of smooth principal $\mathcal{G}$-bundles over N, finishing the proof. □

Combining this corollary with Corollary 2.3.8, we obtain:

Corollary 2.3.24 (Differentiable stacks are 1-stacks)**.** *Every differentiable stack $\mathcal{X}$ is a 1-stack, in the sense that it factors through the subcategory* Grpd $\subseteq$ An *of groupoids. In particular, the ∞-category* DiffStk *is a $(2,1)$-category.* □

2.4 Local properties of maps of stacks

Various properties of smooth maps between smooth manifolds have immediate generalizations to the context of maps between differentiable stacks.

Definition 2.4.1 (Local property for smooth manifolds)**.** Let P be a property of smooth maps of smooth manifolds. We say that P is a *local property* if for every pullback diagram

$$\begin{tikzcd} M' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d, "f"] \\ N' \arrow[r, "g", two heads] & N \end{tikzcd}$$

in Diff in which g is a submersion, we have:

(1) If f has property P, then also f' has property P.

(2) If g is surjective and f' has property P, then also f has property P.

Definition 2.4.2 (Local property for stacks)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism between stacks on Diff, and let P be a local property of smooth maps of smooth manifolds.

Then we say that *has property* P if for every representable submersion $g\colon N' \to \mathcal{Y}$ and every pullback diagram

$$\begin{tikzcd} M' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "f"] \\ N' \arrow[r, two heads, "g"] & \mathcal{Y} \end{tikzcd}$$

the smooth map $f'\colon M' \to N'$ of smooth manifolds has property P.

It is immediate from the definition that if $f\colon M \to N$ is a smooth map of smooth manifolds, then f has property P as a map of smooth manifolds if and only if it has property P when regarded as a morphism of stacks.

Before giving examples of local properties, let us address a potential point of confusion. In Definition 2.1.4, we defined the notion of a representable submersion. Now, since the property for a map between smooth manifolds to be a smooth submersion is a local property, Definition 2.4.2 gives an a priori different definition for what it means for a representable morphism of stacks to be a smooth submersion. The following lemma shows that these two notions agree in the case its source and target are differentiable stacks:

Lemma 2.4.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be differentiable stacks. Then a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is a representable submersion, in the sense of Definition 2.1.4, if and only if it is both representable and a smooth submersion in the sense of Definition 2.4.2.*

Proof. It is clear that a representable submersion is both representable and a smooth submersion, so we will prove the converse. Let $M \twoheadrightarrow \mathcal{Y}$ be a representable atlas for $\mathcal{Y}$. By part (3) of Proposition 2.1.10, the morphism f is a representable submersion if and only if its base change $M \times_{\mathcal{Y}} \mathcal{X} \to M$ is a representable submersion. Since $M \twoheadrightarrow \mathcal{Y}$ is a representable submersion and f is representable, the pullback $M \times_{\mathcal{Y}} \mathcal{X}$ is a smooth manifold. Furthermore, by assumption on f, the base change $M \times_{\mathcal{Y}} \mathcal{X} \to M$ is a smooth submersion between smooth manifolds. It follows in particular that the morphism $M \times_{\mathcal{Y}} \mathcal{X} \to M$ is a representable submersion, which finishes the proof. □

Lemma 2.4.4 (Open substacks)**.** *A morphism $\mathcal{U} \hookrightarrow \mathcal{X}$ of differentiable stacks is an open substack, in the sense of Definition 2.1.7, if and only if it is both representable and an open embedding, in the sense of Definition 2.4.2.*

Proof. The proof is completely analogous to that of Lemma 2.4.3 and will be omitted. □

In analogy with Lemma 2.4.4 we get the following more general definition of embeddings of stacks.

Definition 2.4.5 (Embeddings of stacks)**.** A smooth map $N \to M$ of smooth manifolds is called an *embedding* if it is a smooth immersion which induces a homeomorphism onto its image in M. It is called a *closed (resp. open) embedding* if the image in M is closed (resp. open).

Specializing Definition 2.4.2, we obtain the notions of *embedded*, *closed embedding* and *open embedding*.

Example 2.4.6. Other examples of properties P of smooth maps $f\colon M \to N$ of smooth manifolds are the following:

- f is injective, f is surjective;
- f is a submersion, f is a surjective submersion;
- f is an open map;
- f is *étale*, or a *local diffeomorphism*, meaning that for every $x \in M$ there exists an open neighborhood $x \in U \subseteq M$ such that $f|_U\colon U \to N$ is an open embedding;
- f is *proper*, meaning that for every compact subspace $K \subseteq N$ the preimage $f^{-1}(K) \subseteq M$ is also compact;
- f admits local sections, in the sense of Definition 2.1.2;
- f is a finite covering space, meaning that there is an open cover $\{U_i \subseteq N\}_{i \in I}$ such that every restriction $f|_{U_i}\colon f^{-1}(U_i) \to U_i$ is isomorphic to $U_i^{\sqcup n_i} \to U_i$ for some integer $n_i \geq 0$.

The main feature of local properties of maps of stacks is that they can be tested after pulling back along a representable surjective submersion.

Lemma 2.4.7 (Local properties are local)**.** *Let P be a local property, and consider a pullback diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, two heads, "g"] & \mathcal{Y} \end{tikzcd}$$

of stacks on Diff *such that g is an representable submersion and such that f and f' are representable.*

(1) If f has property P, then so does f'.

(2) If g is a representable surjective submersion and f' has property P, then so does f.

Proof. Part (1) is immediate from the definitions. For part (2), let $M \to \mathcal{Y}$ be a representable submersion from some smooth manifold M. By base changing the whole situation along this map, we may assume that $\mathcal{Y} = M$ is a smooth manifold. Since all four maps are representable, it follows that also $\mathcal{X}$, $\mathcal{Y}'$ and $\mathcal{Y}$ are smooth manifolds. Since P is a local property of maps between smooth manifolds, it follows that f has property P. □

In practice, we use Lemma 2.4.7 mostly in the situation where $\mathcal{Y}'$ is a representable atlas for $\mathcal{Y}$:

Corollary 2.4.8. *Let $\mathcal{Y}$ be a differentiable stack with representable atlas $M \twoheadrightarrow \mathcal{Y}$ and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of stacks. Let P be a local property of maps of smooth manifolds. Then f has property P if and only if the map $M \times_{\mathcal{Y}} \mathcal{X} \to M$ has property P.* □

It follows from Corollary 2.4.8 that representable morphisms between differentiable stacks behave very similar to smooth maps between smooth manifolds. We collect some results in this direction.

Lemma 2.4.9. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable submersion between differentiable stacks. Then there exists an essentially unique factorization*

$$\mathcal{Y} \twoheadrightarrow \mathcal{U} \hookrightarrow \mathcal{X}$$

of f into a surjective representable submersion $\mathcal{Y} \twoheadrightarrow \mathcal{U}$ and an open embedding $\mathcal{U} \hookrightarrow \mathcal{X}$.

We will also write $f(\mathcal{Y}) \hookrightarrow \mathcal{X}$ for the essentially unique open substack of $\mathcal{X}$ determined by the lemma, and refer to it as the *image of f*.

Proof. We may factor f as an effective epimorphism $\mathcal{Y} \twoheadrightarrow \mathcal{U}$ followed by a monomorphism $\mathcal{U} \hookrightarrow \mathcal{X}$. It remains to show that the map $\mathcal{Y} \twoheadrightarrow \mathcal{U}$ is a surjective representable submersion and the map $\mathcal{U} \hookrightarrow \mathcal{X}$ is an open embedding. By Corollary 2.4.8, this may be checked after pulling back the situation along an atlas $M \twoheadrightarrow \mathcal{X}$, and since pulling back preserves the unique factorization we may assume that $f\colon N \to M$ is a smooth submersion between smooth manifolds. In that case, the factorization is given as the factorization $N \twoheadrightarrow f(N) \hookrightarrow M$ of f through its (open) image in M. As the map $N \to f(N)$ is a surjective submersion and the inclusion $f(N) \hookrightarrow N$ is an open embedding, this proves the claim. □

Lemma 2.4.10. *Consider a commutative triangle of differentiable stacks:*

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "f"'] \arrow[dr, "gf"] & \\ \mathcal{Y} \arrow[rr, "g"] & & \mathcal{Z}. \end{tikzcd}$$

(1) If both gf and g are representable, then so is f;

(2) If gf is an embedding and g is representable, then f is an embedding;

(3) If gf is proper and g is representable, then f is proper;

(4) If gf is a closed embedding and g is representable, then f is a closed embedding.

Proof. Let $O \twoheadrightarrow \mathcal{Z}$ be a representable atlas for $\mathcal{Z}$ and consider the following pullback diagram:

$$\begin{tikzcd} M \arrow[r, two heads] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "f"] \\ N \arrow[r, two heads] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y} \arrow[d, "g"] \\ O \arrow[r, two heads] & \mathcal{Z}. \end{tikzcd}$$

By Lemma 2.2.3, the map $N \twoheadrightarrow \mathcal{Y}$ is a representable atlas for $\mathcal{Y}$, and thus by Proposition 2.1.10 and Lemma 2.4.7, it suffices to prove that f' is representable (resp. an embedding / proper / a closed embedding). Part (1) is now immediate, as f' is a morphism between smooth manifolds and thus representable. Parts (2), (3) and (4) follow immediately from the analogous cancellation property for smooth maps between smooth manifolds, see Lemma C.7. □

2.5 Vector bundles over stacks

The notion of a vector bundle over a smooth manifold admits a natural extension to the setting of stacks on Diff.

Definition 2.5.1 (Vector bundle)**.** Let $\mathcal{X}$ be a stack on Diff. For $n \geq 0$, we define an *n-dimensional vector bundle* π on $\mathcal{X}$ to be a morphism of stacks $\pi\colon \mathcal{X} \to \mathbb{B}\mathrm{GL}(n)$. The *total space* $\mathcal{E}_\pi$ of the bundle π is defined as the following pullback of stacks:

$$\begin{tikzcd} \mathcal{E}_\pi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{R}^n//\mathrm{GL}(n) \arrow[d] \\ \mathcal{X} \arrow[r, "\pi"] & \mathbb{B}\mathrm{GL}(n). \end{tikzcd}$$

We will often abuse notation and refer to a map $\mathcal{E} \to \mathcal{X}$ as a vector bundle, leaving the classifying map $\pi\colon X \to \mathbb{B}\,\mathrm{GL}(n)$ implicit in the notation. We denote the groupoid of n-dimensional vector bundles over $\mathcal{X}$ by

$$\mathrm{Vect}_n^{\simeq}(\mathcal{X}) := \mathrm{Hom}_{\mathrm{Shv(Diff)}}(\mathcal{X}, \mathbb{B}\,\mathrm{GL}(n)).$$

We also define

$$\mathrm{Vect}^{\simeq}(\mathcal{X}) := \mathrm{Hom}_{\mathrm{Shv(Diff)}}(\mathcal{X}, \coprod_{n=0}^{\infty} \mathbb{B}\,\mathrm{GL}(n)).$$

This determines a limit-preserving functor $\mathrm{Vect}^{\simeq}\colon \mathrm{Shv(Diff)}^{\mathrm{op}} \to \mathrm{Grpd}$.

Definition 2.5.2 (Associated sphere bundle)**.** Given a vector bundle $\mathcal{E} \to \mathcal{X}$ classified by $\pi\colon \mathcal{X} \to \mathbb{B}\,\mathrm{GL}(n)$, we define its *associated sphere bundle* $S^{\mathcal{E}} \to \mathcal{X}$ as the pullback

$$\begin{tikzcd}
S^{\mathcal{E}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S^n//\mathrm{GL}(n) \arrow[d] \\
\mathcal{X} \arrow[r, "\pi"] & \mathbb{B}\,\mathrm{GL}(n).
\end{tikzcd}$$

The groupoid $\mathrm{Vect}^{\simeq}(\mathcal{X})$ is the core of a category $\mathrm{Vect}(\mathcal{X})$ of vector bundles:

Definition 2.5.3 (Category of vector bundles)**.** Consider the functor $\mathrm{Vect}\colon \mathrm{Diff}^{\mathrm{op}} \to \mathrm{Cat}$ which sends a smooth manifold M to the abelian category of vector bundles over M. This functor satisfies the sheaf condition with respect to the open cover topology, and thus extends uniquely to a limit-preserving functor

$$\mathrm{Vect}\colon \mathrm{Shv(Diff)}^{\mathrm{op}} \to \mathrm{Cat}.$$

For a stack $\mathcal{X}$ on Diff, we refer to the rsulting abelian category $\mathrm{Vect}(\mathcal{X})$ as the *category of vector bundles over* $\mathcal{X}$.

Since the functor $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{Grpd}$ which assigns to an ordinary category C its underlying groupoid $C^{\simeq}$ is a right adjoint and thus preserves limits, we see that there is a natural equivalence of groupoids

$$\mathrm{Vect}(\mathcal{X})^{\simeq} \simeq \mathrm{Vect}^{\simeq}(\mathcal{X}).$$

Morphisms of vector bundles can be made explicit as follows: given two vector bundles $\mathcal{E}_1, \mathcal{E}_1$ over a stack $\mathcal{X}$, a morphism of vector bundles from $\mathcal{E}_1$ to $\mathcal{E}_2$ is a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[rr] \arrow[dr] & & \mathcal{E}_2 \arrow[dl] \\
& \mathcal{X} &
\end{tikzcd}$$

such that for every point $x\colon \mathrm{pt} \to \mathcal{X}$ the induced map $(\mathcal{E}_1)_x \to (\mathcal{E}_2)_x$ on fibers is a linear map between vector spaces.

2.5.1 Homotopy invariance of vector bundles

The category of vector bundles $\mathrm{Vect}(\mathcal{X})$ is a *homotopy invariant* of the stack $\mathcal{X}$:

Lemma 2.5.4. *For every stack $\mathcal{X}$ on* Diff*, pulling back along the projection $\mathcal{X}\times\mathbb{R}\to\mathcal{X}$ induces an equivalence of categories*

$$\mathrm{Vect}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Vect}(\mathcal{X}\times\mathbb{R}).$$

For every $r\in\mathbb{R}$, an inverse is given by pulling back along the inclusion $i_r\colon \mathcal{X}=\mathcal{X}\times\{r\}\hookrightarrow \mathcal{X}\times\mathbb{R}$. In particular, for every $s,r\in\mathbb{R}$ there is an equivalence $i_r^*\simeq i_s^*$.

Proof. Regarding both sides as functors $\mathrm{Shv}(\mathrm{Diff})^{\mathrm{op}}\to\mathrm{Cat}_\infty$ in the variable $\mathcal{X}$, they both preserve limits, and hence it suffices to prove the claim when $\mathcal{X}=M$ is a smooth manifold. This is well-known; see for instance [Hus93, Corollary 4.4.4] □

Definition 2.5.5. Let $f,g\colon\mathcal{X}\to\mathcal{Y}$ be morphisms of stacks on Diff. A *homotopy* between f and g is a map of stacks $H\colon\mathcal{X}\times\mathbb{R}\to\mathcal{Y}$ satisfying $H_0\simeq f$ and $H_1\simeq g$, where $H_r\colon\mathcal{X}\to\mathcal{Y}$ denotes the composite

$$\mathcal{X}\xhookrightarrow{i_r}\mathcal{X}\times\mathbb{R}\xrightarrow{H}\mathcal{Y}.$$

Corollary 2.5.6. *Let $\mathcal{E}\to\mathcal{Y}$ be a vector bundle and let $H\colon\mathcal{X}\times\mathbb{R}\to\mathcal{Y}$ be a homotopy of morphisms $\mathcal{X}\to\mathcal{Y}$. Then there is an isomorphism $H_0^*\mathcal{E}\simeq H_1^*\mathcal{E}$ of vector bundles on $\mathcal{X}$, natural in $\mathcal{E}$.*

Proof. Using Lemma 2.5.4, we obtain an equivalence $f_0^*\mathcal{E}=i_0^*f^*\mathcal{E}\simeq i_1^*f^*\mathcal{E}=f_1^*\mathcal{E}$. □

Corollary 2.5.7. *Let $\pi\colon\mathcal{E}\to\mathcal{X}$ be a vector bundle. Then the functor $\pi^*\colon\mathrm{Vect}(\mathcal{X})\to\mathrm{Vect}(\mathcal{E})$ is an equivalence of categories. An inverse is given by $s_0^*\colon\mathrm{Vect}(\mathcal{E})\to\mathrm{Vect}(\mathcal{X})$, where $s_0\colon\mathcal{X}\to\mathcal{E}$ is the zero-section of π.*

Proof. Since $s_0\pi=\mathrm{id}_{\mathcal{X}}$, it remains to show that $\pi^*s_0^*\simeq\mathrm{id}$. Let $H\colon\mathcal{E}\times\mathbb{R}\to\mathcal{E}$ be the homotopy given by $(e,r)\mapsto re$, so that $H_0=s_0\circ\pi$ while $H_1=\mathrm{id}_{\mathcal{E}}$. The claim then follows from Corollary 2.5.6. □

Corollary 2.5.8. *Let $\pi\colon\mathcal{E}\to\mathcal{X}$ and $\pi'\colon\mathcal{E}'\to\mathcal{E}$ be two vector bundles. Then the composite $\pi'\circ\pi\colon\mathcal{E}'\to\mathcal{X}$ is a vector bundle which is isomorphic to the direct sum $\mathcal{E}\oplus s_0^*\mathcal{E}'$ of the vector bundle $\mathcal{E}$ and the base change of $\mathcal{E}'$ along the zero-section $s_0\colon\mathcal{X}\to\mathcal{E}$.*

Proof. It follows from Corollary 2.5.7 that $\mathcal{E}'$ is isomorphic to the vector bundle $p^*s_0^*\mathcal{E}'$. It follows that the composite $\mathcal{E}'\to\mathcal{E}\to\mathcal{X}$ is isomorphic to the fiber product $\mathcal{E}\times_{\mathcal{X}}s_0^*\mathcal{E}'$, which is indeed a vector bundle over $\mathcal{X}$ isomorphic to the direct sum of $\mathcal{E}$ and $s_0^*\mathcal{E}'$. □

2.5.2 VB-groupoids

We will now discuss the notion of a *VB-groupoid*, which is the analogue of a vector bundle in the setting of Lie groupoids.

Definition 2.5.9 (VB-groupoid, [GM17])**.** Given a Lie groupoid $\mathcal{G}$, a *VB-groupoid over* $\mathcal{G}$ is a morphism $\pi\colon \mathcal{E} \to \mathcal{G}$ of Lie groupoids such that the maps $\pi_0\colon \mathcal{E}_0 \to \mathcal{G}_0$ and $\pi_1\colon \mathcal{E}_1 \to \mathcal{G}_1$ are vector bundles and the structure maps of $\mathcal{E}$ are morphisms of vector bundles.

In terms of differentiable stacks, VB-groupoids correspond to what are known as *2-vector bundles*, see for example [HO20] for a discussion. In order to get an actual vector bundle of differentiable stacks, one needs to assume that the VB-groupoid $\mathcal{E} \to \mathcal{G}$ is *cartesian*, in the following sense:

Definition 2.5.10 (Cartesian morphism of Lie groupoids)**.** A morphism $f\colon \mathcal{H} \to \mathcal{G}$ of Lie groupoids is called *cartesian* if the commutative squares

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r, "s"] \arrow[d, "f_1"'] & \mathcal{H}_0 \arrow[d, "f_0"] \\ \mathcal{G}_1 \arrow[r, "s"] & \mathcal{G}_0 \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{H}_1 \arrow[r, "t"] \arrow[d, "f_1"'] & \mathcal{H}_0 \arrow[d, "f_0"] \\ \mathcal{G}_1 \arrow[r, "t"] & \mathcal{G}_0 \end{tikzcd}$$

are pullback squares.[4]

Lemma 2.5.11. *Let $\pi\colon \mathcal{E} \to \mathcal{G}$ be a VB-groupoid, and assume that the morphism π is cartesian. Then the induced map $\mathbb{B}\pi\colon \mathbb{B}\mathcal{E} \to \mathbb{B}\mathcal{G}$ is naturally a vector bundle of differentiable stacks.*

Proof. By passing to nerves, the groupoid map $\pi\colon \mathcal{E} \to \mathcal{G}$ induces a morphism $N\pi\colon N\mathcal{E} \to N\mathcal{G}$ of simplicial manifolds. As π is cartesian, $N\pi$ is a cartesian natural transformation of functors $\Delta^{\mathrm{op}} \to \mathrm{Diff}$. It follows that $N\pi$ defines a section of the functor $\Delta \to \mathrm{Cat}, [n] \mapsto \mathrm{Vect}(\mathcal{G}_n)$, and thus defines an object of the limit $\lim_{[n]\in\Delta} \mathrm{Vect}(\mathcal{G}_n)$. Since the assignment $\mathcal{X} \mapsto \mathrm{Vect}(\mathcal{X})$ satisfies descent, we see that this limit is $\mathrm{Vect}(\mathbb{B}\mathcal{G})$, giving a vector bundle over $\mathbb{B}\mathcal{G}$. Since the forgetful functors $\mathrm{Vect}(\mathcal{X}) \to \mathrm{Sub}_{/\mathcal{X}}$ are compatible with pullback, it follows that the underlying submersion of this vector bundle is the map $\mathbb{B}\mathcal{E} \to \mathbb{B}\mathcal{G}$, finishing the proof. $\square$

[4] It suffices to check this for just one of the squares.

II.3 Geometry of differentiable stacks

In this chapter, we will have a closer look at the geometrical aspects of differentiable stacks. In Section 3.1, we introduce the *coarse moduli space* $|\mathcal{X}|_{\mathrm{mod}}$ of a stack $\mathcal{X}$ on Diff and show that the open subsets of $|\mathcal{X}|_{\mathrm{mod}}$ are in one-to-one correspondence with the open substacks of $\mathcal{X}$. This allows us to construct the *open complement* $\mathcal{X} \setminus \mathcal{Z}$ of a closed embedding $\mathcal{Z} \hookrightarrow \mathcal{X}$ in Section 3.2. In Section 3.3, we introduce the notion of a *separated* differentiable stack, which in terms of Lie groupoids corresponds to that of a *proper* Lie groupoid. The geometry of separated differentiable stacks is better behaved than that of arbitrary differentiable stacks, and hence separated differentiable stacks will play an important role throughout this article. In Section 3.4 we show that a morphism of separated separated differentiable stack is representable if and only if it induces injections on *isotropy groups*. In Section 3.5 we introduce the notions of relative tangent bundles and normal bundles for representable morphisms of differentiable stacks, and we will show in Section 3.6 that every embedding of separated differentiable stacks admits a tubular neighborhood. Finally, we will show in Section 3.7 that all separated differentiable stacks are locally isomorphic to a quotient stack $\mathbb{R}^n//G$ for some smooth linear action of a compact Lie group G on a Euclidean space $\mathbb{R}^n$, which leads to several important structure theorems for morphisms between separated differentiable stacks.

3.1 The coarse moduli space of a differentiable stack

We start by defining the *coarse moduli space* $|\mathcal{X}|_{\mathrm{mod}}$ of a stack $\mathcal{X}$ on Diff.

Definition 3.1.1. As the category Top of topological spaces admits colimits, the inclusion Diff $\hookrightarrow$ Top uniquely extends to a colimit-preserving functor PSh(Diff) $\to$ Top. Since an open cover $\bigsqcup_\alpha U_\alpha \twoheadrightarrow M$ of a smooth manifold M is an effective epimorphism in Top, this descends to a functor

$$|-|_{\mathrm{mod}} \colon \mathrm{Shv}(\mathrm{Diff}) \to \mathrm{Top}.$$

We call $|\mathcal{X}|_{\mathrm{mod}}$ the *coarse moduli space* of the stack $\mathcal{X}$. The functor $|-|_{\mathrm{mod}}$ admits a right adjoint $\mathrm{N}^{\mathrm{Diff}}\colon \mathrm{Top} \longrightarrow \mathrm{Shv}(\mathrm{Diff})$ given by

$$\mathrm{N}^{\mathrm{Diff}}(X)(M) := \mathrm{Hom}_{\mathrm{Top}}(M, X),$$

which equips a topological space X with its *continuous diffeology*.

Remark 3.1.2. The stack $\mathrm{N}^{\mathrm{Diff}}(X)$ is in fact *diffeological space*, that is, a concrete sheaf of sets on the site Diff. We shall not make use of this fact.

Example 3.1.3. For a smooth manifold M, we have $|M|_{\mathrm{mod}} = M$.

Example 3.1.4. For a Lie groupoid $\mathcal{G}$, the coarse moduli space of its classifying stack $\mathbb{B}\mathcal{G}$ is its given by the quotient space of $\mathcal{G}$:

$$|\mathbb{B}\mathcal{G}|_{\mathrm{mod}} \cong \mathrm{coeq}(\mathcal{G}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{G}_0).$$

In particular, given the action of a Lie group G on a smooth manifold M, the coarse moduli space of the quotient stack $M//G$ is its strict quotient M/G:

$$|M//G|_{\mathrm{mod}} \cong M/G.$$

Given a stack $\mathcal{X}$ on Diff, there is an explicit construction of the topological space $|\mathcal{X}|_{\mathrm{mod}}$. We follow the treatment of Noohi [Noo05, Section 4.3], who discusses this construction in the context of topological stacks.

Construction 3.1.5. Given a stack $\mathcal{X}$ on Diff, we will define a topological space $\mathcal{X}_{\mathrm{mod}}$. The underlying set of $\mathcal{X}_{\mathrm{mod}}$ is the set of path components of the anima of global sections $\Gamma(\mathcal{X}) = \mathrm{Hom}_{\mathrm{Shv}(\mathrm{Diff})}(\mathrm{pt}, \mathcal{X})$ of $\mathcal{X}$:

$$\mathcal{X}_{\mathrm{mod}} := \pi_0(\Gamma(\mathcal{X})).$$

We define a topology on $\mathcal{X}_{\mathrm{mod}}$ as follows: a subset of $\mathcal{X}_{\mathrm{mod}}$ is open precisely if it is of the form $\pi_0(\Gamma(\mathcal{U})) \subseteq \pi_0(\Gamma(\mathcal{X}))$ for some open substack $\mathcal{U} \hookrightarrow \mathcal{X}$; this inclusion makes sense as the map $\Gamma(\mathcal{U}) \hookrightarrow \Gamma(\mathcal{X})$ is an inclusion of path components. To see that this defines a topology, observe first that arbitrary unions and finite intersections of open substacks are again open substacks: given a morphism $M \to \mathcal{X}$ from a smooth manifold M, the pullback functor $\mathrm{Shv}(\mathrm{Diff})_{/\mathcal{X}} \to \mathrm{Shv}(\mathrm{Diff})_{/M}$ preserves limits and colimits, and unions and finite intersections of open subsets of M are again open subsets. Since all morphisms of animae involved are inclusions of path components, applying π_0 commutes with finite intersections and arbitrary unions, showing that the topology on $\mathcal{X}_{\mathrm{mod}}$ is well-defined.

If $f\colon \mathcal{X} \to \mathcal{Y}$ is a map of stacks, then any open substack $\mathcal{U} \hookrightarrow \mathcal{Y}$ pulls back to an open substack $f^{-1}(\mathcal{U}) \hookrightarrow \mathcal{X}$, so the map $f_{\mathrm{mod}}\colon \mathcal{X}_{\mathrm{mod}} \to \mathcal{Y}_{\mathrm{mod}}$ induced by $\Gamma(f)\colon \Gamma(\mathcal{X}) \to \Gamma(\mathcal{Y})$ is continuous. It follows that the formation of $\mathcal{X}_{\mathrm{mod}}$ gives rise to a functor

$$(-)_{\mathrm{mod}}\colon\ \mathrm{Shv}(\mathrm{Diff}) \to \mathrm{Top}.$$

If $\mathcal{X} = M \in \mathrm{Diff}$ is a smooth manifold, its anima of global sections $\Gamma(M) = \mathrm{Hom}_{\mathrm{Diff}}(\mathrm{pt}, M)$ is already a set, and is in canonical bijection with the underlying set of M. Since the open substacks $\mathcal{U} \hookrightarrow M$ of M, regarded as a stack, are in one-to-one correspondence with the open subsets $U \subseteq M$, the induced topology on the set $\mathrm{Hom}_{\mathrm{Diff}}(\mathrm{pt}, M)$ is just the topology on M. We will henceforth identify M_{mod} with M.

Lemma 3.1.6. *The functor $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{mod}}$ is left adjoint to the functor $\mathrm{N}^{\mathrm{Diff}}\colon \mathrm{Top} \to \mathrm{Shv}(\mathrm{Diff})$. In particular, there is a natural equivalence $\mathcal{X}_{\mathrm{mod}} \simeq |\mathcal{X}|_{\mathrm{mod}}$.*

Proof. We start by constructing the unit map $\eta\colon \mathcal{X} \to \mathrm{N}^{\mathrm{Diff}}(\mathcal{X}_{\mathrm{mod}})$ of stacks for every stack $\mathcal{X} \in \mathrm{Shv}(\mathrm{Diff})$. For $M \in \mathrm{Diff}$, the map $\mathcal{X}(M) \to \mathrm{N}^{\mathrm{Diff}}(\mathcal{X}_{\mathrm{mod}})(M) = \mathrm{Hom}_{\mathrm{Top}}(M, \mathcal{X}_{\mathrm{mod}})$ sends a map $f\colon M \to \mathcal{X}$ of stacks to the map $M = M_{\mathrm{mod}} \xrightarrow{f_{\mathrm{mod}}} \mathcal{X}_{\mathrm{mod}}$ of topological spaces. Since $(-)_{\mathrm{mod}}$ is a functor, it follows that this defines a map of stacks $\eta\colon \mathcal{X} \to \mathrm{N}^{\mathrm{Diff}}(\mathcal{X}_{\mathrm{mod}})$ which is natural in $\mathcal{X} \in \mathrm{Shv}(\mathrm{Diff})$.

We next construct the counit map $\varepsilon\colon\ \mathrm{N}^{\mathrm{Diff}}(T)_{\mathrm{mod}} \to T$ for every topological space T. Note that as a set we have $\mathrm{N}^{\mathrm{Diff}}(T)_{\mathrm{mod}} = \pi_0 \mathrm{Hom}_{\mathrm{Top}}(\mathrm{pt}, T) = T$. Furthermore, if $U \subseteq T$ is an open subspace, then $f^{-1}(U) \subseteq M$ is an open submanifold for every continuous map $f\colon M \to T$, and it follows that $\mathrm{N}^{\mathrm{Diff}}(U) \hookrightarrow \mathrm{N}^{\mathrm{Diff}}(T)$ is an open substack. Evaluating this map at $\mathrm{pt} \in \mathrm{Diff}$ gives the inclusion $U \subseteq T$. This shows that the topology on $\mathrm{N}^{\mathrm{Diff}}(T)_{\mathrm{mod}}$ is finer than that of T, giving a continuous map $\varepsilon\colon\ \mathrm{N}^{\mathrm{Diff}}(T)_{\mathrm{mod}} \to T$.

Now we show that the maps $\eta\colon \mathcal{X} \to \mathrm{N}^{\mathrm{Diff}}(\mathcal{X}_{\mathrm{mod}})$ and $\varepsilon\colon\ \mathrm{N}^{\mathrm{Diff}}(T)_{\mathrm{mod}} \to T$ satisfy the triangle identities:

$$\begin{tikzcd}
 & \mathrm{N}^{\mathrm{Diff}}(\mathrm{N}^{\mathrm{Diff}}(T)_{\mathrm{mod}}) \arrow[dr, "\varepsilon"] & \\
\mathrm{N}^{\mathrm{Diff}}(T) \arrow[ur, "\eta"] \arrow[rr, equal] & & \mathrm{N}^{\mathrm{Diff}}(T)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & \mathrm{N}^{\mathrm{Diff}}(\mathcal{X}_{\mathrm{mod}})_{\mathrm{mod}} \arrow[dr, "\varepsilon"] & \\
\mathcal{X}_{\mathrm{mod}} \arrow[ur, "\eta"] \arrow[rr, equal] & & \mathcal{X}_{\mathrm{mod}}.
\end{tikzcd}$$

For the left triangle, we may check this after mapping in from a smooth manifold M, where it is clear. The right triangle is also clear, as the second map is the identity map on underlying sets. ☐

Corollary 3.1.7. *For a stack $\mathcal{X}$, the underlying set of the topological space $|\mathcal{X}|_{\mathrm{mod}}$ is the set of path components of the anima of global sections of $\mathcal{X}$:*

$$|\mathcal{X}|_{\mathrm{mod}} \cong \pi_0(\Gamma(\mathcal{X})) = \pi_0 \mathrm{Hom}_{\mathrm{Shv}(\mathrm{Diff})}(\mathrm{pt}, \mathcal{X}). \qquad ☐$$

Corollary 3.1.8. *Let $\mathcal{U} \hookrightarrow \mathcal{X}$ be an open substack. Then the induced map $|\mathcal{U}|_{\mathrm{mod}} \hookrightarrow |\mathcal{X}|_{\mathrm{mod}}$ is an open subspace.* □

Our next goal is to show that the above assignment $\mathcal{U} \mapsto |\mathcal{U}|_{\mathrm{mod}}$ defines a 1-to-1 correspondence between open substacks of $\mathcal{X}$ and open subspaces of its coarse moduli space $|\mathcal{X}|_{\mathrm{mod}}$.

Lemma 3.1.9 (cf. [Noo05, Lemma 4.16])**.** *Let $\mathcal{X}' \hookrightarrow \mathcal{X}$ be an embedded substack. Assume the induced map $\mathcal{X}'_{\mathrm{mod}} \to \mathcal{X}_{\mathrm{mod}}$ is a bijection of sets, or equivalently, the induced monomorphism $\Gamma(\mathcal{X}') \hookrightarrow \Gamma(\mathcal{X})$ of animae is in fact an equivalence. Then the map $\mathcal{X}' \to \mathcal{X}$ is an equivalence.*

Proof. We follow the proof of [Noo05, Lemma 4.16] in the topological context. We have to show that for every smooth manifold M, the inclusion $\mathcal{X}'(M) \hookrightarrow \mathcal{X}(M)$ of animae is in fact an equivalence, i.e., is a surjection on path components. If this is not the case, we can find a smooth manifold M and a map $f\colon M \to \mathcal{X}$ which does not lift along $\mathcal{X}' \hookrightarrow \mathcal{X}$. Define the subspace $M' \subseteq M$ by the pullback diagram

$$\begin{tikzcd} M' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d, "f"] \\ \mathcal{X}' \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

It follows that the embedding $M' \subseteq M$ does not have a section, and so we conclude that there is a point $w\colon \mathrm{pt} \to M$ which is not contained in M'. But this means that the point $f(w)\colon \mathrm{pt} \to \mathcal{X}$ does not lift to $\mathcal{X}'$, which is in contradiction with the assumption that the map $\pi_0(\Gamma(\mathcal{X}')) \to \pi_0(\Gamma(\mathcal{X}))$ is a bijection. □

Warning 3.1.10. Although the counit map $\mathrm{N}^{\mathrm{Diff}}(T)_{\mathrm{mod}} \to T$ is the identity on underlying sets, it is not in general a homeomorphism. For example, if $T = \mathbb{Q}$ is the set of rational numbers equipped with the subspace topology from $\mathbb{R}$, then $\mathrm{N}^{\mathrm{Diff}}(T)_{\mathrm{mod}}$ is the set $\mathbb{Q}$ equipped with the discrete topology. Topological spaces T for which the counit map $\mathrm{N}^{\mathrm{Diff}}(T)_{\mathrm{mod}} \to T$ is a homeomorphism are known as Δ-*generated topological spaces*, and it follows that the functor $\mathrm{N}^{\mathrm{Diff}}\colon \mathrm{Top} \to \mathrm{Shv}(\mathrm{Diff})$ becomes fully faithful when restricted to the Δ-generated topological spaces.

Corollary 3.1.11. *For an open substack $\mathcal{U} \hookrightarrow \mathcal{X}$, the naturality square*

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[d, "\eta"'] & \mathcal{X} \arrow[d, "\eta"] \\ \mathrm{N}^{\mathrm{Diff}}(\mathcal{U}_{\mathrm{mod}}) \arrow[r] & \mathrm{N}^{\mathrm{Diff}}(\mathcal{X}_{\mathrm{mod}}) \end{tikzcd}$$

of the unit transformation η is a pullback square.

Proof. We have to prove that the map $\mathcal{U} \to \mathcal{X} \times_{\mathrm{N}^{\mathrm{Diff}}(\mathcal{X}_{\mathrm{mod}})} \mathrm{N}^{\mathrm{Diff}}(\mathcal{U}_{\mathrm{mod}})$ induced by the square is an equivalence. Note that this map is an open embedding, since the map $\mathcal{U} \hookrightarrow \mathcal{X}$ is an open embedding by assumption, the projection $\mathcal{X} \times_{\mathrm{N}^{\mathrm{Diff}}(\mathcal{X}_{\mathrm{mod}})} \mathrm{N}^{\mathrm{Diff}}(\mathcal{U}_{\mathrm{mod}}) \to \mathcal{X}$ is an open embedding by Corollary 3.1.8, and the class of open embeddings of stacks is closed under left cancellation. By Lemma 3.1.9 it thus suffices to show this map induces an equivalence on global sections. Note that $\Gamma(\mathrm{N}^{\mathrm{Diff}}(\mathcal{U}_{\mathrm{mod}})) = \pi_0(\Gamma(\mathcal{U}))$ and $\Gamma(\mathrm{N}^{\mathrm{Diff}}(\mathcal{X}_{\mathrm{mod}})) = \pi_0(\Gamma(\mathcal{X}))$, and since the global section functor Γ preserves pullbacks we need to show that the following square of animae is a pullback square:

$$\begin{tikzcd} \Gamma(\mathcal{U}) \arrow[r, hook] \arrow[d] & \Gamma(\mathcal{X}) \arrow[d] \\ \pi_0(\Gamma(\mathcal{U})) \arrow[r, hook] & \pi_0(\Gamma(\mathcal{X})). \end{tikzcd}$$

But this is clear since $\Gamma(\mathcal{U}) \hookrightarrow \Gamma(\mathcal{X})$ is an inclusion of path components. $\square$

For a stack $\mathcal{X}$, let $\mathrm{Open}(\mathcal{X}) \subseteq \mathrm{Shv}(\mathrm{Diff})_{/\mathcal{X}}$ denote the full subcategory spanned by the open substacks $\mathcal{U} \hookrightarrow \mathcal{X}$. Similarly, for a topological space X, let $\mathrm{Open}(X) \subseteq \mathrm{Top}_{/X}$ denote the full subcategory spanned by the open subspaces $U \hookrightarrow X$. Since open embeddings are monomorphisms, it follows that both of these subcategories are in fact posets.

Corollary 3.1.12 (cf. [Noo05, Proposition 4.17])**.** *Pullback along the map $\mathcal{X} \to |\mathcal{X}|_{\mathrm{mod}}$ defines an equivalence of posets*

$$\mathrm{Open}(|\mathcal{X}|_{\mathrm{mod}}) \xrightarrow{\sim} \mathrm{Open}(\mathcal{X}),$$

with inverse given by $\mathcal{U} \mapsto |\mathcal{U}|_{\mathrm{mod}}$.

Proof. Corollary 3.1.11 shows that every open substack $\mathcal{U}$ of $\mathcal{X}$ is naturally equivalent to the inverse image of the open subspace $|\mathcal{U}|_{\mathrm{mod}} \subseteq |\mathcal{X}|_{\mathrm{mod}}$. Conversely, if $U \subseteq \mathcal{X}_{\mathrm{mod}}$ is an open subspace and $\mathcal{U} \hookrightarrow \mathcal{X}$ is its preimage in $\mathcal{X}$, it is easy to see that $U = \mathcal{U}_{\mathrm{mod}}$. $\square$

Observation 3.1.13. Let $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ be an effective epimorphism. Then the induced map $f_{\mathrm{mod}} : \mathcal{X}_{\mathrm{mod}} \to \mathcal{Y}_{\mathrm{mod}}$ is surjective.

Proof. We need to prove that any map $x\colon \mathrm{pt} \to \mathcal{Y}$ lifts to $\mathcal{X}$. In other words, we need to prove that the effective epimorphism $f^{-1}(\mathrm{pt}) \twoheadrightarrow \mathrm{pt}$ admits a section. But this follows immediately from the fact that effective epimorphisms have local sections, Proposition 2.1.3, and the fact that the only open cover of the point is given by the point. $\square$

3.2 Open complements

In this section, we define the *open complement* $\mathcal{X} \setminus \mathcal{Z}$ of a closed embedding $\mathcal{Z} \hookrightarrow \mathcal{X}$.

Proposition 3.2.1. *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks. Then the induced map of topological spaces*

$$|i|_{\mathrm{mod}}\colon |\mathcal{Z}|_{\mathrm{mod}} \to |\mathcal{X}|_{\mathrm{mod}}$$

is the inclusion of a closed subspace.

Proof. Let $M \twoheadrightarrow \mathcal{X}$ be a representable atlas for $\mathcal{X}$. Form the pullback square

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, two heads] \\ \mathcal{Z} \arrow[r, hook, "i"] & \mathcal{X}, \end{tikzcd}$$

providing a representable atlas $N \twoheadrightarrow \mathcal{Z}$ for $\mathcal{Z}$, see Lemma 2.2.3. Let $R \subseteq M \times M$ denote the image of the map $M \times_{\mathcal{X}} M \to M \times M$. This is an equivalence relation defining $|\mathcal{X}|_{\mathrm{mod}}$: we have

$$|\mathcal{X}|_{\mathrm{mod}} = M/R.$$

Similarly we have $|\mathcal{Z}|_{\mathrm{mod}} = N/S$, where S is the image of $N \times_{\mathcal{Z}} N \to N \times N$. We claim that that S is the intersection of R and $N \times N$ inside $M \times M$. Indeed, this follows from the pullback square

$$\begin{tikzcd} N \times_{\mathcal{Z}} N \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M \times_{\mathcal{X}} M \arrow[d] \\ N \times N \arrow[r, hook] & M \times M. \end{tikzcd}$$

The fact that this square is a pullback square in turn follows from the pasting law of pullback squares, applied to the following squares:

$$\begin{tikzcd} N \times_{\mathcal{Z}} N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Z} \arrow[r] \arrow[d, "\Delta"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "\Delta"] & M \times_{\mathcal{X}} M \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ N \times N \arrow[r] & \mathcal{Z} \times \mathcal{Z} \arrow[r, hook] & \mathcal{X} \times \mathcal{X} & M \times M, \arrow[l] \end{tikzcd}$$

where the middle square is a pullback square as $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ is a monomorphism. It follows that the closed subspace $N \subseteq M$ is preserved under the equivalence relation R, and in particular its image $|\mathcal{Z}|_{\mathrm{mod}}$ in $|\mathcal{X}|_{\mathrm{mod}}$ is again a closed subspace. This finishes the proof. $\square$

Definition 3.2.2 (Open complement)**.** Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks. We say an open embedding $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ *exhibits $\mathcal{U}$ as the open complement of $\mathcal{Z}$ in $\mathcal{X}$* if at the level of coarse moduli spaces we have

$$|\mathcal{U}|_{\mathrm{mod}} = |\mathcal{X}|_{\mathrm{mod}} \setminus |\mathcal{Z}|_{\mathrm{mod}}.$$

By Corollary 3.1.12 and Proposition 3.2.1, such open complement exists and is unique, and we will denote it by $\mathcal{U} = \mathcal{X} \setminus \mathcal{Z}$. More explicitly, given a smooth manifold M, the subspace

$$\mathcal{U}(M) \subseteq \mathcal{X}(M) = \mathrm{Hom}_{\mathrm{Shv(Diff)}}(M, \mathcal{X})$$

is given by the collection of path components corresponding to those maps $f\colon M \to \mathcal{X}$ such that the underlying map of topological spaces $M = |M|_{\mathrm{mod}} \xrightarrow{|f|_{\mathrm{mod}}} |\mathcal{X}|_{\mathrm{mod}}$ factors through the open complement $|\mathcal{X}|_{\mathrm{mod}} \setminus |\mathcal{Z}|_{\mathrm{mod}}$.

As a consequence, we observe that every open substack of $\mathcal{Z}$ must be of the form $\mathcal{U} \cap \mathcal{Z}$ for some open substack $\mathcal{U}$ of $\mathcal{X}$:

Corollary 3.2.3. *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks. Let $\mathcal{V} \hookrightarrow \mathcal{Z}$ be an open embedding. Then there exists an open embedding $\mathcal{U} \hookrightarrow \mathcal{X}$ whose pullback along $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ is $\mathcal{V} \hookrightarrow \mathcal{Z}$.*

Proof. By Corollary 3.1.12, the open substack $\mathcal{V}$ of $\mathcal{Z}$ corresponds to an open subspace $|\mathcal{V}|_{\mathrm{mod}}$ of $|\mathcal{Z}|_{\mathrm{mod}}$. Since $|\mathcal{Z}|_{\mathrm{mod}} \subseteq |\mathcal{X}|_{\mathrm{mod}}$ carries the subspace topology by Proposition 3.2.1, there exists an open subspace $U \subseteq |\mathcal{X}|_{\mathrm{mod}}$ such that $U \cap |\mathcal{Z}|_{\mathrm{mod}} = |\mathcal{V}|_{\mathrm{mod}}$. The desired substack is then given by $\mathcal{U} = U \times_{|\mathcal{X}|_{\mathrm{mod}}} \mathcal{X} \subseteq \mathcal{X}$. □

3.3 Separated differentiable stacks

A well-behaved class of differentiable stacks are the *separated* differentiable stacks.

Definition 3.3.1 (Separated morphism)**.** A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks on Diff is called *separated* if its diagonal $\Delta_f\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is a proper map of stacks. We say that a stack $\mathcal{X}$ is *separated* if the morphism $\mathcal{X} \to \mathrm{pt}$ is separated. We denote by

$$\mathrm{SepStk} \subseteq \mathrm{DiffStk}$$

the full subcategory of separated differentiable stacks.

Example 3.3.2 (Proper Lie groupoids)**.** For a Lie groupoid $\mathcal{G}$, the classifying stack $\mathbb{B}\mathcal{G}$ is separated if and only if $\mathcal{G}$ is a *proper Lie groupoid* (that is, the map $(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is a proper map, Definition D.13). Indeed, this follows from Corollary 2.4.8, using the observation that the base change of the diagonal $\mathbb{B}\mathcal{G} \to \mathbb{B}\mathcal{G} \times \mathbb{B}\mathcal{G}$ along the representable atlas $\mathcal{G}_0 \times \mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G} \times \mathbb{B}\mathcal{G}$ is the map $(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$.

In particular, for a Lie group G the stack $\mathbb{B}G$ is separated if and only if G is compact. Similarly, the quotient stack $M//G$ of a smooth G-manifold M is separated if and only if G acts properly on M.

Lemma 3.3.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be differentiable stacks. Then any representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is separated.*

Proof. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of differentiable stacks. Let $M \twoheadrightarrow \mathcal{Y}$ be a representable atlas for $\mathcal{Y}$ and let $N := M \times_{\mathcal{Y}} \mathcal{X} \twoheadrightarrow \mathcal{X}$ denote the associated representable atlas for $\mathcal{X}$. Consider the diagram

$$\begin{tikzcd}
N \arrow[r, two heads] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "\Delta"] \\
N \times_M N \arrow[r, two heads] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[d, "\mathrm{pr}_1"] \\
N \arrow[r, two heads] & \mathcal{X}.
\end{tikzcd}$$

As the outer and bottom squares are pullbacks, so is the top square by the pasting law. The map $N \times_M N \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable and an effective epimorphism, since it is a pullback of $N \twoheadrightarrow \mathcal{X}$. As N and M are Hausdorff, the diagonal $\Delta\colon N \to N \times_M N$ is a closed embedding and hence proper. Since properness is a local property, it follows that $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is also proper, showing that f is separated. □

Definition 3.3.4 (Orbifold, cf. [Met03, Proposition 75, Definition 76])**.** A differentiable stack $\mathcal{X}$ is called an *étale stack* if it admits an étale atlas, i.e. there is a representable atlas $p\colon M \twoheadrightarrow \mathcal{X}$ which is a local diffeomorphism, in the sense of Definition 2.4.2. We say that $\mathcal{X}$ is an *orbifold* or a *differentiable Deligne-Mumford stack* if it is a separated étale stack.

Example 3.3.5. Let $\mathcal{G}$ be a proper *étale Lie groupoid*, meaning that the source and target maps $s,t\colon \mathcal{G}_1 \to \mathcal{G}_0$ are étale maps. Then the classifying stack $\mathbb{B}\mathcal{G}$ is a differentiable Deligne-Mumford stack. Conversely, every differentiable Deligne-Mumford stack $\mathcal{X}$ is of this form $\mathbb{B}\mathcal{G}$ for some proper étale Lie groupoid $\mathcal{G}$: take the Čech groupoid of any étale atlas $M \twoheadrightarrow \mathcal{X}$.

3.4 Isotropy groups

In this section, we introduce the *isotropy groups* of a differentiable stack $\mathcal{X}$, and show that a separated map $\mathcal{X} \to \mathcal{Y}$ of differentiable stacks is representable if and only if it induces injections at the level of isotropy groups. In particular, a separated differentiable stack is a smooth manifold if and only if its isotropy groups are trivial.

Definition 3.4.1 (Isotropy group)**.** Let $\mathcal{X}$ be a differentiable stack. A *point* of $\mathcal{X}$ is a map $x\colon \mathrm{pt} \to \mathcal{X}$ in DiffStk; we will sometimes write $x \in \mathcal{X}$ for short. The *isotropy group* G_x of x in $\mathcal{X}$ is defined as the pullback

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{pt} \arrow[d, "x"] \\ \mathrm{pt} \arrow[r, "x"] & \mathcal{X}. \end{tikzcd}$$

As $G_x = \Omega_x(\mathcal{X})$ is the loop object of $\mathcal{X}$, it is automatically a group object in $\mathrm{Shv(Diff)}$. Note that, as a stack, G_x is equivalent to the fiber of the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ over (x, x).

Lemma 3.4.2. *For every point $x\colon \mathrm{pt} \to \mathcal{X}$ of a differentiable stack $\mathcal{X}$, the isotropy group G_x is a Lie group.*

Proof. Let $M \twoheadrightarrow \mathcal{X}$ be a representable atlas for $\mathcal{X}$. Since this atlas is surjective, the point $x\colon \mathrm{pt} \to \mathcal{X}$ may be lifted to a map $x\colon \mathrm{pt} \to M$. By forming the Čech nerve of the map $M \to \mathcal{X}$, we may present $\mathcal{X}$ as the classifying stack of a Lie groupoid $\mathcal{G}$, where $\mathcal{G}_0 = M$ and $\mathcal{G}_1 = M \times_{\mathcal{X}} M$. We claim that G_x is isomorphic to the isotropy group of $\mathcal{G}$ at $x \in M$. Indeed, this follows from the following pullback diagram:

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & t^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{pt} \arrow[d, "x"] \\ s^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{G}_1 \arrow[r, "t"] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d, two heads] \\ \mathrm{pt} \arrow[r, "x"] & M \arrow[r, two heads] & \mathcal{X}. \end{tikzcd}$$

Since the isotropy groups of a Lie groupoid are Lie groups, see for example Proposition D.11, this finishes the proof. $\square$

Lemma 3.4.3. *Let $\mathcal{X}$ be a separated differentiable stack. Then the isotropy group G_x is compact for every point $x\colon \mathrm{pt} \to \mathcal{X}$.*

Proof. By Corollary 2.3.8, we may assume $\mathcal{X} = \mathbb{B}\mathcal{G}$ is the classifying stack of a Lie groupoid $\mathcal{G}$. By Example 3.3.2, $\mathcal{X}$ is separated if and only if the morphism $(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is a proper map of manifolds. In particular, the fiber of this map over the point (x,x) is a compact subspace of $\mathcal{G}_1$. Since this fiber is isomorphic to the isotropy group G_x, this proves the claim. □

Proposition 3.4.4 (Crainic and Moerdijk [CM01, Theorem 1])**.** *A differentiable stack $\mathcal{X}$ is an étale stack, in the sense of Definition 3.3.4, if and only if for every point $x\colon \mathrm{pt} \to \mathcal{X}$ the isotropy group G_x is a discrete group.* □

A Lie groupoid whose isotropy groups are discrete are called *foliation groupoids*. By Proposition 2.3.14, the previous proposition is equivalent to the statement that every foliation groupoid is Morita equivalent to an étale Lie groupoid, which is the way the result is stated in [CM01].

For a separated differentiable stack $\mathcal{X}$, the isotropy groups can detect whether it is a smooth manifold:

Proposition 3.4.5 (cf. [Met03, Proposition 74])**.** *A separated differentiable stack $\mathcal{X}$ is a smooth manifold if and only if all its isotropy groups are trivial.*

Proof. If $\mathcal{X} = M$ is a smooth manifold, it is clear that all isotropy groups are trivial as the pullbacks defining them may be taken in Diff.

Conversely, let $\mathcal{X}$ be a separated differentiable stack with trivial isotropy groups, and let $M \twoheadrightarrow \mathcal{X}$ be a representable atlas for $\mathcal{X}$. Define the smooth manifold $R := M \times_{\mathcal{X}} M$, and consider the canonical map

$$R \to M \times M.$$

Since this map is a base change of the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ and $\mathcal{X}$ is separated, the map $R \to M \times M$ is proper. We claim that it is also injective. Indeed, the fiber over a pair $(x,y) \in M \times M$ is only non-empty if x and y are identified in $\mathcal{X}$, and in that case the fiber is isomorphic to the isotropy group G_x, which is assumed to be trivial. Since any proper injective map of smooth manifolds is a closed embedding, it follows that $R \hookrightarrow M \times M$ determines a closed equivalence relation on M.

Define N as the quotient space M/R, i.e. the topological space obtained from M by quotienting out the equivalence relation R. By Proposition C.6, this is again a smooth manifold, and the quotient map $M \to N$ is a surjective smooth submersion. The universal property of the stack quotient $\mathcal{X} \simeq M//R$ provides a canonical map $\mathcal{X} \to N$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[r, equal] \arrow[d, two heads] & M \arrow[d] \\ \mathcal{X} \arrow[r] & N. \end{tikzcd}$$

We claim that this map $\mathcal{X} \to N$ is an equivalence, finishing the proof. Since the map $M \to N$ is an effective epimorphism by Remark 2.1.6, it will suffice to show that the induced map $\check{C}(M \twoheadrightarrow \mathcal{X}) \to \check{C}(M \twoheadrightarrow N)$ on Čech nerves is an equivalence. But this is clear: since the inclusion $\mathrm{Diff} \hookrightarrow \mathrm{Shv}(\mathrm{Diff})$ preserves finite limits, the Čech nerve of $M \to N$ is given by the Lie groupoid $(R \rightrightarrows M)$, which by definition of R is also the Čech nerve of the map $M \twoheadrightarrow \mathcal{X}$. ◻

Corollary 3.4.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a separated morphism in* DiffStk. *The following conditions are equivalent:*

(1) The morphism f is representable;

(2) For every point $x \in \mathcal{X}$, the induced map on isotropy groups $G_x \to G_{f(x)}$ is injective.

Proof. Assume first that f is representable. Arguing as in Lemma 3.3.3, we see that the diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ of f is a closed embedding, so in particular a monomorphism. But the following diagram shows that the map $G_x \to G_{f(x)}$ is a pullback of Δ_f, and thus also a monomorphism:

$$\begin{tikzcd} G_x \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, hook, "\Delta_f"] & \\ G_{f(x)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y} \arrow[d, "\Delta_{\mathcal{Y}}"] \\ \mathrm{pt} \arrow[r, "{(x,x)}"] & \mathcal{X} \times \mathcal{X} \arrow[r, "f \times f"] & \mathcal{Y} \times \mathcal{Y}. \end{tikzcd}$$

Conversely, assume that f is injective on isotropy groups. We show that f is representable. Letting $M \twoheadrightarrow \mathcal{Y}$ be a representable atlas for $\mathcal{Y}$, it will suffice to show that the pullback $\mathcal{N} := M \times_{\mathcal{Y}} \mathcal{X}$ is a smooth manifold. The isotropy groups of $\mathcal{N}$ are given by the pullback of the isotropy groups of M, $\mathcal{X}$ and $\mathcal{Y}$. But since M has trivial isotropy groups and the map $\mathcal{X} \to \mathcal{Y}$ induces injections on isotropy groups, it follows that $\mathcal{N}$ also has trivial isotropy groups. Since the map $\mathcal{N} \to M$ is separated, being a base change of f, we see that $\mathcal{N}$ is a separated differentiable stack, and thus it is a smooth manifold by Proposition 3.4.5. ◻

Lemma 3.4.7. *Let $f : \mathcal{X} \hookrightarrow \mathcal{Y}$ be a monomorphism of differentiable stacks. Then for every point $x \in \mathcal{X}$, the induced map of Lie groups $G_x \to G_{f(x)}$ is an isomorphism.*

Proof. Since f is a monomorphism, its diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an equivalence, and thus we have the following pullback square of differentiable stacks:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "\Delta"] \\ \mathcal{X}\times\mathcal{X} \arrow[r, "f\times f"] & \mathcal{Y}\times\mathcal{Y}. \end{tikzcd}$$

In particular, f induces isomorphisms between the fibers of the two vertical maps. Since the induced map on fibers is the map $G_x \to G_{f(x)}$, this finishes the proof. □

3.5 Relative tangent bundles and normal bundles

In this section, we introduce the notions of *relative tangent bundle* and *normal bundle* for morphisms of differentiable stacks. We start by defining the tangent groupoid of a Lie groupoid $\mathcal{G}$.

Definition 3.5.1 (Tangent groupoid)**.** Let $\mathcal{G}$ be a Lie groupoid. We define its *tangent groupoid* $T\mathcal{G}$ as the Lie groupoid

$$T\mathcal{G} := \left(T\mathcal{G}_1 \overset{ds}{\underset{dt}{\rightrightarrows}} T\mathcal{G}_0 \right),$$

whose structure maps are the derivatives of the structure maps of $\mathcal{G}$. It comes with a natural map of Lie groupoids $\pi\colon T\mathcal{G} \to \mathcal{G}$.

The morphism $\pi\colon T\mathcal{G} \to \mathcal{G}$ is naturally a *VB-groupoid*, in the sense of Definition 2.5.9. However, it is not a *cartesian* morphism in general, in the sense of Definition 2.5.10, and so does not necessarily define a vector bundle over $\mathbb{B}\mathcal{G}$. This problem goes away when looking at *relative* versions of the construction: the relative tangent bundles, discussed in Subsection 3.5.1, and the normal bundles, discussed in Subsection 3.5.2.

3.5.1 Relative tangent bundles

Given a smooth map $f\colon M \to N$ between smooth manifolds, its *relative tangent bundle* is defined as the kernel of the derivative of f:

$$T_f := \ker(df\colon TM \to f^*TN) \quad \in \quad \mathrm{Vect}(M).$$

This has a direct generalization to the setting of Lie groupoids:

Definition 3.5.2 (Relative tangent groupoid)**.** Let $f\colon \mathcal{H} \to \mathcal{G}$ be a morphism of Lie groupoids. Passing to tangent groupoids gives a groupoid morphism $df\colon T\mathcal{H} \to T\mathcal{G}$ which is fiberwise linear. As it lives over f, it corresponds to a morphism $df\colon T\mathcal{H} \to f^*T\mathcal{G}$ of VB-groupoids over $\mathcal{H}$. We define the *relative tangent groupoid* Tf as the fiberwise kernel of this map:

$$Tf := \ker(df\colon T\mathcal{H} \to f^*T\mathcal{G}).$$

More explicitly, Tf_0 is the relative tangent bundle of the map $f_0\colon \mathcal{H}_0 \to \mathcal{G}_0$, Tf_1 is the relative tangent bundle of $f_1\colon \mathcal{H}_1 \to \mathcal{G}_1$, and the structure maps of Tf are obtained by restricting those from $T\mathcal{H}$. The bundle projections $Tf_0 \to \mathcal{H}_0$ and $Tf_1 \to \mathcal{H}_1$ define a morphism of Lie groupoids $Tf \to \mathcal{H}$, which is a VB-groupoid in the sense of Definition 2.5.9.

Lemma 3.5.3. *Let $f\colon \mathcal{H} \to \mathcal{G}$ be a cartesian morphism of Lie groupoids, in the sense of Definition 2.5.10. Then the VB-groupoid $Tf \to \mathcal{H}$ is also cartesian.*

Proof. We have to show that the commutative squares

$$\begin{tikzcd} Tf_1 \arrow[r, "s"] \arrow[d] & Tf_0 \arrow[d] \\ \mathcal{H}_1 \arrow[r, "s"] & \mathcal{H}_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Tf_1 \arrow[r, "t"] \arrow[d] & Tf_0 \arrow[d] \\ \mathcal{H}_1 \arrow[r, "t"] & \mathcal{H}_0 \end{tikzcd}$$

are pullback squares, which is a special case of Corollary C.13. □

Combining the previous lemma with Lemma 2.5.11 we see that the relative tangent groupoid defines a vector bundle $\mathbb{B}(Tf) \to \mathbb{B}\mathcal{H}$. We will now show that this vector bundle is Morita-invariant, in the sense that it only depends on the underlying map $\mathbb{B}f\colon \mathbb{B}\mathcal{H} \to \mathbb{B}\mathcal{G}$ of differentiable stacks.

Proposition 3.5.4. *Consider a pullback diagram of Lie groupoids*

$$\begin{tikzcd} \mathcal{H}' \arrow[r, two heads, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{H} \arrow[d, "f"] \\ \mathcal{G}' \arrow[r, two heads, "g"] & \mathcal{G}, \end{tikzcd} \tag{II.3.1}$$

and assume that g and h are Morita fibrations, in the sense of Definition 2.3.10. Then the induced map $dh\colon Tf' \to Tf$ is again a Morita fibration.

Proof. As the map $Tf'_0 \to \mathcal{H}'_0$ is the base change of $Tf_0 \to \mathcal{H}_0$ along the surjective submersion $h_0\colon \mathcal{H}'_0 \to \mathcal{H}_0$, the map $Tf'_0 \to Tf_0$ is also a surjective submersion and thus dh is strongly surjective. To see that dh is also fully faithful, consider the following commutative diagram:

$$\begin{tikzcd}
Tf_1' \arrow[rr] \arrow[dr] \arrow[dd] & & Tf_0 \times Tf_0' \arrow[dr] \arrow[dd] & \\
& \mathcal{H}_1' \arrow[rr, crossing over] & & \mathcal{H}_0' \times \mathcal{H}_0' \arrow[dd, "h_0 \times h_0"] \\
Tf_1 \arrow[rr] \arrow[dr] & & Tf_0 \times Tf_0 \arrow[dr] & \\
& \mathcal{H}_1 \arrow[rr, "{(s,t)}"] \arrow[from=uu, crossing over, "h_1"'] & & \mathcal{H}_0 \times \mathcal{H}_0
\end{tikzcd}$$

As $h\colon \mathcal{H}' \to \mathcal{H}$ is fully faithful, the front square is a pullback square in Diff. Since the given square of Lie groupoids is a pullback square, it follows from Corollary C.13 that also the left and right squares are pullback squares in Diff. By the pasting law for pullback squares, it follows that the back square is a pullback square in Diff. This finishes the proof. $\square$

Morita-invariance allows us to define the relative tangent bundle of a morphism of differentiable stacks:

Definition 3.5.5 (Relative tangent bundle)**.** Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism of differentiable stacks. We define its *relative tangent bundle* $T_f \to \mathcal{Y}$ as follows:

- Choose a representable atlas $p\colon M \twoheadrightarrow \mathcal{X}$ and let $\mathcal{G}$ denote the associated Lie groupoid given as the Čech nerve of p. We obtain an equivalence $\mathcal{X} \simeq \mathbb{B}\mathcal{G}$.

- Define an atlas $q\colon N \twoheadrightarrow \mathcal{Y}$ for $\mathcal{Y}$ as in Lemma 2.2.3 by pulling back p along f:

$$\begin{tikzcd}
N \arrow[r, "f_0"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, two heads, "p"] \\
\mathcal{Y} \arrow[r, "f"] & \mathcal{X}.
\end{tikzcd}$$

  Let $\mathcal{H}$ denote the Čech nerve of q, so that $\mathcal{Y} \simeq \mathbb{B}\mathcal{H}$.

- There is a canonical morphism of Lie groupoids $f_\bullet\colon \mathcal{H} \to \mathcal{G}$, which is cartesian by Lemma E.16. We let T_f be the classifying stack of the relative tangent groupoid $Tf_\bullet$:

$$T_f := \mathbb{B}T(f_\bullet).$$

- The structure map $Tf_\bullet \to \mathcal{H}$ induces a morphism of stacks $T_f \to \mathcal{Y}$, which by Lemma 3.5.3 and Lemma 2.5.11 admits a canonical structure of a vector bundle.

Lemma 3.5.6. *Up to equivalence, the definition of the vector bundle $T_f \to \mathcal{Y}$ is independent of the choice of atlas $M \twoheadrightarrow \mathcal{X}$.*

Proof. Let $M \twoheadrightarrow \mathcal{X}$ and $M' \twoheadrightarrow \mathcal{X}$ be two representable atlases of $\mathcal{X}$. Going through the above construction gives two vector bundles $\mathbb{B}(Tf_\bullet) \to \mathcal{Y}$ and $\mathbb{B}(Tf'_\bullet) \to \mathcal{Y}$, which we need to show are isomorphic.

Define $M'' := M \times_{\mathcal{X}} M'$, so that the map $M'' \twoheadrightarrow \mathcal{X}$ is another representable atlas. Letting $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}'$ denote the Čech nerves of the maps $M \twoheadrightarrow \mathcal{X}$, $M' \twoheadrightarrow \mathcal{X}$ and $M'' \twoheadrightarrow \mathcal{X}$, we obtain morphisms of Lie groupoids

$$\mathcal{G} \leftarrow \mathcal{G}'' \to \mathcal{G}'$$

which by Proposition 2.3.14 are Morita fibrations. Applying the above construction we obtain three relative tangent Lie groupoids $Tf_\bullet$, $Tf'_\bullet$ and $Tf''_\bullet$. They are connected via morphisms of Lie groupoids

$$Tf_\bullet \leftarrow Tf''_\bullet \to Tf'_\bullet,$$

which by Proposition 3.5.4 are Morita fibrations and thus induce equivalences on differentiable stacks. It follows that $\mathbb{B}(Tf_\bullet) \simeq \mathbb{B}(Tf'_\bullet)$ finishing the proof. □

Remark 3.5.7. The construction of the relative tangent bundle T_f we have provided is somewhat unsatisfactory, as it is not defined intrinsically in terms of the differentiable stacks but requires a (non-coherent) choice of atlas. A more satisfactory definition would start by giving a fully functorial description of the relative tangent bundle construction at the level of smooth manifolds, and then use descent to extend this construction to all representable morphisms of stacks on Diff. We will not spell out the details of such a construction.

3.5.2 Normal bundles

We will now move to the construction of the normal bundle of a morphism of stacks, which is to a large extent analogous to that of the relative tangent bundle. Recall that the normal bundle N_f of a smooth map $f\colon M \to N$ is defined as the cokernel of the derivative of f:

$$N_f := \operatorname{coker}(df\colon TM \to f^*TN) \quad \in \quad \operatorname{Vect}(M).$$

This admits a direct generalization to the setting of Lie groupoids:

Definition 3.5.8 (Normal groupoid)**.** Let $f\colon \mathcal{H} \to \mathcal{G}$ be a morphism of Lie groupoids, and consider its derivative $df\colon T\mathcal{H} \to T\mathcal{G}$ on tangent groupoids, which as before we may regard as a morphism of VB-groupoids $T\mathcal{H} \to f^*T\mathcal{G}$ over $\mathcal{H}$. We define the *normal groupoid* Nf of f as the fiberwise cokernel of this map:

$$Nf := \operatorname{coker}(df\colon T\mathcal{H} \to f^*T\mathcal{G}).$$

This means that Nf_0 is the normal bundle of the map $f_0\colon \mathcal{H}_0 \to \mathcal{G}_0$, Nf_1 is the normal bundle of the map $f_1\colon \mathcal{H}_1 \to \mathcal{G}_1$, and the structure maps of Nf are obtained as a quotient of the structure maps of $f^*T\mathcal{G}$. The structure maps of the normal bundles determine a VB-groupoid $Nf \to \mathcal{H}$.

Lemma 3.5.9. *If $f\colon \mathcal{H} \to \mathcal{G}$ is a cartesian morphism of Lie groupoids, then also the VB-groupoid $Nf \to \mathcal{H}$ is cartesian.*

Proof. Just like in Lemma 3.5.3, this follows from Corollary C.13. □

Remark 3.5.10 (Linearization of a Lie groupoid)**.** Let $\mathcal{G}$ be a Lie groupoid and let $S \subseteq \mathcal{G}_0$ be a saturated submanifold, in the sense that S is closed under the $\mathcal{G}$-action, Definition D.12. Define the Lie groupoid $\mathcal{G}_S$ over S as the restriction of $\mathcal{G}$ to S:

$$(\mathcal{G}_S)_1 = \{g \in \mathcal{G}_1 \mid s(g), t(g) \in S\} = s^{-1}(S) = t^{-1}(S),$$

where the last two equalities use the fact that S is saturated. There is an inclusion of Lie groupoids $i\colon \mathcal{G}_S \hookrightarrow \mathcal{G}$, and the fact that S is saturated guarantees that i is a cartesian morphism. In this case, the normal groupoid Ni of i is known in the literature as the *linearization of $\mathcal{G}$ at S* and is denoted $N_S\mathcal{G}$. Its space of objects is $N_S = \operatorname{coker}(TS \to T\mathcal{G}_0|_S)$, the normal bundle of S inside $\mathcal{G}_0$.

Example 3.5.11. Assume that $x \in \mathcal{G}_0$ is a fixed point of $\mathcal{G}$, and let $S = \{x\}$. Then $\mathcal{N}_S\mathcal{G}$ is the action groupoid $G_x \ltimes T_x\mathcal{G}_0$ of the isotropy group action on the tangent space of $\mathcal{G}_0$ at x.

Example 3.5.12. Assume that $\mathcal{G} = G \ltimes M$ is the action groupoid for some Lie group G and some smooth G-manifold M. Let $S = \mathcal{O}_x$ denote the orbit of a point $x \in M$ and let $N_x := T_xM/T_xS$. Then $\mathcal{N}_S\mathcal{G}$ is isomorphic to the action groupoid $G \ltimes (G \times_{G_x} N_x)$.

Morita equivalent Lie groupoids have Morita equivalent linearizations:

Proposition 3.5.13 ([HF19, Proposition 6.4.1])**.** *Consider a pullback of Lie groupoids as in* (II.3.1) *and assume that g and h are Morita fibrations. Then the induced morphism of Lie groupoids $Nf' \to Nf$ is a Morita fibration.*

Proof. The proof is entirely analogous to that of Proposition 3.5.4 and we will leave it to the reader. □

Definition 3.5.14 (Normal bundle)**.** Let $i\colon \mathcal{S} \hookrightarrow \mathcal{X}$ be an embedding of differentiable stacks. We define a *normal bundle* $\mathcal{N}_i \to \mathcal{S}$ as follows:

- Choose a representable atlas $p\colon M \twoheadrightarrow \mathcal{X}$ and let $\mathcal{G}$ denote the associated Lie groupoid given as the Čech nerve of p. We obtain an equivalence $\mathcal{X} \simeq \mathbb{B}\mathcal{G}$.
- Define the submanifold $S \subseteq M$ via the pullback square

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, two heads, "p_S"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, two heads, "p"] \\ \mathcal{S} \arrow[r, hook, "i"] & \mathcal{X}. \end{tikzcd}$$

  This determines a Lie groupoid $\mathcal{H} = \mathcal{G}_S \overset{i_\bullet}{\hookrightarrow} \mathcal{G}$ presenting $\mathcal{S}$.

- We define the differentiable stack $\mathcal{N}_i$ as the classifying stack of the normal groupoid $Ni_\bullet = \mathcal{N}_S\mathcal{G}$. The structure map $\mathcal{N}_S\mathcal{G} \to \mathcal{G}_S$ is cartesian by Lemma 3.5.9 and thus induces a vector bundle of differentiable stacks $\mathcal{N}i \to \mathcal{S}$ by Lemma 2.5.11.

Lemma 3.5.15. *Up to equivalence, the definition of $\mathcal{N}_i \to \mathcal{S}$ is independent of the choice of atlas.*

Proof. This follows from Proposition 3.5.13. The proof is entirely analogous to that of Lemma 3.5.6 and will be omitted. □

3.5.3 Compatibilities

We record various compatibilities between normal bundles and relative tangent bundles.

Lemma 3.5.16 (Normal bundle of composite embedding)**.** *Given embeddings $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ and $j\colon \mathcal{X} \hookrightarrow \mathcal{Y}$, there is a canonical short exact sequence of vector bundles over $\mathcal{Z}$:*

$$\mathcal{N}_i \to \mathcal{N}_{ji} \to i^*\mathcal{N}_j.$$

Proof. When $\mathcal{Z}$, $\mathcal{X}$ and $\mathcal{Y}$ are smooth manifolds, this is immediate from the definitions using the second isomorphism theorem for groups. This directly implies the analogous statement for cartesian morphisms of Lie groupoids, which then gives the desired result by picking an atlas for $\mathcal{Y}$ and choosing the atlases for $\mathcal{Z}$ and $\mathcal{X}$ via pullback. □

Lemma 3.5.17 (Normal bundle of pullback)**.** *Consider a pullback diagram of differentiable stacks*

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "g"] \arrow[d, "i'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Z} \arrow[d, "i"] \\ \mathcal{X}' \arrow[r, "f"] & \mathcal{X}, \end{tikzcd}$$

where i and i' are embeddings. Then there is a preferred isomorphism $\mathcal{N}_{i'} \cong g^*\mathcal{N}_i$ of vector bundles over $\mathcal{Z}'$.

Proof. If all four of the stacks are smooth manifolds, this is Corollary C.13. By choosing a representable atlas for $\mathcal{X}$ and choosing the atlases for the other three stacks via pullback, this directly implies the general claim. □

Lemma 3.5.18 (Normal bundle of isotopic maps)**.** *Let $i\colon \mathcal{Z}\times\mathbb{R} \to \mathcal{X}$ be a map of stacks such that the map $(i,\mathrm{id})\colon \mathcal{Z}\times\mathbb{R} \to \mathcal{X}\times\mathbb{R}$ is an embedding. Then there is an isomorphism of vector bundles*

$$\mathcal{N}_{i_0} \cong \mathcal{N}_{i_1} \quad \in \quad \mathrm{Vect}(\mathcal{Z}).$$

Proof. Let $\mathcal{N}_{(i,\mathrm{id})} \to \mathcal{Z}\times\mathbb{R}$ denote the normal bundle of (i,id). As the embeddings maps $i_0, i_1\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ are base changes of the map (i,id) along the inclusions $\mathcal{X}\times\{0\} \hookrightarrow \mathcal{X}\times\mathbb{R}$ and $\mathcal{X}\times\{1\} \hookrightarrow \mathcal{X}\times\mathbb{R}$, Lemma 3.5.17 provides isomorphisms of vector bundles

$$\mathcal{N}_{i_0} \cong \mathcal{N}_{(i,\mathrm{id})}|_0 \qquad \text{and} \qquad \mathcal{N}_{i_1} \cong \mathcal{N}_{(i,\mathrm{id})}|_1.$$

But by Lemma 2.5.4, the vector bundle $\mathcal{N}_{(i,\mathrm{id})}$ over $\mathcal{Z}\times\mathbb{R}$ is isomorphic to $\mathcal{E}\times\mathbb{R}$ for some vector bundle $\mathcal{E}$ over $\mathcal{Z}$, and thus $\mathcal{N}_{i_0} \cong \mathcal{E} \cong \mathcal{N}_{i_1}$. □

Proposition 3.5.19. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable submersion of differentiable stacks. Then there is an isomorphism $T_f \cong N_{\Delta_f}$ of vector bundles over $\mathcal{Y}$, where $\Delta_f\colon \mathcal{Y} \to \mathcal{Y}\times_{\mathcal{X}}\mathcal{Y}$ is the diagonal of f.*

Proof. We prove the lemma at progressing levels of generality.

Step 1: We first prove the claim when $\mathcal{X}$ is a point, so that $\mathcal{Y} = M$ is a smooth manifold and $T_f = TM$ is the tangent bundle of M. By Lemma C.12, the derivatives of the two projections $\mathrm{pr}_1, \mathrm{pr}_2\colon M\times M \to M$ induce an isomorphism of vector bundles $T(M\times M) \cong TM\times TM$ over $M\times M$. Under this isomorphism, the differential $d\Delta_M\colon TM \to T(M\times M)$ corresponds to the diagonal $\Delta_{TM}\colon TM \to TM\times TM$. Regarding this as a map of vector bundles over M given fiberwise by the diagonal $\Delta_{T_xM}\colon T_xM \to T_xM\times T_xM$, its cokernel in $\mathrm{Vect}(M)$ admits an explicit isomorphism to TM given by the composite $TM \xrightarrow{(1,-1)} TM\times TM \twoheadrightarrow \mathrm{coker}(d\Delta_M)$. Since this cokernel is N_{Δ_M}, this finishes the proof.

Step 2: Next we show the claim when $\mathcal{X} = N$ is a smooth manifold, so that $f\colon M \to N$ is a smooth submersion between smooth manifolds. Observe that the isomorphisms from Step 1 give rise to a commutative square of vector bundles over M:

$$\begin{tikzcd} TM \arrow[r] \arrow[d, "\cong"'] & f^*T_N \arrow[d, "\cong"] \\ N_{\Delta_M} \arrow[r] & f^*N_{\Delta_N}. \end{tikzcd}$$

Since T_f is defined as the fiber of the top map, it suffices to show that N_{Δ_f} is the fiber of the bottom map. To this end, consider the following diagram:

$$\begin{tikzcd} M \arrow[r, "\Delta_f"] \arrow[dr, "\Delta_M"'] & M \times_N M \arrow[r, "g"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\lrcorner"] & N \arrow[d, hook, "\Delta_N"] \\ & M \times M \arrow[r, "f \times f"] & N \times N, \end{tikzcd}$$

where i is the inclusion. By Lemma 3.5.16, there is the following short exact sequence of vector bundles over M:

$$N_{\Delta_f} \to N_{\Delta_M} \to \Delta_f^* N_i.$$

But by Lemma 3.5.17, the vector bundel N_i is isomorphic to $g^*N_{\Delta_N}$, and since $f = g \circ \Delta_f$ it follows that $\Delta_f^* N_i \cong f^*N_{\Delta_N}$. This finishes the proof of Step 2.

Step 3: We now show the general statement. By choosing an atlas for $\mathcal{X}$, we may assume that $\mathcal{X} = \mathbb{B}\mathcal{G}$ is the classifying stack of a Lie groupoid. By pulling back this atlas to $\mathcal{Y}$ we may similarly assume that $\mathcal{Y} = \mathbb{B}\mathcal{H}$ and that f is induced by a cartesian morphism $f\colon \mathcal{H} \to \mathcal{G}$ of Lie groupoids. It will thus suffice to show that the relative tangent groupoid of f is isomorphic to the normal groupoid of its diagonal. This follows from Step 2. $\square$

Lemma 3.5.20. *Consider a commutative diagram of differentiable stacks*

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "i"] \arrow[dr, "g"'] & \mathcal{Y} \arrow[d, "f"] \\ & \mathcal{X}, \end{tikzcd}$$

where f and g are representable submersions and i is an embedding. Then there is a short exact sequence of vector bundles

$$T_g \to i^*T_f \to \mathcal{N}_i.$$

Proof. As in the previous proofs, we may reduce to the case where the three stacks are smooth manifolds, say S, Y and X. In this case the statement follows from the fact that forming vertical kernels commutes with forming horizontal cokernels in the following diagram of vector bundles over S:

$$\begin{tikzcd} T_g \arrow[r, hook] \arrow[d, hook] & i^*T_f \arrow[r, two heads] \arrow[d, hook] & N_i \arrow[d, equal] \\ TS \arrow[r, hook] \arrow[d, two heads] & i^*TY \arrow[r, two heads] \arrow[d, two heads] & N_i \arrow[d] \\ g^*TX \arrow[r, equal] & i^*f^*TX \arrow[r] & 0. \end{tikzcd}$$

□

3.6 Tubular neighborhoods

In this section, we introduce the notion of a *tubular neighborhood* of an embedding of differentiable stacks. We further show that under separatedness assumptions such tubular neighborhoods always exist.

Definition 3.6.1 (Tubular neighborhood)**.** Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be an embedding of differentiable stacks. A *tubular neighborhood of* $\mathcal{Z}$ *in* $\mathcal{X}$ consists of the data of a commutative diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, hook', "i"'] \arrow[d, hook] \arrow[dr, hook, "s_0"] & \\ \mathcal{X} & \mathcal{U} \arrow[l, hook', "j"'] \arrow[r, hook, "j'"] & \mathcal{N}_i \end{tikzcd}$$

where the maps j and j' are open embeddings of differentiable stacks, and where s_0 is the zero section of the bundle projection $\mathcal{N}_i \to \mathcal{Z}$ of the normal bundle of i.

We might sometimes refer to the data of a tubular neighborhood by just mentioning the open substack $\mathcal{U}$ of $\mathcal{X}$, leaving the rest of the data implicit.

The main input for the existence of tubular neighborhoods is the *linearization theorem for proper Lie groupoids*, due to Zung [Zun06] and Weinstein [Wei02] and later refined and clarified by [CS13; PPT14; HF18].

Definition 3.6.2. Let $\mathcal{G}$ be a Lie groupoid and let $S \subseteq M$ a saturated submanifold. We say that $\mathcal{G}$ is *linearizable around* S if there exist open sets $S \subseteq U \subseteq M$ and $S \subseteq V \subseteq \mathcal{N}_S$ and an isomorphism of Lie groupoids

$$\mathcal{G}|_U \cong \mathcal{N}_S(\mathcal{G})|_V$$

which is the identity on $\mathcal{G}_S$. We say that $\mathcal{G}$ is *linearizable* if it is linearizable around any saturated submanifold.

The notation $\mathcal{G}|_U$ in this definition stands for the *restriction* of the Lie groupoid to the open subspace U, see Example D.7.

Theorem 3.6.3 (Linearization theorem for proper Lie groupoids, [Wei02; Zun06; CS13; PPT14], [HF18, Corollary 5.3.3])**.** *Proper Lie groupoids are linearizable.*

We will use the linearization theorem to show the existence of tubular neighborhoods. We need to following auxiliary lemma:

Lemma 3.6.4. *Let $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows M)$ be a Lie groupoid, let $U \subseteq M$ be an open subspace and let $\mathcal{G}|_U$ be the restriction of $\mathcal{G}$ to U, as in Example D.7. Then the inclusion $\mathcal{G}|_U \hookrightarrow \mathcal{G}$ of Lie groupoids induces an open embedding*

$$\mathbb{B}(\mathcal{G}|_U) \hookrightarrow \mathbb{B}\mathcal{G}$$

at the level of classifying stacks.

Proof. Let $U' := \mathcal{G} \cdot U \subseteq M$ be the *$\mathcal{G}$-saturation of U*, that is, the union of all orbits of $\mathcal{G}$ intersecting U non-trivially. Then U' defines an open subspace of the coarse moduli space $|\mathbb{B}\mathcal{G}|_{\mathrm{mod}} = \mathrm{coeq}(\mathcal{G}_1 \rightrightarrows M)$ of $\mathbb{B}\mathcal{G}$, and hence defines an open substack $\mathcal{U} \subseteq \mathbb{B}\mathcal{G}$, fitting in a pullback square as follows:

$$\begin{tikzcd} U' \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, two heads] \\ \mathcal{U} \arrow[r, hook] & \mathbb{B}\mathcal{G}. \end{tikzcd}$$

We claim that the composite $U \hookrightarrow U' \twoheadrightarrow \mathcal{U}$ is a representable atlas for $\mathcal{U}$. By Lemma 2.2.6 it suffices to check this after pulling back along the atlas $U' \twoheadrightarrow \mathcal{U}$, where this follows from the fact that every $x \in U'$ is of the form gu for some $g \in \mathcal{G}_1$ and $u \in U$. We conclude that $\mathcal{U}$ is equivalent to the classifying stack of the Čech nerve of the map $U \twoheadrightarrow \mathcal{U}$, which is precisely the restricted Lie groupoid $\mathcal{G}|_U$. As the resulting open embedding $\mathbb{B}(\mathcal{G}|_U) \hookrightarrow \mathbb{B}\mathcal{G}$ agrees with the map induced by the inclusion $\mathcal{G}|_U \hookrightarrow \mathcal{G}$ of Lie groupoids, this finishes the proof. $\square$

Corollary 3.6.5 (Existence of tubular neighborhoods)**.** *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be an embedding of separated stacks. Then i admits a tubular neighborhood.*

Proof. By choosing a representable atlas for $\mathcal{X}$, we may assume that $\mathcal{X} = \mathbb{B}\mathcal{G}$ is the classifying stack of a Lie groupoid $\mathcal{G}$. As $\mathcal{X}$ is separated, $\mathcal{G}$ is proper, see Example 3.3.2. The embedding i determines a saturated embedded submanifold $S \subseteq \mathcal{G}_0$. By the linearization theorem 3.6.3 there exist open neighborhoods $S \subseteq U \subseteq \mathcal{G}_0$ and $S \subseteq V \subseteq \mathcal{N}_S$ together with an an isomorphism of Lie groupoids $\mathcal{G}|_U \cong \mathcal{N}_S(\mathcal{G})|_V$. We define

$$\mathcal{U} := \mathbb{B}(\mathcal{G}|_U) \hookrightarrow \mathbb{B}\mathcal{G} = \mathcal{X} \quad \text{and} \quad \mathcal{V} := \mathbb{B}(\mathcal{N}_S(\mathcal{G})|_V) \hookrightarrow \mathbb{B}\mathcal{N}_S(\mathcal{G}) = \mathcal{N}_i.$$

By Lemma 3.6.4 these are open neighborhoods of $\mathcal{Z} = \mathbb{B}\mathcal{G}_S$. There is an equivalence $\mathcal{U} \simeq \mathcal{V}$ which is the identity on $\mathcal{Z}$. This finishes the proof. □

We record two useful corollaries of the existence of tubular neighborhoods.

Corollary 3.6.6. *Every embedding $i \colon \mathcal{Z} \hookrightarrow \mathcal{X}$ of separated differentiable stacks factors as a composite*

$$\mathcal{Z} \overset{i'}{\hookrightarrow} \mathcal{U} \overset{j}{\to} \mathcal{X}$$

of a closed embedding i' followed by an open embedding j, where the map i' admits a retraction $f \colon \mathcal{U} \to \mathcal{Z}$. □

Corollary 3.6.7 (Extension of representable submersions)**.** *Let $i \colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be an embedding of separated differentiable stacks. For every representable submersion $\mathcal{Y} \to \mathcal{Z}$, there exists a pullback square*

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y}' \arrow[d, "f'"] \\ \mathcal{Z} \arrow[r, hook, "i"] & \mathcal{X} \end{tikzcd}$$

where f' is also a representable submersion.

Proof. Choosing a factorization of i as in Corollary 3.6.6, we may define $\mathcal{Y}' := \mathcal{U} \times_{\mathcal{Z}} \mathcal{Y}$, and define f' as the composite $\mathcal{U} \times_{\mathcal{Z}} \mathcal{Y} \xrightarrow{\mathrm{pr}_{\mathcal{U}}} \mathcal{U} \overset{j}{\hookrightarrow} \mathcal{X}$. □

Remark 3.6.8 (Linearization of representable submersions)**.** It is possible to prove the following variant of Corollary 3.6.5: for every pullback diagram

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, hook, "i'"] \arrow[d, "f|_{\mathcal{Z}'}"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X}' \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, hook, "i"] & \mathcal{X} \end{tikzcd}$$

where i and i' are embeddings and f is a representable submersion, one may choose tubular neighbhorhoods $\mathcal{Z} \subseteq \mathcal{U} \subseteq \mathcal{X}$, $\mathcal{Z} \subseteq \mathcal{V} \subseteq \mathcal{N}_i$ and $\mathcal{Z}' \subseteq \mathcal{U}' \subseteq \mathcal{X}'$, $\mathcal{Z}' \subseteq \mathcal{V}' \subseteq \mathcal{N}_{i'}$ fitting in a commutative diagram as follows:

$$\begin{tikzcd} \mathcal{N}_{i'} \arrow[d, "df"'] & \mathcal{V}' \arrow[l, hook'] \arrow[r, "\simeq"] & \mathcal{U}' \arrow[r, hook] & \mathcal{X}' \arrow[d, "f"] \\ \mathcal{N}_i & \mathcal{V} \arrow[l, hook'] \arrow[r, "\simeq"] & \mathcal{U} \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

Just as in the proof of Corollary 3.6.5, this can be deduced from the analogous linearization result for proper Lie groupoids, which was proved by Hoyo and Fernandes [HF19, Theorem 4.2.3].

3.6.1 Relative tubular neighborhoods

The notion of tubular neighborhood admits a *relative* version, in which the embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ lives over some base stack $\mathcal{S}$.

Definition 3.6.9 (Relative tubular neighborhood)**.** Consider a commutative diagram of differentiable stacks

$$\begin{tikzcd} \mathcal{Z} \arrow[rr, "i"] \arrow[dr, "g"'] & & \mathcal{X} \arrow[dl, "f"] \\ & \mathcal{S}, & \end{tikzcd}$$

where the morphisms $g_{\mathcal{Z}}$ and $g_{\mathcal{X}}$ are representable submersions and the morphism i is an embedding. A *tubular neighborhood of $\mathcal{Z}$ in $\mathcal{X}$ relative to $\mathcal{S}$* is a commutative diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, hook', "i"'] \arrow[d, hook] \arrow[dr, hook, "s_0"] & \\ \mathcal{X} \arrow[dr, "f"'] & \mathcal{U} \arrow[l, hook', "j"'] \arrow[r, hook, "j'"] \arrow[d] & \mathcal{N}_i \arrow[dl, "g'"] \\ & \mathcal{S}, & \end{tikzcd}$$

where the maps j and j' are open embeddings of differentiable stacks, and g' is the composite of the bundle projection $\mathcal{N}_i \to \mathcal{Z}$ followed by the map $g\colon \mathcal{Z} \to \mathcal{S}$.

Relative tubular neighborhoods always exist if the stack $\mathcal{X}$ is separated: as in Corollary 3.6.5, one may reduce to the analogous statement in the context of Lie groupoids, where the proof strategy of [HF18, Corollary 5.3.3] applies. In the case that $\mathcal{S}$, $\mathcal{Z}$ and $\mathcal{X}$ are smooth manifold, this has been proved by [Mat12]. Unfortunately, we were unable to find a reference for the general relative statement for differentiable stacks. We will not give a complete proof either, but we will give a detailed proof sketch indicating the main ingredients of a proof. We thank Shachar Carmeli, Florian Naef and Maarten Mol for useful discussions.

Proposition 3.6.10 (Existence of relative tubular neighborhoods)**.** *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be an embedding of differentiable stacks over some base stack $\mathcal{S}$, as in Definition 3.6.9, and assume that $\mathcal{X}$ is separated. Then i admits a tubular neighborhood relative to $\mathcal{S}$.*

Detailed proof sketch. Broadly speaking, the proof proceeds just like the ordinary proof of the existence of tubular neighborhoods for differentiable manifolds: one chooses a Riemannian metric and shows that the resulting exponential map restricts to a diffeomorphism on a small open neighborhood inside the normal bundle of the embedding. The subtlety lies in making precise the notions of 'Riemannian metric' and 'exponential map' in the context of differentiable stacks $\mathcal{X}$ relative to some base stack $\mathcal{S}$. We will indicate one possible approach, which follows the usual two-step procedure of reducing to the setting of smooth manifolds.

Step 1: One first considers the case where the base stack $\mathcal{S}$ is a smooth manifold S, so that also $\mathcal{Z} = Z$ and $\mathcal{X} = X$ are smooth manifolds. By Lemma 3.5.20, the normal bundle of i fits in a short exact sequence of vector bundles

$$T_g \hookrightarrow i^*T_f \to N_i$$

over Z. Observe that the Lie bracket on TX restricts to T_f by naturality of the Lie bracket, so that T_f becomes a Lie algebroid over X with the inclusion $T_f \hookrightarrow TX$ serving as the anchor map. If we choose a Riemannian metric on X, we may restrict it to the relative tangent bundle T_f, thus turning T_f into a *Riemannian Lie algebroid*. Generalizing the usual notion of connections on smooth manifolds, there is a notion of a *connection* on a Lie algebroid, and one can show that every Riemannian Lie algebroid admits a unique connection which is both *metric* and *torsion-free*; this is known as the *Levi-Civita-connection*, see [Bou11, Section 3.1]. One obtains a corresponding notion of *geodesics* with respect to this connection. In the case of the relative tangent bundle T_f, it follows from [Bou11, Proposition 3.2] that the geodesics with respect to the Levi-Civita connection on T_f are precisely those curves $\gamma\colon [t_0, t_1] \to X$ such that the composite $f \circ \gamma\colon [t_0, t_1] \to S$ is constant, say with value $s \in S$, and such that the resulting curve $\gamma\colon [t_0, t_1] \to X_s = f^{-1}(s)$ into the fiber of f is a geodesic when X_s is equipped with the Riemannian metric inherited from X. The usual construction of the exponential map carries over to this relative setting and produces a *relative exponential map*

$$\exp_f\colon \mathcal{E}_f \to X,$$

where $\mathcal{E}_f \subseteq T_f$ is the open neighborhood of the zero section on which the relevant geodesics are defined. From the fact that the relative geodesics live in a single fiber, it follows the map $\exp_f$ lives over S, in the sense that $f(\exp(x, v)) = f(x)$ for every relative tangent vector v at $x \in X$.

Since i^*T_f admits a Riemannian metric, we can embed N_i into i^*T_f as the orthogonal complement of T_g. We may restrict the relative exponential map to the intersection $N_i \cap$

$i^*\mathcal{E}_f \subseteq i^*T_f$, and a calculation of the derivative of this restriction shows that it is a local diffeomorphism. An adaptation of the standard argument for Riemannian manifolds shows that this map becomes an actual diffeomorphism when restricted to a sufficiently small open neighborhood of the zero section in N_i, see for example [Lan95, Theorem IV.5.1]. This finishes Step 1.

Step 2: In the general case, we may choose an atlas for $\mathcal{S}$ and equip $\mathcal{Z}$ and $\mathcal{X}$ with the atlases obtained by pullback, as in Lemma 2.2.3. This means that we may represent $\mathcal{S}$, $\mathcal{Z}$ and $\mathcal{X}$ by Lie groupoids $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{G}$, and the maps i, f and g correspond to cartesian morphisms of Lie groupoids $i\colon \mathcal{H} \hookrightarrow \mathcal{G}$, $g\colon \mathcal{Z} \to \mathcal{K}$ and $f\colon \mathcal{G} \to \mathcal{K}$.

Our goal is to reduce the claim to the statement from Step 1. We will use the proof strategy from [HF18, Theorem 5.3.1]. Since $\mathcal{X}$ is separated, the corresponding Lie groupoid $\mathcal{G}$ is proper by Example 3.3.2. It was shown by Hoyo and Fernandes [HF18, Theorem 4.3.4] that every proper Lie groupoid $\mathcal{G}$ admits a *simplicial Riemannian metric*[1]: a collection of Riemannian metrics on each of the smooth manifolds $\mathcal{G}_n$ for $[n] \in \Delta$ such that for every injective map $\varphi\colon [n] \to [m]$ in Δ the induced map $\mathcal{G}_\varphi\colon \mathcal{G}_m \to \mathcal{G}_n$ is a Riemannian submersion. We may restrict these Riemannian metrics to the relative tangent bundles $Tf_n \to \mathcal{G}_n$. By Lemma 3.5.3, the relative tangent groupoid $Tf_\bullet \to \mathcal{G}$ is cartesian, so that for every map $[n] \to [m]$ in Δ we obtain a pullback square

$$\begin{tikzcd} Tf_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{G}_m \arrow[d] \\ Tf_n \arrow[r] & \mathcal{G}_n. \end{tikzcd}$$

Due to the uniqueness of the geodesics, it follows that the relative exponential maps $\exp_{f_n}\colon \mathcal{E}_{f_n} \to \mathcal{G}_n$ produced in Step 1 are all compatible: for every injection $\varphi\colon [n] \to [m]$ in Δ, the square

$$\begin{tikzcd} \mathcal{E}_{f_m} \arrow[r, "\exp_{f_m}"] \arrow[d, "d\mathcal{G}_\varphi"'] & \mathcal{G}_m \arrow[d, "\mathcal{G}_\varphi"] \\ \mathcal{E}_{f_n} \arrow[r, "\exp_{f_n}"] & \mathcal{G}_n \end{tikzcd}$$

commutes. As in [HF18, Theorem 5.3.1], one can show that there exist compatible open neighborhood $U_n \subseteq N_{i_n} \cap i_n^*\mathcal{E}_{f_n}$ of the zero section on which the relative exponential map $\exp_{f_n}$ restricts to a diffeomorphism, where the word 'compatible' means that the U_n form a simplicial diagram in Diff constituting a Lie groupoid. Passing to classifying stacks then produces the desired relative tubular neighborhood. □

[1] In fact, del Hoyo and Fernandes restrict attention to *2-metrics*. We will work with simplicial metrics for ease of exposition.

3.7 Local structure of separated differentiable stacks

An important consequence of the linearization theorem for proper Lie groupoids mentioned above is the simple local structure of separated differentiable stacks: locally they look like a quotient stack $V//G$ for a compact Lie group G and an orthogonal G-representation V. We shall discuss this result and deduce various consequences. The starting point is the following theorem on the local structure of proper Lie groupoids:

Theorem 3.7.1 ([CS13, Corollary 3.9], [PPT14, Corollary 3.11]). *Let $\mathcal{G}$ be a proper Lie groupoid over M, let $x \in M$ and let $\mathcal{O} = \mathcal{O}_x \subseteq M$ be the orbit of x. Then there exists an open neighborhood $\mathcal{O} \subseteq U \subseteq M$ such that the restriction $\mathcal{G}|_U$ is Morita equivalent to the action groupoid $G_x \ltimes \mathcal{N}_x$, where $\mathcal{N}_x = T_xM/T_x(\mathcal{O})$ is the normal tangent space at x of the orbit $\mathcal{O}$ inside M. Under this Morita equivalence, the orbit $\mathcal{O} \subseteq M$ corresponds to the point $0 \in \mathcal{N}_x$.* □

We deduce from this the local structure theorem for separated differentiable stacks:

Theorem 3.7.2 (Local structure of separated differentiable stacks). *Let $\mathcal{X}$ be a separated differentiable stack and let $x \in \mathcal{X}$ be a point of $\mathcal{X}$. Then there exists an open substack $\mathcal{U} \subseteq \mathcal{X}$ containing x, a G_x-representation V, and an equivalence*

$$\mathcal{U} \simeq V//G_x$$

sending the point $x \in \mathcal{U}$ to the point $0 \in V//G_x$.

Proof. Choose an atlas $f\colon M \twoheadrightarrow \mathcal{X}$, and let $\mathcal{G} := \check{C}(f)$ denote the associated Čech groupoid. The point x of $\mathcal{X}$ can be lifted to a point x of M. Let $\mathcal{O} = \mathcal{O}_x$ denote the orbit of x in M, which is an embedded submanifold by Proposition D.11, and let $\mathcal{N}_x := T_xM/T_x\mathcal{O}$ denote the normal space x of $\mathcal{O}$ inside M. By Theorem 3.7.1, there exist an open neighbhorhood U of $\mathcal{O}$ in M such that the restricted Lie groupoid $\mathcal{G}|_U$ is Morita equivalent to the action groupoid $G_x \ltimes \mathcal{N}_x$ and such that the orbit $\mathcal{O}$ corresponds to the zero-vector $0 \in \mathcal{N}_x$ under this Morita equivalence. By Proposition 2.3.14, this Morita equivalence induces an equivalence of stacks

$$\mathbb{B}(\mathcal{G}|_U) \simeq \mathbb{B}(G_x \ltimes \mathcal{N}_x) = \mathcal{N}_x//G_x$$

which sends $x \in \mathbb{B}(\mathcal{G}|_U)$ to $0 \in \mathcal{N}_x//G_x$. By Lemma 3.6.4, the inclusion $\mathcal{G}|_U \hookrightarrow \mathcal{G}$ then induces an open embedding

$$\mathcal{N}_x//G_x \simeq \mathbb{B}(\mathcal{G}|_U) \hookrightarrow \mathbb{B}\mathcal{G} \simeq \mathcal{X},$$

finishing the proof. □

Using the structure theorem, one can generalize various results about the differential geometry of smooth G-manifolds for a compact Lie group G to the setting of separated differentiable stacks.

3.7.1 Factorization of proper maps

We will show that every proper map of separated differentiable stacks can locally be factored as a closed embedding followed by a proper representable submersion.

Proposition 3.7.3. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a proper map between separated differentiable stacks. Around every point $y \in \mathcal{Y}$ there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{Y}$ such that the restriction $p_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{U}$ factors as*

$$\mathcal{X}_{\mathcal{U}} \overset{i}{\hookrightarrow} S^{\mathcal{E}} \overset{\pi}{\to} \mathcal{U},$$

where $\pi\colon \mathcal{E} \to \mathcal{U}$ is a vector bundle and where i is a closed embedding factoring through $\mathcal{E} \hookrightarrow S^{\mathcal{E}}$.

Proof. Let $G = G_y$ denote the isotropy group of $\mathcal{Y}$ at y. By Theorem 3.7.2, there is an open neighborhood $\mathcal{U}'$ of y in $\mathcal{Y}$ of the form $V//G$ for some orthogonal G-representation V. The restriction $\mathcal{X}_{\mathcal{U}'}$ of $\mathcal{X}$ is then of the form $M//G$ for some G-manifold M, and the map p is induced by a G-equivariant proper map $q\colon M \to V$. Let $D \subseteq V$ denote the open unit disk inside V. Since the closure of D inside V is compact and the map q is proper, it follows that the closure of the preimage $p'^{-1}(D) \subseteq M$ is also compact. Hence by [Bre72, Theorem VI.4.1], there exists a G-representation W and a G-equivariant closed embedding $q^{-1}(D) \hookrightarrow W$. Setting $\mathcal{U} = D//G \hookrightarrow \mathcal{Y}$, we thus obtain a closed embedding $\mathcal{X}_{\mathcal{U}} = q^{-1}(D)//G \hookrightarrow W//G$.

Define a vector bundle $\mathcal{E} \to \mathcal{U}$ via the following pullback square:

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner"] & W//G \arrow[d] \\
\mathcal{U} = D//G \arrow[r] & \mathbb{B}G.
\end{tikzcd}$$

Then we get $S^{\mathcal{E}} \simeq S^W//G \times_{\mathbb{B}G} \mathcal{U}$. The maps $\mathcal{X}_{\mathcal{U}} \to \mathcal{U}$ and $\mathcal{X}_{\mathcal{U}} \hookrightarrow W//G \hookrightarrow S^W//G$ then determine a commutative diagram

$$\begin{tikzcd}
\mathcal{X}_{\mathcal{U}} \arrow[r, hook, "i"] \arrow[dr, "p_{\mathcal{U}}"'] & S^{\mathcal{E}} \arrow[d, "\pi"] \\
& \mathcal{U}.
\end{tikzcd}$$

The map i is an embedding, and it is proper by Lemma 2.4.10, since the map $p_{\mathcal{U}}$ is proper. Since any proper embedding is a closed embedding, this finishes the proof. $\square$

3.7.2 Locally extending sections

As our final result in this chapter, we show that a partial section of a representable submersion defined on a closed substack can locally be extended to a section defined on an open neighborhood of the closed substack. Furthermore, this section can be chosen to be the zero section of a vector bundle. We start wtih an auxiliary lemma.

Lemma 3.7.4. *Let G be a compact Lie group, let $f: Y \to X$ be a G-equivariant submersion between smooth G-manifolds and let $s: Y \to X$ be a section of f. For every $x \in X$, there exist G-invariant open neighborhoods $x \in N \subseteq X$ and $s(x) \in E \subseteq Y$ such that:*

- *We have $f(E) \subseteq N$ and $s(N) \subseteq E$;*
- *The restriction $f|_E: E \to N$ is isomorphic to a G-equivariant vector bundle with zero-section $s|_N: N \to E$.*

Proof. Let $H := G_x \subseteq G$ denote the isotropy group of x in X, which is also the isotropy group of $s(x)$ in Y. Consider the closed submanifold $S := s(X) \subseteq Y$. The restriction $p|_S: S \to X$ is a diffeomorphism and in particular a smooth submersion. By [PW19, Corollary 2.8], there exist orthogonal H-representations V, W and W' and G-equivariant open embeddings $\Phi: G \times_H (V \times W \times W') \hookrightarrow Y$ and $\Psi: G \times_H V \hookrightarrow X$ such that:

- We have $\Theta([e,0,0,0]) = s(x)$ and $\Psi([e,0]) = x$;
- The map
$$\Theta: G \times_H (V \times W) \to M, \quad [(g,v,w)] \mapsto \Phi[(g,v,w,0)]$$
has image in $S = s(X)$ and comprises a G-equivariant diffeomorphism onto an open neighborhood of the orbit Gx in S.
- The following diagram commutes:
$$\begin{tikzcd}
G \times_H (V \times W \times W') \arrow[r, hook, "\Phi"] \arrow[d, "\mathrm{id}_G \times \mathrm{pr}_V"'] & M \arrow[d, "f"] \\
G \times_H V \arrow[r, hook, "\Psi"] & N.
\end{tikzcd}$$

Since $p|_S\colon S \to X$ is a diffeomorphism, it follows that $W = 0$ is the trivial H-representation. Observe that the left vertical map is a G-equivariant vector bundle. Defining $s_0\colon G \times_H V \to G \times_H (V \times W \times W')$ by $t([g,v]) = [g,v,0,0]$, we see that the diagram

$$\begin{tikzcd}
G \times_H (V \times W \times W') \arrow[r, hook, "\Phi"] & M \\
G \times_H V \arrow[u, "s_0"] \arrow[r, hook, "\Psi"] & N \arrow[u, "s"']
\end{tikzcd}$$

commutes. Since s_0 is the zero section of the vector bundle, this proves the claim. ◻

Proposition 3.7.5. *Consider a commutative diagram of separated differentiable stacks*

$$\begin{tikzcd}
& \mathcal{Y} \arrow[d, "p"] \\
\mathcal{Z} \arrow[ur, hook, "t"] \arrow[r, hook, "i"] & \mathcal{X},
\end{tikzcd}$$

where p is a representable submersion and i is a closed embedding. Then for every point $z \in \mathcal{Z}$, there is a commutative diagram

$$\begin{tikzcd}
& \mathcal{E} \arrow[r, hook, "j_{\mathcal{E}}"] \arrow[d, shift left, "\pi"] & \mathcal{Y} \arrow[d, "p"] \\
z \in \mathcal{Z} \cap \mathcal{N} \arrow[ur, hook, "t"] \arrow[r, hook, "i"] & \mathcal{N} \arrow[u, shift left, "s_0"] \arrow[r, hook, "j_{\mathcal{N}}"] & \mathcal{X},
\end{tikzcd}$$

where $j_{\mathcal{E}}\colon \mathcal{E} \hookrightarrow \mathcal{Y}$ is an open neighbhorhood of $t(z)$ inside $\mathcal{Y}$, $j_{\mathcal{N}}\colon \mathcal{N} \hookrightarrow \mathcal{X}$ is an open neighbhorhood of z inside $\mathcal{X}$, and the map $\pi\colon \mathcal{E} \to \mathcal{N}$ is a vector bundle with zero section $s_0\colon \mathcal{N} \to \mathcal{E}$.

Proof. We will prove the statement in three steps:

Step 1 We first reduce to the case where $\mathcal{X} \simeq X//G$ is the quotient stack of a smooth action of a compact Lie group G on a smooth manifold X, and the point $z \in \mathcal{X}$ lifts to a G-fixed point $z' \in X$;

Step 2 We then further reduce to the case where the partial section $s\colon \mathcal{Z} \to \mathcal{Y}$ of p is the restriction along $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ of a section $s\colon \mathcal{X} \to \mathcal{Y}$ of p;

Step 3 In this case, we prove that, locally in $\mathcal{Y}$, s can be chosen to be the zero-section of a vector bundle.

Step 1. Let $G := G_z$ denote the isotropy group of z in $\mathcal{X}$. By Theorem 3.7.2, we can find an open neighborhood $\mathcal{U}$ of z in $\mathcal{X}$ of the form $X//G$ for some smooth G-manifold X such that the point $z \in \mathcal{X}$ lifts to a G-fixed point $z' \in X$. The restrictions $\mathcal{Z}_{\mathcal{U}}$ and $\mathcal{Y}_{\mathcal{U}}$ over $\mathcal{U}$ are of the form $Z//G$ and $Y//G$ for smooth G-manifolds Z and Y-respectively. As z' is G-fixed in X, it follows that z' is also G-fixed in Z and thus $t(z)$ is G-fixed in Y.

Step 2. By Step 1, it will suffice to prove the analogous statement in the setting of smooth G-manifolds. So assume given a commutative triangle

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, "s"] \arrow[r, hook, "i"] & X \end{tikzcd}$$

of smooth G-manifolds, and a G-fixed point $z \in Z$. We will show that the partial section t of p extends to a section $s\colon U \to Y$ on some open neighborhood U of Z inside X.

By [PW19, Proposition 2.7], there exist orthogonal G-representations V and W, and G-equivariant embeddings $\Phi\colon V \times W \hookrightarrow Y$ and $\Psi\colon V \hookrightarrow X$ such that $\Phi(0,0) = t(z)$, $\Psi(0) = i(z)$, and the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[r, hook, "\Phi"] \arrow[d, "\mathrm{pr}_V"'] & Y \arrow[d, "p"] \\ V \arrow[r, hook, "\Psi"] & X. \end{tikzcd}$$

By scaling down the embedding Ψ if needed, we may assume that the section $t\colon Z \cap V \to Y$ factors through $V \times W \hookrightarrow Y$. The second component $Z \to W$ can be extended to a small open neighborhood U of $Z \cap V$ inside X, which leads to a section $s\colon U \to U \times W$ extending t.

Step 3: From Step 2, we have obtained a commutative diagram

$$\begin{tikzcd} & U' \arrow[r, hook] \arrow[d, shift left, "p|_{U'}"] & Y \arrow[d, "p"] \\ z \in Z \cap U \arrow[ur, "t"] \arrow[r, hook, "i"] & U \arrow[r, hook] \arrow[u, shift left, "s"] & X \end{tikzcd}$$

of smooth G-manifolds. By Lemma 3.7.4, there exist G-invariant open neighborhoods $z \in N \subseteq U$ and $t(z) \in E \subseteq U'$ such that $p(E) \subseteq N$, $s(N) \subseteq E$, and the restriction $p|_E\colon E \to N$ is isomorphic to a G-equivariant vector bundle with zero-section $s|_N\colon N \to E$. In other words, we obtain a commutative diagram

$$\begin{tikzcd} & E \arrow[r, hook] \arrow[d, shift left, "\pi"] & U' \arrow[r, hook] \arrow[d, shift left, "p|_{U'}"] & Y \arrow[d, "p"] \\ z \in Z \cap N \arrow[ur, "t"] \arrow[r, hook, "i"] & N \arrow[r, hook] \arrow[u, shift left, "s_0"] & U \arrow[r, hook] \arrow[u, shift left, "s"] & X, \end{tikzcd}$$

where π is a G-equivariant vector bundle with zero-section s_0. Passing to quotient stacks then gives the claim. $\square$

II.4 Genuine sheaves

In this chapter, we will introduce for every separated differentiable stack $\mathcal{X}$ the ∞-category $\mathrm{H}(\mathcal{X})$ of *genuine sheaves of animae on* $\mathcal{X}$ and the ∞-category $\mathrm{SH}(\mathcal{X})$ of *genuine sheaves of spectra on* $\mathcal{X}$.

The ∞-categories $\mathrm{H}(\mathcal{X})$ and $\mathrm{SH}(\mathcal{X})$ may be thought of as 'genuine' refinements of the ∞-categories $\mathrm{Shv}(\mathcal{X})$ and $\mathrm{Shv}(\mathcal{X};\mathrm{Sp})$ of ordinary sheaves of animae/spectra on $\mathcal{X}$. While the latter two ∞-categories are easy to define using descent, they do not fully capture all of the "stacky" behavior of $\mathcal{X}$. To illustrate this, consider the case of a classifying stack $\mathcal{X} = \mathbb{B}G$ of a compact Lie group G. Using descent for the effective epimorphism $\mathrm{pt} \twoheadrightarrow \mathbb{B}G$, one can show that sheaves on $\mathbb{B}G$ correspond to animae/spectra with a G-action: there are equivalences of ∞-categories

$$\mathrm{Shv}(\mathbb{B}G) \simeq \mathrm{An}^{BG} \qquad \text{and} \qquad \mathrm{Shv}(\mathbb{B}G;\mathrm{Sp}) \simeq \mathrm{Sp}^{BG},$$

see Lemma 4.1.21 below. In equivariant homotopy theory, one frequently wants to work instead with the more refined ∞-categories An_G and Sp_G of *genuine G-animae* (a.k.a. 'G-spaces') and *genuine G-spectra*, which retain additional geometric fixed point data for closed subgroups of G. Forcing descent with respect to the map $\mathrm{pt} \twoheadrightarrow \mathbb{B}G$ destroys this geometric information, since it inverts all G-equivariant maps which are an equivalence after forgetting the G-action.

The goal of this chapter is to extend the above story to the case of an arbitrary separated differentiable stack $\mathcal{X}$ by introducing a notion of a 'genuine sheaf' on $\mathcal{X}$, which compared to an ordinary sheaf on $\mathcal{X}$ retains additional isotropy information. The resulting ∞-categories $\mathrm{H}(\mathcal{X})$ and $\mathrm{SH}(\mathcal{X})$ come equipped with forgetful functors to $\mathrm{Shv}(\mathcal{X})$ and $\mathrm{Shv}(\mathcal{X};\mathrm{Sp})$, respectively, which are simultaneously localizing and colocalizing, inverting precisely those maps which become equivalences after forgetting all this isotropy information. When $\mathcal{X}$ is the classifying stack $\mathbb{B}G$, these two functors specialize to the forgetful functors $\mathrm{An}_G \to \mathrm{An}^{BG}$ and $\mathrm{Sp}_G \to \mathrm{Sp}^{BG}$ discussed above. In the case of a smooth manifold M, which has trivial isotropy groups, there is no difference between genuine sheaves and ordinary sheaves.

The approach we will take for defining genuine sheaves closely resembles the definition of the motivic categories $\mathrm{H}(S)$ and $\mathrm{SH}(S)$ for a scheme S, introduced by Morel and Voevodsky [MV99] in their development of *motivic homotopy theory* (also referred to as $\mathbb{A}^1$-*homotopy theory*). The starting point is the observation that there is an equivalence between the ∞-category An of animae and the ∞-category $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff})$ of homotopy invariant sheaves of animae on the site of smooth manifolds. An analogue for a separated differentiable stack $\mathcal{X}$ is obtained by replacing the site Diff of smooth manifolds by the site $\mathrm{Sub}_{/\mathcal{X}}$ of representable submersions over $\mathcal{X}$, where we think of a representable submersion $\mathcal{Y} \to \mathcal{X}$ as an '$\mathcal{X}$-indexed family of smooth manifolds'. The ∞-category $\mathrm{H}(\mathcal{X})$ of genuine sheaves of animae on $\mathcal{X}$ is then defined as the ∞-category of homotopy invariant sheaves on $\mathrm{Sub}_{/\mathcal{X}}$. The ∞-category $\mathrm{SH}(\mathcal{X})$ of genuine sheaves of *spectra* on $\mathcal{X}$ is defined, at least locally in $\mathcal{X}$, by monoidally inverting the sphere bundles $S^{\mathcal{E}}$ in $\mathrm{H}(\mathcal{X})_*$, where $\mathcal{E}$ runs over all vector bundles over $\mathcal{X}$.

The chapter is organized as follows. In Section 4.1, we introduce the site $\mathrm{Sub}_{/\mathcal{X}}$ for a differentiable stack $\mathcal{X}$ and study the ∞-category of sheaves on $\mathrm{Sub}_{/\mathcal{X}}$. In Sections 4.2 and 4.3, we introduce the ∞-categories $\mathrm{H}(\mathcal{X})$ and $\mathrm{SH}(\mathcal{X})$ of genuine sheaves of animae/spectra, respectively. In Section 4.4, we establish the connection with equivariant homotopy theory by producing equivalences of ∞-categories $\mathrm{H}(\mathbb{B}G) \simeq \mathrm{An}_G$ and $\mathrm{SH}(\mathbb{B}G) \simeq \mathrm{Sp}_G$. Finally, we show in Section 4.5 that the assignments $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ and $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ admit universal properties, phrased in terms of the notion of a *pullback formalism* $\mathcal{C}\colon \mathrm{DiffStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ due to Drew and Gallauer [DG22].

4.1 Sheaves on $\mathrm{Sub}_{/\mathcal{X}}$

In this section, we study the ∞-category $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ of sheaves on the category $\mathrm{Sub}_{/\mathcal{X}}$ of representable submersions over $\mathcal{X}$.

Definition 4.1.1. Let $\mathcal{X} \in \mathrm{DiffStk}$ be a differentiable stack. We let $\mathrm{Sub}_{/\mathcal{X}} \subseteq \mathrm{DiffStk}_{/\mathcal{X}}$ denote the full subcategory spanned by the representable submersions $\mathcal{Y} \to \mathcal{X}$.

Recall that for a Lie groupoid $\mathcal{G}$, the ∞-category $\mathrm{Sub}_{/\mathbb{B}\mathcal{G}}$ of representable submersions into the classifying stack $\mathbb{B}\mathcal{G}$ is naturally equivalent to the category $\mathrm{Diff}_{\mathcal{G}}$ of smooth $\mathcal{G}$-manifolds, see Corollary 2.3.19. In particular, $\mathrm{Sub}_{/\mathcal{X}}$ is an ordinary category for every differentiable stack $\mathcal{X}$.

The category $\mathrm{Sub}_{/\mathcal{X}}$ admits a Grothendieck topology given by the open covers:

Definition 4.1.2 (Open covers)**.** Given a differentiable stack $\mathcal{Y}$, a collection of morphisms $j_\alpha\colon \mathcal{U}_\alpha \to \mathcal{Y}$ in DiffStk is called an *open cover of* $\mathcal{Y}$ if the following two conditions are satisfied:

(1) Each morphism $j_\alpha\colon \mathcal{U}_\alpha \hookrightarrow \mathcal{Y}$ is an open embedding of differentiable stacks;

(2) The morphism $\bigsqcup_\alpha \mathcal{U}_\alpha \to \mathcal{Y}$ is an effective epimorphism in $\mathrm{Shv}(\mathrm{Diff})$.

A sieve $\{f_\beta\colon \mathcal{Y}_\beta \to \mathcal{Y}\}$ in DiffStk is said to be a *covering sieve* of $\mathcal{Y}$ if it contains an open cover of $\mathcal{Y}$.

Proposition 4.1.3. *The covering sieves of Definition 4.1.2 equip* DiffStk *with the structure of a Grothendieck topology.*

Proof. The identity on $\mathcal{Y}$ is an open cover, hence generates a covering sieve of $\mathcal{Y}$. If $\{f_j\colon \mathcal{Y}_j \to \mathcal{Y}\}$ is a covering sieve of $\mathcal{Y}$ containing an open cover $\{\iota_i\colon \mathcal{U}_i \hookrightarrow \mathcal{Y}\}$ and $\mathcal{Y}' \to \mathcal{Y}$ is a morphism of differentiable stacks, then the pullback sieve $\{\mathcal{Y}_j \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'\}$ on $\mathcal{Y}'$ contains the open cover $\{\mathcal{V}_i := \mathcal{U}_i \times_{\mathcal{Y}} \mathcal{Y}' \hookrightarrow \mathcal{Y}'\}$, where we use that $\bigsqcup_i \mathcal{V}_i \to \mathcal{Y}'$ is an effective epimorphism since it is a base change of the map $\bigsqcup_i \mathcal{U}_i \to \mathcal{Y}$. Finally, let $\{\iota_i\colon \mathcal{U}_i \to \mathcal{Y}\}$ be an open cover of $\mathcal{Y}$ and let $\{f_j\colon \mathcal{Y}_j \to \mathcal{Y}\}$ be an arbitrary sieve. Assume that for every i, the pullback sieve $\{\mathcal{U}_i \times_{\mathcal{Y}} \mathcal{Y}_j \to \mathcal{U}_i\}$ is a covering sieve, so that it contains an open cover $\{\mathcal{V}_{i,j} \hookrightarrow \mathcal{U}_i\}$. We claim that the collection of composites $\{\mathcal{V}_{i,j} \hookrightarrow \mathcal{U}_i \hookrightarrow \mathcal{Y}\}$ forms an open cover of $\mathcal{Y}$, contained in the original sieve. Indeed, these maps are open embeddings and the composite map

$$\bigsqcup_i \bigsqcup_j \mathcal{V}_{i,j} \twoheadrightarrow \bigsqcup_i \mathcal{U}_i \twoheadrightarrow \mathcal{Y}$$

is an effective epimorphism. $\square$

The open covers in DiffStk directly induce a Grothendieck topology on $\mathrm{Sub}_{/\mathcal{X}}$:

Definition 4.1.4. We define a Grothendieck topology on $\mathrm{Sub}_{/\mathcal{X}}$ in which a collection of morphisms $\{f_i\colon \mathcal{Y}_i \to \mathcal{Y}\}$ in $\mathrm{Sub}_{/\mathcal{X}}$ is a cover of $\mathcal{Y}$ if and only if it is cover of $\mathcal{Y}$ in the ∞-category DiffStk, in the sense of Definition 4.1.2. Given a presentable ∞-category $\mathcal{C}$, we let $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}; \mathcal{C})$ denote the ∞-category of $\mathcal{C}$-sheaves on the resulting site $(\mathrm{Sub}_{/\mathcal{X}}, \mathrm{open})$.

The sheaf condition can be made very explicit: a presheaf on $\mathrm{Sub}_{/\mathcal{X}}$ is a sheaf if and only if it satisfies a form of excision and turns countable unions of open substacks into limits. More precisely:

Proposition 4.1.5 (cf. [BBP19, Theorem 5.1], [ADH21, Theorem 3.6.1])**.** *Let $\mathcal{X}$ be a differentiable stack and let $\mathcal{C}$ be a presentable ∞-category. Then a $\mathcal{C}$-valued presheaf $\mathcal{F}\colon \mathrm{Sub}^{\mathrm{op}}_{/\mathcal{X}} \to \mathcal{C}$ is a sheaf if and only if $\mathcal{F}$ satisfies the following conditions:*

(1) The object $\mathcal{F}(\emptyset)$ is terminal in $\mathcal{C}$;

(2) For every representable submersion $\mathcal{Y} \to \mathcal{X}$ and every pair of open substacks $\mathcal{U}, \mathcal{V} \subseteq \mathcal{Y}$ satisfying $\mathcal{Y} = \mathcal{U} \cup \mathcal{V}$, the induced square

$$\begin{tikzcd} \mathcal{F}(\mathcal{Y}) \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{V}) \arrow[d] \\ \mathcal{F}(\mathcal{U}) \arrow[r] & \mathcal{F}(\mathcal{U} \cap \mathcal{V}) \end{tikzcd}$$

is a pullback square in $\mathcal{C}$;

(3) For every representable submersion $\mathcal{Y} \to \mathcal{X}$ and every $\mathbb{N}$-indexed sequence of open substacks

$$\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \cdots \subseteq \mathcal{Y}$$

such that $\mathcal{Y} = \bigcup_{n \geq 0} \mathcal{U}_n = \mathcal{Y}$, the induced morphism

$$\mathcal{F}(\mathcal{Y}) \to \lim_{n \geq 0} \mathcal{F}(\mathcal{U}_n)$$

is an equivalence in $\mathcal{C}$.

Proof. Given a representable submersion $\mathcal{Y} \to \mathcal{X}$, any open substack $\mathcal{U}$ of $\mathcal{Y}$ is naturally an object of $\mathrm{Sub}_{/\mathcal{X}}$ via the composite $\mathcal{U} \hookrightarrow \mathcal{Y} \to \mathcal{X}$, and this provides a functor $\mathrm{Open}(\mathcal{Y}) \to \mathrm{Sub}_{/\mathcal{X}}$. Observe that $\mathcal{F}$ is a sheaf if and only if for every $\mathcal{Y}$ the composite functor

$$\mathrm{Open}(\mathcal{Y})^{\mathrm{op}} \to \mathrm{Sub}^{\mathrm{op}}_{/\mathcal{X}} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a sheaf with respect to the open cover topology on $\mathrm{Open}(\mathcal{Y})$. Recall from Corollary 3.1.12 that the poset $\mathrm{Open}(\mathcal{Y})$ of open substacks of $\mathcal{Y}$ is equivalent to the poset $\mathrm{Open}(|\mathcal{Y}|_{\mathrm{mod}})$ of open subspaces of the coarse moduli space $|\mathcal{Y}|_{\mathrm{mod}}$. Since $\mathcal{Y}$ admits a representable atlas by a smooth manifold, the topological space $|\mathcal{Y}|_{\mathrm{mod}}$ is the quotient of a smooth manifold, and thus is in particular *hereditarily Lindelöf*: every open subspace $U \subseteq |\mathcal{Y}|_{\mathrm{mod}}$ is Lindelöf, meaning that every open cover of U admits a countable subcover. It is a general fact that a presheaf $\mathcal{F}\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$ on a hereditarily Linfelöf topological space X is a sheaf if and only if it satisfies the conditions analogous to (1), (2) and (3), see for example [ADH21, Proposition 3.6.6]. Applying this result to $X = |\mathcal{Y}|_{\mathrm{mod}}$ finishes the proof. □

Corollary 4.1.6. *The subcategory* $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \subseteq \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$ *is closed under* ω_1*-filtered colimits.*

Proof. This is immediate from Proposition 4.1.5 and the fact that ω_1-filtered colimits commute with countable (that is, ω_1-small) limits in the ∞-category of animae by [Lur09, Proposition 5.3.3.3]. □

4.1.1 Pullback and pushforward functors

Let $f : \mathcal{X}' \to \mathcal{X}$ be a morphism of differentiable stacks. Our goal in this subsection is to construct the *pullback* and *pushforward* functors

$$f^* : \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \rightleftarrows \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) : f_*,$$

and prove various basic properties about them.

We start with the construction of f_*. Since base changes of representable submersions exist in DiffStk and are again representable submersions, there is a functor $\mathrm{Sub}_{/\mathcal{X}} \to \mathrm{Sub}_{/\mathcal{X}'}$ given by $\mathcal{Y} \mapsto \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$. Precomposition with this functor then defines a pushforward functor at the level of presheaf categories:

$$\begin{aligned} f_* : \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}'}) &\to \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}), \\ f_*\mathcal{F}(\mathcal{Y}) &:= \mathcal{F}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'). \end{aligned}$$

Lemma 4.1.7. *The functor* f_* *restricts to sheaves:*

$$f_* : \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}).$$

Proof. Observe that the functor $- \times_{\mathcal{X}} \mathcal{X}' : \mathrm{Sub}_{/\mathcal{X}} \to \mathrm{Sub}_{/\mathcal{X}'}$ preserves covering sieves. Since coverings consists of open embeddings, $\mathrm{Sub}_{/\mathcal{X}}$ admits pullbacks along open embeddings and the functor $- \times_{\mathcal{X}} \mathcal{X}'$ preserves such pullbacks, it follows that this is a continuous functor in the sense of Definition E.39, and thus the restriction functor f_* preserves sheaves. □

The functor $f_* : \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ admits a left adjoint

$$f^* : \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'})$$

given as the composite $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \hookrightarrow \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{PSh}(\mathrm{Sub}_{\mathcal{X}/}) \xrightarrow{L_{\mathrm{open}}} \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'})$, where the middle functor is left Kan extension along the functor $- \times_{\mathcal{X}} \mathcal{X}' : \mathrm{Sub}_{/\mathcal{X}} \to \mathrm{Sub}_{/\mathcal{X}'}$. In case f is a representable submersion, f^* admits a further left adjoint $f_\sharp$:

Lemma 4.1.8. *Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a representable submersion.*

(1) The functor $-\times_{\mathcal{X}} \mathcal{X}'\colon \mathrm{Sub}_{/\mathcal{X}} \to \mathrm{Sub}_{/\mathcal{X}'}$ admits a left adjoint $f_\sharp\colon \mathrm{Sub}_{/\mathcal{X}'} \to \mathrm{Sub}_{/\mathcal{X}}$ given by postcomposition with f.

(2) The functor $f_\sharp\colon \mathrm{Sub}_{/\mathcal{X}'} \to \mathrm{Sub}_{/\mathcal{X}}$ is a continuous functor, so that restriction along it preserves sheaves.

(3) The functor $f^*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'})$ admits a left adjoint

$$f_\sharp\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$$

which on representables restricts to $f_\sharp\colon \mathrm{Sub}_{/\mathcal{X}'} \to \mathrm{Sub}_{/\mathcal{X}}$.

Proof. Assertion (1) is clear. For assertion (2), note that $f_\sharp$ preserves covering sieves and preserves all pullbacks that exist in $\mathrm{Sub}_{/\mathcal{X}'}$. In particular, the functor $\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}'})$ given by restriction along $f_\sharp$ preserves sheaves. As $f_\sharp$ is left adjoint to $-\times_{\mathcal{X}} \mathcal{X}'\colon \mathrm{Sub}_{/\mathcal{X}} \to \mathrm{Sub}_{/\mathcal{X}'}$, restriction along $f_\sharp$ is left adjoint to the functor $f_*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$, and therefore is equivalent to f^*. But this means that f^* admits a left adjoint given by the composite

$$\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) \hookrightarrow \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}'}) \to \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{L_{\mathrm{open}}} \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}),$$

where the middle functor is left Kan extension along $f_\sharp$. Since the left Kan extension reduces to the functor $f_\sharp$ on representable objects, the last claim follows. This finishes the proof. □

To have more control over the functoriality of the construction $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$, we now provide an alternative description of the pullback functors f^* in terms of the ∞-topos $\mathrm{Shv}(\mathrm{DiffStk})$ of sheaves on the site DiffStk from Proposition 4.1.3.

Lemma 4.1.9. *For a differentiable stack $\mathcal{X}$, the inclusion $\mathrm{Sub}_{/\mathcal{X}} \hookrightarrow \mathrm{DiffStk}_{/\mathcal{X}}$ induces a fully faithful functor*

$$\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \hookrightarrow \mathrm{Shv}(\mathrm{DiffStk}_{/\mathcal{X}}) \simeq \mathrm{Shv}(\mathrm{DiffStk})_{/\mathcal{X}},$$

given by left Kan extension followed by sheafification. The essential image is the subcategory of $\mathrm{Shv}(\mathrm{DiffStk})_{/\mathcal{X}}$ generated under colimits by the representable submersions $f\colon \mathcal{Y} \to \mathcal{X}$. For a morphism $f\colon \mathcal{X}' \to \mathcal{X}$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \arrow[r, "f^*"] \arrow[d, hook] & \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) \arrow[d, hook] \\
\mathrm{Shv}(\mathrm{DiffStk})_{/\mathcal{X}} \arrow[r, "f^*"] & \mathrm{Shv}(\mathrm{DiffStk})_{/\mathcal{X}'}
\end{tikzcd}$$

where the bottom map is given by pullback along f in $\mathrm{Shv}(\mathrm{DiffStk})$.

Proof. Given a representable submersion $\mathcal{Y} \to \mathcal{X}$ and an open cover $\{\mathcal{U}_i \hookrightarrow \mathcal{Y}\}$ of $\mathcal{Y}$, each of the composite morphisms $\mathcal{U}_i \to \mathcal{Y} \to \mathcal{X}$ is again a representable submersion. It follows that the inclusion $\mathrm{Sub}_{/\mathcal{X}} \hookrightarrow \mathrm{DiffStk}_{/\mathcal{X}}$ is a cocontinuous functor. It follows from Corollary E.47 that the right Kan extension functor along $\mathrm{Sub}_{/\mathcal{X}} \hookrightarrow \mathrm{DiffStk}_{/\mathcal{X}}$ preserves sheaves and is fully faithful.

Since this inclusion also preserves covering sieves, it is also a continuous functor, so that restriction along it preserves sheaves. It follows that this restriction functor admits a further left adjoint given by the functor in the statement. As this is a double left adjoint of a fully faithful functor, it is itself fully faithful, proving the first claim. The description of the essential image is immediate.

To see that the claimed commutative diagram exists, observe that all four functors preserve colimits, and thus it suffices to prove the claim at on representables. But here the claim is clear: by definition, the functor f^* restricts on representables to the functor $- \times_{\mathcal{X}} \mathcal{X}' \colon \mathrm{Sub}_{/\mathcal{X}} \to \mathrm{Sub}_{/\mathcal{X}'}$, which in turn is just defined as a restriction of the functor $- \times_{\mathcal{X}} \mathcal{X}' \colon \mathrm{DiffStk}_{/\mathcal{X}} \to \mathrm{DiffStk}_{/\mathcal{X}'}$. □

By the previous lemma, we can turn the assigment $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ into a functor $\mathrm{DiffStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ by regarding it as a subfunctor of the slice functor $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{SepStk})_{/\mathcal{X}}$.

Lemma 4.1.10. *The functor* $\mathrm{DiffStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *given by* $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ *is a sheaf of* ∞*-categories on* $\mathrm{DiffStk}$.

Proof. It follows directly from Remark 4.5.2 below and Theorem E.3 that the assignment $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{SepStk})_{/\mathcal{X}}$ is a sheaf of ∞-categories on SepStk. Since the assignment $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ is a subfunctor of this functor, it thus remains to show that the condition for an object to be contained in this subcategory can be checked locally in $\mathcal{X}$. To this end, let $f \colon A \to \mathcal{X}$ be an object of $\mathrm{Shv}(\mathrm{SepStk})_{/\mathcal{X}}$ satisfying the assumption that each of the base changes $A \times_{\mathcal{X}} \mathcal{U}_i$ can be written as iterated colimits of representable submersion $\mathcal{Y}_{i,j} \to \mathcal{U}_i$. By descent, it follows that the map $A \to \mathcal{X}$ can be written as an iterated colimit of the composites $\mathcal{Y}_{i,j} \to \mathcal{U}_i \hookrightarrow \mathcal{X}$. Since these are again representable submersions, this finishes the proof of the claim. □

Lemma 4.1.11. *The inclusion* $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \hookrightarrow \mathrm{Shv}(\mathrm{DiffStk}_{/\mathcal{X}})$ *preserves finite products.*

Proof. In both source and target finite products commute with colimits in each variable. In particular, the statement may be checked at the level of representables. There it is clear as the category $\mathrm{Sub}_{/\mathcal{X}}$ admits products and these are preserved by the inclusion $\mathrm{Sub}_{/\mathcal{X}} \hookrightarrow \mathrm{DiffStk}_{/\mathcal{X}}$. □

Corollary 4.1.12 (Symmetric monoidality)**.** *For every morphism $f: \mathcal{X}' \to \mathcal{X}$ of differentiable stacks, the pullback functor $f^*: \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'})$ preserves finite products. In particular, the assignment $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ defines a limit-preserving functor $\mathrm{DiffStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.*

Proof. The first statement follows from Lemma 4.1.11 and Lemma 4.1.9. The second statement is then a direct consequence from Lemma 4.1.10 as the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Cat}_\infty$ preserves limits. □

4.1.2 Sheaves on the coarse moduli space

In Section 3.1 we assigned to every differentiable stack $\mathcal{X}$ a topological space $|\mathcal{X}|_{\mathrm{mod}}$, the coarse moduli space of $\mathcal{X}$. The goal of this section is to relate its sheaf ∞-topos $\mathrm{Shv}(|\mathcal{X}|_{\mathrm{mod}})$ to the ∞-topos $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$.

Recall from Corollary 3.1.12 that the poset $\mathrm{Open}(|\mathcal{X}|_{\mathrm{mod}})$ of open subsets of the coarse moduli space is equivalent to the poset $\mathrm{Open}(\mathcal{X})$ of open substacks of $\mathcal{X}$, giving an equivalence

$$\mathrm{Shv}(|\mathcal{X}|_{\mathrm{mod}}) = \mathrm{Shv}(\mathrm{Open}(|\mathcal{X}|_{\mathrm{mod}})) \simeq \mathrm{Shv}(\mathrm{Open}(\mathcal{X})).$$

Henceforth, we will work with the sheaf category $\mathrm{Shv}(\mathrm{Open}(\mathcal{X}))$ rather than $\mathrm{Shv}(|\mathcal{X}|_{\mathrm{mod}})$. Since every open embedding $\mathcal{U} \hookrightarrow \mathcal{X}$ is in particular a representable submersion, there is an inclusion of categories $\iota: \mathrm{Open}(\mathcal{X}) \hookrightarrow \mathrm{Sub}_{/\mathcal{X}}$.

Proposition 4.1.13. *Let $\mathcal{X}$ be a differentiable stack.*

(1) The inclusion $\iota: \mathrm{Open}(\mathcal{X}) \hookrightarrow \mathrm{Sub}_{/\mathcal{X}}$ is a cocontinuous functor, in the sense of Definition E.43. In particular, right Kan extension along ι preserves sheaves and defines a fully faithful geometric morphism of ∞-topoi

$$\iota_*: \mathrm{Shv}(\mathrm{Open}(\mathcal{X})) \hookrightarrow \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}).$$

(2) The inclusion $\iota: \mathrm{Open}(\mathcal{X}) \hookrightarrow \mathrm{Sub}_{/\mathcal{X}}$ admits a left adjoint $\mathrm{im}: \mathrm{Sub}_{/\mathcal{X}} \to \mathrm{Open}(\mathcal{X})$ which is also cocontinuous. In particular, the restriction functor $- \circ \iota: \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{PSh}(\mathrm{Open}(\mathcal{X}))$ preserves sheaves, and thus restricts to a functor

$$\iota^*: \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(\mathrm{Open}(\mathcal{X}))$$

which is left adjoint to ι_*.

(3) *The functor* ι^* *admits a fully faithful left-exact left adjoint*

$$\iota_\sharp\colon \operatorname{Shv}(\operatorname{Open}(\mathcal{X})) \hookrightarrow \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}}).$$

(4) *The functor* ι^* *preserves limits and colimits.*

(5) *For every morphism* $f\colon \mathcal{Y} \to \mathcal{X}$ *of differentiable stacks, there are preferred commutative diagrams as follows:*

$$\begin{tikzcd} \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{Y}}) \arrow[r, "\iota^*"] \arrow[d, "f_*"'] & \operatorname{Shv}(\operatorname{Open}(\mathcal{Y})) \arrow[d, "f_*"] \\ \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}}) \arrow[r, "\iota^*"] & \operatorname{Shv}(\operatorname{Open}(\mathcal{X})), \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Shv}(\operatorname{Open}(\mathcal{X})) \arrow[r, "\iota_\sharp"] \arrow[d, "f^*"'] & \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}}) \arrow[d, "f^*"] \\ \operatorname{Shv}(\operatorname{Open}(\mathcal{Y})) \arrow[r, "\iota_\sharp"] & \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{Y}}). \end{tikzcd}$$

(6) *For every sheaf* $\mathcal{F} \in \operatorname{Shv}(\operatorname{Open}(\mathcal{X}))$*, the sheaf* $\iota_\sharp\mathcal{F} \in \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}})$ *is given at a representable submersion* $f\colon \mathcal{Y} \to \mathcal{X}$ *by*

$$(\iota_\sharp\mathcal{F})(\mathcal{Y}) \simeq \Gamma_{\mathcal{Y}}(f^*\mathcal{F}),$$

the anima of global sections of the pullback sheaf $f^*\mathcal{F} \in \operatorname{Shv}(\operatorname{Open}(\mathcal{Y}))$.

Proof. We start with part (1). The fact that ι is cocontinuous is clear as any open cover in $\operatorname{Sub}_{\mathcal{X}}$ of an open substack $\mathcal{U} \hookrightarrow \mathcal{X}$ consists of smaller open substacks $\mathcal{V} \hookrightarrow \mathcal{U} \hookrightarrow \mathcal{X}$. The 'in particular' follows from Corollary E.47.

Now we prove part (2). By Lemma 2.4.9, every representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ factors uniquely as a representable surjective submersion $\mathcal{Y} \twoheadrightarrow f(\mathcal{Y})$ followed by an open embedding $f(\mathcal{Y}) \hookrightarrow \mathcal{X}$. Since effective epimorphisms and monomorphisms are part of a factorization system on $\operatorname{Shv}(\operatorname{Diff})$, it follows immediately that sending $(f\colon \mathcal{Y} \to \mathcal{X}) \in \operatorname{Sub}_{/\mathcal{X}}$ to $\operatorname{im}(f) := (f(\mathcal{Y}) \hookrightarrow \mathcal{X}) \in \operatorname{Open}(\mathcal{X})$ defines a left adjoint im of ι.

To see that im is cocontinuous, consider a representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ and consider a collection of open substacks $\mathcal{U}_i \hookrightarrow f(\mathcal{Y})$ covering its image in $\mathcal{X}$. Then the preimages $\mathcal{V}_i := f^{-1}(\mathcal{U}_i) \subseteq \mathcal{Y}$ form an open cover of $\mathcal{Y}$ satisfying $f(\mathcal{V}_i) = \mathcal{U}_i \subseteq f(\mathcal{Y})$. This proves that $\operatorname{im}\colon \operatorname{Sub}_{/\mathcal{X}} \to \operatorname{Open}(\mathcal{X})$ is cocontinuous.

As in part (1), it follows from Corollary E.47 that right Kan extension along im preserves sheaves and defines a geometric morphism of ∞-topoi. Since right Kan extension along im is equivalent to restriction along ι, this proves (2).

For (3), we know that $\iota^* \simeq \operatorname{im}_*$ admits a left-exact left adjoint $\iota_\sharp \simeq \operatorname{im}^*$ by part (2). Furthermore, since ι_* is fully faithful, it follows that also $\iota_\sharp$ is fully faithful.

Part (4) is immediate, since ι^* admits both a left and a right adjoint.

For part (5), the right square is obtained from the left square by passing to left adjoints. The left square is obtained by restriction along the following commutative diagram:

$$\begin{tikzcd} \mathrm{Open}(\mathcal{Y}) \arrow[r, "\iota"] \arrow[d, "\mathcal{X}\times_{\mathcal{Y}}-"'] & \mathrm{Sub}_{/\mathcal{Y}} \arrow[d, "\mathcal{X}\times_{\mathcal{Y}}-"] \\ \mathrm{Open}(\mathcal{X}) \arrow[r, "\iota"] & \mathrm{Sub}_{/\mathcal{X}}\,. \end{tikzcd}$$

For part (6), observe that the value of $\iota_\sharp\mathcal{F}$ at $\mathcal{Y}$ is the same as the anima of global sections of the sheaf $f^*\iota_\sharp\mathcal{F} \in \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Y}})$, as the functor $f^*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Y}})$ is given by precomposition with the functor $f \circ -\colon \mathrm{Sub}_{/\mathcal{Y}} \to \mathrm{Sub}_{/\mathcal{X}}$. By part (5), this is the same as the global sections of the sheaf $\iota_\sharp f^*\mathcal{F} \in \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Y}})$, which in turn is the same as the global sections of $\iota^*\iota_\sharp f^*\mathcal{F} \in \mathrm{Shv}(\mathrm{Open}(\mathcal{Y}))$. As $\iota_\sharp$ is fully faithful, the latter sheaf is equivalent to $f^*\mathcal{F}$, proving the claim. □

Corollary 4.1.14. *Let $\mathcal{X}$ be a differentiable stack. For every representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$, consider the composite*

$$\iota^* \circ f^*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{f^*} \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Y}}) \xrightarrow{\iota^*} \mathrm{Shv}(\mathrm{Open}(\mathcal{Y})).$$

Then the functors $\iota^* \circ f^*$ preserve all limits and colimits and are jointly conservative.

Proof. The functor f^* preserves limits and colimits by Lemma 4.1.8 and the functor ι^* by Proposition 4.1.13. It thus remains to show these functors are jointly conservative. But this is clear: for a sheaf $\mathcal{F} \in \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$, the value of $\mathcal{F}$ at some object $(f\colon \mathcal{Y} \to \mathcal{X}) \in \mathrm{Shv}(\mathrm{Sub}_{\mathcal{Y}})$ is equivalent to the anima of global sections of $(\iota^* \circ f^*)\mathcal{F} = \iota^* f^*\mathcal{F}$:

$$\mathcal{F}(\mathcal{Y}) \simeq \Gamma_{\mathcal{Y}}(\iota^* f^* \mathcal{F}).$$ □

The previous result can be used to prove properties of the ∞-category $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ by reducing them to analogous properties of the ∞-categories $\mathrm{Shv}(\mathrm{Open}(\mathcal{Y}))$. The following result on hypercompleteness of $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ is an important illustration of this method. We thank Marc Hoyois for useful discussions concerning the proof of this result.

Proposition 4.1.15. *For a separated differentiable stack $\mathcal{X}$, the ∞-topos $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ is hypercomplete.*

Proof. Let $\varphi\colon \mathcal{F} \to \mathcal{F}'$ be an ∞-connected morphism in $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$. We have to show that φ is an equivalence. By Corollary 4.1.14, we may show that the image of φ under the composite

$$\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{f^*} \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Y}}) \xrightarrow{\iota^*} \mathrm{Shv}(\mathrm{Open}(\mathcal{Y}))$$

is an equivalence for every $\mathcal{Y} \in \mathrm{Sub}_{/\mathcal{X}}$. As $\iota^* \circ f^*$ preserves limits and colimits, it preserves ∞-connected morphisms, and it follows that the morphism $\iota^* f^* \varphi$ is again ∞-connected. Hence it will suffice to show that the ∞-topos $\mathrm{Shv}(\mathrm{Open}(\mathcal{Y}))$ is hypercomplete.

As $\mathcal{X}$ is separated by assumption and the map $f\colon \mathcal{Y} \to \mathcal{X}$ is representable and thus separated by Lemma 3.3.3, it follows that also $\mathcal{Y}$ is separated. By Theorem 3.7.2, $\mathcal{Y}$ is locally isomorphic to a quotient stack $M//G$ for a compact Lie group G and a smooth G-manifold M. As $\mathrm{Shv}(\mathrm{Open}(\mathcal{Y}))$ satisfies descent for open covers, a morphism in $\mathrm{Shv}(\mathrm{Open}(\mathcal{Y}))$ is an equivalence if and only if it is so locally in $\mathcal{Y}$, and we may thus assume that $\mathcal{Y}$ is of the form $M//G$.

Observe that pullback along the quotient map $M \twoheadrightarrow M//G$ defines an inclusion of posets $\mathrm{Open}(M//G) \hookrightarrow \mathrm{Open}(M)$ whose image consists of the G-invariant open subspaces of M. This inclusion admits a left adjoint given by sending an open subspace U to the G-saturation $G \cdot U = \{g \cdot u \mid g \in G, u \in U\}$, and this functor preserves open coverings. In particular, the inclusion $\mathrm{Open}(M//G) \hookrightarrow \mathrm{Open}(M)$ is a morphism of sites by Example E.42, and the resulting pullback functor $\mathrm{Shv}(\mathrm{Open}(M//G)) \hookrightarrow \mathrm{Shv}(\mathrm{Open}(M)) = \mathrm{Shv}(M)$ preserves ∞-connected morphisms by Remark E.51. It will thus suffice to show that $\mathrm{Shv}(M)$ is hypercomplete. This is a special case of [Lur09, Theorem 7.2.3.6, Corollary 7.2.1.12], see also Example E.54. ◻

4.1.3 Ordinary sheaves on differentiable stacks

In this subsection, we recall the definition of the ∞-category $\mathrm{Shv}(\mathcal{X})$ of (ordinary) sheaves on a differentiable stack $\mathcal{X}$, and compare it with the ∞-topos $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$. The material of this subsection will not play a significant role in the remainder of the article and may be skipped on first reading.

Lemma 4.1.16. *Let $\mathcal{C}$ be a presentable ∞-category. Then the functor* $\mathrm{Shv}(-;\mathcal{C})\colon \mathrm{Diff}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *if a sheaf of ∞-categories with respect to the open cover topology on* Diff.

Proof. Any open cover of a smooth manifold M consists of objects of $\mathrm{Open}(M)$ and hence it suffices that for every M the composite

$$\mathrm{Open}(M)^{\mathrm{op}} \hookrightarrow (\mathrm{Diff}_{/M})^{\mathrm{op}} \xrightarrow{\mathrm{fgt}} \mathrm{Diff}^{\mathrm{op}} \xrightarrow{\mathrm{Shv}(-;\mathcal{C})} \mathrm{Cat}_\infty$$

is a sheaf of ∞-categories on M. For every open $U \subseteq M$, the inclusion $\mathrm{Shv}(U) \hookrightarrow \mathrm{Shv}(M)$ induces an equivalence $\mathrm{Shv}(U) \xrightarrow{\sim} \mathrm{Shv}(M)_{/U}$, and thus we obtain an equivalence

$$\mathrm{Shv}(U;\mathcal{C}) \simeq \mathrm{Shv}(M)_{/U} \otimes \mathcal{C},$$

where the tensor product is Lurie's tensor product of presentable ∞-categories. It is clear that for a smaller open subset $V \subseteq U \subseteq M$ the restriction functor $(-)|_V \colon \mathrm{Shv}(U) \to \mathrm{Shv}(V)$ corresponds under this equivalence to the functor $\mathrm{Shv}(M)_{/U} \to \mathrm{Shv}(M)_{/V}$ given by pullback along the inclusion $V \hookrightarrow U$, hence we see that the above composite is equivalent to the composite

$$\mathrm{Open}(M)^{\mathrm{op}} \xrightarrow{\mathrm{Shv}(M)_{/-}} \mathrm{Pr}^{\mathrm{R}} \xrightarrow{-\otimes\mathcal{C}} \mathrm{Pr}^{\mathrm{R}} \xrightarrow{\mathrm{fgt}} \mathrm{Cat}_\infty.$$

Since the ∞-topos $\mathrm{Shv}(M)$ satisfies descent, the first functor sends covering sieves to limits. As the functors $-\otimes\mathcal{C} \colon \mathrm{Pr}^{\mathrm{R}} \to \mathrm{Pr}^{\mathrm{R}}$ and $\mathrm{fgt} \colon \mathrm{Pr}^{\mathrm{R}} \to \mathrm{Cat}_\infty$ preserves limits, this finishes the proof. ◻

Definition 4.1.17 (Sheaves on a stack)**.** Let $\mathcal{C}$ be a presentable ∞-category. By the previous lemma, the functor $\mathrm{Shv}(-;\mathcal{C}) \colon \mathrm{Diff}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ uniquely extends to a limit-preserving functor

$$\mathrm{Shv}(-;\mathcal{C}) \colon \mathrm{Shv}(\mathrm{Diff})^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

For a stack $\mathcal{X}$ on Diff, we refer to $\mathrm{Shv}(\mathcal{X};\mathcal{C})$ as the *∞-category of $\mathcal{C}$-valued sheaves on $\mathcal{X}$*.

Remark 4.1.18. We will frequently refer to a sheaf on $\mathcal{X}$ as an *ordinary sheaf*, to emphasize the contrast with the notion of a *genuine sheaf* which will be introduced in Section 4.2.

There are comparison functors between $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ and $\mathrm{Shv}(\mathcal{X})$ analogous to the ones for $\mathrm{Shv}(|\mathcal{X}|_{\mathrm{mod}})$ from Proposition 4.1.13:

Construction 4.1.19. Given $\mathcal{X}$ be a differentiable stack, we construct a functor

$$\gamma^* \colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(\mathcal{X})$$

which is natural in $\mathcal{X}$. Since the assignment $\mathcal{X} \mapsto \mathrm{Shv}(\mathcal{X})$ is by definition right Kan extended from the subcategory $\mathrm{Diff} \subseteq \mathrm{DiffStk}$, it suffices to define this when $\mathcal{X} = M$ is a smooth manifold, naturally in M. In this case we let γ^* be the functor $\iota^* \colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(M)$ from Proposition 4.1.13.

Since the ∞-category $\mathrm{Shv}(\mathcal{X})$ is a limit of $\mathrm{Shv}(M)$ where M ranges over all smooth manifolds M equipped with a map of stacks $f \colon M \to \mathcal{X}$, the functor γ^* is essentially determined by the existence of commutative squares

$$\begin{tikzcd}
\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \arrow[r, "\gamma^*"] \arrow[d, "f^*"'] & \mathrm{Shv}(\mathcal{X}) \arrow[d, "f^*"] \\
\mathrm{Shv}(\mathrm{Sub}_{/M}) \arrow[r, "\iota^*"] & \mathrm{Shv}(M),
\end{tikzcd}$$

naturally in M.

Proposition 4.1.20. *Let $\mathcal{X}$ be a differentiable stack.*

(1) The functor $\gamma^* \colon \operatorname{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \operatorname{Shv}(\mathcal{X})$ admits a fully faithful right adjoint

$$\gamma_* \colon \operatorname{Shv}(\mathcal{X}) \hookrightarrow \operatorname{Shv}(\mathrm{Sub}_{/\mathcal{X}})).$$

(2) The functor γ^* admits a fully faithful left adjoint

$$\gamma_\sharp \colon \operatorname{Shv}(\mathcal{X}) \hookrightarrow \operatorname{Shv}(\mathrm{Sub}_{/\mathcal{X}})).$$

(3) For every morphism $g \colon \mathcal{Y} \to \mathcal{X}$ of differentiable stacks, there are preferred commutative diagrams as follows:

$$\begin{tikzcd} \operatorname{Shv}(\mathrm{Sub}_{/\mathcal{Y}}) \arrow[r, "\gamma^*"] \arrow[d, "g_*"'] & \operatorname{Shv}(\mathcal{Y}) \arrow[d, "g_*"] \\ \operatorname{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \arrow[r, "\gamma^*"] & \operatorname{Shv}(\mathcal{X}), \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Shv}(\mathcal{X}) \arrow[r, hook, "\gamma_\sharp"] \arrow[d, "g^*"'] & \operatorname{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \arrow[d, "g^*"] \\ \operatorname{Shv}(\mathcal{Y}) \arrow[r, hook, "\gamma_\sharp"] & \operatorname{Shv}(\mathrm{Sub}_{/\mathcal{Y}}). \end{tikzcd}$$

(4) For every sheaf $\mathcal{F} \in \operatorname{Shv}(\mathcal{X})$, the sheaf $\gamma_\sharp\mathcal{F} \in \operatorname{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ is given at a representable submersion $f \colon \mathcal{Y} \to \mathcal{X}$ by

$$(\gamma_\sharp\mathcal{F})(\mathcal{Y}) \simeq \Gamma_{\mathcal{Y}}(f^*\mathcal{F}),$$

the anima of global sections of the pullback sheaf $f^*\mathcal{F} \in \operatorname{Shv}(\mathcal{Y})$.

Proof. For parts (1) and (2), we first show that the functor γ^* preserves limits and colimits. Let $N \twoheadrightarrow \mathcal{X}$ be a representable atlas for $\mathcal{X}$. By descent there is an equivalence $\operatorname{Shv}(\mathcal{X}) \simeq \lim_{[n]\in\Delta} \operatorname{Shv}(N^{\times^n_{\mathcal{X}}})$. Since each map $N^{\times^n_{\mathcal{X}}} \to \mathcal{X}$ is a representable submersion, it will suffice to show that for every representable submersion $f \colon M \to \mathcal{X}$, the composite

$$\operatorname{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{\gamma^*} \operatorname{Shv}(\mathcal{X}) \xrightarrow{f^*} \operatorname{Shv}(M)$$

preserves limits and colimits. But by definition of γ^*, this functor agrees with the composite

$$\operatorname{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{f^*} \operatorname{Shv}(\mathrm{Sub}_{/M}) \xrightarrow{\iota^*} \operatorname{Shv}(M),$$

which preserve limits and colimits by Corollary 4.1.14. It follows in particular that γ^* admits both a right adjoint γ_* and a left adjoint $\gamma_\sharp$.

For part (3) follows directly from part (5) of Proposition 4.1.13, since the both functors g_* commute with the pullback functors f^* for representable submersions $M \to \mathcal{X}$.

We may now prove that the functors γ_* and $\gamma_\sharp$ are fully faithful, finishing the proof of parts (1) and (2). It suffices to prove that $\gamma_\sharp$ is fully faithful, as then so is its double right adjoint γ_*. To show that the unit $\mathrm{id} \to \gamma^*\gamma_\sharp$ is an equivalence, it suffices to do so after applying the pullback functor $f^*\colon \mathrm{Shv}(\mathcal{X}) \to \mathrm{Shv}(M)$ for every representable submersion $f\colon M \to \mathcal{X}$. To this end, consider the following diagram:

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{X}) \arrow[r, "\gamma_\sharp"] \arrow[d, "f^*"'] & \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \arrow[r, "\gamma^*"] \arrow[d, "f^*"] & \mathrm{Shv}(\mathcal{X}) \arrow[d, "f^*"] \\
\mathrm{Shv}(M) \arrow[r, "\iota_\sharp"] & \mathrm{Shv}(\mathrm{Sub}_{/M}) \arrow[r, "\iota^*"] & \mathrm{Shv}(M).
\end{tikzcd}$$

The right square commutes by definition of γ^* while the left square commutes by part (3) we have just proved. We thus obtain an equivalence $f^*\gamma^*\gamma_\sharp \simeq \iota^*\iota_\sharp f^*$ and the claim thus follows from fully faithfulness of $\iota_\sharp$.

Finally, part (4) follows from (3) just as in the proof of part (6) of Proposition 4.1.13. □

The following result, which identifies C-valued sheaves on the classifying stack $\mathbb{B}G$ with objects in C with a G-action, was pointed out to the author by Dustin Clausen.

Lemma 4.1.21 (Sheaves on classifying stacks)**.** *Let G be a Lie group and let C be a presentable ∞-category. Then there is an equivalence*

$$\mathrm{Shv}(\mathbb{B}G; C) \simeq C^{BG}$$

between the ∞-category of C-valued sheaves on the classifying stack $\mathbb{B}G$ and the ∞-category of C-valued local system on the classifying space $BG \in$ An.

Proof. It will suffice to show the claim when C is the ∞-category of animae, as the general case may be obtained by tensoring both sides with C in Pr^{L}. By definition, the stack $\mathbb{B}G$ is the colimit of the simplicial diagram $\Delta^{\mathrm{op}} \to \mathrm{Shv}(\mathrm{Diff})$, $[n] \mapsto G^n$, hence by descent there is an equivalence

$$\mathrm{Shv}(\mathbb{B}G) \simeq \lim_{[n]\in\Delta} \mathrm{Shv}(G^n).$$

For a topological space X, let $\mathrm{Loc}(X) \subseteq \mathrm{Shv}(X)$ denote the subcategory of *locally constant sheaves*, as in [Lur17, Definition A.1.12]. By [Lur17, Theorem A.1.15], there is a functorial equivalence $\mathrm{Loc}(X) \simeq \mathrm{An}^{\mathrm{shp}(X)}$, where $\mathrm{shp}(X) \in \mathrm{An}$ denotes the *shape* of X. Since G is a smooth manifold, its shape agrees with its homotopy type, and thus we obtain an equivalence

$$\lim_{[n]\in\Delta} \mathrm{Loc}(G^n) \simeq \lim_{[n]\in\Delta} \mathrm{An}^{G^n} \simeq \mathrm{An}^{\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} G^n} \simeq \mathrm{An}^{BG}.$$

To finish the proof, it thus remains to observe that the inclusion

$$\lim_{[n]\in\Delta} \mathrm{Loc}(G^n) \hookrightarrow \lim_{[n]\in\Delta} \mathrm{Shv}(G^n)$$

is essentially surjective: given an object $(\mathcal{F}_n)_{[n]\in\Delta}$ of the target, the sheaf $\mathcal{F}_n \in \mathrm{Shv}(G^n)$ is the pullback along $G^n \to \mathrm{pt}$ of the sheaf $\mathcal{F}_0 \in \mathrm{Shv}(\mathrm{pt}) = \mathrm{Loc}(\mathrm{pt})$ and thus is a locally constant sheaf. $\square$

4.2 Genuine sheaves of animae

In this section, we will introduce and study the ∞-category $\mathrm{H}(\mathcal{X})$ of genuine sheaves of animae on a differentiable stack $\mathcal{X}$: homotopy invariant sheaves of animae on the site $\mathrm{Sub}_{/\mathcal{X}}$.

4.2.1 Homotopy invariant presheaves

We start by defining the notion of a *homotopy invariant presheaf* and study the homotopy localization functor $L_{\mathbb{R}}$. Throughout this subsection, we fix a differentiable stack $\mathcal{X}$ and a presentable ∞-category $\mathcal{C}$.

Definition 4.2.1 (Homotopy invariance)**.** A $\mathcal{C}$-valued presheaf $\mathcal{F} \in \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}};\mathcal{C})$ is said to be *homotopy invariant*[1] if for every object $\mathcal{Y} \in \mathrm{Sub}_{/\mathcal{X}}$, the projection map $\mathrm{pr}\colon \mathcal{Y}\times\mathbb{R} \to \mathcal{Y}$ induces an equivalence $\mathrm{pr}^*\colon \mathcal{F}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{F}(\mathcal{Y}\times\mathbb{R})$ in $\mathcal{C}$. We denote by

$$\mathrm{PSh}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}};\mathcal{C}) \subseteq \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}};\mathcal{C})$$

the full subcategory spanned by the homotopy invariant presheaves.

Since a presheaf $\mathcal{F}$ is homotopy invariant if and only if it is local with respect to the projections $\mathrm{pr}\colon \mathcal{Y}\times\mathbb{R} \to \mathcal{Y}$, the ∞-category $\mathrm{PSh}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}};\mathcal{C})$ is a localization of $\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}};\mathcal{C})$ at these maps and in particular $\mathrm{PSh}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}};\mathcal{C})$ is a presentable ∞-category. We recall an explicit description of the localization functor, due to Morel and Voevodsky [MV99] and already known to Suslin. Our discussion takes inspiration from [ADH21, Chapter 5].

Construction 4.2.2 (The Morel–Suslin–Voevodsky construction)**.** We define the *algebraic n-simplex* Δ^n_{alg} as the hyperplane in $\mathbb{R}^{n+1}$ defined by

$$\Delta^n_{\mathrm{alg}} := \{(x_0,x_1,\ldots,x_n) \in \mathbb{R}^{(n+1)} \mid \sum_{i=0}^{n} x_i = 1\}.$$

[1] This is sometimes also referred to as $\mathbb{R}$-*invariance* or *concordance invariance*.

The algebraic simplices form a cosimplicial smooth manifold $\Delta^\bullet_{\mathrm{alg}} \colon \Delta \to \mathrm{Diff}$. Given a presheaf $\mathcal{F} \in \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}; \mathcal{C})$, we define another presheaf $L_{\mathbb{R}}\mathcal{F}$ by

$$L_{\mathbb{R}}\mathcal{F}(\mathcal{Y}) := \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathcal{F}(\mathcal{Y} \times \Delta^n_{\mathrm{alg}}),$$

the geometric realization in $\mathcal{C}$ of the simplicial object $[n] \mapsto \mathcal{F}(\mathcal{Y} \times \Delta^n_{\mathrm{alg}})$. This defines a functor

$$L_{\mathbb{R}} \colon \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}; \mathcal{C}) \to \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}; \mathcal{C}).$$

As $\Delta^0_{\mathrm{alg}} \cong \mathrm{pt}$, there is a canonical map of presheaves $\eta_{\mathcal{F}} \colon \mathcal{F} \to L_{\mathbb{R}}\mathcal{F}$.

Proposition 4.2.3. *For any presheaf $\mathcal{F} \in \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}; \mathcal{C})$, the presheaf $L_{\mathbb{R}}\mathcal{F}$ is homotopy invariant. Furthermore, the resulting functor*

$$L_{\mathbb{R}} \colon \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}; \mathcal{C}) \to \mathrm{PSh}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}}; \mathcal{C})$$

is left adjoint to the inclusion $\mathrm{PSh}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}}; \mathcal{C}) \hookrightarrow \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}; \mathcal{C})$, with unit given by the map $\eta_{\mathcal{F}} \colon \mathcal{F} \to L_{\mathbb{R}}\mathcal{F}$.

Proof. The proof is in essence identical to that of [ADH21, Proposition 5.1.2], where the case $\mathcal{X} = \mathrm{pt}$ is treated. We spell out the core ingredients of the proof and leave some details to the reader.

First we show that the presheaf $L_{\mathbb{R}}\mathcal{F}$ is homotopy invariant. Given a smooth manifold $\mathcal{Y}$, we have to show that the functor $\mathrm{pr}^* \colon L_{\mathbb{R}}\mathcal{F}(\mathcal{Y}) \to L_{\mathbb{R}}\mathcal{F}(\mathcal{Y} \times \mathbb{R})$ induced by the projection $\mathrm{pr} \colon \mathcal{Y} \times \mathbb{R} \to \mathcal{Y}$ is an equivalence. We claim that an inverse is given by the map $i_0^* \colon L_{\mathbb{R}}\mathcal{F}(\mathcal{Y} \times \mathbb{R}) \to L_{\mathbb{R}}\mathcal{F}(\mathcal{Y})$, where $i_0 \colon \mathcal{Y} \hookrightarrow \mathcal{Y} \times \mathbb{R}$ denotes the inclusion of $\mathcal{Y} \times \{0\}$. Since $\mathrm{pr} \circ i_0 = \mathrm{id}$, it remains to show that $\mathrm{pr}^* i_0^* \simeq \mathrm{id}$. This is a standard argument which goes back at least to [MV99]: the composite map

$$\mathcal{Y} \times \mathbb{R} \times \Delta^\bullet_{\mathrm{alg}} \xrightarrow{\mathrm{pr} \times \mathrm{id}} \mathcal{Y} \times \Delta^\bullet_{\mathrm{alg}} \xrightarrow{i_0 \times \mathrm{id}} \mathcal{Y} \times \mathbb{R} \times \Delta^\bullet_{\mathrm{alg}}$$

is simplicially homotopic to the identity of the simplicial object $\mathcal{Y} \times \mathbb{R} \times \Delta^\bullet_{\mathrm{alg}}$ and since simplicially homotopic maps between simplicial objects induce homotopic maps on realizations this provides the desired homotopy $\mathrm{pr}^* i_0^* \simeq \mathrm{id}^* = \mathrm{id}$.

In order to show that the map $\eta_{\mathcal{F}} \colon \mathcal{F} \to L_{\mathbb{R}}\mathcal{F}$ is the unit of an adjunction, we apply [Lur09, Proposition 5.2.7.4]. Note that if $\mathcal{F}$ is already homotopy invariant, each of the maps $\mathcal{F}(\mathcal{Y}) \to \mathcal{F}(\mathcal{Y} \times \Delta^n_{\mathrm{alg}})$ is an equivalence and hence the map $\eta_{\mathcal{F}}$ is an equivalence. For arbitrary $\mathcal{F}$, we showed above that $L_{\mathbb{R}}\mathcal{F}$ is homotopy invariant, and thus the map $\eta_{L_{\mathbb{R}}\mathcal{F}} \colon L_{\mathbb{R}}\mathcal{F} \to L_{\mathbb{R}}L_{\mathbb{R}}\mathcal{F}$ is an equivalence for arbitrary $\mathcal{F}$. It follows that also the map

$L_{\mathbb{R}}\eta_{\mathcal{F}}\colon L_{\mathbb{R}}\mathcal{F} \to L_{\mathbb{R}}L_{\mathbb{R}}\mathcal{F}$ is an equivalence, as this agrees with $\eta_{L_{\mathbb{R}}\mathcal{F}}$ up to swapping the two indices $[n]$ and $[m]$ in the double colimit

$$L_{\mathbb{R}}L_{\mathbb{R}}\mathcal{F} = \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} \operatorname{colim}_{[m]\in\Delta^{\mathrm{op}}} \mathcal{F}(\mathcal{Y} \times \Delta^n_{\mathrm{alg}} \times \Delta^m_{\mathrm{alg}}).$$

This shows that the conditions of part (3) of [Lur09, Proposition 5.2.7.4] are satisfied, finishing the proof. □

Corollary 4.2.4 (cf. Hoyois [Hoy17, Proposition 3.4])**.** *Assume that geometric realizations commute with finite products in C (e.g. C is an ∞-topos or C is stable). Then the localization functor $L_{\mathbb{R}}$ preserves finite products.*

Proof. Since the functor $\mathcal{F} \mapsto \mathcal{F}(- \times \Delta^n_{\mathrm{alg}})$ preserves limits and geometric realizations commute with finite products in $\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}; C)$, this is clear from the definition of $L_{\mathbb{R}}$. □

Corollary 4.2.5 (cf. Hoyois [Hoy17, Proposition 3.4])**.** *When $C = \mathrm{An}$ is the ∞-category of animae, the localization functor $L_{\mathbb{R}}\colon \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{PSh}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}})$ is locally cartesian: for a pair of maps $A \to B$ and $C \to B$ in $\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$ such that A and B are homotopy invariant, the exchange map*

$$L_{\mathbb{R}}(A \times_B C) \to A \times_B L_{\mathbb{R}}(C)$$

is an equivalence in $\mathrm{PSh}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}})$.

Proof. We have to show that for all $\mathcal{Y} \in \mathrm{Sub}_{/\mathcal{Y}}$ the square

$$\begin{tikzcd}
\operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} \Big(A(\mathcal{Y} \times \Delta^n_{\mathrm{alg}}) \times_{B(\mathcal{Y}\times\Delta^n_{\mathrm{alg}})} C(\mathcal{Y} \times \Delta^n_{\mathrm{alg}}) \Big) \arrow[r] \arrow[d] & A(\mathcal{Y}) \arrow[d] \\
\operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} C(\mathcal{Y} \times \Delta^n_{\mathrm{alg}}) \arrow[r] & B(\mathcal{Y})
\end{tikzcd}$$

is a pullback square. As A and B are homotopy invariant, the top left corner is equivalent to the realization of the simplicial object $[n] \mapsto A(\mathcal{Y}) \times_{B(\mathcal{Y})} C(\mathcal{Y} \times \Delta^n_{\mathrm{alg}})$, and thus this is an instance of universality of colimits in $\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$. □

4.2.2 Genuine sheaves of animae

The following is the main definition of this section.

Definition 4.2.6 (Genuine sheaves)**.** Let $\mathcal{X}$ be a differentiable stack. A *genuine sheaf* on $\mathcal{X}$ is a sheaf $\mathcal{F} \in \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ which is homotopy invariant as a presheaf, in the sense of Definition 4.2.1. We denote by

$$\mathrm{H}(\mathcal{X}) := \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}}) \subseteq \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$$

the full subcategory spanned by the genuine sheaves on $\mathcal{X}$. More generally, if $\mathcal{C}$ is a presentable ∞-category, we let

$$\mathrm{H}(\mathcal{X};\mathcal{C}) := \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathcal{X}};\mathcal{C}) \subseteq \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}};\mathcal{C})$$

be the subcategory of homotopy invariant $\mathcal{C}$-valued sheaves on $\mathrm{Sub}_{/\mathcal{X}}$. Note that there is an equivalence $\mathrm{H}(\mathcal{X},\mathcal{C}) \simeq \mathrm{H}(\mathcal{X}) \otimes \mathcal{C}$.

As in the case of presheaves, $\mathrm{H}(\mathcal{X})$ is a localization of $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ at the projections $\mathcal{Y} \times \mathbb{R} \to \mathcal{Y}$, and the inclusion admits a left adjoint

$$L_{\mathrm{htp}} \colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{H}(\mathcal{X}),$$

which we will refer to as the *homotopy localization functor*. We will study this functor in more detail in Subsection 4.2.3 below.

Unsurprisingly, a consequence of forcing homotopy invariance is that fiberwise homotopic maps over $\mathcal{X}$ become equivalent in $\mathrm{H}(\mathcal{X})$.

Definition 4.2.7. Let $\mathcal{X}$ be a differentiable stack, let $\mathcal{Y},\mathcal{Z} \in \mathrm{Sub}_{/\mathcal{X}}$ be submersions over $\mathcal{X}$, and let $f_0, f_1 \colon \mathcal{Y} \to \mathcal{Z}$ be two maps in $\mathrm{Sub}_{/\mathcal{X}}$. A *homotopy over* $\mathcal{X}$ is a homotopy $f \colon \mathcal{Y} \times \mathbb{R} \to \mathcal{Z}$ in $\mathrm{Sub}_{/\mathcal{X}}$ whose restriction to $i_r \colon \mathcal{Y} \times \{r\} \hookrightarrow \mathcal{Y} \times \mathbb{R}$ is the map f_r, for $r \in \{0,1\}$.

A map $f \colon \mathcal{Y} \to \mathcal{Z}$ in $\mathrm{Sub}_{/\mathcal{X}}$ is a *strict homotopy equivalence* if there is a map $g \colon \mathcal{Y} \to \mathcal{Z}$ over $\mathcal{X}$ such that the maps gf and fg are homotopic over $\mathcal{X}$ to the respective identities, in the sense of Definition 2.5.5.

Lemma 4.2.8. *Let $\mathcal{X}$ be a differentiable stack and let $f \colon \mathcal{Y} \times \mathbb{R} \to \mathcal{Z}$ be a homotopy over $\mathcal{X}$ of maps $\mathcal{Y} \to \mathcal{Z}$ in* $\mathrm{Sub}_{/\mathcal{X}}$*. Then the maps $L_{\mathrm{htp}}(f_0), L_{\mathrm{htp}}(f_1) \colon L_{\mathrm{htp}}(\mathcal{Y}) \to L_{\mathrm{htp}}(\mathcal{Z})$ are equivalent as maps in* $\mathrm{H}(\mathcal{X})$.

Proof. It will suffice to prove this in the case that $\mathcal{Z} = \mathcal{Y} \times \mathbb{R}$ and f is the identity, so that f_0 and f_1 are the inclusions $i_r \colon \mathcal{Y} \times \{r\} \hookrightarrow \mathcal{Y} \times \mathbb{R}$. In that case, $L_{\mathrm{htp}}(i_0)$ and $L_{\mathrm{htp}}(i_1)$ are both inverse to the equivalence $L_{\mathrm{htp}}(\mathrm{pr}) \colon L_{\mathrm{htp}}(\mathcal{Y} \times \mathbb{R}) \xrightarrow{\sim} L_{\mathrm{htp}}(\mathcal{Y})$, and are hence equivalent. □

Corollary 4.2.9. *Let $f: \mathcal{Y} \to \mathcal{Z}$ be a strict homotopy equivalence over $\mathcal{X}$. Then the morphism $L_{\mathrm{htp}}(f): L_{\mathrm{htp}}(\mathcal{Y}) \to L_{\mathrm{htp}}(\mathcal{Z})$ is an equivalence in* $\mathrm{H}(\mathcal{X})$. □

Example 4.2.10. Every vector bundle $\mathcal{E} \to \mathcal{Y}$ over $\mathcal{X}$ is a strict homotopy equivalence in $\mathrm{Sub}_{/\mathcal{X}}$.

4.2.3 The homotopy localization functor

The homotopy localization functor $L_{\mathrm{htp}}: \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{H}(\mathcal{X})$ admits an explicit formula as a transfinite iteration of the localization functor $L_{\mathbb{R}}$ and the sheafification functor L_{open}. The goal of this subsection is to make this precise and to deduce some consequences.

Construction 4.2.11. For an ordinal α, let $[\alpha]$ denote the linearly ordered set of ordinals $\beta \leq \alpha$. We will construct for every ordinal α a functor

$$L: [\alpha] \to \mathrm{Fun}(\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}), \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})),$$

which we informally denote by $\beta \mapsto L^{\beta}$ for $\beta \leq \alpha$. We proceed via transfinite induction:

- If $\alpha = 0$, we set $L^0 = \mathrm{id}$.
- If $\alpha = \beta + 1$ is a sucessor ordinal, we define

  $$L^{\beta+1} := L_{\mathrm{open}} \circ L_{\mathbb{R}} \circ L^{\beta}: \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}).$$

  The units $\mathrm{id} \to L_{\mathrm{open}}$ and $\mathrm{id} \to L_{\mathbb{R}}$ define a natural transformation $L^{\beta} \to L^{\beta+1}$ which thus defines the extension $L: [\beta+1] \to \mathrm{Fun}(\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}), \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}))$.
- If α is a limit ordinal, we define

  $$L^{\alpha} := \mathrm{colim}_{\beta < \alpha} L^{\beta}$$

  and let $L: [\alpha] = \alpha^{\triangleright} \to \mathrm{Fun}(\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}), \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}))$ denote the cocone for this colimit.

We may think of the functor L^{α} as the α-fold power of the functor $L^1 = L_{\mathrm{open}} \circ L_{\mathbb{R}}$.

Lemma 4.2.12. *For every ordinal α, the functor $L^{\alpha}: \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$ preserves finite products and is locally cartesian.*

Proof. We proceed via transfinite induction. For $\alpha = 0$ we have $L^\alpha = \mathrm{id}$ and the claim is obvious. When $\alpha = \beta + 1$ is a successor ordinal, we have $L^\alpha = L_{\mathrm{open}} \circ L_{\mathbb{R}} \circ L^\beta$ and the claim follows from a combination of the induction hypothesis, Corollary 4.2.4 and left-exactness of the sheafification functor L_{open}. When α is a limit ordinal, the claim is immediate as finite limits commute with filtered colimits in $\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$. $\square$

Lemma 4.2.13. *For any uncountable limit ordinal α, the functor $L^\alpha \colon \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{PSh}(\mathrm{Shv}_{/\mathcal{X}})$ lands in the subcategory $\mathrm{H}(\mathcal{X})$ of homotopy invariant sheaves. Moreover, the transformation $\mathrm{id} = L^0 \to L^\alpha$ exhibits the functor L^α as a left adjoint to the inclusion $\mathrm{H}(\mathcal{X}) \hookrightarrow \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$.*

Proof. We start by showing that L^α lands in homotopy invariant sheaves. It follows immediately from Corollary 4.1.6 that $L^\alpha \mathcal{F}$ is a sheaf for every preheaf $\mathcal{F} \in \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$, as for every $\beta < \alpha$ the functor $L^{\beta+1} = L_{\mathrm{open}} \circ L_{\mathbb{R}} \circ L^\beta$ lands in sheaves. Since $L_{\mathbb{R}}^2 \simeq L_{\mathbb{R}}$, we obtain an equivalent definition of L^α if for successor ordinals we had defined $L^{\beta+1} = L_{\mathbb{R}} \circ L_{\mathrm{open}} \circ L_{\mathbb{R}} \circ L^\beta$, and since homotopy invariant presheaves are closed under colimits in $\mathrm{PSh}(\mathrm{Shv}_{/\mathcal{X}})$ it follows that L^α lands in homotopy invariant presheaves. This proves the first claim.

To prove that L^α is left adjoint to the inclusion, let $\mathcal{F}, \mathcal{G} \in \mathrm{PSh}(\mathrm{Shv}_{/\mathcal{X}})$ be presheaves and assume that $\mathcal{G}$ is a homotopy invariant sheaf. We will prove by transfinite induction that for every ordinal $\beta \leq \alpha$, the map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Shv}_{/\mathcal{X}})}(L^\beta \mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Shv}_{/\mathcal{X}})}(\mathcal{F}, \mathcal{G})$$

given by precomposition with $\mathcal{F} \to L^\beta \mathcal{F}$ is an equivalence. For $\beta = 0$ we have $L^\beta \mathcal{F} = \mathcal{F}$ and there is nothing to show. When $\beta = \gamma + 1$ is a successor ordinal, the claim follows from the induction hypothesis for γ, the fact that $L^\beta \mathcal{F} = L_{\mathrm{open}}(L_{\mathbb{R}}(L^\gamma \mathcal{F}))$ and the fact that $\mathcal{G}$ is a homotopy invariant sheaf. When β is a limit ordinal, this is immediate from the induction hypothesis.

Taking $\beta = \alpha$ then finishes the proof. $\square$

Corollary 4.2.14. *The functor $L_{\mathrm{htp}} \colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{H}(\mathcal{X})$ preserves finite products and is locally cartesian: for every pair of maps $A \to B$ and $C \to B$ in $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ such that A and B are homotopy invariant, the exchange map*

$$L_{\mathrm{htp}}(A \times_B C) \to A \times_B L_{\mathrm{htp}}(C)$$

is an equivalence in $\mathrm{H}(\mathcal{X})$.

Proof. Let α be an uncountable ordinal. By Lemma 4.2.13 there is an equivalence $L_{\mathrm{htp}} \simeq L^\alpha$, and thus the claim follows from Lemma 4.2.12. $\square$

4.2.4 Pullback and pushforward functors

The ∞-categories $\mathrm{H}(\mathcal{X})$ come equipped with pullback and pushforward functors analogous to those for the ∞-categories $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$.

Lemma 4.2.15. *For a morphism $f\colon \mathcal{X}' \to \mathcal{X}$ of differentiable stacks, the pushforward functor $f_*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ preserves homotopy invariant sheaves:*

$$f_*\colon \mathrm{H}(\mathcal{X}') \to \mathrm{H}(\mathcal{X}).$$

Proof. The functor f_* is given by precomposition with the base change functor $-\times_{\mathcal{X}} \mathcal{X}'\colon \mathrm{Sub}_{/\mathcal{X}} \to \mathrm{Sub}_{/\mathcal{X}'}$, which sends the projection map $\mathcal{Y}\times\mathbb{R} \to \mathcal{Y}$ to the projection $(\mathcal{Y}\times_{\mathcal{X}} \mathcal{X}')\times\mathbb{R} \to \mathcal{Y}\times_{\mathcal{X}}\mathcal{X}'$. The claim follows immediately. □

The pushforward functor $f_*\colon \mathrm{H}(\mathcal{X}') \to \mathrm{H}(\mathcal{X})$ admits a left adjoint

$$f^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{H}(\mathcal{X}')$$

given as the composite $\mathrm{H}(\mathcal{X}) \hookrightarrow \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{f^*} \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) \xrightarrow{L_{\mathrm{htp}}} \mathrm{H}(\mathcal{X}')$. Here we abuse notation by writing both f^* for the pullback functor both before and after applying homotopy localization; when potential confusion arises we make the source and target of the functor f^* explicit. In case f is a representable submersion, the two functors in fact agree:

Lemma 4.2.16. *For every representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$, the pullback functor $f^*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Y}})$ preserves homotopy invariant sheaves.*

Proof. The functor f^* is given by precomposition with the forgetful functor $\mathrm{Sub}_{/\mathcal{Y}} \to \mathrm{Sub}_{/\mathcal{X}}$. Since this preserves the projection maps $\mathcal{Z}\times\mathbb{R} \to \mathcal{Z}$, the claim follows. □

Corollary 4.2.17. *For every representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$, the pullback functor $f^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{H}(\mathcal{Y})$ admits a left adjoint $f_\sharp\colon \mathrm{H}(\mathcal{Y}) \to \mathrm{H}(\mathcal{X})$ given as the composite*

$$\mathrm{H}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Y}}) \xrightarrow{f_\sharp} \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{L_{\mathrm{htp}}} \mathrm{H}(\mathcal{X}).$$ □

Again, we abuse notation by writing $f_\sharp$ for the left adjoint to f^* both before and after applying homotopy localization.

Proposition 4.2.18 (Symmetric monoidality)**.** *For a morphism $f\colon \mathcal{X}' \to \mathcal{X}$ of differentiable stacks, the pullback functor $f^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{H}(\mathcal{X}')$ preserves finite products.*

Proof. By Corollary 4.2.14, this follows directly from Corollary 4.1.12. □

Proposition 4.2.19 (Smooth base change)**.** *Consider a pullback square of differentiable stacks*

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X}' \arrow[r, "g"] & \mathcal{X}, \end{tikzcd}$$

where f (and thus f') is a representable submersion. Then the Beck-Chevalley map

$$\mathrm{BC}_\sharp \colon f'_\sharp h^* \Rightarrow g^* f_\sharp$$

of functors $\mathrm{H}(\mathcal{Y}) \to \mathrm{H}(\mathcal{X}')$ *is an equivalence.*

Proof. The map is obtained by applying the homotopy localization functor

$$L_{\mathrm{htp}} \colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'}) \to \mathrm{H}(\mathcal{X}')$$

to the analogous exchange map $f'_\sharp h^* \Rightarrow g^* f_\sharp$ as functors from $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Y}})$ to $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}'})$, and thus it will suffice to show the latter is an equivalence. As all functors in sight preserve colimits, it suffices to check this on representables $\mathcal{Z} \in \mathrm{Sub}_{/\mathcal{Y}}$. But in this case, this is simply the equivalence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{Y}' \xrightarrow{\sim} \mathcal{Z} \times_{\mathcal{X}} \mathcal{X}'$ over $\mathcal{X}'$, obtained from the pullback square. □

Corollary 4.2.20. *Let $j \colon \mathcal{U} \hookrightarrow \mathcal{X}$ be an open embedding of differentiable stacks. Then the functors*

$$j_\sharp \colon \mathrm{H}(\mathcal{U}) \to \mathrm{H}(\mathcal{X}) \qquad \textit{and} \qquad j_* \colon \mathrm{H}(\mathcal{U}) \to \mathrm{H}(\mathcal{X})$$

are fully faithful.

Proof. Applying Proposition 4.2.19 to the pullback square

$$\begin{tikzcd} \mathcal{U} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & \mathcal{U} \arrow[d, "j"] \\ \mathcal{U} \arrow[r, "j"] & \mathcal{X} \end{tikzcd}$$

shows that the unit $\mathrm{id} \to j^* j_\sharp$ is an equivalence, showing that $j_\sharp$ is fully faithful. The claim for j_* follows immediately. □

Proposition 4.2.21 (Smooth projection formula)**.** *Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a representable submersion of differentiable stacks. Then for all objects $A \in \mathrm{H}(\mathcal{X})$ and $B \in \mathrm{H}(\mathcal{Y})$, the exchange map*

$$\mathrm{PF}_\sharp \colon f_\sharp(f^* A \times B) \to A \times f_\sharp B$$

is an equivalence in $\mathrm{H}(\mathcal{X})$

Proof. By Corollary 4.2.14, it suffices to prove the analogous statement for the functor $f_\sharp \colon \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{Y}}) \to \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}})$. As both sides preserve colimits in A and B, we may assume that $A \in \operatorname{Sub}_{/\mathcal{X}}$ and $B \in \operatorname{Sub}_{/\mathcal{Y}}$ are representables. In that case, this map is simply the canonical equivalence

$$(A \times_{\mathcal{X}} \mathcal{Y}) \times_{\mathcal{Y}} B \xrightarrow{\sim} A \times_{\mathcal{X}} B. \qquad \square$$

Proposition 4.2.22 (Homotopy invariance). *Let $\pi \colon \mathcal{E} \to \mathcal{X}$ be a vector bundle over a differentiable stack $\mathcal{X}$. Then the functor $\pi^* \colon \mathrm{H}(\mathcal{X}) \to \mathrm{H}(\mathcal{E})$ is fully faithful.*

Proof. We show that for every object $A \in \mathrm{H}(\mathcal{X})$, the counit map $\pi_\sharp \pi^* A \to A$ is an equivalence. Since both sides preserve colimits in the variable A, it suffices to prove this when A is a representable $(f \colon \mathcal{Y} \to \mathcal{X}) \in \operatorname{Sub}_{/\mathcal{X}}$. In that case, there is an equivalence $\pi_\sharp \pi^* A \simeq L_{\mathrm{htp}}(\mathcal{E} \times_{\mathcal{X}} \mathcal{Y})$, and the counit is given by the map $L_{\mathrm{htp}}(\mathrm{pr}_{\mathcal{Y}}) \colon L_{\mathrm{htp}}(\mathcal{E} \times_{\mathcal{X}} \mathcal{Y}) \to L_{\mathrm{htp}}(\mathcal{Y})$ induced by the projection onto $\mathcal{Y}$. Since this projection is a strict homotopy equivalence, the claim thus follows from Corollary 4.2.9. $\square$

We end the section by making precise the functoriality of the assignment $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ and showing that it forms a sheaf of ∞-categories on DiffStk.

Construction 4.2.23. We will turn the assignment $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ into a functor $\mathrm{DiffStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Consider again the functor $\operatorname{Shv}(\operatorname{Sub}_{/-}) \colon \mathrm{DiffStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ from Corollary 4.1.12. As the value at the terminal object $\mathrm{pt} \in \mathrm{DiffStk}$ is $\operatorname{Shv}(\mathrm{Diff})$, this functor admits a canonical enhancement to a functor $\operatorname{Shv}(\operatorname{Sub}_{/-}) \colon \mathrm{DiffStk}^{\mathrm{op}} \to \mathrm{CAlg}_{\operatorname{Shv}(\mathrm{Diff})}(\mathrm{Pr}^{\mathrm{L}})$. To obtain $\mathrm{H}(\mathcal{X})$ from $\operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}})$, we have to invert the morphisms $\mathcal{Y} \times \mathbb{R} \to \mathcal{Y}$, but note that these morphisms are obtained by tensoring $\mathcal{Y} \in \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}})$ with the map $\mathbb{R} \to \mathrm{pt}$ in $\operatorname{Shv}(\mathrm{Diff})$. As localizing the ∞-category $\operatorname{Shv}(\mathrm{Diff})$ at the map $\mathbb{R} \to \mathrm{pt}$ gives the ∞-category An by Proposition 4.2.27, it follows that for every differentiable stack $\mathcal{X}$ there is an equivalence

$$\mathrm{H}(\mathcal{X}) \simeq \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}}) \otimes_{\operatorname{Shv}(\mathrm{Diff})} \mathrm{An} \quad \in \quad \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

For every morphism $f \colon \mathcal{X}' \to \mathcal{X}$, the pullback functor $f^* \colon \mathrm{H}(\mathcal{X}) \to \mathrm{H}(\mathcal{X}')$ is obtained by localizing the pullback functor $f^* \colon \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}}) \to \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}'})$, and it follows that the functor

$$\mathrm{H}(-) := \operatorname{Shv}(\operatorname{Sub}_{/-}) \otimes_{\operatorname{Shv}(\mathrm{Diff})} \mathrm{An} \colon \mathrm{DiffStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

is a functorial extension of the assignment $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$.

Lemma 4.2.24. *The functor $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ is a sheaf of ∞-categories on* DiffStk.

Proof. It suffices to show that for every differentiable stack $\mathcal{X}$, the composition of H with the forgetful functor $\mathrm{Open}(\mathcal{X})^{\mathrm{op}} \to \mathrm{DiffStk}^{\mathrm{op}}$ is a sheaf of ∞-categories on $\mathrm{Open}(\mathcal{X})$. As each inclusion $j\colon \mathcal{U} \hookrightarrow \mathcal{V}$ in $\mathrm{Open}(\mathcal{X})$ is a representable submersion, it follows from Lemma 4.2.16 that the pullback functor $j^*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{V}}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{U}})$ sends the subcategory $\mathcal{H}(\mathcal{V})$ into the subcategory $\mathcal{H}(\mathcal{U})$. This determines a unique subdiagram $\mathrm{H}(-) \subseteq \mathrm{Shv}(\mathrm{Sub}_{/-})$ on $\mathrm{Open}(\mathcal{X})^{\mathrm{op}}$. Since each composite

$$\mathrm{H}(\mathcal{U}) \hookrightarrow \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{L_{\mathrm{htp}}} \mathrm{H}(\mathcal{U})$$

is an equivalence, the functoriality of this subdiagram agrees with the restricted functoriality from the functor $\mathrm{H}\colon \mathrm{DiffStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. Since the assignment $\mathcal{U} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{U}})$ is a sheaf of ∞-categories by Lemma 4.1.10, it will thus suffice to prove the following statement: given an open cover $\{\mathcal{U}_i \hookrightarrow \mathcal{X}\}$ of a differentiable stack $\mathcal{X}$, a sheaf $\mathcal{F} \in \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ is homotopy invariant if and only if its restriction to every $\mathcal{U}_i$ is homotopy invariant. But this is immediate from the definition. □

4.2.5 From ordinary sheaves to genuine sheaves

In this subsection, we will show that the forgetful functor $\gamma^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X})$ from genuine sheaves to ordinary sheaves, defined in Construction 4.1.19, exhibits $\mathrm{Shv}(\mathcal{X})$ as both a localization as well as a colocalization of $\mathrm{H}(\mathcal{X})$. In particular, every ordinary sheaf on a differentiable stack $\mathcal{X}$ naturally gives rise to two distinct genuine sheaves on $\mathcal{X}$.

Notation 4.2.25. For a differentiable stack $\mathcal{X}$, we abuse notation and write $\gamma^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X})$ for the restriction of the functor $\gamma^*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(\mathcal{X})$ from Construction 4.1.19.

Proposition 4.2.26. *Let $\mathcal{X}$ be a differentiable stack. Then the fully faithful functor $\gamma_\sharp\colon \mathrm{Shv}(\mathcal{X}) \hookrightarrow \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ from Proposition 4.1.20 lands in the subcategory of genuine sheaves:*

$$\gamma_\sharp\colon \mathrm{Shv}(\mathcal{X}) \hookrightarrow \mathrm{H}(\mathcal{X}).$$

In particular, the functor $\gamma^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X})$ admits a fully faithful left adjoint.

Proof. Given an ordinary sheaf $\mathcal{F} \in \mathrm{Shv}(\mathcal{X})$, we show that the associated sheaf $\gamma_\sharp\mathcal{F} \in \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ is homotopy invariant. Recall from Proposition 4.1.20 that, for a representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$, the ∞-groupoid $(\iota_\sharp\mathcal{F})(\mathcal{Y})$ is obtained by first pulling back $\mathcal{F}$ to a sheaf $f^*\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ over $\mathcal{Y}$, and then passing to global sections:

$$(\iota_\sharp\mathcal{F})(\mathcal{Y}) \simeq \Gamma_{\mathcal{Y}}(f^*\mathcal{F}).$$

In a similar way, one sees that the restriction map $\mathrm{pr}^*\colon (\iota_\sharp\mathcal{F})(\mathcal{Y}) \to (\iota_\sharp\mathcal{F})(\mathcal{Y}\times\mathbb{R})$ induced by the projection map $\mathrm{pr}\colon \mathcal{Y}\times\mathbb{R} \to \mathcal{Y}$ is equivalent to the map $\Gamma_{\mathcal{Y}}(f^*\mathcal{F}) \to \Gamma_{\mathcal{Y}\times\mathbb{R}}((f\circ \mathrm{pr})^*\mathcal{F}) = \Gamma_{\mathcal{Y}}(\mathrm{pr}_*\,\mathrm{pr}^*\,f^*\mathcal{F})$ induced by the unit map $\mathrm{id} \to \mathrm{pr}_*\,\mathrm{pr}^*$. It will thus suffice to show that this unit map is an equivalence, i.e., that the pullback functor

$$\mathrm{pr}^*\colon\ \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}\times\mathbb{R})$$

is fully faithful. Since both sides satisfy descent in the variable $\mathcal{Y}$, it will suffice to prove this in the case where $\mathcal{Y} = M$ is a smooth manifold, which is a special case of [Lur17, Lemma A.2.9]. This finishes the proof. $\square$

As we will see in Section 4.4, the functor $\gamma^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X})$ is not an equivalence in general: the ∞-category $\mathrm{H}(\mathcal{X})$ of genuine sheaves captures more geometric fixed point data than the ∞-category $\mathrm{Shv}(\mathcal{X})$ of ordinary sheaves. The following result shows that this difference between genuine and ordinary sheaves vanishes when $\mathcal{X}$ has trivial isotropy:

Proposition 4.2.27. *Let M be a smooth manifold. Then the functor*

$$\gamma^*\colon \mathrm{H}(M) \to \mathrm{Shv}(M)$$

from Notation 4.2.25 is an equivalence of ∞-categories.

Proof. Since γ^* admits a fully faithful left adjoint $\gamma_\sharp$, it will suffice to show that the functor γ^* is conservative. To this end, let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of genuine sheaves on $\mathcal{X}$, i.e., a morphism of homotopy invariant sheaves on $\mathrm{Sub}_{/\mathcal{X}}$, and assume that the induced map $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of animae for every open subspace $U \subseteq M$. By homotopy invariance of $\mathcal{F}$ and $\mathcal{G}$, it follows that also the map $\varphi(U\times\mathbb{R}^n)\colon \mathcal{F}(U\times\mathbb{R}^n) \to \mathcal{G}(U\times\mathbb{R}^n)$ is an equivalence for every $U \in \mathrm{Open}(M)$ and $n \in \mathbb{N}$. Consider now an arbitrary smooth submersion $f\colon Y \to M$ and let $\mathcal{U}$ be the poset of open subspaces V of Y for which the restriction $f|_V\colon V \to f(V) =: U$ is isomorphic in $\mathrm{Sub}_{/M}$ to the projection $\mathrm{pr}\colon U\times\mathbb{R}^n \to U$ for some natural number n. By Theorem C.8, $\mathcal{U}$ forms a *complete open cover* of Y, in the sense of Definition E.58, and thus it follows from Proposition E.60 that Y admits a hypercover consisting of elements of $\mathcal{U}$. Since $\mathrm{Shv}(\mathrm{Sub}_{/M})$ is hypercomplete by Proposition 4.1.15 and since the map $\varphi(V)\colon \mathcal{F}(V) \to \mathcal{G}(V)$ is an equivalence for every $V \in \mathcal{U}$, we deduce that also the map $\varphi(Y)\colon \mathcal{F}(V) \to \mathcal{G}(V)$ is an equivalence. Since this holds for every $Y \in \mathrm{Sub}_{/M}$, it follows that the map φ is an equivalence in $\mathrm{H}(M)$, finishing the proof. $\square$

Remark 4.2.28. By taking $M = \mathrm{pt}$ in the previous proposition, we recover the well-known fact that the constant sheaf functor $\Gamma^*\colon \mathrm{An} \to \mathrm{Shv}(\mathrm{Diff})$ induces an equivalence $\mathrm{An} \simeq \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff})$. We refer to [ADH21, Proposition 4.3.1] for a more direct proof of this fact.

We saw in Proposition 4.2.26 that the functor $\gamma^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X})$ admits a fully faithful left adjoint $\gamma_\sharp$ and thus exhibits $\mathrm{Shv}(\mathcal{X})$ as a colocalization of $\mathrm{H}(\mathcal{X})$. The next lemma shows that it also admits a full faithful *right* adjoint:

Lemma 4.2.29. *Let $\mathcal{X}$ be a differentiable stack. Then the functor $\gamma^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X})$ admits a fully faithful right adjoint*

$$\gamma_*\colon \mathrm{Shv}(\mathcal{X}) \hookrightarrow \mathrm{H}(\mathcal{X}).$$

Proof. To show that γ^* admits a right adjoint, it suffices to show that it preserves small colimits. Choosing a representable atlas $f\colon M \twoheadrightarrow \mathcal{X}$ for $\mathcal{X}$, the functor $f^*\colon \mathrm{Shv}(\mathcal{X}) \to \mathrm{Shv}(M)$ is conservative and preserves colimits, so it suffices to show that the composite $f^* \circ \gamma^*$ preserves colimits. By naturality of γ^*, this composite is equivalent to the composite

$$\mathrm{H}(\mathcal{X}) \xrightarrow{f^*} \mathrm{H}(M) \xrightarrow{\gamma^*} \mathrm{Shv}(M).$$

The first functor preserves colimits as it admits a right adjoint f_*, and the second functor preserves colimits as it is an equivalence by Proposition 4.2.27. This shows that γ^* preserves colimits and thus admits a right adjoint γ_*. As γ_* is a double right adjoint of the fully faithful functor $\gamma_\sharp\colon \mathrm{Shv}(\mathcal{X}) \hookrightarrow \mathrm{H}(\mathcal{X})$, it follows that also γ_* is fully faithful, finishing the proof. □

We may now make precise the heuristic mentioned in the introduction of this chapter that a morphism is inverted by the localization functor $\gamma^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X})$ if and only if it becomes an equivalence after forgetting the isotropy information:

Corollary 4.2.30. *Let $\mathcal{X}$ be a differentiable stack and let $f\colon M \twoheadrightarrow \mathcal{X}$ be a representable atlas for $\mathcal{X}$. Then a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of genuine sheaves on $\mathcal{X}$ is sent to an equivalence under the functor $\gamma^*\colon \mathrm{H}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X})$ if and only if the morphism $f^*\gamma\colon f^*\mathcal{F} \to f^*\mathcal{G}$ in $\mathrm{H}(M)$ is an equivalence.*

Proof. By Proposition 4.2.27, the bottom horizontal map in the commutative diagram

$$\begin{tikzcd} \mathrm{H}(\mathcal{X}) \arrow[r, "\gamma^*"] \arrow[d, "f^*"'] & \mathrm{Shv}(\mathcal{X}) \arrow[d, "f^*"] \\ H(M) \arrow[r, "\gamma^*", "\sim"'] & \mathrm{Shv}(M) \end{tikzcd}$$

is an equivalence. As the functor $f^*\colon \mathrm{Shv}(\mathcal{X}) \to \mathrm{Shv}(M)$ is conservative, the claim follows. □

4.3 Genuine sheaves of spectra

In this section, we define for every separated differentiable stack $\mathcal{X}$ an ∞-category $\mathrm{SH}(\mathcal{X})$ of *genuine sheaves of spectra on* $\mathcal{X}$.

4.3.1 Pointed genuine sheaves

Fix a differentiable stack $\mathcal{X}$.

Definition 4.3.1 (Pointed genuine sheaves)**.** We define the ∞-category $\mathrm{H}_\bullet(\mathcal{X})$ of *pointed genuine sheaves on* $\mathcal{X}$ as the ∞-category of pointed objects in $\mathrm{H}(\mathcal{X})$:

$$\mathrm{H}_\bullet(\mathcal{X}) := \mathrm{H}(\mathcal{X})_*.$$

Observe that this is equivalent to the ∞-category $\mathrm{H}(\mathcal{X}, \mathrm{An}_*)$ of An_*-valued genuine sheaves on $\mathcal{X}$.

A priori, the ∞-category $\mathrm{H}_\bullet(\mathcal{X})$ contains two kinds of spheres S^n: the geometric spheres, obtained by regarding S^n as a smooth manifold, and the homotopical spheres, obtained by regarding S^n as an ∞-groupoid. We will show that they are in fact equivalent.

Lemma 4.3.2. *For any vector bundle* $p\colon \mathcal{E} \to \mathcal{X}$*, there is a cofiber sequence in* $\mathrm{H}(\mathcal{X})$

$$L_{\mathrm{htp}}(\mathcal{E} \setminus s(\mathcal{X})) \hookrightarrow L_{\mathrm{htp}}(\mathcal{E}) \to L_{\mathrm{htp}}(S^{\mathcal{E}}).$$

Proof. There are pullback squares

$$\begin{tikzcd}
\mathcal{E} \setminus s(\mathcal{E}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S^{\mathcal{E}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "p"] \\
(\mathbb{R}^n \setminus \{0\})//\mathrm{GL}(n) \arrow[r, hook] & \mathbb{R}^n//\mathrm{GL}(n) \arrow[r] & S^n//\mathrm{GL}(n) \arrow[r] & \mathbb{B}\,\mathrm{GL}(n).
\end{tikzcd}$$

By descent, it will suffice to prove the statement for the universal case $\mathbb{R}^n//\mathrm{GL}(n) \to \mathbb{B}\,\mathrm{GL}(n)$. Identify the subspace $\mathbb{R}^n \subseteq S^n$ with the open complement of the point $\infty \in S^n$. Then $\mathbb{R}^n$ and $S^n \setminus \{0\}$ form an open cover of S^n whose intersection is $\mathbb{R}^n \setminus \{0\}$. By passing to quotient stacks, we obtain an analogous open cover of $S^n//\mathrm{GL}(n)$. By descent, we thus obtain a pushout square

$$\begin{tikzcd}
L_{\mathrm{htp}}(\mathbb{R}^n \setminus \{0\}//\mathrm{GL}(n)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & L_{\mathrm{htp}}(\mathbb{R}^n//\mathrm{GL}(n)) \arrow[d] \\
L_{\mathrm{htp}}(S^n \setminus \{0\}//\mathrm{GL}(n)) \arrow[r] & L_{\mathrm{htp}}(S^n//\mathrm{GL}(n))
\end{tikzcd}$$

in $\mathrm{H}(\mathbb{B}\,\mathrm{GL}(n))$. As $S^n \setminus \{0\} \cong \mathbb{R}^n$ is strictly homotopy equivalent to the point, the lower left corner is terminal in $\mathrm{H}(\mathbb{B}\,\mathrm{GL}(n))$, and hence this pushout square constitutes the desired cofiber sequence. $\square$

Corollary 4.3.3. *For every natural number $n \geq 0$, the following two objects in $\mathrm{H}_\bullet(\mathcal{X})$ are equivalent:*

(1) The geometric n-sphere, defined as the image of $S^n \in \mathrm{Diff}$ under the functor

$$\mathrm{Diff} \xrightarrow{\mathcal{X}\times -} \mathrm{Sub}_{/\mathcal{X}} \hookrightarrow \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{L_{\mathrm{htp}}} \mathrm{H}(\mathcal{X}),$$

with base point given by the map $\{\infty\} \to S^n$.

(2) The homotopical n-sphere, defined as the image of $S^n \in \mathrm{An}_*$ under the unique symmetric monoidal colimit-preserving functor $\mathrm{An}_* \to \mathrm{H}_\bullet(\mathcal{X})$.

Proof. This is clear when $n = 0$. For $n > 1$, this follows by induction: applying Lemma 4.3.2 to the vector bundle $\mathcal{X} \times \mathbb{R}^n$ and using the homotopy equivalences $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ and $\mathbb{R}^n \simeq \mathrm{pt}$, we obtain a cofiber sequence

$$L_{\mathrm{htp}}(\mathcal{X} \times S^{n-1}) \to * \to L_{\mathrm{htp}}(\mathcal{X} \times S^n),$$

showing that $L_{\mathrm{htp}}(\mathcal{X} \times S^n) \simeq \Sigma(L_{\mathrm{htp}}(\mathcal{X} \times S^{n-1}))$. $\square$

4.3.2 Genuine sheaves of spectra on global quotient stacks

We will now define the ∞-category $\mathrm{SH}(\mathcal{X})$ in the case where $\mathcal{X}$ is a *global quotient stack*, i.e. of the form $M//G$ for some compact Lie group G and some smooth G-manifold M. Equivalently, by Corollary 2.3.21, this means that the stack $\mathcal{X}$ comes equipped with a representable map $\mathcal{X} \to \mathbb{B}G$ for some compact Lie group G.

Definition 4.3.4. Let $\mathcal{X}$ be a global quotient stack, and let $\mathrm{Sph}(\mathcal{X}) \subseteq \mathrm{H}_\bullet(\mathcal{X})$ denote the subcategory spanned by the sphere bundles $L_{\mathrm{htp}}(S^{\mathcal{E}})$ for all vector bundles $\mathcal{E} \to \mathcal{X}$. We define the ∞-category $\mathrm{SH}(\mathcal{X})$ of *genuine sheaves of spectra over $\mathcal{X}$* by formally inverting the objects of $\mathrm{Sph}(\mathcal{X})$:

$$\mathrm{SH}(\mathcal{X}) := \mathrm{H}_\bullet(\mathcal{X})[\mathrm{Sph}(\mathcal{X})^{-1}] \quad \in \quad \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

More explicitly, this means that $\mathrm{SH}(\mathcal{X})$ comes equipped with a functor $\Sigma^\infty \colon \mathrm{H}_\bullet(\mathcal{X}) \to \mathrm{SH}(\mathcal{X})$ such that for any presentable ∞-category $\mathcal{E}$ precomposition with Σ^∞ induces an inclusion of path components

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})}(\mathrm{SH}(\mathcal{X}), \mathcal{E}) \to \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})}(\mathrm{H}_\bullet(\mathcal{X}), \mathcal{E})$$

hitting those symmetric monoidal left adjoints $F\colon \mathrm{H}_\bullet(\mathcal{X}) \to \mathcal{E}$ which sends every sphere bundle $L_{\mathrm{htp}}(S^{\mathcal{E}}) \in \mathrm{H}_\bullet(\mathcal{X})$ to an invertible object in $\mathcal{E}$.

We let $\Sigma^\infty_+\colon \mathrm{H}(\mathcal{X}) \to \mathrm{SH}(\mathcal{X})$ denote the composite

$$\mathrm{H}(\mathcal{X}) \xrightarrow{(-)_+} \mathrm{H}_\bullet(\mathcal{X}) \xrightarrow{\Sigma^\infty} \mathrm{SH}(\mathcal{X}).$$

Remark 4.3.5. We shall show in Proposition 4.4.17 below that for a compact Lie group G the ∞-category $\mathrm{SH}(\mathbb{B}G)$ of genuine sheaves of spectra over the classifying stack $\mathbb{B}G$ is equivalent to the ∞-category Sp_G of genuine G-spectra, explaining the terminology.

Lemma 4.3.6. *For every global quotient stack $\mathcal{X}$, the ∞-category $\mathrm{SH}(\mathcal{X})$ is stable.*

Proof. As $\mathrm{SH}(\mathcal{X})$ is stable if and only if the homotopical 1-sphere S^1 is invertible, this is a direct consequence from Corollary 4.3.3, since the object $L_{\mathrm{htp}}(\mathcal{X}\times S^1) \in H_\bullet(\mathcal{X})$ is the sphere bundle of the vector bundle $\mathcal{X}\times\mathbb{R} \to \mathcal{X}$ and thus invertible in $\mathrm{SH}(\mathcal{X})$. □

Warning 4.3.7. The ∞-category $\mathrm{SH}(\mathcal{X})$ should not be confused with the ∞-category $\mathrm{H}(\mathcal{X};\mathrm{Sp})$ of Sp-valued genuine sheaves on $\mathcal{X}$, which is obtained from $\mathrm{H}_\bullet(\mathcal{X})$ by only inverting the object $L_{\mathrm{htp}}(\mathcal{X}\times S^1)$.

Lemma 4.3.8. *For every vector bundle $\mathcal{E} \to \mathcal{X}$, the sphere bundle $S = L_{\mathrm{htp}}(S^{\mathcal{E}}) \in \mathrm{H}_\bullet(\mathcal{X})$ is a symmetric object, in the sense that the cyclic permutation $\sigma_{123}\colon S\wedge S\wedge S \to S\wedge S\wedge S$ is homotopic to the identity.*

Proof. It suffices to prove this in the universal case $\mathcal{E} = \mathbb{R}^n//\mathrm{GL}(n) \to \mathbb{B}\mathrm{GL}(n)$. Note that the twist map $\tau = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^n$ is homotopic to the linear map $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ via the rotation homotopy $t \mapsto \left(\begin{smallmatrix} -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \\ \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \end{smallmatrix}\right)$. It follows that the cyclic permutation $\sigma_{123}\colon \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ is homotopic to the identity. As all of the linear maps in the homotopy are invertible, they extend to maps $S^{n+n+n} \to S^{n+n+n}$, giving the claim. □

Proposition 4.3.9. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism of global quotient stacks. Then the functor*

$$\mathrm{H}(\mathcal{Y}) \otimes_{\mathrm{H}(\mathcal{X})} \mathrm{SH}(\mathcal{X}) \to \mathrm{SH}(\mathcal{Y}),$$

obtained by the tensoring up the functor $\Sigma^\infty_+\colon \mathrm{H}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{Y})$ from $\mathrm{H}(\mathcal{X})$ to $\mathrm{SH}(\mathcal{Y})$, is an equivalence.

Proof. By assumption we may write $\mathcal{X}$ in the form $N//G$ for a compact Lie group G and a smooth G-manifold N. Since f is representable, it follows that we may write $\mathcal{Y}$ as $M//G$ for some smooth G-manifold M and that f is induced by a G-equivariant map $M \to N$.

Observe that if we know the statement for both maps $M//G \to \mathbb{B}G$ and $N//G \to \mathbb{B}G$, it also follows for f. This means we may assume that $\mathcal{X} = \mathbb{B}G$ and $\mathcal{Y} = M//G$.

Let $E \to M$ be a G-equivariant vector bundle over M. By [Seg68, Proposition 2.4] there exists a finite-dimensional G-representation V and a G-equivariant vector bundle $E' \to M$ such that there is an isomorphism $E \oplus E' \cong V \times M$ as vector bundles over M. This implies that $S^{V\times M} \simeq S^E \otimes S^{E'}$, and thus S^E is a factor of the pullback of $S^V \in \mathrm{H}_\bullet(\mathbb{B}G)$ along the map $M//G \to \mathbb{B}G$. Since this holds for all $E \in \mathrm{Vect}^G(M)$, the claim now follows from Lemma I.2.45 and Lemma I.2.44 in Part I. □

Corollary 4.3.10 (Sheaves of spectra on smooth manifolds)**.** *When $\mathcal{X} = M$ is a smooth manifold, there is an equivalence*

$$\mathrm{SH}(M) \simeq \mathrm{Shv}(M; \mathrm{Sp})$$

between the ∞-categories of genuine and ordinary sheaves of spectra on M.

Proof. Recall the equivalence $\mathrm{H}(M) \simeq \mathrm{Shv}(M)$ from Proposition 4.2.27. By Proposition 4.3.9 we thus have $\mathrm{SH}(M) \simeq \mathrm{Shv}(M) \otimes_{\mathrm{An}} \mathrm{Sp} = \mathrm{Shv}(M; \mathrm{Sp})$. □

We will now make precise the functoriality of the assignment $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ and show that it is a sheaf of ∞-categories.

Construction 4.3.11. Let $\mathrm{QtStk} \subseteq \mathrm{DiffStk}$ denote the full subcategory of global quotient stacks. We will turn the assignment $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ into a functor $\mathrm{SH}\colon \mathrm{QtStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Recall from Definition 2.41 in Chapter I.2 the ∞-category $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ of augmented presentably symmetric monoidal ∞-categories: presentably symmetric monoidal ∞-categories C equipped with a fully faithful subcategory S. The assignment $\mathcal{X} \mapsto \mathrm{H}_\bullet(\mathcal{X})$ admits a lift to a functor $\mathrm{H}_\bullet\colon \mathrm{QtStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ by equipping $\mathrm{H}_\bullet(\mathcal{X})$ with the subcategory $\mathrm{Sph}(\mathcal{X})$ of sphere bundles from Definition 4.3.4; this is well-defined as the pullback functor f^* sends sphere bundles to sphere bundles for any morphism of stacks $f\colon \mathcal{Y} \to \mathcal{X}$. The functor $\mathrm{SH}\colon \mathrm{QtStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is now given as the composite

$$\mathrm{SH}\colon \mathrm{QtStk}^{\mathrm{op}} \xrightarrow{\mathrm{H}_\bullet} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}} \xrightarrow{\mathcal{L}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}),$$

where $\mathcal{L}$ is the formal inversion functor $(C, S) \mapsto C[S^{-1}]$ from Lemma 2.42 in Chapter I.2.

More explicitly, given a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of global quotient stacks, the functor $f^*\colon \mathrm{SH}(\mathcal{X}) \to \mathrm{SH}(\mathcal{Y})$ is the unique symmetric monoidal colimit-presering left adjoint which makes the following diagram commute:

$$\begin{tikzcd} \mathrm{H}(\mathcal{X}) \arrow[r, "f^*"] \arrow[d, "\Sigma^\infty_+"'] & \mathrm{H}(\mathcal{Y}) \arrow[d, "\Sigma^\infty_+"] \\ \mathrm{SH}(\mathcal{X}) \arrow[r, dashed, "f^*"] & \mathrm{SH}(\mathcal{Y}). \end{tikzcd}$$

Lemma 4.3.12. *The functor* $\mathrm{SH}\colon \mathrm{QtStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *is a sheaf of* ∞*-categories on* QtStk.

Proof. It suffices to show that for every compact Lie group G, the functor $\mathrm{Diff}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending M to $\mathrm{SH}(M//G)$ is a sheaf of ∞-categories on Diff_G. By Proposition 4.3.9, this functor is equivalent to the functor $M \mapsto \mathrm{H}_\bullet(M//G) \otimes_{\mathrm{H}_\bullet(\mathbb{B}G)} \mathrm{SH}(\mathbb{B}G)$. As the objects $L_{\mathrm{htp}}(S^V) \in H(\mathbb{B}G)$ are symmetric objects by Lemma 4.3.8, the ∞-category $\mathrm{SH}(\mathbb{B}G)$ may be expressed as a colimit in Pr^{L} of the sequence

$$\mathrm{H}_\bullet(\mathbb{B}G) \xrightarrow{-\otimes S^{V_1}} \mathrm{H}_\bullet(\mathbb{B}G) \xrightarrow{-\otimes S^{V_2}} \mathrm{H}_\bullet(\mathbb{B}G) \xrightarrow{-\otimes S^{V_3}} \dots,$$

where $V_1, V_2, V_3, \dots$ is a sequence of irreducible G-representations containing every irreducible G-representation infinitely many times, see [Rob15, Corollary 2.22] and [Hoy17, Section 6.1]. It follows that for every smooth G-manifold M the ∞-category $\mathrm{SH}(M//G)$ is expressed as the colimit in Pr^{L} of the sequence

$$\mathrm{H}_\bullet(M//G) \xrightarrow{-\otimes S^{V_1}} \mathrm{H}_\bullet(M//G) \xrightarrow{-\otimes S^{V_2}} \mathrm{H}_\bullet(M//G) \xrightarrow{-\otimes S^{V_3}} \dots,$$

which means that the underlying ∞-category of $\mathrm{SH}(M//G)$ is a limit in Cat_∞ of the diagram of right adjoints $\underline{\mathrm{Hom}}(S^{V_i}, -)\colon \mathrm{H}_\bullet(M//G) \to \mathrm{H}_\bullet(M//G)$. Since the functor $\mathrm{H}_\bullet(-//G)\colon \mathrm{Diff}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is a sheaf of ∞-categories by Lemma 4.2.24 and limits commute with limits, it follows that also the functor $\mathrm{SH}(-//G)\colon \mathrm{Diff}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is a sheaf of ∞-categories. This finishes the proof. $\square$

4.3.3 Genuine sheaves of spectra on separated stacks

We will now define the ∞-category $\mathrm{SH}(\mathcal{X})$ for an arbitrary separated differentiable stack $\mathcal{X}$. Observe that every global quotient stack $\mathcal{X} = M//G$ is in particular separated, since we assume that the Lie group G is compact. This provides an inclusion of QtStk into the $(2,1)$-category SepStk of separated differentiable stacks.

Lemma 4.3.13. *Every sheaf of* ∞*-categories on* QtStk *extends uniquely to a sheaf of* ∞*-categories on* SepStk *via right Kan extension.*

Proof. Since every open substack of a global quotient stack is again a global quotient stack, right Kan extension provides a fully faithful functor from sheaves on QtStk to sheaves on SepStk, see Corollary E.47. Since every separated differentiable stack is locally a global quotient stack by Theorem 3.7.2, this functor is an equivalence. $\square$

Definition 4.3.14 (Genuine sheaves of spectra)**.** We define the functor

$$\mathrm{SH}\colon \mathrm{SepStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

as the unique extension of the functor $\mathrm{SH}\colon \mathrm{QtStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ from Construction 4.3.11. For a separated differentiable stack $\mathcal{X}$ we refer to the ∞-category $\mathrm{SH}(\mathcal{X})$ as the *∞-category of genuine sheaves of spectra on $\mathcal{X}$*. We similarly define a functor $\Sigma^\infty_+\colon \mathrm{H}(\mathcal{X}) \to \mathrm{SH}(\mathcal{X})$ by extension from QtStk.

The ∞-category $\mathrm{SH}(\mathcal{X})$ admits the following explicit description:

$$\mathrm{SH}(\mathcal{X}) \simeq \lim_{\mathcal{U}\in\mathrm{Open}^{\mathrm{QtStk}}(\mathcal{X})} \mathrm{SH}(\mathcal{U}) \quad \in \quad \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

Here the limit runs over the poset $\mathrm{Open}^{\mathrm{QtStk}}(\mathcal{X})$ of open substacks $\mathcal{U} \subseteq \mathcal{X}$ of $\mathcal{X}$ which are global quotient stacks.

Proposition 4.3.15. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism of separated differentiable stacks. Then the functor*

$$\mathrm{H}(\mathcal{Y}) \otimes_{\mathrm{H}(\mathcal{X})} \mathrm{SH}(\mathcal{X}) \to \mathrm{SH}(\mathcal{Y}),$$

obtained by the tensoring up the functor $\Sigma^\infty_+\colon \mathrm{H}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{Y})$ from $\mathrm{H}(\mathcal{X})$ to $\mathrm{SH}(\mathcal{Y})$, is an equivalence.

Proof. By Proposition 4.3.9 this holds after pulling back along any inclusion $\mathcal{U} \hookrightarrow \mathcal{X}$ of a global quotient stack $\mathcal{U}$. The general statement then follows by descent. $\square$

Proposition 4.3.16. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable submersion of separated differentiable stacks.*

(1) (Left adjoint) The pullback functor $f^*\colon \mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$ admits a left adjoint, denoted by $f_\sharp\colon \mathrm{SH}(\mathcal{X}) \to \mathrm{SH}(\mathcal{Y})$;

(2) (Smooth base change) These left adjoints satisfy smooth base change, as in Proposition 4.2.19;

(3) (Smooth projection formula) These left adjoints satisfy the smooth projection formula, as in Proposition 4.2.21.

Proof. By Proposition 4.3.15, the functor $f^*\colon \mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$ is given by tensoring the functor $f^*\colon \mathrm{H}(\mathcal{Y}) \to \mathrm{H}(\mathcal{X})$ with $\mathrm{SH}(\mathcal{X})$ over $\mathrm{H}(\mathcal{X})$. By Corollary 4.2.17 the latter functor admits a left adjoint $f_\sharp\colon \mathrm{H}(\mathcal{X}) \to \mathrm{H}(\mathcal{Y})$ which is $\mathrm{H}(\mathcal{X})$-linear by Proposition 4.2.21, and thus we may tensor it with $\mathrm{SH}(\mathcal{X})$ over $\mathrm{H}(\mathcal{X})$ to obtain a $\mathrm{SH}(\mathcal{X})$-linear left adjoint $f_\sharp\colon \mathrm{H}(\mathcal{X}) \to \mathrm{H}(\mathcal{Y})$, proving parts (1) and (3). For part (2), consider a pullback square of separated differentiable stacks

$$\begin{tikzcd}
\mathcal{Y}' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "f"] \\
\mathcal{X}' \arrow[r, "g"] & \mathcal{X}.
\end{tikzcd}$$

By Proposition 4.2.19, the induced square

$$\begin{tikzcd}
\mathrm{H}(\mathcal{X}) \arrow[r, "f^*"] \arrow[d, "g^*"'] & \mathrm{H}(\mathcal{Y}) \arrow[d, "h^*"] \\
\mathrm{H}(\mathcal{X}') \arrow[r, "f'^*"] & \mathrm{H}(\mathcal{X})
\end{tikzcd}$$

is horizontally left adjointable. By tensoring with $\mathrm{SH}(\mathcal{X})$ over $\mathrm{H}(\mathcal{X})$ and using the equivalence $\mathrm{H}(\mathcal{Y}) \otimes_{\mathrm{H}(\mathcal{X})} \mathrm{SH}(\mathcal{X}) \to \mathrm{SH}(\mathcal{Y})$ from Proposition 4.3.15, we see that the top square in the following diagram is horizontally left adjointable:

$$\begin{tikzcd}
\mathrm{SH}(\mathcal{X}) \arrow[r, "f^*"] \arrow[d, "g^*"'] & \mathrm{SH}(\mathcal{Y}) \arrow[d, "h^*"] \\
\mathrm{H}(\mathcal{X}') \otimes_{\mathrm{H}(\mathcal{X})} \mathrm{SH}(\mathcal{X}) \arrow[r, "f'^*"] \arrow[d] & \mathrm{SH}(\mathcal{X}) \otimes_{\mathrm{H}(\mathcal{X})} \mathrm{SH}(\mathcal{X}) \arrow[d] \\
\mathrm{H}(\mathcal{X}') \otimes_{\mathrm{H}(\mathcal{X}')} \mathrm{SH}(\mathcal{X}') \arrow[r, "f'^*"] \arrow[d, equal] & \mathrm{SH}(\mathcal{X}) \otimes_{\mathrm{H}(\mathcal{X}')} \mathrm{SH}(\mathcal{X}') \arrow[d, "\simeq"] \\
\mathrm{SH}(\mathcal{X}') \arrow[r, "f'^*"] & \mathrm{SH}(\mathcal{Y}').
\end{tikzcd}$$

The middle square is also clearly horizontally left adjointable, and the equivalence on the bottom right is another instance of Proposition 4.3.15. It follows that the outer square is horizontally left adjointable, which is what we needed to prove. □

4.4 Comparison with genuine equivariant spectra

Recall from the introduction of this chapter that our main motivation for introducing the ∞-categories $\mathrm{H}(\mathcal{X})$ and $\mathrm{SH}(\mathcal{X})$ of genuine sheaves of animae/spectra on a differentiable stack $\mathcal{X}$ is its relation to equivariant homotopy theory. For a compact Lie group G, there are ∞-categories An_G and Sp_G of *genuine G-animae* (a.k.a. 'G-spaces') and *genuine G-spectra*, respectively. The goal of this section is to show that there are equivalences of ∞-categories

$$\mathrm{H}(\mathbb{B}G) \simeq \mathrm{An}_G \qquad \text{and} \qquad \mathrm{SH}(\mathbb{B}G) \simeq \mathrm{Sp}_G,$$

explaining the terminology 'genuine sheaves'. We thank Marc Hoyois and Sil Linskens for useful discussions and valuable input concerning the results of this section.

We start by analyzing the ∞-category $\mathrm{H}(\mathbb{B}G)$. Recall from Corollary 2.3.20 that the category $\mathrm{Sub}_{/\mathbb{B}G}$ of representable submersions over the classifying stack $\mathbb{B}G$ is equivalent to the category Diff_G of smooth G-manifolds. As a consequence, we get an equivalence

$$H(\mathbb{B}G) = \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Sub}_{/\mathbb{B}G}) \xrightarrow{\sim} \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$$

between the ∞-category of genuine sheaves on $\mathbb{B}G$ and the ∞-category of homotopy invariant sheaves on the site Diff_G, where the Grothendieck topology is given by G-equivariant open covers. We shall show that the latter is equivalent to An_G in two steps:

(a) We first show that the ∞-category $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ is equivalent to a presheaf category, indexed by the full subcategory $\mathrm{Orb}'_G \subseteq \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ spanned by the objects $L_{\mathrm{htp}}(G/H)$;

(b) We then produce a comparison functor $R\colon \mathrm{An}_G \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ and show that it preserves colimits and restricts to an equivalence $\mathrm{Orb}_G \xrightarrow{\sim} \mathrm{Orb}'_G$.

By combining (a) and (b), and using Elmendorf's theorem, it follows that R is an equivalence, leading to the desired equivalence $H(\mathbb{B}G) \simeq \mathrm{An}_G$. The equivalence $\mathrm{SH}(\mathbb{B}G) \simeq \mathrm{Sp}_G$ is then immediate an immediate consequence. We shall now spell out these steps in more detail.

4.4.1 Genuine sheaves form a presheaf category

We start by showing that the ∞-category $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ is equivalent to a presheaf category.

Proposition 4.4.1. *The evaluation functors*

$$\mathrm{ev}_{G/H}\colon \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) \to \mathrm{An}$$

for closed subgroups $H \leqslant G$ are jointly conservative.

Proof. Let $\varphi\colon \mathcal{F} \to \mathcal{F}'$ be a morphism of homotopy invariant sheaves on Diff_G and assume that $\varphi(G/H)\colon \mathcal{F}(G/H) \to \mathcal{F}'(G/H)$ is an equivalence for every closed subgroup $H \leqslant G$.

We have to show that $\varphi(M)\colon \mathcal{F}(M) \to \mathcal{F}'(M)$ is an equivalence for every smooth G-manifold. To this end, recall that M is locally of the form $G \times_H \mathbb{R}^n$ for some closed subgroup $H \subseteq G$ and some H-action on $\mathbb{R}^n$, and thus it follows from Proposition E.60 that M admits a hypercover by G-manifolds of this form. Since the ∞-topos $\mathrm{Shv}(\mathrm{Diff}_G)$ is hypercomplete by Proposition 4.1.15, it follows that M is a colimit of objects of the form $G \times_H \mathbb{R}^n$. Consequently, it will suffice to show that the map

$$\varphi(G \times_H \mathbb{R}^n)\colon \mathcal{F}(G \times_H \mathbb{R}^n) \to \mathcal{F}'(G \times_H \mathbb{R}^n)$$

is an equivalence for all $H \leqslant G$ and all $n \in \mathbb{N}$. But since $\mathcal{F}$ and $\mathcal{F}'$ are homotopy invariant and $G \times_H \mathbb{R}^n$ is G-equivariantly homotopy equivalent to G/H, this follows directly from the assumption on φ. $\square$

Corollary 4.4.2. *The ∞-category $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ is generated under colimits by the objects $L_{\mathrm{htp}}(G/H)$ for all closed subgroups $H \leqslant G$.*

Proof. By adjunction and the Yoneda lemma, there is a natural equivalence

$$\mathrm{ev}_{G/H} \simeq \mathrm{Hom}_{\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)}(L_{\mathrm{htp}}(G/H), -)$$

of functors $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) \to \mathrm{An}$. By Proposition 4.4.1, these functors are conservative. It follows that the objects $L_{\mathrm{htp}}(G/H)$ generate $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ under colimits, see for example [Yan22, Corollary 2.5]. $\square$

Definition 4.4.3 (Fixed point functor)**.** For a closed subgroup $H \leqslant G$, consider the functor $G/H \times -\colon \mathrm{Diff} \to \mathrm{Diff}_G$ from smooth manifolds to smooth G-manifolds. We suggestively denote by

$$(-)^H\colon \mathrm{PSh}(\mathrm{Diff}_G) \to \mathrm{PSh}(\mathrm{Diff})$$

the functor given by precomposition with this functor, and refer to it as the *H-fixed point functor*.

Remark 4.4.4. For a smooth G-manifold M, the sheaf $M^H\colon \mathrm{Diff}^{\mathrm{op}} \to \mathrm{An}$ is a subsheaf of the representable sheaf on M, given at a smooth manifold N by those smooth maps $N \to M$ which factor through the subset of H-fixed points $M^H \subseteq M$. This justifies the notation and terminology for the functor $(-)^H$.

The functor $(-)^H$ is fully compatible with sheafification and homotopy localization:

Proposition 4.4.5. *Let $H \leqslant G$ be a closed subgroup.*

(1) The functor $(-)^H\colon \mathrm{PSh}(\mathrm{Diff}_G) \to \mathrm{PSh}(\mathrm{Diff})$ *preserves sheaves.*

(2) The functor $(-)^H$ *commutes with sheafification:*

$$\begin{tikzcd} \mathrm{PSh}(\mathrm{Diff}_G) \arrow[r, "(-)^H"] \arrow[d, "L_{\mathrm{open}}"'] & \mathrm{PSh}(\mathrm{Diff}) \arrow[d, "L_{\mathrm{open}}"] \\ \mathrm{Shv}(\mathrm{Diff}_G) \arrow[r, "(-)^H"] & \mathrm{Shv}(\mathrm{Diff}). \end{tikzcd}$$

(3) The functor $(-)^H\colon \mathrm{PSh}(\mathrm{Diff}_G) \to \mathrm{PSh}(\mathrm{Diff})$ *preserves homotopy invariant presheaves.*

(4) The functor $(-)^H$ *commutes with homotopy localization:*

$$\begin{tikzcd} \mathrm{PSh}(\mathrm{Diff}_G) \arrow[r, "(-)^H"] \arrow[d, "L_{\mathbb{R}}"'] & \mathrm{PSh}(\mathrm{Diff}) \arrow[d, "L_{\mathbb{R}}"] \\ \mathrm{PSh}^{\mathrm{htp}}(\mathrm{Diff}_G) \arrow[r, "(-)^H"] & \mathrm{PSh}^{\mathrm{htp}}(\mathrm{Diff}). \end{tikzcd}$$

(5) The functor $(-)^H\colon \mathrm{PSh}(\mathrm{Diff}_G) \to \mathrm{PSh}(\mathrm{Diff})$ *restricts to a functor*

$$(-)^H\colon \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff})$$

which preserves limits and colimits.

Proof. Part (1) is clear from the fact that the functor $G/H\times -\colon \mathrm{Diff} \to \mathrm{Diff}_G$ preserves pullbacks along open embeddings and that it sends open covers to G-equivariant open covers.

For part (2), it suffices by Corollary E.45 to show that the functor $G/H\times -\colon \mathrm{Diff} \to \mathrm{Diff}_G$ is a cocontinuous functor of sites, in the sense of Definition E.43. This is immediate from the fact that for a smooth manifold M, the G-equivariant subsets of $G/H\times M$ are precisely those of the form $G/H\times U$ for open subsets $U \subseteq M$.

Part (3) is obvious from the definition of homotopy invariance. Part (4) is immediate from the formula for the homotopy localization functor provided in Construction 4.2.2.

For part (5), the H-fixed point functor restricts to homotopy invariant sheaves by parts (1) and (3). At the level of presheaves, the H-fixed point functor $(-)^H\colon \mathrm{PSh}(\mathrm{Diff}_G) \to \mathrm{PSh}(\mathrm{Diff})$ preserves limits and colimits as these are computed pointwise. It follows at once that also the functor $(-)^H\colon \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff})$ preserves limits as these are computed in presheaves. Since colimits in homotopy invariant sheaves are computed by first computing the colimit in presheaves and then reflecting back into homotopy invariant sheaves, it then follows from parts (2) and (4) that the functor $(-)^H\colon \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff})$ also preserves colimits, finishing the proof. □

Corollary 4.4.6. *For every closed subgroup $H \leqslant G$, the evaluation functor*

$$\mathrm{ev}_{G/H}\colon \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) \to \mathrm{An}$$

preserves colimits.

Proof. This evaluation functor is given by the composite

$$\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) \xrightarrow{(-)^H} \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}) \xrightarrow{\Gamma_*} \mathrm{An},$$

where Γ_* denotes the global section functor. The first functor preserves colimits by part (5) of Proposition 4.4.5. The second functor preserves colimits because it is an equivalence: it is the right adjoint of the equivalence $\gamma_\sharp\colon \mathrm{An} \xrightarrow{\sim} \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff})$ of Proposition 4.2.27, cf. Remark 4.2.28. □

Corollary 4.4.7. *Let* $\mathrm{Orb}'_G \subseteq \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ *denote the full subcategory spanned by the objects* $L_{\mathrm{htp}}(G/H)$*. Then the unique colimit-preserving extension*

$$\mathrm{PSh}(\mathrm{Orb}'_G) \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$$

of the inclusion $\mathrm{Orb}'_G \hookrightarrow \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ *is an equivalence.*

Proof. The objects $L_{\mathrm{htp}}(G/H)$ are completely compact (or *tiny*) by Corollary 4.4.6: by adjunction and representability there is an equivalence of functors

$$\mathrm{Hom}_{\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)}(L_{\mathrm{htp}}(G/H), -) \simeq \mathrm{ev}_{G/H}\colon \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) \to \mathrm{An}.$$

Furthermore, the objects $L_{\mathrm{htp}}(G/H)$ generate $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ under colimits by Corollary 4.4.2. The statement is now an instance of [Lur09, Corollary 5.1.6.11]. □

4.4.2 Comparison with genuine equivariant animae

Having established that $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ is a presheaf category, we will now move to the construction of a comparison functor $R\colon \mathrm{An}_G \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ and prove that it is an equivalence.

Definition 4.4.8 (Equivariant homotopy type)**.** Consider the composite functor

$$\Pi_G\colon \mathrm{Diff}_G \xrightarrow{\mathrm{fgt}} \mathrm{Top}_G \to \mathrm{An}_G,$$

where the first functor is the forgetful functor from smooth G-manifolds to topological G-spaces and where the second is the Dwyer-Kan localization at the weak G-homotopy equivalences defining the ∞-category An_G of genuine G-animae (a.k.a 'G-spaces'). Observe that Π_G sends open covers of smooth G-manifolds to colimit-diagrams, and hence it uniquely extends to a colimit-preserving functor

$$\Pi_G\colon \mathrm{Shv}(\mathrm{Diff}_G) \to \mathrm{An}_G.$$

We refer to the functor Π_G as the *G-equivariant homotopy type functor*.

Definition 4.4.9. As Π_G is colimit-preserving, it admits a right adjoint, which we will denote by $R\colon \mathrm{An}_G \to \mathrm{Shv}(\mathrm{Diff}_G)$. It follows immediately from the adjunction that for a genuine G-anima X, the sheaf $R(X)\colon \mathrm{Diff}_G^{\mathrm{op}} \to \mathrm{An}$ is given by

$$R(X)(M) = \mathrm{Hom}_{\mathrm{An}_G}(\Pi_G(M), X).$$

Observation 4.4.10. The functor $\Pi_G\colon \mathrm{Diff}_G \to \mathrm{An}_G$ is homotopy invariant: for a smooth G-manifold M we have $\Pi_G(M\times\mathbb{R}) \xrightarrow{\sim} \Pi_G(M)$. It follows that the functor $R\colon \mathrm{An}_G \to \mathrm{Shv}(\mathrm{Diff}_G)$ lands in homotopy invariant sheaves, so that R defines a functor

$$R\colon \mathrm{An}_G \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G).$$

Lemma 4.4.11. *The functor $R\colon \mathrm{An}_G \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ preserves colimits.*

Proof. Since the evaluation functors $\mathrm{ev}_{G/H}\colon \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) \to \mathrm{An}$ are jointly conservative by Proposition 4.4.1 and preserve colimits by Corollary 4.4.6, it will suffice to show that each of the composites $\mathrm{ev}_{G/H}\circ R$ preserves colimits. Observe that this composite is given by sending X to

$$R(X)(G/H) = \mathrm{Hom}_{\mathrm{An}_G}(\Pi_G(G/H), X) = \mathrm{Hom}_{\mathrm{An}_G}(G/H, X) = X^H,$$

the anima of H-fixed points of X. This functor is well-known to preserve colimits. $\square$

To show that R is an equivalence, it will suffice to show that R restricts to an equivalence between $\mathrm{Orb}_G \subseteq \mathrm{An}_G$ and $\mathrm{Orb}'_G \subseteq \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$. The main ingredient for this will be the comparison between the simplicial enrichments of Diff_G and Top_G.

Construction 4.4.12. Recall from Construction 4.2.2 the cosimplicial object $\Delta_{\mathrm{alg}}\colon \Delta \to \mathrm{Diff}_G$, sending $[n]$ to the algebraic n-simplex Δ^n_{alg} equipped with the trivial G-action. It gives rise to a simplicial enrichment of Diff_G given by

$$\mathrm{Hom}^{\Delta}_{\mathrm{Diff}_G}(M,N)_n := \mathrm{Hom}_{\mathrm{Diff}_G}(M\times\Delta^n_{\mathrm{alg}}, N).$$

In a similar way, we obtain a simplicial enrichment of Top_G:

$$\mathrm{Hom}^{\Delta}_{\mathrm{Top}_G}(X,Y)_n := \mathrm{Hom}_{\mathrm{Top}_G}(X \times \Delta^n_{\mathrm{alg}}, Y).$$

The forgetful functor $\mathrm{Diff}_G \to \mathrm{Top}_G$ induces a morphism of simplicial sets

$$\varphi\colon\ \mathrm{Hom}^{\Delta}_{\mathrm{Diff}_G}(M,N) \to \mathrm{Hom}^{\Delta}_{\mathrm{Top}_G}(M,N),$$

natural in M and N.

Our next goal is to show that the map φ is a simplicial homotopy equivalence of Kan simplicial sets.

Lemma 4.4.13. *For every $n,k \geq 0$, every simplicial subset $A \subseteq \Delta^n \times \Delta^k$ and every solid commutative diagram of simplicial sets*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & \mathrm{Hom}^{\Delta}_{\mathrm{Diff}_G}(M,N) \arrow[d, "\varphi"] \\ \Delta^n \times \Delta^k \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & \mathrm{Hom}^{\Delta}_{\mathrm{Top}_G}(M,N), \end{tikzcd}$$

there exists a diagonal filler $\tilde{H}$ which makes the top triangle commute strictly and which makes the bottom triangle commute up to a homotopy which is constant on A.

Proof. The map H corresponds to a G-equivariant continuous map $H\colon M \times \Delta^n_{\mathrm{alg}} \times \Delta^k_{\mathrm{alg}} \to N$, and the map f expresses that H is a smooth map when restricted to the (closed) subset of $M \times \Delta^n_{\mathrm{alg}} \times \Delta^k_{\mathrm{alg}}$ corresponding to $A \subseteq \Delta^n \times \Delta^k$. By [Bre72, Theorem VI.4.2], H is G-equivariantly homotopic to a smooth map $\tilde{H}\colon M \times \Delta^n_{\mathrm{alg}} \times \Delta^k_{\mathrm{alg}} \to N$ which agrees with H on the subset of $M \times \Delta^n_{\mathrm{alg}} \times \Delta^k_{\mathrm{alg}}$ corresponding to $A \subseteq \Delta^n \times \Delta^k$, and the homotopy can be chosen to be constant on this subset. $\square$

Proposition 4.4.14. *For every two smooth G-manifolds M and N, the map*

$$\varphi\colon\ \mathrm{Hom}^{\Delta}_{\mathrm{Diff}_G}(M,N) \to \mathrm{Hom}^{\Delta}_{\mathrm{Top}_G}(M,N)$$

is a simplicial homotopy equivalence of Kan simplicial sets.

Proof. We first show that the source and target are Kan simplicial sets. This is clear for $\mathrm{Hom}^{\Delta}_{\mathrm{Top}_G}(M,N)$. For $\mathrm{Hom}^{\Delta}_{\mathrm{Diff}_G}(M,N)$, this follows immediately from Lemma 4.4.13 by taking $k=0$ and letting $A = \Lambda^n_l \subseteq \Delta^n$ be a horn.

To show that the map φ is a simplicial homotopy equivalence, it will thus suffice to show that φ induces a bijection on path components and isomorphisms on all simplicial homotopy groups. Surjectivity follows from Lemma 4.4.13 using $A = \partial\Delta^n \subseteq \Delta^n$. Injectivity follows from Lemma 4.4.13 using $A = \Delta^n \times \partial\Delta^1 \subseteq \Delta^n \times \Delta^1$. $\square$

The above comparsion map leads to an equivalence between $L_{\mathrm{htp}}(G/H)$ and $R(G/H)$.

Construction 4.4.15. We produce an equivalence $\psi\colon L_{\mathrm{htp}}(G/H) \to R(G/H)$ of homotopy invariant sheaves on Diff_G. Let M be a smooth G-manifold. By Proposition 4.2.3, there is an equivalence

$$L_{\mathrm{htp}}(G/H)(M) \simeq |\mathrm{Hom}^{\Delta}_{\mathrm{Diff}_G}(M, G/H)| \quad \in \quad \mathrm{An}.$$

Furthermore, since the ∞-category An_G is the homotopy coherent nerve of the simplicially enriched category CW_G of G-CW-complexes, which is a full sipmlicial subcategory of Top_G, we have by [HK20] a natural equivalence

$$R(G/H)(M) \simeq \mathrm{Hom}_{\mathrm{An}_G}(\Pi_G(M), G/H) \simeq |\mathrm{Hom}^{\Delta}_{\mathrm{Top}_G}(M, G/H)| \quad \in \quad \mathrm{An}.$$

The map $\varphi\colon \mathrm{Hom}^{\Delta}_{\mathrm{Diff}_G}(M, G/H) \to \mathrm{Hom}^{\Delta}_{\mathrm{Top}_G}(M, G/H)$ from Construction 4.4.12 thus produces the desired map $\psi\colon L_{\mathrm{htp}}(G/H) \to R(G/H)$, which is an equivalence by Proposition 4.4.14.

Theorem 4.4.16. *Let G be a Lie group. Then the functor $R\colon \mathrm{An}_G \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G) = \mathrm{H}(\mathbb{B}G)$ is an equivalence of ∞-categories.*

Proof. Consider the full subcategory $\mathrm{Orb}_G \subseteq \mathrm{An}_G$ spanned by the orbits G/H and the full subcategory $\mathrm{Orb}'_G \subseteq \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G)$ spanned by the objects $L_{\mathrm{htp}}(G/H)$. The equivalence $R(G/H) \simeq L_{\mathrm{htp}}(G/H)$ of Construction 4.4.15 shows that the functor R restricts to a functor $R|\colon \mathrm{Orb}_G \to \mathrm{Orb}'_G$. Consider now the following diagram:

$$\begin{tikzcd}
\mathrm{PSh}(\mathrm{Orb}_G) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{PSh}(\mathrm{Orb}'_G) \arrow[d, "\simeq"] \\
\mathrm{An}_G \arrow[r, "R"] & \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}_G).
\end{tikzcd}$$

The top horizontal functor is the colimit-extension of $R|$. The vertical functors are the colimit-extensions of the respective inclusions; the left vertical functor is an equivalence by Elmendorf's theorem [Elm83] while the right vertical functor is an equivalence by Corollary 4.4.7. It thus remains to show that the restriction $R|\colon \mathrm{Orb}_G \to \mathrm{Orb}'_G$ is an equivalence of ∞-categories. The equivalence $R(G/H) \simeq L_{\mathrm{htp}}(G/H)$ shows it is essentially surjective. For fully faithfulness, let $H \leqslant G$ be a closed subgroup and X a genuine G-anima, and consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{An}_G}(G/H, X) \arrow[r, "R"] \arrow[dr] & \mathrm{Hom}_{\mathrm{Shv}(\mathrm{Diff})}(R(G/H), R(X)) \arrow[d, "-\circ\psi"] \\
& \mathrm{Hom}_{\mathrm{Shv}(\mathrm{Diff}_G)}(L_{\mathrm{htp}}(G/H), R(X)).
\end{tikzcd}$$

We have to show that the horizontal map is an equivalence. Since ψ is an equivalence, we may equivalently show that the diagonal composite is an equivalence. We may further compose this diagonal with the following chain of equivalences:

$$\mathrm{Hom}_{\mathrm{Shv}(\mathrm{Diff}_G)}(L_{\mathrm{htp}}(G/H), R(X)) \simeq R(X)(G/H) \simeq \mathrm{Hom}_{\mathrm{An}_G}(G/H, X).$$

The resulting map $\mathrm{Hom}(G/H, X) \to \mathrm{Hom}(G/H, X)$ is natural in X and thus by the Yoneda-lemma it is given by precomposition with some endomorphism $G/H \to G/H$. Since every such endomorphism is an equivalence, this finishes the proof. □

4.4.3 Comparison with genuine equivariant spectra

Having established the equivalence between $\mathrm{H}(\mathbb{B}G)$ and An_G in Theorem 4.4.16, we will now move to its stable analogue. The equivalence $R\colon \mathrm{An}_G \xrightarrow{\sim} \mathrm{H}(\mathbb{B}G)$ induces an equivalence $R_*\colon \mathrm{An}_{G,*} \xrightarrow{\sim} \mathrm{H}(\mathbb{B}G)_* = \mathrm{H}_\bullet(\mathbb{B}G)$. This functor sends the representation sphere S^V of a G-representation V to the sphere bundle $S^{V//G}$ of the vector bundle $V//G \to \mathbb{B}G$ associated to V, and thus induces a functor

$$\mathrm{Sp}_G = \mathrm{An}_{G,*}[\{(S^V)^{-1}\}] \to \mathrm{H}_\bullet(\mathbb{B}G)[\{(S^{\mathcal{E}})^{-1}\}] = \mathrm{SH}(\mathbb{B}G).$$

Proposition 4.4.17. *Let G be a compact Lie group. Then the functor* $\mathrm{Sp}_G \to \mathrm{SH}(\mathbb{B}G)$ *is an equivalence of* ∞*-categories.*

Proof. This is immediate as every vector bundle $\mathcal{E} \to \mathbb{B}G$ is of the form $V//G \to \mathbb{B}G$ for some G-representation V. □

4.5 Pullback formalisms and universal properties

In this section, we study the assignments $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ and $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ and show that they can be characterized by universal properties. The universal properties will be phrased within the setting of *pullback formalisms* on SepStk, a concept we will introduce momentarily. We will show:

(1) The assignment $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ is the initial pullback formalism on SepStk, see Proposition 4.5.12;

(2) The assignment $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ is the initial pullback formalism on SepStk , see Proposition 4.5.21;

(3) The assignment $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ is the initial pullback formalism on SepStk satisfying both homotopy invarianceand *genuinely stability*, see Proposition 4.5.27.

These results will not be used later in this paper and thus may be skipped on first reading.

4.5.1 Sheaves of ∞-categories on SepStk

We start by making explicit the notion of a *sheaf of ∞-categories on* SepStk.

Definition 4.5.1. A *sheaf of ∞-categories on* SepStk is a functor $C\colon \mathrm{SepStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which satisfies the sheaf condition with respect to the open cover topology: for every open cover $\{\mathcal{U}_i\}_{i\in I}$ of a separated differentiable stack $\mathcal{X}$, the map

$$C(\mathcal{X}) \to \lim\Big(\textstyle\prod_{i\in I} C(\mathcal{U}_i) \rightrightarrows \prod_{i,j\in I} C(\mathcal{U}_i \times_{\mathcal{X}} \mathcal{U}_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \;\; \dots \;\Big)$$

is an equivalence. Similarly, a *sheaf of presentably symmetric monoidal ∞-categories* is a functor $C\colon \mathrm{SepStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ satisfying the sheaf condition. Since the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Cat}_\infty$ preserves limits, this may be tested on underlying ∞-categories.

Remark 4.5.2. A sheaf of ∞-categories on SepStk may equivalently be encoded by a limit-preserving functor $\mathrm{Shv}(\mathrm{SepStk})^{\mathrm{op}} \to \mathrm{Cat}_\infty$. In other words, letting $\mathcal{B}$ denote the ∞-topos $\mathrm{Shv}(\mathrm{SepStk})$, this is the same data as a *$\mathcal{B}$-category* in the sense of [Mar21; MW21], see also Chapter I.2.

Example 4.5.3. The three assignments $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$, $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ and $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ form sheaves of presentably symmetric monoidal ∞-categories on SepStk. For the first two, see Corollary 4.1.12 and Lemma 4.2.24. For the third, we showed in Lemma 4.3.12 that the assignment $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ is a sheaf of ∞-categories on QtStk, and this then uniquely extends to a sheaf of ∞-categories on all of SepStk.

Lemma 4.5.4 (Disjoint union)**.** *Let C be a sheaf of ∞-categories on* SepStk.

(1) The ∞-category $C(\emptyset)$ is equivalent to the terminal ∞-category;

(2) For every collection of differentiable stacks $\{\mathcal{X}_i\}_{i\in I}$, the pullback functors $C(\bigsqcup_{i\in I} \mathcal{X}_i) \to C(\mathcal{X}_i)$ induced by the inclusions $\mathcal{X}_i \hookrightarrow \bigsqcup_{i\in I} \mathcal{X}_i$ constitute an equivalence of ∞-categories

$$C(\bigsqcup_{i\in I} \mathcal{X}_i) \to \prod_{i\in I} C(\mathcal{X}_i).$$

Proof. Part (1) follows from applying the sheaf condition to the empty cover of the empty stack. For part (2), note that the disjoint union $\bigsqcup_{i\in I} \mathcal{X}_i$ is covered by the collection of open substacks $\mathcal{X}_i \hookrightarrow \bigsqcup_{i\in I} \mathcal{X}_i$, which have empty intersection. By part (1), the sheaf condition for this cover thus provides the claimed equivalence. □

4.5.2 Pullback formalisms on differentiable stacks

We now introduce the notion of a *pullback formalism* on the site SepStk. Our definition is essentially a specialization of that of Drew and Gallauer [DG22], except for some differences in conventions explained in Remark 4.5.6 below.

Definition 4.5.5 (Pullback formalism, cf. [DG22, Definition 2.11])**.** Let $C\colon \mathrm{SepStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a sheaf of presentably symmetric monoidal ∞-categories on SepStk. We will say that C is a *pullback formalism* on SepStk if the following conditions are satisfied:

(1) For every representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of differentiable stacks, the pullback functor $f^*\colon C(\mathcal{X}) \to C(\mathcal{Y})$ admits a left adjoint $f_\sharp\colon C(\mathcal{Y}) \to C(\mathcal{X})$;

(2) (Smooth base change) For every pullback square

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X}' \arrow[r, "g"] & \mathcal{X} \end{tikzcd}$$

of differentiable stacks, where f (and thus f') is a representable submersion, the Beck-Chevalley transformation

$$\mathrm{BC}_\sharp\colon f_\sharp h^* \to g^* i'_\sharp\colon C(\mathcal{Y}) \to C(\mathcal{X}')$$

is an equivalence;

(3) (Smooth projection formula) For every representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of differentiable stacks, every $A \in C(\mathcal{X})$ and every $B \in C(\mathcal{Y})$, the projection formula map

$$\mathrm{PF}_\sharp\colon f_\sharp(f^*A \otimes B) \to A \otimes f_\sharp B \quad \in \quad C(\mathcal{X})$$

is an equivalence.

Let $\mathcal{D}$ be another pullback formalism on SepStk. A natural transformation $F\colon C \to \mathcal{D}$ of functors $\mathrm{SepStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is called a *morphism of pullback formalisms* if for every representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of differentiable stacks, the naturality square

$$\begin{tikzcd} \mathcal{C}(\mathcal{Y}) \arrow[r, "F_{\mathcal{Y}}"] \arrow[d, "f^*"'] & \mathcal{D}(\mathcal{Y}) \arrow[d, "f^*"] \\ \mathcal{C}(\mathcal{X}) \arrow[r, "F_{\mathcal{X}}"] & \mathcal{D}(\mathcal{X}) \end{tikzcd}$$

is vertically left adjointable, in the sense that the Beck-Chevalley transformation $f_\sharp \circ F_{\mathcal{X}} \Rightarrow F_{\mathcal{Y}} \circ f_\sharp \colon \mathcal{C}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$ is an equivalence. We let

$$\mathrm{PB}(\mathrm{SepStk}) \subseteq \mathrm{Fun}(\mathrm{SepStk}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))$$

denote the (non-full) subcategory spanned by the pullback formalisms on SepStk and the morphisms of pullback formalisms.

Remark 4.5.6. In contrast to [DG22], we include both the presentability condition as well as the descent condition with respect to the Grothendieck topology on SepStk into the definition of a pullback formalism. Hence the ∞-category $\mathrm{PB}(\mathrm{SepStk})$ would be written as $\mathrm{PB}^{\mathrm{pr}}_{\mathrm{open}}(\mathrm{SepStk})$ in [DG22]. Closely related is also the notion of a $(*, \sharp, \otimes)$*-formalism* by Khan and Ravi [KR21, Definition 5.5].

Remark 4.5.7. By Drew and Gallauer [DG22, Proposition 2.12, Proposition 4.4, Proposition 5.6], the (very large) ∞-category $\mathrm{PB}(\mathrm{SepStk})$ is presentable and the inclusion $\mathrm{PB}(\mathrm{SepStk}) \hookrightarrow \mathrm{Fun}(\mathrm{SepStk}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Cat}_\infty))$ admits a left adjoint.

Example 4.5.8. Each of the three functors $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$, $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ and $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ is a pullback formalism, see Corollary 4.2.17, Proposition 4.2.19, Proposition 4.2.21 and Proposition 4.3.16.

The condition for a sheaf of ∞-categories on SepStk to be a pullback formalism may be neatly formulated in the language of parametrized category theory.

Construction 4.5.9. Let $\mathcal{B}$ denote the ∞-topos $\mathrm{Shv}(\mathrm{SepStk})$. By Remark 4.5.2, a sheaf $\mathcal{C}$ of presentably symmetric monoidal ∞-categories on SepStk may equivalently be encoded by a fiberwise presentably symmetric monoidal $\mathcal{B}$-category. In particular, the sheaf $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ from Example 4.5.3 corresponds to a $\mathcal{B}$-category $\mathbf{U}$. By Lemma 4.1.9 this defines a parametrized subcategory of $\mathcal{X} \mapsto \mathrm{Shv}(\mathrm{SepStk})_{/\mathcal{X}}$, and thus it defines a *class of* $\mathcal{B}$*-groupoids* in the sense of [MW21, Remark 2.7.5]. We say that a morphism $f\colon \mathfrak{Y} \to \mathfrak{X}$ in $\mathrm{Shv}(\mathrm{SepStk})$ is a *morphism in* $\mathbf{U}$ if it is an object of the subcategory $\mathbf{U}(\mathfrak{X}) \subseteq \mathrm{Shv}(\mathrm{SepStk})_{/\mathfrak{X}}$.

Lemma 4.5.10. *With the above notations, let* $\mathcal{C}\colon \mathrm{Shv}(\mathrm{SepStk})^{\mathrm{op}} = \mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ *be a fiberwise presentably symmetric monoidal $\mathcal{B}$-category. Then its restriction to* $\mathrm{SepStk}^{\mathrm{op}}$ *is a pullback formalism if and only if the $\mathcal{B}$-category $\mathcal{C}$ admits* $\mathbf{U}$*-colimits and the tensor product $\mathcal{B}$-functor* $-\otimes-\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ *preserves* $\mathbf{U}$*-colimits in both variables.*

Proof. We refer to [MW21, Definition 4.1.3] for the definition of parametrized colimits, and to [MW21, Example 4.1.9] for the characterization of parametrized colimits along $\mathcal{B}$-groupoids. It follows from this characterization that $\mathcal{C}$ admits $\mathbf{U}$-colimits if and only if the following two conditions hold:

(1') For every morphism $f\colon \mathfrak{Y}\to\mathfrak{X}$ in $\mathbf{U}$, the pullback functor $f^*\colon \mathcal{C}(\mathfrak{X})\to\mathcal{C}(\mathfrak{Y})$ admits a left adjoint $f_\sharp\colon \mathcal{C}(\mathfrak{Y})\to\mathcal{C}(\mathfrak{X})$;

(2') For every pullback square

$$\begin{tikzcd} \mathfrak{Y}' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{Y} \arrow[d, "f"] \\ \mathfrak{X}' \arrow[r, "g"] & \mathfrak{X} \end{tikzcd}$$

in $\mathrm{Shv}(\mathrm{SepStk})$ where f (and thus f') is in $\mathbf{U}$, the Beck-Chevalley transformation $f_\sharp h^* \to g^* i'_\sharp$ is an equivalence.

Similarly, the condition that the $\mathcal{B}$-functor $-\otimes-\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ to preserve $\mathbf{U}$-colimits in both variables is characterized by the following condition:

(3') For every $f\colon \mathfrak{Y}\to\mathfrak{X}$ in $\mathbf{U}$, every $A\in\mathcal{C}(\mathfrak{X})$ and every $B\in\mathcal{C}(\mathfrak{Y})$, the projection formula map $f_\sharp(f^*A\otimes B)\to A\otimes f_\sharp B$ is an equivalence.

Since $\mathbf{U}$ is generated under fiberwise colimits by the representable submersions $f\colon \mathcal{Y}\to\mathcal{X}$, Lemma 4.1.9, and since $\mathcal{B} = \mathrm{Shv}(\mathrm{SepStk})$ is the localization of a presheaf category, it follows from [MW22, Proposition A.2.1] and [MW21, Corollary 3.2.10] that it suffices to check the conditions (1')-(3') in the case where $f\colon \mathcal{Y}\to\mathcal{X}$ is a representable submersion between differentiable stacks. But these are precisely conditions (1)-(3) in the definition of a pullback formalism, finishing the proof. □

Given an ∞-topos $\mathcal{B}$ and a class of $\mathcal{B}$-groupoids $\mathbf{U}$, Martini and Wolf [MW22, Section 8] defined a *tensor product of* $\mathbf{U}$*-cocomplete $\mathcal{B}$-categories*, giving rise to a symmetric monoidal ∞-category $\mathrm{Cat}^{\mathbf{U}-\mathrm{cc}}(\mathcal{B})$ of $\mathbf{U}$-cocomplete $\mathcal{B}$-categories. The above characterization of pullback formalisms can then be reformulated as follows:

Corollary 4.5.11. *Let $\mathcal{B} = \mathrm{Shv}(\mathrm{SepStk})$ and let $\mathbf{U}'$ be the class of $\mathcal{B}$-categories containing both $\mathbf{U}$ and the locally constant $\mathcal{B}$-categories. There is a fully faithful inclusion*

$$\mathrm{PB}(\mathrm{SepStk}) \hookrightarrow \mathrm{CAlg}(\mathrm{Cat}^{\mathbf{U}'-\mathrm{cc}}(\mathcal{B}))$$

whose essential image consists of those C such that $C(X)$ is presentable for every $X \in$ SepStk.

Proof. Recall that a $\mathcal{B}$-category $K \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is called *locally constant* if it is equivalent to the sheafification of a constant functor with value some ∞-category K. A $\mathcal{B}$-category C admits K-indexed parametrized colimits if and only if each ∞-category $C(X)$ admits K-indexed colimits and each pullback functo f^* preserves K-indexed colimits, see [MW21, Example 4.2.6]. In particular, a commutative algebra in $\mathrm{Cat}^{\mathbf{U}'-\mathrm{cc}}(\mathcal{B})$ consists of symmetric monoidal $\mathcal{B}$-category $C \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ which, in addition to satisfying the conditions (1)-(3) from Definition 4.5.5, also factors through the subcategory $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{cc}})$ of symmetric monoidal ∞-categories admitting small colimits and whose tensor product preserves small colimits in each variable. Since the condition on a transformation $F \colon C \to \mathcal{D}$ of functors $\mathrm{SepStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ to satisfy the Beck-Chevalley condition in Definition 4.5.5 corresponds to the condition for the associated $\mathcal{B}$-functor to preserve $\mathbf{U}$-indexed colimits, it follows that ∞-category PB(SepStk) is equivalent to the subcategory of $\mathrm{CAlg}(\mathrm{Cat}^{\mathbf{U}'-\mathrm{cc}}(\mathcal{B}))$ spanned by those functors $\mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ which in fact factor through the full subcategory $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \subseteq \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{cc}})$. This finishes the proof. □

4.5.3 The initial pullback formalism

Proposition 4.5.12. *The assignment $X \mapsto \mathrm{Shv}(\mathrm{Sub}_{/X})$ is the initial pullback formalism on* SepStk.

Proof. Let $\mathcal{B} = \mathrm{Shv}(\mathrm{SepStk})$ and recall from the previous subsection that we defined the $\mathcal{B}$-category $\mathbf{U}$ as the $\mathcal{B}$-category corresponding to the pullback formalism $X \mapsto \mathrm{Shv}(\mathrm{Sub}_{/X})$. By Corollary 4.5.11, it will now suffice to prove that $\mathbf{U}$ is an initial object of the ∞-category $\mathrm{CAlg}(\mathrm{Cat}^{\mathbf{U}'-\mathrm{cc}}(\mathcal{B}))$, or equivalently that $\mathbf{U}$ is the monoidal unit of the symmetric monoidal ∞-category $\mathrm{Cat}^{\mathbf{U}'-\mathrm{cc}}(\mathcal{B})$. But since $\mathbf{U}$ is already closed under fiberwise colimits, we see that $\mathbf{U}$ is the smallest $\mathcal{B}$-subcategory of $\Omega_{\mathcal{B}}$ containing the point and closed under $\mathbf{U}'$-colimits, and thus the theorem is a consequence of [MW21, Theorem 7.1.11]. □

Notation 4.5.13. Let C be a pullback formalism on SepStk. By the previous proposition, there exists a unique morphism $h^C \colon \mathrm{Shv}(\mathrm{Sub}_{/-}) \Rightarrow C$ of pullback formalisms. In particular, for every differentiable stack $\mathcal{X}$, there is a symmetric monoidal left adjoint

$$h^{\mathcal{C}}_{\mathcal{X}} \colon \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}}) \to \mathcal{C}(\mathcal{X}).$$

Because these functors are required to commute with the functors $f_\sharp$ for all representable submersions f, it follows that $h^{\mathcal{C}}_{\mathcal{X}}$ is given at a representable object $(f\colon \mathcal{Y} \to \mathcal{X}) \in \operatorname{Sub}_{/\mathcal{X}}$ by

$$h^{\mathcal{C}}_{\mathcal{X}}(\mathcal{Y}) = f_\sharp \mathbb{1}_{\mathcal{Y}} \quad \in \quad \mathcal{C}(\mathcal{X}).$$

We will often just write $h_{\mathcal{X}}(\mathcal{Y})$ for this object when $\mathcal{C}$ is clear from context.

Definition 4.5.14. A pullback formalism $\mathcal{C}$ on SepStk is called *pointed* if the ∞-category $\mathcal{C}(\mathcal{X})$ is pointed for every differentiable stack $\mathcal{X}$. We say that $\mathcal{C}$ is *stable* if $\mathcal{C}(\mathcal{X})$ is stable for every differentiable stack $\mathcal{X}$.

Remark 4.5.15. For a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of differentiable stacks, the pullback functor $f^*\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{Y})$ preserves all colimits. Hence if $\mathcal{C}$ is pointed, f^* is a pointed functor. Similarly, if $\mathcal{C}$ is stable the pullback functors are exact.

Corollary 4.5.16. *The assignment* $\mathcal{X} \mapsto \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}}; \mathrm{An}_*)$ *is the initial pointed pullback formalism on* SepStk. *The assignment* $\mathcal{X} \mapsto \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}}; \mathrm{Sp})$ *is the initial stable pullback formalism on* SepStk.

Proof. These pullback functors are obtained from $\mathcal{X} \mapsto \operatorname{Shv}(\operatorname{Sub}_{/\mathcal{X}})$ by pointwise tensoring with An_* and Sp, respectively, and hence the result follows from Proposition 4.5.12 and the fact that the functors $- \otimes \mathrm{An}_*\colon \operatorname{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \operatorname{CAlg}(\mathrm{Pr}^{\mathrm{L}}_*)$ and $- \otimes \mathrm{Sp}\colon \operatorname{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \operatorname{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ are left adjoint to the respective inclusions. □

4.5.4 Homotopy invariance

We introduce the notion of homotopy invariance for a pullback formalism, and show that the assignment $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ is the initial pullback formalism on SepStk satisfying homotopy invariance.

Definition 4.5.17 (Homotopy invariance)**.** We say that a pullback formalism $\mathcal{C}$ on SepStk satisfies *homotopy invariance* if for every differentiable stack $\mathcal{X}$ the pullback functor $\mathrm{pr}^*\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{X} \times \mathbb{R})$ associated to the projection $\mathrm{pr}\colon \mathcal{X} \times \mathbb{R} \to \mathcal{X}$ is fully faithful.

Warning 4.5.18. We emphasize that homotopy invariance does not imply that the ∞-category $\mathcal{C}(\mathcal{X})$ is a homotopy invariant of the stack $\mathcal{X}$.

The condition of homotopy invariance is in essence the condition that the real line becomes contractible in C:

Lemma 4.5.19. *A pullback formalism C on* SepStk *satisfies homotopy invariance if and only if the object $h_{\mathrm{pt}}(\mathbb{R}) \in C(\mathrm{pt})$ is terminal.*

Proof. Note that C satisfies homotopy invariance if and only if for every object $A \in C(\mathcal{X})$ the counit map $\mathrm{pr}_\sharp \mathrm{pr}^* A \to A$ is an equivalence. By the smooth projection formula and smooth base change, one sees that there is an equivalence $\mathrm{pr}_\sharp \mathrm{pr}^* A \simeq h_{\mathcal{X}}(\mathcal{X} \times \mathbb{R}) \otimes A$, and the counit map is obtained by tensoring A with the projection $h_{\mathcal{X}}(\mathcal{X} \times \mathbb{R}) \to h_{\mathcal{X}}(\mathcal{X})$. Since this map is the pullback along $\mathcal{X} \to \mathrm{pt}$ of the map of $h_{\mathrm{pt}}(\mathbb{R}) \to h_{\mathrm{pt}}(\mathrm{pt}) = *$ in $C(\mathrm{pt})$, we get the claim. $\square$

For a separated differentiable stack $\mathcal{X}$, consider the symmetric monoidal functor

$$\mathrm{Shv}(\mathrm{Diff}) \xrightarrow{p^*} \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \xrightarrow{h_{\mathcal{X}}} C(\mathcal{X}),$$

where $p \colon \mathcal{X} \to \mathrm{pt}$ is the map to the point. Homotopy invariance of C demands that this functor inverts the map $\mathbb{R} \to \mathrm{pt}$, or in other words, that it factors through the homotopy localization functor $L_{\mathrm{htp}} \colon \mathrm{Shv}(\mathrm{Diff}) \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff}) \simeq \mathrm{An}$. Conversely, this means that we can enforce homotopy invariance by tensoring with An over $\mathrm{Shv}(\mathrm{Diff})$:

Lemma 4.5.20. *The inclusion* $\mathrm{PB}^{\mathrm{htp}}(\mathrm{SepStk}) \hookrightarrow \mathrm{PB}(\mathrm{SepStk})$ *of homotopy invariant pullback formalisms into all pullback formalisms admits a left adjoint $C \mapsto C^{\mathrm{htp}}$ given by*

$$C^{\mathrm{htp}}(\mathcal{X}) := C(\mathcal{X}) \otimes_{\mathrm{Shv}(\mathrm{Diff})} \mathrm{An}.$$

Proof. As argued above, C satisfies homotopy invariance if and only if the $\mathrm{Shv}(\mathrm{Diff})$-algebra $C(\mathcal{X})$ in Pr^{L} is the restriction of an $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff})$-algebra in Pr^{L}, and a left adjoint to the inclusion $\mathrm{Pr}^{\mathrm{L}} \simeq \mathrm{CAlg}_{\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Diff})} \hookrightarrow \mathrm{CAlg}_{\mathrm{Shv}(\mathrm{Diff})}(\mathrm{Pr}^{\mathrm{L}})$ is given by $- \otimes_{\mathrm{Shv}(\mathrm{Diff})} \mathrm{An}$. It thus remains to show that C^{htp} is in fact a pullback formalism.

The fact that C^{htp} is a sheaf of ∞-categories can be proved just like in Lemma 4.2.24. The left adjoints $f_\sharp$ for C^{htp} are inherited from C, as are smooth base change and the smooth projection formula, where for the latter we invoke Corollary 4.2.14. It follows that C^{htp} is a pullback formalism, which satisfies homotopy invariance by Lemma 4.5.19. $\square$

Proposition 4.5.21. *The assignment $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ is the initial homotopy invariant pullback formalism on* SepStk.

Proof. As the functor $\mathrm{H}(-)$ was defined in Construction 4.2.23 as C^{htp} for $C(\mathcal{X}) = \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$, this is immediate from Lemma 4.5.20 and Proposition 4.5.12. $\square$

It follows that for every homotopy invariant pullback formalism C on SepStk there is a unique morphism $\mathrm{H} \to C$ of pullback formalisms. We will abuse notation and denote this morphism again by h^C, just like in Notation 4.5.13. So for every $\mathcal{X} \in$ SepStk, there is a symmetric monoidal left adjoint

$$h^C_{\mathcal{X}} \colon \mathrm{H}(\mathcal{X}) \to C(\mathcal{X});$$

it sends the object $L_{\mathrm{htp}}(\mathcal{Y})$ to $h^C_{\mathcal{X}} = g_\sharp \mathbb{1}_{\mathcal{Y}}$ for every $(g \colon \mathcal{Y} \to \mathcal{X}) \in \mathrm{Sub}_{/\mathcal{X}}$. We will often write $h_{\mathcal{X}}(\mathcal{Y})$ for $h^C_{\mathcal{X}}(\mathcal{Y})$ when C is clear from context.

Corollary 4.5.22. *Let $\mathcal{X}$ be a differentiable stack and let C be a pullback formalism on* SepStk *satisfying homotopy invariance.*

(1) If $f \colon \mathcal{Y} \times \mathbb{R} \to \mathbb{Z}$ is a homotopy over $\mathcal{X}$ of maps $\mathcal{Y} \to \mathcal{Z}$ in $\mathrm{Sub}_{/\mathcal{X}}$*, then the maps $h_{\mathcal{X}}(f_0), h_{\mathcal{X}}(f_1) \colon h_{\mathcal{X}}(\mathcal{Y}) \to h_{\mathcal{X}}(\mathcal{Z})$ are equivalent as maps in $C(\mathcal{X})$;*

(2) Let $f \colon \mathcal{Y} \to \mathcal{Z}$ be a strict homotopy equivalence over $\mathcal{X}$. Then the morphism

$$h_{\mathcal{X}}(f) \colon h_{\mathcal{X}}(\mathcal{Y}) \to h_{\mathcal{X}}(\mathcal{Z})$$

is an equivalence in $C(\mathcal{X})$.

Proof. This is immediate from Lemma 4.2.8 Corollary 4.2.9. □

Corollary 4.5.23. *Assume that C is a homotopy invariant pullback formalism on* SepStk*. Then for every vector bundle $\pi \colon \mathcal{E} \to \mathcal{X}$, the functor $\pi^* \colon C(\mathcal{X}) \to C(\mathcal{E})$ is fully faithful.*

Proof. We need to show that the counit map $\pi_\sharp \pi^* \to \mathrm{id}$ is a homotopy equivalence. By the smooth projection formula, this map is given by tensoring with $h_{\mathcal{X}}(\mathcal{E}) \to h_{\mathcal{X}}(\mathcal{X})$. As the map $\mathcal{E} \to \mathcal{X}$ is a strict homotopy equivalence, this map is an equivalence by Corollary 4.5.22. □

Corollary 4.5.24. *The assignment $\mathcal{X} \mapsto \mathrm{H}_\bullet(\mathcal{X})$ is the initial pointed homotopy invariant pullback formalism on* SepStk.

Proof. This is immediate from Proposition 4.5.21. □

4.5.5 Genuine stability

We introduce the notion of genuine stability for a pullback formalism C, and show that the assignment $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ of genuine sheaves of spectra form the universal example of a pullback formalism which satisfies both homotopy invariance and genuinely stability.

Notation 4.5.25 (Sphere bundles in a pullback formalism)**.** Let C be a pointed pullback formalism on SepStk satisfying homotopy invariance. Let $\mathcal{E} \to \mathcal{X}$ be a vector bundle of separated differentiable stacks. Then we define the *sphere bundle of* $\mathcal{E}$ *in* C as the cofiber

$$S^{\mathcal{E}} := \operatorname{cofib}(h_{\mathcal{X}}(\mathcal{X}) \to h_{\mathcal{X}}(S^{\mathcal{E}})) \quad \in \quad C(\mathcal{X}),$$

where the map is induced from the canonical section $\mathcal{X} \to S^{\mathcal{E}}$ at ∞.

Definition 4.5.26 (Genuine stability)**.** A pullback formalism C on SepStk is called *genuinely stable* if it is pointed and for every vector bundle $\pi\colon \mathcal{E} \to \mathcal{X}$ its sphere bundle $S^{\mathcal{E}} \in C(\mathcal{X})$ is monoidally invertible. In this case, we let $S^{-\mathcal{E}}$ denote a monoidal inverse of $S^{\mathcal{E}}$.

Note that every morphism $F\colon C \to \mathcal{D}$ of pullback formalisms preserves the sphere bundles $S^{\mathcal{E}}$. In particular, if C is genuinely stable, then so is $\mathcal{D}$.

Proposition 4.5.27. *The assignment* $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ *is the initial homotopy invariant genuinely stable pullback formalism on* SepStk.

Proof. Observe that a pullback formalism C on SepStk is homotopy invariant or genuinely stable if and only if its restriction to the subcategory QtStk of global quotient stacks is homotopy invariant or genuinely stable, hence we may as well restrict to pullback formalisms on QtStk. In that case, SH is obtained from $\mathrm{H}_{\bullet}$ by inverting the sphere bundles, hence the statement is a consequence of Corollary 4.5.24. □

II.5 Localization sequences

Our goal in this chapter is to prove a version of the localization theorem for genuine sheaves on differentiable stacks. This roughly states that, for a closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ with open complement $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$, the inclusions

$$\mathrm{SH}(\mathcal{Z}) \xhookrightarrow{i_*} \mathrm{SH}(\mathcal{X}) \xhookleftarrow{j_*} \mathrm{SH}(\mathcal{U})$$

form a *recollement* in the sense of [Lur17, Definition A.8.1]. Our proof strategy will closely follow that of [Hoy17, Section 4.3] and [Kha19], which in turn are based on the original reference [MV99, Theorem 3.2.21]. Other sources are [CD19, Section 2.3] and [Vol21, Section 4].

5.1 The localization axiom

In the spirit of [CD19, Section 2.3], we start by introducing the localization axiom for an arbitrary pullback formalism $C\colon \mathrm{SepStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ on SepStk (see Definition 4.5.5).

Lemma 5.1.1. *Let C be a pointed pullback formalism on* SepStk *and let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks with complementary open embedding $j\colon \mathcal{U} \to \mathcal{X}$. Then the following properties are satisfied:*

(a1) The unit $\mathrm{id} \implies j^* j_\sharp$ *is an equivalence;*

(a2) The counit $j^* j_* \implies \mathrm{id}$ *is an equivalence;*

(a3) The functor $j_\sharp\colon C(\mathcal{U}) \to C(\mathcal{X})$ *is fully faithful;*

(a4) The functor $j_*\colon C(\mathcal{U}) \to C(\mathcal{X})$ *is fully faithful;*

(b1) The composite functor $i^* j_\sharp\colon C(\mathcal{U}) \to C(\mathcal{Z})$ *is zero;*

(b2) The composite functor $j^* i_*\colon C(\mathcal{Z}) \to C(\mathcal{U})$ *is zero;*

(c) The composite map $j_\sharp j^* \xRightarrow{\text{counit}} \mathrm{id} \xRightarrow{\text{unit}} i_* i^*$ *admits a preferred null-homotopy.*

Proof. Note that (a1)-(a4) are all equivalent, and also (b1) and (b2) are equivalent. Properties (a1) and (b1) follow from smooth base change, applied to the following two pullback squares:

$$\begin{tikzcd} \mathcal{U} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & \mathcal{U} \arrow[d, hook, "j"] \\ \mathcal{U} \arrow[r, hook, "j"] & \mathcal{X} \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{U} \arrow[d, hook, "j"] \\ \mathcal{Z} \arrow[r, hook, "i"] & \mathcal{X}. \end{tikzcd}$$

For (c), note that by naturality the counit $j_\sharp j^* \to \mathrm{id}$ and the unit $\mathrm{id} \to i_* i^*$ give rise to a commutative square

$$\begin{tikzcd} j_\sharp j^* \arrow[r] \arrow[d] & \mathrm{id} \arrow[d] \\ j_\sharp j^* i_* i^* \arrow[r] & i_* i^*. \end{tikzcd}$$

By (b2) the lower left corner is the zero functor, hence this square produces the required null-homotopy. □

Definition 5.1.2 (Localization axiom, cf. [CD19, Section 2.3])**.** Let C be a pointed pullback formalism on SepStk. Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks with complementary open embedding $j\colon \mathcal{U} \to \mathcal{X}$. We say that C *satisfies the localization*[1] *axiom for i,* (Loc$_i$)*,* if the following two conditions are satisfied:

(1) The functor $i_*\colon C(\mathcal{Z}) \to C(\mathcal{X})$ is conservative;

(2) The sequence

$$j_\sharp j^* \xRightarrow{\text{counit}} \mathrm{id} \xRightarrow{\text{unit}} i_* i^*,$$

equipped with the null-homotopy from (c) above, is a cofiber sequence of natural transformations $C(\mathcal{X}) \to C(\mathcal{X})$.

We say that C *satisfies the localization axiom* if C satisfies (Loc$_i$) for every closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ of separated differentiable stacks.

The localization property implies that the functor i_* is well-behaved: it is fully faithful, satisfies base change, satisfies the projection formula and commutes with the pushforward functors $f_\sharp$ for representable submersions f:

[1] This is sometimes referred to as the *gluing axiom.*

Proposition 5.1.3 (Properties closed pushforwards)**.** *Let C be a pointed pullback formalism on* SepStk *which satisfies property* (Loc$_i$) *for a closed embedding $i\colon \mathcal{Z} \to \mathcal{X}$. Then the following conditions are satisfied:*

(1) (Fully faithfulness) The functor $i_*\colon C(\mathcal{Z}) \to C(\mathcal{X})$ is fully faithful.

(2) (Closed projection formula) For objects $A \in C(\mathcal{Z})$ and $B \in C(\mathcal{X})$, the exchange map

$$\mathrm{PF}_*\colon i_*(A) \otimes B \to i_*(A \otimes i^* B)$$

is an equivalence in $C(\mathcal{X})$.

(3) (Closed base change) For every pullback square of differentiable stacks

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, hook, "i'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}' \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, hook, "i"] & \mathcal{X}, \end{tikzcd}$$

the exchange map

$$\mathrm{BC}_*\colon f^* i_* \Rightarrow i'_* f'^* \colon C(\mathcal{Z}) \to C(\mathcal{X}')$$

is an equivalence.

(4) (Smooth-closed base change) Consider a pullback square of differentiable stacks

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, hook, "i'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, hook, "i"] & \mathcal{X}, \end{tikzcd}$$

where f is a representable submersion. Assume that C satisfies property (Loc$_{i'}$)*. Then the exchange map*

$$\mathrm{BC}_{\sharp,*}\colon f_\sharp i'_* \Rightarrow i_* f'_\sharp \colon C(\mathcal{Z}') \to C(\mathcal{X})$$

is an equivalence.

Proof. To see that i_* is fully faithful, we need to show that the counit map $i^* i_* Z \to Z$ is an equivalence for all $Z \in C(\mathcal{Z})$. Since i_* is conservative, it suffices to show that $i_* i^* i_* Z \to i_* Z$ is an equivalence, and by the triangle identities it will suffice to show that the unit map $i_* Z \to i_* i^* i_* Z$ is an equivalence. By (Loc$_i$), there is a cofiber sequence

$$j_\sharp j^* i_* Z \to i_* Z \to i_* i^* i_* Z$$

in $\mathcal{C}(\mathcal{X})$. But by (b2), the object $j_\sharp j^* i_* Z$ is zero, and thus the map $i_* Z \to i_* i^* i_* Z$ is an equivalence. This proves part (1).

We now show part (2). As i_* is fully faithful, the unit map $A \to i^* i_* A$ is an equivalence and thus it suffices to prove the claim when $A = i^* A'$ for some object $A' \in \mathcal{C}(\mathcal{X})$. In this case, consider the following diagram:

$$\begin{tikzcd}
j_\sharp j^*(A') \otimes B \arrow[r] & A' \otimes B \arrow[r] \arrow[dd, equal] & i_* i^*(A') \otimes B \arrow[d, "\mathrm{PF}_*"] \\
j_\sharp(j^*(A') \otimes j^* B) \arrow[u, "\simeq"', "\mathrm{PF}_\sharp"] \arrow[d, "\simeq"'] & & i_*(i^* A \otimes i^* B) \arrow[d, "\simeq"] \\
j_\sharp j^*(A' \otimes B) \arrow[r] & A' \otimes B \arrow[r] & i_* i^*(A' \otimes B).
\end{tikzcd}$$

The diagram is commutative by Lemma F.18. By assumption, the top and bottom sequences are cofiber sequences. The left vertical composite is an equivalence by the smooth projection formula, and thus also the right vertical composite is an equivalence. It follows that the map $i_* i^*(A') \otimes B \to i_*(i^* A' \otimes i^* B)$ is an equivalence, proving part (2).

The proofs of parts (3) and (4) are similar to that of part (2), and hence we will be more brief. For (3), it suffices by fully faithfulness of i_* to show that the composite $f^* i_* i^* \Rightarrow i'_* f'^* i^* \simeq i'_* i'^* f^*$ is an equivalence. This follows from the following diagram of cofiber sequences:

$$\begin{tikzcd}
f^* j_\sharp j^* \arrow[r] \arrow[d, "\simeq"'] & f^* \arrow[r] \arrow[d, equal] & f^* i_* i^* \arrow[d] \\
j'_\sharp j'^* f^* \arrow[r] & f^* \arrow[r] & i'_* i'^* f^*,
\end{tikzcd}$$

where the left equivalence is obtained from smooth base change. The diagram commutes by Lemma F.5. For (4), it suffices to show that the composite $f_\sharp i'_* i'^* \Rightarrow i_* f'_\sharp i'^* \simeq i_* i^* f_\sharp$ is an equivalence. This follows from the following diagram of cofiber sequences:

$$\begin{tikzcd}
f_\sharp j'_\sharp j'^* \arrow[r] \arrow[d, "\simeq"'] & f_\sharp \arrow[r] \arrow[d, equal] & f_\sharp i'_* i'^* \arrow[d] \\
j_\sharp j^* f_\sharp \arrow[r] & f_\sharp \arrow[r] & i_* i^* f_\sharp,
\end{tikzcd}$$

where the left equivalence is obtained from smooth base change. Again the diagram commutes by Lemma F.5. □

The proof strategy from the previous lemma also applies to the following very general statement:

Lemma 5.1.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be pointed pullback formalisms on* SepStk *which satisfy property* (Loc$_i$) *for a closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism of pullback formalisms. Then the Beck-Chevalley transformation*

$$F_{\mathcal{X}} \circ i_* \Rightarrow i_* \circ F_{\mathcal{Z}} \quad \in \quad \mathrm{Fun}(\mathcal{C}(\mathcal{Z}), \mathcal{D}(\mathcal{X}))$$

is an equivalence.

Proof. As $i_*\colon \mathcal{C}(\mathcal{Z}) \hookrightarrow \mathcal{C}(\mathcal{X})$ is fully faithful, it suffices to show that the map $F_{\mathcal{X}} i_* i^* \to i_* F_{\mathcal{Z}} i^*$ is an equivalence. As F is a morphism of pullback formalisms, the latter functor is equivalent to $i_* i^* F_{\mathcal{X}}$. We may now contemplate the following two cofiber sequences:

$$\begin{tikzcd}
F_{\mathcal{X}} j_\sharp j^* \arrow[r] \arrow[d, "\simeq"'] & F_{\mathcal{X}} \arrow[r] \arrow[d, equal] & F_{\mathcal{X}} i_* i^* \arrow[d] \\
j_\sharp j^* F_{\mathcal{X}} \arrow[r] & F_{\mathcal{X}} \arrow[r] & i_* i^* F_{\mathcal{X}}.
\end{tikzcd}$$

The vertical equivalence on the left is given by the equivalences $F_{\mathcal{X}} j_\sharp \simeq j_\sharp F_{\mathcal{Z}}$ and $F_{\mathcal{Z}} j^* \simeq j^* F_{\mathcal{Z}}$. As the diagram commutes by Lemma F.5, it follows that also the right vertical map on cofibers is an equivalence, finishing the proof. □

Corollary 5.1.5 (Closed exceptional pullback)**.** *For a closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$, the functor $i_*\colon \mathcal{C}(\mathcal{Z}) \hookrightarrow \mathcal{C}(\mathcal{X})$ preserves colimits, and thus admits a right adjoint*

$$i^!\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{Z}),$$

called the exceptional pullback functor.

Proof. Given an indexing ∞-category, we have to show that i_* preserves I-indexed colimits. This is a special case of Lemma 5.1.4 applied to the morphism of pullback formalisms $\mathrm{colim}_I\colon \mathcal{C}^I \to \mathcal{C}$. □

In the stable setting, we obtain the following alternative characterization of the localization axiom:

Lemma 5.1.6. *Let $\mathcal{C}$ be a stable pullback formalism on* SepStk *and let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding with open complement $j\colon \mathcal{U} \to \mathcal{X}$. Then $\mathcal{C}$ satisfies* (Loc$_i$) *if and only if the following two conditions hold:*

(d) The functor $i_*\colon \mathcal{C}(\mathcal{Z}) \to \mathcal{C}(\mathcal{X})$ is fully faithful;

(e) The functors $j^*\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{U})$ and $i^*\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{Z})$ are jointly conservative.

Proof. First assume that C satisfies the localization axiom (Loc_i) for i. We have seen in Proposition 5.1.3 that i_* is fully faithful, proving (d). To see that the functors j^* and i^* are jointly conservative, let $f\colon X \to Y$ be a morphism in $C(\mathcal{X})$ such that i^*f and j^*f are equivalences. It follows that $j_\sharp j^*f$ and i_*i^*f are equivalences, and hence so is f by the bifiber sequence $j_\sharp j^*f \to f \to i_*i^*f$. This proves (e).

Conversely, assume that conditions (d) and (e) are satisfied. We will show that C satisfies (Loc_i). As every fully faithful functor is conservative, it remains to show that for every $X \in C(\mathcal{X})$, the sequence

$$j_\sharp j^*X \to X \to i_*i^*X$$

is a cofiber sequence in $C(\mathcal{X})$. By (e), we may test this after pulling back along i and j. Using (a1), (b1), (b2) and (d), the pulled back sequences are equivalent to

$$j^*X \to j^*X \to 0 \qquad \text{and} \qquad 0 \to i^*X \to i^*X,$$

which are clearly cofiber sequences. □

Remark 5.1.7 (Stable recollement)**.** If C is a stable pullback formalism on SepStk which satisfies the localization axiom for $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$, then Lemma 5.1.6 shows that the full subcategories $i_*\colon C(\mathcal{Z}) \hookrightarrow C(\mathcal{X})$ and $j_*\colon C(\mathcal{U}) \hookrightarrow C(\mathcal{Z})$ of $C(\mathcal{X})$ are part of a *stable recollement*, in the sense of [Cal+20, Definition A.2.9]:

$$\begin{tikzcd} C(\mathcal{Z}) \arrow[r, "i_*"] & C(\mathcal{X}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & C(\mathcal{U}). \arrow[l, bend right, "j_\sharp"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

5.2 The localization theorem for genuine sheaves

In the previous section, we introduced the localization axiom for a pointed pullback formalism C on SepStk. The goal of this section is to verify this axiom in the case $C = \mathrm{H}_\bullet$ of pointed genuine sheaves.

5.2.1 Exactness of the closed pushforward functor

Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks. In this subsection, we will show that the pushforward functor $i_*\colon \mathrm{H}(\mathcal{Z}) \to \mathrm{H}(\mathcal{X})$ preserves all *weakly contractible* colimits, meaning those colimits indexed by weakly contractible[2] ∞-categories. This exactness property of i_* follows from a general criterion on a morphism of sites. We follow Khan [Kha19, Section 3.1].

Definition 5.2.1 ([Kha19, Definition 3.1.5]). Let $\mathbf{C}$ and $\mathbf{D}$ be essentially small ∞-categories equipped with Grothendieck topologies $\tau_{\mathbf{C}}$ and $\tau_{\mathbf{D}}$. Assume that $\mathbf{D}$ admits an initial object $\emptyset_{\mathbf{D}}$. A functor $u\colon \mathbf{C} \to \mathbf{D}$ is called *topologically quasi-cocontinuous* if the following condition holds:

> For every $c \in \mathbf{C}$ and every $\tau_{\mathbf{D}}$-covering sieve $R' \hookrightarrow y(u(c))$ in $\mathbf{D}$, the sieve $R \hookrightarrow y(c)$, generated by morphisms $c' \to c$ such that either $u(c')$ is initial or $y(u(c')) \to y(u(c))$ factors through $R' \hookrightarrow y(u(c))$, is a covering in $\mathbf{C}$.

Lemma 5.2.2 ([Kha19, Lemma 3.1.6]). *Consider a topologically quasi-cocontinuous functor $u\colon \mathbf{C} \to \mathbf{D}$, as in Definition 5.2.1. Assume further that the following two conditions on $\mathbf{D}$ are satisfied:*

(1) The initial object $\emptyset_{\mathbf{D}}$ is strict: given an object $d \in \mathbf{D}$, any morphism $d \to \emptyset_{\mathbf{D}}$ is invertible.

(2) For any object $d \in \mathbf{D}$, the sieve $\emptyset_{\mathrm{PSh}(\mathbf{D})} \hookrightarrow y(d)$ is a covering in $\mathbf{D}$ if and only if d is initial (where $\emptyset_{\mathrm{PSh}(\mathbf{D})}$ denotes the initial object of $\mathrm{PSh}(\mathbf{D})$).

Then the functor $\mathrm{Shv}_{\tau_{\mathbf{D}}}(\mathbf{D}) \to \mathrm{Shv}_{\tau_{\mathbf{C}}}(\mathbf{C})$ given by the assignment $\mathcal{F} \mapsto L_{\tau_{\mathbf{C}}}(u^*(\mathcal{F}))$ commutes with contractible colimits. □

We apply this to the situation of differentiable stacks.

Proposition 5.2.3. *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks. Then the functor $i_*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Z}}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ commutes with contractible colimits.*

Proof. We apply Lemma 5.2.2 to the morphism of sites $i^*\colon \mathrm{Sub}_{/\mathcal{X}} \to \mathrm{Sub}_{/\mathcal{Z}}$. It is clear that the initial object $\emptyset \to \mathcal{Z}$ of $\mathrm{Sub}_{/\mathcal{Z}}$ is strict and that the only object of $\mathrm{Sub}_{/\mathcal{Z}}$ covered by the empty sieve is the empty stack. It thus remains to show that i^* is topologically quasi-cocontinuous.

Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable submersion of differentiable stacks and consider an open cover $\{\mathcal{V}_i \hookrightarrow i^*\mathcal{Y}\}_{i \in I}$ of the pullback $i^*\mathcal{Y} \to \mathcal{Z}$. We have to show that the sieve

$$T = \{\mathcal{W} \hookrightarrow \mathcal{Y} \mid i^*\mathcal{W} = \emptyset \text{ or } i^*\mathcal{W} \hookrightarrow \mathcal{V}_i \text{ for some } i \in I\}$$

[2] Recall that an ∞-category I is called *weakly contractible* if the ∞-groupoid $|I|$ formed by inverting all morphisms in I is contractible

covers $\mathcal{Y}$. For every $i \in I$, let $\mathcal{W}_i \hookrightarrow \mathcal{Y}$ be an open substack such that $\mathcal{V}_i = i^*\mathcal{W}_i \subseteq i^*\mathcal{Y}$; this exists by Corollary 3.2.3. Further, let $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ denote the open complement of $\mathcal{Z}$. It is clear that the inclusion $\mathcal{Y}_{\mathcal{U}} \hookrightarrow \mathcal{Y}$ and the inclusions $\mathcal{W}_i \hookrightarrow \mathcal{Y}$ are all in T and that they cover $\mathcal{Y}$, proving the claim. □

In order to deduce something about the pushforward functor $i_*\colon \mathrm{H}(\mathcal{Z}) \to \mathrm{H}(\mathcal{X})$ at the level of genuine sheaves, we need to understand the interaction with the localization functor $L_{\mathrm{htp}}\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{H}(\mathcal{X})$. We introduce the following auxiliary terminology:

Definition 5.2.4 (Weak equivalence)**.** A morphism $f\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ is called a *weak equivalence* if it is inverted by the localization functor $L_{\mathrm{htp}}\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{H}(\mathcal{X})$. A square

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}' \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{G}' \end{tikzcd}$$

in $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ is called *weakly cocartesian* if its image in $\mathrm{H}(\mathcal{X})$ is a cocartesian square.

If follows from Corollary 4.2.9 that every strict homotopy equivalence is a weak equivalence. We thank Adeel Khan for a discussion about the proof of the following result.

Proposition 5.2.5. *Let $f\colon \mathcal{Z} \to \mathcal{X}$ be a morphism of differentiable stacks. Then the functor $f_*\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{Z}}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ preserves weak equivalences.*

Proof. It will suffice to show that the functor $f_*\colon \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{Z}}) \to \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$ preserves $L_{\mathbb{R}}$-local morphisms. As this functor preserves colimits, it suffices to show that it sends morphisms of the form $\mathrm{pr}\colon \mathcal{Y} \times \mathbb{R} \to \mathcal{Y}$ to weak equivalences, where $\mathcal{Y} \in \mathrm{Sub}_{/\mathcal{Z}}$. The projection map admits a section given by the map $i_0\colon \mathcal{Y} \to \mathcal{Y} \times \mathbb{R}$, and the composite $i_0 \circ \mathrm{pr}\colon \mathcal{Y} \times \mathbb{R} \to \mathcal{Y} \times \mathbb{R}$ is homotopic to the identity map. It will thus suffice to show that f_* sends homotopic maps to homotopic maps. But this is clear: if $H\colon \mathcal{Y}' \times \mathbb{R} \to \mathcal{X}'$ is a homotopy between two maps H_0 and H_1, then a homotopy between $f_*(H_0)$ and $f_*(H_1)$ is given by the composite $f_*(\mathcal{Y}') \times \mathbb{R} \to f_*(\mathcal{Y}' \times \mathbb{R}) \xrightarrow{f_*(H)} f_*(\mathcal{X}')$, where the first map is the projection formula map. □

Corollary 5.2.6. *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of differentiable stacks. Then the functor $i_*\colon \mathrm{H}(\mathcal{Z}) \to \mathrm{H}(\mathcal{X})$ preserves weakly contractible colimits.*

Proof. Colimits in $\mathrm{H}(\mathcal{X})$ are computed by first forming them in $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$ and then applying the homotopy localization functor $L_{\mathrm{htp}}\colon \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{H}(\mathcal{X})$. The result is thus immediate from Proposition 5.2.3 and Proposition 5.2.5. □

Corollary 5.2.7 (Exceptional pushforward)**.** *Let* $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ *be a closed embedding of differentiable stacks. Then the functor* $i_*\colon \mathrm{H}_\bullet(\mathcal{Z}) \to \mathrm{H}_\bullet(\mathcal{X})$ *preserves all small colimits, and thus admits a right adjoint* $i^!\colon \mathrm{H}_\bullet(\mathcal{X}) \to \mathrm{H}_\bullet(\mathcal{Z})$.

Proof. Since $\mathrm{H}_\bullet(\mathcal{Z})$ and $\mathrm{H}_\bullet(\mathcal{X})$ are pointed, it is clear that i_* preserves the initial object. By Corollary 5.2.6 it preserves all contractible colimits. The claim follows: given any small ∞-category I and a diagram $F\colon I \to \mathrm{H}(\mathcal{Z})_\bullet$, we may lift F to a functor $\widetilde{F}\colon I^\triangleleft \to \mathrm{H}(\mathcal{Z})_\bullet$ which sends the cone point to the zero object. Note that $\widetilde{F}$ has the the same colimit as F, but has a weakly contractible indexing diagram. As i_* preserves the initial object, we have $i_* \circ F^\triangleleft = (i_* \circ F)^\triangleleft$, and it follows that i_* preserves the colimit of F. $\square$

5.2.2 Presheaves of $\mathcal{Z}$-trivialized morphisms

To streamline the proof of the localization theorem for $C = \mathrm{H}_\bullet$, we introduce some auxiliary notation regarding certain presheaves of $\mathcal{Z}$-trivialized morphisms. Our definitions, statements and proofs are direct analogues of those of Khan [Kha19, Section 4.1] and go back to Morel and Voevodsky [MV99].

In the following, we denote for every differentiable stack $\mathcal{X}$ the Yoneda embedding of $\mathrm{Sub}_{/\mathcal{X}}$ by $h_\mathcal{X}\colon \mathrm{Sub}_{/\mathcal{X}} \hookrightarrow \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$.

Definition 5.2.8. Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding with open complement $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$. For any $\mathcal{Y} \in \mathrm{Sub}_{/\mathcal{X}}$, we define the presheaf $h_\mathcal{X}^\mathcal{Z}(\mathcal{Y}) \in \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$ as the following pushout:

$$\begin{tikzcd} h_\mathcal{X}(\mathcal{Y}_\mathcal{U}) \arrow[r] \arrow[d] & h_\mathcal{X}(\mathcal{Y}) \arrow[d] \\ h_\mathcal{X}(\mathcal{U}) \arrow[r] & h_\mathcal{X}^\mathcal{Z}(\mathcal{Y}). \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

Note that for $\mathcal{W} \in \mathrm{Sub}_{/\mathcal{X}}$, the anima $\Gamma(\mathcal{W}, h_\mathcal{X}(\mathcal{U}))$ is either empty or contractible depending on whether $\mathcal{W}_\mathcal{Z}$ is empty or not. It follows that the anima of sections $\Gamma(\mathcal{W}, h_\mathcal{X}^\mathcal{Z}(\mathcal{Y}))$ is contractible when $\mathcal{W}_\mathcal{Z}$ is empty, and otherwise is given by the set $\mathrm{Hom}_{/\mathcal{X}}(\mathcal{W}, \mathcal{Y})$.

Since i^* commutes with colimits, we have $i^*(h_\mathcal{X}^\mathcal{Z}(\mathcal{Y})) \simeq h_\mathcal{Z}(\mathcal{Y}_\mathcal{Z})$.

Construction 5.2.9 (Presheaf of $\mathcal{Z}$-trivialized morphisms)**.** Let $\mathcal{Y} \in \mathrm{Sub}_{/\mathcal{X}}$ and let $t\colon \mathcal{Z} \hookrightarrow \mathcal{Y}$ be a morphism over $\mathcal{X}$, i.e., a partially defined section of $\mathcal{Y} \to \mathcal{X}$. We define the presheaf $h_\mathcal{X}^\mathcal{Z}(\mathcal{Y}, t) \in \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$, referred to as the *presheaf of $\mathcal{Z}$-trivialized morphisms*, as the following pullback:

$$\begin{tikzcd} h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y},t) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y}) \arrow[d, "u_i^*"] \\ \mathrm{pt}_{\mathcal{X}} \arrow[r, "\tau"] & i_*i^*h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y}). \end{tikzcd}$$

Here the right vertical map is the unit map and the bottom horizontal map is given by

$$\mathrm{pt}_{\mathcal{X}} \simeq i_*h_{\mathcal{Z}}(\mathcal{Z}) \xrightarrow{t} i_*h_{\mathcal{Z}}(\mathcal{Y}_{\mathcal{Z}}) \simeq i_*i^*h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y}).$$

In other words, the anima $\Gamma(\mathcal{W}, h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y},t))$ is contractible when $\mathcal{W}_{\mathcal{Z}}$ is empty, and otherwise is given by the fiber of the restriction map

$$\mathrm{Hom}_{/\mathcal{X}}(\mathcal{W},\mathcal{Y}) \to \mathrm{Hom}_{/\mathcal{Z}}(\mathcal{W}_{\mathcal{Z}},\mathcal{Y}_{\mathcal{Z}})$$

at the point defined by the composite $\mathcal{W}_{\mathcal{Z}} \to \mathcal{Z} \xrightarrow{t} \mathcal{X}_{\mathcal{Z}}$.

Remark 5.2.10. For a representable submersion $p\colon \mathcal{X}' \to \mathcal{X}$, there is a natural equivalence

$$p^*h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y},t) \simeq h_{\mathcal{X}'}^{p^*\mathcal{Z}}(p^*\mathcal{Y}, p^*t),$$

where $p^*t\colon p^*\mathcal{Z} \to p^*\mathcal{Y}$ is the base change of t along p. This is immediate from the fact that p^* preserves limits and colimits.

Our goal in this subsection will be to show that the presheaf $h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y},t)$ is weakly contractible, see Proposition 5.2.13 below.

Lemma 5.2.11. *Let $\pi\colon \mathcal{E} \to \mathcal{X}$ be a vector bundle with zero section $s_0\colon \mathcal{X} \hookrightarrow \mathcal{E}$. Let $t\colon \mathcal{Z} \to \mathcal{E}$ denote the composite $t = s_0 \circ i$. Then the presheaf $h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{E},t)$ is weakly contractible.*

Proof. The map π induces a map $\pi\colon h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{E},t) \to h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{X},i)$, and conversely s_0 induces a map $s_0\colon h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{X},i) \to h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{E},t)$. We have $\pi \circ s_0 = \mathrm{id}$, and $s_0 \circ \pi \simeq \mathrm{id}$ via the homotopy

$$\mathbb{R} \times h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{E},t) \to h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{E},t), \quad (a,f) \mapsto af.$$

We thus have a weak equivalence $h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{E},t) \simeq h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{X},i)$. The claim now follows since the presheaf $h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{X},i)$ is the terminal presheaf. □

Lemma 5.2.12. *Let $j'\colon \mathcal{V} \hookrightarrow \mathcal{Y}$ be an open embedding in $\mathrm{Sub}_{/\mathcal{X}}$. Let $t_{\mathcal{V}}\colon \mathcal{Z} \to \mathcal{V}$ be a partial section of the map $\mathcal{V} \to \mathcal{X}$ and let $t_{\mathcal{Y}} = j' \circ t_{\mathcal{V}}$. Then the induced map*

$$j'\colon h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{V},t_{\mathcal{V}}) \to h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y},t_{\mathcal{Y}})$$

in $\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$ is inverted by the sheafification functor $L_{\mathrm{open}}\colon \mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}}) \to \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$.

Proof. We will show the map $L_{\mathrm{open}}(j')$ is both a monomorphism as well as an effective epimorphism in $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$.

Step 1: We first show that $L_{\mathrm{open}}(j')$ is a monomorphism. Since the sheafification preserves pullbacks, we may show that the map $j' \colon h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{V}, t_{\mathcal{V}}) \to h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{Y}, t_{\mathcal{Y}})$ is a monomorphism. Given $\mathcal{W} \in \mathrm{Sub}_{/\mathcal{X}}$, we will show that the map

$$\Gamma(\mathcal{W}, h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{V}, t_{\mathcal{V}})) \to \Gamma(\mathcal{W}, h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{Y}, t_{\mathcal{Y}}))$$

of animae is a monomorphism. This is clear if $\mathcal{W}_{\mathcal{Z}}$ is empty, so assume it is not. Then this map is the map induced on fibers in the following diagram:

$$\begin{tikzcd}
\Gamma(\mathcal{W}, h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{V}, t_{\mathcal{V}})) \arrow[r] \arrow[d, dashed] & \mathrm{Map}_{/\mathcal{X}}(\mathcal{W}, \mathcal{V}) \arrow[r] \arrow[d] & \mathrm{Map}_{/\mathcal{Z}}(\mathcal{W}_{\mathcal{Z}}, \mathcal{V}_{\mathcal{Z}}) \arrow[d] \\
\Gamma(\mathcal{W}, h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{Y}, t_{\mathcal{Y}})) \arrow[r] & \mathrm{Map}_{/\mathcal{X}}(\mathcal{W}, \mathcal{Y}) \arrow[r] & \mathrm{Map}_{/\mathcal{Z}}(\mathcal{W}_{\mathcal{Z}}, \mathcal{Y}_{\mathcal{Z}}).
\end{tikzcd}$$

Since the middle and right vertical morphisms are monomorphisms of animae, so is the induced map on fibers.

Step 2: We show that the map $L_{\mathrm{open}}(j')$ is an effective epimorphism in $\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$. It suffices to show that for every $\mathcal{W} \in \mathrm{Sub}_{/\mathcal{X}}$, any $\mathcal{W}$-section of $h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{Y}, t)$ can locally be lifted along j'. We may again assume that $\mathcal{W}_{\mathcal{Z}}$ is non-empty, since the claim is clear otherwise. In that case, let f be a $\mathcal{W}$-section of $h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{Y}, t)$, i.e., a $\mathcal{Z}$-trivialized morphism $f \colon \mathcal{W} \to \mathcal{Y}$. We claim that the substacks

$$\mathcal{W}_{\mathcal{U}} \hookrightarrow \mathcal{W}, \qquad \mathcal{W}_{\mathcal{V}} \hookrightarrow \mathcal{W}$$

cover $\mathcal{W}$. Indeed, $\mathcal{W}_{\mathcal{U}}$ is the open complement of the closed substack $\mathcal{W}_{\mathcal{Z}}$, and the map $f_{\mathcal{Z}} \colon \mathcal{W}_{\mathcal{Z}} \to \mathcal{Y}_{\mathcal{Z}}$ factors through $\mathcal{Z}$ by assumption, and thus in particular through $\mathcal{V}_{\mathcal{Z}}$.

It will thus suffice to show that both $f_{\mathcal{U}}$ as well as $f_{\mathcal{V}}$ admit lifts along j'. For $f_{\mathcal{V}}$ this is true by construction, as $\mathcal{Y}_{\mathcal{V}} = \mathcal{V}$. For $f_{\mathcal{U}}$ this is automatic as $\mathcal{U}_{\mathcal{Z}} = \emptyset$. □

Proposition 5.2.13. *Let $i \colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of separated differentiable stacks with open complement $j \colon \mathcal{U} \hookrightarrow \mathcal{X}$. Let $p \colon \mathcal{Y} \to \mathcal{X}$ be a representable submersion, and let $t \colon \mathcal{Z} \hookrightarrow \mathcal{Y}$ be a section of p over $\mathcal{Z}$. Then the presheaf $h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{Y}, t)$ is weakly contractible, in the sense that its image in $\mathrm{H}(\mathcal{X})$ is the terminal object.*

Proof. The question is local in $\mathcal{X}$, in light of Remark 5.2.10. Around points in $\mathcal{U} = \mathcal{X} \setminus \mathcal{Z}$ the statement is trivially true, so it suffices to prove the claim locally around points $z \in \mathcal{Z}$.

By Proposition 3.7.5, we can therefore reduce to the case where there is a commutative diagram

$$\begin{tikzcd} & \mathcal{E} \arrow[r, hook, "j"] \arrow[d, shift left, "\pi"] & \mathcal{Y} \arrow[d, "p"] \\ \mathcal{Z} \arrow[ur, hook, "t"] \arrow[r, hook, "i"] & \mathcal{X} \arrow[u, shift left, "s_0"] \arrow[r, equal] & \mathcal{X}, \end{tikzcd}$$

where j is an open embedding and the map $\pi\colon \mathcal{E} \to \mathcal{X}$ is a vector bundle with zero section $s_0\colon \mathcal{X} \to \mathcal{E}$. By Lemma 5.2.12, the map j induces a weak equivalence

$$j\colon h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{E}, t) \xrightarrow{\sim} h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y}, t).$$

But by Lemma 5.2.11, the presheaf $h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{E}, t)$ is weakly contractible, finishing the proof. □

5.2.3 Proof of the localization theorem

We are now ready for the proof of the localization theorem for genuine sheaves. Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding with open complement $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$. For any sheaf $\mathcal{F} \in \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$, consider the functorial commutative square

$$\begin{tikzcd} j_\sharp j^*(\mathcal{F}) \arrow[r, "c_j^\sharp"] \arrow[d, "u_i^*"] & \mathcal{F} \arrow[d, "u_i^*"] \\ j_\sharp j^* i_* i^*(\mathcal{F}) \arrow[r, "c_j^\sharp"] & i_* i^*(\mathcal{F}), \end{tikzcd}$$

where $c_j^\sharp\colon j_\sharp j^* \to \mathrm{id}$ is the adjunction counit and $u_i^*\colon \mathrm{id} \to i_* i^*$ is the adjunction unit. By smooth base change, $j^* i_*$ is the constant functor with value the terminal object $\mathcal{U} \in \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{U}})$. Hence the lower left corner $j_\sharp j^* i_* i^*(\mathcal{F})$ of the diagram is simply $j_\sharp \mathcal{U} \in \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})$.

Theorem 5.2.14 (Localization theorem for genuine sheaves)**.** *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding of separated differentiable stacks with open complement $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$. Then for every $\mathcal{F} \in \mathrm{H}(\mathcal{X})$ the square*

$$\begin{tikzcd} j_\sharp j^*(\mathcal{F}) \arrow[r, "c_j^\sharp"] \arrow[d] & \mathcal{F} \arrow[d, "u_i^*"] \\ j_\sharp(\mathcal{U}) \arrow[r] & i_* i^*(\mathcal{F}) \end{tikzcd}$$

is cocartesian in $\mathrm{H}(\mathcal{X})$.

Proof. Given an object $\mathcal{F} \in \mathrm{H}(\mathcal{X})$, we have to show that the square is weakly cocartesian in $\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$, in the sense of Definition 5.2.4. Observe that $\mathcal{F}$ can be written as a weakly contractible colimit of representables, indexed by the ∞-category

$$(\mathrm{Sub}_{/\mathcal{X}})_{/\mathcal{F}} = \mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})_{/\mathcal{F}} \times_{\mathrm{Shv}(\mathrm{Sub}_{/\mathcal{X}})} \mathrm{Sub}_{/\mathcal{X}},$$

which is weakly contractible as it has an initial object $\emptyset \to \mathcal{F}$. By Corollary 5.2.6, all four functors in the square commute with contractible colimits, and thus it suffices to prove the claim when $\mathcal{F} = L_{\mathrm{htp}}(\mathcal{Y})$ for some representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$. In other words, we have to show that canonical map

$$h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y}) = h_{\mathcal{X}}(\mathcal{Y}) \sqcup_{h_{\mathcal{X}}(\mathcal{Y}_{\mathcal{U}})} h_{\mathcal{X}}(\mathcal{U}) \to i_* h_{\mathcal{Z}}(\mathcal{Y}_{\mathcal{Z}})$$

is a weak equivalence of presheaves on $\mathrm{Sub}_{/\mathcal{X}}$. By universality of colimits in $\mathrm{PSh}(\mathrm{Sub}_{/\mathcal{X}})$, it will suffice to prove that for any other $(p\colon \mathcal{W} \to \mathcal{X}) \in \mathrm{Sub}_{/\mathcal{X}}$ and any morphism $h_{\mathcal{X}}(\mathcal{W}) \to i_* h_{\mathcal{Z}}(\mathcal{Y}_{\mathcal{Z}})$, the base change

$$h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y}) \times_{i_* h_{\mathcal{Z}}(\mathcal{Y}_{\mathcal{Z}})} h_{\mathcal{X}}(\mathcal{W}) \to h_{\mathcal{X}}(\mathcal{W})$$

is a weak equivalence. Note that we have $h_{\mathcal{X}}(\mathcal{W}) \simeq p_\sharp h_{\mathcal{W}}(\mathcal{W})$, and thus by the smooth projection formula we may identify this morphism with

$$p_\sharp(p^* h_{\mathcal{X}}^{\mathcal{Z}}(\mathcal{Y}) \times_{p^* i_* h_{\mathcal{Z}}(\mathcal{Y}_{\mathcal{Z}})} h_{\mathcal{W}}(\mathcal{W})) \to p_\sharp h_{\mathcal{X}}(\mathcal{W}).$$

Consider the pullback square

$$\begin{tikzcd} \mathcal{W}_{\mathcal{Z}} \arrow[r, hook, "k"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{W} \arrow[d, "p"] \\ \mathcal{Z} \arrow[r, hook, "i"] & \mathcal{X}. \end{tikzcd}$$

By smooth base change we have $p^* i_* \simeq k_* q^*$, and under this equivalence, the above morphism is the image of $p_\sharp$ of the morphism

$$h_{\mathcal{W}}^{\mathcal{W}_{\mathcal{Z}}}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{W}) \times_{k_* h_{\mathcal{W}_{\mathcal{Z}}}((\mathcal{Y} \times_{\mathcal{X}} \mathcal{W})_{\mathcal{Z}})} h_{\mathcal{W}}(\mathcal{W}) \to h_{\mathcal{W}}(\mathcal{W}).$$

But the source of this morphism is precisely the presheaf $h_{\mathcal{W}}^{\mathcal{W}_{\mathcal{Z}}}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{W}, t_{\mathcal{W}})$, where $t_{\mathcal{W}}\colon \mathcal{Z} \times_{\mathcal{X}} \mathcal{W} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{W}$ is the base change of $t\colon \mathcal{Z} \to \mathcal{Y}$ along $p\colon \mathcal{W} \to \mathcal{X}$. We conclude by Proposition 5.2.13. $\square$

Corollary 5.2.15. *For every closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ of separated differentiable stacks with open complement $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ and every pointed genuine sheaf $\mathcal{F} \in \mathrm{H}_\bullet(\mathcal{X})$, the sequence*

$$j_\sharp j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}$$

is a cofiber sequence in $\mathrm{H}_\bullet(\mathcal{X})$.

Proof. For the purpose of this proof, let us make a distinction in notation between the functors $j_\sharp^{\mathrm{H}}\colon \mathrm{H}(\mathcal{U}) \hookrightarrow \mathrm{H}(\mathcal{X})$ and $j_\sharp^{\mathrm{H}_\bullet}\colon \mathrm{H}_\bullet(\mathcal{U}) \hookrightarrow \mathrm{H}_\bullet(\mathcal{X})$. We may express $j_\sharp^{\mathrm{H}_\bullet}$ in terms of $j_\sharp^{\mathrm{H}}$ as follows: for a pointed object $\mathcal{G} \in \mathrm{H}_\bullet(\mathcal{U}) = \mathrm{H}(\mathcal{U})_*$, regarded as a morphism $\mathcal{U} \to \mathcal{G}$ in $\mathrm{H}(\mathcal{U})$, we have a pushout square

$$\begin{tikzcd}
j_\sharp^{\mathrm{H}}(\mathcal{U}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & j_\sharp^{\mathrm{H}}(\mathcal{G}) \arrow[d] \\
y(\mathcal{X}) \arrow[r] & j_\sharp^{\mathrm{H}_\bullet}(\mathcal{G}).
\end{tikzcd}$$

In particular, the object $j_\sharp^{\mathrm{H}_\bullet} j^*\mathcal{F}$ is given by the top left pushout square in the following commutative diagram:

$$\begin{tikzcd}
j_\sharp^{\mathrm{H}}(\mathcal{U}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & y(\mathcal{X}) \arrow[d] & \\
j_\sharp j^*\mathcal{F} \arrow[r] \arrow[d] & j_\sharp^{\mathrm{H}_\bullet} j^*\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
j_\sharp(\mathcal{U}) \arrow[r] & y(\mathcal{X}) \arrow[r] & i_*i^*\mathcal{F}.
\end{tikzcd}$$

It follows by the pasting rule for pushout squares that the bottom left square is also a pushout. As the outer bottom rectangle is a pushout by Theorem 5.2.14, it follows from another application of the pasting rule that also the bottom right square is a pushout square. This square is precisely the underlying null-homotopy of the sequence $j_\sharp^{\mathrm{H}_\bullet}\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F}$ in the statement, so since the forgetful functor $\mathrm{H}_\bullet(\mathcal{X}) \to \mathrm{H}(\mathcal{X})$ preserves pushouts this finishes the proof. □

Theorem 5.2.16 (Localization theorem for pointed genuine sheaves)**.** *The pullback formalism $C = \mathrm{H}_\bullet$ satisfies the localization axiom.*

Proof. In light of Corollary 5.2.15, it remains to show that for every closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$, the pushforward functor $i_*\colon \mathrm{H}_\bullet(\mathcal{Z}) \hookrightarrow \mathrm{H}_\bullet(\mathcal{X})$ is conservative. This may be tested after forgetting the base point. So let $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism in $\mathrm{H}(\mathcal{Z})$ such that the induced map $i_*(\varphi)$ is invertible in $\mathrm{H}(\mathcal{X})$. We need to show that φ itself is already an equivalence, i.e., that for every $\mathcal{Y} \in \mathrm{Sub}_{/\mathcal{Z}}$ the map $\varphi(\mathcal{Y})\colon \mathcal{F}_1(\mathcal{Y}) \to \mathcal{F}_2(\mathcal{Y})$ is an equivalence. By Corollary 3.6.7, there exists an object $\mathcal{Y}' \in \mathrm{Sub}_{/\mathcal{X}}$ such that $\mathcal{Y} \simeq \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}'$. It thus then follows that $\mathcal{F}_k(\mathcal{Y}) \simeq \mathcal{F}_k(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}') = i^*\mathcal{F}_k(\mathcal{Y}')$ and similarly $\varphi(\mathcal{Y}) \simeq i_*\varphi(\mathcal{Y}')$. This proves the claim. □

Corollary 5.2.17. *The functors* $i_* \colon \mathrm{H}_\bullet(\mathcal{Z}) \to \mathrm{H}_\bullet(\mathcal{X})$ *for closed embeddings between separated differentiable stacks are fully faithful and satisfy the closed projection formula, closed base change and smooth-closed base change.*

Proof. This is a special case of Proposition 5.1.3. □

Corollary 5.2.18 (Localization theorem for genuine sheaves of spectra)**.** *The pullback formalism* $\mathcal{C} = \mathrm{SH}$ *satisfies the localization axiom.*

Proof. In light of Lemma 5.1.4, this is an immediate consequence of Theorem 5.2.16. □

II.6 Relative Poincaré duality for differentiable stacks

The goal of this chapter is to establish a relative version of Poincaré duality for separated differentiable stacks: for every proper representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of separated differentiable stacks, we will establish an equivalence

$$f_\sharp(-) \simeq f_*(- \otimes \omega_f)$$

of functors $\mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$, where $\omega_f \in \mathrm{SH}(\mathcal{Y})$ is the *relative dualizing sheaf* of f. Our treatment is similar to that of Hoyois [Hoy17], who proves the analogous result in the context of equivariant motivic homotopy theory.

6.1 Statement of Main Theorem

The proof of relative Poincaré duality only relies on three crucial properties satisfied by the ∞-categories $\mathrm{SH}(\mathcal{X})$: homotopy invariance, the localization axiom, and the stability axiom. We have chosen to write all constructions and proofs of this chapter in this level of generality, as we feel that removing the specifics of the construction of $\mathrm{SH}(\mathcal{X})$ from the discussion clarifies the nature of the argument. Following [KR21, Definition 5.5], we will refer to these three properties as the *Voevodsky conditions*, as they were first singled out by Voevodsky in the case of schemes.

Definition 6.1.1 (Voevodsky conditions)**.** Consider a pullback formalism $\mathcal{C}\colon \mathrm{SepStk}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ on SepStk, in the sense of Definition 4.5.5. We say that $\mathcal{C}$ *satisfies the Voevodsky conditions* if it satisfies homotopy invariance, genuine stability and the localization axiom:

(1) (Homotopy invariance, Definition 4.5.17) For every separated differentiable stack $\mathcal{X}$, the pullback functor $\mathrm{pr}^*\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{X} \times \mathbb{R})$ associated to the projection map $\mathrm{pr}\colon \mathcal{X} \times \mathbb{R} \to \mathcal{X}$ is fully faithful;

(2) (Genuine stability, Definition 4.5.26) The pullback formalism C is pointed and for every vector bundle $\pi\colon \mathcal{E} \to \mathcal{X}$ over a separated differentiable stack $\mathcal{X}$, the associated sphere bundle $S^{\mathcal{E}} \in C(\mathcal{X})$ is monoidally invertible.

(3) (Localization axiom, Definition 5.1.2) For every closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ with open complement $j\colon \mathcal{U} \hookrightarrow \mathcal{Y}$, the functors $i_*\colon C(\mathcal{Z}) \hookrightarrow C(\mathcal{X})$ and $j_*\colon C(\mathcal{U}) \hookrightarrow C(\mathcal{X})$ are fully faithful, and there is a cofiber sequence $j_\sharp j^* \to \mathrm{id} \to i_* i^*$ in $\mathrm{Fun}(C(\mathcal{X}), C(\mathcal{X}))$. Equivalently, $C(\mathcal{X})$ is a recollement of the full subcategories $C(\mathcal{U})$ and $C(\mathcal{Z})$.

Example 6.1.2. The pullback formalism $C = \mathrm{SH}$ satisfies the Voevodsky conditions: homotopy invariance and genuine stability hold by definition, while the localization axiom holds by Corollary 5.2.18.

For the remainder of the section, we fix a pullback formalism C satisfying the Voevodsky conditions. We will now give the statement of relative Poincaré duality in this level of generality.

Definition 6.1.3 (Dualizing object)**.** Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable submersion. We define its *dualizing object* $\omega_f \in C(\mathcal{Y})$ as

$$\omega_f := \mathrm{pr}_{1\sharp}\Delta_* \mathbb{1}_{\mathcal{Y}} \quad \in \quad C(\mathcal{Y}),$$

where $\mathrm{pr}_1\colon \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is the projection to the first factor and $\Delta\colon \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is the diagonal of f.

We will see below in Corollary 6.2.11 that the object ω_f is the incarnation in $C(\mathcal{Y})$ of the tangent sphere bundle S^{T_f} of f over $\mathcal{Y}$, i.e. the fiberwise one-point compactification of the relative tangent bundle $T_f \to \mathcal{Y}$ of f.

Remark 6.1.4 (Twist functor)**.** We refer to the functor $-\otimes\omega_f\colon C(\mathcal{Y}) \to C(\mathcal{Y})$ as a 'twist functor'. By the smooth and closed projection formulas, it is equivalent to the composite $\mathrm{pr}_{1\sharp}\Delta_*\colon C(\mathcal{Y}) \to C(\mathcal{Y})$:

$$X \otimes \omega_f = X \otimes \mathrm{pr}_{1\sharp}\Delta_* \mathbb{1}_{\mathcal{Y}} \simeq \mathrm{pr}_{1\sharp}(\mathrm{pr}_1^* X \otimes \Delta_* \mathbb{1}_{\mathcal{Y}}) \simeq \mathrm{pr}_{1\sharp}\Delta_*(\Delta^* \mathrm{pr}_1^* X \otimes \mathbb{1}_{\mathcal{Y}}) \simeq \mathrm{pr}_{1\sharp}\Delta_* X.$$

Construction 6.1.5 (Poincaré duality map, cf. [CD19, Section 2.4.b])**.** Given a representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$, we construct a natural transformation $\mathfrak{p}_f\colon f_\sharp(-) \to f_*(-\otimes\omega_f)$, called the *Poincaré duality map*. Consider the following pullback square:

$$\begin{tikzcd} \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X}. \end{tikzcd}$$

There is a a *double Beck-Chevalley transformation* $\mathrm{BC}_{\sharp,*}\colon f_\sharp \mathrm{pr}_{2*} \to f_* \mathrm{pr}_{1\sharp}$ associated to this diagram, see Appendix F.2 for details. We then define $\mathfrak{p}_f$ as the following composite:

$$\mathfrak{p}_f\colon f_\sharp \simeq f_\sharp \mathrm{pr}_{2*}\Delta_* \xrightarrow{\mathrm{BC}_{\sharp,*}} f_* \mathrm{pr}_{1\sharp}\Delta_* \simeq f_*(-\otimes \omega_f),$$

where the last equivalence holds by Remark 6.1.4.

Remark 6.1.6. There seems to be no standard terminology for the transformation $\mathfrak{p}_f$. Cisinski and Déglise use the term 'purity equivalence'. Bachmann and Hoyois [BH21, Section 5.4] use the term 'ambidexterity equivalence'. Although we certainly think this map as a geometric form of ambidexterity, we prefer to reserve the term 'ambidexterity' for the homotopical notion introduced in Part I. As explained in the introduction, we prefer the terminology 'relative Poincaré duality': we want to think of the representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ as an $\mathcal{X}$-indexed family of smooth manifolds over $\mathcal{X}$, and we think of the functors $f_\sharp$ and f_* as the $\mathcal{X}$-indexed homology and cohomology, respectively, of this family.

The following is our main theorem, which is a refined version of Theorem A stated in the introduction:

Theorem 6.1.7 (Relative Poincaré duality)**.** *Let $\mathcal{C}$ be a pullback formalism on* SepStk *satisfying the Voevodsky conditions. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a proper representable submersion between separated differentiable stacks. Then the transformation*

$$\mathfrak{p}_f\colon f_\sharp(-) \to f_*(-\otimes \omega_f)$$

is an equivalence of functors $\mathcal{C}(\mathcal{Y}) \to \mathcal{C}(\mathcal{X})$.

The proof strategy for Theorem 6.1.7 is a generalization of the usual proof of Atiyah duality for a compact smooth manifold M. A crucial ingredient in the proof is the construction of a *Pontryagin-Thom collapse map* associated to a closed embedding of differentiable stacks, discussed in Section 6.2. To simplify the required bookkeeping in the proof of Theorem 6.1.7, we introduce in Section 6.3 the notion of a *kernel operator*, which is closely related to the notion of Fourier-Mukai transforms in algebraic geometry. Using these ingredients, we give a proof of Theorem 6.1.7 in Section 6.4. Finally, we discuss various important consequences of relative Poincaré duality in Section 6.5, like relative Atiyah duality, proper base change and smooth-proper base change.

6.2 Relative Thom spaces and the Pontryagin-Thom construction

Fixing a pullback formalism C satisfying the Voevodsky conditions, we start by introducing the analogues of Thom spaces and the Pontryagin-Thom construction in C. Throughout, we fix a base stack $\mathcal{S}$, and all other stacks are assumed to live in the category $\mathrm{Sub}_{/\mathcal{S}}$: every differentiable stack $\mathcal{X}$ comes equipped with a representable submersion $g_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{S}$ and every morphism $f\colon \mathcal{Y} \to \mathcal{X}$ is assumed to commute with the structure maps to $\mathcal{S}$.

6.2.1 Relative Thom spaces

We start by defining relative Thom spaces.

Definition 6.2.1 (Relative Thom space)**.** Consider a commutative diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[rr, "i"] \arrow[dr, "g_{\mathcal{Z}}"'] & & \mathcal{X} \arrow[dl, "g_{\mathcal{X}}"] \\ & \mathcal{S}, & \end{tikzcd}$$

where $g_{\mathcal{Z}}$ and $g_{\mathcal{X}}$ are representable submersions and i is a closed embedding. We define the *(relative) Thom space of $\mathcal{Z}$ in $\mathcal{X}$ over $\mathcal{S}$* as

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}) := (g_{\mathcal{X}})_{\sharp} i_* \mathbb{1}_{\mathcal{Z}} \quad \in \quad C(\mathcal{S}).$$

We might sometimes write $\mathrm{Th}^{C}_{\mathcal{S}}(\mathcal{X},\mathcal{Z})$ to emphasize the dependence on C.

Remark 6.2.2 (Change of base stack)**.** If $h\colon \mathcal{S} \to \mathcal{S}'$ is a representable submersion of differentiable stacks, then every differentiable stack over $\mathcal{S}$ may be regarded as a differentiable stack over $\mathcal{S}'$ by composing with h. It is clear from the definitions that we have

$$\mathrm{Th}_{\mathcal{S}'}(\mathcal{X},\mathcal{Z}) \simeq h_{\sharp} \mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}) \quad \in \quad C(\mathcal{S}').$$

Example 6.2.3. If $\mathcal{Z} = \mathcal{X}$ and $i = \mathrm{id}_{\mathcal{X}}$, we get $\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{X}) = h_{\mathcal{S}}(\mathcal{X})$.

Example 6.2.4. Given a representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$, the dualizing object $\omega_f \in C(\mathcal{Y})$ is an example of a relative Thom space: we have

$$\omega_f = \mathrm{pr}_{1\sharp} \Delta_* \mathbb{1}_{\mathcal{Y}} = \mathrm{Th}_{\mathcal{Y}}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}, \mathcal{Y}) \quad \in \quad C(\mathcal{Y}),$$

where the diagonal $\Delta\colon \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ of f is a closed embedding which we regard as a morphism in $\mathrm{Sub}_{/\mathcal{X}}$ via the first projection map $\mathrm{pr}_1\colon \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$.

The relative Thom space $\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z})$ may be thought of as a homotopy quotient of the stack $\mathcal{X}$ by the open substack $\mathcal{X}\setminus\mathcal{Z}$, taken fiberwise over $\mathcal{S}$. This is made precise by the following lemma:

Lemma 6.2.5. *For a closed embedding $i\colon \mathcal{Z}\hookrightarrow\mathcal{X}$ in $\mathrm{Sub}_{/\mathcal{X}}$, there is a preferred cofiber sequence*

$$h_{\mathcal{S}}(\mathcal{X}\setminus\mathcal{Z})\to h_{\mathcal{S}}(\mathcal{X})\to \mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z})\quad\in\quad C(\mathcal{S}).$$

Proof. Let $j\colon \mathcal{X}\setminus\mathcal{Z}\hookrightarrow\mathcal{X}$ denote the open complement of i. By the localization axiom, there is a preferred cofiber sequence

$$j_\sharp\mathbb{1}_{\mathcal{X}\setminus\mathcal{Z}}\to\mathbb{1}_{\mathcal{X}}\to i_*\mathbb{1}_{\mathcal{Z}}\quad\in\quad C(\mathcal{X}).$$

Applying the colimit-preserving functor $(g_{\mathcal{X}})_\sharp\colon C(\mathcal{X})\to C(\mathcal{Z})$, we get a cofiber sequence

$$(g_{\mathcal{X}\setminus\mathcal{Z}})_\sharp\mathbb{1}_{\mathcal{X}\setminus\mathcal{Z}}\to(g_{\mathcal{X}})_\sharp\mathbb{1}_{\mathcal{X}}\to(g_{\mathcal{X}})_\sharp i_*\mathbb{1}_{\mathcal{Z}}=\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z})\quad\in\quad C(\mathcal{Z}).$$

Since $h_{\mathcal{S}}(\mathcal{X}\setminus\mathcal{Z})=(g_{\mathcal{X}\setminus\mathcal{Z}})_\sharp\mathbb{1}_{\mathcal{X}\setminus\mathcal{Z}}$ and $h_{\mathcal{S}}(\mathcal{X})=(g_{\mathcal{X}})_\sharp\mathbb{1}_{\mathcal{X}}$, this finishes the proof. $\square$

We will frequently be interested in the case where $\mathcal{S}=\mathcal{Z}$, so that $i\colon\mathcal{S}\hookrightarrow\mathcal{X}$ is the section of a representable submersion $g\colon\mathcal{X}\to\mathcal{S}$. In the case g is a vector bundle, the relative Thom space is given by the associated sphere bundle:

Corollary 6.2.6. *Let $\pi\colon\mathcal{E}\to\mathcal{S}$ be a vector bundle and let $s\colon\mathcal{S}\to\mathcal{E}$ be its zero section. Then the Thom space of s over $\mathcal{S}$ is the sphere bundle associated to $\mathcal{E}$:*

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{E},\mathcal{S})\simeq S^{\mathcal{E}}\quad\in\quad C(\mathcal{S}).$$

Proof. Consider the unique morphism $h\colon\mathrm{H}\to C$ of pullback formalisms. The induced functor $h_{\mathcal{S}}\colon\mathrm{H}(\mathcal{S})\to C(\mathcal{S})$ preserves colimits and sends the object $L_{\mathrm{htp}}(\mathcal{Y})$ to $h_{\mathcal{S}}(\mathcal{Y})$ for every $\mathcal{Y}\in\mathrm{Sub}_{/\mathcal{S}}$. By Lemma 4.3.2 we thus obtain for every vector bundle $\pi\colon\mathcal{E}\to\mathcal{S}$ a pushout square

$$\begin{tikzcd}
h_{\mathcal{S}}(\mathcal{E}\setminus\mathcal{S}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & h_{\mathcal{S}}(\mathcal{E}) \arrow[d] \\
h_{\mathcal{S}}(\mathcal{S}) \arrow[r] & h_{\mathcal{S}}(\mathcal{S}^{\mathcal{E}})
\end{tikzcd}$$

in $C(\mathcal{S})$, which gives rise to an equivalence between the cofibers in $C(\mathcal{S})$ of the two horizontal maps. As the object $S^{\mathcal{E}}$ was defined in Notation 4.5.25 to be the cofiber of the bottom map and the relative Thom space $\mathrm{Th}_{\mathcal{S}}(\mathcal{E},\mathcal{S})$ is by Lemma 6.2.5 the cofiber of the top map, this finishes the proof. $\square$

We prove various basic properties of the relative Thom space construction. We start by showing that the relative Thom space $\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z})$ only depends on an arbitrarily small open neighborhood of $\mathcal{Z}$ in $\mathcal{X}$.

Lemma 6.2.7 (Invariance under open neighborhoods)**.** *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding over $\mathcal{S}$, and let $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ be an open embedding over $\mathcal{S}$ such that i factors through a map $i'\colon \mathcal{Z} \hookrightarrow \mathcal{U}$. Then there is a preferred equivalence*

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{U},\mathcal{Z}) \xrightarrow{\sim} \mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}) \quad \in \quad \mathcal{C}(\mathcal{S}).$$

Proof. Applying smooth-closed base change to the pullback square

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "i'"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & \mathcal{U} \arrow[d, "j"] \\ \mathcal{Z} \arrow[r, "i"] & \mathcal{X}, \end{tikzcd}$$

we obtain an equivalence $j_\sharp i'_* \xrightarrow{\sim} i_*$. We thus obtain the desired equivalence as the following composite:

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{U},\mathcal{Z}) = (g_{\mathcal{U}})_\sharp i'_* \mathbb{1}_{\mathcal{Z}} = (g_{\mathcal{X}})_\sharp j_\sharp i'_* \mathbb{1}_{\mathcal{Z}} \xrightarrow{\sim} (g_{\mathcal{X}})_\sharp i_* \mathbb{1}_{\mathcal{Z}} = \mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}). \qquad \square$$

Due to the existence of relative tubular neighborhoods from Proposition 3.6.10, the previous lemma implies that the relative Thom space of a closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ in fact only depends on its normal bundle:

Corollary 6.2.8. *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding over $\mathcal{S}$ and let $\mathcal{N}_i \to \mathcal{Z}$ be its normal bundle. There is an equivalence*

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}) \simeq \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_i,\mathcal{Z}) \quad \in \quad \mathcal{C}(\mathcal{S}).$$

Proof. By Proposition 3.6.10, the substack $\mathcal{Z}$ admits a tubular neighborhood $\mathcal{U}$ inside $\mathcal{X}$ relative to $\mathcal{S}$, meaning that there exists a commutative diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, hook', "i"'] \arrow[d, hook] \arrow[dr, hook, "s_0"] & \\ \mathcal{X} & \mathcal{U} \arrow[l, hook', "j"'] \arrow[r, hook, "j'"] & \mathcal{N}_i \end{tikzcd}$$

in $\mathrm{Sub}_{/\mathcal{S}}$, where $\mathcal{N}_i$ lives over $\mathcal{S}$ by composing the bundle projection $\mathcal{N}_i \to \mathcal{Z}$ with the map $g_{\mathcal{Z}}\colon \mathcal{Z} \to \mathcal{S}$ and where the maps j and j' are open embeddings of differentiable stacks. The claim thus follows from two instances of Lemma 6.2.7: equivalences

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}) \xleftarrow{\sim} \mathrm{Th}_{\mathcal{S}}(\mathcal{U},\mathcal{Z}) \xrightarrow{\sim} \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_i,\mathcal{Z}). \qquad \square$$

Corollary 6.2.9 (Invertibility of Thom spaces)**.** *Let $p\colon \mathcal{X} \to \mathcal{S}$ be a representable submersion and assume p admits a section $i\colon \mathcal{S} \hookrightarrow \mathcal{X}$, which is automatically a closed embedding. Then there is an equivalence*

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{S}) \simeq S^{\mathcal{N}_i} \quad \in \quad \mathcal{C}(\mathcal{S}).$$

In particular, the relative Thom space $\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{S})$ *is an invertible object of* $\mathcal{C}(\mathcal{S})$.

Proof. The first statement is immediate from Corollary 6.2.8 and Corollary 6.2.6. The second statement follows from the genuine stability assumption on $\mathcal{C}$. □

Warning 6.2.10. In Corollary 6.2.9, it is crucial that $\mathcal{Z} = \mathcal{S}$: the relative Thom space $\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z})$ is generally not invertible when $\mathcal{Z} \neq \mathcal{S}$.

Corollary 6.2.11 (Dualizing object is tangent sphere bundle)**.** *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable submersion. Then there is an equivalence $\omega_f \simeq S^{T_f} \in \mathcal{C}(\mathcal{Y})$.*

Proof. By Corollary 6.2.9, there is an equivalence $\omega_f = \mathrm{Th}_{\mathcal{Y}}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}, \mathcal{Y}) \simeq S^{\mathcal{N}_\Delta}$ in $\mathcal{C}(\mathcal{Y})$. Since Proposition 3.5.19 provides an isomorphism of vector bundles $\mathcal{N}_\Delta \cong T_f$ over $\mathcal{Y}$, this finishes the proof. □

Lemma 6.2.12 (Multiplicativity of Thom spaces)**.** *Consider representable submersions $g_i\colon \mathcal{X}_i \to \mathcal{S}$ for $i = 1,2,3$, equipped with sections $s_i\colon \mathcal{S} \to \mathcal{X}_i$. Assume there exists a pullback square*

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "a"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}_2 \arrow[d, "b"] \\ \mathcal{S} \arrow[r, "s_3"] & \mathcal{X}_3 \end{tikzcd}$$

over $\mathcal{S}$, where b is a representable submersion and such that $a \circ s_1 = s_2\colon \mathcal{S} \to \mathcal{X}_2$. Then there is an equivalence

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{X}_2) \simeq \mathrm{Th}_{\mathcal{S}}(\mathcal{X}_1) \otimes \mathrm{Th}_{\mathcal{S}}(\mathcal{X}_3) \quad \in \quad \mathcal{C}(\mathcal{S}).$$

In particular, taking $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{S}} \mathcal{X}_3$ gives $\mathrm{Th}_{\mathcal{S}}(\mathcal{X}_1 \times_{\mathcal{S}} \mathcal{X}_3) \simeq \mathrm{Th}_{\mathcal{S}}(\mathcal{X}_1) \otimes \mathrm{Th}_{\mathcal{S}}(\mathcal{X}_3)$.

Proof. Since $g_2 = g_3 \circ b$ and $s_2 = a \circ s_1$, we have

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{X}_2) = g_{2\sharp}s_{2*} = g_{3\sharp}b_\sharp a_* s_{1*} \xrightarrow{\mathrm{BC}_{\sharp,*}} g_{3\sharp}s_{3*}g_{1\sharp}s_{1*} \simeq \mathrm{Th}_{\mathcal{S}}(\mathcal{X}_1) \otimes \mathrm{Th}_{\mathcal{S}}(\mathcal{X}_3).$$

Here the double Beck-Chevalley map $\mathrm{BC}_{\sharp,*}$ associated to the pullback square is an equivalence by smooth-closed base change and the last equivalence follows from the smooth and closed porjection formulas. □

Remark 6.2.13. One can in fact prove more generally that for closed embeddings $i\colon \mathcal{Z} \to \mathcal{X}$ and $i'\colon \mathcal{Z}' \to \mathcal{X}'$ over $\mathcal{S}$ there is an equivalence

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}) \otimes \mathrm{Th}_{\mathcal{S}}(\mathcal{X}',\mathcal{Z}') \simeq \mathrm{Th}_{\mathcal{S}}(\mathcal{X} \times_{\mathcal{S}} \mathcal{X}', \mathcal{Z} \times_{\mathcal{S}} \mathcal{Z}') \quad \in \quad C(\mathcal{S}).$$

We will not need this and leave the verification to the reader.

6.2.2 Pontryagin-Thom construction

Given a closed embedding of smooth manifolds $i\colon Z \hookrightarrow M$, one can define a *Pontryagin-Thom collapse map* $\mathrm{PT}(i)\colon M \to \mathrm{Th}_Z(N_i)$, where N_i denotes the normal bundle of Z in M, and $\mathrm{Th}_Z(N_i)$ is the Thom space of this normal bundle. This map is usually constructed at the point-set level by collapsing the complement of a tubular neighborhood U of Z inside M to a point, and identifying the resulting quotient with the Thom space $\mathrm{Th}_Z(N_i)$. From a more homotopical perspective, we may also think of this map as a composite of the homotopy quotient map $M \to M/(M \setminus Z)$ with the string of identifications

$$M/(M \setminus Z) \simeq U/(U \setminus Z) \simeq N_i/(N_i \setminus Z) \simeq \mathrm{Th}_Z(N_i);$$

here we abuse notation by writing X/A for the *homotopy quotient* of a topological space X by a subspace A, that is, the cofiber of the map $A \hookrightarrow X$ in the ∞-category An of animae. The advantage of the latter formulation is that it admits a simple generalization to the general context of a pullback formalism C over differentiable stacks satisfying the Voevodsky conditions.

Remark 6.2.14. Another approach to a generalization of the Pontryagin-Thom construction to differentiable stacks can be found in [EG11, Section 3].

Construction 6.2.15 (Quotient map)**.** Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{Z}'$ and $i'\colon \mathcal{Z}' \hookrightarrow \mathcal{X}$ be two closed embeddings over $\mathcal{S}$. Then the unit map $\mathbb{1}_{\mathcal{Z}'} \to i_* i^* \mathbb{1}_{\mathcal{Z}'} \simeq i_* \mathbb{1}_{\mathcal{Z}}$ gives rise to a map

$$u_i^*\colon \ \mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}') = (g_{\mathcal{X}})_\sharp i'_* \mathbb{1}_{\mathcal{Z}'} \to (g_{\mathcal{X}})_\sharp i'_* i_* \mathbb{1}_{\mathcal{Z}} = \mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}).$$

Construction 6.2.16 (Pontryagin-Thom construction)**.** Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding, and let $\mathcal{U}$ be a relative tubular neighborhood of $\mathcal{Z}$ in $\mathcal{X}$ relative to $\mathcal{S}$ in the sense of Definition 3.6.9: a choice of a commutative diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, hook', "i"'] \arrow[d, hook] \arrow[dr, hook, "s_0"] & \\ \mathcal{X} & \mathcal{U} \arrow[l, hook', "j"'] \arrow[r, hook, "j'"] & N_i \end{tikzcd}$$

of stacks over $\mathcal{S}$, where the maps j and j' are open embeddings. We define the *Pontryagin-Thom collapse map* $\mathrm{PT}_{\mathcal{S}}(i)\colon h_{\mathcal{S}}(\mathcal{X}) \to \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_i, \mathcal{Z})$ (with respect to $\mathcal{U}$) as the following composite:

$$\mathrm{PT}_{\mathcal{S}}(i)\colon h_{\mathcal{S}}(\mathcal{X}) \overset{6.2.3}{\simeq} \mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{X}) \xrightarrow{u_i^*} \mathrm{Th}_{\mathcal{S}}(\mathcal{X},\mathcal{Z}) \overset{6.2.7}{\simeq} \mathrm{Th}_{\mathcal{S}}(\mathcal{U},\mathcal{Z}) \overset{6.2.7}{\simeq} \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_i,\mathcal{Z}).$$

Here the map u_i^* is the quotient map from Construction 6.2.15.

Informally speaking, this map may be thought of as the composite

$$\mathcal{X} \to \mathcal{X}/(\mathcal{X}\setminus\mathcal{Z}) \simeq \mathcal{U}/(\mathcal{U}\setminus\mathcal{Z}) \simeq \mathcal{N}_i/(\mathcal{N}_i\setminus\mathcal{Z}) = \mathrm{Th}(\mathcal{N}_i,\mathcal{Z}),$$

where the first map is the homotopy quotient map $\mathcal{X} \to \mathcal{X}/(\mathcal{X}\setminus\mathcal{Z})$.

Example 6.2.17. Let $\pi\colon \mathcal{E} \to \mathcal{S}$ be a vector bundle and let $s\colon \mathcal{S} \to \mathcal{E}$ be its zero section. Let i be the composite inclusion $\mathcal{S} \hookrightarrow \mathcal{E} \hookrightarrow S^{\mathcal{E}}$, and pick the open neighborhood $\mathcal{E}$ as a tubular neighborhood of $\mathcal{S}$ inside $S^{\mathcal{E}}$. Then by Corollary 6.2.6 there is an equivalence $\mathrm{Th}_{\mathcal{S}}(\mathcal{E},\mathcal{S}) \simeq S^{\mathcal{E}}$, and unwinding that equivalence we see that Pontryagin-Thom map $\mathrm{PT}_{\mathcal{S}}(i)$ fits in a cofiber sequence

$$h_{\mathcal{S}}(\mathcal{S}) \to h_{\mathcal{S}}(S^{\mathcal{E}}) \xrightarrow{\mathrm{PT}_{\mathcal{S}}(i)} \mathrm{Th}_{\mathcal{S}}(\mathcal{E},\mathcal{S}).$$

We will in fact need a slight generalization of the above construction where $\mathcal{X}$ is allowed to be embedded in some larger ambient stack $\mathcal{X}'$:

Construction 6.2.18 (Generalized Pontryagin-Thom construction)**.** Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ and $i'\colon \mathcal{X} \hookrightarrow \mathcal{X}'$ be closed embeddings. Choose a tubular neighborhood

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, hook, bend right, "i'"'] \arrow[d, hook] \arrow[dr, hook, bend left, "s_0"] & \\ \mathcal{X}' & \mathcal{U}' \arrow[l, hook', "j'"'] \arrow[r, hook] & \mathcal{N}_{i'} \end{tikzcd}$$

of i' relative to $\mathcal{S}$, and choose another tubular neighborhood

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, hook, bend right, "j'\circ i"'] \arrow[d, hook] \arrow[dr, hook, bend left, "s_0"] & \\ \mathcal{U}' & \mathcal{U} \arrow[l, hook'] \arrow[r, hook] & \mathcal{N}_{j'\circ i} \cong \mathcal{N}_{i'\circ i} \end{tikzcd}$$

of the composite $j'\circ i\colon \mathcal{Z} \hookrightarrow \mathcal{U}'$ relative to $\mathcal{S}$. We define the map $\mathrm{PT}_{\mathcal{S}}(i,i')$ as the following composite:

$$\mathrm{PT}_{\mathcal{S}}(i,i')\colon \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'},\mathcal{X}) \xrightarrow{u_i^*} \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'},\mathcal{Z}) \simeq \mathrm{Th}_{\mathcal{S}}(\mathcal{U}',\mathcal{Z}) \simeq \mathrm{Th}_{\mathcal{S}}(\mathcal{U},\mathcal{Z}) \simeq \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'\circ i},\mathcal{Z}).$$

Here the last three equivalences are instances of Lemma 6.2.7.

It is clear that Construction 6.2.18 specializes to Construction 6.2.16 when $\mathcal{X}' = \mathcal{X}$ and $i' = \mathrm{id}_{\mathcal{X}}$.

Lemma 6.2.19 (Compatibility with composition)**.** *In the situation of Construction 6.2.18, the following diagram commutes up to preferred homotopy:*

$$\begin{tikzcd}
h_{\mathcal{S}}(\mathcal{X}') \arrow[r, "\mathrm{PT}_{\mathcal{S}}(i')"] \arrow[d, "\mathrm{PT}_{\mathcal{S}}(i'\circ i)"'] & \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'}, \mathcal{X}) \arrow[d, "\mathrm{PT}_{\mathcal{S}}(i,i')"] \\
\mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'\circ i}, \mathcal{Z}) \arrow[r, equal] & \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'\circ i}, \mathcal{Z}).
\end{tikzcd}$$

Proof. This follows from the following commutative diagram:

$$\begin{tikzcd}
h_{\mathcal{S}}(\mathcal{X}') \arrow[d, "u_{i'}^*"] \arrow[dr, "\mathrm{PT}_{\mathcal{S}}(i')"] \arrow[drr, bend left, "\mathrm{PT}_{\mathcal{S}}(i'\circ i)"] \arrow[dd, bend right=60, "u_{i'\circ i}^*"'] & & \\
\mathrm{Th}_{\mathcal{S}}(\mathcal{X}', \mathcal{X}) \arrow[r, "\simeq"'] \arrow[d, "u_i^*"] & \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'}, \mathcal{X}) \arrow[d, "u_i^*"] \arrow[dr, "\mathrm{PT}_{\mathcal{S}}(i,i')"] & \\
\mathrm{Th}_{\mathcal{S}}(\mathcal{X}', \mathcal{Z}) \arrow[r, "\simeq"'] & \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'}, \mathcal{Z}) \arrow[r, "\simeq"'] & \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'\circ i}, \mathcal{Z})
\end{tikzcd}$$

Note that the bottom composite is given by the composite of the equivalences

$$\mathrm{Th}_{\mathcal{S}}(\mathcal{X}', \mathcal{Z}) \xleftarrow{\sim} \mathrm{Th}_{\mathcal{S}}(\mathcal{U}, \mathcal{Z}) \xrightarrow{\sim} \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i'\circ i}, \mathcal{Z}).$$

from Lemma 6.2.7. The two small triangles and the outer triangle then commute by definition and the lower left square commutes by naturality. □

Lemma 6.2.20 (Compatibility with base change)**.** *Let $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding over $\mathcal{S}$. Let $f\colon \mathcal{S}' \to \mathcal{S}$ be a morphism of differentiable stacks and write $f^*\colon \mathrm{Sub}_{\mathcal{S}} \to \mathrm{Sub}_{\mathcal{S}'}$ for the pullback functor.*

(1) If $\mathcal{U}$ is a tubular neighborhood of $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ relative to $\mathcal{S}$, then $f^*\mathcal{U}$ is a tubular neighborhood of $f^*i\colon f^*\mathcal{Z} \hookrightarrow f^*\mathcal{X}$ relative to $\mathcal{S}'$.

(2) There are preferred equivalences

$$h_{\mathcal{S}'}(f^*\mathcal{X}) \simeq f^* h_{\mathcal{S}}(\mathcal{X}) \qquad \textit{and} \qquad \mathrm{Th}_{\mathcal{S}'}(\mathcal{N}_{f^*i}, f^*\mathcal{Z}) \simeq f^*\,\mathrm{Th}_{\mathcal{S}}(\mathcal{N}_i, \mathcal{Z})$$

(3) With respect to the choice of tubular neighborhood from (1), there is a preferred homotopy making the following diagram commute:

$$\begin{tikzcd}
h_{\mathcal{S}'}(f^*\mathcal{X}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{PT}_{\mathcal{S}'}(f^*(i))"'] & f^* h_{\mathcal{S}}(\mathcal{X}) \arrow[d, "f^*\mathrm{PT}_{\mathcal{S}}(i)"] \\
\mathrm{Th}_{\mathcal{S}'}(\mathcal{N}_{f^*i}, f^*\mathcal{Z}) \arrow[r, "\simeq"] & f^*\,\mathrm{Th}_{\mathcal{S}}(\mathcal{N}_i, \mathcal{Z}).
\end{tikzcd}$$

Proof. This is immediate from spelling out the definitions and using that by smooth and closed base change the functor f^* commutes with all the constructions that appear. ◻

Lemma 6.2.21 (Invariance under homotopy)**.** *Let $i\colon \mathcal{Z}\times\mathbb{R}\hookrightarrow\mathcal{X}\times\mathbb{R}$ be closed embedding over $\mathcal{S}\times\mathbb{R}$, and let $\mathcal{Z}\times\mathbb{R}\hookrightarrow\mathcal{U}\hookrightarrow\mathcal{X}\times\mathbb{R}$ be a tubular neighborhood. Then:*

(1) For every $r\in\mathbb{R}$, the restriction $\mathcal{Z}\hookrightarrow\mathcal{U}_r\hookrightarrow\mathcal{X}$ is a tubular neighborhood of the closed embedding $i_r\colon\mathcal{Z}\hookrightarrow\mathcal{X}$;

(2) For every $r\in R$, there is an isomorphism $\alpha\colon\mathcal{N}_{i_r}\cong\mathcal{N}_{i_0}$ of vector bundles over $\mathcal{Z}$ making the following diagram commute up to homotopy:

$$\begin{tikzcd} & h_{\mathcal{S}}(\mathcal{X}) \arrow[dl, "\mathrm{PT}(i_r)"'] \arrow[dr, "\mathrm{PT}(i_0)"] & \\ \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i_r},\mathcal{Z}) \arrow[rr, "\alpha", "\simeq"'] & & \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i_0},\mathcal{Z}). \end{tikzcd}$$

Proof. Part (1) is an instance of part (1) of Lemma 6.2.20. The first part of (2) is a special case of Lemma 3.5.18. For the convenience of the reader, we shall recall how the isomorphism α is constructed. Considering the normal bundle $\mathcal{N}_i\to\mathcal{Z}\times\mathbb{R}$ of i, it follows from Lemma 2.5.4 that $\mathcal{N}_i$ is of the form $\pi\times\mathbb{R}\colon\mathcal{N}\times\mathbb{R}\to\mathcal{Z}\times\mathbb{R}$ for some vector bundle $\pi\colon\mathcal{N}\to\mathcal{Z}$ over $\mathcal{Z}$. As $\mathcal{N}_{i_r}$ is obtained by pulling back $\mathcal{N}_i$ along the inclusion $\mathcal{Z}\times\{r\}\hookrightarrow\mathcal{Z}\times\mathbb{R}$, we obtain an isomorphism $\mathcal{N}_{i_r}\cong\mathcal{N}$ for every $r\in\mathbb{R}$, and thus we may define α as the composite $\mathcal{N}_{i_r}\cong\mathcal{N}\cong\mathcal{N}_{i_0}$.

Next consider the Pontryagin-Thom collapse map for the embedding i relative to $\mathcal{S}\times\mathbb{R}$, which takes the form

$$\mathrm{PT}_{\mathcal{S}\times\mathbb{R}}(i)\colon h_{\mathcal{S}\times\mathbb{R}}(\mathcal{X}\times\mathbb{R})\to\mathrm{Th}_{\mathcal{S}\times\mathbb{R}}(\mathcal{N}_i,\mathcal{Z}\times\mathbb{R})\quad\in\quad\mathcal{C}(\mathcal{S}\times\mathbb{R}).$$

Letting $\mathrm{pr}\colon\mathcal{S}\times\mathbb{R}\to\mathcal{S}$ denote the projection, observe that both source and target of this map lie in the image of the fully faithful functor $\mathrm{pr}^*\colon\mathcal{C}(\mathcal{S})\hookrightarrow\mathcal{C}(\mathcal{S}\times\mathbb{R})$, so that the map $\mathrm{PT}_{\mathcal{S}\times\mathbb{R}}(i)$ is of the form $\mathrm{pr}^*(\varphi)$ for some morphism $\varphi\colon h_{\mathcal{S}}(\mathcal{X})\to\mathrm{Th}_{\mathcal{S}}(\mathcal{N},\mathcal{Z})$ in $\mathcal{C}(\mathcal{S})$. Applying Lemma 6.2.20 to the inclusion $\mathcal{S}=\mathcal{S}\times\{r\}\hookrightarrow\mathcal{S}\times\mathbb{R}$ and using that this map is a section of the projection map pr, we obtain for every $r\in\mathbb{R}$ a commutative square as follows:

$$\begin{tikzcd} h_{\mathcal{S}}(\mathcal{X}) \arrow[r, equal] \arrow[d, "\mathrm{PT}_{\mathcal{S}}(i_r)"'] & h_{\mathcal{S}}(\mathcal{X}) \arrow[d, "\varphi"] \\ \mathrm{Th}_{\mathcal{S}}(\mathcal{N}_{i_r},\mathcal{Z}) \arrow[r, "\simeq"] & \mathrm{Th}_{\mathcal{S}}(\mathcal{N},\mathcal{Z}). \end{tikzcd}$$

Combining this square with the analogous square for $r=0$ then proves the claim. ◻

6.3 Kernel operators

We continue fixing a pullback formalism C satisfying the Voevodsky conditions. In this section, we will study the behavior of functors $C(\mathcal{X}) \to C(\mathcal{Y})$ of a specific form, which we will call *kernel operators*. As in the previous section, we fix a base stack $\mathcal{S}$ over which all other stacks live.

Definition 6.3.1 (Kernel operator)**.** Let $\mathcal{X}, \mathcal{Y} \in \mathrm{Sub}_{/\mathcal{S}}$ and let $D \in C(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})$. We define the *kernel operator F_D of D (relative to $\mathcal{S}$)* as the functor $F_D\colon C(\mathcal{X}) \to C(\mathcal{Y})$ given as the following composite:

$$F_D : C(\mathcal{X}) \xrightarrow{(\mathrm{pr}_{\mathcal{X}})^*} C(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}) \xrightarrow{-\otimes D} C(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}) \xrightarrow{(\mathrm{pr}_{\mathcal{Y}})_\sharp} C(\mathcal{Y}).$$

A functor $F\colon C(\mathcal{X}) \to C(\mathcal{Y})$ is called a *kernel operator (relative to $\mathcal{S}$)* if it comes equipped with an equivalence $F \simeq F_D$ for some $D \in C(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})$. The object D is called the *kernel* of F.

If $D' \in C(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})$ is another kernel, then every morphism $\alpha\colon D \to D'$ induces a natural transformation $F_\alpha\colon F_D \to F_{D'}$ on kernel operators.

Warning 6.3.2. It is not necessarily true that the functor F_D uniquely determines the object D. For this reason, the kernel D is required as data.

The notion of a kernel operator is formally analogous to the notion of a *Fourier-Mukai transform* in algebraic geometry, where one would instead consider functors of the form $(\mathrm{pr}_{\mathcal{Y}})_*((\mathrm{pr}_{\mathcal{X}})^*(-) \otimes D)\colon C(\mathcal{X}) \to C(\mathcal{Y})$, using the right adjoint $(\mathrm{pr}_{\mathcal{Y}})_*$ rather than the left adjoint $(\mathrm{pr}_{\mathcal{Y}})_\sharp$. All the results below on kernel operators are straightforward adaptations of well-known results for Fourier-Mukai transforms. A similar discussion appears in [FS21, p.263].

The following are the main examples of kernel operators that we will use.

Example 6.3.3 (Pullback)**.** Let $f\colon \mathcal{X} \to \mathcal{S}$ be a representable submersion and let $\mathcal{Y} = \mathcal{S}$. Then the functor $f^*\colon C(\mathcal{S}) \to C(\mathcal{X})$ is a kernel operator relative to $\mathcal{S}$, with kernel $\mathbb{1}_{\mathcal{X}} \in C(\mathcal{X})$.

Example 6.3.4 (Pushforward)**.** Let $g\colon \mathcal{Y} \to \mathcal{S}$ be a representable submersion and let $\mathcal{X} = \mathcal{S}$. Then the functor $g_\sharp\colon C(\mathcal{Y}) \to C(\mathcal{S})$ is a kernel operator relative to $\mathcal{S}$, with kernel $\mathbb{1}_{\mathcal{Y}} \in C(\mathcal{Y})$.

Example 6.3.5. Let $f: \mathcal{X} \to \mathcal{S}$ be a representable submersion. Then the composite $f^* f_\sharp: C(\mathcal{X}) \to C(\mathcal{X})$ is equivalent to $\mathrm{pr}_{1\sharp}\mathrm{pr}_2^*: C(\mathcal{X}) \to C(\mathcal{X})$ by smooth base change and thus is a kernel operator with kernel $\mathbb{1}_{\mathcal{X}\times_{\mathcal{S}}\mathcal{X}} \in C(\mathcal{X} \times_{\mathcal{S}} \mathcal{X})$. Given the previous two examples, this may also be deduced from Lemma 6.3.9 below.

Example 6.3.6 (Tensor product)**.** Letting $\mathcal{X} = \mathcal{Y} = \mathcal{S}$, every object $D \in C(\mathcal{S})$ gives rise to a kernel operator $F_D = - \otimes D: C(\mathcal{S}) \to C(\mathcal{S})$.

Example 6.3.7 (Suspension by a vector bundle)**.** Let $\pi: \mathcal{E} \to \mathcal{S}$ be a vector bundle and let $s: \mathcal{S} \to \mathcal{E}$ be its zero section. We define the *suspension functor* $\Sigma^{\mathcal{E}}$ as

$$\Sigma^{\mathcal{E}} := \pi_\sharp s_*: C(\mathcal{S}) \to C(\mathcal{S}).$$

Note that $\Sigma^{\mathcal{E}}\mathbb{1}_{\mathcal{S}} = \mathrm{Th}_{\mathcal{S}}(\mathcal{E},\mathcal{S})$, which by Corollary 6.2.6 is equivalent to the sphere bundle $S^{\mathcal{E}}$. By the smooth and closed projection formulas, it then follows that there is a natural equivalence

$$\Sigma^{\mathcal{E}} \simeq - \otimes S^{\mathcal{E}},$$

so that $\Sigma^{\mathcal{E}}$ is a kernel functor relative to $\mathcal{S}$.

Example 6.3.8 (Twist functor)**.** Let $f: \mathcal{Y} \to \mathcal{S}$ be a representable submersion and consider its dualizing object $\omega_f \in C(\mathcal{Y})$ from Definition 6.1.3. We claim that the twist functor

$$- \otimes \omega_f: C(\mathcal{Y}) \to C(\mathcal{Y})$$

is a kernel operator over $\mathcal{S}$ with kernel $\Delta_* \mathbb{1}_{\mathcal{Y}} \in C(\mathcal{Y} \times_{\mathcal{S}} \mathcal{Y})$. Indeed, by the smooth and closed projection formulas we obtain natural equivalences

$$\begin{aligned} A \otimes \omega_f = A \otimes \mathrm{pr}_{1\sharp}\Delta_* \mathbb{1}_{\mathcal{Y}} &\simeq \mathrm{pr}_{1\sharp}(\mathrm{pr}_1^* A \otimes \Delta_* \mathbb{1}_{\mathcal{Y}}) \\ &\simeq \mathrm{pr}_{1\sharp}\Delta_*(\Delta^* \mathrm{pr}_1^* A \otimes \mathbb{1}_{\mathcal{Y}}) \\ &\simeq \mathrm{pr}_{1\sharp}\Delta_*(\Delta^* \mathrm{pr}_2^* A \otimes \mathbb{1}_{\mathcal{Y}}) \\ &\simeq \mathrm{pr}_{1\sharp}(\mathrm{pr}_2^* A \otimes \Delta_* \mathbb{1}_{\mathcal{Y}}) = F_{\Delta_* \mathbb{1}_{\mathcal{Y}}}(A), \end{aligned}$$

where we use that $\mathrm{pr}_1 \circ \Delta = \mathrm{pr}_2 \circ \Delta$ as both are the identity on $\mathcal{Y}$.

Kernel operators behave well under composition:

Lemma 6.3.9. *Consider objects $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \mathrm{Sub}_{/\mathcal{S}}$ and consider kernels $D \in C(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})$ and $D' \in C(\mathcal{Y} \times_{\mathcal{S}} \mathcal{Z})$. Then the composite $F_{D'} \circ F_D: C(\mathcal{X}) \to C(\mathcal{Z})$ is again a kernel operator, with kernel given by*

$$D' \odot D := (\mathrm{pr}_{13})_\sharp(\mathrm{pr}_{12}^* D \otimes \mathrm{pr}_{23}^* D') \quad \in \quad C(\mathcal{X} \times_{\mathcal{S}} \mathcal{Z}),$$

where $\mathrm{pr}_{12}\colon \mathcal{X}\times_{\mathcal{S}}\mathcal{Y}\times_{\mathcal{S}}\mathcal{Z}\to\mathcal{X}\times_{\mathcal{S}}\mathcal{Y}$ *denotes the projection on the first two factors, and similarly for* pr_{23} *and* pr_{13}.

Proof. This is a straightforward computation using smooth base change and the smooth projection formula, which we will leave to the reader. See Lemma I.3.23 in Part I for an analogous computation. □

Lemma 6.3.10. *Let* $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}\in\mathrm{Sub}_{/\mathcal{S}}$ *and consider a kernel* $D\in C(\mathcal{X}\times_{\mathcal{S}}\mathcal{Y})$ *with associated kernel operator* $F_D\colon C(\mathcal{X})\to C(\mathcal{Y})$.

(1) For a representable submersion $f\colon\mathcal{Y}\to\mathcal{Z}$*, the composite* $f_\sharp\circ F_D\colon C(\mathcal{X})\to C(\mathcal{Z})$ *is again a kernel operator with kernel* $(\mathcal{X}\times f)_\sharp D\in C(\mathcal{X}\times_{\mathcal{S}}\mathcal{Z})$.

(2) For a representable submersion $g\colon\mathcal{Z}\to\mathcal{Y}$*, the composite* $g^*\circ F_D\colon C(\mathcal{X})\to C(\mathcal{Z})$ *is again a kernel operator with kernel* $(\mathcal{X}\times g)^*D\in C(\mathcal{X}\times_{\mathcal{S}}\mathcal{Z})$.

(3) For a representable submersion $h\colon\mathcal{W}\to\mathcal{X}$*, the composite* $F_D\circ h_\sharp\colon C(\mathcal{W})\to C(\mathcal{Y})$ *is again a kernel operator with kernel* $(h\times\mathcal{Y})^*D\in C(\mathcal{W}\times_{\mathcal{S}}\mathcal{Y})$.

(4) For a representable submersion $k\colon\mathcal{X}\to\mathcal{W}$*, the composite* $F_D\circ k^*\colon C(\mathcal{W})\to C(\mathcal{Y})$ *is again a kernel operator with kernel* $(k\times\mathcal{Y})_\sharp D\in C(\mathcal{W}\times_{\mathcal{S}}\mathcal{Y})$.

Proof. This is a straightforward computation using smooth base change and the smooth projection formula, which we will leave to the reader. □

Warning 6.3.11. We warn the reader that the identity $\mathrm{id}_{C(\mathcal{X})}\colon C(\mathcal{X})\to C(\mathcal{X})$ is not necessarily a kernel operator relative to $\mathcal{S}$, unlike in the situation for Fourier-Mukai transforms. The problem is that the diagonal $\Delta\colon\mathcal{X}\to\mathcal{X}\times_{\mathcal{S}}\mathcal{X}$ of $\mathcal{X}$ over $\mathcal{S}$ is usually not a representable submersion. If it is, then the identity *is* a kernel operator with kernel given by $\Delta_\sharp\mathbb{1}_{\mathcal{X}}\in C(\mathcal{X}\times_{\mathcal{S}}\mathcal{X})$.

As a consequence, it is not necessarily true that for a representable submersion $f\colon\mathcal{X}\to\mathcal{Y}$ over $\mathcal{S}$, the functors $f_\sharp\colon C(\mathcal{X})\to C(\mathcal{Y})$ and $f^*\colon C(\mathcal{Y})\to C(\mathcal{X})$ are kernel functors relative to $\mathcal{S}$.

6.4 Proof of Main Theorem

In this section, we will prove relative Poicaré duality for separated differentiable stacks, Theorem 6.1.7. Recall the setup: we are given a pullback formalism C satisfying the

Voevodsky conditions and a proper representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of separated differentiable stacks. Our goal is to show that the natural transformation

$$\mathfrak{p}_f\colon f_\sharp(-) \Rightarrow f_*(- \otimes \omega_f),$$

constructed in Construction 6.1.5, is an equivalence. The proof will consist of two steps: in Subsection 6.4.1 we prove the theorem in the special case where $\mathcal{Y}$ is a closed substack of a vector bundle over $\mathcal{X}$, and in Section 6.4.2 we argue how to reduce the general case to this special case.

6.4.1 A special case

We will start by proving relative Poincaré duality under the following additional assumption on the proper submersion f:

(A) The stack $\mathcal{Y}$ is a closed substack of a vector bundle over $\mathcal{X}$: there is a commutative diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathcal{E} \arrow[d, "\pi"] \\ & \mathcal{X}, \end{tikzcd}$$

where i is a closed embedding and π is a vector bundle.

Our proof strategy in this case will be as follows: we will use the embedding i to construct explicit unit and counit transformations that exhibit the functor $f_\sharp(- \otimes \omega_f^{-1})$ as a right adjoint of f^*, and then show that the resulting equivalence $f_* \simeq f_\sharp(- \otimes \omega_f^{-1})$ is compatible with the Poincaré duality map $\mathfrak{p}_f$.

Let $\pi_i\colon \mathcal{N}_i \to \mathcal{Y}$ denote the normal bundle of the embedding i and let $\pi_\Delta\colon \mathcal{N}_\Delta \to \mathcal{Y}$ denote the normal bundle of the diagonal map $\Delta = \Delta_f\colon \mathcal{Y} \to \mathcal{Y} \times_\mathcal{X} \mathcal{Y}$. Via Example 6.3.7, these bundles give rise to suspension functors $\Sigma^{\mathcal{N}_\Delta}, \Sigma^{\mathcal{N}_i}\colon C(\mathcal{Y}) \to C(\mathcal{Y})$. The following lemma shows that their composite is given by suspension along the pullback bundle $f^*\pi\colon f^*\mathcal{E} \to \mathcal{Y}$:

Lemma 6.4.1. *There is a short exact sequence of vector bundles over $\mathcal{Y}$ of the form $\mathcal{N}_\Delta \to f^*\mathcal{E} \to \mathcal{N}_i$. In particular, there is an equivalence $S^{f^*\mathcal{E}} \simeq S^{\mathcal{N}_\Delta} \otimes S^{\mathcal{N}_i}$ in $C(\mathcal{Y})$, giving rise to natural equivalences of functors*

$$\Sigma^{\mathcal{N}_i}\Sigma^{\mathcal{N}_\Delta} \simeq \Sigma^{f^*\mathcal{E}} \simeq \Sigma^{\mathcal{N}_\Delta}\Sigma^{\mathcal{N}_i}\colon C(\mathcal{Y}) \to C(\mathcal{Y}).$$

Proof. Observe that the second statement follows from the first using Lemma 6.2.12 and Corollary 6.2.6, while the final statement then follows from the fact that $\Sigma^{\mathcal{E}}\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{X})$ is given by tensoring with $S^{\mathcal{E}} \in \mathcal{C}(\mathcal{X})$ for a vector bundle $\mathcal{E}$ over $\mathcal{X}$, see Example 6.3.7. It thus remains to prove the first statement.

Consider the embedding $i'\colon \mathcal{Y} \hookrightarrow S^{\mathcal{E}}$, given as the composite of $i\colon \mathcal{Y} \hookrightarrow \mathcal{E}$ with the inclusion $\mathcal{E} \hookrightarrow S^{\mathcal{E}}$. Since the map f is proper and $S^{\mathcal{E}} \to \mathcal{X}$ is representable, it follows from Lemma 2.4.10 that the map i' is proper, and thus it is a closed embedding. Consider now the composite

$$(i', \mathrm{id})\colon \mathcal{Y} \overset{\Delta}{\hookrightarrow} \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \xhookrightarrow{i' \times_{\mathcal{X}} \mathcal{Y}} S^{\mathcal{E}} \times_{\mathcal{X}} \mathcal{Y}.$$

Observe that the normal bundle of the map $i' \times_{\mathcal{X}} \mathcal{Y}$ is $\mathcal{N}_i \times_{\mathcal{X}} \mathcal{Y}$, and the pullback of this bundle along the diagonal Δ is simply the map $\pi_i\colon \mathcal{N}_i \to \mathcal{Y}$. By Lemma 3.5.16, we thus obtain a short exact sequence

$$\mathcal{N}_\Delta \to \mathcal{N}_{(i', \mathrm{id})} \to \mathcal{N}_i$$

of vector bundles over $\mathcal{Y}$. It thus remains to show that the normal bundle of the map (i', id) is isomorphic to $f^*\mathcal{E}$.

Letting $s_0\colon \mathcal{X} \to \mathcal{E}$ denote the zero-section of $\pi\colon \mathcal{E} \to \mathcal{X}$, observe that the embedding $i\colon \mathcal{Y} \hookrightarrow \mathcal{E}$ is homotopic to the *zero map* $0\colon \mathcal{Y} \xrightarrow{f} \mathcal{X} \xrightarrow{s_0} \mathcal{E} \hookrightarrow S^{\mathcal{E}}$ via the straight-line homotopy $\mathcal{Y} \times \mathbb{R} \to S^{\mathcal{E}}, (y, r) \mapsto r \cdot i(y)$. It follows that the map (i', id) is isotopic to the closed embedding $(0, \mathrm{id})\colon \mathcal{Y} \hookrightarrow S^{\mathcal{E}} \times_{\mathcal{X}} \mathcal{Y}$. Lemma 3.5.18 thus provides an isomorphism of normal bundles

$$\mathcal{N}_{(i', \mathrm{id})} \cong \mathcal{N}_{(0, \mathrm{id})} \quad \in \quad \mathrm{Vect}(\mathcal{Y}).$$

Since $(0, \mathrm{id})$ is the zero-section of the vector bundle $f^*\pi\colon f^*\mathcal{E} \to \mathcal{Y}$, its normal bundle is simply given by the map $f^*\mathcal{E} \to \mathcal{Y}$, finishing the proof. $\square$

We now move to the construction of the aforementioned unit and counit transformations.

Construction 6.4.2. We define a natural transformations $\varepsilon\colon f^* f_\sharp \to \Sigma^{\mathcal{N}_\Delta}$ and $\eta\colon \Sigma^{\mathcal{E}} \to f_\sharp \Sigma^{\mathcal{N}_i} f^*$:

- For ε, recall from Example 6.3.5 and Example 6.3.8 that the functors $f^* f_\sharp$ and $\Sigma^{\mathcal{N}_\Delta} \simeq \mathrm{pr}_{1\sharp} \Delta_*$ are kernel functors relative to $\mathcal{X}$, in the sense of Definition 6.3.1, with kernels given by with kernels given by $\mathbb{1}_{\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}}$ and $\Delta_* \mathbb{1}_{\mathcal{Y}}$, respectively. We define ε as the transformation induced by the unit map $u_\Delta^*\colon \mathbb{1}_{\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}} \to \Delta_* \Delta^* \mathbb{1}_{\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}} = \Delta_* \mathbb{1}_{\mathcal{Y}}$ in $\mathcal{C}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y})$. More explicitly, ε is given by the following composite:

$$f^* f_\sharp \simeq \mathrm{pr}_{1\sharp} \mathrm{pr}_2^* \xrightarrow{u_\Delta^*} \mathrm{pr}_{1\sharp} \Delta_* \Delta^* \mathrm{pr}_2^* \simeq \mathrm{pr}_{1\sharp} \Delta_* \simeq \Sigma^{\mathcal{N}_\Delta} \quad \in \quad \mathrm{Fun}(\mathcal{C}(\mathcal{Y}), \mathcal{C}(\mathcal{Y})).$$

- For η, its source and target are also kernel functors relative to $\mathcal{X}$, as they are given by tensoring with the objects $S^{\mathcal{E}}$ and $f_\sharp S^{\mathcal{N}_i}$, respectively. We define η as the transformation induced by the Pontryagin-Thom map $\mathrm{PT}(i')$ of the closed embedding $i' \colon \mathcal{Y} \hookrightarrow S^{\mathcal{E}}$:

$$\mathrm{PT}(i') \colon S^{\mathcal{E}} \to \mathrm{Th}_{\mathcal{X}}(\mathcal{N}_{i'}, \mathcal{Y}) = \mathrm{Th}_{\mathcal{X}}(\mathcal{N}_i, \mathcal{Y}) \simeq f_\sharp S^{\mathcal{N}_i}.$$

The following result is the main ingredient for relative Poincaré duality in the special case.

Proposition 6.4.3 (cf. [Hoy17, Theorem 5.22]). *The composites*

$$f^*\Sigma^{\mathcal{E}} \xrightarrow{f^*\eta} f^* f_\sharp \Sigma^{\mathcal{N}_i} f^* \xrightarrow{\varepsilon\Sigma^{\mathcal{N}_i} f^*} \Sigma^{\mathcal{N}_\Delta}\Sigma^{\mathcal{N}_i} f^* \overset{6.4.1}{\simeq} \Sigma^{f^*\mathcal{E}} f^* \simeq f^*\Sigma^{\mathcal{E}} \tag{II.6.1}$$

$$\Sigma^{\mathcal{E}} f_\sharp \xrightarrow{\eta f_\sharp} f_\sharp \Sigma^{\mathcal{N}_i} f^* f_\sharp \xrightarrow{f_\sharp\Sigma^{\mathcal{N}_i}\varepsilon} f_\sharp \Sigma^{\mathcal{N}_i}\Sigma^{\mathcal{N}_\Delta} \overset{6.4.1}{\simeq} f_\sharp \Sigma^{f^*\mathcal{E}} \simeq \Sigma^{\mathcal{E}} f_\sharp \tag{II.6.2}$$

are homotopic to the identity.

Proof. As ε and η are defined as transformations between kernel functors induced by a map on kernels, it follows from Lemma 6.3.9 that also each of the functors appearing in (II.6.1) and (II.6.2) are kernel functors relative to $\mathcal{X}$ and that each of the transformations is induced by a morphism on kernels. It will thus suffice to show that the composites at the level of the kernels are homotopic to the respective identities.

It follows from Lemma 6.3.10 that the transformations $f^*\eta$ and $\eta f_\sharp$ are both induced by the $f^*\mathrm{PT}(i') \colon f^*S^{\mathcal{E}} \to f^*\mathrm{Th}_{\mathcal{X}}(\mathcal{N}_i, \mathcal{Y})$ on kernels. Similarly, it follows from Lemma 6.3.9 that the transformation $\varepsilon\Sigma^{\mathcal{N}_i f^*}$ is induced by the composite

$$f^* f_\sharp S^{\mathcal{N}_i} \simeq \mathrm{pr}_{1\sharp}\mathrm{pr}_2^* S^{\mathcal{N}_i} \xrightarrow{u_\Delta^*} \mathrm{pr}_{1\sharp}\Delta_*\Delta^*\mathrm{pr}_2^* S^{\mathcal{N}_i} \simeq S^{\mathcal{N}_\Delta} \otimes S^{\mathcal{N}_i}$$

on kernels, and the transformation $f_\sharp\Sigma^{\mathcal{N}_i}\varepsilon$ is induced by the analogous map $f^* f_\sharp S^{\mathcal{N}_i} \to S^{\mathcal{N}_\Delta} \otimes S^{\mathcal{N}_i}$ where the roles of the two projection maps pr_1 and pr_2 are swapped. It follows that up to swapping the two components of $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ the two composites (II.6.1) and (II.6.2) are induced by the same map on kernels, and thus it suffices to show the statement for (II.6.1). To this end, consider the following diagram:

$$
\begin{tikzcd}
f^*S^{\mathcal{E}} \arrow[r, "f^*\mathrm{PT}_{\mathcal{X}}(i')"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "(1)"] & f^*\mathrm{Th}_{\mathcal{X}}(\mathcal{N}_i, \mathcal{Y}) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] \arrow[drr, phantom, "(2)"] & f^*f_\sharp S^{\mathcal{N}_i} \arrow[r, "\varepsilon"] & S^{\mathcal{N}_\Delta} \otimes S^{\mathcal{N}_i} \arrow[d, "\simeq"] \\
S^{\mathcal{E}} \times_{\mathcal{X}} \mathcal{Y} \arrow[r, "\mathrm{PT}_{\mathcal{Y}}(f^*i')"] \arrow[d, equal] \arrow[drrr, phantom, "(3)"] & \mathrm{Th}_{\mathcal{Y}}(\mathcal{N}_i \times_{\mathcal{X}} \mathcal{Y}, \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}) \arrow[rr, "\mathrm{PT}(\Delta, f^*i')"] & & S^{\mathcal{N}_{(i',\mathrm{id})}} \arrow[d, equal] \\
S^{\mathcal{E}} \times_{\mathcal{X}} \mathcal{Y} \arrow[rrr, "\mathrm{PT}((i',\mathrm{id}))" description] \arrow[d, equal] \arrow[drrr, phantom, "(4)"] & & & S^{\mathcal{N}_{(i',\mathrm{id})}} \arrow[d, "\simeq"] \\
S^{\mathcal{E}} \times_{\mathcal{X}} \mathcal{Y} \arrow[rrr, "\mathrm{PT}((0,\mathrm{id}))"'] & & & S^{f^*\mathcal{E}}.
\end{tikzcd}
$$

Note that by Example 6.2.17 the bottom map is the canonical equivalence, as the map $(0, \mathrm{id}) \colon \mathcal{Y} \to \mathcal{E} \times_{\mathcal{X}} \mathcal{Y}$ is the zero-section of the vector bundle $f^*\mathcal{E}$ over $\mathcal{Y}$. Also note that the right vertical composite $S^{\mathcal{N}_\Delta} \otimes S^{\mathcal{N}_i} \simeq S^{f^*\mathcal{E}}$ is precisely the equivalence from Lemma 6.4.1. It follows that the composite around the top, right and bottom is the map on kernels inducing the composite (II.6.1). Since the left side of the diagram is the identity, it thus remains to show that the diagram commutes up to homotopy. The square (1) commutes by Lemma 6.2.20. Square (3) commutes by Lemma 6.2.19. Square (4) commutes by Lemma 6.2.21. It thus remains to show that square (2) commutes.

From the description for the map ε obtained before, we see that we may write the square (2) as follows:

$$
\begin{tikzcd}
h_{\mathcal{Y}}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}) \otimes \mathrm{Th}_{\mathcal{Y}}(\mathcal{N}_i, \mathcal{Y}) \arrow[r, "\mathrm{PT}_{\mathcal{Y}}(\Delta)\otimes \mathrm{id}"] \arrow[d, "\simeq"'] \arrow[drr, phantom, "(2)"] & \mathrm{Th}_{\mathcal{Y}}(\mathcal{N}_\Delta, \mathcal{Y}) \otimes \mathrm{Th}_{\mathcal{Y}}(\mathcal{N}_i, \mathcal{Y}) \arrow[r, "\simeq"] & S^{\mathcal{N}_\Delta} \otimes S^{\mathcal{N}_i} \arrow[d, "\simeq"] \\
\mathrm{Th}_{\mathcal{Y}}(\mathcal{N}_i \times_{\mathcal{X}} \mathcal{Y}, \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}) \arrow[rr, "\mathrm{PT}_{\mathcal{Y}}(\Delta, f^*i')"] & & S^{\mathcal{N}_{(i,\mathrm{id})}},
\end{tikzcd}
$$

where the left equivalence is the equivalence from Lemma 6.2.12 using the identification of $\mathcal{N}_i \times_{\mathcal{X}} \mathcal{Y}$ with $\mathcal{N}_i \times_{\mathcal{Y}} (\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y})$. Both of the Pontryagin-Thom maps $\mathrm{PT}_{\mathcal{Y}}(\Delta)$ and $\mathrm{PT}_{\mathcal{Y}}(\Delta, f^*i')$ start with the quotient map u_Δ^*. On the top, we use a choice of tubular neighborhood $\mathcal{U}_\Delta$ for the diagonal embedding Δ, while on the bottom we use a choice of tubular neighborhood $\mathcal{U}_{(i,\mathrm{id})}$ for the embedding $(i, \mathrm{id}) \colon \mathcal{Y} \hookrightarrow \mathcal{E} \times_{\mathcal{X}} \mathcal{Y}$. We thus need to show the commutativity of the following diagram:

$$
\begin{tikzcd}
\mathrm{Th}_{\mathcal{Y}}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}, \mathcal{Y}) \otimes \mathrm{Th}_{\mathcal{Y}}(N_i, \mathcal{Y}) \arrow[r, "(1)", "\simeq"'] \arrow[d, "(5)"', "\simeq"] & \mathrm{Th}_{\mathcal{Y}}(\mathcal{U}_\Delta, \mathcal{Y}) \otimes \mathrm{Th}_{\mathcal{Y}}(N_i, \mathcal{Y}) \arrow[r, "(2)", "\simeq"'] & \mathrm{Th}_{\mathcal{Y}}(N_\Delta, \mathcal{Y}) \otimes \mathrm{Th}_{\mathcal{Y}}(N_i, \mathcal{Y}) \arrow[d, "\simeq"', "(6)"] \\
\mathrm{Th}_{\mathcal{Y}}(\mathcal{N}_i \times_{\mathcal{X}} \mathcal{Y}, \mathcal{Y}) \arrow[r, "(3)", "\simeq"'] & \mathrm{Th}_{\mathcal{Y}}(\mathcal{U}_{(i,\mathrm{id})}, \mathcal{Y}) \arrow[r, "(4)", "\simeq"'] & \mathrm{Th}_{\mathcal{Y}}(N_{(i,\mathrm{id})}, \mathcal{Y}).
\end{tikzcd}
$$

Each of the six equivalences appearing in the above square are specific instances of double Beck-Chevalley maps. The equivalences labeled (1)-(4) are obtained from Lemma 6.2.7, and are thus given by the double Beck-Chevalley maps $\mathrm{BC}_{\sharp,*}$ associated to the following four pullback squares:

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, equal] \arrow[d, hook, "\Delta"'] \arrow[dr, phantom, "(1)"] & \mathcal{Y} \arrow[r, equal] \arrow[d, hook] \arrow[dr, phantom, "(2)"] & \mathcal{Y} \arrow[d, hook, "s_0"] \\
\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} & \mathcal{U}_\Delta \arrow[l, hook'] \arrow[r, hook] & \mathcal{N}_\Delta
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{Y} \arrow[r, equal] \arrow[d, hook, "(i', \mathrm{id})"'] \arrow[dr, phantom, "(3)"] & \mathcal{Y} \arrow[r, equal] \arrow[d, hook] \arrow[dr, phantom, "(4)"] & \mathcal{Y} \arrow[d, hook] \\
\mathcal{N}_i \times_{\mathcal{X}} \mathcal{Y} & \mathcal{U}_{(i', \mathrm{id})} \arrow[l, hook'] \arrow[r, hook] & \mathcal{N}_{(i', \mathrm{id})}.
\end{tikzcd}$$

The two vertical equivalences are both instances of the multiplicativity of Thom spaces, Lemma 6.2.12, which in turn are obtained from the double Beck-Chevalley equivalences $\mathrm{BC}_{\sharp,*}$ associated to the following two pullback squares:

$$\begin{tikzcd}
\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \arrow[r, hook, "(s_0, \mathrm{id})"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "(5)"] & \mathcal{N}_i \times_{\mathcal{X}} \mathcal{Y} \arrow[d, "\mathrm{pr}_1"] \\
\mathcal{Y} \arrow[r, hook, "s_0"'] & \mathcal{N}_i
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{N}_\Delta \arrow[r, hook] \arrow[d, "\pi_\Delta"'] \arrow[dr, phantom, "(6)"] & \mathcal{N}_{(i, \mathrm{id})} \arrow[d] \\
\mathcal{Y} \arrow[r, hook, "s_0"'] & \mathcal{N}_i.
\end{tikzcd}$$

The trick now is to observe that the tubular neighborhood $\mathcal{U}_\Delta$ of Δ can be chosen as a pullback along $s_0 \colon \mathcal{Y} \hookrightarrow \mathcal{N}_i$ of a tubular neighborhood $\mathcal{U}_{(i, \mathrm{id})}$ of (i, id). To see this, consider the following pullback square:

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, hook, "\Delta"] \arrow[d, "s_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \arrow[d, "s_0 \times \mathrm{id}"] \\
\mathcal{N}_i \arrow[r, hook, "\Delta'"] & \mathcal{N}_i \times_{\mathcal{X}} \mathcal{Y};
\end{tikzcd}$$

here the map $\Delta' = (\mathrm{id}, \pi_i)$ is the identity in the first component and is the bundle projection $\pi_i \colon \mathcal{N}_i \to \mathcal{Y}$ in the second component. By Lemma 6.2.20, may obtain a tubular neighborhood $\mathcal{U}_\Delta$ of Δ relative to $\mathcal{Y}$ by pulling back a tubular neighborhood $\mathcal{U}_{\Delta'}$ of Δ' along the inclusion $s_0 \colon \mathcal{Y} \to \mathcal{N}_i$. We claim that the composite $\mathcal{N}_\Delta \to \mathcal{N}_i \xrightarrow{\pi_i} \mathcal{Y}$ is isomorphic to the normal bundle of the map (i, id). Indeed, note that the normal bundle of (i, id) is the same as that of $(s_0, \mathrm{id}) \colon \mathcal{Y} \to \mathcal{N}_i \times_{\mathcal{X}} \mathcal{Y}$, which is the diagonal composite in the previous pullback square. In turn, the normal bundle of (s_0, id) is isomorphic to the normal bundle of the composite $\mathcal{Y} \xrightarrow{s_0} \mathcal{N}_i \xrightarrow{s_0} \mathcal{N}_\Delta$, which is the zero-section of the composite bundle $\mathcal{N}_\Delta \to \mathcal{N}_i \xrightarrow{\pi_i} \mathcal{Y}$, thus showing the claim. It follows that $\mathcal{U}_{\Delta'}$ also serves as a tubular neighborhood $\mathcal{U}_{(i, \mathrm{id})}$ for (i, id).

All in all, we see that the six pullback squares (1)-(6) above all fit in a large pullback diagram of the following form:

$$\begin{tikzcd}
\mathcal{Y} \arrow[rrrr, equal] \arrow[dr, hook, "\Delta"'] & & & & \mathcal{Y} \arrow[rrrr, equal] \arrow[dr] & & & & \mathcal{Y} \arrow[dr, hook, "s_0"] & & \\
& \mathcal{Y}\times_{\mathcal{X}}\mathcal{Y} \arrow[dr, "\mathcal{Y}\times s_0"'] \arrow[dd, "\mathrm{pr}_1"'] & & & & \mathcal{U}_\Delta \arrow[llll, hook'] \arrow[rrrr, hook] \arrow[dr, hook] \arrow[dd] & & & & \mathcal{N}_\Delta \arrow[dr, hook, "s_0"] \arrow[dd, "\pi_\Delta"'] & \\
& & N_i\times_{\mathcal{X}}\mathcal{Y} \arrow[dd, "\mathrm{pr}_1"'] & & & & \mathcal{U}_{(i,\mathrm{id})} \arrow[llll, hook'] \arrow[rrrr, hook] \arrow[dd] & & & & \mathcal{N}_{(i,\mathrm{id})} \arrow[dd] \\
& \mathcal{Y} \arrow[rrrr, equal] \arrow[dr, hook, "s_0"] & & & & \mathcal{Y} \arrow[rrrr, equal] \arrow[dr, hook, "s_0"] & & & & \mathcal{Y} \arrow[dr, hook, "s_0"] & \\
& & \mathcal{N}_i \arrow[rrrr, equal] & & & & \mathcal{N}_i \arrow[rrrr, equal] & & & & \mathcal{N}_i .
\end{tikzcd}$$

The result now follows from the pasting law for double Beck-Chevalley transformations, Lemma F.13. □

We are now ready to prove the special case of relative Poincaré duality, Theorem 6.1.7.

Proposition 6.4.4 (Special case of relative Poincaré duality)**.** *Consider a commutative diagram of separated differentiable stacks*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathcal{E} \arrow[d, "\pi"] \\
& \mathcal{X},
\end{tikzcd}$$

where f is a proper representable submersion, i is a closed embedding and π is a vector bundle. Then the Poincaré duality map $\mathfrak{p}_f\colon f_\sharp \to f_*(-\otimes\omega_f)$ is an equivalence.

Proof. We first show that the transformation $\varepsilon\colon f^*f_\sharp \to \Sigma^{\mathcal{N}_\Delta}$ adjoins over to an equivalence $f_\sharp \simeq f_*\circ\Sigma^{\mathcal{N}_\Delta}$. Equivalently, since $\Sigma^{\mathcal{N}_\Delta}$ is invertible, we may show that the induced map $\varepsilon\colon f^*f_\sharp\Sigma^{-\mathcal{N}_\Delta} \to \mathrm{id}$ exhibits the functor $f_\sharp\Sigma^{-\mathcal{N}_\Delta}\colon C(\mathcal{Y}) \to C(\mathcal{X})$ as a right adjoint to $f^*\colon C(\mathcal{X}) \to C(\mathcal{Y})$. Indeed, it is a direct consequence of the triangle identities from Proposition 6.4.3 that a compatible unit transformation $\mathrm{id} \to f_\sharp\Sigma^{-\mathcal{N}_\Delta}f^*$ is given by whiskering the transformation $\eta\colon \Sigma^{\mathcal{E}} \to f_\sharp\Sigma^{\mathcal{N}_i}f^*$ with $\Sigma^{-\mathcal{E}}\colon C(\mathcal{Y}) \to C(\mathcal{Y})$.

It thus remains to show that the transformation $\mathfrak{p}_f$ is obtained under adjunction from the transformation $\varepsilon\colon f^*f_\sharp \to \mathrm{pr}_{1\sharp}\Delta_* \simeq -\otimes\omega_f$. It follows from the explicit description of ε from Construction 6.4.2 that the mate transformation arising in this way is given by the following composite:

$$f_\sharp \xrightarrow{u_f^*} f_* f^* f_\sharp \simeq f_* \mathrm{pr}_{1\sharp} \mathrm{pr}_2^* \xrightarrow{u_\Delta^*} f_* \mathrm{pr}_{1\sharp} \Delta_* \Delta^* \mathrm{pr}_2^* \simeq f_* \mathrm{pr}_{1\sharp} \Delta_* \simeq f_*(- \otimes \omega_f).$$

To unwind the definition of $\mathfrak{p}_f$, recall that the double Beck-Chevalley transformation $\mathrm{BC}_{\sharp,*}$ is given by the composite

$$f_\sharp \mathrm{pr}_{2*} \xrightarrow{u_f^*} f_* f^* f_\sharp \mathrm{pr}_{2*} \simeq f_* \mathrm{pr}_{1\sharp} \mathrm{pr}_2^* \mathrm{pr}_{2*} \xrightarrow{c_{\mathrm{pr}_2}^*} f_* \mathrm{pr}_{1\sharp}.$$

Plugging this into the definition of $\mathfrak{p}_f$, we thus see that it unwinds to the composite

$$f_\sharp \xrightarrow{u_f^*} f_* f^* f_\sharp \simeq f_* \mathrm{pr}_{1\sharp} \mathrm{pr}_2^* \simeq f_* \mathrm{pr}_{1\sharp} \mathrm{pr}_2^* \mathrm{pr}_{2*} \Delta_* \xrightarrow{c_{\mathrm{pr}_2}^*} f_* \mathrm{pr}_{1\sharp} \Delta_* \simeq f_*(- \otimes \omega_f).$$

The claim thus follows from the commutativity of the following square:

$$\begin{tikzcd}
\mathrm{pr}_2^* \arrow[r, "u_\Delta^*"] \arrow[d, "\simeq"'] & \Delta_* \Delta^* \mathrm{pr}_2^* \arrow[d, "\simeq"] \\
\mathrm{pr}_2^* \mathrm{pr}_{2*} \Delta_* \arrow[r, "c_{\mathrm{pr}_2}^*"] & \Delta_*,
\end{tikzcd}$$

which holds by the triangle identity and the fact that the equivalence $\mathrm{id} \simeq (\mathrm{pr}_2 \Delta)_* \simeq \mathrm{pr}_{2*} \Delta_*$ can be taken to be the following composite:

$$\mathrm{id} \xrightarrow{\text{unit}} \mathrm{pr}_{2*} \mathrm{pr}_2^* \xrightarrow{\text{unit}} \mathrm{pr}_{2*} \Delta_* \Delta^* \mathrm{pr}_2^* \simeq \mathrm{pr}_{2*} \Delta_* (\mathrm{pr}_2 \circ \Delta)^* = \mathrm{pr}_{2*} \Delta_*. \qquad \square$$

6.4.2 The general case

We will now prove relative Poincaré duality for an arbitrary proper representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of separated differentiable stacks. The key ingredient is the fact that the property of the Poincaré map $\mathfrak{p}_f$ to be an equivalence can be tested locally in the base stack $\mathcal{X}$, see Corollary 6.4.6 below, which lets us reduce to the special case treated in the previous subsection. This in turn is a consequence of the following compatibility between the Poincaré duality transformations and pullbacks of stacks:

Lemma 6.4.5 (Poincaré map commutes with base change)**.** *Consider a pullback square of differentiable stacks*

$$\begin{tikzcd}
\mathcal{Y}' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y} \arrow[d, "f"] \\
\mathcal{X}' \arrow[r, "g"] & \mathcal{X}
\end{tikzcd}$$

such that the morphisms f and f' are representable submersions.

(1) There is an equivalence

$$\alpha\colon \omega_{f'} \simeq h^*\omega_f \quad \in \quad \mathcal{C}(\mathcal{Y}').$$

(2) There is a commutative diagram

$$\begin{tikzcd}
f'_\sharp h^* \arrow[r, "\mathfrak{p}_{f'}"] \arrow[dd, "\mathrm{BC}_\sharp"'] & f'_*(h^*(-)\otimes\omega_{f'}) \\
 & f'_* h^*(-\otimes\omega_f). \arrow[u, "\alpha"'] \\
g^* f_\sharp \arrow[r, "\mathfrak{p}_f"'] & g^* f_*(-\otimes\omega_f) \arrow[u, "\mathrm{BC}_*"']
\end{tikzcd}$$

Proof. Consider the following commutative diagram, in which all faces are pullback squares:

$$\begin{tikzcd}
 & \mathcal{Y}'\times_{\mathcal{X}'}\mathcal{Y}' \arrow[rrr, "k"] \arrow[dl, "p'_1"'] \arrow[dd, "p'_2"] & & & \mathcal{Y}\times_{\mathcal{X}}\mathcal{Y} \arrow[dl, "p_1"'] \arrow[dd, "p_2"] \\
\mathcal{Y}' \arrow[rrr, "h"] \arrow[dd, "f'"'] & & & \mathcal{Y} \arrow[dd, "f"] & \\
 & \mathcal{Y}' \arrow[rrr, "h"] \arrow[dl, "f'"'] & & & \mathcal{Y} \arrow[dl, "f"] \\
\mathcal{X}' \arrow[rrr, "g"'] & & & \mathcal{X}. &
\end{tikzcd}$$

Part (1) follows from the following sequence of equivalences:

$$h^*\omega_f = h^*\mathrm{pr}_{1\sharp}\Delta_{f*}\mathbb{1}_{\mathcal{Y}} \simeq \mathrm{pr}_{1\sharp}k^*\Delta_{f*}\mathbb{1}_{\mathcal{Y}} \simeq \mathrm{pr}_{1\sharp}\Delta_* h^*\mathbb{1}_{\mathcal{Y}} = \mathrm{pr}_{1\sharp}\Delta_*\mathbb{1}_{\mathcal{Y}'} = \omega_{f'},$$

where we used smooth and closed base change associated to the above pullback squares.

For part (2), we consider the following large diagram:

$$
\begin{tikzcd}
f'_\sharp h^* \arrow[rrr, "\mathfrak{p}_{f'}"] \arrow[ddd, equal] \arrow[dr, "\simeq"] & & & f'_*(h^*(-)\otimes \omega_{p'}) \arrow[dl, "\simeq"'] \\
& f_\sharp p'_{2*}\Delta'_* h^* \arrow[r, "\mathrm{BC}_{\sharp,*}"] & f'_* p'_{1\sharp}\Delta'_* h^* & \\
\arrow[r, phantom, "(1)"] & f'_\sharp p'_{2*} k^* \Delta_* \arrow[u, "\mathrm{BC}_*"] \arrow[r, "\mathrm{BC}_{\sharp,*}"] & f'_* p'_{1\sharp} k^* \Delta_* \arrow[u, "\mathrm{BC}_*"'] \arrow[d, "\mathrm{BC}_\sharp"'] \arrow[r, phantom, "(3)"] & \\
f'_\sharp h^* \arrow[r, dashed, "\simeq"] \arrow[ddd, "\mathrm{BC}_\sharp"'] & f'_\sharp h^* p_{2*}\Delta_* \arrow[u, "\mathrm{BC}_*"] \arrow[d, "\mathrm{BC}_\sharp"] \arrow[r, phantom, "(2)"] & f'_* h^* p_{1\sharp}\Delta_* & f'_* h^*(-\otimes \omega_p) \arrow[l, dashed, "\simeq"'] \arrow[uuu, "\alpha"'] \\
& g^* f_\sharp p_{2*}\Delta_* \arrow[r, "\mathrm{BC}_{\sharp,*}"'] & g^* f_* p_{1\sharp}\Delta_* \arrow[u, "\mathrm{BC}_*"'] & \\
& & & \\
g^* f_\sharp \arrow[uur, "\simeq"] \arrow[rrr, "\mathfrak{p}_f"'] & & & g^* f_*(-\otimes \omega_f). \arrow[uul, "\simeq"'] \arrow[uuu, "\mathrm{BC}_*"']
\end{tikzcd}
$$

Apart from the faces labeled (1), (2) and (3), all faces either by naturality or by definition. Commutativity of (1) holds by Lemma F.6. Commutativity of (2) holds by Proposition F.14. Commutativity of (3) is immediate from unwinding the constructions of the equivalences from part (1) and Remark 6.1.4. □

Corollary 6.4.6. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable submersion between separated differentiable stacks. Then the property that the Poincaré duality transformation $\mathfrak{p}_f\colon f_\sharp \to f_*(-\otimes \omega_f)$ is an equivalence can be checked locally in $\mathcal{X}$.*

Proof. Let $\{j_i\colon \mathcal{U}_i \hookrightarrow \mathcal{X}\}_{i\in I}$ be an open cover of $\mathcal{X}$ and for every $i \in I$ and let $f_i\colon \mathcal{Y}\times_{\mathcal{X}}\mathcal{U}_i \to \mathcal{U}_i$ denote the base change of f along the inclusion $j_i\colon \mathcal{U}_i \hookrightarrow \mathcal{X}$. Assume that for every $i \in I$, the transformation $\mathfrak{p}_{f_i}\colon f_{i\sharp}(-) \Rightarrow f_{i*}(-\otimes \omega_f)$ is an equivalence. Our goal is to show that also the transformation $\mathfrak{p}_f\colon f_\sharp(-) \Rightarrow f_*(-\otimes\omega_f)$ is an equivalence.

Since C is a sheaf of ∞-categories with respect to the open cover topology on SepStk, the collection of pullback functors $j_i^*\colon C(\mathcal{X}) \to C(\mathcal{U}_i)$ is jontly conservative. It will thus suffice to show that the transformation $j_i^*\mathfrak{p}_f\colon j_i^* f_\sharp \Rightarrow j_i^* f_*(-\otimes\omega_f)$ is an equivalence for every $i \in I$. By Lemma 6.4.5, this follows directly from the condition that $\mathfrak{p}_{f_i}$ is an equivalence for every $i \in I$, finishing the proof. □

We may now finish the proof of Theorem 6.1.7: for every proper representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$, the Poincaré duality map $\mathfrak{p}_f\colon f_\sharp(-) \to f_*(-\otimes\omega_f)$ is an equivalence.

Proof of Theorem 6.1.7. By Proposition 3.7.3, we may find an open cover $\{j_\alpha \colon \mathcal{U}_\alpha \hookrightarrow \mathcal{X}\}_{\alpha\in I}$ of $\mathcal{X}$ such that for every $\alpha \in I$ the restriction $f_\alpha \colon \mathcal{X}_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha$ factors as

$$\mathcal{X}_{\mathcal{U}_\alpha} \xrightarrow{i_\alpha} S^{\mathcal{E}_\alpha} \xrightarrow{\pi_\alpha} \mathcal{U}_\alpha,$$

where $\pi_\alpha \colon \mathcal{E}_\alpha \to \mathcal{U}_\alpha$ is a vector bundle and where i_α is a closed embedding factoring through $\mathcal{E} \hookrightarrow S^{\mathcal{E}}$. By Proposition 6.4.4 we get that the transformation $\mathfrak{p}_{f_\alpha}$ is an equivalence for every α. It then follows from Corollary 6.4.6 that also the transformation $\mathfrak{p}_f$ is an equivalence, finishing the proof. □

6.5 Consequences of relative Poincaré duality

This section will discuss several important consequences of relative Poincaré duality. We continue working in the setting of a pullback formalism C on SepStk satisfying the Voevodsky conditions.

6.5.1 Relative Atiyah duality

Given a proper representable submersion $\mathcal{Y} \to \mathcal{X}$, it follows from relative Poincaré duality that the image of $\mathcal{Y}$ in $C(\mathcal{X})$ is dualizable, with dual given by the Thom object of the inverse tangent sphere bundle:

Proposition 6.5.1 (Relative Atiyah duality)**.** *Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a proper representable submersion of separated differentiable stacks. Then the object $h_{\mathcal{X}}(\mathcal{Y}) \in C(\mathcal{X})$ is dualizable with dual given by $f_\sharp(S^{-T_f})$.*

Proof. The functor $h_{\mathcal{X}}(\mathcal{Y}) \otimes - \colon C(\mathcal{X}) \to C(\mathcal{X})$ is equivalent to $f_\sharp f^*$, regarded as a $C(\mathcal{X})$-linear functor. Its right adjoint is equivalent to $f_* f^*$, which is given by tensoring with $f_* \mathbb{1}_{\mathcal{Y}}$ by the projection formula for f_*. It follows that $h_{\mathcal{X}}(\mathcal{Y})$ is dualizable with dual $f_* \mathbb{1}_{\mathcal{Y}}$. But by relative Poincaré duality there is an equivalence $f_* \mathbb{1}_{\mathcal{Y}} \simeq f_\sharp(S^{-T_f})$ in $C(\mathcal{X})$, finishing the proof. □

6.5.2 Proper base change

As another consequence of relative Poincaré duality, we obtain good properties of the pushforward functors $p_* \colon C(\mathcal{Y}) \to C(\mathcal{X})$ for proper morphisms $p \colon \mathcal{Y} \to \mathcal{X}$ of separated differentiable stacks: proper base change, the proper projection formula and smooth-proper base change.

Proposition 6.5.2 (Proper base change)**.** *Let $p\colon \mathcal{Y} \to \mathcal{X}$ be a proper morphism of separated differentiable stacks. For every pullback square of differentiable stacks*

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "h"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "p"] \\ \mathcal{X}' \arrow[r, "g"] & \mathcal{X}, \end{tikzcd}$$

the right Beck-Chevalley map

$$\mathrm{BC}_*\colon g^*p_* \Rightarrow p'_*h^*$$

is an equivalence in $\mathrm{Fun}(\mathcal{C}(\mathcal{Y}), \mathcal{C}(\mathcal{X}'))$.

Proof. By Proposition 3.7.3, the map p factors as a closed embedding followed by a proper representable submersion, hence by Lemma F.6 it suffices to prove the claim in these two cases. For closed embeddings this is closed base change, see Proposition 5.1.3. So assume that p is a proper representable submersion. Since the functor $-\otimes\omega_p\colon \mathcal{C}(\mathcal{Y}) \to \mathcal{C}(\mathcal{Y})$ is an equivalence by Corollary 6.2.11, it will suffice to show that the transformation $\mathrm{BC}_*\colon g^*p_*(-\otimes\omega_p) \to p'_*h^*(-\otimes\omega_p)$ is an equivalence. This follows from Lemma 6.4.5, since the map $\mathrm{BC}_\sharp$ is an equivalence by smooth base change and the transformations $\mathfrak{p}_f$ and $\mathfrak{p}_{f'}$ are equivalences by relative Poincaré duality, Theorem 6.1.7. This finishes the proof. $\square$

Proposition 6.5.3 (Proper projection formula)**.** *For every proper map $p\colon \mathcal{X} \to \mathcal{Y}$ of separated differentiable stacks and objects $A \in \mathcal{C}(\mathcal{X})$, $B \in \mathcal{C}(\mathcal{Y})$, the exchange map*

$$\mathrm{PF}_*\colon p_*(A)\otimes B \to p_*(A\otimes p^*B)$$

is an equivalence in $\mathcal{C}(\mathcal{Y})$.

Proof. The proof is analogous to Proposition 6.5.2. By Proposition 3.7.3, the map p factors as a closed embedding followed by a proper representable submersion, hence by Lemma F.17 it suffices to prove the claim in these two cases. For closed embeddings this is the closed projection formula, see Proposition 5.1.3. So assume that p is a proper representable submersions. By relative Poincaré duality and the smooth projection formula, it suffices to show that the following diagram commutes:

$$\begin{tikzcd} p_\sharp(A)\otimes B \arrow[r, "\mathfrak{p}_p"] & p_*(A\otimes\omega_p)\otimes B \arrow[d, "\mathrm{PF}_*"] \\ p_\sharp(A\otimes p^*B) \arrow[u, "\mathrm{PF}_\sharp"] \arrow[r, "\mathfrak{p}_p"'] & p_*(A\otimes p^*B\otimes\omega_p). \end{tikzcd}$$

This is a consequence of the following large commutative diagram:

$$\begin{tikzcd}[column sep=small]
p_\sharp A \otimes B \arrow[rrrr, "\mathfrak{p}_p"] \arrow[dr, "\simeq"] & & & & p_*(A\otimes\omega_p)\otimes B \arrow[dd, "\mathrm{PF}_*"] \\
& p_\sharp p_{1*}\Delta_* A \otimes B \arrow[rr, "\mathrm{BC}_{\sharp,*}"] & & p_* p_{1\sharp}\Delta_* A \otimes B \arrow[ur, "\simeq"] \arrow[d, "\mathrm{PF}_*"] & \\
p_\sharp(A\otimes p^*B) \arrow[uu, "\mathrm{PF}_\sharp"] \arrow[r, dashed, "\simeq"] \arrow[dddd, equal] & p_\sharp(p_{1*}\Delta_*A\otimes p^*B) \arrow[u, "\mathrm{PF}_\sharp"] \arrow[d, "\mathrm{PF}_*"] & (2) & p_*(p_{1\sharp}\Delta_*A\otimes p^*B) \arrow[r, dashed, "\simeq"] & p_*(A\otimes\omega_p\otimes p^*B) \arrow[dddd, "\simeq"] \\
& p_\sharp p_{1*}(\Delta_*A\otimes p_1^*p^*B) \arrow[rr, "\mathrm{BC}_{\sharp,*}"] \arrow[d, "\mathrm{PF}_*"] & & p_*p_{1\sharp}(\Delta_*A\otimes p_1^*p^*B) \arrow[u, "\mathrm{PF}_\sharp"] \arrow[d, "\mathrm{PF}_*"] & \\
(1) & p_\sharp p_{1*}\Delta_*(A\otimes\Delta^*p_1^*p^*B) \arrow[d, "\simeq"] & & p_*p_{1\sharp}\Delta_*(A\otimes\Delta^*p_1^*p^*B) \arrow[d, dashed, "\simeq"] & (3) \\
& p_\sharp p_{1*}\Delta_*(A\otimes p^*B) \arrow[rr, "\mathrm{BC}_{\sharp,*}"] & & p_*p_{1\sharp}\Delta_*(A\otimes p^*B) \arrow[dr, "\simeq"] & \\
p_\sharp(A\otimes p^*B) \arrow[ur, "\simeq"] \arrow[rrrr, "\mathfrak{p}_p"] & & & & p_*(A\otimes p^*B\otimes\omega_p).
\end{tikzcd}$$

The unlabeled faces of the diagram commute either by naturality or by definition. Face (1) commutes by Lemma F.17. Face (2) commutes by Lemma F.20. The commutativity of face (3) follows from unwinding the definition of the equivalence from Remark 6.1.4. ◻

Proposition 6.5.4 (Smooth-proper base change)**.** *For every pullback square of separated differentiable stacks*

$$\begin{tikzcd}
\mathcal{Y}' \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y} \arrow[d, "p"] \\
\mathcal{X}' \arrow[r, "f"] & \mathcal{X}
\end{tikzcd}$$

where p is a proper morphism and f is a representable submersion, the double Beck-Chevalley map

$$\mathrm{BC}_{\sharp,*}\colon f_\sharp p'_* \Rightarrow p_* f'_\sharp$$

is an equivalence in $\mathrm{Fun}(\mathcal{C}(\mathcal{Y}),\mathcal{C}(\mathcal{X}))$.

Proof. The proof is analogous to Proposition 6.5.2. By Proposition 3.7.3, the map p factors as a closed embedding followed by a proper representable submersion, hence by Lemma F.13 it suffices to prove the claim in these two cases. For closed embeddings this holds by smooth-closed base change, see Proposition 5.1.3, so assume that p is a proper representable submersions. Since the functor $-\otimes\omega_{p'}$ is invertible, it will suffice to show that the map $\mathrm{BC}_{\sharp,*}\colon f_\sharp p'_*(-\otimes\omega_{p'}) \to p_* f'_\sharp(-\otimes\omega_{p'})$ is an equivalence. By relative Poincaré duality and smooth base change, this follows from the following large commutative diagram:

$$
\begin{tikzcd}
f_\sharp p'_\sharp \arrow[rrrr, "\mathfrak{p}_{p'}"] \arrow[dd, "\simeq"'] \arrow[dr, "\simeq"] & & & & f_\sharp p'_*(-\otimes\omega_{p'}) \arrow[dd, "\mathrm{BC}_{\sharp,*}"] \\
& f_\sharp p'_\sharp p'_{2*}\Delta'_* \arrow[r, "\mathrm{BC}_{\sharp,*}"] \arrow[d, dashed, "\simeq"] & f_\sharp p'_* p'_{1\sharp}\Delta'_* \arrow[urr, "\simeq"] \arrow[d, "\mathrm{BC}_{\sharp,*}"] & & \\
p_\sharp f'_\sharp \arrow[r, dashed, "\simeq"] \arrow[ddd, equal] & p_\sharp f'_\sharp p'_{2*}\Delta'_* \arrow[r, phantom, "(2)"] \arrow[d, "\mathrm{BC}_{\sharp,*}"'] & p_* f'_\sharp p'_{1\sharp}\Delta'_* \arrow[rr, dashed, "\simeq"] \arrow[d, dashed, "\simeq"] & & p_* f'_\sharp(-\otimes\omega_{p'}) \arrow[d, "\simeq"] \\
\arrow[d, phantom, "(1)"] & p_\sharp p_{2*} f''_\sharp \Delta'_* \arrow[r, "\mathrm{BC}_{\sharp,*}"] \arrow[d, "\mathrm{BC}_{\sharp,*}"'] & p_* p_{1\sharp} f''_\sharp \Delta'_* \arrow[r, phantom, "(3)"] \arrow[d, "\mathrm{BC}_{\sharp,*}"] & & p_* f'_\sharp(-\otimes f'^*\omega_p) \arrow[dd, "\mathrm{PF}_\sharp"] \\
& p_\sharp p_{1\sharp}\Delta_* f'_\sharp \arrow[r, "\mathrm{BC}_{\sharp,*}"] & p_* p_{1\sharp}\Delta_* f'_\sharp \arrow[drr, "\simeq"] & & \\
p_\sharp f'_\sharp \arrow[ur, "\simeq"] \arrow[rrrr, "\mathfrak{p}_p"'] & & & & p_*(f'_\sharp(-)\otimes\omega_p).
\end{tikzcd}
$$

The unlabeled faces of the diagram commute either by naturality or by definition. Faces (1) and (2) commute by Lemma F.13. The commutativity of face (3) is somewhat involved and we will only give the core idea of the proof. Spelling out the definitions of the equivalences from Remark 6.1.4 and Lemma 6.4.5, the claim will follow from the following commutative diagram:

$$
\begin{tikzcd}
f'_\sharp p'_{1\sharp}\Delta'_* \arrow[r, "\tilde{\simeq}"] \arrow[d, "\simeq"'] & f'_\sharp p'_{1\sharp}\Delta'_*(f'^*\mathbb{1}\otimes\Delta'^*{p'_1}^*(-)) & f'_\sharp p'_{1\sharp}(\Delta'_*f'^*\mathbb{1}\otimes{p'_1}^*(-)) \arrow[l, "\mathrm{PF}_*"'] \arrow[r, "\mathrm{PF}_\sharp"] \arrow[dd, bend left, "\mathrm{BC}_*"] & f'_\sharp(p'_{1\sharp}\Delta'_*f'^*\mathbb{1}\otimes -) \arrow[dd, "\mathrm{BC}_*"] \\
p_{1\sharp}f''_\sharp\Delta'_* \arrow[r, phantom, "\simeq"] \arrow[ddd, "\mathrm{BC}_{\sharp,*}"'] & p_{1\sharp}f''_\sharp\Delta'_*(f'^*\mathbb{1}\otimes\Delta'^*{p'_1}^*(-)) \arrow[u, "\simeq"'] \arrow[d, "\mathrm{BC}_{\sharp,*}"'] & p_{1\sharp}f''_\sharp(\Delta'_*f'^*\mathbb{1}\otimes{p'_1}^*(-)) \arrow[l, "\mathrm{PF}_*"'] \arrow[u, "\simeq"'] \arrow[dd, bend right, "\mathrm{BC}_*"'] & \\
& p_{1\sharp}\Delta_*f'_\sharp(f'^*\mathbb{1}\otimes\Delta'^*{p'_1}^*(-)) \arrow[d, "\mathrm{PF}_\sharp"'] & f'_\sharp p'_{1\sharp}(f''^*\Delta_*\mathbb{1}\otimes{p'_1}^*(-)) \arrow[r, "\mathrm{PF}_\sharp"] & f'_\sharp(p'_{1\sharp}f''^*\Delta_*\mathbb{1}\otimes -) \arrow[dd, "\mathrm{BC}_\sharp"] \\
p_{1\sharp}\Delta_*f'_\sharp \arrow[r, dashed, "\simeq"] \arrow[dd, equal] \arrow[ddr, phantom, "(1)"] & p_{1\sharp}\Delta_*(\mathbb{1}\otimes f'_\sharp\Delta'^*{p'_1}^*(-)) \arrow[d, "\mathrm{BC}_\sharp"'] \arrow[r, phantom, "(2)"] & p_{1\sharp}f''_\sharp(f''^*\Delta_*\mathbb{1}\otimes{p'_1}^*(-)) \arrow[u, "\simeq"'] \arrow[d, "\mathrm{PF}_\sharp"] \arrow[r, phantom, "(3)"] & \\
& p_{1\sharp}\Delta_*(\mathbb{1}\otimes\Delta^*f''_\sharp{p'_1}^*(-)) \arrow[d, "\mathrm{BC}_\sharp"'] & p_{1\sharp}(\Delta_*\mathbb{1}\otimes f''_\sharp{p'_1}^*(-)) \arrow[l, "\mathrm{PF}_*"'] \arrow[d, "\mathrm{BC}_\sharp"] & f'_\sharp(f'^*p_{1\sharp}\Delta_*\mathbb{1}\otimes -) \arrow[d, "\mathrm{PF}_\sharp"] \\
p_{1\sharp}\Delta_*f'_\sharp \arrow[r, "\simeq"'] & p_{1\sharp}\Delta_*(\mathbb{1}\otimes\Delta^*p_1^*f'_\sharp(-)) & p_{1\sharp}(\Delta_*\mathbb{1}\otimes p_1^*f'_\sharp(-)) \arrow[l, "\mathrm{PF}_*"] \arrow[r, "\mathrm{PF}_\sharp"'] & p_{1\sharp}\Delta_*\mathbb{1}\otimes f'_\sharp(-).
\end{tikzcd}
$$

Except for the labeled faces, all faces commute by naturality. Face (1) commutes by Lemma F.6. Faces (2) and (3) are formal properties of mate transformations, whose verification we will leave to the reader. $\square$

Proposition 6.5.5 (Proper exceptional pullback)**.** *Let $p\colon \mathcal{Y}\to\mathcal{X}$ be a proper map of separated differentiable stacks. Then the functor $p_*\colon C(\mathcal{Y})\to C(\mathcal{X})$ admits a right adjoint $p^!\colon C(\mathcal{X})\to C(\mathcal{Y})$.*

Proof. By Proposition 3.7.3, the map p factors as a closed embedding followed by a proper representable submersion. If p is a closed embedding, the claim holds by Corollary 5.1.5. If p is a proper representable submersion, there is an equivalence $p_*(-)\simeq p_\sharp(-\otimes\omega_f^{-1})$ and the latter functor admits a right adjoint $\omega_f\otimes f^*(-)$. $\square$

6.5.3 Relative Poincaré duality for stacks on SepStk

Thus far, our discussion of the categorical properties for the pullback functors $f^*\colon C(\mathcal{X})\to C(\mathcal{Y})$ has been restricted to morphisms $f\colon\mathcal{Y}\to\mathcal{X}$ between differentiable stacks. As we will see now, these properties formally extend to more general morphisms in Shv(SepStk). This generalization will be used in Section III.III.4 to deduce relative Poincaré duality for proper genuine sheaves of spectra.

Recall that, since $\mathcal{C}$ is a sheaf of ∞-categories on SepStk, it admits a unique extension to a limit-preserving functor $\mathcal{C}\colon \mathrm{Shv}(\mathrm{SepStk})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. We will refer to objects $\mathfrak{X} \in \mathrm{Shv}(\mathrm{SepStk})$ as *stacks on* SepStk. We will identify a differentiable stack $\mathcal{X}$ with its image under the Yoneda embedding.

Definition 6.5.6 (Representable submersions and proper maps of stacks on SepStk)**.** Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a morphism of stacks on SepStk.

(1) We say that f is a *representable submersion* if for every separated differentiable stack $\mathcal{X}$ and every pullback square

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

in $\mathrm{Shv}(\mathrm{SepStk})$, the object $\mathcal{Y}$ is also a separated differentiable stack and the base change morphism $f'\colon \mathcal{Y} \to \mathcal{X}$ is a representable submersion;

(2) We say that f is an *open embedding* if it is a representable submersion and the base change morphism $f'\colon \mathcal{Y} \to \mathcal{X}$ in part (1) is even an open embedding for all $\mathcal{X} \to \mathfrak{X}$;

(3) We say that f is *representable* if for every separated differentiable stack $\mathcal{X}$ and every pullback square

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

in $\mathrm{Shv}(\mathrm{SepStk})$ in which the bottom map $\mathcal{X} \to \mathfrak{X}$ is a representable submersion, the object $\mathcal{Y}$ is also a separated differentiable stack;

(4) We say that f is *proper* if it is representable and the base change morphism $f'\colon \mathcal{Y} \to \mathcal{X}$ in the previous point is proper for every $\mathcal{X} \to \mathfrak{X}$.

Remark 6.5.7. By definition, the condition that the base change morphism $f'\colon \mathcal{Y} \to \mathcal{X}$ is a representable submersion or a proper morphism can be tested by further pulling back along a map $M \to \mathcal{X}$ from a smooth manifold M, hence the same could be done in Definition 6.5.6. We have chosen this formulation to enhance the analogy with Definition 2.1.4.

Proposition 6.5.8. *Let* $\mathcal{C}\colon \mathrm{Shv}(\mathrm{SepStk})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ *be a pullback formalism on* SepStk *satisfying the Voevodsky conditions. Then the following hold:*

(1) *For every representable submersion* $f: \mathfrak{Y} \to \mathfrak{X}$ *of stacks on* SepStk*, the pullback functor* $f^*: C(\mathfrak{X}) \to C(\mathfrak{Y})$ *admits a left adjoint* $f_\sharp: C(\mathfrak{Y}) \to C(\mathfrak{X})$*. Furthermore, these left adjoints satisfy smooth base change and the smooth projection formula.*

(2) *For every proper morphism* $p: \mathfrak{Y} \to \mathfrak{X}$ *of stacks on* SepStk*, the pullback functor* $p^*: C(\mathfrak{X}) \to C(\mathfrak{Y})$ *admits a right adjoint* $p_*: C(\mathfrak{Y}) \to C(\mathfrak{X})$*. Furthermore, these right adjoints satisfy proper base change and the proper projection formula.*

(3) *The left adjoints* $f_\sharp$ *and right adjoints* p_* *from parts (1) and (2) satisfy smooth-proper base change.*

(4) *For every proper representable submersion* $f: \mathfrak{Y} \to \mathfrak{X}$ *of stacks on* SepStk *the Poincaré duality map* $\mathfrak{p}_f: f_\sharp(-) \to f_*(- \otimes \omega_f)$ *is an equivalence of functor* $C(\mathfrak{Y}) \to C(\mathfrak{X})$*.*

Proof. Each of these properties are local in the morphism $f: \mathfrak{Y} \to \mathfrak{X}$, in the sense that they may be tested after pulling back along an effective epimorphism $\mathfrak{X}' \twoheadrightarrow \mathfrak{X}$ in $\mathrm{Shv}(\mathrm{SepStk})$. They then follow immediately from the analogous properties for morphisms of differentiable stacks:

(1) Smooth base change and the smooth projection formula hold by assumption on C;

(2) Proper base change and the proper projection formula hold by Proposition 6.5.2 and Proposition 6.5.3, respectively;

(3) Smooth-proper base change holds by Proposition 6.5.4;

(4) Relative Poincaré duality holds by Theorem 6.1.7.

This finishes the proof. □

Part III

Outlook

III.1 Introduction

In the final part of this dissertation, we indicate several potential directions for future research following up on the results from parts I and II. In Section III.2, we explain in some detail the expected close relation between twisted ambidexterity from Part I and relative Poincaré duality from Part II. In Section III.3, we discuss six-functor formalisms on the site SepStk and formulate a conjecture concerning a six-functor formalism of genuine sheaves of spectra. In Section III.4, we discuss the notion of *proper genuine sheaves* on a differentiable stack.

We emphasize that the contents of this final part are much more speculative in nature than the contents of parts I and II.

III.2 From Poincaré duality to twisted ambidexterity

The goal of this section is to explain the close connection between twisted ambidexterity for orbispectra, the main topic of Part I of this dissertation, and relative Poincaré duality for differentiable stacks, the main topic of Part II of this dissertation.

We start in Subsection 2.1 by discussing the underlying orbispace of a differentiable stack. In Subsection 2.2 we propose a notion of *locally constant* genuine sheaves on a stack and indicate why the category of such should only depend on the underlying orbispace of the stack. In Subsection 2.3 we come to the comparison of twisted ambidexterity and Poincaré duality.

2.1 From differentiable stacks to orbispaces

Given a differentiable stack $\mathcal{X}$ and a compact Lie group G, the *space of G-fixed points of $\mathcal{X}$* is the homotopy type of the internal mapping stack $\mathbb{H}\mathrm{om}(\mathbb{B}G, \mathcal{X})$ in $\mathrm{Shv}(\mathrm{Diff})$. These fixed point spaces can be assembled into a presheaf on the global indexing category Glo from Definition I.4.11, thus forming a *global space*. It was recognized by Gepner and Henriques [GH07], who worked in the topological rather than the differentiable setting, that this passage from stacks to global spaces is precisely capturing the *homotopy theory of stacks*.[1]

The philosophy that global spaces constitute the homotopy theory of stacks admits a perspective using the notion of *homotopy invariant sheaves*, introduced in Section II.4.2.

[1] Both orbispaces and global spaces are called 'orbispaces' by Gepner and Henriques

Consider the ∞-category $\mathrm{Shv}(\mathrm{SepStk})$ of sheaves of spaces on the site SepStk of separated differentiable stacks, equipped with the open covering topology from Definition II.4.1.2. By inverting the projection maps $\mathcal{X} \times \mathbb{R} \to \mathcal{X}$ for all separated differentiable stacks, we obtain a homotopy localization functor

$$L_{\mathrm{htp}} \colon \mathrm{Shv}(\mathrm{SepStk}) \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{SepStk}).$$

The target of this map can be shown to be equivalent to a *presheaf* category, whose indexing category is the full subcategory $\mathrm{Glo}' \subseteq \mathrm{Shv}^{\mathrm{htp}}(\mathrm{SepStk})$ spanned by the homotopy localizations $L_{\mathrm{htp}}(\mathbb{B}G)$ of the classifying stacks $\mathbb{B}G$ for all compact Lie groups G:

Proposition 2.1. *There is a unique equivalence* $\mathrm{PSh}(\mathrm{Glo}') \xrightarrow{\simeq} \mathrm{Shv}^{\mathrm{htp}}(\mathrm{SepStk})$ *extending the inclusion* $\mathrm{Glo}' \hookrightarrow \mathrm{Shv}^{\mathrm{htp}}(\mathrm{SepStk})$.

The proof is entirely analogous to that of Corollary II.4.4.7 in the G-equivariant situation, the only subtle point being the use of Theorem II.3.7.2 to deduce that every separated differentiable stack is locally homotopy equivalent to a stack of the form $\mathbb{B}G$.

The ∞-categories Glo' and Glo are very similar to each other: both have one object for every compact Lie group G, and the space of morphisms from H to G can in both cases be computed to be $\Pi_\infty(\mathbb{H}\mathrm{om}(\mathbb{B}H, \mathbb{B}G))$, the homotopy type of the internal mapping object $\mathbb{H}\mathrm{om}(\mathbb{B}H, \mathbb{B}G)$ in $\mathrm{Shv}(\mathrm{Diff})$. It seems likely that the two ∞-categories are in fact equivalent:

Conjecture 2.2. *There is an equivalence of ∞-categories* $\mathrm{Glo} \simeq \mathrm{Glo}'$. *In particular, there exists a functor*

$$\Pi_{\mathrm{Glo}} \colon \mathrm{Shv}(\mathrm{SepStk}) \to \mathrm{GloSpc}$$

which exhibits the ∞-category GloSpc *as the Bousfield localization of* $\mathrm{Shv}(\mathrm{SepStk})$ *at the homotopy equivalences.*

Given a separated differentiable stack $\mathcal{X}$, we refer to the global space $\Pi_{\mathrm{Glo}}(\mathcal{X})$ as the *global homotopy type* of $\mathcal{X}$.

Remark 2.3. A variant of this conjecture has previously been considered by Adrian Clough in personal writings.

From the characterization of representable maps between separated differentiable stacks from Corollary II.3.4.6, one may deduce that for any representable map $f \colon \mathcal{Y} \to \mathcal{X}$ between separated differentiable stacks, the induced map $\Pi_{\mathrm{Glo}}(f) \colon \Pi_{\mathrm{Glo}}(\mathcal{Y}) \to \Pi_{\mathrm{Glo}}(\mathcal{X})$ is a *faithful* map of global spaces. Since every separated differentiable stack is a colimit of global quotient stacks along representable maps, it follows that the global homotopy type $\Pi_{\mathrm{Glo}}(\mathcal{X})$

of a separated stack $\mathcal{X}$ is in fact an *orbispace*, meaning that it is contained in the essential image of the left Kan extension functor $\mathrm{OrbSpc} = \mathrm{PSh}(\mathrm{Orb}) \hookrightarrow \mathrm{PSh}(\mathrm{Glo}) = \mathrm{GloSpc}$.

2.2 Locally constant genuine sheaves

Given an ∞-topos $\mathcal{B}$, one may consider *sheaves of ∞-categories on $\mathcal{B}$*, i.e. limit-preserving functors $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. In Part I of this dissertation, our focus was on examples of *homotopical* nature, like the ∞-topoi OrbSpc and GloSpc of orbispaces and global spaces, respectively. In Part II of this dissertation, our focus was instead on examples of *geometric* nature, in particular the ∞-topos Shv(SepStk). In light of Conjecture 2.2, these two situations are closely related:

Definition 2.4 (Strong homotopy invariance)**.** A sheaf of ∞-categories $\mathcal{C}$ on Shv(SepStk) is said to be *strongly homotopy invariant* if for every separated differentiable stack $\mathcal{X}$ the pullback functor $\mathrm{pr}^*\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{X} \times \mathbb{R})$ is an equivalence.

Corollary 2.5. *Assuming that Conjecture 2.2 holds, precomposition with the global homotopy type functor* $\Pi_{\mathrm{Glo}}\colon \mathrm{Shv}(\mathrm{SepStk}) \to \mathrm{GloSpc}$ *induces an equivalence*

$$\mathrm{Cat}(\mathrm{GloSpc}) \xrightarrow{\simeq} \mathrm{Cat}^{\mathrm{s.htp}}(\mathrm{Shv}(\mathrm{SepStk})$$

between the ∞-category of global ∞-categories, i.e. sheaves of ∞-categories on GloSpc*, and the ∞-category of strongly homotopy invariant sheaves of ∞-categories on* SepStk.

The main examples of sheaves of ∞-categories on SepStk studied in Part II are the assignments $\mathcal{X} \mapsto \mathrm{H}(\mathcal{X})$ and $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ of *genuine sheaves of spaces/spectra*. However, just like the ∞-category $\mathrm{Shv}(M)$ of sheaves on a smooth manifold M does not solely depend on the homotopy type of M, also the ∞-categories $\mathrm{H}(\mathcal{X})$ and $\mathrm{SH}(\mathcal{X})$ depend on more than just the underlying global homotopy type of the differentiable stack $\mathcal{X}$, and thus these assignments are not strongly homotopy invariant. Nevertheless, we expect that, just like in the situation for smooth manifolds, one can define subcategories

$$\mathrm{H}^{\mathrm{Loc}}(\mathcal{X}) \subseteq \mathrm{H}(\mathcal{X}) \qquad \text{and} \qquad \mathrm{SH}^{\mathrm{Loc}}(\mathcal{X}) \subseteq \mathrm{SH}(\mathcal{X})$$

of *locally constant* genuine sheaves which *do* satisfy strong homotopy invariance. The following definition is an attempt of defining such notion of locally constant genuine sheaves:

Predefinition 2.6 (Locally constant objects)**.** Let $\mathcal{C}\colon \mathrm{SepStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be a sheaf of ∞-categories on SepStk which is homotopy invariant, in the sense of Definition II.4.5.17. Let $\mathcal{X}$ be a separated differentiable stack.

- Assume $\mathcal{X}$ comes equipped with a map of stacks $p\colon \mathcal{X} \to \mathbb{B}G$ for some compact Lie group G. An object $A \in \mathcal{C}(\mathcal{X})$ is called *G-constant* if it is in the essential image of the functor $p^*\colon \mathcal{C}(\mathbb{B}G) \to \mathcal{C}(\mathcal{X})$.

- An object $A \in \mathcal{C}(\mathcal{X})$ is called *locally constant* if for every point $x \in \mathcal{X}$ admits an open neighborhood $\mathcal{U} \subseteq \mathcal{X}$ equipped with a map $\mathcal{U} \to \mathbb{B}G$ for some compact Lie group G such that $A|_{\mathcal{U}}$ is G-constant.

We let $\mathcal{C}^{\mathrm{Loc}}(\mathcal{X}) \subseteq \mathcal{C}(\mathcal{X})$ denote the full subcategory spanned by the locally constant objects.

The reason for calling this a 'predefinition' is to emphasize that we view this as a *suggestion* for a definition, but that more work should be done to see whether this definition is actually well-behaved. The idea is that every separated stack $\mathcal{X}$ is locally of the form $V//G$ for some representation V of a compact Lie group G, and the locally constant objects over $V//G$ should all be pulled back from $\mathbb{B}G$. Informally speaking: "locally constant objects are locally captured by their isotropy".

Example 2.7. Vector bundles are locally constant and thus so are their associated sphere bundles. It thus follows from Corollary II.6.2.11 that the dualizing sheaf $\omega_f \in \mathrm{SH}(\mathcal{Y})$ of a representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ is locally constant.

We leave to the reader the proof that the pullback functors $f^*\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{Y})$ preserves locally constant objects for every morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of separated differentiable stacks, so that $\mathcal{C}^{\mathrm{Loc}}$ defines a subfunctor

$$\mathcal{C}^{\mathrm{Loc}}\colon \mathrm{SepStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

of $\mathcal{C}$. Since being a locally constant object is clearly a local condition, this subfunctor $\mathcal{C}^{\mathrm{Loc}}$ is again a sheaf of ∞-categories on SepStk. We conjecture this sheaf is strongly homotopy invariant:

Conjecture 2.8. *Let $\mathcal{C}$ be a homotopy invariant sheaf of ∞-categories on* SepStk. *Then the subsheaf $\mathcal{C}^{\mathrm{Loc}}$ is strongly homotopy invariant.*

This conjecture seems plausible: as the pullback functor $\mathrm{pr}^*\colon \mathcal{C}^{\mathrm{Loc}}(\mathcal{X}) \to \mathcal{C}^{\mathrm{Loc}}(\mathcal{X} \times \mathbb{R})$ is fully faithful by assumption, it will suffice to show that any locally constant object in $\mathcal{C}(\mathcal{X} \times \mathbb{R})$ is pulled back from $\mathcal{X}$. Using a technique like the one from [Lur17, Proposition A.2.1], one should be able to reduce to the case where the object is in fact *constant*, i.e. obtained via pullback along a map $\mathcal{X} \times \mathbb{R} \to \mathbb{B}G$ for some G. But this map must factor through $\mathcal{X}$ by strong homotopy invariance of principal G-bundles, giving the claim.

Assuming both Conjecture 2.2 and Conjecture 2.8 are true, the locally constant objects in C give rise to a global ∞-category $C^{\mathrm{Loc}} \colon \mathrm{GloSpc}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. In particular, taking $C = \mathrm{SH}$ would provide a global ∞-category $\mathrm{SH}^{\mathrm{Loc}}$ of genuine sheaves of spectra. At the level of objects, $\mathrm{SH}^{\mathrm{Loc}}$ agrees with the global ∞-category $\underline{\mathrm{Sp}}_\bullet$ of genuine spectra, in light of the equivalences $\mathrm{SH}(\mathbb{B}G) \simeq \mathrm{Sp}_G$ from Proposition 4.4.17. One can similarly check that they agree at the level of morphism. This leads to the following conjecture:

Conjecture 2.9. *There is an equivalence of global ∞-categories $\mathrm{SH}^{\mathrm{Loc}} \simeq \underline{\mathrm{Sp}}_\bullet$ between the global ∞-category of locally constant genuine sheaves and the global ∞-category $\mathbb{B}G \mapsto \mathrm{Sp}_G$ of genuine equivariant spectra.*

In fact, we expect this equivalence to hold as *presentably symmetric monoidal* global ∞-categories.

2.3 From Poincaré duality to twisted ambidexterity

We will now explain the conjectural connection between the (homotopical) twisted ambidexterity equivalences obtained in Part I and the (geometric) Poincaré duality equivalences obtained in Part II.

Recall from Theorem II.6.1.7 that for every proper representable submersion $f \colon \mathcal{Y} \to \mathcal{X}$, there is a *relative Poincaré duality* equivalence $f_\sharp(-) \simeq f_*(- \otimes \omega_f)$ of functors $\mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$. For an arbitrary representable submersion f, the functors $f_\sharp$ and f_* do not in general restrict to locally constant genuine sheaves. However, they do if we assume that f is *locally trivial*, in the following sense:

Predefinition 2.10 (Locally trivial submersions)**.** Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a morphism of separated differentiable stacks.

- Assume $\mathcal{X} = N//G$ is a global quotient stack of some smooth G-manifold N, where G is a compact Lie group. We say that f is *G-trivial* if there exists another G-manifold M such that $\mathcal{Y} \to \mathcal{X}$ is equivalent to the projection map $(M \times N)//G \to N//G$.

- For general $\mathcal{X}$, we say that f is *locally trivial* if around every point $x \in \mathcal{X}$ there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{X}$ of x of the form $\mathcal{U} \cong N//G$ for some compact Lie group G and a smooth G-manifold N such that the restriction $f|_{\mathcal{U}} \colon \mathcal{Y}|_{\mathcal{U}} \to \mathcal{U}$ is G-trivial.

Since every G-trivial morphism is a representable submersion, it follows that every locally trivial morphism is a representable submersion.

Conjecture 2.11. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a locally trivial submersion. Then the functor $f_\sharp\colon \mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$ preserves locally constant objects. If f is proper, then also the functor $f_*\colon \mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$ preserves locally constant objects.*

Proof sketch. By smooth/proper base change, the statement can be checked locally in $\mathcal{X}$, so that we may assume that f is given as the projection map $(M \times N)//G \to N//G$ for some compact Lie group G and smooth G-manifolds M and N. By [Die87, Theorem 5.6] we may further assume $N = G \times_H V$ for a closed subgroup $H \leqslant G$ and an H-representation V. Replacing G by H we may in fact assume that $N = V$ is a G-representation. It follows from Conjecture 2.8 that every locally constant object A in $\mathrm{SH}((M \times V)//G)$ is pulled back from $\mathrm{SH}(M//G)$ and it thus follows from smooth and proper base change that the objects $f_\sharp A$ and $f_* A$ in $\mathcal{C}(V//G)$ are pulled back from $\mathcal{C}(\mathbb{B}G)$, showing that they are locally constant. □

A stacky version of the classical Ehresmann's theorem says that any proper representable submersion is locally trivial:

Proposition 2.12 (Ehresmann's theorem for differentiable stacks, Hoyo and Fernandes [HF19, Corollary 6.4.5])**.** *Any proper representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ is locally trivial.* □

In particular, given a proper representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ the relative Poincaré duality equivalence $f_\sharp(-) \simeq f_*(- \otimes \omega_f)$ restricts to an equivalence of functors $\mathrm{SH}^{\mathrm{Loc}}(\mathcal{Y}) \to \mathrm{SH}^{\mathrm{Loc}}(\mathcal{X})$. This equivalence only depends on the under lying map $\Pi_{\mathrm{Glo}}(f)\colon \Pi_{\mathrm{Glo}}(\mathcal{Y}) \to \Pi_{\mathrm{Glo}}(\mathcal{X})$ of orbispaces. We expect that the resulting equivalence is essentially the twisted ambidexterity equivalence for orbispectra from I.4.20:

Conjecture 2.13. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a proper representable submersion between separated differentiable stacks.*

(1) Under the conjectured equivalence $\mathrm{SH}^{\mathrm{Loc}}(\mathcal{Y}) \simeq \mathrm{OrbSp}(\Pi_{\mathrm{Glo}}(\mathcal{Y}))$ from Conjecture 2.9, the dualizing sheaf $\omega_f \in \mathrm{SH}^{\mathrm{Loc}}(\mathcal{Y})$ from Definition II.6.1.3 is inverse to the dualizing object $D_f \in \mathrm{OrbSp}(\Pi_{\mathrm{Glo}}(\mathcal{Y}))$ from Definition I.3.1:

$$D_f \simeq \omega_f^{-1} \quad \in \quad \mathrm{SH}^{\mathrm{Loc}}(\mathcal{Y}).$$

(2) Under the conjectured equivalences from Conjecture 2.9, the relative Poincaré duality equivalence

$$\mathfrak{p}_f\colon f_\sharp(- \otimes \omega_f^{-1}) \simeq f_*(-)\colon \mathrm{SH}^{\mathrm{Loc}}(\mathcal{Y}) \to \mathrm{SH}^{\mathrm{Loc}}(\mathcal{X})$$

agrees with the twisted ambidexterity equivalence

$$\mathrm{Nm}_f\colon f_!(-\otimes D_f) \simeq f_*(-)\colon \mathrm{OrbSp}(\Pi_{\mathrm{Glo}}(\mathcal{Y})) \to \mathrm{OrbSp}(\Pi_{\mathrm{Glo}}(\mathcal{X})).$$

Proof sketch. In the proof of relative Poincaré duality, we have produced a map $\eta\colon \mathbb{1}_{\mathcal{X}} \to f_\sharp S^{-T_f}$ which induced the unit transformation $\mathrm{id}_{C(\mathcal{X})} \to f_\sharp(f^*(-)\otimes S^{-T_f})$ of an adjunction $f^* \dashv f_\sharp(-\otimes S^{-T_f})$. Considering the global ∞-category $C = \mathrm{SH}^{\mathrm{Loc}}$, it then follows from Lemma 3.26 that the map η exhibits S^{-T_f} as a left Costenoble-Waner dual, in the sense of Definition 3.24. Due to the formulation of twisted ambidexterity in terms of Costenoble-Waner duality from Proposition 3.28, it follows that the dualizing object D_f is equivalent to S^{-T_f}, which is the inverse of the dualizing sheaf ω_f, and the resulting twisted norm map $\mathrm{Nm}_f\colon f_\sharp(-\otimes S^{-T_f}) \to f_*$ agrees with the relative Poincaré duality equivalence $\mathfrak{p}_f$. □

III.3 A six-functor formalism of genuine sheaves on differentiable stacks

In this section, we discuss the notion of a *six-functor formalism* on the site SepStk of separated differentiable stacks. We speculate that the assignment $\mathcal{X} \mapsto \mathrm{SH}(\mathcal{X})$ should enhance to such a six-functor formalism, assigning an exceptional pushforward functor $f_!\colon \mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$ to every representable morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of separated differentiable stacks. We illustrate this by giving some partial results in this direction.

3.1 Six-functor formalisms

We start by recalling Mann's definition of a six-functor formalism.

Definition 3.1 ([Man22, Definition A.5.1])**.** A *geometric setup* is a pair $(\mathcal{B}, E)$ where $\mathcal{B}$ is an ∞-category and $E \subseteq \mathcal{B}$ is a subcategory satisfying the following conditions:

(i) The subcategory E contains all equivalences;

(ii) Pullbacks of morphisms in E exist in $\mathcal{B}$ and are again in E.

Recall that for any geometric setup $(\mathcal{B}, E)$, there exists an ∞-category $\mathrm{Span}(\mathcal{B}, E)$, called the *span category of* $(\mathcal{B}, E)$ or the *∞-category of correspondences in* $(\mathcal{B}, E)$, which may informally be described as follows:

- The objects of $\mathrm{Span}(\mathcal{B}, E)$ are the objects of $\mathcal{B}$;

- A morphism from X to Y in $\mathrm{Span}(\mathcal{B},E)$ is a triple (U,f,e), where $U\in\mathcal{B}$ and f and e are maps

$$\begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

such that $e\in E$. We refer to such triple as a *span* from X to Y.

- The identity span on X is given by the triple $(X,\mathrm{id}_X,\mathrm{id}_X)$.

- Composition of spans is given by pullback in $\mathcal{B}$, as indicated by the following diagram:

$$\begin{tikzcd} & & W \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[dd, phantom, very near start, "\lrcorner"] \arrow[ddll, bend right, "fu"'] \arrow[ddrr, bend left, "e'v"] & & \\ & U \arrow[dl, "f"'] \arrow[dr, "e"] & & V \arrow[dl, "f'"'] \arrow[dr, "e'"] & \\ X & & Y & & Z. \end{tikzcd}$$

Remark 3.2. If the ∞-category $\mathcal{B}$ is modeled by a quasicategory, the span category $\mathrm{Span}(\mathcal{B},E)$ may be modeled by the quasicategory whose n-simplices are those maps of quasicategories $C(\Delta^n)\to\mathcal{B}$ which send vertical morphisms to morphisms in E and send exact squares in $C(\Delta^n)$ to pullback squares in $\mathcal{B}$. Here $C(\Delta^n)\subseteq\Delta^n\times(\Delta^n)^{\mathrm{op}}$ is the full subcategory spanned by the pairs $([i],[j])$ with $i\le j$, a morphism in $C(\Delta^n)$ is called *vertical* if its projection to the second component is an equivalence, and a square in $C(\Delta^n)$ is called *exact* if it is both a pullback and a pushout square.

The span-category $\mathrm{Span}(\mathcal{B},E)$ is symmetric monoidal, where the symmetric monoidal structure is inherited from the cartesian product in $\mathcal{B}$.

Definition 3.3 ([Man22, Definition A.5.6])**.** Let $(\mathcal{B},E)$ be a geometric setup. A *pre-six-functor formalism* on $(\mathcal{B},E)$ is a lax symmetric monoidal functor

$$\mathcal{D}\colon\ \mathrm{Span}(\mathcal{B},E)\to\mathrm{Cat}_\infty,$$

where Cat_∞ is equipped with the cartesian symmetric monoidal structure.

Let us unpack some of the information contained in a pre-six-functor formalism $\mathcal{D}$ on $(\mathcal{B},E)$:

(a) (Symmetric monoidality) By restricting $\mathcal{D}$ along the symmetric monoidal inclusion $\mathcal{B}^{\mathrm{op}}\hookrightarrow\mathrm{Span}(\mathcal{B},E)$, we obtain a lax symmetric monoidal functor $\mathcal{D}^*\colon\mathcal{B}^{\mathrm{op}}\to\mathrm{Cat}_\infty$.

Since $\mathcal{B}^{\mathrm{op}}$ carries the cocartesian symmetric monoidal structure, this corresponds to a functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ by [Lur17, Theorem 2.4.3.18]. In particular, the ∞-category $\mathcal{D}(\mathcal{X}) = \mathcal{D}^*(\mathcal{X})$ comes equipped with a symmetric monoidal structure $(\mathcal{D}(\mathcal{X}), \otimes_{\mathcal{X}}, \mathbb{1}_{\mathcal{X}})$ for every object $\mathcal{X} \in \mathcal{B}$;

(b) (Pullback functors) For every morphism $f \colon \mathcal{Y} \to \mathcal{X}$ in $\mathcal{B}$, we obtain a symmetric monoidal functor

$$f^* := \mathcal{D}^*(f) \colon \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y}),$$

which we will call the *pullback functor*.

(c) (Exceptional pushforward functors) Restricting $\mathcal{D}$ to the subcategory $E \subseteq \mathrm{Span}(\mathcal{B}, E)$ produces a functor $\mathcal{D}_! \colon E \to \mathrm{Cat}_\infty$. For a morphism $f \colon \mathcal{Y} \to \mathcal{X}$ in E, we will write

$$f_! := \mathcal{D}(f)_! \colon \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$$

for the resulting functor, and refer to $f_!$ as the *exceptional pushforward functor*.

(d) (Base change) By unwinding the functoriality of $\mathcal{D}$ on compositions of spans, it follows that the exceptional pushforward functors satisfy base change: for every pullback square

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X}' \arrow[r, "g"] & \mathcal{X} \end{tikzcd}$$

in $\mathcal{B}$, there is a natural equivalence

$$g^* f_! \simeq f'_! h^*$$

of functors $\mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X}')$.

(e) (Projection formula) By unwinding the lax symmetric monoidality of $\mathcal{D}$, one observes that for every morphism $f \colon \mathcal{Y} \to \mathcal{X}$ in E and every pair of objects $A \in \mathcal{D}(\mathcal{Y})$ and $B \in \mathcal{D}(\mathcal{X})$, there is a natural equivalence

$$f_!(A \otimes f^* B) \simeq f_! A \otimes B \quad \in \quad \mathcal{C}(\mathcal{X}).$$

Definition 3.4 (Six-functor formalism, [Man22, Definition A.5.6])**.** Let $(\mathcal{B}, E)$ be a geometric setup. A pre-six-functor formalism $\mathcal{D} \colon \mathrm{Span}(\mathcal{B}, E) \to \mathrm{Cat}_\infty$ on $(\mathcal{B}, E)$ is called a *six-functor formalism* if it satisfies the following three conditions:

(1) For every object $\mathcal{X} \in \mathcal{B}$, the symmetric monoidal ∞-category $\mathcal{D}(\mathcal{X})$ is closed monoidal: there exists an internal hom functor $\underline{\mathrm{Hom}}(-,-) \colon \mathcal{D}(\mathcal{X})^{\mathrm{op}} \times \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X})$.

(2) For every morphism $f: \mathcal{Y} \to \mathcal{X}$ in $\mathcal{B}$, the pullback functor f^* admits a right adjoint $f_*: \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$, called the *pushforward functor*.

(3) For every morphism $f: \mathcal{Y} \to \mathcal{X}$ in E, the exceptional pushforward functor $f_!$ admits a right adjoint $f^!: \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$, called the *exceptional pullback functor*.

Definition 3.5. Let $(\mathcal{B}, E)$ be a geometric setup. A *presentable six-functor formalism* is a lax symmetric monoidal functor $\mathcal{D}: \operatorname{Span}(\mathcal{B}, E) \to \operatorname{Pr}^{\mathrm{L}}$, where $\operatorname{Pr}^{\mathrm{L}}$ is equipped with the Lurie tensor product.

Remark 3.6. Every presentable six-functor formalism gives rise to a pre-six-functor formalism by postcomposing with the lax symmetric monoidal forgetful functor $\operatorname{Pr}^{\mathrm{L}} \to \operatorname{Cat}_\infty$, and it follows from the adjoint functor theorem that this pre-six-functor formalism is in fact a six-functor formalism. A six-functor formalism $\mathcal{D}: \operatorname{Span}(\mathcal{B}, E) \to \operatorname{Cat}_\infty$ is obtained in this way if and only if the ∞-category $\mathcal{D}(\mathcal{X})$ is presentable for all $\mathcal{X} \in \mathcal{B}$, see [Lur17, Remark 4.8.1.9].

3.2 A six-functor formalism of genuine sheaves of spectra

We expect that the functor $\mathrm{SH}: \operatorname{Shv}(\mathrm{SepStk})^{\mathrm{op}} \to \operatorname{CAlg}(\operatorname{Pr}^{\mathrm{L}})$ of genuine sheaves of spectra should extend to a six-functor formalism on the ∞-category $\operatorname{Shv}(\mathrm{SepStk})$, where E consists of the representable morphisms from Definition II.6.5.6:

Conjecture 3.7. *Set* $\mathcal{B} := \operatorname{Shv}(\mathrm{SepStk})$*, and let* $\mathcal{C}: \mathcal{B}^{\mathrm{op}} \to \operatorname{CAlg}(\operatorname{Pr}^{\mathrm{L}})$ *be a pullback formalism on* SepStk *satisfying the Voevodsky conditions: homotopy invariance, genuine stability and the localization axiom. Let* $E \subseteq \mathcal{B}$ *denote the wide subcategory spanned by the representable morphisms. Then* $\mathcal{C}$ *extends to a presentable six-functor formalism* $\mathcal{C}^!_*: \operatorname{Span}(\mathcal{B}, E) \to \operatorname{Pr}^{\mathrm{L}}$*, satisfying the following conditions:*

(1) For every open embedding $j: \mathcal{U} \hookrightarrow \mathcal{X}$ *of separated differentiable stacks, there is an equivalence* $j_! \simeq j_\sharp$*;*

(2) For every proper morphism $p: \mathcal{Y} \hookrightarrow \mathcal{X}$ *of separated differentiable stacks, there is an equivalence* $p_! \simeq p_*$*.*

We currently do not have the technology to prove this conjecture in the stated generality. Nevertheless, we can obtain some partial results using a general existence result for six-functor formalisms due to Mann [Man22, Proposition A.5.10]. The idea of this result is that one starts off with a functor $\mathcal{D}: \mathcal{B}^{\mathrm{op}} \to \operatorname{CAlg}(\operatorname{Cat}_\infty)$ such that $\mathcal{D}$ satisfies the analogues of smooth base change, smooth projection formula, proper base change, proper projection formula and smooth-proper base change. These conditions are formulated with respect to two classes of morphisms I and P of $\mathcal{B}$, thought of as the 'open immersions'[2] and the 'proper morphisms' respectively; I and P need to satisfy various compatibility conditions, among which is the condition that every morphism of E factors as an open immersion followed by a proper morphism. Under these assumptions, Mann's result says that $\mathcal{D}$ extends to a pre-six-functor formalism $\mathcal{D}\colon \mathrm{Span}(\mathcal{B}, E) \to \mathrm{Cat}_\infty$. The condition that this an actual six-functor formalism translates to conditions on the original functor $\mathcal{D}$.

While Mann's result does not immediately seem to apply in the case of Conjecture 3.7, it can be used to prove a weaker result where we obtain exceptional pushforwards $f_!$ for *compactifiable* morphisms. In the remainder of this section, we fix a separated differentiable stack $\mathcal{S}$ which serves as our base stack.

Definition 3.8 (Separated $\mathcal{S}$-stacks)**.** We let $\mathrm{SepStk}_{\mathcal{S}}$ be the full subcategory of $\mathrm{DiffStk}_{/\mathcal{S}}$ spanned by the separated morphisms $\mathcal{X} \to \mathcal{S}$ of differentiable stacks. We will refer to such morphism as a *separated $\mathcal{S}$-stack*. We will often drop the structure map to $\mathcal{S}$ from the notation.

The ∞-category $\mathrm{SepStk}_{\mathcal{S}}$ inherits a Grothendieck topology from SepStk given by the open coverings.

Definition 3.9 (Compactifiable morphisms)**.** A morphism $f\colon \mathfrak{Y} \to \mathfrak{X}$ in $\mathrm{Shv}(\mathrm{SepStk}_{\mathcal{S}})$ is called *$\mathcal{S}$-compactifiable* if there exists a proper morphism $\mathcal{P} \to \mathcal{S}$ of separated differentiable stacks and a morphism $j\colon \mathfrak{Y} \to \mathcal{P}$ in $\mathrm{Shv}(\mathrm{SepStk}_{\mathcal{S}})$ such that the map $(f, j)\colon \mathfrak{Y} \to \mathfrak{X} \times_{\mathcal{S}} \mathcal{P}$ is an open embedding. In particular, f factors as an open embedding (f, j) followed by a proper morphism $\mathrm{pr}_1\colon \mathfrak{X} \times_{\mathcal{S}} \mathcal{P} \to \mathfrak{X}$.

We say that f is *source-locally $\mathcal{S}$-compactifiable* if there exists an open cover $\{j_\alpha\colon \mathfrak{U}_\alpha \hookrightarrow \mathfrak{Y}\}_{\alpha \in I}$ in $\mathrm{Shv}(\mathrm{SepStk}_{\mathcal{S}})$ such that the composite $\mathfrak{U}_\alpha \hookrightarrow \mathfrak{Y} \to \mathfrak{X}$ is $\mathcal{S}$-compactifiable for every α.

It is clear from the definition that both the $\mathcal{S}$-compactifiable and the source-locally $\mathcal{S}$-compactifiable morphisms define geometric setups on $\mathrm{Shv}(\mathrm{SepStk}_{\mathcal{S}})$.

Proposition 3.10. *For a separated differentiable stack $\mathcal{S}$, consider the ∞-category $\mathcal{B} = \mathrm{Shv}(\mathrm{SepStk}_{\mathcal{S}})$. Let $C\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a pullback formalism satisfying the Voevodsky conditions: homotopy invariance, genuine stability and the localization axiom. Let $E \subseteq \mathcal{B}$ denote the wide subcategory spanned by all the source-locally $\mathcal{S}$-compactifiable morphisms.*

[2] Mann refers to them as the 'local isomorphisms'

Then $\mathcal{C}$ extends to a presentable six-functor formalism $\mathcal{C}^!_*\colon \operatorname{Span}(\mathcal{B}, E) \to \mathrm{Pr}^{\mathrm{L}}$, satisfying the following conditions:

(1) For every open embedding $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ of separated $\mathcal{S}$stacks, there is an equivalence $j_! \simeq j_\sharp$;

(2) For every proper morphism $p\colon \mathcal{Y} \hookrightarrow \mathcal{X}$ of separated $\mathcal{S}$-stacks, there is an equivalence $p_! \simeq p_*$.

Proof. We first show the weaker claim that $\mathcal{C}$ extends to a presentable six-functor formalism $\operatorname{Span}(\mathcal{B}, E') \to \mathrm{Pr}^{\mathrm{L}}$, where E' is the wide subcategory of $\mathcal{B} = \operatorname{Shv}(\operatorname{SepStk}_{\mathcal{S}})$ spanned by the $\mathcal{S}$-compactifiable morphisms. For this, we apply [Man22, Proposition A.5.10]. Let I denote the collection of open embeddings in $\operatorname{Shv}(\operatorname{SepStk}_{\mathcal{S}})$ and let P denote the collection of proper morphisms in $\operatorname{Shv}(\operatorname{SepStk}_{\mathcal{S}})$. These two collections of morphisms satisfy the conditions of [Man22, Definition A.5.9]:

(a) By definition, every morphism in E' is of the form $f = p \circ i$ for some $i \in I$ and $p \in P$;

(b) The morphisms in I and P are 0-truncated;

(c) The collections of morphisms I and P contain all equivalences and are closed under composition and base change in $\operatorname{Shv}(\operatorname{SepStk}_{\mathcal{S}})$.

(d) The collections of morphisms I and P are left-cancellable: given morphisms $f\colon \mathcal{Y} \to \mathcal{X}$ and $g\colon \mathcal{Z} \to \mathcal{Y}$ in $\operatorname{Shv}(\operatorname{SepStk}_{\mathcal{S}})$ such that $f \in I$ and $fg \in I$ (resp. $f \in P$ and $fg \in P$) then also $g \in I$ (resp. $g \in P$).

It then remains to check the conditions of [Man22, Proposition A.5.10]. This is precisely the content of Proposition 6.5.8.

This gives the extension of $\mathcal{C}$ to $\operatorname{Span}(\mathcal{B}, E') \to \mathrm{Pr}^{\mathrm{L}}$. The further extension to the source-locally compactifiable morphisms is then an immediate consequence of [Man22, Proposition A.5.14], since by assumption $\mathcal{C}$ satisfies descent with respect to open covers of differentiable stacks. □

III.4 Proper genuine sheaves

When defining genuine sheaves of spectra on differentiable stacks in Chapter II.4, we restricted our attention to the *separated* differentiable stacks. Every such stack is locally equivalent to a quotient stack $M//G$ of a compact Lie group G and a smooth G-manifold

M, and on such stacks there is a good supply of vector bundles whose associated sphere bundles can be inverted. In stable equivariant homotopy theory one usually restricts to *compact* Lie groups G rather than arbitrary Lie groups for essentially the same reason.

If the Lie group G is non-compact, one can still obtain a well-behaved notion of G-spectra by restricting attention only to the *compact subgroups* of G. This leads to the notion of *proper genuine G-spectra*, introduced and studied by Degrijse et al. [Deg+19]. As made precise in Section I.4.4, one can rephrase this definition in terms of parametrized orbispectra: one associates to G an orbispace $\mathbb{B}G$ described informally by 'probing' G by all compact Lie groups K, and the ∞-category of proper genuine G-spectra is equivalent to the ∞-category of orbispectra parametrized over the orbispace $\mathbb{B}G$.

The goal of this section is to indicate how one probing an arbitrary differentiable stack $\mathcal{X}$ by separated differentiable stacks similarly similarly leads to an ∞-category $\mathrm{SH}^{\mathrm{pr}}(\mathcal{X})$ of *proper genuine sheaves of spectra.*

Definition 4.1. We define a functor $(-)^{\mathrm{pr}}\colon \mathrm{Shv}(\mathrm{Diff}) \to \mathrm{Shv}(\mathrm{SepStk})$ as follows: for a stack $\mathcal{X}$ on Diff, we define $\mathcal{X}^{\mathrm{pr}}\colon \mathrm{SepStk}^{\mathrm{op}} \to \mathrm{Spc}$ by $\mathcal{X}^{\mathrm{pr}}(\mathcal{Y}) := \mathrm{Hom}_{\mathrm{Shv}(\mathrm{Diff})}(\mathcal{Y}, \mathcal{X})$ for $\mathcal{Y} \in \mathrm{SepStk} \subseteq \mathrm{Shv}(\mathrm{Diff})$. Since every open cover $\bigsqcup_\alpha \mathcal{U}_\alpha \to \mathcal{Y}$ in SepStk is in particular an effective epimorphism in $\mathrm{Shv}(\mathrm{Diff})$, $\mathcal{X}^{\mathrm{pr}}$ is indeed a sheaf.

It is immediate that the functor $(-)^{\mathrm{pr}}$ preserves limits and that composing $(-)^{\mathrm{pr}}$ with the inclusion $\mathrm{SepStk} \hookrightarrow \mathrm{Shv}(\mathrm{Diff})$ results in the Yoneda embedding for SepStk.

Predefinition 4.2 (Proper genuine sheaves of spectra)**.** We define the functor

$$\mathrm{SH}^{\mathrm{pr}}\colon \mathrm{Shv}(\mathrm{Diff})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

by $\mathrm{SH}^{\mathrm{pr}}(\mathcal{X}) := \mathrm{SH}(\mathcal{X}^{\mathrm{pr}})$. Given a sheaf $\mathcal{X} \in \mathrm{Shv}(\mathrm{Diff})$, we refer to $\mathrm{SH}^{\mathrm{pr}}(\mathcal{X})$ as the ∞-category of *proper genuine sheaves of spectra on $\mathcal{X}$.*

Warning 4.3. The functor $\mathrm{SH}^{\mathrm{pr}}$ is *not* a sheaf of ∞-categories on $\mathrm{Shv}(\mathrm{Diff})$.

Note that if $f\colon \mathcal{X}' \to \mathcal{X}$ is a morphism in $\mathrm{Shv}(\mathrm{Diff})$ which is a representable submersion, an open/closed embedding or a a proper morphism, in the sense of Definition 2.4.2, then the map $f^{\mathrm{pr}}\colon \mathcal{X}'^{\mathrm{pr}} \to \mathcal{X}^{\mathrm{pr}}$ in $\mathrm{Shv}(\mathrm{SepStk})$ has the analogous property, now interpreted in the sense of Definition 6.5.6. It thus follows from Proposition 6.5.8 that the ∞-categories $\mathrm{SH}^{\mathrm{pr}}(\mathcal{X})$ inherit all of the functoriality properties from $\mathrm{SH}(\mathcal{Y})$: left adjoints $f_\sharp$ for representable submersions satisfying smooth base change and the smooth projection formula, right adjoints p_* for proper morphisms satisfying proper base change, the proper projection formula and smooth-proper base change. In particular:

Proposition 4.4 (Relative Poincaré duality for proper genuine sheaves)**.** *For every proper representable submersion $f\colon \mathcal{Y} \to \mathcal{X}$ of stacks on* Diff*, there is an equivalence*

$$\mathfrak{p}_f\colon f_\sharp(-) \xrightarrow{\sim} f_*(-\otimes\omega_f)$$

of functors $\mathrm{SH}^{\mathrm{pr}}(\mathcal{Y}) \to \mathrm{SH}^{\mathrm{pr}}(\mathfrak{X})$.

We expect that the aforementioned functorial properties of the ∞-categories $\mathrm{SH}^{\mathrm{pr}}(\mathcal{X})$ should be part of a six-functor formalism:

Conjecture 4.5. *The functor* $\mathrm{SH}^{\mathrm{pr}}\colon \mathrm{Shv}(\mathrm{Diff})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ *extends to a six-functor formalism* $\mathrm{Span}(\mathrm{Shv}(\mathrm{Diff}), E) \to \mathrm{Pr}^{\mathrm{L}}$*, where E consists of the representable morphisms in* $\mathrm{Shv}(\mathrm{Diff})$.

Just like genuine sheaves of spectra on the classifying stack $\mathbb{B}G$ of a *compact* Lie group G recover genuine G-spectra, we expect the proper genuine sheaves of spectra on the classifying stack of an *arbitrary* Lie group to recover *proper* genuine G-spectra:

Conjecture 4.6. *For a Lie group G, there is an equivalence of ∞-categories* $\mathrm{SH}^{\mathrm{pr}}(\mathbb{B}G) \simeq \mathrm{Sp}_G^{\mathrm{pr}}$.

The idea is that both ∞-categories are a limit of the ∞-categories $\mathrm{SH}(\mathbb{B}K)$ as K ranges over all compact subgroups of G. For $\mathrm{Sp}_G^{\mathrm{pr}}$ this holds by [LNP22, Theorem 12.11], using the equivalences $\mathrm{SH}(\mathbb{B}K) \simeq \mathrm{Sp}_K$ from Proposition 4.4.17. For $\mathrm{SH}^{\mathrm{pr}}(\mathbb{B}G)$ one would a priori need to take the limit of the ∞-categories $\mathrm{SH}(\mathcal{X})$ over the larger indexing diagram of separated differentiable stacks $\mathcal{X}$ equipped with a morphism $\mathcal{X} \to \mathbb{B}G$ of stacks. However, since any separated stack is locally a quotient stack $M//G$ for some compact Lie group K and some smooth K-manifold M, and since every map $M//K \to \mathbb{B}G$ factors through $\mathbb{B}K$, we may instead form the limit over maps $\mathbb{B}K \to \mathbb{B}G$. Since every group homomorphism $K \to G$ factors as a surjection followed by an injection, it will suffice to take the limit over those maps $\mathbb{B}K \to \mathbb{B}G$ corresponding to compact subgroups $K \leqslant G$.

Appendix

A Symmetric monoidal unstraightening

Let $\mathcal{B}$ be an ∞-topos and let $\mathcal{O}^{\otimes}$ be an ∞-operad. In this appendix, we recall unstraightening techniques from Lurie [Lur17] and Drew and Gallauer [DG22, Appendix A] to describe $\mathcal{O}$-algebras in the ∞-category $\mathrm{Cat}(\mathcal{B})$ of $\mathcal{B}$-categories in terms of suitable cocartesian fibrations.

We start by recalling the situation for $\mathcal{O} = \mathcal{C}\mathrm{omm}^{\otimes}$ from [DG22, Appendix A].

Definition A.1. Let $\mathcal{D}^{\otimes}$ be a symmetric monoidal ∞-category. A *$\mathcal{D}^{\otimes}$-monoidal ∞-category* is a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ equipped with a symmetric monoidal cocartesian fibration $p^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$. A *$\mathcal{D}^{\otimes}$-monoidal functor* is a symmetric monoidal functor over $\mathcal{D}^{\otimes}$ which preserves $\mathcal{D}^{\otimes}$-cocartesian edges. We let $\mathrm{Cat}^{\mathcal{D}^{\otimes}} \subseteq \mathrm{CAlg}(\mathrm{Cat}_{\infty})_{/\mathcal{D}}$ denote the (non-full) subcategory of $\mathcal{D}^{\otimes}$-monoidal ∞-categories.

Remark A.2. By [DG22, Remark A.3], a functor $p^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a $\mathcal{D}^{\otimes}$-monoidal ∞-category Definition A.1 if and only if it is a $\mathcal{D}^{\otimes}$-monoidal ∞-category in the sense of [Lur17, p. 2.1.2.13].

Proposition A.3 ([DG22, Corollary A.12])**.** *Assume $\mathcal{D}^{\otimes}$ is a cocartesian monoidal structure. Straightening / unstraightening induces an equivalence*

$$\mathrm{Cat}_{\infty}^{\mathcal{D}^{\otimes}} \simeq \mathrm{Fun}(\mathcal{D}, \mathrm{CAlg}(\mathrm{Cat}_{\infty})).$$

Let $\mathcal{B}$ be an ∞-topos, and consider the symmetric monoidal ∞-category $\mathcal{B}^{\mathrm{op},\sqcup}$, the opposite of $\mathcal{B}$ equipped with the cocartesian monoidal structure. In this case, the equivalence of Proposition A.3 may informally be described as follows. Given a functor $\mathcal{C}\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op},\sqcup}$, the resulting $\mathcal{B}^{\mathrm{op},\sqcup}$-monoidal ∞-category

$$p^{\otimes} : \mathcal{C}^{\boxtimes} \to \mathcal{B}^{\mathrm{op},\sqcup}$$

which may informally be described as follows:

- The objects of $\mathcal{C}^{\boxtimes}$ are pairs (B, X) where B is an object in $\mathcal{B}$, and X is an object in $\mathcal{C}(B)$.
- A morphism $(B, X) \to (B', X')$ in $\mathcal{C}^{\boxtimes}$ consists of a morphism $f : B' \to B$ in $\mathcal{B}$, and a morphism $f^*X \to X'$ in $\mathcal{C}(B')$.
- The tensor product of (B, X) and (B', X') is the "external product"

$$X \boxtimes X' := \mathrm{pr}_B^* X \otimes_{B \times B'} \mathrm{pr}_{B'}^* X' \quad \in \quad \mathcal{C}(B \times B'),$$

  where $\mathrm{pr}_B : B \times B' \to B$ and $\mathrm{pr}_{B'} : B \times B' \to B'$ are the canonical projections in $\mathcal{B}$.

Conversely, if $p^{\otimes} : \mathcal{C}^{\boxtimes} \to \mathcal{B}^{\mathrm{op},\sqcup}$ is a $\mathcal{B}^{\mathrm{op},\sqcup}$-monoidal ∞-category, we may straighten the underlying cocartesian fibration $p : (\mathcal{C}^{\boxtimes})_1 \to \mathcal{B}^{\mathrm{op}}$ to a functor $\mathcal{C} : \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$, sending $B \in \mathcal{B}$ to the fiber of p over B. The symmetric monoidal structure on this fiber $\mathcal{C}(B)$ may be described as follows: given $X, X' \in \mathcal{C}(B)$, their tensor product is the object $\Delta^*(X \boxtimes X')$ where Δ denotes the diagonal map $B \to B \times B$ in $\mathcal{B}$.

As there is an equivalence $\mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Cat}_{\infty})) \simeq \mathrm{CAlg}(\mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_{\infty}))$, it follows from Proposition A.3 that we may regard every symmetric monoidal $\mathcal{B}$-category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat}(\mathcal{B}))$ as a cocartesian fibration $\mathcal{C}^{\boxtimes} \to \mathcal{B}^{\mathrm{op},\sqcup}$. We will now discuss how one may describe $\mathcal{O}$-algebras in $\mathrm{Cat}(\mathcal{B})$ for an arbitrary ∞-operad $\mathcal{O}^{\otimes}$ in a similar fashion.

Definition A.4. We define the ∞-category $\mathcal{B}^{\mathrm{op},\mathcal{O}}$ via the following pullback diagram:

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op},\mathcal{O}} \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B}^{\mathrm{op},\sqcup} \arrow[d, "p"] \\
\mathcal{O}^{\otimes} \arrow[r] & \mathrm{Fin}_*.
\end{tikzcd}$$

Being the pullback of a cocartesian fibration, the map $q \colon \mathcal{B}^{\mathrm{op},\mathcal{O}} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration.

Proposition A.5. *There is an equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_{\infty})) \simeq \mathrm{Cat}_{\infty}^{\mathcal{B}^{\mathrm{op},\mathcal{O}}},$$

natural in $\mathcal{O}^{\otimes}$, which for $\mathcal{O}^{\otimes} = \mathcal{C}\mathrm{omm}^{\otimes}$ reduces to the equivalence of Proposition A.3.

Proof. This follows from the following equivalences:

$$\begin{aligned}
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_{\infty})) &\simeq \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}_{\infty})) \\
&\simeq \mathrm{Alg}_{\mathcal{B}^{\mathrm{op},\mathcal{O}}}(\mathrm{Cat}_{\infty}) && \text{[Lur17, Theorem 2.4.3.18]} \\
&\simeq \mathrm{Cat}_{\infty}^{\mathcal{B}^{\mathrm{op},\mathcal{O}}}. && \text{[Lur17, Remark 2.4.2.6]}
\end{aligned}$$

For $\mathcal{O}^{\otimes} = \mathcal{C}\mathrm{omm}^{\otimes}$, this reduces to the equivalence of Proposition A.3 given in [DG22, Corollary A.12]. ◻

We are mainly interested in the case $\mathcal{O}^{\otimes} = \mathcal{LM}^{\otimes}$ in order to describe left $\mathcal{C}$-modules in $\mathrm{Cat}(\mathcal{B})$.

Corollary A.6. *Consider $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat}(\mathcal{B}))$ and let $\mathcal{C}^{\boxtimes} \in \mathrm{Cat}_{\infty}^{\mathcal{B}^{\mathrm{op},\sqcup}}$ denote the associated $\mathcal{B}^{\mathrm{op},\sqcup}$-monoidal ∞-category. The equivalence of Proposition A.5 restricts to an equivalence*

$$\mathrm{LMod}_{\mathcal{C}}(\mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_{\infty})) \simeq \mathrm{Cat}_{\infty}^{\mathcal{B}^{\mathrm{op},\mathcal{LM}}} \times_{\mathrm{Cat}_{\infty}^{\mathcal{B}^{\mathrm{op},\sqcup}}} \{\mathcal{C}^{\boxtimes}\}.$$

The full subcategory $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}(\mathcal{B}))$ of the left-hand side is equivalent to a full subcategory of the right-hand side spanned by those $\mathcal{B}^{\mathrm{op},\mathcal{LM}}$-monoidal ∞-categories $\mathcal{M}^{\boxtimes} \to \mathcal{B}^{\mathrm{op},\mathcal{LM}}$ whose pullback along the inclusion $\mathcal{B}^{\mathrm{op}} \hookrightarrow \mathcal{B}^{\mathrm{op},\mathcal{LM}}$ is corresponds to a $\mathcal{B}$-category (i.e., its straightening $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$ preserves limits).

From this description of $\mathcal{C}$-modules in $\mathrm{Cat}(\mathcal{B})$ in terms of cocartesian fibrations, we can deduce a non-parametrized criterion for a $\mathcal{C}$-linear $\mathcal{B}$-functor $F\colon \mathcal{D} \to \mathcal{E}$ to have a $\mathcal{C}$-linear right adjoint.[1]

Proposition A.7. *Let $\mathcal{C}$ be a symmetric monoidal $\mathcal{B}$-category, let $\mathcal{D}$ and $\mathcal{E}$ be $\mathcal{B}$-categories tensored over $\mathcal{C}$, and let $F\colon \mathcal{D} \to \mathcal{E}$ be a $\mathcal{C}$-linear functor. Assume that F admits a parametrized right adjoint $G\colon \mathcal{E} \to \mathcal{D}$ satisfying the following projection formula: for every $B \in \mathcal{B}$, every $C \in \mathcal{C}(B)$ and every $E \in \mathcal{E}(B)$, the map $C \otimes_B G(E) \to G(C \otimes_B E)$ adjoint to the composite*

$$F(C \otimes_B G(E)) \simeq C \otimes_B F(G(E)) \xrightarrow{C \otimes_B \mathrm{unit}} C \otimes_B E$$

is an equivalence in $\mathcal{D}(B)$. Then the right adjoint G admits canonical $\mathcal{C}$-linear structure and the adjunction enhances to an adjunction in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}(\mathcal{B}))$.

Proof. By Corollary A.6, we may identify the $\mathcal{C}$-module structures on $\mathcal{D}$ and $\mathcal{E}$ with cocartesian fibrations over $\mathcal{B}^{\mathrm{op},\mathcal{LM}}$ whose restriction to $\mathcal{B}^{\mathrm{op},\sqcup}$ is $\mathcal{C}^{\boxtimes}$. The map F thus corresponds to a map

$$\begin{tikzcd} \mathcal{D}^{\boxtimes} \arrow[rr, "F^{\boxtimes}"] \arrow[dr] & & \mathcal{E}^{\boxtimes} \arrow[dl] \\ & \mathcal{B}^{\mathrm{op},\mathcal{LM}} & \end{tikzcd}$$

[1] In this appendix, contrary to the convention used in the body of the text, $\mathcal{C}$-linear $\mathcal{B}$-functors are not assumed to preserve colimits.

of cocartesian fibrations. If $F\colon \mathcal{D} \to \mathcal{E}$ has a parametrized right adjoint, then in particular $F^{\boxtimes}$ has fiberwise right adjoints, and by [Lur17, Proposition 7.3.2.1] these assemble into a relative right adjoint $G^{\boxtimes}\colon \mathcal{E}^{\boxtimes} \to \mathcal{D}^{\boxtimes}$ over $\mathcal{B}^{\mathrm{op},\mathcal{LM}}$ which moreover is a map of ∞-operads. In particular G preserves inert maps. Our goal is to show that G in fact preserves all cocartesian edges. By the product description of the mapping spaces in an ∞-operad, we may restrict attention to cocartesian morphism in $\mathcal{E}^{\boxtimes}$ whose target lies over $\langle 1\rangle \in \mathrm{Fin}_*$. Given such morphism, let $(B_1, \ldots, B_n) \to B$ be the image in $\mathcal{B}^{\mathrm{op},\mathcal{LM}}$.[2] This map is a composite of two maps $(B_1, \ldots, B_n) \to B_1 \times \cdots \times B_n \to B$, so we may assume without loss of generality that either $n = 1$ or that the map is cocartesian for the cocartesian fibration $\mathcal{B}^{\mathrm{op},\mathcal{LM}} \to \mathrm{Fin}_*$. When $n = 1$, this is just the condition that $G\colon \mathcal{E} \to \mathcal{D}$ is a parametrized functor, by assumption on F. So assume the map $(B_1, \ldots, B_n) \to B$ is cocartesian for $\mathcal{B}^{\mathrm{op},\mathcal{LM}} \to \mathrm{Fin}_*$. When B lies over $\mathfrak{a} \in \mathcal{LM}$, the claim is clear since F is the identity over $\mathcal{A}\mathrm{ssoc} \subseteq \mathcal{LM}$. So we may assume without loss of generality that B lies over $\mathfrak{m} \in \mathcal{LM}$, and moreover that B_n lies over $\mathfrak{m}$ while all the other B_i lie over $\mathfrak{a}$. By induction, we may assume that $n = 2$, so that the map in $\mathcal{B}^{\mathrm{op},\mathcal{LM}}$ takes the form $(B_1, B_1) \to B_1 \times B_2$ and lies over the map $(\mathfrak{a}, \mathfrak{m}) \to \mathfrak{m}$ in $\mathcal{LM}$. A cocartesian edge in $\mathcal{E}^{\boxtimes}$ over this map has the form $(C, E) \to C \boxtimes E$. The condition that the induced map $G(C, E) = (C, G(E)) \to G(C \boxtimes E)$ is again cocartesian is equivalent to the condition that the map $C \boxtimes G(E) \to G(C \boxtimes E)$ is an equivalence in $\mathcal{D}(A \times B)$. Since for $C \in \mathcal{C}(A)$, $E \in \mathcal{E}(B)$ and $E' \in \mathcal{E}(A)$ there are equivalences

$$C \boxtimes E \simeq \pi_A^* C \otimes_{A \times B} \pi_B^* E$$
$$C \otimes_A E' \simeq {\Delta_A}^*(C \boxtimes E'),$$

and since G commutes with base change, it follows that this condition is equivalent to the assumption on G. $\square$

[2] This notation is abusive: the $\mathcal{LM}^{\otimes}$-component is hidden in the notation.

B Duality in equivariant stable homotopy theory

Let G be a compact Lie group. In his PhD dissertation, Campion [Cam23] proved a universal property of the ∞-category Sp^G of genuine G-spectra: the suspension spectrum functor $\Sigma^\infty \colon \mathrm{Spc}^G_* \to \mathrm{Sp}^G$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is initial among symmetric monoidal left adjoints $F\colon \mathrm{Spc}^G_* \to \mathcal{D}$ into a stable presentably symmetric monoidal ∞-category $\mathcal{D}$ such that $F(X)$ is dualizable for every compact pointed G-space X. Since we know from [GM20, Corollary C.7] that Σ^∞ is initial among functors F that *invert representation spheres*, the crux of Campion's result is that these two conditions on $F\colon \mathrm{Spc}^G_* \to \mathcal{D}$ are in fact equivalent:

Theorem B.1 (cf. Campion [Cam23, Section 5])**.** *Let G be a compact Lie group, let $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a stable presentably symmetric monoidal ∞-category, and let $F\colon \mathrm{Spc}^G_* \to \mathcal{D}$ be a symmetric monoidal left adjoint. Then the following are equivalent:*

(1) For every orthogonal G-representation V, the functor F sends the representation sphere S^V to an invertible object of $\mathcal{D}$;

(2) For every compact pointed G-space $X \in \mathrm{Spc}^G_*$, the object $F(X)$ is dualizable in $\mathcal{D}$;

(3) For every orthogonal G-representation V, the functor F sends the representation sphere S^V to a dualizable object of $\mathcal{D}$.

For completeness, we will include a proof of this theorem. While we present both the statement and the proof slightly differently, the core ideas are taken from Campion's thesis [Cam23]. The first core idea is that certain dualizable objects are already close to being invertible:

Definition B.2 (Campion [Cam23, Definition 2.1.2])**.** Let $\mathcal{D}$ be a presentably symmetric monoidal ∞-category, let $T \in \mathcal{D}$ be an object and let $t\colon T \to T$ be an endomorphism of T. We say that T has *t-twisted trivial braiding* if the twist morphism $\sigma_{T,T}\colon T \otimes T \to T \otimes T$ is homotopic to the map $1 \otimes t\colon T \otimes T \to T \otimes T$.

Proposition B.3 (cf. Campion [Cam23, Proposition 2.3.1]). *Let $T \in \mathcal{D}$ be a dualizable object. Assume that T has t-twisted trivial braiding for some endomorphism $t\colon T \to T$. Then the morphism*

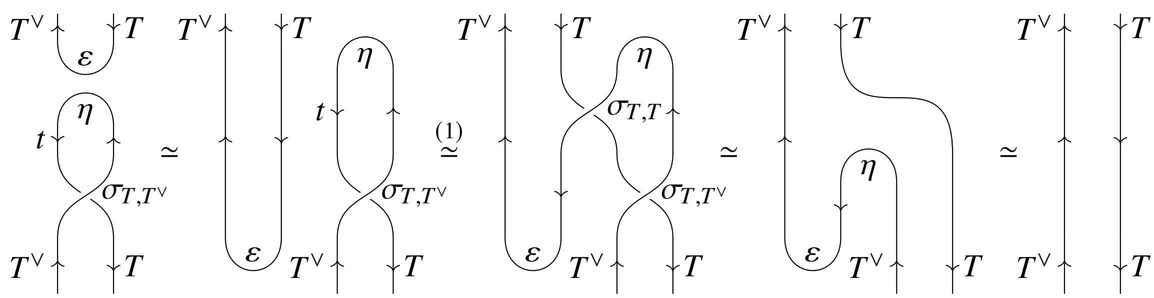
$$T \otimes T^\vee \otimes T \xrightarrow{1\otimes \mathrm{ev}} T$$

is an equivalence.

Proof. Since $1 \otimes \mathrm{ev}_T$ always admits a right-inverse given by $\mathrm{coev}_T \otimes 1\colon T \to T \otimes T^\vee \otimes T$, it remains to show that $1 \otimes \mathrm{ev}_T$ also admits a left-inverse. We claim that in fact already the evaluation map ev_T itself admits a left-inverse, given by the following composite:

$$\mathbb{1}_{\mathcal{D}} \xrightarrow{\mathrm{coev}} T \otimes T^\vee \xrightarrow{t\otimes T^\vee} T \otimes T^\vee \xrightarrow{\sigma_{T,T^\vee}} T^\vee \otimes T.$$

Indeed, this follows from the following string diagram:



Here we have abbreviated $\varepsilon = \mathrm{ev}_T$ and $\eta = \mathrm{coev}_T$. Each of the five diagrams represents a composite of morphisms in $\mathcal{D}$, where each downward pointing arrow denotes a tensor factor of $T^\vee$ while an upward pointing arrow denotes a tensor factor of T. For example, the third diagram is to be read as the following composite:

$$T^\vee \otimes T \xrightarrow{1\otimes\eta} T^\vee \otimes T \otimes T \otimes T^\vee \xrightarrow{1\otimes\sigma_{T,T}\otimes 1} T^\vee \otimes T \otimes T \otimes T^\vee \xrightarrow{1\otimes\sigma_{T,T^\vee}} T^\vee \otimes T \otimes T^\vee \otimes T \xrightarrow{\varepsilon\otimes 1} T^\vee \otimes T$$

The equivalence labelled (1) holds because of the assumption that T has a t-twisted trivial braiding. The other equivalences are easy manipulations of string diagrams. This finishes the proof. ◻

Lemma B.4 (Campion [Cam23, Example 2.1.6]). *For every G-representation V, the representation sphere $S^V \in \mathrm{Spc}_*^G$ has (-1)-twisted trivial braiding, where $-1\colon S^V \to S^V\colon v \mapsto -v$.*

Proof. Consider the family of linear G-equivariant maps $H\colon [0,1] \to \mathrm{End}_G(V \times V)$ sending t to the matrix

$$\begin{pmatrix} -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \\ \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \end{pmatrix}\colon V \times V \to V \times V.$$

We have $H_0(v_1, v_2) = (v_2, v_1)$ and $H_1(v_1, v_2) = (-v_1, v_2)$. Since H_t is invertible at all times, it induces a pointed map $S^{H_t}\colon S^{V\times V} \to S^{V\times V}$. As $S^{V\times V} \cong S^V \wedge S^V$, it follows that the twist map of S^V in Spc_*^G is homotopic to the map $(-1) \wedge \mathrm{id}$ as desired. $\square$

To continue, we need the notion of G-spaces concentrated at a single conjugacy class. We fix a compact Lie group G and a conjugacy class (H) of subgroups of G.

Definition B.5. We say that a pointed G-space $X \in \mathrm{Spc}_*^G$ is *concentrated at H* if we have $X^K = *$ for any subgroup $K \subseteq G$ that is not in the conjugacy class of H.

Let $W = W_G(H) = N_G(H)/H$ denote the Weyl group of H in G. Observe that the full subcategory of the orbit category Orb_G spanned by G/H is equivalent to the classifying space BW of W, as W is equivalent to the endomorphism space of G/H in Orb_G. It follows that the H-fixed point functor $(-)^H\colon \mathrm{Spc}_*^G = \mathrm{Fun}(\mathrm{Orb}_G, \mathrm{Spc}_*) \to \mathrm{Fun}(BW, \mathrm{Spc}_*)$ becomes an equivalence when restricted to the pointed G-spaces concentrated at H.

We let $S^{\{H\}}$ denote a copy of S^0 concentrated at H corresponding:

$$(S^{\{H\}})^K = \begin{cases} S^0 & K \text{ conjugate to } H; \\ * & \text{otherwise.} \end{cases}$$

Note that the functor $- \wedge S^{\{H\}}\colon \mathrm{Spc}_*^G \to \mathrm{Spc}_*^G$ is a localization onto the pointed G-spaces concentrated at H.

Lemma B.6. *Let V be a G-representation and let $n := \dim(V^H)$. Then there is an equivalence of pointed G-spaces $S^V \wedge S^{\{H\}} \wedge G/H_+ \simeq \Sigma^n(S^{\{H\}} \wedge G/H_+)$.*

Proof. Since both sides are concentrated at H, the equivalence can be checked after forgetting to $\mathrm{Fun}(BW, \mathrm{Spc}_*)$ after passing to H-fixed points. As the H-fixed points of G/H_+ are W_+, this follows from the sequence of equivalences

$$(S^V)^H \wedge W_+ \simeq S^{V^H} \wedge W_+ \simeq S^n \wedge W_+ = \Sigma^n(W_+).$$

Here the second equivalence uses the shear isomorphism for W-spaces, using that the underlying pointed space of S^{V^H} is S^n. $\square$

We are now ready for the proof of Theorem B.1.

Proof of Theorem B.1. Let $F\colon \mathrm{Spc}_*^G \to \mathcal{D}$ be a symmetric monoidal left adjoint as in the statement of the theorem. We prove that 1), 2) and 3) are equivalent. The implication (2) $\implies$ (3) is obvious. For the implication (1) $\implies$ (2), note that (1) implies by the universal property of $\mathrm{Sp}^G = (\mathrm{Spc}_*^G)[\{S^V\}^{-1}]$ that F uniquely extends to a symmetric monoidal left adjoint $\widetilde{F}\colon \mathrm{Sp}^G \to \mathcal{D}$. Since the suspension functor $\Sigma_+^\infty\colon \mathrm{Spc}_*^G \to \mathrm{Sp}^G$ sends compact G-spaces to dualizable objects, the same follows for F.

It remains to show that 3) implies 1). Let V be a G-representation and assume that $F(S^V)$ is dualizable. To prove that $F(S^V)$ is invertible, we need to show that the evaluation map $\mathrm{ev}_{F(S^V)}\colon F(S^V) \otimes D(F(S^V)) \to \mathbb{1}_{\mathcal{D}}$ is an equivalence, or equivalently that the cofiber

$$Q := \mathrm{cofib}(F(S^V) \otimes D(F(S^V)) \xrightarrow{\mathrm{ev}_{F(S^V)}} \mathbb{1}_{\mathcal{D}})$$

is zero in $\mathcal{D}$. We will inductively show that $Q \otimes F(X)$ is zero for a large family of G-spaces X.

Step 1: We have seen in Lemma B.4 that $X = S^V$ has twisted trivial braiding, and since the functor F is symmetric monoidal it follows that $F(S^V)$ also has twisted-trivial braiding. Proposition B.3 then gives us that the map $\mathrm{ev}_{F(S^V)}\colon F(S^V) \otimes D(F(S^V)) \to \mathbb{1}_{\mathcal{D}}$ becomes an equivalence after tensoring with $F(S^V)$. In particular, $Q \otimes F(S^V) = 0$.

Step 2: We will show that $Q \otimes F(S^{\{H\}}) = 0$ for every subgroup H of G. By step 1, we know that $Q \otimes F(S^V \wedge S^{\{H\}} \wedge G/H_+) \simeq Q \otimes F(S^V) \otimes F(S^{\{H\}} \wedge G/H_+) = 0$. From the equivalence of Lemma B.6 and exactness of $Q \otimes F(-)$, it follows that

$$0 = Q \otimes F(S^V \wedge S^{\{H\}} \wedge G/H_+) \simeq \Sigma^n(Q \otimes F(S^{\{H\}} \wedge G/H_+)),$$

so that also $Q \otimes F(S^{\{H\}} \wedge G/H_+) = 0$ by stability of $\mathcal{D}$. Since $Q \otimes F(-)$ commutes with homotopy orbits, we get

$$Q \otimes F(S^{\{H\}}) \simeq Q \otimes F\left((S^{\{H\}} \wedge G/H_+)_{hW}\right) \simeq \left(Q \otimes F(S^{\{H\}} \wedge G/H_+)\right)_{hW} = 0.$$

Step 3: We show that $Q = 0$. For a family $\mathcal{F}$ of subgroups of G we let $E_{\mathcal{F}}$ denote the universal G-space with $\mathcal{F}$-isotropy, characterized by the property that

$$(E_{\mathcal{F}})^K = \begin{cases} * & K \in \mathcal{F}; \\ \emptyset & \text{otherwise.} \end{cases}$$

We let $E_{\mathcal{F},+} := (E_{\mathcal{F}})_+$. Consider $\mathcal{U}$ be the poset of all families $\mathcal{F}$ of subgroups of G for which $Q \otimes F(E_{\mathcal{F},+}) = 0$. Our goal is to show that the family of *all* subgroups is contained in $\mathcal{U}$, as in that case we have $E_{\mathcal{F},+} = S^0$ so that $Q \otimes F(E_{\mathcal{F},+}) = Q \otimes F(S^0) \simeq Q \otimes \mathbb{1}_{\mathcal{D}} \simeq Q$.

Note that $\mathcal{U}$ contains $\mathcal{F} = \emptyset$ since $E_{\mathcal{F}} = \emptyset$ and $Q \otimes F(-)$ preserves the initial object. Furthermore, for a chain $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \dots$ of nested families in $\mathcal{U}$ with union $\mathcal{F}$, we have $E_{\mathcal{F}} = \operatorname{colim}_i E_{\mathcal{F}_i}$, as is easily checked on K-fixed points for every $K \leqslant G$. As $Q \otimes F(-)$ preserves colimits, it follows that also $\mathcal{F} \in \mathcal{U}$. Thus $\mathcal{U}$ satisfies the conditions of Zorn's lemma and thus contains a maximal element $\mathcal{F}_m$.

We claim that $\mathcal{F}_m$ must contain all subgroups of G. Assume that a subgroup H is not contained in $\mathcal{F}_m$. Since the poset of subgroups of G is well-founded, we may assume that H is minimal with this property, i.e. every strict subgroup of H is in $\mathcal{F}_m$. We now define $\mathcal{F}'$ as the family of subgroups either contained in $\mathcal{F}_m$ or conjugate to H. This is indeed a family by assumption on H. We then have a cofiber sequence

$$E_{\mathcal{F}_m,+} \to E_{\mathcal{F}',+} \to S^{\{H\}}.$$

in Spc_*^G. Since $Q \otimes F(-)$ preserves cofiber sequences, the sequence

$$Q \otimes F(E_{\mathcal{F}_m,+}) \to Q \otimes F(E_{\mathcal{F}',+}) \to Q \otimes F(S^{\{H\}})$$

is again a cofiber sequence in $\mathcal{D}$. Since $\mathcal{F}_m \in \mathcal{U}$, the left term is zero, and by step 2) also the right term is zero. It follows that the middle term is zero and thus $\mathcal{F}' \in \mathcal{U}$. This contradicts the maximality of $\mathcal{F}_m$. We conclude that H must have already be contained in $\mathcal{F}_m$ and thus that $\mathcal{F}_m$ contains all subgroups of G. This finishes the proof of the implication (3) $\implies$ (1), thus finishing the proof of the theorem. $\square$

C Smooth manifolds

We collect some basic results on smooth manifolds. Let Diff denote the category of smooth manifolds and smooth maps. It will be convenient for us to allow *impure* smooth manifolds, meaning that different path components might have distinct dimensions.

Definition C.1. Let $f\colon M \to N$ be a smooth map between smooth manifolds.

- The map f is called a *submersion* if for every $x \in M$ the map $T_x f\colon T_x M \to T_{f(x)} N$ on tangent spaces is surjective.
- The map f is called an *immersion* if for every $x \in M$ the map $T_x f\colon T_x M \to T_{f(x)} N$ on tangent spaces is injective.
- The map f is called an *embedding* if it is an injective immersion such that f restricts to a homeomorphism $M \xrightarrow{\cong} f(M)$.
- The map f is called *proper* if the preimage $f^{-1}(K) \subseteq M$ of a compact subspace $K \subseteq N$ is compact.

Definition C.2 (Embedded submanifold)**.** A subset $N \subseteq M$ of an n-dimensional smooth manifold M is called an *embedded k-submanifold* for $k \leq n$ if around every point $x \in N$ there exists a chart $\varphi\colon U \xrightarrow{\cong} V \subseteq \mathbb{R}^n$ such that

$$\varphi(U \cap N) = \{(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \in V \cap x_{k+1} = \cdots = x_n = 0\}.$$

This definition is local in M and hence gives a notion of an *embedded submanifold* also for impure smooth manifolds.

Proposition C.3 ([Lee02, Proposition 5.5])**.** *Let $N \subseteq M$ be an embedded submanifold, equipped with the subspace topology. Then N admits a unique structure of a smooth manifold such that the inclusion $N \hookrightarrow M$ is a smooth embedding.* □

Proposition C.4 ([Mic08, Theorem 1.15])**.** *The category* Diff *is idempotent complete: given a smooth manifold M and a smooth endomorphism $f\colon M \to M$ satisfying $f \circ f = f$, the image $f(M) \subseteq M$ is an embedded submanifold of M.* □

Corollary C.5. *The image of the Yoneda embedding $y\colon \mathrm{Diff} \hookrightarrow \mathrm{PSh}(\mathrm{Diff})$ is closed under retracts.* □

Smooth manifolds are closed under closed equivalence relations.

Proposition C.6 (Godement, [Ser92, Section III.12, Theorem 2], [Bou67, Section 5.9.5])**.** *Let M be a smooth manifold and let $R \subseteq M \times M$ be a closed smooth submanifold defining an equivalence relation on M. Assume that the first projection $\mathrm{pr}_1\colon R \to M$ is a smooth submersion. Then the quotient space M/R admits a unique structure of a smooth manifold making the quotient map $M \to M/R$ a smooth submersion.* □

The following lemma discusses various cancellation properties of smooth maps between smooth manifolds:

Lemma C.7. *Consider a commutative triangle of smooth maps between smooth manifolds:*

$$\begin{tikzcd} & M \arrow[dl, "f"'] \arrow[dr, "gf"] & \\ N \arrow[rr, "g"] & & O. \end{tikzcd}$$

(1) If gf is an embedding, then f is an embedding;

(2) If gf is proper, then f is proper;

(3) If gf is a closed embedding, then f is a closed embedding.

(4) If f is a surjective submersion, then gf is a surjective submersion if and only if g is a surjective submersion.

Proof. For (1), assume that gf is an embedding. As gf is injective, also f is injective. As the composite

$$T_xM \xrightarrow{T_xf} T_{f(x)}N \xrightarrow{T_{f(x)}g} T_{gf(x)}O$$

is injective, so is T_xf. So f is an injective immersion. If $h\colon gf(M) \to M$ is a continuous inverse to the homeomorphism $gf\colon M \to gf(M)$, then the composite

$$f(M) \xrightarrow{g} gf(M) \xrightarrow{h} M$$

will be a continuous inverse to $f\colon M \to f(M)$.

For (2), assume that gf is proper. Let $K \subseteq N$ be a compact subspace. Since gf is proper, the subspace $(gf)^{-1}(g(K)) \subseteq N$ is compact. The preimage $f^{-1}(K)$ is a closed subspace of $(gf)^{-1}(g(K))$, hence it is also compact.

For (3), it suffices to observe that a smooth map is a closed embedding if and only if it is a proper embedding.

For (4), it is clear that if f and g are surjective submersions then so is gf, so assume that f and gf are surjective submersions. For any $y \in N$, take $x \in M$ such that $f(x) = y$. Then both $T_x f\colon T_x M \to T_{f(x)}N$ as well as the composite

$$T_x M \xrightarrow{T_x f} T_y N \xrightarrow{T_y g} T_{gy} O$$

are surjective, and thus so is $T_y g$ as desired. □

Every smooth submersion locally admits a simple 'normal form':

Theorem C.8 (Local submersion theorem)**.** *Let $f\colon M \to N$ be a smooth submersion between smooth manifolds. Then for every point $x \in M$ there exists an open neighborhood $x \in U \subseteq M$ equipped with an isomorphism $U \cong f(U) \times \mathbb{R}^n$ for some natural number n such that the following diagram commutes:*

$$\begin{tikzcd} f(U) \times \mathbb{R}^n \arrow[r, "\cong"] \arrow[dr, "\mathrm{pr}_{f(U)}"'] & U \arrow[r, hook] \arrow[d, "f|_U"] & M \arrow[d, "f"] \\ & f(U) \arrow[r, hook] & N. \end{tikzcd} \tag{C.1}$$

Proof. This is a special case of the Constant Rank Theorem of [Lee02, Theorem 5.13]: as f has constant rank $k = \dim(N)$, one may choose local coordinates on M and N such that f is of the form $f(x_1,\ldots,x_k,x_{k+1},\ldots,x_{k+n}) = (x_1,\ldots,x_k)$. Choosing small open balls around x and $f(x)$ thus shows that f is locally given by a projection $\mathbb{R}^{k+n} \to \mathbb{R}^k$ onto the first k coordinates. □

It is a folklore theorem that base changes of smooth submersions exist in Diff and are again smooth immersions. More precisely:

Proposition C.9. *Let M, N and N' be smooth manifolds, let $f\colon M \to N$ be a smooth submersion, and let $g\colon N' \to N$ be a smooth map. Then the pullback space $M \times_N N'$ admits a unique smooth structure such that a map $h\colon X \to M \times_N N'$ is smooth if and only if both composites $\mathrm{pr}_M \circ h\colon X \to M$ and $\mathrm{pr}_{M'} \circ h\colon X \to N'$ are smooth. Furthermore, the projection $M \times_N N' \to N'$ is a smooth submersion.*

Proof. This is a special case of the fact that pullbacks in Diff of transversal maps exist. Although this result is widely known, we were not able to find a proof in the literature, hence we will provide one for the special case f is a smooth submersion.

Observe that the statement is local in M, in the sense that if $\{U_\alpha\}$ is an open cover of M such that the statement holds for the composite $f_\alpha = U_\alpha \hookrightarrow M \xrightarrow{f} N$, then it also holds for f. By Theorem C.8, it will thus suffice to prove the claim when f is either an open embedding, where the statement is clear, or a projection $N \times \mathbb{R}^n \to N$, where the pullback is the projection $N' \times \mathbb{R}^n \to N'$. □

Lemma C.10. *Let $f\colon M \to N$ be a submersion between smooth manifolds. Then it is surjective if and only if it admits local sections.*

Proof. If f admits local sections, it is clear that f is surjective. Conversely, consider a point $y \in N$. We have to find an open neighborhood around y on which f admits a section. By surjectivity of f, we may pick $x \in f^{-1}(\{y\})$. Using Theorem C.8, we may find an open neighborhood $x \in U \subseteq M$ equipped with an isomorphism $U \cong f(U) \times \mathbb{R}^n$ for some natural number n such that the diagram (C.1) commutes. Since the left diagonal map $f(U) \times \mathbb{R}^n \to f(U)$ admits a section $y' \mapsto (y', 0)$, it follows that f admits a partial section defined on the open neighborhood $f(U)$ around y, proving that f admits local sections. □

We finish the chapter by discussing relative tangent bundles and normal bundles.

Definition C.11. Let $f\colon M \to N$ be a smooth map between smooth manifolds. We define its *relative tangent bundle* T_f as the kernel of the derivative df of f:

$$T_f := \ker(df\colon TM \to f^*TN) \quad \in \quad \mathrm{Vect}(M).$$

We define its *normal bundle* as the cokernel of df:

$$N_f := \mathrm{coker}(df\colon TM \to f^*TN) \quad \in \quad \mathrm{Vect}(M).$$

Relative tangent bundles and normal bundles behave well under pullback:

Lemma C.12. *Consider a pullback square of smooth manifolds*

$$\begin{tikzcd} M' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d, "f"] \\ N' \arrow[r, "g"] & N \end{tikzcd}$$

such that g is a smooth submersion. Then the induced square on tangent bundles

$$\begin{tikzcd} TM' \arrow[r, "dh"] \arrow[d, "df'"'] & TM \arrow[d, "df"] \\ TN' \arrow[r, "dg"] & TN \end{tikzcd}$$

is again a pullback square.

Proof. The statement is clear when g is an open embedding. Theorem C.8 then allows us to reduce to the case where $N' = N \times \mathbb{R}^n$ and $g = \mathrm{pr}_N \colon N \times \mathbb{R}^n \to N$ is the projection. In this case we have $M' = M \times \mathbb{R}^n$, and the statement now follows as $T(N \times \mathbb{R}^n) \cong TN \times \mathbb{R}^{n+n}$ and similarly for $T(M \times \mathbb{R}^n)$. □

Corollary C.13. *For a pullback square of smooth manifolds as in Lemma C.12, the following square is also a pullback square:*

$$\begin{tikzcd} T_{f'} \arrow[r, "dh"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T_f \arrow[d] \\ M' \arrow[r, "h"] & M. \end{tikzcd}$$

If f and (thus) f' are immersions, than also the following square is a pullback square:

$$\begin{tikzcd} N_{f'} \arrow[r, "dg"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & N_f \arrow[d] \\ M' \arrow[r, "h"] & M. \end{tikzcd}$$

Proof. For every $x \in M'$, the induced square of vector bundles

$$\begin{tikzcd} T_x M' \arrow[r, "dh"] \arrow[d, "df'"'] \arrow[dr, phantom, "\lrcorner"] & T_{h(x)} M \arrow[d, "df"] \\ T_{f'(x)} N' \arrow[r, "dg"] & T_{f(h(x))} N \end{tikzcd}$$

is a pullback square by Lemma C.12. Passing to vertical kernels shows that the map $dh \colon T_{f'} \to T_f$ induces isomorphisms on fibers, proving that the first square is a pullback square. If f and f' are immersions, then df and df' are injections. Passing to vertical cokernels in the above square shows that the map $dg \colon N_{f'} \to N_f$ induces isomorphisms on fibers, proving that the second square is a pullback square. □

D Lie groupoids

In this appendix, we recall some background material on Lie groupoids freely used in the body of the text.

D.1 Definition and examples

We recall the definition and some basic examples of Lie groupoids.

Definition D.1 (Lie groupoid)**.** A *Lie groupoid* $\mathcal{G}$ is a groupoid $(\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ in which $\mathcal{G}_1$ and $\mathcal{G}_0$ are smooth manifolds, the structure maps $s,t\colon \mathcal{G}_1 \to \mathcal{G}_0$, $m\colon \mathcal{G}_1 \times_{s,\mathcal{G}_0,t} \mathcal{G}_1 \to \mathcal{G}_1$ and $u\colon \mathcal{G}_0 \to \mathcal{G}_1$ are all smooth, and the source and target maps s,t are smooth submersions. A *morphism of Lie groupoids* $f\colon \mathcal{G} \to \mathcal{H}$ is a smooth functor, i.e. a pair of smooth maps $f_0\colon \mathcal{G}_0 \to \mathcal{H}_0$ and $f_1\colon \mathcal{G}_1 \to \mathcal{H}_1$ commuting with the structuer maps. We denote by LieGrpd the category of Lie groupoids and morphisms of Lie groupoids.

Given a smooth manifold M, we say that $\mathcal{G}$ is a *Lie groupoid over M* if $\mathcal{G}_0 = M$.

Warning D.2. While one sometimes reads that 'Lie groupoids are groupoid objects internal to the category Diff of smooth manifolds', this statement needs to be taken with care, as the category Diff does not admit all pullbacks. The assumption that source and target maps are smooth submersions is needed to get a well-behaved theory.

Example D.3 (Classifying groupoid)**.** Let G be a Lie group. We let $G \ltimes \mathrm{pt}$ be the Lie groupoid given by $(G \ltimes \mathrm{pt})_0 = \mathrm{pt}$ and $(G \ltimes \mathrm{pt})_1 = G$, with composition and inversion given by multiplication and inversion in G. We call $G \ltimes \mathrm{pt}$ the *classifying groupoid* of G.

Example D.4 (Action groupoid)**.** More generally, let M be a smooth G-manifold, i.e., a smooth manifold equipped with a smooth G-action. We let $G \ltimes M$ be the Lie groupoid given by $(G \ltimes M)_0 = M$ and $(G \ltimes M)_1 = G \times M$. The structure maps are given by

$$\begin{aligned} s(g,x) &= x, & t(g,x) &= g\cdot x, & e(x) &= (e,x), \\ i(g,x) &= (g^{-1}, gx) & m((g',gx),(g,x)) &= (g'g,x). \end{aligned}$$

We call $G \ltimes M$ the *action groupoid* of M. It defines a functor $G \ltimes -\colon \mathrm{Diff}_G \to \mathrm{LieGrpd}$, where Diff_G denotes the category of smooth G-manifolds.

Example D.5 (Čech groupoid)**.** Let $p\colon M \to N$ be a smooth submersion between smooth manifolds. Then its *Čech groupoid* $\check{C}(p)$ is defined as the Lie groupoid with $\check{C}(p)_0 = M$, $\check{C}(p)_1 = M \times_N M$, and structure maps

$$\begin{aligned} s(x,x') &= x, & t(x,x') &= x', & e(x) &= (x,x), \\ i(x,x') &= (x',x), & m((x,x'),(x',x'')) &= (x,x''). \end{aligned}$$

When $N = \mathrm{pt}$ is a point, the resulting Lie groupoid $M \times M \rightrightarrows M$ is known as the *pair groupoid* of M.

Example D.6 (Gauge groupoid)**.** Let $P \to M$ be a principal G-bundle. The *gauge groupoid* is the Lie groupoid $(P \times P)/G \rightrightarrows M$, where G acts diagonally on $P \times P$. The composition is defined as in the pair groupoid.

Example D.7 (Pullback groupoid)**.** Let $\mathcal{G}$ be a Lie groupoid over M, let N be a smooth manifold and let $f : N \to M$ be a smooth submersion. We define the *pullback groupoid* $f^*\mathcal{G}$ by letting $(f^*\mathcal{G})_0 = N$ and defining

$$(f^*\mathcal{G})_1 = N \times_M \mathcal{G}_1 \times_M N = \{(x,g,y) \in N \times \mathcal{G}_1 \times N \mid s(g) = f(x), t(g) = f(y)\},$$

with structure maps inherited from $\mathcal{G}$. As a special case, we get for every open subspace $U \subseteq M$ a groupoid $\mathcal{G}|_{\mathcal{U}}$, called the *restriction of* $\mathcal{G}$ *to* U:

$$\mathcal{G}|_U = \left(\{g \in \mathcal{G}_1 \mid s(g), t(g) \in U\} \overset{s}{\underset{t}{\rightrightarrows}} U \right).$$

More generally, we may define a *topological* groupoid $\mathcal{G}|_X$ for any subspace $X \subseteq M$. However, it is not always true that $\mathcal{G}|_X$ is a Lie groupoid.

We recall the definition of the orbits and isotropy groups of a Lie groupoid.

Definition D.8 (Orbits)**.** Let $\mathcal{G}$ be a Lie groupoid over M. Two points $x, y \in M$ are said to *lie in the same orbit of* $\mathcal{G}$ if there exists an arrow $g\colon x \to y$, i.e., a point $g \in \mathcal{G}_1$ with $s(g) = x$ and $t(g) = y$. This gives rise to a partition of M into the *orbits of* $\mathcal{G}$: every point $x \in M$ is contained in a unique orbit:

$$\mathcal{O}_x := \{t(g) \in M \mid g \in \mathcal{G}_1, s(g) = x\} \subseteq M.$$

Definition D.9 (Isotropy group)**.** Let $\mathcal{G}$ be a Lie groupoid over M and let $x, y \in M$. We define

$$\begin{aligned} \mathcal{G}(x,-) &:= s^{-1}(x) = \{g \in \mathcal{G}_1 \mid s(g) = x\} \\ \mathcal{G}(-,y) &:= t^{-1}(y) = \{g \in \mathcal{G}_1 \mid t(g) = y\} \\ \mathcal{G}(x,y) &:= \mathcal{G}(x,-) \cap \mathcal{G}(-,y) = \{g \in \mathcal{G}_1 \mid s(g) = x, t(g) = y\} \\ G_x &:= \mathcal{G}(x,x). \end{aligned}$$

The composition in $\mathcal{G}$ restricts to a group structure on G_x, and we call G_x the *isotropy group of $\mathcal{G}$ at x.*

Example D.10. The isotropy group of the action groupoid $G \ltimes M$ at a point $x \in M$ is the usual isotropy group $G_x = \{g \in G \mid gx = x\}$ of the G-action on M.

As s and t are smooth submersions, the fibers $\mathcal{G}(x,-)$ and $\mathcal{G}(-,y)$ are closed submanifolds of $\mathcal{G}_1$. One can prove that also the subspaces $\mathcal{G}(x,y) \subseteq \mathcal{G}_1$ and $\mathcal{O}_x \subseteq M$ are smooth manifolds, implying in particular that the isotropy group G_x is a Lie group:

Proposition D.11 (Moerdijk and Mrčun [MM03, Theorem 5.4])**.** *Let $\mathcal{G}$ be a Lie groupoid over M and let $x, y \in M$.*

(1) The subspace $\mathcal{G}(x,y)$ is a closed submanifold of $\mathcal{G}_1$;

(2) The isotropy group G_x is a Lie group;

(3) The orbit $\mathcal{G} \cdot x$ is an immersed submanifold of M;

(4) The target map $t\colon \mathcal{G}(x,-) \to \mathcal{O}_x$ is a principal G_x-bundle.

Definition D.12. Let $\mathcal{G}$ be a Lie groupoid over M and let $S \subseteq M$ be a subspace. We say that X is *saturated* if it contains all orbits that intersect S: if $x \in S$, then also $gx \in S$ for all $g \in \mathcal{G}_1$ with $s(g) = x$. For an arbitrary subspace $S \subseteq M$, we define the *$\mathcal{G}$-saturation $\mathcal{G} \cdot S$ of S* as the union in M of all orbits of $\mathcal{G}$ intersecting S non-trivially:

$$\mathcal{G} \cdot S := \bigcup_{x \in S} \mathcal{O}_x = \{gx \in M \mid g \in \mathcal{G}_1, s(g) = x \in X\}.$$

Definition D.13 (Proper Lie groupoids)**.** A Lie groupoid $\mathcal{G}$ is called *proper* if the map $(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is a proper map.

An action groupoid $G \ltimes M$ is proper if and only if the action of the Lie group G on the smooth manifold M is proper. Note that the isotropy groups G_x of a proper Lie groups are compact, being given as the fibers of the map $(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ at (x,x).

D.2 Principal bundles

We recall the notions left actions of Lie groupoids and of smooth principal bundles for Lie groupoids.

Definition D.14 (Left $\mathcal{G}$-action)**.** Let $\mathcal{G}$ be a Lie groupoid, let M be a smooth manifold and let $M \to \mathcal{G}_0$ be a smooth submersion. A *left $\mathcal{G}$-action on $M \to \mathcal{G}_0$* consists of a smooth map $a\colon \mathcal{G}_1 \times_{s,\mathcal{G}_0} M \to M$ over $\mathcal{G}_0$ which is associative and unital in the sense that the diagrams

$$\begin{tikzcd} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} M \arrow[r, "m\times 1"] \arrow[d, "1\times a"] & \mathcal{G}_1 \times_{\mathcal{G}_0} M \arrow[d, "a"] \\ \mathcal{G}_1 \times_{\mathcal{G}_0} M \arrow[r, "a"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{G}_0 \times_{\mathcal{G}_0} M \arrow[r, "i\times 1"] \arrow[d, "\cong"] & \mathcal{G}_1 \times_{\mathcal{G}_0} M \arrow[d, "a"] \\ M \arrow[r, equal] & M \end{tikzcd}$$

commute. A *smooth $\mathcal{G}$-manifold* is a smooth manifold $M \to \mathcal{G}_0$ over $\mathcal{G}_0$ equipped with a left $\mathcal{G}$-action.

Given another map $M' \to \mathcal{G}_0$ equipped with a left $\mathcal{G}$-action, a *morphism of smooth $\mathcal{G}$-manifolds $\varphi\colon M \to M'$* is a map $\varphi\colon M \to M'$ which commutes with the action maps $a\colon \mathcal{G}_1 \times_{\mathcal{G}_0} M \to M$ and $a'\colon \mathcal{G}_1 \times_{\mathcal{G}_0} M' \to M'$. We denote the resulting category of smooth $\mathcal{G}$-manifolds by $\mathrm{Diff}_{\mathcal{G}}$.

Note that for a Lie group G, a smooth $(G \ltimes \mathrm{pt})$-manifold is simply a smooth G-manifold, i.e. a smooth manifold M equipped with a smooth G-action. We let Diff_G denote the category of smooth G-manifolds.

Definition D.15 (Smooth principal $\mathcal{G}$-bundle)**.** Let $\mathcal{G}$ be a Lie groupoid and let N be a smooth manifold. A *smooth principal $\mathcal{G}$-bundle* is a smooth manifold P equipped with smooth maps $P \to \mathcal{G}_0$ and $p\colon P \to N$ and equipped with a left $\mathcal{G}$-action $a\colon \mathcal{G}_1 \times_{\mathcal{G}_0} P \to P$ on P satisfying the following conditions:

(1) The map $p\colon P \twoheadrightarrow N$ is a surjective smooth submersion.

(2) The action lives over N, in that the following diagram commutes:

$$\begin{tikzcd} \mathcal{G}_1 \times_{\mathcal{G}_0} P \arrow[rr, "a"] \arrow[dr, "p\circ \mathrm{pr}_2"'] & & P \arrow[dl, "p"] \\ & N. & \end{tikzcd}$$

(3) The map

$$\mathcal{G}_1 \times_{\mathcal{G}_0} P \xrightarrow{(a,\mathrm{pr}_2)} P \times_N P$$

is a diffeomorphism.

If $P' \to N$ is another smooth principal $\mathcal{G}$-bundle over N, a *morphism $P \to P'$ of smooth principal $\mathcal{G}$-bundles over N* consists of a map $\varphi\colon P \to P'$ of smooth $\mathcal{G}$-manifolds which lies over N. We let $\mathrm{PrnBdl}_{\mathcal{G}}(N)$ denote the resulting category of smooth principal $\mathcal{G}$-bundles over N.

Construction D.16. Let $p\colon P \twoheadrightarrow N$ be a smooth principal $\mathcal{G}$-bundle and consider a point $x \in N$. As p is a surjective smooth submersion, it admits local sections, and thus there exists an open neighborhood $x \in U \subseteq N$ and a smooth map $s\colon U \hookrightarrow P$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[d, "p"] \\ U \arrow[ur, hook, "s"] \arrow[r, hook] & N. \end{tikzcd}$$

Now, pulling back the diffeomorphism

$$\mathcal{G}_1 \times_{\mathcal{G}_0} P \xrightarrow{(a,\mathrm{pr}_2)} P \times_N P$$

along the map $s\colon U \hookrightarrow P$, we obtain a diffeomorphism

$$\mathcal{G}_1 \times_{\mathcal{G}_0} U \xrightarrow{\cong} P|_U$$

over U, where the map $U \to \mathcal{G}_0$ is the composite of $s\colon U \hookrightarrow P$ and the structure map $P \to \mathcal{G}_0$. In particular, for a point $x \in U \subseteq N$ the fiber P_x of p over x is given by $\mathcal{G}(x', -)$, where x' is the image of x under the map $U \to \mathcal{G}_0$.

We may think of the diffeomorphisms $\mathcal{G}_1 \times_{\mathcal{G}_0} U \xrightarrow{\cong} P|_U$ as analogue to the classical local triviality condition on principal bundles.

Corollary D.17. *Every morphism $\varphi\colon P \to P'$ of smooth principal $\mathcal{G}$-bundles over N is invertible. In other words, the category $\mathrm{PrnBdl}_{\mathcal{G}}(N)$ is a groupoid.*

Proof. It suffices to check this locally on N. Choosing a point $x \in N$ and proceeding as in Construction D.16, we find an open neighborhood $x \in U \subseteq N$ and a commutative diagram

$$\begin{tikzcd} & \mathcal{G}_1 \times_{\mathcal{G}_0} U \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ P|_U \arrow[rr, "\varphi|_U"] & & P'|_U, \end{tikzcd}$$

proving that the restriction of φ to U is a diffeomorphism. The claim follows. $\square$

D.3 Morita equivalence

We recall the notion of Morita equivalence of Lie groupoids. The relevance of this notion of equivalence is that it corresponds to an equivalence between the underlying differentiable stacks of the two Lie groupoids, see Proposition II.2.3.14.

Definition D.18 (Morita equivalence)**.** Let $\mathcal{G}$ be a Lie groupoid over M and let $\mathcal{H}$ be a Lie groupoid over N. A *Morita equivalence* between $\mathcal{G}$ and $\mathcal{H}$ consists of:

(1) Surjective smooth submersions $\alpha\colon P \to M$ and $\beta\colon P \to N$;

(2) A left $\mathcal{G}$-action $a_{\mathcal{G}}\colon \mathcal{G}_1 \times_M P \to P$ on P over M making $\beta\colon P \to N$ into a principal $\mathcal{G}$-bundle;

(3) A left $\mathcal{H}$-action $a_{\mathcal{H}}\colon \mathcal{H}_1 \times_N P \to P$ on P over N making $\alpha\colon P \to M$ into a principal $\mathcal{H}$-bundle;

(4) We require that the actions $a_{\mathcal{G}}$ and $a_{\mathcal{H}}$ commute with each other.

We say that $\mathcal{G}$ and $\mathcal{H}$ are *Morita equivalent* if there exists a Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$.

We mention some standard examples of Morita equivalences, whose proofs we leave to the reader.

Example D.19. Let G and H be Lie groups. Then the Lie groupoids $G \ltimes \mathrm{pt}$ and $H \ltimes \mathrm{pt}$ are Morita equivalent if and only if G and H are isomorphic.

Example D.20. Let $\mathcal{G}$ be a Lie groupoid over M with a single orbit. Then there is a Morita equivalence between $\mathcal{G}$ and $G_x \ltimes \mathrm{pt}$ for any $x \in M$.

Example D.21. Let $p\colon M \to N$ be a smooth submersion. Then the Čech groupoid $\check{C}(p)$ from Example D.5 is Morita equivalent to the submanifold $p(M) \subseteq N$, regarded as a Lie groupoid with only identity arrows.

Example D.22. Given a smooth, free and proper action of a Lie group G on a smooth manifold M, the action groupoid $G \ltimes M$ is equivalent to the quotient $N = M/G$.

Example D.23. Let $\mathcal{G}$ be a Lie groupoid over M and let $U \subseteq M$ be an open subset. Let $U' := \mathcal{G} \cdot U \subseteq M$ be the $\mathcal{G}$-saturation of U. Then the restriction $\mathcal{G}|_U$ of $\mathcal{G}$ to U is Morita equivalent to the restriction $\mathcal{G}|_{U'}$ of $\mathcal{G}$ to U'.

E Recollections on ∞-topoi

Throughout this dissertation, we make extensive use of the theory of ∞-topoi, due to Rezk, Toën-Vezzosi and Lurie. For the convenience of the reader, we will provide in this appendix a thorough treatment of this theory, covering all the foundations that we will need in this dissertation.

The appendix is organized as follows. We start in Appendix E.1 by recalling the definition and most important characterizations of ∞-topoi. The correspondence between groupoid objects and effective epimorphisms is discussed in Appendix E.2. In Appendix E.3 we define groupoid actions and principal bundles in ∞-topoi and show how they can be classified using the classifying stack of the groupoid, following the treatment of [NSS15] and [SS21, Section 3.2] for *group* actions. We discuss sheaf topoi in Appendix E.4 and hypercompleteness and hyperdescent in Appendix E.5.

E.1 Definition and characterizations of ∞-topoi

Recall the definition of an ∞-topos and of a geometric morphism between ∞-topoi.

Definition E.1 (∞-topos)**.** An ∞-category $\mathcal{B}$ is called an ∞-*topos* if it is equivalent to an accessible left exact Bousfield localization of a presheaf category $\mathrm{PSh}(C)$, for some small ∞-category C.

Definition E.2 (Geometric morphism)**.** A functor $\psi_* \colon \mathcal{B} \to \mathcal{B}'$ between ∞-topoi is called a *geometric morphism* if it admits a left exact left adjoint $\psi^* \colon \mathcal{B}' \to \mathcal{B}$.

The following theorem provides criteria for testing whether an ∞-category is an ∞-topos:

Theorem E.3 (Lurie [Lur09, Proposition 6.1.3.9, Proposition 6.1.0.6])**.** *Let $\mathcal{B}$ be an ∞-category. The following conditions are equivalent:*

(1) The ∞-category $\mathcal{B}$ is an ∞-topos;

(2) The ∞-category $\mathcal{B}$ satisfies the following ∞-categorical analogues of the Giraud's axioms:

(i) The ∞-category $\mathcal{B}$ is presentable;

(ii) Colimits in $\mathcal{B}$ are universal, see Definition E.4;

(iii) Coproducts in $\mathcal{B}$ are disjoint, see Definition E.6;

(iv) Every groupoid object of $\mathcal{B}$ is effective, see Definition E.7.

(3) The ∞-category $\mathcal{B}$ is presentable and the cartesian fibration $t\colon \operatorname{Ar}(\mathcal{B}) \to \mathcal{B}$ is classified by a limit preserving functor

$$\mathcal{B}_{/-}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}},\ X \mapsto \mathcal{B}_{/X}.$$

Condition (3) is often referred to as *descent for ∞-topoi*.

We will now explain the meaning of some of the conditions in the above theorem.

Universal colimits

Let $f: T \to S$ be a morphism in a presentable ∞-category $\mathcal{B}$. Since $\mathcal{B}$ has all limits, the functor $f_!\colon \mathcal{B}_{/T} \to \mathcal{B}_{/S}$ given by composition with f admits a right adjoint

$$\begin{gathered} f^*\colon \mathcal{B}_{/S} \to \mathcal{B}_{/T} \\ (X \to S) \mapsto (X \times_S T \to T). \end{gathered}$$

Definition E.4. Let $\mathcal{B}$ be a presentable ∞-category. We say that *colimits in $\mathcal{B}$ are universal* if, for any morphism $f: T \to S$ in $\mathcal{B}$, the associated pullback functor

$$f^*\colon \mathcal{B}_{/S} \to \mathcal{B}_{/T}$$

preserves colimits.

By the adjoint functor theorem, f^* preserves all colimits if and only if it admits a right adjoint $f_*\colon \mathcal{B}_{/T} \to \mathcal{B}_{/S}$.

Proposition E.5 (Lurie [Lur09, Proposition 6.1.1.4])**.** *Let $\mathcal{B}$ be an ∞-category which admits finite limits. The following conditions are equivalent:*

(1) The ∞-category $\mathcal{B}$ is presentable, and colimits in $\mathcal{B}$ are universal;

(2) The cartesian fibration $t\colon \operatorname{Ar}(\mathcal{B}) \to \mathcal{B}$ is classified by a functor $\mathcal{B}_{/-}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$ into the ∞-category of presentable ∞-categories and colimit preserving functors.

Disjoint coproducts

Definition E.6. If $\mathcal{C}$ is an ∞-category that admits finite coproducts, then we will say that *coproducts in* $\mathcal{C}$ *are disjoint* if every cocartesian diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \sqcup Y \end{tikzcd}$$

is also cartesian, provided that $\emptyset$ is an initial object of $\mathcal{C}$. More informally this says that the intersection of X and Y inside the union $X \sqcup Y$ is empty.

Effective groupoid objects

We refer to Appendix E.2 for the definition of a groupoid object in an ∞-category and for the definition of the Čech nerve of a morphism in an ∞-category.

Definition E.7. Let $\mathcal{G}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a groupoid object in a presentable ∞-category $\mathcal{C}$. We say that $\mathcal{G}$ is *effective* if it is equivalent to the Čech nerve of the canonical map $\mathcal{G}_0 \to |\mathcal{G}| := \operatorname{colim}_{[n]\in\Delta} \mathcal{G}_n$.

E.2 Groupoids and effective epimorphisms

We recall from [Lur09, Section 6.2.3] the correspondence between groupoid objects and effective epimorphisms in an ∞-topos $\mathcal{B}$. Throughout this section, $\mathcal{B}$ will stay fixed.

Definition E.8 (Groupoid object, [Lur09, Definition 6.1.2.7])**.** Let $\mathcal{C}$ be an ∞-category. A simplicial object $\mathcal{G} : \Delta^{\mathrm{op}} \to \mathcal{C}$ is called a *groupoid object* if for every $n \in \mathbb{N}$ and every (not necessarily order-preserving) partition

$$[n] \simeq \{i_0, \dots, i_k\} \sqcup_{\{i_k\}} \{i_k, \dots, i_n\},$$

the induced diagram

$$\begin{tikzcd} \mathcal{G}_n \arrow[r, "(i_0,\dots,i_k)^*"] \arrow[d, "(i_k,\dots,i_n)^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_k \arrow[d, "i_k^*"] \\ \mathcal{G}_{n-k} \arrow[r, "i_k^*"] & \mathcal{G}_0 \end{tikzcd}$$

is a pullback in $\mathcal{C}$. We let $\mathrm{Grpd}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ denote the full subcategory spanned by the groupoid objects in $\mathcal{C}$.

Definition E.9. Let C be an ∞-category admitting simplicial colimits and let $U\colon \Delta^{\mathrm{op}} \to C$ be a simplicial object. We define is *geometric realization* $|U|$ of U as its colimit in C:

$$|U| := \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} U_n.$$

It comes equipped with a structure map $U_0 \to |U|$.

Construction E.10 (Čech nerve)**.** Let C be a presentable ∞-topos and let $f\colon X \to Y$ be a morphism C. We will construct a groupoid object $\check{C}(f)$ called the *Čech nerve* of f, informally given by

$$\check{C}(f)_n := X \times_Y X \times_Y \cdots \times_Y X,$$

the n-fold fiber product of X over Y. More formally, we let Δ_+ denote the augmented simplex category and let $\Delta_+^{\leq 0} \subseteq \Delta_+$ the full subcategory on the objects $[-1]$ and $[0]$. As $\Delta_+^{\leq 0}$ has a single non-identity morphism $[-1] \to [0]$, we may regard f as as a functor $f\colon (\Delta_+^{\leq 0})^{\mathrm{op}} \to C$, sending $[0]$ to X, $[-1]$ to Y and the morphism $[-1] \to [0]$ to $f\colon X \to Y$. Let $\iota_*(f)\colon \Delta_+^{\mathrm{op}} \to C$ denote the right Kan extension of f along the inclusion $\iota\colon \Delta_+^{\leq 0} \hookrightarrow \Delta_+$. Then the *Čech nerve* $\check{C}(f)$ of f is defined as the restriction of $\iota_*(f)$ to Δ^{op}.

Lemma E.11. *For every morphism $f\colon X \to Y$ in a presentable ∞-category C, the Čech nerve $\check{C}(f)$ is a groupoid object in C.*

Proof. This follows immediately from [Lur09, Proposition 6.1.2.11]. □

Definition E.12 (Effective epimorphism, cf. [Lur09, Corollary 6.2.3.5])**.** Consider a morphism $f\colon X \to Y$ in $\mathcal{B}$. The diagram $\iota_*(f)\colon \Delta_+^{\mathrm{op}} \to \mathcal{B}$ from Construction E.10 defines a cocone from the Čech nerve $\check{C}(f)$ to the object Y, giving rise to a comparison map

$$|\check{C}(f)| = \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} X^{\times_Y^n} \to Y.$$

We call f an *effective epimorphism* if this comparison map is an equivalence. In this case, we will also refer to f as exhibiting X as an *atlas* for Y. We write

$$\mathrm{Atl}(\mathcal{B}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{B})$$

for the full subcategory spanned by the effective epimorphisms $f\colon X \twoheadrightarrow Y$ in $\mathcal{B}$.

The above constructions of geometric realization and Čech nerve are inverse to each other:

Proposition E.13 (cf. [Lur09, Discussion below Corollary 6.2.3.5])**.** *Let $\mathcal{B}$ be a ∞-topos. Then the Čech nerve construction restricts to an equivalence of ∞-categories*

$$\check{C}\colon \operatorname{Atl}(\mathcal{B}) \xrightarrow{\sim} \operatorname{Grpd}(\mathcal{B}), \quad f \mapsto \check{C}(f).$$

The inverse is given by sending a groupoid object $\mathcal{G}$ to the map $\mathcal{G}_0 \to |\mathcal{G}|$.

Proof. By definition, the effective epimorphisms are precisely those morphisms $f\colon X \to Y$ for which the canonical comparison map

$$\begin{tikzcd} \check{C}(f)_0 \arrow[r, equal] \arrow[d] & X \arrow[d, "f"] \\ |\check{C}(f)| \arrow[r] & Y \end{tikzcd}$$

in $\operatorname{Fun}(\Delta^1, \mathcal{B})$ is an equivalence. Conversely, given a groupoid object $\mathcal{G}$ in $\mathcal{B}$, the colimit cone defining $|\mathcal{G}|$ corresponds to an extension of $\mathcal{G}$ to Δ_+^{op}, which in turn gives rise to a comparison map $\mathcal{G} \to \check{C}(\mathcal{G}_0 \to |\mathcal{G}|)$ of groupoid objects. A groupoid object $\mathcal{G}$ is called *effective* if this comparison map is an equivalence. But in an ∞-topos, every groupoid object is effective, see [Lur09, Theorem 6.1.0.6(3)]. It follows in particular that the morphism $\mathcal{G}_0 \to |\mathcal{G}|$ is an effective epimorphism. □

Definition E.14. If $\mathcal{G}$ is a groupoid object in an ∞-topos, we denote its geometric realization alternatively by $\mathbb{B}\mathcal{G}$ and refer to it as the *classifying stack* of $\mathcal{G}$:

$$\mathbb{B}\mathcal{G} := |\mathcal{G}| = \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathcal{G}_n.$$

Corollary E.15. *The structure map $\mathcal{G}_0 \to \mathbb{B}\mathcal{G}$ of the colimit is an effective epimorphism.*

Proof. This is immediate from Proposition E.13. □

We finish this subsection by showing that the equivalence $\operatorname{Atl}(\mathcal{B}) \simeq \operatorname{Grpd}(\mathcal{B})$ of Proposition E.13 sends cartesian squares on the left to cartesian natural transformations on the right, and vice versa.

Lemma E.16. *Let $\alpha\colon \mathcal{G} \to \mathcal{H}$ be a morphism of groupoid objects in $\mathcal{B}$, and let*

$$\begin{tikzcd} \mathcal{G}_0 \arrow[r, "\alpha_0"] \arrow[d, two heads] & \mathcal{H}_0 \arrow[d, two heads] \\ |\mathcal{G}| \arrow[r, "|\alpha|"] & |\mathcal{H}| \end{tikzcd} \tag{E.1}$$

be the map in $\operatorname{Atl}(\mathcal{B})$ *induced by the equivalence of Proposition E.13. Then α is cartesian as a natural transformation of simplicial objects in $\mathcal{B}$ if and only if the diagram* (E.1) *is a cartesian square.*

Proof. If α is cartesian, then the square (E.1) is cartesian by [Lur09, Theorem 6.1.3.9(4)]. For the converse, it suffices by the pasting law of pullback squares to show that the diagram

$$\begin{tikzcd} \mathcal{G}_n \arrow[r, "\alpha_n"] \arrow[d] & \mathcal{H}_n \arrow[d] \\ \mathcal{G}_0 \arrow[r, "\alpha_0"] & \mathcal{H}_0 \end{tikzcd}$$

is a pullback square for each $n \geq 0$. For $n = 0$ this is clear. For $n = 1$, this follows by applying the pasting law of pullback squares to the following diagram:

$$\begin{tikzcd}[row sep=small, column sep=small] & \mathcal{G}_1 \arrow[rr, "\alpha_1"] \arrow[dl] \arrow[dd] & & \mathcal{H}_1 \arrow[dl] \arrow[dd] \\ \mathcal{G}_0 \arrow[rr, "\alpha_0", near start] \arrow[dd] & & \mathcal{H}_0 \arrow[dd] & \\ & \mathcal{G}_0 \arrow[rr, "\alpha_0", near start] \arrow[dl] & & \mathcal{H}_0 \arrow[dl] \\ |\mathcal{G}| \arrow[rr, "|\alpha|"] & & |\mathcal{H}|. & \end{tikzcd}$$

The general case then follows by induction by applying the pasting law of pullback squares to the diagram

$$\begin{tikzcd}[row sep=small, column sep=small] & \mathcal{G}_n \arrow[rr, "\alpha_n"] \arrow[dl] \arrow[dd] & & \mathcal{H}_n \arrow[dl] \arrow[dd] \\ \mathcal{G}_{n-1} \arrow[rr, "\alpha_{n-1}", near start] \arrow[dd] & & \mathcal{H}_{n-1} \arrow[dd] & \\ & \mathcal{G}_1 \arrow[rr, "\alpha_1", near start] \arrow[dl] & & \mathcal{H}_1 \arrow[dl] \\ \mathcal{G}_0 \arrow[rr, "\alpha_0"] & & \mathcal{H}_0. & \end{tikzcd}$$

This finishes the proof. $\square$

E.3 Groupoid actions and principal bundles

In this subsection we discuss $\mathcal{G}$-actions for a groupoid object $\mathcal{G}$ in an ∞-topos $\mathcal{B}$ and prove that the ∞-category of $\mathcal{G}$-actions is equivalent to the slice of $\mathcal{B}$ over the classifying stack $\mathbb{B}\mathcal{G}$ of $\mathcal{G}$. We further recall the notion of a principal $\mathcal{G}$-bundle over an object $B \in \mathcal{B}$, and show that these are classified by morphisms $B \to \mathbb{B}\mathcal{G}$. Our treatment is based on [NSS15] and [SS21, Section 3.2], where the analogous situation for *group* actions is discussed.

Actions by groupoid objects

We introduce the notion of an action of a groupoid object in an ∞-topos $\mathcal{B}$, generalizing the treatment of Sati and Schreiber [SS21] of group actions in ∞-topoi.

Definition E.17. Let $\mathcal{G}$ be a groupoid object in an ∞-topos $\mathcal{B}$, and consider an object $X \in \mathcal{B}$. Then an *action* of $\mathcal{G}$ on X consists of the following data:

- a groupoid object $\mathcal{G} \ltimes X$ in $\mathcal{B}$;
- an equivalence $(\mathcal{G} \ltimes X)_0 \simeq X$;
- a *cartesian* natural transformation $c\colon \mathcal{G} \ltimes X \to \mathcal{G}$ of simplicial objects in $\mathcal{B}$.

We let $\mathrm{Act}_{\mathcal{G}}(\mathcal{B}) \subseteq \mathrm{Grpd}(\mathcal{B})_{/\mathcal{G}}$ denote the full subcategory of $\mathcal{G}$-actions in $\mathcal{B}$.

Remark E.18. Let $\mathcal{G} \ltimes X$ be an action of $\mathcal{G}$ on X. Since $(\mathcal{G} \ltimes X)_0 \simeq X$, the map $c\colon \mathcal{G} \ltimes X \to \mathcal{G}$ induces a structure map $c_0\colon X \to \mathcal{G}_0$. Since c is a cartesian natural transformation, there are cartesian squares

$$\begin{tikzcd}
\cdots \arrow[r] & (\mathcal{G} \ltimes X)_2 \arrow[r, "d_0"] \arrow[d, "c_2"] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{G} \ltimes X)_1 \arrow[r, "d_0"] \arrow[d, "c_1"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "c_0"] \\
\cdots \arrow[r] & \mathcal{G}_2 \arrow[r, "d_0"] & \mathcal{G}_1 \arrow[r, "t=d_0"] & \mathcal{G}_0,
\end{tikzcd}$$

and in particular we obtain equivalences

$$\begin{aligned}
(\mathcal{G} \ltimes X)_1 &\xrightarrow{\sim} \mathcal{G}_1 \times_{\mathcal{G}_0} X; \\
(\mathcal{G} \ltimes X)_2 &\xrightarrow{\sim} \mathcal{G}_2 \times_{\mathcal{G}_0} X \simeq \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} X; \\
&\vdots
\end{aligned}$$

Under these equivalences, the map $s = d_1\colon (\mathcal{G} \ltimes X)_1 \to (\mathcal{G} \ltimes X)_0$ corresponds to a map $a\colon \mathcal{G}_1 \times_{\mathcal{G}_0} X \to X$, thought of as the *action map*. The rest of the simplicial diagram $\mathcal{G} \ltimes X$ should be thought of the data witnessing that this action is unital and associative up to coherent homotopy.

Definition E.19. If $\mathcal{G} \ltimes X$ is an action of a groupoid object $\mathcal{G}$ on an object $X \in \mathcal{B}$, we define the *quotient* $X//\mathcal{G}$ as the geometric realization of $\mathcal{G} \ltimes X$:

$$X//\mathcal{G} := |\mathcal{G} \ltimes X| \simeq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathcal{G}_n \times_{\mathcal{G}_0} X.$$

The map $\mathcal{G} \ltimes X \to \mathcal{G}$ of groupoids induces a map $X//\mathcal{G} = |\mathcal{G} \ltimes X| \to |\mathcal{G}| = \mathbb{B}\mathcal{G}$, and this defines a functor $-//\mathcal{G}\colon \mathrm{Act}_{\mathcal{G}}(\mathcal{B}) \to \mathcal{B}_{/\mathbb{B}\mathcal{G}}$.

Proposition E.20 (Classification of groupoid-actions, cf. [SS21, Proposition 3.2.63])**.** *The quotient functor $-//\mathcal{G}$ is an equivalence of ∞-categories:*

$$-//\mathcal{G}\colon \mathrm{Act}_{\mathcal{G}}(\mathcal{B}) \xrightarrow{\sim} \mathcal{B}_{/\mathbb{B}\mathcal{G}}.$$

Proof. By definition, $\mathrm{Act}_{\mathcal{G}}(\mathcal{B})$ is a full subcategory of the slice $\mathrm{Grpd}(\mathcal{B})_{/\mathcal{G}}$ spanned by those morphisms into $\mathcal{G}$ which are cartesian. Similarly, the slice $\mathcal{B}_{/\mathbb{B}\mathcal{G}}$ embeds as a full subcategory of the slice $\mathrm{Atl}(\mathcal{B})_{/(\mathcal{G}_0 \to \mathbb{B}\mathcal{G})}$ by sending a morphism $B \to \mathbb{B}\mathcal{G}$ to the square

$$\begin{tikzcd}
B \times_{\mathbb{B}\mathcal{G}} \mathcal{G}_0 \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_0 \arrow[d, two heads] \\
B \arrow[r] & \mathbb{B}\mathcal{G}.
\end{tikzcd}$$

The essential image of this embedding $\mathcal{B}_{/\mathbb{B}\mathcal{G}} \hookrightarrow \mathrm{Atl}(\mathcal{B})_{/(\mathcal{G}_0 \to \mathbb{B}\mathcal{G})}$ precisely consists of the pullback squares. Under these identifications, the functor $-//\mathcal{G}\colon \mathrm{Act}_{\mathcal{G}}(\mathcal{B}) \to \mathcal{B}_{/\mathbb{B}\mathcal{G}}$ is given by the restriction of the equivalence $\mathrm{Grpd}(\mathcal{G})_{/\mathcal{G}} \simeq \mathrm{Atl}(\mathcal{B})_{/(\mathcal{G}_0 \to \mathbb{B}\mathcal{G})}$ of Proposition E.13. By Lemma E.16, the subcategory $\mathrm{Act}_{\mathcal{G}}(\mathcal{B})$ on the left precisely corresponds to the subcategory $\mathcal{B}_{/\mathbb{B}\mathcal{G}}$ on the right, finishing the proof □

Free and transitive actions

Given a $\mathcal{G}$-action in $\mathcal{B}$, one can define when the action is *free* or *transitive*. Actions which are both free and transitive will correspond to principal $\mathcal{G}$-bundles over the terminal object of $\mathcal{B}$.

Definition E.21 (Higher shear maps, cf. [SS21, Definition 3.2.73])**.** Let $\mathcal{G} \ltimes X$ be an action of a groupoid object $\mathcal{G}$ on an object X in the fixed ∞-topos $\mathcal{B}$. We define the *shear map* of $\mathcal{G} \ltimes X$ as the map

$$\mathrm{shear}_1 = (a, \mathrm{pr}_X)\colon \mathcal{G}_1 \times_{\mathcal{G}_0} X \to X \times X,$$

where $a\colon \mathcal{G}_1 \times_{\mathcal{G}_0} X \to X$ is the action map and pr_X is the projection onto X. More generally, we define the *higher shear maps* of $\mathcal{G} \ltimes X$ as follows. Consider the following commutative square in $\mathcal{B}$:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, two heads] & X \arrow[d] \\
X//\mathcal{G} \arrow[r] & \mathrm{pt}.
\end{tikzcd}$$

Regarding this as a morphism from $X \twoheadrightarrow X//\mathcal{G}$ to $X \to \mathrm{pt}$ in $\mathrm{Fun}(\Delta^1, \mathcal{B})$ and passing to Čech nerves, we obtain a morphism of groupoid objects

$$\mathrm{shear}\colon (\mathcal{G} \ltimes X) \to \check{C}(X \to \mathrm{pt}).$$

Under the identifications of Remark E.18, it may be displayed as follows:

$$\begin{tikzcd}
\mathcal{G} \ltimes X \arrow[d, "\mathrm{shear}"] & = & \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathcal{G}_2 \times_{\mathcal{G}_0} X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\mathrm{shear}_2"] & \mathcal{G}_1 \times_{\mathcal{G}_0} X \arrow[r, shift left, "a"] \arrow[r, shift right, "\mathrm{pr}_X"'] \arrow[d, "\mathrm{shear}_1"] & X \arrow[d, equal, "\mathrm{shear}_0"] \\
\check{C}(X \to \mathrm{pt}) & = & \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X \times X \times X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X \times X \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X.
\end{tikzcd}$$

Definition E.22 (Free, transitive and regular ∞-actions, cf. [SS21, Definition 3.2.75])**.** Let $\mathcal{G} \ltimes X$ be an action of a groupoid object $\mathcal{G}$ on an object $X \in \mathcal{B}$. We say that this action is

- *free* if its shear map shear_1 is a monomorphism:
$$\mathcal{G} \ltimes X \text{ is free} \quad \Leftrightarrow \quad \mathcal{G}_1 \times_{\mathcal{G}_0} X \xhookrightarrow{\mathrm{shear}_1} X \times X\,;$$
- *transitive* if its shear map shear_1 is an effective epimorphism:
$$\mathcal{G} \ltimes X \text{ is transitive} \quad \Leftrightarrow \quad \mathcal{G}_1 \times_{\mathcal{G}_0} X \xrightarrow{\mathrm{shear}_1} \!\!\!\!\rightarrow X \times X\,;$$
- *regular* if its shear map shear_1 is an equivalence:
$$\mathcal{G} \ltimes X \text{ is regular} \quad \Leftrightarrow \quad \mathcal{G}_1 \times_{\mathcal{G}_0} X \xrightarrow[\sim]{\mathrm{shear}_1} X \times X\,.$$

Note that $\mathcal{G} \ltimes X$ is regular if and only if it is free and transitive.

In case the morphism $X \to \mathrm{pt}$ is an effective epimorphism in $\mathcal{B}$, regularity of $\mathcal{G} \ltimes X$ can be expressed as the triviality of the quotient $X//\mathcal{G}$:

Lemma E.23 (cf. [SS21, Proposition 3.2.77])**.** *Let $X \in \mathcal{B}$ be an object such that the terminal morphism $X \to \mathrm{pt}$ is an effective epimorphism in $\mathcal{B}$. Let $\mathcal{G} \ltimes X$ be an action of a groupoid object on X. Then the action $\mathcal{G} \ltimes X$ is regular if and only if the quotient $X//\mathcal{G}$ is the terminal object of $\mathcal{B}$.*

Proof. First assume that the map $X//\mathcal{G} \to \mathrm{pt}$ is an equivalence. Then the map $X \twoheadrightarrow X//\mathcal{G}$ is equivalent to the map $X \to \mathrm{pt}$. It follows that the shear map $\mathrm{shear}\colon (\mathcal{G} \ltimes X) \to \check{C}(X \to \mathrm{pt})$ is an equivalence, being defined by applying Čech nerves to these two morphisms.

Conversely, assume that $\mathcal{G} \times X$ is regular and that $X \twoheadrightarrow \mathrm{pt}$ is an effective epimorphism. Then the map $X//G \to \mathrm{pt}$ is obtained from the higher shear map $\mathrm{shear}\colon (\mathcal{G} \ltimes X) \to \check{C}(X \to \mathrm{pt})$ by passing to colimits. It will thus suffice to show that each of the higher shear maps $\mathrm{shear}_n\colon \mathcal{G}_n \times_{\mathcal{G}_0} X \to X^n$ is an equivalence. We proceed by induction. The case $n = 1$ is assumed, so assume that shear_{n-1} is an equivalence. Consider the following diagram:

$$
\begin{tikzcd}
& \mathcal{G}_n \times_{\mathcal{G}_0} X \arrow[rr, "\mathrm{shear}_n"] \arrow[dl] \arrow[dd] & & X^{n+1} \arrow[dl] \arrow[dd] \\
\mathcal{G}_{n-1} \times_{\mathcal{G}_0} X \arrow[rr, "\mathrm{shear}_{n-1}", "\simeq"'] \arrow[dd] & & X^n \arrow[dd] & \\
& \mathcal{G}_1 \times_{\mathcal{G}_0} X \arrow[rr, "\mathrm{shear}_1", "\simeq"'] \arrow[dl] & & X^2 \arrow[dl] \\
X \arrow[rr, equal, "\mathrm{shear}_0"] & & X &
\end{tikzcd}
$$

Since the left and right squares are pullback squares, it follows that shear_n is an equivalence as well, finishing the proof. □

Principal bundles

Given a groupoid object $\mathcal{G}$ in $\mathcal{B}$ and an object $B \in \mathcal{B}$, we obtain a notion of *principal $\mathcal{G}$-bundles over B* by applying the above definition of regular actions to the slice topos $\mathcal{B}_{/B}$. Observe that $\mathcal{G}$ gives rise to a groupoid $\mathcal{G} \times B$ in the slice $\mathcal{B}_{/B}$ by applying the pullback-preserving functor $B \times -\colon \mathcal{B} \to \mathcal{B}_{/B}$. For every object $X \in \mathcal{B}_{/B}$, there is an equivalence

$$(\mathcal{G}_n \times B) \times_{\mathcal{G}_0 \times B} X \simeq \mathcal{G}_n \times_{\mathcal{G}_0} X \quad \in \quad \mathcal{B}_{/B},$$

and thus a $(\mathcal{G} \times B)$-action on X is nothing but a $\mathcal{G}$-action $\mathcal{G} \ltimes X\colon \Delta^{\mathrm{op}} \to \mathcal{B}$ of $\mathcal{G}$ on the underlying object of X together with a lift of $\mathcal{G} \ltimes X$ along the forgetful functor $\mathcal{B}_{/B} \to \mathcal{B}$. For this reason, we will refer to a $(\mathcal{G} \times B)$-action in the slice $\mathcal{B}_{/B}$ simply as a *$\mathcal{G}$-action over B*.

Definition E.24 (Principal bundles, cf. [SS21, Definition 3.2.79])**.** Consider an object $B \in \mathcal{B}$ and let $\mathcal{G} \in \mathrm{Grpd}(\mathcal{B})$ be a groupoid in $\mathcal{B}$.

(1) Let $p\colon P \to B$ be an object of $\mathcal{B}_{/B}$ equipped with a $\mathcal{G}$-action over B. We say that the map p is a *formally principal $\mathcal{G}$-bundle over B* if this action corresponds to a regular $(\mathcal{G} \times B)$-action in the slice $\mathcal{B}_{/B}$:

$$\mathcal{G}_1 \times_{\mathcal{G}_0} P \xrightarrow[\sim]{\mathrm{shear}_1} P \times_B P\,.$$

(2) A formally principal $\mathcal{B}$-bundle $p\colon P \to B$ is called a *principal $\mathcal{G}$-bundle* if it is an effective epimorphism.

(3) We write

$$\mathrm{PrnBdl}_{\mathcal{G}}(\mathcal{B})_B \hookrightarrow \mathrm{Act}_{\mathcal{G}}(\mathcal{B}_{/B})$$

for the full subcategory of principal $\mathcal{G}$-bundles over B.

Remark E.25. One observes that for a morphism $f\colon B' \to B$ in $\mathcal{B}$, the pullback functor $f^*\colon \mathcal{B}_{/B} \to \mathcal{B}_{/B'}$ sends principal $\mathcal{G}$-bundles to principal $\mathcal{G}$-bundles.

We end this section with a classification of principal $\mathcal{G}$-bundles over B in terms of morphisms $B \to \mathbb{B}\mathcal{G}$ in $\mathcal{B}$.

Lemma E.26. *Let $p\colon P \twoheadrightarrow B$ be an effective epimorphism in $\mathcal{B}$ which comes equipped with a $\mathcal{G}$-action over B. Let $P//\mathcal{G} \in \mathcal{B}_{/B}$ denote the quotient of the $\mathcal{G}$-action. Then the map $p\colon P \to B$ is a principal $\mathcal{G}$-bundle if and only if the canonical map $P//\mathcal{G} \to B$ is an equivalence.*

Proof. This is an instance of Lemma E.23. □

It follows from Lemma E.26 that the forgetful functor from principal $\mathcal{G}$-bundles over B to $\mathcal{G}$-actions in $\mathcal{B}$ lands in the fiber over $B \in \mathcal{B}$ of the quotient functor $-//\mathcal{G}\colon \mathrm{Act}_{\mathcal{G}}(\mathcal{B}) \to \mathcal{B}$.

Proposition E.27. *The functor*

$$\mathrm{PrnBdl}_{\mathcal{G}}(\mathcal{B})_B \to \mathrm{Act}_{\mathcal{G}}(\mathcal{B}) \times_{\mathcal{B}} \{B\}$$

is an equivalence of ∞-categories.

Proof. We will construct an explicit inverse of this functor. Consider a $\mathcal{G}$-action $\mathcal{G} \ltimes P$ in $\mathcal{B}$ equipped with an equivalence $B \simeq P//\mathcal{G}$. The cocone $(\Delta^{\mathrm{op}})^{\triangleright} \to \mathcal{B}$ which exhibits $B \simeq P//\mathcal{G}$ as a colimit of $\mathcal{G} \ltimes P$ then adjoints over to provide a lift of the simplicial object $\mathcal{G} \ltimes P\colon \Delta^{\mathrm{op}} \to \mathcal{B}$ along the forgetful functor $\mathcal{B}_{/B} \to \mathcal{B}$; we will abuse notation and again denote this lift by $\mathcal{G} \ltimes P$. By adjunction, this lift comes equipped with a groupoid map to $\mathcal{G} \times B$, and since the forgetful functor $\mathcal{B}_{/B} \to B$ preserves pullbacks this is a cartesian morphism of groupoids, thus defining a $(\mathcal{G} \times B)$-action in $\mathcal{G}_{/B}$. This construction thus defines a functor

$$\mathrm{Act}_{\mathcal{G}}(\mathcal{B}) \times_{\mathcal{B}} \{B\} \to \mathrm{Act}_{\mathcal{G}}(\mathcal{B}_{/B}).$$

It follows directly from Lemma E.26 that this functor factors through the subcategory $\mathrm{PrnBdl}_{\mathcal{G}}(\mathcal{B})_B$ of principal $\mathcal{G}$-bundles over B. It is clear that the two functors we have constructed are inverse to each other, finishing the proof. □

Theorem E.28 (Classification of principal $\mathcal{G}$-bundles, cf. [SS21, Theorem 3.2.82]). *Given a groupoid object $\mathcal{G} \in \mathrm{Grpd}(\mathcal{B})$ and an object $B \in \mathcal{B}$, there is an equivalence of ∞-categories*

$$\mathrm{Hom}_{\mathcal{B}}(B, \mathbb{B}\mathcal{G}) \xrightarrow{\sim} \mathrm{PrnBdl}_{\mathcal{G}}(\mathcal{B})_B,$$

given on objects by sending a morphism $c\colon B \to \mathbb{B}\mathcal{G}$ to the principal $\mathcal{G}$-bundle $P \to B$ defined by the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_0 \arrow[d, two heads] \\ B \arrow[r, "c"] & \mathbb{B}\mathcal{G}. \end{tikzcd}$$

Proof. Recall from Proposition E.20 the equivalence $-//\mathcal{G}\colon \mathrm{Act}_{\mathcal{G}}(\mathcal{B}) \xrightarrow{\sim} \mathcal{B}_{/\mathbb{B}\mathcal{G}}$, which induces an equivalence on fibers over $\mathcal{B}$:

$$\mathrm{Act}_{\mathcal{G}}(\mathcal{B}) \times_{\mathcal{B}} \{B\} \simeq \mathcal{B}_{/\mathbb{B}\mathcal{G}} \times_{\mathcal{B}} \{B\} \simeq \mathrm{Hom}_{\mathcal{B}}(B, \mathbb{B}\mathcal{G}).$$

Its inverse sends a morphism $c\colon B \to \mathbb{B}\mathcal{G}$ to the base change $P = B \times_{\mathbb{B}\mathcal{G}} \mathcal{G}_0 \twoheadrightarrow B$ along c of the map $\mathcal{G}_0 \twoheadrightarrow \mathbb{B}\mathcal{G}$. Combining this equivalence with the equivalence from Proposition E.27 then finishes the proof. □

Corollary E.29. *For every groupoid object $\mathcal{G}$ in $\mathcal{B}$ and every object $B \in \mathcal{B}$, the ∞-category* $\mathrm{PrnBdl}_{\mathcal{G}}(\mathcal{B})_B$ *is an ∞-groupoid.* □

E.4 Sheaf topoi

We recall the definition of a Grothendieck topology τ on an ∞-category C and the definition of the associated ∞-topos $\mathrm{Shv}_\tau(C)$ of τ-sheaves on C.

Definition E.30 (Sieve, [Lur09, Definition 6.2.2.1]). Let C be an ∞-category. A *sieve on* C is a full subcategory $C^{(0)} \subseteq C$ having the property that if $f\colon X \to Y$ is a morphism in C and Y belongs to $C^{(0)}$, then X also belongs to $C^{(0)}$. If $X \in C$ is an object, then a *sieve on X* is a sieve on the ∞-category $C_{/X}$. Given a morphism $f\colon X \to Y$ and a sieve $C^{(0)}_{/Y}$ on Y, we let $f^*C^{(0)}_{/Y} \subseteq C_{/X}$ denote subcategory spanned by those morphisms $(g\colon Z \to X) \in C_{/X}$ such that $f \circ g \in C^{(0)}_{/Y}$.

Remark E.31. By [Lur09, Lemma 6.2.2.5], the data of a sieve $C(0)$ on C is equivalenc to the data of a subterminal object R of $\mathrm{PSh}(C)$: $C^{(0)}$ defines the presheaf

$$R(X) = \begin{cases} \mathrm{pt} & X \in C^{(0)}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Because of the equivalence $\mathrm{PSh}(C_{/X}) \simeq \mathrm{PSh}(C)_{/y(X)}$, it follows that the data of a sieve on X is the same as the data of a monomorphism $R \hookrightarrow y(X)$ of presheaves on C, where $y(X) := \mathrm{Hom}_C(-, X) \colon C^{\mathrm{op}} \to \mathrm{Spc}$ is the presheaf represented by X.

Definition E.32 (Grothendieck topology, [Lur09, Definition 6.2.2.1])**.** Let C be an ∞-category. A *Grothendieck topology* on C consists of a specification, for each object X of C, of a collection of sieves on C which we will refer to as *covering sieves*. The collections of covering sieves are required to satisfy the following properties:

(1) For every object X of C, the sieve $C_{/X} \subseteq C_{/X}$ on X is a covering sieve;

(2) For every morphism $f\colon X \to Y$ in C and every covering sieve $C^{(0)}_{/Y}$ on Y, the sieve $f^*C^{(0)}_{/Y}$ is a covering sieve on X;

(3) Let X be an object of C, $C^{(0)}_{/X}$ a covering sieve on X, and $C^{(1)}_{/X}$ an arbitrary sieve on X. Suppose that, for every morphism $f\colon Y \to X$ belonging to the sieve $C^{(0)}_{/X}$, the pullback sieve $f^*C^{(1)}_{/X}$ is a covering sieve on Y. Then $C^{(1)}_{/X}$ is a covering sieve on X.

A *site* is an ∞-category C equipped with a Grothendieck topology.

Definition E.33 (Sheaf category, [Lur09, Definition 6.2.2.6])**.** Let C be an ∞-category equipped with a Grothendieck topology τ and let S denote the collections of monomorphisms $\{R \hookrightarrow y(X)\}$ which corresponds to covering sieves on X. An object $\mathcal{F} \in \mathrm{PSh}(C)$ is called a *τ-sheaf*, or simply a *sheaf* if τ is clear from context, if it is S-local: for every covering sieve $R \hookrightarrow y(X)$, the induced map of spaces

$$\mathcal{F}(X) = \mathrm{Hom}_{\mathrm{PSh}(C)}(y(X), \mathcal{F}) \to \mathrm{Hom}_{\mathrm{PSh}(C)}(R, \mathcal{F})$$

is an equivalence. We let $\mathrm{Shv}_\tau(C) \subseteq \mathrm{PSh}(C)$ denote the full subcategory of sheaves with respect to τ.

Proposition E.34 ([Lur09, Proposition 6.2.2.7])**.** *The inclusion* $\mathrm{Shv}_\tau(C) \subseteq \mathrm{PSh}(C)$ *admits a left exact accessible left adjoint* $L_\tau\colon \mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C)$. *In particular,* $\mathrm{Shv}_\tau(C)$ *is an* ∞*-topos.* □

The functor L_τ is known as the *sheafification functor*.

An equivalent description of the ∞-category $\mathrm{Shv}_\tau(C)$ is in terms of *Čech descent*.

Definition E.35 (Čech descent)**.** Let C be an ∞-category and let $\mathcal{U} = \{U_i \to X\}_{i \in I}$ be a collection of morphisms. We will denote by $\check{C}(\mathcal{U})$ the Čech nerve of the morphism

$$\bigsqcup_{i \in I} y(U_i) \to y(X)$$

in $\mathrm{PSh}(C)$. A presheaf $\mathcal{F} \in \mathrm{PSh}(C)$ is said to *satisfy Čech descent with respect to* $\mathcal{U}$ if the map

$$\mathcal{F}(X) = \mathrm{Hom}_{\mathrm{PSh}(C)}(y(X), \mathcal{F}) \to \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{PSh}(C)}(\check{C}(\mathcal{U})_n, \mathcal{F})$$

is an equivalence. More concretely, $\mathcal{F}$ satisfies Čech descent with respect to $\mathcal{U}$ if the diagram

$$\mathcal{F}(X) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_X U_j) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots$$

is a limit diagram.

Proposition E.36 (cf. [Lur09, Lemma 6.2.3.18])**.** *Let $\mathcal{U} = \{U_i \to X\}_{i\in I}$ be a collection of morphisms in an ∞-category C and let $R \hookrightarrow X$ be the covering sieve generated by $\mathcal{U}$. Then a presheaf $\mathcal{F}$ on C satisfies Čech descent with respect to $\mathcal{U}$ if and only if it is local with respect to $R \hookrightarrow X$.*

Proof. The morphism $\bigsqcup_{i\in I} y(U_i) \to y(X)$ factors as an effective epimorphism followed by a monomorphism:

$$\bigsqcup_{i\in I} y(U_i) \twoheadrightarrow R \hookrightarrow y(X).$$

By [Lur09, Lemma 6.2.3.18], the map $R \hookrightarrow y(X)$ is precisely the monomorphism corresponding to the covering sheaf generated by $\mathcal{U}$. It follows that R is equivalent to the colimit of the Čech nerve $\check{C}(\mathcal{U})$ of $\mathcal{U}$, and we obtain for every $\mathcal{F} \in \mathrm{PSh}(C)$ an equivalence

$$\mathrm{Hom}_{\mathrm{PSh}(C)}(R, \mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{PSh}(C)}(\mathrm{colim}_{[n]\in\Delta} \check{C}(\mathcal{U})_n, \mathcal{F}) \simeq \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{PSh}(C)}(\check{C}(\mathcal{U})_n, \mathcal{F}).$$

As this map is compatible with the map from $\mathcal{F}(X)$, the claim follows. $\square$

The effective epimorphisms in a sheaf topos $\mathrm{Shv}_\tau(C)$ can be characterized as those morphisms which *admit local sections*, in the following sense:

Definition E.37. Let C be an ∞-category equipped with a Grothendieck topology τ. Let $f\colon X \to Y$ be a morphism in $\mathrm{Shv}_\tau(C)$ and assume that $Y = y(C)$ lies in the image of the sheafified Yoneda functor $y\colon C \to \mathrm{Shv}_\tau(C)$. We say that f *admits local sections* if there exists a covering family $\{U_i \to C\}_{i\in I}$ of C such that the base change $f'\colon X \times_Y y(U_i) \to y(U_i)$ of f along the map $y(U_i) \to Y$ admits a section for every $i \in I$.

If $f\colon X \to Y$ is an arbitrary morphism in $\mathrm{Shv}_\tau(C)$, we say that f *admits local sections* if its base change along any map $y(C) \to Y$ from a representable admits local sections.

Lemma E.38. *A morphism $f\colon X \to Y$ in $\mathrm{Shv}_\tau(C)$ is an effective epimorphism if and only if it admits local sections.*

Proof. By [Lur09, Proposition 6.2.3.15], a morphism $f\colon X \to Y$ is an effective epimorphism if and only if its base change along any map $y(C) \to Y$ is an effective epimorphism. We may thus assume that $Y = y(C)$ is a representable object.

Consider the sieve $C^{(f)}_{/C} \subseteq C_{/C}$ on C generated by the map $f\colon X \to C$:

$$C^{(f)}_{/C} := \{g\colon C' \to C \mid y(g)\colon y(C') \to y(C) \text{ factors through } f\colon X \to y(C)\}.$$

Notice that a morphism $g\colon C' \to C$ factors through f if and only if the base change $y(C') \times_{y(C)} X \to y(C')$ admits a section. Hence we have to prove that f is an effective epimorphism if and only if $C^{(f)}_{/C}$ contains an covering family of C, i.e., if $C^{(f)}_{/C}$ is a covering sieve. Under the equivalence of sieves over C and monomorphisms $Z \hookrightarrow y(C)$ from Remark E.31, the sieve $C^{(f)}_{/C}$ corresponds to a monomorphism $\tau_{\leq -1}(f) \hookrightarrow y(C)$ which by (a slight generalization of) [Lur09, Lemma 6.2.3.18] is equivalent to the (-1)-truncation of f inside the slice $\mathrm{PSh}(C)_{/C}$. By [Lur09, Lemma 6.2.2.16] the sieve $C^{(f)}_{/C}$ is a covering sieve if and only if the sheafification of this map $\tau_{\leq -1}(f) \hookrightarrow C$ is an equivalence in $\mathrm{Shv}_\tau(C)$. By [Lur09, Proposition 6.2.3.4], the (-1)-truncation $\tau_{\leq -1}(f)$ of f is computed by the colimit of its Čech nerve inside $\mathrm{PSh}(C)$. Since colimits in $\mathrm{Shv}_\tau(C)$ are computed by sheafifying the colimit in $\mathrm{PSh}(C)$, it follows that $C^{(f)}_{/C}$ is a covering sieve if and only if the colimit of the Čech nerve $\check{C}(f)$ inside $\mathrm{Shv}_\tau(C)$ is equivalent to c, which is by definition what it means for $f\colon X \to y(C)$ to be an effective epimorphism. This finishes the proof. □

Morphisms of sites

We discuss continuous and cocontinuous functors between Grothendieck sites.

Definition E.39 (Continuous functor between sites)**.** Let (C, τ) and $(\mathcal{D}, \tau')$ be ∞-categories equipped with Grothendieck topologies. A functor $u\colon C \to \mathcal{D}$ is called *continuous* if the functor $\mathrm{PSh}(\mathcal{D}) \to \mathrm{PSh}(C)$ given by precomposition with u restricts to a functor

$$u_*\colon \mathrm{Shv}_{\tau'}(\mathcal{D}) \to \mathrm{Shv}_\tau(C),$$

i.e., sends τ'-sheaves on $\mathcal{D}$ to τ-sheaves on C. In this case, u_* admits a left adjoint

$$u^*\colon \mathrm{Shv}_\tau(C) \to \mathrm{Shv}_{\tau'}(\mathcal{D})$$

given by the composite $\mathrm{Shv}_\tau(C) \hookrightarrow \mathrm{PSh}(C) \xrightarrow{\mathrm{LKE}_u} \mathrm{PSh}(\mathcal{D}) \xrightarrow{L_{\tau'}} \mathrm{Shv}_{\tau'}(\mathcal{D})$, where LKE_u denotes left Kan extension along u.

Remark E.40. Assume that C and $\mathcal{D}$ admit pullbacks and that u preserves pullbacks. Then $u\colon C \to \mathcal{D}$ is continuous if and only if u sends τ-covering sieves to τ'-covering sieves.

Definition E.41 (Morphism of sites)**.** Let (C, τ) and $(\mathcal{D}, \tau')$ be ∞-categories equipped with Grothendieck topologies. A functor $u: C \to \mathcal{D}$ is called a *morphism of sites* $\mathcal{D} \to C$ if it is a continuous functor and the pullback functor

$$u^*: \operatorname{Shv}_\tau(C) \to \operatorname{Shv}_{\tau'}(\mathcal{D})$$

preserves finite limits. In other words, the functor $u_*: \operatorname{Shv}_{\tau'}(\mathcal{D}) \to \operatorname{Shv}_\tau(C)$ is a geometric morphism of ∞-topoi.

Example E.42. Let $u: C \to \mathcal{D}$ be a continuous functor and assume that u admits a left adjoint $u': \mathcal{D} \to C$. Then the functor $\operatorname{LKE}_u: \operatorname{PSh}(C) \to \operatorname{PSh}(\mathcal{D})$ given by left Kan extension along u is equivalent to the restriction functor along u'. In particular, the functor u^* is a composite of three left exact functors and thus is itself left exact. It follows that u is a morphism of sites.

Definition E.43 (Cocontinuous functor between sites)**.** Let (C, τ) and $(\mathcal{D}, \tau')$ be ∞-categories equipped with Grothendieck topologies. A functor $v: C \to \mathcal{D}$ is called *cocontinuous* if for every object $X \in C$ and every covering sieve $\mathcal{D}^{(0)}_{/v(X)} \subseteq \mathcal{D}_{/v(X)}$ on $v(X) \in \mathcal{D}$, the pullback sieve

$$v^*\mathcal{D}^{(0)}_{/X} := \{(\psi: Y \to X) \in C_{/X} \mid (v(\psi): v(Y) \to v(X)) \in \mathcal{D}^{(0)}_{/v(X)}\} \subseteq C_{/X}$$

is a covering sieve on X.

Lemma E.44. *Let $v: (C, \tau) \to (\mathcal{D}, \tau')$ be a functor between sites. Then v is cocontinuous if and only if the right Kan extension functor* $\operatorname{PSh}(C) \to \operatorname{PSh}(\mathcal{D})$ *restricts to a functor*

$$v_*: \operatorname{Shv}_\tau(C) \to \operatorname{Shv}_{\tau'}(\mathcal{D}),$$

i.e., sends τ-sheaves on C to τ'-sheaves on $\mathcal{D}$. In this case, v_* admits a left exact left adjoint v^*, so that v_* is a geometric morphism of ∞-topoi.

Proof. The right Kan extension functor sends τ-sheaves on C to τ'-sheaves on $\mathcal{D}$ if and only if the composite

$$\operatorname{PSh}(\mathcal{D}) \xrightarrow{-\circ v} \operatorname{PSh}(C) \xrightarrow{L_\tau} \operatorname{Shv}_\tau(C)$$

sends every covering sieve $R \hookrightarrow y(X)$ to an equivalence. Since this composite preserves finite limits, it preserves monomorphisms, and thus the image of $R \hookrightarrow y(X)$ is again a monomorphism. It follows that the map $R \hookrightarrow y(X)$ is sent to an equivalence in $\operatorname{Shv}_\tau(C)$ if and only if it is sent to an effective epimorphism. By Lemma E.38, this is in turn equivalent to the statement that this map admits local sections. Spelling out the definition, this is precisely the condition that f is cocontinuous.

The last statement holds because the left adjoint v^* of v_* is given by the composite

$$\mathrm{Shv}_{\tau'}(\mathcal{D}) \hookrightarrow \mathrm{PSh}(\mathcal{D}) \xrightarrow{-\circ v} \mathrm{PSh}(C) \xrightarrow{L_\tau} \mathrm{Shv}_\tau(C),$$

and each of these three functors is left-exact. □

Corollary E.45. *Let $v\colon (C,\tau) \to (\mathcal{D},\tau')$ be a cocontinuous functor between sites. Then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{D}) \arrow[r, "-\circ v"] \arrow[d, "L_{\tau'}"'] & \mathrm{PSh}(C) \arrow[d, "L_\tau"] \\ \mathrm{Shv}_{\tau'}(\mathcal{D}) \arrow[r, "v^*"] & \mathrm{Shv}_\tau(C). \end{tikzcd}$$

Proof. This diagram is obtained by passing to left adjoints from the fact that right Kan extension along v preserves sheaves by Lemma E.44. □

Warning E.46. Let $u\colon (C,\tau) \hookrightarrow (\mathcal{D},\tau')$ is a functor between sites which is both continuous and cocontinuous, then the notation u_* is overloaded: it could both stand for restriction along u as well as for right Kan extension along u. In such cases, we will explicitly state what the intended meaning of u_* is.

Corollary E.47. *Let $v\colon (C,\tau) \hookrightarrow (\mathcal{D},\tau')$ be a functor between sites. Assume that v is fully faithful and cocontinuous. Then the geometric morphism*

$$v_*\colon \mathrm{Shv}_\tau(C) \hookrightarrow \mathrm{Shv}_{\tau'}(\mathcal{D})$$

of Lemma E.44 is fully faithful.

Proof. This is immediate from the fact that the right Kan extension functor along a fully faithful functor is again fully faithful. □

Corollary E.48. *Let $v\colon (C,\tau) \hookrightarrow (\mathcal{D},\tau')$ be a functor between sites and assume that v admits a right adjoint $u\colon \mathcal{D} \to C$. Then v is cocontinuous if and only if u is a morphism of sites. In this case, we have $v_* \simeq u_*$ and $v^* \simeq u^*$.*

Proof. Observe that right Kan extension along v is equivalent to restriction along u, and thus the former preserves sheaves if and only if the latter does. This shows that v is cocontinuous if and only if u is continuous, and that in this case we have $v_* = u_*$ and thus $v^* = u^*$. It follows from Example E.42 that u is continuous if and only if it is a morphism of sites. This finishes the proof. □

E.5 Hypercompleteness

We discuss the notion of hypercomplete ∞-topoi. Throughout, we work in a fixed ∞-topos $\mathcal{B}$.

Definition E.49 (n-truncated morphisms)**.** A morphism $f: X \to Y$ in $\mathcal{B}$ is called (-2)-*truncated* if it is an equivalence. It is called *n-truncated*, for $n \geq -1$, if its diagonal $\Delta_f: X \to X \times_Y X$ is $(n-1)$-truncated. An object X of $\mathcal{B}$ is *n-truncated* if the map $X \to \mathrm{pt}$ to the terminal object is n-truncated.

Definition E.50 (n-connective morphisms, cf. [Lur09, Proposition 6.5.1.18])**.** A morphism $f: X \to Y$ is called 0-*connective* if it is an effective epimorphism. It is called *n-connective*, for $n \geq 1$, if it is an effective epimorphism and its diagonal $\Delta_f: X \to X \times_Y X$ is $(n-1)$-connective.

Remark E.51. Let $\psi_*: \mathcal{B} \to \mathcal{B}'$ be a geometric morphism of ∞-topoi. Then both the functor ψ_* as well as its left adjoint ψ^* are left-exact and thus preserve n-truncated morphisms for every n. The left adjoint ψ^* further preserves effective epimorphisms by [Lur09, Remark 6.2.3.6], and thus also preserves n-connected morphisms for every n.

Definition E.52 (Hypercomplete ∞-topoi, [Lur09, Section 6.5.2])**.** A morphism $f: X \to Y$ in an ∞-topos $\mathcal{B}$ is called *∞-connected* if it is n-connective for every $n \geq 0$. The ∞-topos $\mathcal{B}$ is called *hypercomplete* if every ∞-connected morphism in $\mathcal{B}$ is an equivalence.

Example E.53 ([Lur09, Theorem 7.2.3.6, Corollary 7.2.1.12])**.** Let X be a paracompact topological space of covering dimension $\leq n$. Then the ∞-topos $\mathrm{Shv}(X)$ of sheaves on X is hypercomplete.

Example E.54. Let M be a smooth manifold. Then M has finite covering dimension, and thus by the previous example the ∞-topos $\mathrm{Shv}(M)$ of sheaves on M is hypercomplete.

Hypercovers

Given an effective epimorphism $f: X \twoheadrightarrow Y$ in an ∞-topos $\mathcal{B}$, we may think of X as a *cover* of Y. The Čech nerve $\check{C}(f)$ f encodes all the iterated self-intersections $X^{\times^n_Y}$ of X over Y, and since f is an effective epimorphism this expresses Y as a colimit of these iterated self-intersections.

In certain situations, it is convenient to work with a more general notion of cover, known as a *hypercover*. A hypercover of Y also comes with an effective epimorphism $X \twoheadrightarrow Y$, but rather than simply using the iterated self-intersections $X^{\times_Y^n}$ one is allowed to further refine this: there is some object X_1 covering the intersection $X \times_Y X$, some object X_2 covering the double intersections of X_1 and X, and so forth. The colimit of a hypercover of Y admits a map to Y which one can show is always ∞-connected. In particular, if the ∞-topos is hypercomplete then the hypercover expresses Y as a colimit of the objects X_i. We will now recall the precise definitions and statements.

Notation E.55. For each natural number $n \geq 0$, let $\Delta^{\leq n}$ denote the full subcategory of Δ spanned by the set of objects $\{[0], [1], \ldots, [n]\}$. If C is a presentable ∞-category, the restriction functor

$$\mathrm{sk}_n \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, C) \to \mathrm{Fun}((\Delta^{\leq n})^{\mathrm{op}}, C)$$

admits a right adjoint given by right Kan extension along the inclusion functor $(\Delta^{\leq n})^{\mathrm{op}} \hookrightarrow \Delta^{\mathrm{op}}$. We let

$$\mathrm{cosk}_n \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, C) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, C)$$

denote the composition of sk_n with its right adjoint and refer to cosk_n as the *n-coskeleton functor*.

Definition E.56 (Hypercover, [Lur09, Definition 6.5.3.2])**.** Let $\mathcal{B}$ be an ∞-topos. A simplicial object $U_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{B})$ is called a *hypercover* of $\mathcal{B}$ if, for each $n \geq 0$, the unit map

$$U_n \to (\mathrm{cosk}_{n-1} U_\bullet)_n$$

is an effective epimorphism. We will say that $U_\bullet$ is an *effective hypercover* of $\mathcal{B}$ if the colimit of $U_\bullet$ is a terminal object of $\mathcal{B}$. For an object $B \in \mathcal{B}$, a *hypercover of B* is a hypercover of the slice topos $\mathcal{B}_{/B}$.

For $n = 0$, the map $U_n \to (\mathrm{cosk}_{n-1} U_\bullet)_n$ is simply the map $U_0 \to 1_{\mathcal{B}}$ to the terminal object of $\mathcal{B}$. For $n = 1$, it is the map $(d_1, d_0) \colon U_1 \to U_0 \times U_0$. For general n, the object $(\mathrm{cosk}_{n-1} U_\bullet)_n$ is known as the *n-th matching object*.

As mentioned above, one can show that every hypercover is effective in a hypercomplete ∞-topos. In fact, this property completely characterizes the hypercomplete ∞-topoi:

Theorem E.57 ([Lur09, Theorem 6.5.3.12])**.** *Let $\mathcal{B}$ be an ∞-topos. Then the following conditions are equivalent:*

(1) The ∞-topos $\mathcal{B}$ is hypercomplete;

(2) For every object $B \in \mathcal{B}$, every hypercover $U_\bullet$ of B is effective. □

Complete covers

An important example of a family of hypercovers comes from the notion of a *complete cover* of a topological space. We follow the discussion of [DI04, Subsection 4.4].

Definition E.58 ([DI04, Definition 4.5])**.** Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. We say that $\mathcal{U}$ is a *complete cover* if for every finite collection $\sigma = \{i_1, \ldots, i_n\}$ of indices, the intersection

$$U_\sigma := U_{i_1} \cap \cdots \cap U_{i_n}$$

is covered by elements of $\mathcal{U}$. It is called a *Čech cover* if each of the intersections U_σ is again in $\mathcal{U}$.

In what follows, we identify an open cover of X with the associated subposet of the poset $\mathrm{Open}(X)$ of open subsets of X.

Construction E.59 (Dugger-Isaksen)**.** Given a cover $\mathcal{U}$ of a topological space X, we define a simplicial topological space $\Omega(\mathcal{U})_\bullet$. For any $n \geq 0$, let P_n denote the poset of nonempty subsets of $\{0, \ldots, n\}$ and inclusions. The assignment $[n] \mapsto P_n$ defines a cosimplicial poset in the obvious way. We then define $\Omega(\mathcal{U})_\bullet$ to be the simplicial topological space

$$[n] \mapsto \coprod_{F\colon P_n^{\mathrm{op}} \to \mathcal{U}} F(\{0, \ldots, n\}),$$

where the coproduct runs over all maps of posets $F\colon P_n^{\mathrm{op}} \to \mathcal{U}$. The face and degeneracy maps are induced by those in $P_\bullet$ in the expected way.

To illustrate the definition, observe that a point in $\Omega(\mathcal{U})_3$ is given by the following data:

(1) A sequence of open subsets $U_0, \ldots, U_3$ of X which are in $\mathcal{U}$;

(2) Six open subsets $U_{01}, U_{02}, \ldots, U_{23}$ in $\mathcal{U}$ such that $U_{ij} \subseteq U_i \cap U_j$ for all $i < j$;

(3) Four open subsets $U_{012}, \ldots, U_{123}$ in $\mathcal{U}$ such that $U_{ijk} \subseteq U_{ij} \cap U_{jk} \cap U_{ik}$ for all $i < j < k$;

(4) An open subset U_{0123} in $\mathcal{U}$ which is contained in all the U_{ijk};

(5) A point in U_{0123}.

It is usually helpful to think of these open sets as indexed by the faces of a 3-simplex.

Proposition E.60 ([DI04, Proposition 4.6])**.** *Let $\mathcal{U}$ be a complete cover of a topological space X. Then the simplicial topological space $\Omega(\mathcal{U})_\bullet$ is a hypercover of X, in the sense that for every $n \geq 0$ the matching morphism $\Omega(\mathcal{U})_n \to (\mathrm{cosk}_{n-1}\, \Omega(\mathcal{U})_\bullet)_n$ is an open cover.*

F The calculus of mates

Throughout this thesis, we often make use of various kinds of 'exchange maps', also known as 'Beck-Chevalley maps' or 'mate transformations'. In this appendix, we clarify some of the terminology and prove some of the basic properties of these exchange maps. A standard reference for the calculus of mates is [KS74, Section 2.2]. Some aspects of our treatment are inspired by [CSY22, Section 2.2] and [CD19, Section 1.1]. We thank Tobias Lenz for various useful conversations concerning the contents of this appendix.

F.1 Beck-Chevalley transformations

We recall the definition of Beck-Chevalley transformations.

Convention F.1. Units and counits of adjunctions are denoted by u and c, respectively. We will often use indices to indicate to which adjunction the units and counits belong, but will sometimes drop the indices when the context is clear.

Definition F.2 (Beck-Chevalley transformation)**.** Let $\alpha\colon HF \Rightarrow F'G$ be a natural transformation of functors as displayed in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}"] & \mathcal{D} \arrow[d, "H"] \\ \mathcal{C}' \arrow[r, "F'"'] & \mathcal{D}'. \end{tikzcd}$$

(1) If the functors F and F' have left adjoints $F_\sharp\colon \mathcal{D} \to \mathcal{C}$ and $F'_\sharp\colon \mathcal{D}' \to \mathcal{C}'$, we define the *left Beck-Chevalley transformation* $\mathrm{BC}_\sharp\colon F'_\sharp H \Rightarrow GF_\sharp$ as the composite

$$\mathrm{BC}_\sharp(\alpha)\colon F'_\sharp H \xRightarrow{u_F} F'_\sharp HFF_\sharp \overset{\alpha}{\Rightarrow} F'_\sharp F'GF_\sharp \xRightarrow{c_{F'}} GF_\sharp.$$

(2) If the functors G and H have right adjoints $G_*\colon \mathcal{C}' \to \mathcal{C}$ and $H_*\colon \mathcal{D}' \to \mathcal{D}$, we define the *right Beck-Chevalley transformation* $\mathrm{BC}_*\colon FG_* \Rightarrow H_*F'$ as the composite

$$\mathrm{BC}_*(\alpha)\colon FG_* \xRightarrow{u_H} H_*HFG_* \overset{\alpha}{\Rightarrow} H_*F'GG_* \xRightarrow{c_G} H_*F'.$$

Diagrammatically, the two Beck-Chevalley transformations may be displayed as follows:

$$
\mathrm{BC}_\sharp(\alpha) = \begin{tikzcd}
\mathcal{D} \arrow[d, dashed, "F_\sharp"'] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\swarrow u"' near start] \\
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\swarrow \alpha"] & \mathcal{D} \arrow[d, "H"] \\
\mathcal{C}' \arrow[r, "F'"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\swarrow c"] & \mathcal{D}' \arrow[d, dashed, "F'_\sharp"] \\
& \mathcal{C}'
\end{tikzcd}
\qquad \text{and} \qquad
\mathrm{BC}_*(\alpha) = \begin{tikzcd}
\mathcal{C}' \arrow[r, "G_*"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\swarrow c"] & \mathcal{C} \arrow[r, "F"] \arrow[d, "G"] \arrow[dr, phantom, "\swarrow \alpha"] & \mathcal{D} \arrow[d, "H"] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\swarrow u"] \\
& \mathcal{C}' \arrow[r, "F'"] & \mathcal{D}' \arrow[r, "H_*"] & \mathcal{D}.
\end{tikzcd}
$$

We will often employ these diagrammatic ways of displaying Beck-Chevalley transformations in proofs, as it is often easier to parse than the more symbolic way of writing these transformations.

Warning F.3. The notation of Definition F.2 is somewhat abusive: if G and H have left adjoints $G_\sharp$ and $H_\sharp$ we may flip the diagram and obtain another transformation $\mathrm{BC}_\sharp(\alpha)\colon H_\sharp F' \Rightarrow FG_\sharp$. Analogously if F and F' have right adjoints F_* and F'_* one obtains a transformation $\mathrm{BC}_*(\alpha)\colon GF_* \Rightarrow F'_*H$. It will always be clear from the source and target of the maps $\mathrm{BC}_\sharp(\alpha)$ and $\mathrm{BC}_*(\alpha)$ which version we mean to use.

Lemma F.4. *The assignment* $\alpha \mapsto \mathrm{BC}_\sharp(\alpha)$ *defines an equivalence of spaces*

$$\mathrm{BC}_\sharp\colon\ \mathrm{Nat}(HF, F'G) \xrightarrow{\sim} \mathrm{Nat}(F'_\sharp H, GF_\sharp),$$

whose inverse is given by $\mathrm{BC}_*\colon\ \mathrm{Nat}(F'_\sharp H, GF_\sharp) \to \mathrm{Nat}(HF, F'G)$.

Proof. It follows directly from the triangle identities that one may recover α from $\mathrm{BC}_\sharp(\alpha)$ by forming the right Beck-Chevalley transformation:

$$
\mathrm{BC}_*(\mathrm{BC}_\sharp(\alpha)) = \begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\swarrow c"] & \mathcal{D} \arrow[d, dashed, "F_\sharp"] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\swarrow u"] \\
& \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\swarrow \alpha"] & \mathcal{D} \arrow[d, "H"] \\
& \mathcal{C}' \arrow[r, "F'"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\swarrow c"] & \mathcal{D}' \arrow[d, dashed, "F'_\sharp"] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\swarrow u"] \\
& & \mathcal{C}' \arrow[r, "F'"] & \mathcal{D}'
\end{tikzcd}
\simeq
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\swarrow \alpha"] & \mathcal{D} \arrow[d, "H"] \\
\mathcal{C}' \arrow[r, "F'"'] & \mathcal{D}'.
\end{tikzcd}
= \alpha.
$$

One may similarly prove that $\mathrm{BC}_\sharp(\mathrm{BC}_*(\beta)) \simeq \beta$ for a transformation $\beta\colon F'_\sharp H \Rightarrow GF_\sharp$. $\square$

Lemma F.5. *Consider the situation of Definition F.2.*

(1) *If the functors F and F' have left adjoints $F_\sharp \colon \mathcal{D} \to C$ and $F'_\sharp \colon \mathcal{D}' \to C'$, then the following diagrams commute:*

$$\begin{tikzcd} F'_\sharp HF \arrow[r, "\alpha"] \arrow[d, "\mathrm{BC}_\sharp(\alpha)"'] & F'_\sharp F'G \arrow[d, "c_{F'}"] \\ GF_\sharp F \arrow[r, "c_F"] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H \arrow[r, "u_{F'}"] \arrow[d, "u_F"'] & F'F'_\sharp H \arrow[d, "\mathrm{BC}_\sharp(\alpha)"] \\ HFF_\sharp \arrow[r, "\alpha"] & F'GF_\sharp. \end{tikzcd}$$

(2) *If the functors G and H have right adjoints $G_* \colon C' \to C$ and $H_* \colon \mathcal{D}' \to \mathcal{D}$, then the following diagrams commute:*

$$\begin{tikzcd} HFG_* \arrow[r, "\alpha"] \arrow[d, "\mathrm{BC}_*(\alpha)"'] & F'GG_* \arrow[d, "c_G"] \\ HH_*F' \arrow[r, "c_H"] & F' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[r, "u_G"] \arrow[d, "u_H"'] & FG_*G \arrow[d, "\mathrm{BC}_*(\alpha)"] \\ H_*HF \arrow[r, "\alpha"] & H_*F'G. \end{tikzcd}$$

Proof. Just like the proof of Lemma F.4, this follows directly from the triangle identities. □

Lemma F.6 (Pasting laws for Beck-Chevalley transformations)**.** *Consider natural transformations α, β and γ as in the following diagram:*

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{D} \arrow[r, "E"] \arrow[d, dashed, "H"] \arrow[dr, phantom, "\swarrow\beta"] & \mathcal{E} \arrow[d, "K"] \\ C' \arrow[r, dashed, "F'"] \arrow[d, "G'"'] \arrow[dr, phantom, "\swarrow\gamma"] & \mathcal{D}' \arrow[r, "E'"'] \arrow[d, "H'"] & \mathcal{E}' \\ C' \arrow[r, "F''"'] & \mathcal{D}''. \end{tikzcd}$$

(1) *If the functors F, F', E and E' have left adjoints $F_\sharp$, $F'_\sharp$, $E_\sharp$ and $E'_\sharp$, then the composite*

$$(E'F')_\sharp K \simeq F'_\sharp E'_\sharp K \xrightarrow{\mathrm{BC}_\sharp(\beta)} F'_\sharp HE_\sharp \xrightarrow{\mathrm{BC}_\sharp(\alpha)} GF_\sharp E_\sharp \simeq G(EF)_\sharp$$

is homotopic to $\mathrm{BC}_\sharp(\alpha\beta)$.

(2) *If the functors G, H and K admit right adjoints G_*, H_* and K_*, then the composite*

$$EFG_* \xrightarrow{\mathrm{BC}_*(\beta)} EH_*F' \xrightarrow{\mathrm{BC}_*(\alpha)} K_*E'F'$$

is homotopic to $\mathrm{BC}_*(\alpha\beta)$.

(3) *If the functors F, F' and F'' have left adjoints $F_\sharp$, $F'_\sharp$ and $F''_\sharp$, then the composite*

$$F''_\sharp H'H \xrightarrow{\mathrm{BC}_\sharp(\gamma)} G'F'_\sharp H \xrightarrow{\mathrm{BC}_\sharp(\alpha)} G'GF_\sharp$$

is homotopic to $\mathrm{BC}_\sharp(\alpha\gamma)$.

(4) *If the functors G, G', H and H' admit right adjoints G_*, G'_*, H_* and H'_*, then the composite*

$$F(G'G)_* = FG_*G'_* \xrightarrow{\mathrm{BC}_*(\gamma)} H_*F'G'_* \xrightarrow{\mathrm{BC}_*(\alpha)} H_*H'_*F'' = (H'H)_*F''$$

is homotopic to the $\mathrm{BC}_*(\alpha\gamma)$.

Proof. This follows directly from the definitions and the triangle identities. □

Total mates

We may specialize the definition of Beck-Chevalley transformation to the case where some of the functors are the identity. In this case, the resulting transformation is known as the *total mate*:

Definition F.7 (Total mate transformation)**.** Consider two functors $L, L' \colon \mathcal{C} \to \mathcal{D}$ that admit right adjoints $R, R' \colon \mathcal{D} \to \mathcal{C}$.

(1) Given a transformation $\alpha \colon L \to L'$, we obtain a *(right) total mate* $\overline{\alpha} = \mathrm{BC}_*(\alpha) \colon R' \to R$;

(2) Given a transformation $\beta \colon R' \to R$, we obtain a *(left) total mate* $\overline{\beta} = \mathrm{BC}_\sharp(\beta) \colon L \to L'$.

It follows from Lemma F.4 that $\overline{\alpha} \simeq \beta$ if and only if $\overline{\beta} \simeq \alpha$, in which case we will say that α and β are *total mates* of each other.

Lemma F.8 (Total mates of composites)**.** *Consider three functors $L, L', L'' \colon \mathcal{C} \to \mathcal{D}$ with right adjoints R, R' and R''. Then for all transformations $\alpha \colon L \to L'$ and $\alpha' \colon L \to L''$, there is an equivalence*

$$\overline{\alpha' \circ \alpha} \simeq \overline{\alpha'} \circ \overline{\alpha} \quad \in \quad \mathrm{Nat}(R'', R).$$

Proof. This is a special case of Lemma F.6. □

Corollary F.9 (Total mates of inverses)**.** *Consider functors $L, L' \colon \mathcal{C} \to \mathcal{D}$ with right adjoints R and R'. Then a transformation $\alpha \colon L \to L'$ is an equivalence if and only if its total mate $\overline{\alpha} \colon R' \to R$ is an equivalence. In this case, the inverse of $\overline{\alpha} \colon R' \to R$ is the total mate of $\alpha^{-1} \colon L' \to L$.*

Proof. This is immediate from Lemma F.8. □

F.2 Double Beck-Chevalley transformations

Given a transformation α as in Definition F.2, if both $\mathrm{BC}_\sharp(\alpha)$ and $\mathrm{BC}_*(\alpha)$ are defined, then these transformations are the total mates of each other. This will lead to the definition of a *double Beck-Chevalley transformation* $\mathrm{BC}_{\sharp,*}$, Definition F.12.

Lemma F.10. *Consider again a natural transformation $\alpha\colon HF \to F'G$ as in Definition F.2, and assume that the following conditions are satisfied:*

(1) The functors F and F' have left adjoints $F_\sharp\colon \mathcal{D} \to C$ and $F'_\sharp\colon \mathcal{D}' \to C'$;

(2) The functors G and H have right adjoints $G_*\colon C' \to C$ and $H_*\colon \mathcal{D}' \to \mathcal{D}$.

Then the left Beck-Chevalley transformation $\mathrm{BC}_\sharp(\alpha)\colon F'_\sharp H \Rightarrow GF_\sharp$ is the total mate of the right Beck-Chevalley transformation $\mathrm{BC}_*(\alpha)\colon FG_* \Rightarrow H_*F'$. In particular the former transformation is an equivalence if and only if the latter is.

Proof. The total mate of $\mathrm{BC}_\sharp(\alpha)\colon F'_\sharp H \Rightarrow GF_\sharp$ is defined as the image of $\mathrm{BC}_\sharp(\alpha)$ under the equivalence

$$\mathrm{BC}_*\colon\ \mathrm{Nat}(F'_\sharp H, GF_\sharp) \xrightarrow{\sim} \mathrm{Nat}(FG_*, H_*F').$$

Since $F'_\sharp H$ and $GF_\sharp$ are composites of adjoints, this map factors as two equivalences

$$\mathrm{Nat}(F'_\sharp H, GF_\sharp) \xrightarrow[\simeq]{\mathrm{BC}_*} \mathrm{Nat}(HF, F'G) \xrightarrow[\simeq]{\mathrm{BC}_*} \mathrm{Nat}(FG_*, H_*F').$$

Since the first map sends $\mathrm{BC}_\sharp(\alpha)$ to α by Lemma F.4, the claim follows. ◻

Lemma F.11 (Double Beck-Chevalley transformation)**.** *In the situation of Lemma F.10, assume that the left Beck-Chevalley transformation $\mathrm{BC}_\sharp(\alpha)\colon F'_\sharp H \Rightarrow GF_\sharp$ is an equivalence, or equivalently that its total mate $\mathrm{BC}_*(\alpha)\colon FG_* \Rightarrow H_*F'$ is an equivalence. Then the following two transformations are homotopic:*

$$\mathrm{BC}_*(\mathrm{BC}_\sharp(\alpha)^{-1})\colon F_\sharp H_* \xRightarrow{u_G} G_*GF_\sharp H_* \xRightarrow{\mathrm{BC}_\sharp(\alpha)^{-1}} G_*F'_\sharp HH_* \xRightarrow{c_H} G_*F'_\sharp;$$

$$\mathrm{BC}_\sharp(\mathrm{BC}_*(\alpha)^{-1})\colon F_\sharp H_* \xRightarrow{u_{F'}} F_\sharp H_*F'F'_\sharp \xRightarrow{\mathrm{BC}_*(\alpha)^{-1}} F_\sharp FG_*F'_\sharp \xRightarrow{c_F} G_*F'_\sharp.$$

Proof. By Lemma F.4, we may equivalently show that the image of $\mathrm{BC}_*(\mathrm{BC}_\sharp(\alpha)^{-1})$ under the equivalence

$$\mathrm{BC}_*\colon\ \mathrm{Nat}(F_\sharp H_*, G_*F'_\sharp) \xrightarrow{\sim} \mathrm{Nat}(H_*F', FG_*)$$

is homotopic to $\mathrm{BC}_*(\alpha)^{-1}\colon H_*F' \Rightarrow FG_*$. Since this image is the total mate of $\mathrm{BC}_\sharp(\alpha)^{-1}$, this follows from a combination of Lemma F.10 and Corollary F.9. ◻

Definition F.12 (Double Beck-Chevalley transformation)**.** We will write $\mathrm{BC}_{\sharp,*}(\alpha)\colon F_\sharp H_* \to G_* F'_\sharp$ for the transformation from Lemma F.11, and refer to it as the *double Beck-Chevalley transformation*.

Lemma F.13 (Pasting laws for double Beck-Chevalley transformations)**.** *Consider natural transformations α, β and γ as in the following diagram:*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Swarrow\alpha"] & \mathcal{D} \arrow[r, "E"] \arrow[d, dashed, "H"] \arrow[dr, phantom, "\Swarrow\beta"] & \mathcal{E} \arrow[d, "K"] \\
\mathcal{C}' \arrow[r, dashed, "F'"] \arrow[d, "G'"'] \arrow[dr, phantom, "\Swarrow\gamma"] & \mathcal{D}' \arrow[r, "E'"'] \arrow[d, "H'"] & \mathcal{E}' \\
\mathcal{C}' \arrow[r, "F''"'] & \mathcal{D}''.
\end{tikzcd}$$

Then the following diagrams commute whenever they are defined:

$$\begin{tikzcd}
(EF)_\sharp K_* \arrow[r, "\simeq"] \arrow[dd, "\mathrm{BC}_{\sharp,*}(\alpha\beta)"'] & F_\sharp E_\sharp K_* \arrow[d, "\mathrm{BC}_{\sharp,*}(\beta)"] \\
& F_\sharp H_* E'_\sharp \arrow[d, "\mathrm{BC}_{\sharp,*}(\alpha)"] \\
G_*(E'F')_\sharp \arrow[r, "\simeq"] & G_* F_\sharp E'_\sharp
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
F''_\sharp (G'G)_* \arrow[r, "\simeq"] \arrow[dd, "\mathrm{BC}_{\sharp,*}(\alpha\gamma)"'] & F''_\sharp G_* G'_* \arrow[d, "\mathrm{BC}_{\sharp,*}(\gamma)"] \\
& H_* F'_\sharp G'_* \arrow[d, "\mathrm{BC}_{\sharp,*}(\alpha)"] \\
F_\sharp (H'H)_* \arrow[r, "\simeq"] & F_\sharp H_* H'_*.
\end{tikzcd}$$

Proof. This follows from a double application of Lemma F.6. □

The following proposition describes an intricate interaction between the transformations $\mathrm{BC}_\sharp$, BC_* and $\mathrm{BC}_{\sharp,*}$.

Proposition F.14. *Consider twelve functors as displayed in the following (non-commutative) cubical diagram:*

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[rr, "F_0"] \arrow[dr, "K"] \arrow[dd, "G_0"'] & & \mathcal{D}_0 \arrow[dr, "L"] \arrow[dd, "H_0" near start] & \\
& \mathcal{C}_1 \arrow[rr, "F_1" near start, crossing over] & & \mathcal{D}_1 \arrow[dd, "H_1"] \\
\mathcal{C}'_0 \arrow[rr, "F'_0" near start] \arrow[dr, "K'"'] & & \mathcal{D}'_0 \arrow[dr, "L'"] & \\
& \mathcal{C}'_1 \arrow[rr, "F'_1"'] \arrow[from=uu, "G_1" near start, crossing over] & & \mathcal{D}'_1.
\end{tikzcd}$$

Given are further six natural transformations on each of the six faces of the cube such that the two total transformations are homotopic:

$$
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F_0"] \arrow[d, equal] & \mathcal{D}_0 \arrow[r, "L"] \arrow[d, phantom, "\Swarrow \beta"] & \mathcal{D}_1 \arrow[d, equal] \\
\mathcal{C}_0 \arrow[r, "K"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Swarrow \gamma'"] & \mathcal{C}_1 \arrow[r, "F_1"] \arrow[d, "G_1"] \arrow[dr, phantom, "\Swarrow \alpha_1"] & \mathcal{D}_1 \arrow[d, "H_1"] \\
\mathcal{C}'_0 \arrow[r, "K'"'] & \mathcal{C}'_1 \arrow[r, "F'_1"'] & \mathcal{D}'_1.
\end{tikzcd}
\simeq
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F_0"] \arrow[d, "G_0"'] \arrow[dr, phantom, "\Swarrow \alpha_0"] & \mathcal{D}_0 \arrow[r, "L"] \arrow[d, "H_0"] \arrow[dr, phantom, "\Swarrow \gamma"] & \mathcal{D}_1 \arrow[d, "H_1"] \\
\mathcal{C}'_0 \arrow[r, "F'_0"] \arrow[d, equal] & \mathcal{D}'_0 \arrow[r, "L'"] \arrow[d, phantom, "\Swarrow \beta'"] & \mathcal{D}'_1 \arrow[d, equal] \\
\mathcal{C}'_0 \arrow[r, "K'"'] & \mathcal{C}'_1 \arrow[r, "F'_1"'] & \mathcal{D}'_1
\end{tikzcd}
$$

Assume that the functors F_0, F'_0, F_1 and F'_1 have left adjoints and that G_0, H_0, G_1 and H_1 have right adjoints. Further assume that the transformations $\mathrm{BC}_\sharp(\alpha_0)$ and $\mathrm{BC}_\sharp(\alpha_1)$ are equivalences, so that the double Beck-Chevalley transformations $\mathrm{BC}_{\sharp,*}(\alpha_0)$ and $\mathrm{BC}_{\sharp,*}(\alpha_1)$ are defined. Then there exists a commutative diagram of natural transformations as follows:

$$
\begin{tikzcd}
& F_{1\sharp}H_{1*}L' \arrow[r, Rightarrow, "\mathrm{BC}_{\sharp,*}(\alpha_1)"] & G_{1*}F'_{1\sharp}L' \arrow[dr, Rightarrow, "\mathrm{BC}_\sharp(\beta')"] & \\
F_{1\sharp}LH_{0*} \arrow[ur, Rightarrow, "\mathrm{BC}_*(\gamma)"] \arrow[dr, Rightarrow, "\mathrm{BC}_\sharp(\beta)"'] & & & G_{1*}K'F'_{0\sharp} \\
& KF_{0\sharp}H_{0*} \arrow[r, Rightarrow, "\mathrm{BC}_{\sharp,*}(\alpha_0)"] & KG_{0*}F'_{0\sharp}. \arrow[ur, Rightarrow, "\mathrm{BC}_*(\gamma')"'] &
\end{tikzcd}
$$

Proof. Applying the pasting law for the right Beck-Chevalley transformations BC_* to the two given diagrams of natural transformations, we obtain

$$
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F_0"] \arrow[d, equal] & \mathcal{D}_0 \arrow[r, "L"] \arrow[d, phantom, "\Searrow \beta"] & \mathcal{D}_1 \arrow[d, equal] \\
\mathcal{C}_0 \arrow[r, "K"] \arrow[dr, phantom, "\Searrow \mathrm{BC}_*(\gamma')"] & \mathcal{C}_1 \arrow[r, "F_1"] \arrow[dr, phantom, "\Searrow \mathrm{BC}_*(\alpha_1)"] & \mathcal{D}_1 \\
\mathcal{C}'_0 \arrow[r, "K'"'] \arrow[u, "G_{0*}"] & \mathcal{C}'_1 \arrow[r, "F'_1"'] \arrow[u, "G_{1*}"] & \mathcal{D}'_1. \arrow[u, "H_{1*}"']
\end{tikzcd}
\simeq
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F_0"] \arrow[dr, phantom, "\Searrow \mathrm{BC}_*(\alpha_0)"] & \mathcal{D}_0 \arrow[r, "L"] \arrow[dr, phantom, "\Searrow \mathrm{BC}_*(\gamma)"] & \mathcal{D}_1 \\
\mathcal{C}'_0 \arrow[r, "F'_0"] \arrow[u, "G_{0*}"] \arrow[d, equal] & \mathcal{D}'_0 \arrow[r, "L'"] \arrow[u, "H_{0*}"] \arrow[d, phantom, "\Searrow \beta'"] & \mathcal{D}'_1 \arrow[u, "H_{1*}"'] \arrow[d, equal] \\
\mathcal{C}'_0 \arrow[r, "K'"'] & \mathcal{C}'_1 \arrow[r, "F'_1"'] & \mathcal{D}'_1
\end{tikzcd}
$$

Reorganizing the diagrams somewhat, we get

$$
\begin{tikzcd}
\mathcal{D}_0 \arrow[r, "L"] & \mathcal{D}_1 \arrow[r, equal] & \mathcal{D}_1 \\
\mathcal{C}_0 \arrow[u, "F_0"] \arrow[r, "K"] \arrow[ur, phantom, "\Downarrow \beta"] & \mathcal{C}_1 \arrow[u, "F_1"'] \arrow[r, phantom, "\Downarrow \mathrm{BC}_*(\alpha_1)"] & \mathcal{D}'_1 \arrow[u, "H_{1*}"'] \\
\mathcal{C}'_0 \arrow[u, "G_{0*}"] \arrow[r, "K'"'] \arrow[ur, phantom, "\Downarrow \mathrm{BC}_*(\gamma')"] & \mathcal{C}'_1 \arrow[u, "G_{1*}"'] \arrow[r, equal] & \mathcal{C}'_1 . \arrow[u, "F'_1"']
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
\mathcal{D}_0 \arrow[r, equal] & \mathcal{D}_0 \arrow[r, "L"] & \mathcal{D}_1 \\
\mathcal{C}_0 \arrow[u, "F_0"] \arrow[r, phantom, "\Downarrow \mathrm{BC}_*(\alpha_0)"] & \mathcal{D}'_0 \arrow[u, "H_{0*}"'] \arrow[r, "L'"] \arrow[ur, phantom, "\Downarrow \mathrm{BC}_*(\gamma)"] & \mathcal{D}'_1 \arrow[u, "H_{1*}"'] \\
\mathcal{C}'_0 \arrow[u, "G_{0*}"] \arrow[r, equal] & \mathcal{C}'_0 \arrow[u, "F'_0"'] \arrow[r, "K'"'] \arrow[ur, phantom, "\Downarrow \beta'"] & \mathcal{C}'_1 \arrow[u, "F'_1"']
\end{tikzcd}
$$

Since $\mathrm{BC}_*(\alpha_0)$ and $\mathrm{BC}_*(\alpha_1)$ are assumed to be invertible, we can bring them to the other side to obtain

$$
\begin{tikzcd}
\mathcal{D}_0 \arrow[r, equal] & \mathcal{D}_0 \arrow[r, "L"] & \mathcal{D}_1 \\
\mathcal{D}'_0 \arrow[u, "H_{0*}"] \arrow[r, phantom, "\Downarrow \mathrm{BC}_*(\alpha_0)^{-1}"] & \mathcal{C}_0 \arrow[u, "F_0"'] \arrow[r, "K"] \arrow[ur, phantom, "\Downarrow \beta"] & \mathcal{C}_1 \arrow[u, "F_1"'] \\
\mathcal{C}'_0 \arrow[u, "F'_0"] \arrow[r, equal] & \mathcal{C}'_0 \arrow[u, "G_{0*}"'] \arrow[r, "K'"'] \arrow[ur, phantom, "\Downarrow \mathrm{BC}_*(\gamma')"] & \mathcal{C}'_1 . \arrow[u, "G_{1*}"']
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
\mathcal{D}_0 \arrow[r, "L"] & \mathcal{D}_1 \arrow[r, equal] & \mathcal{D}_1 \\
\mathcal{D}'_0 \arrow[u, "H_{0*}"] \arrow[r, "L'"] \arrow[ur, phantom, "\Downarrow \mathrm{BC}_*(\gamma)"] & \mathcal{D}'_1 \arrow[u, "H_{1*}"'] \arrow[r, phantom, "\Downarrow \mathrm{BC}_*(\alpha_1)^{-1}"] & \mathcal{C}_1 \arrow[u, "F_1"'] \\
\mathcal{C}'_0 \arrow[u, "F'_0"] \arrow[r, "K'"'] \arrow[ur, phantom, "\Downarrow \beta'"] & \mathcal{C}'_1 \arrow[u, "F'_1"'] \arrow[r, equal] & \mathcal{C}'_1 \arrow[u, "G_{1*}"']
\end{tikzcd}
$$

Again rewriting the diagrams somewhat and applying the pasting-law for the *left* Beck-Chevalley transformations $\mathrm{BC}_\sharp$, we obtain the following equivalence:

$$
\begin{tikzcd}
\mathcal{D}'_0 \arrow[r, "H_{0*}"] \arrow[d, "F'_{0\sharp}"'] \arrow[dr, phantom, "\Swarrow \mathrm{BC}_{\sharp,*}(\alpha_0)"] & \mathcal{D}_0 \arrow[r, "L"] \arrow[d, "F_{0\sharp}"] \arrow[dr, phantom, "\Swarrow \mathrm{BC}_\sharp(\beta)"] & \mathcal{D}_1 \arrow[d, "F_{1\sharp}"] \\
\mathcal{C}'_0 \arrow[r, "G_{0*}"] \arrow[d, equal] & \mathcal{C}_0 \arrow[r, "K"] \arrow[d, phantom, "\Swarrow \mathrm{BC}_*(\gamma')"] & \mathcal{C}_1 \arrow[d, equal] \\
\mathcal{C}'_0 \arrow[r, "K'"'] & \mathcal{C}'_1 \arrow[r, "G_{1*}"'] & \mathcal{C}_1 ,
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
\mathcal{D}'_0 \arrow[r, "H_{0*}"] \arrow[d, equal] & \mathcal{D}_0 \arrow[r, "L"] \arrow[d, phantom, "\Swarrow \mathrm{BC}_*(\gamma)"] & \mathcal{D}_1 \arrow[d, equal] \\
\mathcal{D}'_0 \arrow[r, "L'"] \arrow[d, "F'_{0\sharp}"'] \arrow[dr, phantom, "\Swarrow \mathrm{BC}_\sharp(\beta')"] & \mathcal{D}'_1 \arrow[r, "H_{1*}"] \arrow[d, "F'_{1\sharp}"] \arrow[dr, phantom, "\Swarrow \mathrm{BC}_{\sharp,*}(\alpha_1)"] & \mathcal{D}_1 \arrow[d, "F_{1\sharp}"] \\
\mathcal{C}'_0 \arrow[r, "K'"'] & \mathcal{C}'_1 \arrow[r, "G_{1*}"'] & \mathcal{C}_1
\end{tikzcd}
$$

where we use that the transformation $\mathrm{BC}_{\sharp,*}(\alpha_i)$ is defined as $\mathrm{BC}_\sharp(\mathrm{BC}_*(\alpha_i)^{-1})$ for $i \in \{0, 1\}$. This produces the desired equivalence of natural transformations in the statement. $\square$

F.3 Projection formulas

We recall the definition of the projection formula map and record its interaction with Beck-Chevalley transformations.

Definition F.15 (Projection formula)**.** Let $F^*\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories.

(1) If F admits a left adjoint $F_\sharp\colon \mathcal{D} \to \mathcal{C}$, we say that F^* *satisfies the left projection formula* if for all objects $X \in \mathcal{D}$ and $Y \in \mathcal{C}$ the exchange map

$$\mathrm{PF}^F_\sharp\colon F_\sharp(X \otimes F^*Y) \xrightarrow{u^\sharp_F} F_\sharp(F^*F_\sharp X \otimes F^*Y) \simeq F_\sharp F^*(F_\sharp X \otimes Y) \xrightarrow{c^\sharp_F} F_\sharp X \otimes Y$$

is an equivalence.

(2) If F admits a right adjoint $F_*\colon \mathcal{D} \to \mathcal{C}$, we say that F^* *satisfies the left projection formula* if for all objects $X \in \mathcal{D}$ and $Y \in \mathcal{C}$ the exchange map

$$\mathrm{PF}^F_*\colon F_*X \otimes Y \xrightarrow{u^*_F} F_*F^*(F_*X \otimes Y) \simeq F_*(F^*F_*X \otimes F^*Y) \xrightarrow{c^*_F} F_*(X \otimes F^*Y)$$

is an equivalence.

Remark F.16. By symmetric monoidality of $F^*\colon \mathcal{C} \to \mathcal{D}$, we may consider the following commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F^*"] \arrow[d, "-\otimes_{\mathcal{C}} Y"'] & \mathcal{D} \arrow[d, "-\otimes F^*Y"] \\ \mathcal{C} \arrow[r, "F^*"] & \mathcal{D}. \end{tikzcd}$$

The map $\mathrm{PF}_\sharp$ is precisely the left Beck-Chevalley transformation $\mathrm{BC}_\sharp$ associated to this diagram.

Lemma F.17. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be symmetric monoidal functors.*

(1) If F and G have left adjoints $F_\sharp$ and $G_\sharp$, then for all $X \in \mathcal{E}$ and $Y \in \mathcal{C}$, the composite map

$$G_\sharp F_\sharp(X \otimes F^*G^*Y) \xrightarrow{\mathrm{PF}^F_\sharp} G_\sharp(F_\sharp X \otimes G^*Y) \xrightarrow{\mathrm{PF}^G_\sharp} G_\sharp F_\sharp X \otimes Y$$

is homotopic to $\mathrm{PF}^{GF}_\sharp\colon (GF)_\sharp(X \otimes (GF)^*Y) \to (GF)_\sharp X \otimes Y$.

(2) If F and G have right adjoints F_* and G_*, then for all $X \in \mathcal{E}$ and $Y \in C$, the composite map

$$G_*F_*X \otimes Y \xrightarrow{\mathrm{PF}_*^G} G_*(F_*X \otimes G^*Y) \xrightarrow{\mathrm{PF}_*^G} G_*F_*(X \otimes F^*G^*Y)$$

is homotopic to $\mathrm{PF}_*^{GF} \colon (GF)_*X \otimes Y \to (GF)_*(X \otimes (GF)^*Y)$.

Proof. By Remark F.16, this is a special case of Lemma F.6. □

Lemma F.18. *Let $F^* \colon C \to \mathcal{D}$ be a symmetric monoidal functor.*

(1) Assume that F admits a left adjoint $F_\sharp$. Then for all objects $X \in \mathcal{D}$, $Y \in C$ and $Z \in C$, the diagrams

$$\begin{tikzcd} X \otimes F^*Y \arrow[r, "u_F^\sharp"] \arrow[d, "u_F^\sharp \otimes 1"'] & F^*F_\sharp(X \otimes F^*Y) \arrow[d, "\mathrm{PF}_\sharp"] \\ F^*F_\sharp X \otimes F^*Y \arrow[r, "\simeq"] & F^*(F_\sharp X \otimes Y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_\sharp(F^*Z \otimes F^*Y) \arrow[r, "\simeq"] \arrow[d, "\mathrm{PF}_\sharp"'] & F_\sharp F^*(Z \otimes Y) \arrow[d, "c_F^\sharp"] \\ F_\sharp F^*Z \otimes Y \arrow[r, "c_F^\sharp \otimes 1"] & Z \otimes Y \end{tikzcd}$$

commute.

(2) Assume that F admits a right adjoint F_*. Then for all objects $X \in \mathcal{D}$, $Y \in C$ and $Z \in C$, the diagrams

$$\begin{tikzcd} F^*(F_*X \otimes Y) \arrow[r, "\simeq"] \arrow[d, "\mathrm{PF}_*"'] & F^*F_*X \otimes F^*Y \arrow[d, "c_F^* \otimes 1"] \\ F^*F_*(X \otimes F^*Y) \arrow[r, "c_F^*"] & X \otimes F^*Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z \otimes Y \arrow[r, "u_F^* \otimes 1"] \arrow[d, "u_F^*"'] & F_*F^*Z \otimes Y \arrow[d, "\mathrm{PF}_*"] \\ F_*F^*(Z \otimes Y) \arrow[r, "\simeq"] & F_*(F^*Z \otimes F^*Y) \end{tikzcd}$$

commute.

Proof. By Remark F.16, this is a special case of Lemma F.5. □

We finish this chapter by recording the interaction between Beck-Chevalley transformations and the projection formula.

Lemma F.19. *Consider a commutative square of symmetric monoidal functors*

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[d, "H"] \\ C' \arrow[r, "F'"] & \mathcal{D}'. \end{tikzcd}$$

(1) Assume that F and F' admit left adjoints $F_\sharp$ and $F'_\sharp$. Then the following diagram commutes for all $X \in \mathcal{D}$ and $Y \in C$:

$$\begin{tikzcd}
& F'_\sharp H(X \otimes FY) \arrow[r, "\simeq"] \arrow[dl, "\mathrm{BC}_\sharp"'] & F'_\sharp(HX \otimes F'GY) \arrow[dr, "\mathrm{PF}_\sharp"] & \\
GF_\sharp(X \otimes FY) \arrow[dr, "\mathrm{PF}_\sharp"'] & & & F'_\sharp HX \otimes GY \arrow[dl, "\mathrm{BC}_\sharp"] \\
& G(F_\sharp X \otimes Y) \arrow[r, "\simeq"] & GF_\sharp X \otimes GY. &
\end{tikzcd}$$

(2) Assume that G and H admit right adjoints G_* and H_*. Then the following diagram commutes for all $X \in \mathcal{D}$ and $Y \in C$:

$$\begin{tikzcd}
& G^*(F_\sharp X \otimes Y) \arrow[r, "\simeq"] \arrow[dl, "\mathrm{PF}_*"'] & GF_\sharp X \otimes GY \arrow[dr, "\mathrm{BC}_*"] & \\
GF_\sharp(X \otimes FY) \arrow[dr, "\mathrm{BC}_*"'] & & & F'_\sharp HX \otimes GY \arrow[dl, "\mathrm{PF}_*"] \\
& F'_\sharp H(X \otimes FY) \arrow[r, "\simeq"] & F'_\sharp(HX \otimes F'GY). &
\end{tikzcd}$$

Proof. For (1), it follows from Lemma F.6 that the composites on the left and right are given by left Beck-Chevalley transformations associated to the following two commutative diagrams:

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "G(-\otimes_C Y)"'] & \mathcal{D} \arrow[d, "H(-\otimes FY)"] \\
C \arrow[r, "F'"] & \mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "G(-)\otimes_C GY"'] & \mathcal{D} \arrow[d, "H(-)\otimes GF'Y"] \\
C \arrow[r, "F'"] & \mathcal{D}.
\end{tikzcd}$$

The claim follows, since these diagrams are equivalent to each other. The proof of (2) is similar. □

Lemma F.20. *Consider a commutative diagram of symmetric monoidal functors*

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[d, "H"] \\
C' \arrow[r, "F'"] & \mathcal{D}'.
\end{tikzcd}$$

Assume that F and F' admit left adjoints $F_\sharp$ and $F'_\sharp$, that H and G admit right adjoints G_* and H_*, and that the double Beck-Chevalley map $\mathrm{BC}\sharp, *$ exists. Then for all $X \in \mathcal{D}'$ and

$Y \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
 & F_\sharp H_* X \otimes Y \arrow[r, "\mathrm{BC}_{\sharp,*}"] & G_* F'_\sharp X \otimes Y \arrow[dr, "\mathrm{PF}_*"] & \\
F_\sharp(H_* X \otimes FY) \arrow[ur, "\mathrm{PF}_\sharp"] \arrow[dr, "\mathrm{PF}_*"'] & & & G_*(F'_\sharp \otimes GB) \\
 & F_\sharp H_*(X \otimes HFB) \arrow[r, "\mathrm{BC}_{\sharp,*}"] & G_* F'_\sharp(X \otimes F'GY). \arrow[ur, "\mathrm{PF}_\sharp"'] &
\end{tikzcd}$$

Proof. This is a special case of Proposition F.14. □

List of Symbols

$\hookrightarrow$ A monomorphism or an embedding

$\twoheadrightarrow$ An effective epimorphism in an ∞-topos

$\odot$ The composition product, Definition I.3.22

$\emptyset$ The empty space or empty stack

$\mathbb{1}$ The monoidal unit of a symmetric monoidal ∞-category C

$[1]$ The poset $\{0 < 1\}$, regarded either as a category or as an object of Δ

$[n]$ The poset $\{0 < 1 < \cdots < n\}$, regarded either as a category or as an object of Δ

$[\alpha]$ The poset of ordinals $\beta \leq \alpha$ for an ordinal α

$\mathrm{Act}_{\mathcal{G}}(\mathcal{B})$ The ∞-category of objects of an ∞-category $\mathcal{B}$ equipped with the action of a groupoid object $\mathcal{G}$ in $\mathcal{B}$, Definition E.17

An The ∞-category of animae/∞-groupoids

An_G The ∞-category of genuine G-animae/G-spaces

$\mathrm{Atl}(\mathcal{B})$ The ∞-category of effective epimorphisms in an ∞-topos $\mathcal{B}$, Definition E.12

$\mathrm{Atl}^{\mathrm{rep}}(\mathrm{Shv}(\mathrm{Diff}))$ The ∞-category of representable atlases $M \twoheadrightarrow \mathcal{X}$, Definition II.2.3.6

$\mathcal{B}$ A generic ∞-topos, Definition E.1

$\mathbb{B}G$ Classifying stack of a Lie group G, Example II.2.3.4

$\mathbb{B}\mathcal{G}$ Classifying stack of a Lie groupoid $\mathcal{G}$, Definition II.2.3.2

$\mathrm{BC}_\sharp$ Left Beck-Chevalley transformation, Definition F.2

BC_* Right Beck-Chevalley transformation, Definition F.2

$\mathrm{BC}_{\sharp,*}$ Double Beck-Chevalley transformation, Definition F.12

$\mathrm{CAlg}(\mathcal{C})$ The ∞-category of commutative algebras in a symm. mon. ∞-category $\mathcal{C}$

$\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ The (very large) ∞-category of augmented presentably symmetric monoidal ∞-categories, Definition I.2.41

Cat_∞ The (very large) ∞-category of large ∞-categories

$\mathrm{Cat}(\mathcal{B})$ The (very large) ∞-category of large $\mathcal{B}$-categories, Definition I.2.2

$\mathrm{Cat}_{\mathcal{B}}$ The (very large) $\mathcal{B}$-category of large $\mathcal{B}$-categories, see page 23

$\mathcal{C}^{\mathrm{op}}$ The opposite of an ∞-category $\mathcal{C}$

$\mathcal{C}^{\simeq}$ The core, or underlying ∞-groupoid, of an ∞-category $\mathcal{C}$

$\mathcal{C}^A$ The $\mathcal{C}$-linear $\mathcal{B}$-category given by $\mathcal{C}^A(B) = \mathcal{C}(A \times B)$, Definition I.2.23

$\mathcal{C}[A]$ The free $\mathcal{C}$-linear $\mathcal{B}$-category on $A \in \mathcal{B}$, Definition I.2.23

$\mathcal{C}[S^{-1}]$ The formal inversion of objects S in a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$, Definition I.2.35

$\check{C}(f)$ The Čech nerve of a morphism $f\colon X \to Y$ in an ∞-category, Construction E.10

coev A coevaluation map

$\mathrm{cofib}(f)$ The cofiber of a map f

coind_H^G Coinduction, right adjoint to res_H^G

const A constant functor

cosk_n The n-coskeleton of a simplicial object

D_A The dualizing object of an object $A \in \mathcal{B}$ (with respect to a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$), Definition I.3.1

D_f The relative dualizing object of morphism $f\colon A \to B$ in $\mathcal{B}$, Definition I.3.5

$D_B^{CW}(X)$ The weak Costenoble-Waner dual of an object $X \in \mathcal{C}(B)$, Definition I.3.31

Δ The simplex category

$\Delta^{\leq n}$ The full subcategory of Δ spanned by the set of objects $\{[0], [1], \ldots, [n]\}$

Δ^n_{alg} The algebraic n-simplex, Construction II.4.2.2

Δ_f The diagonal of a morphism f

$\Delta^k(f)$ The k-fold iterated diagonal of a morphism f, Definition I.3.14

Diff The site of smooth manifolds and open covers, Definition II.2.1.1

DiffStk The (2,1)-category of differentiable stacks, Definition II.2.2.1

ev An evaluation map

$\mathcal{E}$ A generic vector bundle over a stack

f^* The pullback functor along a morphism f

f_* The pushforward functor, right adjoint to f^*

$f_!$ In Part I: the left adjoint to f^*. In Part III: the exceptional pushforward functor in a six-functor formalism

$f_\sharp$ In parts II and III: the left adjoint to f^*

fgt A forgetful functor

$\mathrm{fib}(f)$ The fiber of a map f

F_D The $\mathcal{B}$-functor $\mathcal{C}^A \to \mathcal{C}^B$ classified by an object $D \in \mathcal{C}(A \times B)$, Definition I.2.33

$\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ The ∞-category of functors $\mathcal{C} \to \mathcal{D}$

$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ The ∞-category of colimit-presering functors $\mathcal{C} \to \mathcal{D}$

$\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ The ∞-category of limit-presering functors $\mathcal{C} \to \mathcal{D}$

$\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C}, \mathcal{D})$ The parametrized functor category, Example I.2.7

$\mathcal{F}$ A generic sheaf

$\Gamma(\mathcal{C})$ The underlying ∞-category of a parametrized ∞-category $\mathcal{C}$, Definition I.2.2

$\Gamma(\mathcal{F})$ The global sections of a sheaf $\mathcal{F}$

γ^* Comparison functor between genuine and ordinary sheaves on a differentiable stack $\mathcal{X}$, Construction II.4.2.25

$g_{\mathcal{X}}$ The structure map $g_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{S}$ of a stack $\mathcal{X}$ over a base stack $\mathcal{S}$

G_x The isotropy group of a differentiable stack at a point x, Definition II.3.4.1

$G \ltimes M$ The action groupoid of a smooth G-manifold M, Example D.4

Glo The global indexing category, Definition I.4.11

Grpd The ordinary category of groupoids

$\mathrm{Grpd}(\mathcal{C})$ The ∞-category of groupoid objects in an ∞-category $\mathcal{C}$, Definition E.8

$\mathcal{G}$ A generic Lie groupoid, Definition D.1, or a generic groupoid object in an ∞-category, Definition E.8

$\mathcal{G}_n$ The manifold of n-tuples of composable morphisms in a Lie groupoid $\mathcal{G}$, Definition 2.3.1, or the n-th level of a groupoid object in an ∞-category

$\mathcal{G}_U$ The restriction of a Lie groupoid, Example D.7

$h_{\mathcal{X}}(\mathcal{Y})$ The image of $\mathcal{Y} \in \mathrm{Sub}_{/\mathcal{X}}$ in $\mathcal{C}(\mathcal{X})$, Notation II.4.5.13

$h^{\mathcal{Z}}_{\mathcal{X}}(\mathcal{Y}, t)$ The presheaf of $\mathcal{Z}$-trivialized morphisms, Construction II.5.2.9

$\mathrm{H}(\mathcal{X})$ The ∞-category of genuine sheaves on a differentiable stack $\mathcal{X}$, Definition II.4.2.6

$\mathrm{Hom}_{\mathcal{C}}(-,-)$ The space of morphisms in an ∞-category $\mathcal{C}$

$\mathrm{Hom}^{\Delta}_{\mathcal{C}}(-,-)$ The simplicial hom-set of a simplicially enriched category $\mathcal{C}$

$\underline{\mathrm{Hom}}_{\mathcal{C}}$ The internal hom object in a symmetric monoidal ∞-category $\mathcal{C}$

i A generic closed embedding of differentiable stacks

id An identity map

ind^G_H Induction from H-equivariant objects to G-equivariant objects, left adjoint to res^G_H

j A generic open embedding of differentiable stacks

l.b.c. Abbreviation of ‘left base change, Definition I.2.9

l.p.f. Abbreviation of ‘left projection formula, see (2) on page 24

$\mathrm{L}_{\mathrm{open}}$ The sheafification functor $L_{\mathrm{open}}\colon \mathrm{PSh}(\mathrm{Diff}) \to \mathrm{Shv}(\mathrm{Diff})$ with respect to the open cover topology

$\mathrm{L}_{\mathbb{R}}$ The homotopy localization functor at the level of presheaves inverting $\mathbb{R} \to \mathrm{pt}$, Construction II.4.2.2

$\mathrm{L}_{\mathrm{htp}}$ The homotopy localization functor at the level of sheaves inverting $\mathbb{R} \to \mathrm{pt}$, Definition II.4.2.6

L^{α} The α-th iteration of the functor $L^1 = L_{\mathrm{open}} \circ L_{\mathbb{R}}$ for an ordinal α, Construction II.4.2.11

$\mathrm{LConst}(\mathcal{E})$ The locally constant $\mathcal{B}$-category associated to an ∞-category $\mathcal{E}$, Example I.2.5

LieGrpd The ordinary category of Lie groupoids

$\mathrm{LMod}_R(\mathcal{C})$ The ∞-category of left modules over an associative algebra R in a monoidal ∞-category $\mathcal{C}$

$\mathcal{L}$ The formal inversion functor, Lemma I.2.42

M//G The quotient stack of a smooth G-manifold M, Example II.2.3.5

$\mathrm{Mod}_R(\mathcal{C})$ The ∞-category of modules over a commutative algebra R in a symmetric monoidal ∞-category $\mathcal{C}$

$\mathrm{Nat}(F,G)$ The space of natural transformations $F \Rightarrow G$ between two functors $F,G\colon \mathcal{C} \to \mathcal{D}$

$\mathrm{Nat}_{\mathcal{C}}(F,G)$ The space of $\mathcal{C}$-linear natural transformations $F \Rightarrow G$ between two $\mathcal{C}$-linear functors $F,G\colon \mathcal{D} \to \mathcal{E}$,

$\mathrm{N}^{\mathrm{Diff}}$ The continuous diffeology functor, Definition II.3.1.1

Nm_A The twisted norm map of an object $A \in \mathcal{B}$, Definition I.3.3

$\widetilde{\mathrm{Nm}}_A$ The adjoint twisted norm map of an object $A \in \mathcal{B}$, Definition I.3.3

Nm_f The twisted norm map of a morphism f in $\mathcal{B}$, Definition I.3.5

$\mathcal{N}_i$ The normal bundle of an embedding of stacks i, Definition II.3.5.14

$\mathrm{Open}(\mathcal{X})$ The poset of open substacks of a stack $\mathcal{X}$

Orb The global orbit category, the wide subcategory of Glo spanned by the injective group homomorphisms, Definition I.4.12

Orb_G The orbit category of a Lie group G

$\mathrm{Orb}_G^{\mathrm{pr}}$ The proper orbit category of a Lie group G, Definition I.4.21

$\underline{\mathrm{OrbSp}}$ The orbicategory of orbispectra, Definition I.4.15

$\underline{\mathrm{OrbSpc}}$ The orbicategory of orbispaces, Definition I.4.14

ω_f The dualizing object of a morphism of stacks f, Definition II.6.1.3

$\Omega_{\mathcal{B}}$ The $\mathcal{B}$-category of $\mathcal{B}$-groupoids, Definition I.2.11

$\mathrm{PB}(\mathrm{SepStk})$ The (very large) ∞-category of pullback formalisms on SepStk, Definition 4.5.5

$\mathrm{PF}_\sharp$ The left projection formula map, Definition F.15

PF_* The right projection formula map, Definition F.15

$\mathfrak{p}_f$ The Poincaré duality map $f_\sharp \to f_*(- \otimes \omega_f)$, Construction II.6.1.5

pr A projection map

Pr^{L} The (very large) ∞-category of presentable ∞-categories and colimit-preserving functors

Pr^{R} The (very large) ∞-category of presentable ∞-categories and accessible limit-preserving functors

$\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ The ∞-category of stable presentable ∞-categories and colimit-preserving functors

$\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ The ∞-category of presentable $\mathcal{B}$-categories and colimit-preserving $\mathcal{B}$-functors, Definition I.2.12

$\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}$ The $\mathcal{B}$-category of presentable $\mathcal{B}$-categories and colimit-preserving $\mathcal{B}$-functors, see page 23

$\mathrm{Pr}^{\mathrm{L}}_G$ The ∞-category $\mathrm{Pr}^{\mathrm{L}}(\mathrm{Spc}_G)$ of presentable G-categories

$\mathrm{PrnBdl}_{\mathcal{G}}(\mathcal{B})_B$ The ∞-category of principal $\mathcal{G}$-bundles over an object B in an ∞-topos $\mathcal{B}$, Definition E.24

$\mathrm{PrnBdl}_{\mathcal{G}}(N)$ The ∞-category of smooth principal $\mathcal{G}$-bundles over a smooth manifold N, Definition D.15

$\mathrm{PSh}(C)$ The ∞-category of presheaves on a small ∞-category C

$\mathrm{PSh}^{\mathrm{htp}}(-)$ The ∞-category of homotopy invariant presheaves, Definition II.4.2.1

pt The one-point space

$\mathrm{PT}_{\mathcal{S}}(i)$ The Pontryagin-Thom collapse map of a closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ over $\mathcal{S}$, Construction II.6.2.16

$\pi_B^* C$ The pullback $\mathcal{B}_{/B}$-category of a $\mathcal{B}$-category C, Example I.2.6

$(\pi_B)_* C$ The pushforward $\mathcal{B}$-category of a $\mathcal{B}_{/B}$-category C, Example I.2.6

$\Pi_\infty(X)$ The fundamental ∞-groupoid of a topological space X

$\Pi_G(M)$ The G-equivariant homotopy type of a smooth G-manifold M, Definition II.4.4.8

$\Pi_{\mathrm{Glo}}(\mathcal{X})$ The global homotopy type of a differentiable stack, Section III.2.1

QtStk The (2,1)-category of global quotient stacks $M//G$, Example II.2.3.5

Rep_G The category of G-representations

res_H^G Restriction from G-equivariant objects to H-equivariant objects

$\mathrm{RMod}_R(C)$ The ∞-category of right modules over an associative algebra R in a monoidal ∞-category C

$S^{\mathcal{E}}$ Sphere bundle of a vector bundle $\mathcal{E} \to \mathcal{X}$ of stacks, Definition II.2.5.2, Notation II.4.5.25

$\mathbb{S}_G$ The G-equivariant sphere spectrum

SepStk The (2,1)-category of separated differentiable stacks, Definition II.3.3.1

$\mathrm{SH}(\mathcal{X})$ The ∞-category of genuine sheaves of spectra on a separated differentiable stack $\mathcal{X}$, Definition II.4.3.14

shear The shear map, Definition E.21

$\mathrm{Shv}(C)$ The ∞-category of sheaves on a Grothendieck site C, Definition E.33

$\mathrm{Shv}(\mathcal{X})$ The ∞-category of ordinary sheaves on a differentiable stack $\mathcal{X}$, Definition II.4.1.17

sk_n The n-skeleton of a simplicial object

$\mathrm{Span}(\mathcal{B}, E)$ The span category of a geometric setup $(\mathcal{B}, E)$, Definition III.3.1

Sp The ∞-category of spectra

Spc The ∞-category of spaces/∞-groupoids

Sp^G The ∞-category of genuine G-spectra for a compact Lie group G, Definition I.4.1

Spc^G The ∞-category of G-spaces for a compact Lie group G

$\mathrm{Spc}^G_{\mathrm{pr}}$ The ∞-category of proper G-spaces for a Lie group G, Definition I.4.21

$\underline{\mathrm{Sp}}_G$ The G-category of genuine G-spectra, Definition I.4.2

$\underline{\mathrm{Spc}}_G$ The G-category of G-spaces, Definition I.4.2

$\mathrm{Sub}_{/\mathcal{X}}$ The category of representable submersions $\mathcal{Y} \to \mathcal{X}$, Definition II.4.1.1

$\mathrm{Th}_{\mathcal{S}}(\mathcal{X}, \mathcal{Z})$ The relative Thom space of a closed embedding $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ relative to a base stack $\mathcal{S}$, Definition II.6.2.1

$T\mathcal{G}$ The tangent groupoid of a Lie groupoid $\mathcal{G}$, Definition II.3.5.1

T_f The relative tangent bundle of a morphism f of stacks, Definition II.3.5.5

Top The ordinary category of topological spaces

$\mathrm{TopGrpd}_1$ The ordinary category of topological groupoids, Definition I.4.10

TopGrpd The ∞-category of topological groupoids, Definition I.4.10

U_A Unit for the composition product, Definition I.3.22

$\mathcal{U}$ A generic open substack of some stack

$\mathrm{Vect}(\mathcal{X})$ The ordinary category of vector bundles over a differentiable stack $\mathcal{X}$, Definition II.2.5.3

Vect_k The ordinary category of vector spaces over a field k

$\mathcal{X} \setminus \mathcal{Z}$ The open complement of a closed substack $\mathcal{Z} \subseteq \mathcal{X}$, Definition II.3.2.2

$|\mathcal{X}|_{\mathrm{mod}}$ The coarse moduli space of a stack $\mathcal{X}$, Definition II.3.1.1

$\mathfrak{X}$ A generic sheaf on SepStk

y The Yoneda embedding $y\colon C \hookrightarrow \mathrm{PSh}(C)$

$\mathcal{Z}$ A generic closed substack of some stack

Bibliography

[ABG18] Matthew Ando, Andrew J. Blumberg, and David Gepner. "Parametrized spectra, multiplicative Thom spectra and the twisted Umkehr map". *Geom. Topol.* 22.7 (2018), pp. 3761–3825.

[ADH21] Araminta Amabel, Arun Debray, and Peter J Haine. "Differential cohomology: categories, characteristic classes, and connections". *arXiv preprint arXiv:2109.12250* (2021).

[Ati61] Michael F. Atiyah. "Thom complexes". *Proc. Lond. Math. Soc. (3)* 11 (1961), pp. 291–310.

[Ayo07] Joseph Ayoub. *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I.* Vol. 314. Astérisque. Paris: Société Mathématique de France, 2007.

[Bar+16a] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. "Parametrized higher category theory and higher algebra: A general introduction". *arXiv preprint arXiv:1608.03654* (2016).

[Bar+16b] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. "Parametrized higher category theory and higher algebra: Exposé I - Elements of parametrized higher category theory". *arXiv preprint arXiv:1608.03657* (2016).

[Bau04] Tilman Bauer. "p-compact groups as framed manifolds." *Topology* 43.3 (2004), pp. 569–597.

[BBP19] Daniel Berwick-Evans, Pedro Boavida de Brito, and Dmitri Pavlov. "Classifying spaces of infinity-sheaves". *arXiv preprint arXiv:1912.10544* (2019).

[BD20] Paul Balmer and Ivo Dell'Ambrogio. *Mackey 2-functors and Mackey 2-motives*. EMS Monogr. Math. European Mathematical Society (EMS), 2020.

[BD95] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*. Corrected reprint of the 1985 orig. Vol. 98. Grad. Texts Math. Springer, 1995.

[BDS16] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. "Grothendieck-Neeman duality and the Wirthmüller isomorphism". *Compos. Math.* 152.8 (2016), pp. 1740–1776.

[BH21] Tom Bachmann and Marc Hoyois. *Norms in motivic homotopy theory*. Vol. 425. Astérisque. Paris: Société Mathématique de France (SMF), 2021.

[Blu06] Andrew J. Blumberg. "Continuous functors as a model for the equivariant stable homotopy category". English. *Algebr. Geom. Topol.* 6 (2006), pp. 2257–2295.

[Bou11] Mohamed Boucetta. "Riemannian geometry of Lie algebroids". *J. Egypt. Math. Soc.* 19.1-2 (2011), pp. 57–70.

[Bou67] N. Bourbaki. *Éléments de mathématique. XXXIII. Fascicule de résultats. Paragraphes 1 à 7: Variétés différentielles et analytiques*. Actualités scientifiques et industrielles. 1333. Paris: Hermann & Cie. 97 p. (1967). 1967.

[Bre72] Glen E. Bredon. *Introduction to compact transformation groups*. Vol. 46. Academic Press, 1972.

[BX11] Kai Behrend and Ping Xu. "Differentiable stacks and gerbes". *J. Symplectic Geom.* 9.3 (2011), pp. 285–341.

[Cal+20] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. "Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity". *arXiv preprint arXiv:2009.07224* (2020).

[Cam23] Timothy Campion. "Free duals and a new universal property for stable equivariant homotopy theory". *arXiv preprint arXiv:2302.04207v1* (2023).

[Car+22] Schachar Carmeli, Bastiaan Cnossen, Maxime Ramzi, and Lior Yanovski. "Characters and transfer maps via categorified traces". *arXiv preprint arXiv:2210.17364* (2022).

[Car11] David Carchedi. "Categorical properties of topological and differentiable stacks". PhD thesis. Utrecht University, 2011.

[CD19] Denis-Charles Cisinski and Frédéric Déglise. *Triangulated categories of mixed motives*. Springer Monogr. Math. Springer, 2019.

[CLL23] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens. “Parameterized stability and the universal property of global spectra”. *arXiv preprint arXiv:2301.08240* (2023).

[CM01] Marius Crainic and Ieke Moerdijk. “Foliation groupoids and their cyclic homology”. *Adv. Math.* 157.2 (2001), pp. 177–197.

[CS13] Marius Crainic and Ivan Struchiner. “On the linearization theorem for proper Lie groupoids”. *Ann. Sci. Éc. Norm. Supér. (4)* 46.5 (2013), pp. 723–746.

[CS23] Kestutis Cesnavicius and Peter Scholze. “Purity for flat cohomology”. *To appear in Annals of Mathematics* (2023).

[CSY21] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski. “Ambidexterity and height”. *Adv. Math.* 385 (2021), p. 90.

[CSY22] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski. “Ambidexterity in chromatic homotopy theory”. *Invent. Math.* 228.3 (2022), pp. 1145–1254.

[CW16] Steven R. Costenoble and Stefan Waner. *Equivariant ordinary homology and cohomology*. Vol. 2178. Lect. Notes Math. Springer, 2016.

[Deg+19] Dieter Degrijse, Markus Hausmann, Wolfgang Lück, Irakli Patchkoria, and Stefan Schwede. “Proper equivariant stable homotopy theory”. *arXiv preprint arXiv:1908.00779* (2019).

[DG22] Brad Drew and Martin Gallauer. “The universal six-functor formalism”. *Ann. K-Theory* 7.4 (2022), pp. 599–649.

[DI04] Daniel Dugger and Daniel C. Isaksen. “Topological hypercovers and $\mathbb{A}^1$-realizations”. *Math. Z.* 246.4 (2004), pp. 667–689.

[Die87] Tammo tom Dieck. *Transformation groups*. Vol. 8. De Gruyter Stud. Math. De Gruyter, Berlin, 1987.

[EG11] Johannes Ebert and Jeffrey Giansiracusa. “Pontrjagin–Thom maps and the homology of the moduli stack of stable curves”. *Math. Ann.* 349.3 (2011), pp. 543–575.

[Elm83] Anthony D. Elmendorf. “Systems of fixed point sets”. *Trans. Am. Math. Soc.* 277 (1983), pp. 275–284.

[FHM03] Halvard Fausk, Po Hu, and J. Peter May. “Isomorphisms between left and right adjoints”. *Theory Appl. Categ.* 11 (2003), pp. 107–131.

[FS21] Laurent Fargues and Peter Scholze. “Geometrization of the local Langlands correspondence”. *arXiv preprint arXiv:2102.13459* (2021).

[GH07] David Gepner and André Henriques. "Homotopy theory of orbispaces". *arXiv preprint math.AT/0701916* (2007).

[GHK22] David Gepner, Rune Haugseng, and Joachim Kock. "∞-operads as analytic monads". *Int. Math. Res. Not.* 2022.16 (2022), pp. 12516–12624.

[GM17] Alfonso Gracia-Saz and Rajan Amit Mehta. "$\mathcal{VB}$-groupoids and representation theory of Lie groupoids". *J. Symplectic Geom.* 15.3 (2017), pp. 741–783.

[GM20] David Gepner and Lennart Meier. "On equivariant topological modular forms". *arXiv preprint arXiv:2004.10254* (2020).

[GS96] J. P. C. Greenlees and Hal Sadofsky. "The Tate spectrum v_n-periodic complex oriented theories". *Math. Z.* 222.3 (1996), pp. 391–405.

[Har20] Yonatan Harpaz. "Ambidexterity and the universality of finite spans". *Proc. Lond. Math. Soc. (3)* 121.5 (2020), pp. 1121–1170.

[HF18] Matias del Hoyo and Rui Loja Fernandes. "Riemannian metrics on Lie groupoids". *J. Reine Angew. Math.* 735 (2018), pp. 143–173.

[HF19] Matias del Hoyo and Rui Loja Fernandes. "Riemannian metrics on differentiable stacks". *Math. Z.* 292.1-2 (2019), pp. 103–132.

[HHR16] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. "On the nonexistence of elements of Kervaire invariant one". *Ann. Math. (2)* 184.1 (2016), pp. 1–262.

[HK20] Fabian Hebestreit and Achim Krause. "Mapping spaces in homotopy coherent nerves". *arXiv preprint arXiv:2011.09345* (2020).

[HL13] Michael Hopkins and Jacob Lurie. "Ambidexterity in K(n)-local stable homotopy theory". *arXiv preprint arXiv:1510.03304* (2013).

[HO20] Matias del Hoyo and Cristian Ortiz. "Morita equivalences of vector bundles". *Int. Math. Res. Not.* 2020.14 (2020), pp. 4395–4432.

[Hoy13] Matias L. del Hoyo. "Lie groupoids and their orbispaces". *Port. Math. (N.S.)* 70.2 (2013), pp. 161–209.

[Hoy17] Marc Hoyois. "The six operations in equivariant motivic homotopy theory". *Adv. Math.* 305 (2017), pp. 197–279.

[HS96] Mark Hovey and Hal Sadofsky. "Tate cohomology lowers chromatic Bousfield classes". *Proc. Am. Math. Soc.* 124.11 (1996), pp. 3579–3585.

[Hus93] Dale H. Husemoller. *Fibre bundles.* 3rd ed. Vol. 20. Grad. Texts Math. Berlin: Springer-Verlag, 1993.

[Kha19] Adeel Khan. “The Morel-Voevodsky localization theorem in spectral algebraic geometry”. *Geom. Topol.* 23.7 (2019), pp. 3647–3685.

[Kle01] John R. Klein. “The dualizing spectrum of a topological group”. *Math. Ann.* 319.3 (2001), pp. 421–456.

[Kör18] Alexander Körschgen. “A comparison of two models of orbispaces”. *Homology Homotopy Appl.* 20.1 (2018), pp. 329–358.

[KR21] Adeel A Khan and Charanya Ravi. “Generalized cohomology theories for algebraic stacks”. *arXiv preprint arXiv:2106.15001* (2021).

[KS74] G. M. Kelly and Ross Street. *Review of the elements of 2-categories*. Category Sem., Proc., Sydney 1972/1973, Lect. Notes Math. 420, 75-103 (1974). 1974.

[Lan95] Serge Lang. *Differential and Riemannian manifolds.* 3rd ed.; Repr. of the orig. 1972. Vol. 160. Grad. Texts Math. Heidelberg: Springer-Verlag, 1995.

[Lee02] John M. Lee. *Introduction to smooth manifolds*. Vol. 218. Grad. Texts Math. Springer, 2002.

[Ler10] Eugene Lerman. “Orbifolds as stacks?” *Enseign. Math. (2)* 56.3-4 (2010), pp. 315–363.

[LNP22] Sil Linskens, Denis Nardin, and Luca Pol. “Global homotopy theory via partially lax limits”. *arXiv preprint arXiv:2206.01556* (2022).

[LO01] Wolfgang Lück and Bob Oliver. “The completion theorem in K-theory for proper actions of a discrete group”. *Topology* 40.3 (2001), pp. 585–616.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Ann. Math. Stud. Princeton, NJ: Princeton University Press, 2009.

[Lur17] Jacob Lurie. “Higher algebra”. *https://www.math.ias.edu/~lurie/papers/HA.pdf* (2017).

[Man22] Lucas Mann. “A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry”. *arXiv preprint arXiv:2206.02022* (2022).

[Mar21] Louis Martini. “Yoneda’s lemma for internal higher categories”. *arXiv preprint arXiv:2103.17141* (2021).

[Mat12] John Mather. “Notes on topological stability”. *Bull. Am. Math. Soc., New Ser.* 49.4 (2012), pp. 475–506. ISSN: 0273-0979.

[Met03] David Metzler. "Topological and smooth stacks". *arXiv preprint math/0306176* (2003).

[Mic08] Peter W. Michor. *Topics in differential geometry*. Vol. 93. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2008.

[MM02] Michael A. Mandell and J. Peter May. *Equivariant orthogonal spectra and S-modules*. Vol. 755. Mem. Am. Math. Soc. American Mathematical Society (AMS), 2002.

[MM03] Ieke Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*. Vol. 91. Camb. Stud. Adv. Math. Cambridge University Press, 2003.

[Moe02] Ieke Moerdijk. "Orbifolds as groupoids: an introduction." *Orbifolds in mathematics and physics. Proceedings of a conference on mathematical aspects of orbifold string theory, Madison, WI, USA, May 4–8, 2001*. Providence, RI: American Mathematical Society (AMS), 2002, pp. 205–222.

[MS06] J. P. May and J. Sigurdsson. *Parametrized homotopy theory*. Vol. 132. Math. Surv. Monogr. Providence, RI: American Mathematical Society (AMS), 2006.

[MV99] Fabien Morel and Vladimir Voevodsky. "$\mathbb{A}^1$-homotopy theory of schemes". *Publ. Math., Inst. Hautes Étud. Sci.* 90 (1999), pp. 45–143.

[MW21] Louis Martini and Sebastian Wolf. "Limits and colimits in internal higher category theory". *arXiv preprint arXiv:2111.14495* (2021).

[MW22] Louis Martini and Sebastian Wolf. "Presentable categories internal to an ∞-topos". *arXiv preprint arXiv:2209.05103* (2022).

[Nar16] Denis Nardin. "Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category". *arXiv preprint arXiv:1608.07704* (2016).

[Noo05] Behrang Noohi. "Foundations of topological stacks I". *arXiv preprint math/050 3247* (2005).

[NS18] Thomas Nikolaus and Peter Scholze. "On topological cyclic homology". *Acta Math.* 221.2 (2018), pp. 203–409.

[NSS15] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson. "Principal ∞-bundles: general theory". *J. Homotopy Relat. Struct.* 10.4 (2015), pp. 749–801.

[Nui16] Joost Nuiten. "Higher stacks as a category of fractions". *Unpublished, available at https://www.math.univ-toulouse.fr/~jnuiten/Writing/Groupoids.pdf* (2016).

[Pal61] Richard S. Palais. “On the existence of slices for actions of non-compact Lie groups”. *Ann. Math. (2)* 73 (1961), pp. 295–323.

[Par20] John Pardon. “Orbifold bordism and duality for finite orbispectra”. *arXiv preprint arXiv:2006.12702* (2020).

[PPT14] Markus J. Pflaum, Hessel Posthuma, and Xiang Tang. “Geometry of orbit spaces of proper Lie groupoids”. *J. Reine Angew. Math.* 694 (2014), pp. 49–84.

[Pro96] Dorette A. Pronk. “Étendues and stacks as bicategories of fractions”. *Compos. Math.* 102.3 (1996), pp. 243–303.

[PW19] Markus J. Pflaum and Graeme Wilkin. “Equivariant control data and neighborhood deformation retractions”. *Methods Appl. Anal.* 26.1 (2019), pp. 13–36.

[Rez14] Charles Rezk. “Global homotopy theory and cohesion”. *https://faculty.math.illinois.edu/~rezk/global-cohesion.pdf* (2014).

[Rob15] Marco Robalo. “K-theory and the bridge from motives to noncommutative motives”. *Adv. Math.* 269 (2015), pp. 399–550.

[Rog08] John Rognes. *Galois extensions of structured ring spectra. Stably dualizable groups*. Vol. 898. Mem. Am. Math. Soc. Providence, RI: American Mathematical Society (AMS), 2008.

[RS22] Shaul Ragimov and Tomer M. Schlank. “The ∞-Categorical Reflection Theorem and Applications”. *arXiv preprint arXiv:2207.09244* (2022).

[Sch18] Stefan Schwede. *Global homotopy theory*. Vol. 34. New Math. Monogr. Cambridge University Press, 2018.

[Sch20] Stefan Schwede. “Lectures on equivariant stable homotopy theory”. *Available on the author’s website* (2020).

[Seg68] Graeme Segal. “Equivariant K-theory”. *Publ. Math., Inst. Hautes Étud. Sci.* 34 (1968), pp. 129–151.

[Ser92] Jean-Pierre Serre. *Lie algebras and Lie groups. 1964 lectures, given at Harvard University.* 2nd ed. Vol. 1500. Lect. Notes Math. Springer-Verlag, 1992.

[Sha21] Jay Shah. “Parametrized higher category theory”. *to appear in Algebr. Geom. Topol., arXiv:1809.05892* (2021).

[SS21] Hisham Sati and Urs Schreiber. “Equivariant principal infinity-bundles”. *arXiv preprint arXiv:2112.13654* (2021).

[Vol21] Marco Volpe. “Six operations in topology”. *arXiv preprint arXiv:2110.10212* (2021).

[Wei02] Alan Weinstein. “Linearization of regular proper groupoids”. *J. Inst. Math. Jussieu* 1.3 (2002), pp. 493–511.

[Wir74] Klaus Wirthmüller. “Equivariant homology and duality”. *Manuscr. Math.* 11 (1974), pp. 373–390.

[Yan22] Lior Yanovski. “The monadic tower for ∞-categories”. *J. Pure Appl. Algebra* 226.6 (2022), p. 22.

[Zun06] Nguyen Tien Zung. “Proper groupoids and momentum maps: linearization, affinity, and convexity”. *Ann. Sci. Éc. Norm. Supér. (4)* 39.5 (2006), pp. 841–869.