Parametrised noncommutative motives and equivariant cubical descent in algebraic K-theory

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For an atomic orbital base category in the sense of [BDG+16a], we introduce the category of parametrised perfect–stable categories and use it to construct the parametrised version of noncommutative motives in which algebraic K-theory is corepresented. Furthermore, we initiate a rudimentary theory of parametrised cubes which could be of independent interest, generalising some of the elements in [Dot17] beyond the equivariant case. Using this cubical theory, we show that in the equivariant case for finite 2–groups G, the parametrised noncommutative motives canonically refine to G–symmetric monoidal categories. Consequently, this endows the equivariant algebraic K-theory spectra for these groups with the structure of $\mathbb{E}_\infty$–ring spectra equipped with multiplicative norms in the sense of [HHR16]. Along the way, we will also provide a machine to manufacture G–symmetric monoidal categories from symmetric monoidal categories equipped with G–actions and elucidate how the aforementioned parametrised perfect–stable categories relate to Mackey functors valued in perfect–stable categories.

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1 Introduction

Algebraic K-theory, as a functor $\mathrm{K} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$, is an additive spectral invariant on the ∞-category $\mathrm{Cat}^{\mathrm{perf}}$ of small perfect stable ∞-categories by the work of [BGT13] and moreover admits a lax symmetric monoidal refinement by [BGT14]. The methods of these papers were to construct the initial stable category receiving an additive functor (in the sense of sending exact sequences of categories to exact sequences in the target category) called the category of *noncommutative motives* NMot through which the functor K above factors. Using this lax symmetric monoidal structure, [BGS20] has been able to show that equivariant algebraic K–theory for finite groups G – whose precise definition we shall give in due course – naturally admits the so–called *Green functor* structure, in the sense of Mackey functor theory (i.e. $\mathbb{E}_\infty$–algebras in the Day convolution structure on $\mathrm{Sp}_G \coloneqq \mathrm{Mack}_G(\mathrm{Sp})$).

On the other hand, there are more refined multiplicative structures in equivariant homotopy theory for finite groups G in the form of "equivariant power operations" known as the *multiplicative norms*. This extra structure, which has a long history (see [Eve63] who introduced it in the setting of group cohomologies; see also [GM97] for its first appearance in stable homotopy theory), can be extremely valuable but is well–known to be tricky to construct. One such application is the celebrated resolution of the Kervaire invariant one problem in [HHR16]. As such, it would be desirable if equivariant algebraic K–theory could be shown to admit such structures.

The goal of this paper is to investigate the equivariant analogues of all the elements involved in the functor $\mathrm{K} \colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{NMot} \to \mathrm{Sp}$ and to lay the groundwork in studying the equivariant multiplicative refinements of all these. It turns out that the question of these refinements is intimately related to that of *cubical descent* in algebraic K–theory. Therefore, in order to handle these cubical matters in the equivariant setting, we will initiate the study of a theory on what we term as *parametrised cubes*, which could be of independent interest. In particular, the theory put forth here will be the rudiments of a theory of *parametrised functor calculus* which will be the subject of a future article. Employing this rudimentary theory, our main result will then be that equivariant algebraic K–theory can indeed be refined to admit multiplicative norms in the special case when G is a group of order 2^n for any n. We do not think that this result is optimal, and we will at least indicate later as we state the theorem the combinatorial difficulties involved in proving this for an arbitrary finite group G.

In the rest of the introduction, as we highlight some key results from the article, we will also give further details on the motivations from equivariant stable homotopy theory for having these multiplicative norms as well as summarise the relationship between these structures and the phenomenon of cubical descent for algebraic K–theory. This connection will be crucial in our solution for the case when $|G| = 2^n$.

Convention: This paper is written in the language of ∞–categories as developed in [Lur09; Lur17]. As such, in order not to encumber the exposition, by a "category" we will always mean an "∞–category". Hence, for example, we will write Cat for the ∞–category of small ∞–categories, usually written as Cat_∞. Furthermore, our work will heavily draw from the theory of *parametrised higher categories* of [BDG+16a; BDG+16b; Sha23; Sha22; NS22]. As there can be different conventions at times, we point out to the expert reader now that for a fixed base category $\mathcal{T}$, by a $\mathcal{T}$–category we will mean an object in $\mathrm{Cat}_{\mathcal{T}} \coloneqq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$.

Motivations for equivariant multiplicative norms

Let us first recall the notion of multiplicative norms and why it can be a useful piece of extra structure. Let $R \in \mathrm{CAlg}(\mathrm{Sp}_G^{\otimes})$ be an $\mathbb{E}_\infty$–ring in genuine G–spectra. In particular, this means that R is equipped with the structure of a multiplication $\otimes\colon R \otimes R \to R$ which participates in many coherence diagrams. Now for a subgroup $H \leq G$, there is a functor $\mathrm{N}_H^G\colon \mathrm{Sp}_H \to \mathrm{Sp}_G$ called the *norm*, which intuitively is the multiplicative version of the usual additive induction functor $\mathrm{Ind}_H^G\colon \mathrm{Sp}_H \to \mathrm{Sp}_G$ which satisfies a multiplicative version of the usual double–coset formula for Ind_H^G. Taking our cue from the multiplicative map $\otimes$ above, we may then ask if we could endow R with the structure of an "equivariant multiplication", i.e. a ring map

$$\otimes_H^G\colon \mathrm{N}_H^G \operatorname{Res}_H^G R \longrightarrow R$$

participating in appropriate coherence diagrams. These maps are the *multiplicative norms* alluded to above, and if we can supply these structures for all subgroups $H \leq G$ together with all the requisite coherences, then we will say that R has been enhanced to the structure of a G–$\mathbb{E}_\infty$–ring spectrum.

Now, admitting the structure of an $\mathbb{E}_\infty$–algebra object places severe constraints on a spectrum which might be fruitfully leveraged. For example, to show that an $\mathbb{E}_\infty$–ring A is zero, one just has to show that its π_0 vanishes; to show that A is nonzero, one may try to build nontrivial $\mathbb{E}_\infty$–ring maps out of A. This latter observation, while elementary, has been used very successfully in the proof of a key case of the $\mathbb{E}_\infty$–redshift conjecture in [Yua21] as well as the reduction to this key case in [BSY22]. In a similar vein, the multiplicative norms provide a lot extra structure and constraints that can be productively exploited. As cited above, these norm structures were used by Hill, Hopkins, and Ravenel in the celebrated [HHR16] in an essential way to construct a bordism–type spectrum which is sufficiently computable by virtue of having these multiplicative norms and which also sees enough of the Adams spectral sequence to obstruct the existence of Kervaire invariant one elements for all but six of the infinitely many cases. For an example of the rigid demands the multiplicative norms put on a ring G–spectrum, see for instance [PSW22, Ex. 3.28] where it is argued that the so–called "inflated" $H\mathbb{Z}$ C_2–spectral Mackey functor *cannot* admit such structures. Yet another instance where these norms can be illuminating is pointed out in [NS18, Rmk. III.1.5] where they showed how the norms can be used to recover their famed Tate diagonal, a structure now accepted as one of the central pillars of higher algebra at large.

Closer to our interests, however, is the potential use of the norms in proving "completion theorems" in equivariant stable homotopy theory, flagship examples of which are the Atiyah–Segal completion theorem [Ati61; AS69; AHJ+88] and the Segal conjecture [Lin80; AGM85; Car84]. This is a type of descent problem which is deeply tied to the birth of genuine equivariant homotopy theory and remains one of the central types of questions in this field. Roughly speaking, it can be stated as follows: for an object $R \in \mathrm{CAlg}(\mathrm{Sp}_G^{\otimes})$, we define the augmentation ideal I_G to be the kernel of the restriction map $\pi_0^G R \to \pi_0^e R$ where $e \leq G$ is the trivial subgroup. In reasonable situations, there is then a canonically constructed comparison map

$$(R^G)^\wedge_{I_G} \longrightarrow R^{hG}$$

and the completion problem asks to what extent this is an equivalence. For a nice exposition of this type of problem, we refer the reader to [GM95, §6 − §8]. A standard approach to these types of questions is to prove it by induction on the group G, and for this, a technical but often important step is to show that the two ideals I_H and $\operatorname{Res}_H^G I_G$ in $\pi_0^H R$ define the same completion. This, in turn, may be reduced to showing that they have the same radical. Often, this can turn out to be very difficult and rests on quite precise knowledge of the ring theory of $\pi_0^H R$ for all subgroups $H \leq G$ (for example, the various proofs of the Atiyah–Segal completion theorem rest on Atiyah's comprehensive description of the prime ideals in the complex representation ring in [Ati61, §6]). Nevertheless, in the presence of the multiplicative norms, this problem may be painlessly resolved in certain cases by the following type of manoeuvre:

Lemma. *Let G an abelian group and $H \leq G$, and let R be a G–$\mathbb{E}_\infty$–ring spectrum. In this case, we have* $\sqrt{\mathrm{Res}^G_H I_G} = \sqrt{I_H}$.

Proof. We always have $\sqrt{\mathrm{Res}^G_H I_G} \subseteq \sqrt{I_H}$. To see the reverse inclusion, let $a \in \pi_0^H R$ such that $a^n \in I_H$. By the double–coset formula for the norm, we get

$$\mathrm{Res}^G_H N^G_H(a^n) = \prod_{H\backslash G/H} N^H_{H^g \cap H} \mathrm{Res}^{H^g}_{H^g\cap H} g_*(a^n)$$

Since all our groups were abelian, there are no interesting conjugations and so the right hand side looks like $a^m \coloneqq \prod_{H\backslash G/H} a^n$, and hence $a^m \in \mathrm{Res}^G_H I_G$ as required. □

Indeed, a more sophisticated version of this was the main step in the proof of the completion theorem for equivariant MU in [GM97] where the multiplicative norms were first co–opted in stable homotopy theory. The lesson here, as also in [HHR16], is that the norms can be an invaluable tool in the way of constructing classes in equivariant homotopy groups with desirable computational and structural properties.

Having said that, with fantastic structures come serious difficulties. Unsurprisingly, these structures turn out to be rather tricky to formulate precisely and are yet trickier to construct. It was in large part to this end that the theory of *parametrised higher categories* were introduced and studied by Barwick, Dotto, Glasman, Nardin, and Shah in [BDG+16a; BDG+16b; Sha23; Sha22; NS22]. In the next part of the introduction, we shall recall the basic philosophy of this setup and explain what we mean by equivariant algebraic K–theory in this language.

Parametrised higher categories and equivariant algebraic K–theory

The primary ingredient of the parametrised formalism aforementioned is a fixed small base category $\mathcal{T}$, and the basic category of interest is $\mathrm{Cat}_{\mathcal{T}} \coloneqq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ whose objects are termed as *$\mathcal{T}$–categories*. In fact, for most of this article we will restrict to the case when $\mathcal{T}$ is *atomic orbital* (c.f. Definition 2.1.15). Roughly speaking, this is a class of categories isolated by the group [BDG+16b] that are particularly suited to higher algebraic considerations such as parametrised semiadditivity and multiplicative structures. The most important example of an atomic orbital base category for us will be the case of $\mathcal{T}$ being the orbit category $\mathcal{O}_G$ for a finite group G, in which case we will write $\mathrm{Cat}_G \coloneqq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat})$ where the objects are called *G–categories*. The point of this notion is that by having such highly structured objects, a theory of *parametrised (co)limits* can be developed which affords precise meanings to many analogies between equivariant higher algebra and ordinary higher algebra, thus rendering transparent the nonequivariant statements and proofs which readily generalise in the equivariant direction. In other words, the parametrised theory possesses both executive and suggestive powers to find and prove the "correct" generalisations of interesting nonequivariant notions.

As hinted at above, one salient feature of this theory is that it affords the theory of *G–symmetric monoidal categories* by which we may make sense of G–$\mathbb{E}_\infty$–algebras CAlg_G, i.e. $\mathbb{E}_\infty$–algebras equipped with multiplicative norms. In more detail, [Nar17] and its later expansion [NS22] set up the theory of *$\mathcal{T}$–operads* for any atomic orbital $\mathcal{T}$ analogous to Lurie's notion of ∞–operads in [Lur17, §2.1]. As in [Lur17], this provides a scaffolding over which the theory of $\mathcal{T}$–symmetric monoidal categories and $\mathcal{T}$–$\mathbb{E}_\infty$–algebras were developed. Satisfyingly, [NS22, Thm. 2.3.9] gives an equivalent description of the category G–symmetric monoidal categories as $\mathrm{Mack}_G(\mathrm{Cat})$, i.e. G–Mackey functors (in the sense of [Bar17]) valued in Cat.

Notwithstanding the pleasant generality in which all of these concepts can be developed, the task of constructing examples of G–$\mathbb{E}_\infty$–algebras remains by and large a tricky one. Despite this, the general consensus and expectation is that, at least, symmetric monoidal categories equipped with G–actions should certainly induce a valid G–symmetric monoidal category whose G–$\mathbb{E}_\infty$–algebras should be easy to describe. It is therefore in this context that we offer our first main result confirming these expectations:

Theorem A (Precise and full statement in Theorem 2.4.10)**.** *Let $\mathcal{D}^\otimes \in \mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat}))$ be a symmetric monoidal category with a G–action. From this datum, we may construct a G–symmetric monoidal category $\underline{\mathrm{Bor}}(\mathcal{D}^\otimes)$ whose underlying G–category is $\underline{\mathrm{Bor}}(\mathcal{D}) := \{G/H \mapsto \mathcal{D}^{hH}\}$. The multiplicative norm map $\mathrm{N}_H^G\colon \mathcal{D}^{hH} \to \mathcal{D}^{hG}$ can be concretely described as follows: for $X \in \mathcal{D}^{hH}$ a H–object in $\underline{\mathrm{Bor}}(\mathcal{D})$, the G–object $\mathrm{N}_H^G X \in \mathcal{D}^{hG}$ is given by $\bigotimes_{g\in G/H} gX$. Moreover, the category of G–$\mathbb{E}_\infty$–algebras in $\underline{\mathrm{Bor}}(\mathcal{D}^\otimes)$ admits a simple description as $\mathrm{CAlg}_G\big(\underline{\mathrm{Bor}}(\mathcal{D}^\otimes)\big) \simeq \mathrm{CAlg}(\mathcal{D}^\otimes)^{hG}$, i.e. $\mathbb{E}_\infty$–algebras in $\mathcal{D}^\otimes$ equipped with a G–action.*

We call the construction $\underline{\mathrm{Bor}}(-)$ the *Borelification* functor, and in the full version, we also show that *any* G–symmetric monoidal category admits a natural G–symmetric monoidal functor to its Borelification. The proof of this will proceed by first placing everything at the appropriate categorical level, where most of the problem may essentially be reduced to understanding the fully faithful functor $b\colon BG \hookrightarrow \mathcal{O}_G^{\mathrm{op}}$, and later extracting the desired statements by a process of decategorification. In hindsight, this was very much inspired by the philosophy of [GGN15] (which, in turn, was inspired by [Lur17, §4.8.2]) in dealing with monoidal *structures* via the *properties* of categorical products.

Next, we work towards explaining what we shall mean by equivariant (or more generally, $\mathcal{T}$–parametrised) algebraic K–theory in this article. For a fixed finite group G and a category with finite products $\mathcal{C}$, as mentioned before, [Bar17] supplies us the ∞–categorical version of G–Mackey functors valued in $\mathcal{C}$ denoted $\mathrm{Mack}_G(\mathcal{C})$. This is a powerful construction which recovers, for example, genuine G–spectra Sp_G as $\mathrm{Mack}_G(\mathrm{Sp})$ (this was first proved in [GM11], but see also [Nar16, App. A] and [CMN+20, App. A]). Since $\mathrm{Mack}_G(-)$ is functorial on product–preserving functors, we may apply it to the functor $\mathrm{K} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ to obtain

$$\mathrm{Mack}_G(\mathrm{K})\colon\ \mathrm{Mack}_G(\mathrm{Cat}^{\mathrm{perf}}) \longrightarrow \mathrm{Mack}_G(\mathrm{Sp}) = \mathrm{Sp}_G$$

The lax symmetric monoidal refinement of K then induces a lax symmetric monoidal refinement of $\mathrm{Mack}_G(\mathrm{K})$, and hence this functor takes $\mathbb{E}_\infty$–algebras in the domain to an $\mathbb{E}_\infty$–algebra object in Sp_G: this is the content of the Green functor refinement of equivariant algebraic K–theory due to [BGS20]. This construction has been used to great effect, for example, in [CMN+20], where they provided one of the key ingredients for the proof of the chromatic redshift conjecture for $\mathbb{E}_\infty$–rings (c.f. [LMM+22; Yua21; BSY22] for the complementary ingredients).

Unfortunately, this $\mathbb{E}_\infty$–structure lacks the desired multiplicative norms. Worse still, we do not even know if $\mathrm{Mack}_G(\mathrm{Cat}^{\mathrm{perf}})$ may be endowed with a reasonable G–symmetric monoidal structure with which to even begin to speak of G–$\mathbb{E}_\infty$–algebras. Therefore, as a first step, our goal is to provide a replacement for $\mathrm{Mack}_G(\mathrm{Cat}^{\mathrm{perf}})$ which admits a natural G–symmetric monoidal structure. This leads us to introduce the study of perfect–stable categories *internal* to the parametrised framework. In §2.5 we introduce the $\mathcal{T}$–category $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ of $\mathcal{T}$–perfect–stable categories for an arbitrary atomic orbital $\mathcal{T}$. By our work in [Hil22b], this $\mathcal{T}$–category is equivalent to the $\mathcal{T}$–category $\underline{\mathrm{Pr}}_{\mathcal{T},L,\mathrm{st},\omega}$ of $\mathcal{T}$–presentable–stable categories and $\mathcal{T}$–colimit–preserving functors which preserve $\mathcal{T}$–compact objects. The benefit of this equivalence is that there is already a natural $\mathcal{T}$–symmetric monoidal structure on $\underline{\mathrm{Pr}}_{\mathcal{T},L,\mathrm{st},\omega}$ constructed by Nardin in [Nar17], whence a $\mathcal{T}$–symmetric monoidal structure on $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. Moreover, in the same subsection we also prove various basic categorical properties about $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$, including the fact that it is $\mathcal{T}$–semiadditive and $\mathcal{T}$–presentable in Corollary 2.5.8. As a crucial bridge between this notion and the version of equivariant K–theory in [BGS20; CMN+20], we provide there our next main result in the form of:

Theorem B (See Theorem 2.5.11)**.** *We have a conservative $\mathcal{T}$–faithful inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subset \underline{\mathrm{Mack}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$. Moreover, this inclusion preserves and reflects parametrised (co)limits.*

Intuitively, in the case of $\mathcal{T} = \mathcal{O}_G$, the image of the faithful inclusion $\mathrm{Cat}^{\mathrm{perf}}_G \subset \mathrm{Mack}_G(\mathrm{Cat}^{\mathrm{perf}})$ consists of those $\mathrm{Cat}^{\mathrm{perf}}$–valued G–Mackey functors where the abstract transfer maps are both left and right adjoint to the restrictions. The theorem not only allows us to *define* $\mathcal{T}$–parametrised algebraic K–theory as the composite

$$\underline{\mathrm{K}}_{\mathcal{T}} \colon \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}} \hookrightarrow \mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \xrightarrow{\mathrm{Mack}_{\mathcal{T}}(\mathrm{K})} \mathrm{Mack}_{\mathcal{T}}(\mathrm{Sp}),$$

it also allows us to port many known concepts and results about $\mathrm{Cat}^{\mathrm{perf}}$ to the setting of $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$, the most important of which is the theory of *split Verdier sequences*. Recall that a split Verdier sequence in $\mathrm{Cat}^{\mathrm{perf}}$ is a sequence of objects $\mathcal{C} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{E}$ in $\mathrm{Cat}^{\mathrm{perf}}$ which is both a cofibre and a fibre sequence, and where both functors admit both left and right adjoints. Thanks to Theorem B, we can and will define a sequence in $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$ to be a split Verdier sequence if it is so when viewed as a sequence in $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ under the faithful inclusion, and we deduce various properties of this definition in §4.1 from the unparametrised theory. We also hope that this faithful inclusion provides some degree of reassurance and justification that $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$ is a legitimate notion to consider.

Multiplicativity and cubical descent

We now introduce the twin problem of endowing algebraic K–theory with equivariant multiplicative structures and that of equivariant cubical descent. First, recall that there is a very general method, in the form of [Lur17, Def. 2.2.1.6, Prop. 2.2.1.9], by which we may attempt to enhance a Bousfield localisation $L \colon \mathcal{C} \to \mathcal{D}$ to a symmetric monoidal functor, given a symmetric monoidal structure $\mathcal{C}^{\otimes}$ on $\mathcal{C}$. Essentially, we may do this if we can verify that for any finite collection $\{f_i \colon x_i \to y_i\}_{i\in I}$ of morphisms in $\mathcal{C}$ all of which are L–equivalences, the morphism $\otimes_i f_i \colon \otimes_i x_i \to \otimes_i y_i$ is also an L–equivalence, in which case we shall say that L is *compatible with the symmetric monoidal structure* $\mathcal{C}^{\otimes}$. In fact, by [Lur17, Ex. 2.2.1.7], this condition can be drastically simplified just to checking that the collection of L–equivalences is a tensor ideal, i.e. if $f \colon x \to y$ is an L–equivalence, then so is $\mathrm{id}_z \otimes f \colon z \otimes x \to z \otimes y$. This is because the map $f_1 \otimes f_2 \colon x_1 \otimes x_2 \to y_1 \otimes y_2$ can be factored into the composition of maps

$$x_1 \otimes x_2 \xrightarrow{\mathrm{id}_{x_1} \otimes f_2} x_1 \otimes y_2 \xrightarrow{f_1 \otimes \mathrm{id}_{y_2}} y_1 \otimes y_2 \tag{1}$$

both of which are of the form specified in the simplified condition.

Next, we recall how the functor $\mathrm{K} \colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ was enhanced to a lax symmetric monoidal one in [BGT14]. The point of view taken by the authors was a *motivic* one, which, to the best of our knowledge, is also the only perspective in the literature that affords this lax symmetric monoidal enhancement. In more detail, up to set–theoretic considerations which will not trouble us in this vague introduction, it was shown in [BGT13] that algebraic K–theory may be factored as the composition

$$\mathrm{K} \colon \mathrm{Cat}^{\mathrm{perf}} \xrightarrow{y} \mathrm{PSh}(\mathrm{Cat}^{\mathrm{perf}}) \xrightarrow{\mathcal{L}} \mathrm{NMot} \xrightarrow{\mathrm{map}(\mathcal{Z}(\mathrm{Sp}^{\omega}), -)} \mathrm{Sp} \tag{2}$$

where y is the Yoneda embedding, $\mathcal{Z} \coloneqq \mathcal{L}y$, and NMot is the so–called stable category of *noncommutative motives* obtained by stabilising the localisation of $\mathrm{PSh}(\mathrm{Cat}^{\mathrm{perf}})$ against the split Verdier sequences, i.e. inverting the maps $y(\mathcal{D})/y(\mathcal{C}) \to y(\mathcal{E})$ for split Verdier sequences as above. We call maps in $\mathrm{PSh}(\mathrm{Cat}^{\mathrm{perf}})$ that get inverted by $\mathcal{L}$ the *motivic equivalences*. This factorisation (2) allows us to upgrade the functor K to a lax symmetric monoidal one, where the main issue boils down to showing that $\mathcal{L}$ enhances to a symmetric monoidal functor using the Day convolution structure on $\mathrm{PSh}(\mathrm{Cat}^{\mathrm{perf}})$. By the general theory recalled above, in order to do this, one observes that $y(\mathcal{A}) \otimes [y(\mathcal{D})/y(\mathcal{C}) \to y(\mathcal{E})] \simeq [y(\mathcal{A}\otimes\mathcal{D})/y(\mathcal{A}\otimes\mathcal{C}) \to y(\mathcal{A}\otimes\mathcal{E})]$. Since applying $\mathcal{A}\otimes -$ preserves split Verdier sequences, the right–hand–side (and hence also the left–hand–side) is a motivic equivalence, as required.

Coming back to the parametrised setup, it turns out that one may also make sense of the notion of a $\mathcal{T}$–category of $\mathcal{T}$*–noncommutative motives* using the notion of $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ introduced above. In fact, in §4.2, we construct *two* variants $\underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}}$ and $\underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$ of $\mathcal{T}$–noncommutative motives called the pointwise and normed variants, respectively. As suggested by the name, we prove in Proposition 4.2.17 that the functor $\mathcal{Z}_{\mathrm{nm}}\colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$ refines to a $\mathcal{T}$–symmetric monoidal functor. On the other hand, $\underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}}$ is the one which provides the universal property of $\mathcal{T}$–parametrised algebraic K–theory analogous to the one of [BGT13; CDH+] (the universal property we give follows the latter citation which removed the need for additive functors to preserve filtered colimits), as encapsulated by the following theorem:

Theorem C (See Theorems 4.2.11 and 4.2.15)**.** *For any $\mathcal{T}$-presentable-stable category $\underline{\mathcal{E}}$, the precomposition $\mathcal{Z}^*_{\mathrm{pw}}\colon \underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}}, \underline{\mathcal{E}}) \to \underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}})$ is an equivalence, where $\underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}})$ denotes the full subcategory of additive functors, i.e. those that send split Verdier sequences to fibre sequences. Moreover, we have the factorisation*

$$\underline{\mathrm{K}}_{\mathcal{T}}\colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \xrightarrow{\mathcal{Z}_{\mathrm{pw}}} \underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}} \xrightarrow{\underline{\mathrm{map}}(\mathcal{Z}_{\mathrm{pw}}\underline{\mathrm{Sp}},-)} \underline{\mathrm{Sp}}_{\mathcal{T}}$$

of $\mathcal{T}$–parametrised algebraic K–theory.

By the universal property from Theorem C, we obtain a canonical comparison map $\Psi\colon \underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$. This map is an equivalence if and only if the functor $\mathcal{Z}_{\mathrm{pw}}$ is compatible with the $\mathcal{T}$–symmetric monoidal structures in a sense analogous to the unparametrised situation sketched above but which we will not make precise here (this criterion is given for example in [QS22, Lem. 5.27]; see also Propositions 2.3.4 and 2.3.7 where we provide a different proof). If this happens, then $\underline{\mathrm{K}}_{\mathcal{T}}$ refines to a $\mathcal{T}$–lax symmetric monoidal functor. There is no essential mathematical content in the reformulation in terms of the map Ψ, and we constructed $\underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$ only to show that it is always formally possible to define a version of $\mathcal{T}$–parametrised algebraic K–theory which has the multiplicative norms by *defining* it to be the functor corepresented by the multiplicative unit in $\underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$. Incidentally, we should also mention that, apart from its pleasant universal property, such categories of noncommutative motives can also serve as a convenient setting to prove things about additive functors in a uniform way, as for example appears in [CMN+20].

Unfortunately, this is where the breezy transferability from the nonequivariant setting to the equivariant one ends. The core issue in this setting is the lack of a *currying* manoeuvre for G–tensor products. In slightly more detail, recall from (1) that we were able to simplify dramatically the sufficient condition for symmetric monoidality of localisations because we were able to *curry* and separate the problem into each tensor component. This led to a rather easy check to multiplicatively enhance the motivic localisation $\mathcal{Z}$ from (2). When considering G–tensor products, we have no such luxury since tensoring an object G–times and remembering the permutation G–equivariant structure inextricably links the tensor components, precluding any ability to separate and deal with each tensor component in an inductive fashion. This forces us to deal with tensor powers of localisation equivalences head–on, leading naturally to the phenomenon of *K–theoretic cubical descent*, as we shall presently explain.

Suppose we are given two split Verdier sequences $\{\mathcal{C}_i \hookrightarrow \mathcal{D}_i \twoheadrightarrow \mathcal{E}_i\}_{i=0,1}$ in $\mathrm{Cat}^{\mathrm{perf}}$. We would like to show that the tensored map $y(\mathcal{D}_1)/y(\mathcal{C}_1) \otimes y(\mathcal{D}_2)/y(\mathcal{C}_2) \longrightarrow y(\mathcal{E}_1)\otimes y(\mathcal{E}_2)$ is still a motivic equivalence. Since $y(-)$ was symmetric monoidal and $-\otimes-$ commutes with colimits in each variable, this map is equivalent to the horizontal map in

$$\begin{tikzcd} \frac{y(\mathcal{D}_1\otimes\mathcal{D}_2)}{y(\mathcal{D}_1\otimes\mathcal{C}_2)\cup_{y(\mathcal{C}_1\otimes\mathcal{C}_2)}y(\mathcal{C}_1\otimes\mathcal{D}_2)} \arrow[r] \arrow[d] & y(\mathcal{E}_1\otimes\mathcal{E}_2) \\ \frac{y(\mathcal{D}_1\otimes\mathcal{D}_2)}{y(\mathcal{D}_1\otimes\mathcal{C}_2\cup_{\mathcal{C}_1\otimes\mathcal{C}_2}\mathcal{C}_1\otimes\mathcal{D}_2)} \arrow[ur] & \end{tikzcd} \tag{3}$$

For general reasons, the horizontal map factors as displayed above. This breaks up the problem at hand into two parts:

(a) If we could show that the induced map $\mathcal{D}_1\otimes\mathcal{C}_2\cup_{\mathcal{C}_1\otimes\mathcal{C}_2}\mathcal{C}_1\otimes\mathcal{D}_2\rightarrow\mathcal{D}_1\otimes\mathcal{D}_2$ is fully faithful, then by general facts about split Verdier sequences, we would get that $\mathcal{D}_1\otimes\mathcal{C}_2\cup_{\mathcal{C}_1\otimes\mathcal{C}_2}\mathcal{C}_1\otimes\mathcal{D}_2\hookrightarrow\mathcal{D}_1\otimes\mathcal{D}_2\twoheadrightarrow\mathcal{E}_1\otimes\mathcal{E}_2$ is again a split Verdier sequence. Thus, if this is the case, then the diagonal map in (3) is a motivic equivalence.

(b) If we could show that the functor $\mathcal{Z}\colon \mathrm{Cat}^{\mathrm{perf}}\rightarrow\mathrm{NMot}$ preserves pushouts of the form

$$\begin{tikzcd} \mathcal{C}_1\otimes\mathcal{C}_2 \arrow[r, hook] \arrow[d, hook] & \mathcal{C}_1\otimes\mathcal{D}_2 \arrow[d] \\ \mathcal{D}_1\otimes\mathcal{C}_2 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{D}_1\otimes\mathcal{C}_2\cup_{\mathcal{C}_1\otimes\mathcal{C}_2}\mathcal{C}_1\otimes\mathcal{D}_2 \end{tikzcd}$$

then the vertical map in (3) will be seen to be a motivic equivalence.

Analogous questions involving cubes of higher dimensions can of course be formulated when we have more than two split Verdier sequences. For reasons which we hope are clear, we will loosely term problems of type (b) above as the problem of *motivic cubical descent.* As such, upon forgoing currying manoeuvres and up to the technical point (a), endowing coherent multiplicative structures on K–theory from this motivic perspective is equivalent to a certain descent rigidity with regards to special types of cubes. Since all of these serve only to rephrase the problem of enhancing $\mathcal{Z}$ to a symmetric monoidal functor, it is unsurprising that problems (a) and (b) can be solved in the nonequivariant setting by inducting on the number of tensor components via currying. In particular, this implies that K–theory (and in fact, any additive functor) automatically satisfies extra descent with respect to a class of diagrams larger than just the split Verdier sequences, a phenomenon well–observed in the literature.

Motivated by these questions, we initiate in §3 the study of *parametrised cubes.* The main idea here is that, for the very same reason that cubes show up in Goodwillie's seminal theory of functor calculus, generalised versions of cubes which we call parametrised cubes also show up naturally when one considers parametrised category theory. More concretely, said cubes are parametrised categories obtained by taking finite indexed products of Δ^1 and they show up in our work because we are interested in taking indexed tensors, i.e. multiplicative norms, of split Verdier sequences (which are in particular cofibre sequences, and may hence be specified by a Δ^1–diagram in perfect–stable categories). As we shall see in §3, the hypothesis that $\mathcal{T}$ is atomic orbital will be used in an essential way in order to define the "singletons" in a parametrised cube. A variant of such a theory, in the equivariant setting, has for example been investigated in [Dot17] using model categories. Our approach, however, is purely ∞–categorical and model–independent, running on the philosophy that much of cubical theory may be phrased beneficially in terms of the yoga of Kan extensions (see also [Sto22] for another manifestation of this philosophy in the theory of posets in unparametrised higher categories). To the best of our knowledge, a cubical theory in this level of generality is new could be of independent interest. Indeed, we should say that the theory recorded here will serve as the rudiments of a more comprehensive theory of *parametrised functor calculus* which we will be the subject of a separate article.

The guiding example for us (which will in any case be the key example powering our main theorem) will be that of C_2–*pushouts*, assuming for now that $G = C_2$ for simplicity. Namely, suppose in the setting of problem (b) above, instead of tensoring two different split Verdier sequences, we took its C_2–norm instead. As explained above, this needs to be reckoned with if we are to enhance $\mathcal{Z}_{\mathrm{pw}} \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_G \to \underline{\mathrm{NMot}}^{\mathrm{pw}}_G$ to a G–symmetric monoidal functor. In a fashion similar to (b), this will induce a C_2–colimit diagram in $\mathrm{Cat}^{\mathrm{perf}}_{C_2}$

$$\begin{tikzcd} \mathrm{N}^{C_2}_e\mathcal{C} \arrow[r, hook] \arrow[d, hook] & \mathcal{C}\otimes\mathcal{D} \arrow[d] \arrow[dl, dashed, no head] \\ \mathcal{D}\otimes\mathcal{C} \arrow[r] & \mathcal{D}\otimes\mathcal{C}\underline{\sqcup}_{\mathrm{N}^{C_2}_e\mathcal{C}}\mathcal{C}\otimes\mathcal{D} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \tag{4}$$

which we term as a C_2–pushout diagram. While outwardly this looks rather similar to the pushout in (b) above, there are also key differences. In (4), the top left and bottom right terms are C_2–objects in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{C_2}$, but the other two terms are merely objects in $\mathrm{Cat}^{\mathrm{perf}}$. The point is that the group C_2 acts on the *entire* diagram, where it acts by swapping $\mathcal{C}\otimes\mathcal{D}$ with $\mathcal{D}\otimes\mathcal{C}$. In other words, the indexing C_2–diagram, as an object in $\mathrm{Cat}_{C_2} = \mathrm{Fun}(\mathcal{O}^{\mathrm{op}}_{C_2}, \mathrm{Cat})$ looks like the square $\Delta^1\times\Delta^1$ for the underlying category (i.e. the value at C_2/e) whose C_2–fixed points (i.e. the value at C_2/C_2) are only the full subcategory spanned by the initial and terminal objects in $\Delta^1\times\Delta^1$. These types of equivariant diagrams give rise to a wealth of interesting constructions. For example, in C_2–spaces, taking the colimit of the ordinary diagram $(* \leftarrow X \rightarrow *)$ yields the usual suspension ΣX, whereas the colimit of the C_2–diagram $(* \leftarrow X \rightarrow *)$ where the two copies of $*$ are swapped by C_2 yields the the *sign suspension* $\Sigma^\sigma X$ of $X \in \mathcal{S}_{C_2}$.

In the nonequivariant case, we have indicated how one can use currying to show K–theoretic descent for cubes of the sort in problem (b). The idea for the C_2–equivariant case now is to show that the square descent in K–theory induces descent with respect to C_2–colimits of the form (4). To this end, we prove a general re–expression result in Notation 4.3.1 which allows us to rewrite the C_2–pushout diagram (4) into the *ordinary* pushout diagram

$$\begin{tikzcd} \mathrm{Ind}^{C_2}_e\mathcal{C}\otimes\mathcal{C} \simeq \mathrm{Ind}^{C_2}_e\mathrm{Res}^{C_2}_e\mathrm{N}^{C_2}_e\mathcal{C} \arrow[r, "\varepsilon"] \arrow[d, hook] & \mathrm{N}^{C_2}_e\mathcal{C} \arrow[d] \\ \mathrm{Ind}^{C_2}_e\mathcal{C}\otimes\mathcal{D} \arrow[r] & \mathcal{D}\otimes\mathcal{C}\underline{\sqcup}_{\mathrm{N}^{C_2}_e\mathcal{C}}\mathcal{C}\otimes\mathcal{D} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

of C_2–objects in $\mathrm{Cat}^{\mathrm{perf}}_{C_2}$, where the map ε is the counit for the adjunction $\mathrm{Ind} \dashv \mathrm{Res}$. We then show that additive functors satisfy descent with respect to such squares by standard methods, and so solve the C_2–analogue (4) of problem (b) above. Together with this, a dévissage–type argument via the solvability of p–groups (in the case $p = 2$) then yields the following main theorem of this article, providing in the special case of G being a 2–group the desired refinement of the Green functor structure from [BGS20] to G–$\mathbb{E}_\infty$–algebras.

Theorem D (See Theorem 4.3.11 and Corollary 4.3.12)**.** *Let G be a group with $|G| = 2^n$ for some n. The comparison map $\Psi : \underline{\mathrm{NMot}}^{\mathrm{pw}}_G \to \underline{\mathrm{NMot}}^{\mathrm{nm}}_G$ in this case is an equivalence. Consequently, $\underline{\mathrm{K}}_G \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_G \to \underline{\mathrm{Sp}}_G$ canonically refines to a G–lax symmetric monoidal functor for such G and so induces the functor $\underline{\mathrm{K}}_G \colon \mathrm{CAlg}_G(\underline{\mathrm{Cat}}^{\mathrm{perf}}_G) \to \mathrm{CAlg}_G(\underline{\mathrm{Sp}}_G)$.*

This in particular means that for such groups, the equivariant K–theory spectrum $\{\mathrm{K}(\mathbb{S}_H)\}_{H\leq G}$ of the equivariant sphere spectrum canonically assembles to a normed $\mathbb{E}_\infty$–ring spectrum in Sp_G. Our expectation is that, armed with a good categorical and combinatorial control of general parametrised cubes which we presently lack, the theorem should hold for arbitrary atomic orbital base categories $\mathcal{T}$. As far as we know, the descent of K–theory with respect to parametrised cubes such as (4) has never been investigated before and we think that the question of parametrised cubical descent for additive functors could be an independently interesting line of pursuit for a fuller understanding of their rigidity properties.

Lastly, a straightforward combination of Theorem A and Theorem D gives us the following:

Corollary E (See Corollary 4.4.2)**.** *Let G be a group with $|G| = 2^n$ for some n and $\mathcal{C}^{\otimes} \in \mathrm{Fun}(BG, \mathrm{CAlg}((\mathrm{Cat}^{\mathrm{perf}})^{\otimes}))$ be a small symmetric monoidal perfect–stable category equipped with a symmetric monoidal G–action. Then the spectra $\{\mathrm{K}(\mathcal{C}^{hH})\}_{H\leq G}$ naturally assemble to a spectral G–Mackey functor equipped with multiplicative norms.*

The type of equivariant K–theory spectra considered in Corollary E represents an extremely interesting class of examples and are sometimes called Swan K–theory in the literature. In the classical setting, its importance in K–theory's equivariant structure theory has been recognised as early as Swan's groundbreaking work [Swa60] together with its axiomatisation and hermitian elaboration [Dre75] (the latter which is the original source for the Mackey formulation of induction theorems, often called Dress induction theorems today). More recently, they have also been considered, for instance, in [BGS20, §8] and [MM19].

Relation to other work. Most results in this paper are corrections and expansions of Chapters 2–4 of the author's PhD thesis [Hil22a]. Equivariant algebraic K-theory is not a new subject and much work has been done in this area, see for example [BMM+21; Len21; Mer17; Sch19]. A slightly over-simplified but helpful view is that there are two versions of higher algebraic K-theory: on the one hand, there is the *group-completion* K*-theory* whose input is a small symmetric monoidal category $\mathcal{C}$ and one group completes the $\mathbb{E}_\infty$-space $\mathcal{C}^{\simeq}$ to obtain a connective spectrum - classically, this is related to Quillen's $+$-construction and the reader is referred to [GGN15] for an ∞-categorical treatment which gives a highly structure refinement of this construction; on the other hand, there is the Quillen/Segal/Waldhausen K*-theory* whose input is a small stable ∞-category - this corresponds to Quillen's Q-construction and Segal and Waldhausen's $\mathrm{S}_\bullet$-construction. All the literature cited above dealt with the equivariant enhancement of the group-completion K-theory. In this paper, we treat the latter version of K-theory, and is a further refinement of the multiplicative structures treated in [BGS20; CMN+20] to include the multiplicative norms.

Finally, but perhaps most importantly, we mention that our work is not the first to treat the structure of multiplicative norms on equivariant algebraic K–theory. In [EH23], Elmanto and Haugseng have, independently of us, shown that for *all* finite groups G, the equivariant algebraic K–theory *space* enhances to one with norms when the input is endowed with such structure, using as a key input the recent advancement on K–theoretic power operations in [BGM+22]. As a matter of commentary on the differences between our work and theirs, we should say that our methods are totally distinct from theirs and consequently, so too are the results in their finer, but important, points. While their work provides the norms for *all* finite groups, because of its dependence on the much more general polynomiality of [BGM+22], the norms they obtain are maps of *spaces*; whereas, because we work with K–theory *spectra* throughout, the norm structures we obtain are happening at the level of spectra, which is strictly more structure. This distinction can prove to be crucial when one wants to consider modules in G–spectra over these equivariant K–theory ring spectra, for example. Furthermore, we think that the motivic universal property of parametrised K–theory as well as the link to the problem of parametrised cubical descent we provide in this article could be interesting points in their own rights for a fuller understanding of equivariant algebraic K–theory.

Assumptions and outline of paper. This paper builds on the theory of parametrised homotopy theory as introduced and studied in [BDG+16a; BDG+16b; Sha23; Sha22; Nar17] and as further developed in [Hil22b]. Moreover, unless otherwise stated, we will always assume the base category $\mathcal{T}$ to be atomic orbital (cf. Definition 2.1.15).

Since our K–theoretic goals will require a fair bit of parametrised machinery, we will take the opportunity in §2 to contribute to what may be classified as general parametrised theory where we will in particular prove Theorems A and B. Next, we introduce and develop the basics of the theory of parametrised cubes in §3. In the final §4, we will apply all the theory above to construct the parametrised version of noncommutative motives and prove Theorems C and D and Corollary E. Each section will be prefaced with a more detailed outline of its contents.

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2 Aspects of parametrised category theory

This section pertains to supplying miscellaneous results in parametrised higher category theory that may be viewed to be of general utility. We recollect the foundations of the parametrised theory that we shall need in §2.1 as well as take the opportunity to prove some basic categorical generalities in §2.2; in §2.3 (from which point on we will always assume that the base category is atomic orbital), we will prove various elements in the interaction between localisations and multiplicative structures that will be crucial for our construction of multiplicative structures in noncommutative motives in §4.2; next, we will elucidate in §2.4 the G–symmetric monoidal theory associated to so–called "Borel" G–categories, proving Theorem A via a categorification–decategorification procedure; to end the section, we shall introduce in §2.5 the "internal" notion of perfect–stable categories in the parametrised setting, prove their basic categorical properties such as $\mathcal{T}$–semiadditivity and $\mathcal{T}$–presentability, and finish with a proof of Theorem B which relates this internal notion of perfect–stability with the "external" one. This internal notion of perfect–stable categories will be important to us since they will serve as the input of our parametrised algebraic K–theory functor later.

2.1 Basic setup

We provide here an overview of the basic theory that has already appeared in the literature. For the categorical foundations of parametrised homotopy theory, we refer the reader to [BDG+16a; BDG+16b; Sha23; Sha22] (in particular, for a discussion of parametrised adjunctions, which will be the bread and butter of this article, see [Sha23, §8]); for the algebraic theories of semiadditivity and symmetric monoidal structures, we refer the reader to [Nar16; Nar17; NS22]; for the theory of presentability, see [Hil22b]; and finally, for a one–stop survey for many of the basic theory, see for instance [Hil22a]. Expert readers should feel free to skip this subsection.

Definition 2.1.1 ($\mathcal{T}$–categories)**.** Let $\mathcal{T}$ be a small category. The category of *$\mathcal{T}$–categories* $\mathrm{Cat}_{\mathcal{T}}$ is defined to be $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$. An object in $\mathrm{Cat}_{\mathcal{T}}$ will be indicated with the underline notation $\underline{\mathcal{C}}$. Under Lurie's straightening–unstraightening equivalence $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) \simeq \mathrm{coCart}(\mathcal{T}^{\mathrm{op}})$, we will denote by $\int \underline{\mathcal{C}} \to \mathcal{T}^{\mathrm{op}}$ the cocartesian unstraightening of $\underline{\mathcal{C}} \in \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$. Morphisms in $\mathrm{Cat}_{\mathcal{T}}$ are called *$\mathcal{T}$–functors.*

Terminology 2.1.2 (Objects)**.** By an *object* in a parametrised category $\underline{\mathcal{C}}$, we will mean a $\mathcal{T}$–functor $\underline{*} \to \underline{\mathcal{C}}$ where $\underline{*} \in \mathrm{Cat}_{\mathcal{T}}$ is the terminal $\mathcal{T}$–category which is constant with value $*$. For a fixed $V \in \mathcal{T}$, writing $v\colon \mathcal{T}_{/V} \to \mathcal{T}$ for the canonical functor, we may then define a V–object in $\underline{\mathcal{C}}$ to be a $\mathcal{T}$–functor $v_!\underline{*} \to \underline{\mathcal{C}}$. By adjunction, this is the same datum as a $\mathcal{T}_{/V}$–functor $\underline{*} \to v^*\underline{\mathcal{C}}$. Hence, the datum of a V–object in $\underline{\mathcal{C}}$ is the same datum as an object (in the sense of the first sentence above) in $v^*\underline{\mathcal{C}}$.

Notation 2.1.3. For a morphism $f\colon W \to V$ in $\mathcal{T}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$, we will write $f^*\colon \mathcal{C}_W \to \mathcal{C}_V$ for the structure map encoded by $\underline{\mathcal{C}}$.

Notation 2.1.4. We will write $(-)^{\underline{\mathrm{op}}}\colon \mathrm{Cat}_{\mathcal{T}} \xrightarrow{\simeq} \mathrm{Cat}_{\mathcal{T}}$ for the self–equivalence induced by applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to the self–equivalence $(-)^{\mathrm{op}}\colon \mathrm{Cat} \xrightarrow{\simeq} \mathrm{Cat}$.

Construction 2.1.5 (Cofree parametrisation, [Nar16, Thm. 2.8])**.** Let $\mathcal{D}$ be a category. Then there is a $\mathcal{T}$-category $\underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathcal{D})$ classified by

$$\mathcal{T}^{\mathrm{op}} \to \mathrm{Cat} \quad :: \quad V \mapsto \mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{D})$$

called the *cofree $\mathcal{T}$–category on $\mathcal{D}$*. This has the following universal property: if $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$, then there is a natural equivalence $\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathcal{D})) \simeq \mathrm{Fun}(\int \underline{\mathcal{C}}, \mathcal{D})$ of unparametrised categories, where $\mathrm{Fun}_{\mathcal{T}}$ denotes the category of $\mathcal{T}$–functors.

As illustrated by the following examples, for underlined objects, we will often omit the subscripts $\mathcal{T}$ for readability.

Example 2.1.6 (Spaces and categories). An important example is the *$\mathcal{T}$–category of $\mathcal{T}$–spaces* $\underline{\mathcal{S}}$, defined as $\underline{\mathrm{Cofree}}(\mathcal{S})$. There is then a parametrised mapping space $\underline{\mathrm{Map}}(-,-)$ functor landing in $\underline{\mathcal{S}}$ associated to any $\mathcal{T}$–category which induces a parametrised Yoneda embedding satisying the usual universal property of presheaves. We refer the reader to [Sha23, §10] for more details. Similarly, we define the *$\mathcal{T}$–category of $\mathcal{T}$–categories* $\underline{\mathrm{Cat}}$ as $\underline{\mathrm{Cofree}}(\mathrm{Cat})$.

Terminology 2.1.7. A $\mathcal{T}$–functor $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is said to be *$\mathcal{T}$–fully faithful* if it is so fibrewise. Via the notion of parametrised mapping spaces from the example above, this can also be formulated as saying the $\mathcal{T}$–functor induces equivalences on the parametrised mapping spaces.

Recollections 2.1.8 (Adjunctions)**.** Of foundational importance in the parametrised theory is the concept of *$\mathcal{T}$–adjunctions* and this was defined by Shah in [Sha23, §8] building crucially on Lurie's notion of relative adjunctions from [Lur17, §7.3.2]. From the point of view of cocartesian fibrations, a $\mathcal{T}$–adjunction between $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ is the data of $\mathcal{T}$–functors (i.e. maps of cocartesian fibrations over $\mathcal{T}^{\mathrm{op}}$) $L\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and $R\colon \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ together with the data of an ordinary adjunction $\int L\colon \int \underline{\mathcal{C}} \rightleftharpoons \int \underline{\mathcal{D}} : \int R$ where the adjunction (co)units map to the identity in the base category $\mathcal{T}^{\mathrm{op}}$. By [Hil22b, Lem. 2.2.9] for example, this may also be phrased more internally as the datum of a natural equivalence of mapping space functors

$$\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(L-,-) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-,R-)\colon \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{D}} \longrightarrow \underline{\mathcal{S}}$$

as in the usual definition of adjunctions.

Notation 2.1.9 (Cotensors)**.** There is an internal hom functor $\underline{\mathrm{Fun}}$ (also written as $\underline{\mathrm{Fun}}_{\mathcal{T}}$ when we want to be explicit with the base category we are working over) equipped with natural equivalence $\mathrm{Fun}_{\mathcal{T}}(- \times \underline{\mathcal{C}}, -) \simeq \mathrm{Fun}_{\mathcal{T}}(-, \underline{\mathrm{Fun}}(\underline{\mathcal{C}}, -))$ for any $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. Write $p\colon \mathcal{T}^{\mathrm{op}} \to *$ for the unique functor and let I be a small unparametrised category. Then the adjunction $- \times I : \mathrm{Cat} \rightleftarrows \mathrm{Cat} : \mathrm{Fun}(I, -)$ induces the adjunction

$$(- \times I)_* : \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) \rightleftarrows \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) : \mathrm{Fun}(I, -)_*$$

Under the identification $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) = \mathrm{Cat}_{\mathcal{T}}$, it is clear that $(- \times I)_*$ corresponds to the $\mathcal{T}$-functor $- \times p^*I$, whose right adjoint is $\underline{\mathrm{Fun}}(p^*I, -)$. Therefore $\underline{\mathrm{Fun}}(p^*I, -) \simeq \mathrm{Fun}(I, -)_*$ implements the *fibrewise functor construction*. We will often write $\underline{\mathrm{Fun}}(I, -)$ for $\underline{\mathrm{Fun}}(p^*I, -)$. This satisfies the following properties whose proofs are immediate.

1. $\underline{\mathrm{Cat}}_{\mathcal{T}}$ is cotensored over Cat in the sense that for any $\mathcal{T}$-categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ we have

$$\underline{\mathrm{Fun}}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}(I, \underline{\mathcal{D}})) \simeq \underline{\mathrm{Fun}}(I, \underline{\mathrm{Fun}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$$

2. $\underline{\mathrm{Fun}}(I, -)$ preserves $\mathcal{T}$-adjunctions. This is straightforward to deduce from [Sha23, §8].

Observation 2.1.10. Suppose $\mathcal{T}$ has a final object and let $s\colon * \hookrightarrow \mathcal{T}$ be the inclusion of the final object, so that upon passing to the opposites, we have the adjunction $s\colon * \rightleftharpoons \mathcal{T}^{\mathrm{op}} : p$. We claim now that there is a natural equivalence of functors

$$s^*\underline{\mathrm{Fun}}(s_!I, -) \simeq \mathrm{Fun}(I, s^*-)\colon \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}$$

To wit, fixing some $A \in \mathrm{Cat}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$, consider the sequence of natural equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Cat}}\left(A, s^*\underline{\mathrm{Fun}}(s_!I, \underline{\mathcal{C}})\right) &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(s_!A \times s_!I, \underline{\mathcal{C}}) \\ &\simeq \mathrm{Map}_{\mathrm{Cat}}(A \times I, s^*\underline{\mathcal{C}}) \\ &\simeq \mathrm{Map}_{\mathrm{Cat}}\left(A, \mathrm{Fun}(I, s^*\underline{\mathcal{C}})\right)\end{aligned}$$

where we have also used that $s_!\colon \mathrm{Cat} \to \mathrm{Cat}_{\mathcal{T}}$ preserves products since $s_! \simeq p^*$ is a right adjoint by virtue of the adjunction $s \dashv p$.

Recollections 2.1.11 ((Co)limits and indexed (co)products)**.** The notion of parametrised adjunctions and $\underline{\mathrm{Fun}}$ afford us the key concept of *parametrised (co)limits*. That is, for any $\underline{I}, \underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ and writing $\pi\colon \underline{I} \to \underline{*}$ for the unique $\mathcal{T}$–functor, the *$\underline{I}$–shaped (co)limit in $\underline{\mathcal{C}}$* functor, if it exists, may be defined as the parametrised right (resp. left) adjoint π_* (resp. $\pi_!$) to the restriction functor $\pi^*\colon \underline{\mathcal{C}} \to \underline{\mathrm{Fun}}(\underline{I}, \underline{\mathcal{C}})$. The reader should be warned that these parametrised colimits are *not* given by fibrewise taking (co)limit, although this *is* so when the indexing shape is a constant $\mathcal{T}$–category (i.e. those of the form p^*I as in Notation 2.1.9). Furthermore, one can also develop the notion of parametrised Kan extensions etc., and we refer the reader to [Sha23, §§9, 10] for more technical details.

A very important part of the general theory that we will use often in our arguments later is that every parametrised colimit may be decomposed into an unparametrised part and a truly parametrised part. To add precision to this, it would be helpful first to recall some standard terminologies: it is common in the literature to term as *fibrewise (co)limits* those parametrised (co)limits which are indexed by a constant diagram (in the sense explained above). On the other hand, for any fixed $V \in \mathcal{T}$, by basechanging from $\mathcal{T}$ to $\mathcal{T}_{/V}$ (i.e. by considering the functor $\mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}_{/V}}$ induced by the canonical functor $\mathcal{T}_{/V} \to \mathcal{T}$), we may without loss of generality assume that V was a final object in $\mathcal{T}$. In this case, for any $U \in \mathcal{T}$, writing $f\colon U \to V$ in $\mathcal{T}$ for the unique map, we also write $f\colon f_!f^*\underline{*} \to \underline{*}$ for the unique map of $\mathcal{T}$–categories (note the intentional abuse of the notation f). Now for any $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$, we shall term the left (resp. right) adjoint $f_!$ (resp. f_*)$\colon \underline{\mathrm{Fun}}(f_!f^*\underline{*}, \underline{\mathcal{C}}) \simeq f_*f^*\underline{\mathcal{C}} \to \underline{\mathcal{C}}$ to the functor f^*, if it exists, as the *f–indexed coproduct (resp. product)*. These (co)limits play a distinguished role in the parametrised theory in the following manner: much like in the unparametrised setting where every colimit can be rewritten as a geometric realisation all of whose terms are coproducts (this is usually called a "Bousfield–Kan decomposition"), we know by [Sha23, §12] that any parametrised colimit may be rewritten as a fibrewise geometric realisation all of whose terms are indexed coproducts. The upshot of this is that we can often divide a proof into dealing with fibrewise (co)limits and indexed (co)products separately.

Recollections 2.1.12 (Adjointed squares)**.** We recall the notion of Beck–Chevalley transformations and adjointability from [Lur09, §7.3.1]. Suppose we are given a commuting square

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, "\varphi"] \arrow[d, "L"'] & \underline{\widetilde{\mathcal{C}}} \arrow[d, "\widetilde{L}"] \\ \underline{\mathcal{D}} \arrow[r, "\psi"] & \underline{\widetilde{\mathcal{D}}} \end{tikzcd}$$

such that $L, \widetilde{L}$ admit $\mathcal{T}$–right adjoints $R, \widetilde{R}$ respectively. We may then obtain a natural transformation $\varphi R \Rightarrow \widetilde{R}\psi$ via

$$\varphi R \xrightarrow{\widetilde{\eta}_{\varphi R}} \widetilde{R}\widetilde{L}\varphi R \simeq \widetilde{R}\psi L R \xrightarrow{\widetilde{R}\psi\varepsilon} \widetilde{R}\psi$$

This canonically constructed transformation is called the *Beck–Chevalley transformation*. If this map is an equivalence (hence we get an equivalence $\varphi R \simeq \widetilde{R}\psi$), then we say that the original square is *right adjointable*. Similarly, we may also define the notion of left adjointability.

Proposition 2.1.13 (Fibrewise criteria for $\mathcal{T}$-adjunctions, [Hil22b, Cor. 2.2.7])**.** *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$-functor. Then it admits a $\mathcal{T}$-right adjoint if and only if it admits fibrewise right adjoints G_V for all $V \in \mathcal{T}$ and for all morphism $f : W \to V$ in $\mathcal{T}$, the Beck–Chevalley square*

$$\begin{tikzcd} \mathcal{C}_W & \mathcal{D}_W \arrow[l, "G_W"'] \\ \mathcal{C}_V \arrow[u, "f^*"] & \mathcal{D}_V \arrow[l, "G_V"] \arrow[u, "f^*"'] \end{tikzcd}$$

commutes. Similarly for $\mathcal{T}$-left adjoints.

Proposition 2.1.14 ((Co)limit preservation, [Hil22b, Prop. 2.4.2])**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}$-cocomplete categories and $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ a $\mathcal{T}$-functor. Then F preserves $\mathcal{T}$-colimits if and only if it preserves colimits in each fibre and for all $f : W \to V$ in $\mathcal{T}$, the Beck–Chevalley square*

$$\begin{tikzcd} \mathcal{C}_W \arrow[r, "F_W"] \arrow[d, "f_!"] & \mathcal{D}_W \arrow[d, "f_!"] \\ \mathcal{C}_V \arrow[r, "F_V"] & \mathcal{D}_V \end{tikzcd}$$

commutes. Similarly for $\mathcal{T}$-limits.

Definition 2.1.15 ([Nar16, Def. 4.1])**.** Let $\mathcal{T}$ be a small category. We say that it is:

- *atomic* if whenever we have $f : W \to V$ and $g : V \to W$ in $\mathcal{T}$ such that $g \circ f$ is an equivalence, then f and g were already inverse equivalences.
- *orbital* if the finite coproduct cocompletion $\mathrm{Fin}_{\mathcal{T}}$ admits finite pullbacks. Here, by finite coproduct cocompletion, we mean the full subcategory of the presheaf category $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ spanned by finite coproduct of representables. See Recollection 2.1.20 for more details on $\mathrm{Fin}_{\mathcal{T}}$.

Example 2.1.16. The orbit category $\mathcal{O}_G$ for a finite group G is an example of an atomic orbital category.

Observation 2.1.17. It is straightforward to argue by unwinding the definitions that atomic orbitality ensures that for any map $f\colon U \to V$ in $\mathcal{T}$, the orbital decomposition of the pullback $U \times_V U$ contains a copy of U. That is, we have the pullback diagram

$$\begin{tikzcd} U \coprod Z \arrow[r, "\mathrm{id}\sqcup c"] \arrow[d, "\mathrm{id}\sqcup\overline{c}"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "f"] \\ U \arrow[r, "f"] & V \end{tikzcd} \tag{5}$$

This is the key property enjoyed by atomic orbital categories that is crucial for algebraic considerations such as Nardin's definition of $\mathcal{T}$–semiadditivity (which we will recall shortly) as well as the theory of parametrised cubes which we introduce in §3.

Observation 2.1.18. Suppose $\mathcal{T}$ is atomic and has a final object and let $s\colon * \hookrightarrow \mathcal{T}$ be the inclusion of the final object $T \in \mathcal{T}$, and $w\colon W \to T$ be a map in $\mathcal{T}$ which is not an equivalence. For an arbitrary $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}_{/W}}$, we claim that there is a natural equivalence $s^*w_!\underline{\mathcal{C}} \simeq \emptyset \in \mathrm{Cat}$. To see this, note by the usual pointwise right Kan extension formula that, for $\mathcal{D} \in \mathrm{Cat}$, $s_*\mathcal{D} \in \mathrm{Cat}_{\mathcal{T}}$ has value $\mathcal{D}$ at $T \in \mathcal{T}$ and $*$ everywhere else. Hence, since w was not an equivalence, by atomicity we know that there is no morphism $T \to W$ and so pulling back along $w\colon \mathcal{T}_{/W} \to \mathcal{T}$ yields $w^*s_*\mathcal{D} \simeq \underline{*} \in \mathrm{Cat}_{\mathcal{T}_{/W}}$ for any $\mathcal{D} \in \mathrm{Cat}$. Therefore, we obtain $\mathrm{Map}_{\mathrm{Cat}}(s^*w_!\underline{\mathcal{C}}, \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/W}}}(\underline{\mathcal{C}}, w^*s_*\mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/W}}}(\underline{\mathcal{C}}, \underline{*}) \simeq *$, and so $s^*w_!\underline{\mathcal{C}}$ satisfies the universal property of the initial object in Cat.

Terminology 2.1.19 (Left/right Beck–Chevalley conditions)**.** Let $\mathcal{T}$ be an orbital category and let $\underline{\mathcal{C}}$ be a $\mathcal{T}$-category that admits finite fibrewise coproducts (resp. products) such that for each $f : W \to V$ in $\mathcal{T}$, the pullback $f^* : \mathcal{C}_V \to \mathcal{C}_W$ admits a left adjoint $f_!$ (resp. right adjoint f_*). We say that $\underline{\mathcal{C}}$ satisfies the *left Beck-Chevalley condition* (resp. *right Beck-Chevalley condition*) if for every pair of edges $f : W \to V$ and $g : Y \to V$ in $\mathcal{T}$, if we write the pullback (whose orbital decomposition exists by orbitality of $\mathcal{T}$) as

$$\begin{tikzcd} \coprod_a R_a = Y \times_V W \arrow[r, "\coprod_a f_a"] \arrow[d, "\coprod_a g_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ W \arrow[r, "f"] & V \end{tikzcd}$$

then the canonical transformation $\coprod_a g_{a!}f_a^* \Longrightarrow f^*g_!$ (resp. $f^*g_* \Longrightarrow \prod_a g_{a*}f_a^*$) is an equivalence.

We now recall the algebraic aspects of parametrised higher category theory in the presence of the atomic orbitality assumption on $\mathcal{T}$. These were first introduced and studied in [Nar16; Nar17], and later revisited with further developments in [NS22].

Recollections 2.1.20 (Finite $\mathcal{T}$–sets)**.** For every $V \in \mathcal{T}$, we may define the category $\mathrm{Fin}_{/V} \subseteq \mathrm{PSh}(\mathcal{T}_{/V})$ given by the finite coproduct cocompletion of $\mathcal{T}_{/V}$. When no $V \in \mathcal{T}$ is specified, we write $\mathrm{Fin}_{\mathcal{T}}$ for the finite coproduct cocompletion of $\mathcal{T}$. By general category theory, we know that given a map $f\colon U \to V$, the left Kan extension $f_!\colon \mathrm{PSh}(\mathcal{T}_{/U}) \to \mathrm{PSh}(\mathcal{T}_{/V})$ restricts to a functor $f_!\colon \mathrm{Fin}_{/U} \to \mathrm{Fin}_{/V}$. And in our setting, by the orbitality assumption, the right adjoint $f^*\colon \mathrm{PSh}(\mathcal{T}_{/V}) \to \mathrm{PSh}(\mathcal{T}_{/U})$ also restricts to a right adjoint $f^*\colon \mathrm{Fin}_{/V} \to \mathrm{Fin}_{/U}$. These assemble to a $\mathcal{T}$–category $\underline{\mathrm{Fin}}_{\mathcal{T}}$. Similarly, we may construct the pointed version $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ whose fibre over $V \in \mathcal{T}$ is given by $(\mathrm{Fin}_{/V})_{[V=V]/}$. For details, see for example [NS22, Def. 2.1.1]

Write $\mathrm{Cat}_* \subset \mathrm{Cat}$ for the non–full subcategory of pointed categories and morphisms the functors which preserve these.

Definition 2.1.21 (Pointedness)**.** A $\mathcal{T}$–category $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ is said to be $\mathcal{T}$*–pointed* if it lies in the non–full subcategory $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}_*) \subset \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) = \mathrm{Cat}_{\mathcal{T}}$. That is, it is a $\mathcal{T}$–category all of whose fibres are pointed and such that the structure maps preserve the zero objects.

Recollections 2.1.22 (Semiadditivity norm maps, [Nar16, Cons. 5.2])**.** Let $\underline{\mathcal{C}}$ be $\mathcal{T}$–pointed and have finite $\mathcal{T}$-coproducts, and $\underline{\mathcal{D}}$ have finite $\mathcal{T}$-products. Let $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor and $f\colon U \to V$ be a map in $\underline{\mathrm{Fin}}_{\mathcal{T}}$. We would like now to construct a canonical map

$$Ff_! \longrightarrow f_*f^*F \tag{6}$$

called the *semiadditivity norm map*. By atomic orbitality, the pullback square (5) gives us a natural equivalence $f^*f_! \simeq \mathrm{id} \sqcup \overline{c}_!c^*$ and so since $\underline{\mathcal{C}}$ was $\mathcal{T}$–pointed, we may postcompose this with the map $\pi := \mathrm{id} \sqcup 0\colon \mathrm{id} \sqcup \overline{c}_!c^* \to \mathrm{id} \sqcup 0 \simeq \mathrm{id}$. Applying f_* to this and precomposing this with the adjunction unit for $f^* \dashv f_*$ yields the map

$$f_! \xrightarrow{\eta_{f_!}} f_*f^*f_! \xrightarrow{f_*\pi} f_*$$

Finally, applying F to this transformation and postcomposing with the canonical map $Ff_* \to f_*f^*F$ coming from the counit $f^*Ff_* \simeq Ff^*f_* \to F$ yields the map

$$Ff_! \xrightarrow{F\eta_{f_!}} Ff_*f^*f_! \xrightarrow{Ff_*\pi} Ff_* \xrightarrow{\mathrm{can}} f_*f^*F$$

as desired in (6). A little unwinding of definitions shows that, when $U \simeq V \sqcup V$ and $f\colon U \to V$ is the fold map, the construction above specialises to the usual unparametrised canonical comparison map $\coprod \to \prod$ required to be an equivalence in the definition of semiadditive categories.

Definition 2.1.23 (Semiadditivity and stability, [Nar16, Def. 5.3, Def. 7.1])**.** Let $\underline{\mathcal{C}}$ be $\mathcal{T}$–pointed and have finite $\mathcal{T}$-coproducts, and $\underline{\mathcal{D}}$ have finite $\mathcal{T}$-products. Let $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor. We say that it is $\mathcal{T}$*-semiadditive* if for all $f\colon U \to V$ in $\underline{\mathrm{Fin}}_{\mathcal{T}}$, the semiadditivity norm map constructed above is an equivalence. We say that a pointed $\mathcal{T}$-category $\underline{\mathcal{C}}$ with finite $\mathcal{T}$–(co)products is $\mathcal{T}$*-semiadditive* if the identity functor is $\mathcal{T}$-semiadditive. A $\mathcal{T}$–semiadditive category is said to be $\mathcal{T}$*–stable* if it is furthermore fibrewise stable.

If moreover $\underline{\mathcal{C}}$ has fibrewise pushouts and $\underline{\mathcal{D}}$ has fibrewise pullbacks, then we say that F is $\mathcal{T}$*–linear* if it is $\mathcal{T}$-semiadditive and sends fibrewise pushouts to fibrewise pullbacks. We write $\underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ (resp. $\underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$) for the $\mathcal{T}$-full subcategories of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ consisting of the $\mathcal{T}$–semiadditive functors (resp. $\mathcal{T}$–linear functors).

Definition 2.1.24 (Commutative monoids, [Nar16, Def. 5.9, Thm. 6.5])**.** For $\underline{\mathcal{C}}$ with finite $\mathcal{T}$-limits we will denote $\mathcal{T}$*–Mackey functors* by $\underline{\mathrm{Mack}}_{\mathcal{T}}(\underline{\mathcal{C}}) := \underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\underline{\mathrm{Span}}(\underline{\mathrm{Fin}}), \underline{\mathcal{C}})$ and $\mathcal{T}$*–commutative monoids* by $\underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathcal{C}}) := \underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}})$.

Remark 2.1.25. By the proof of [NS22, Thm. 2.3.9], we see that evaluating at any $V \in \mathcal{T}$, we get an equivalence

$$\underline{\mathrm{Mack}}_{\mathcal{T}}(\underline{\mathrm{Cofree}}(\mathcal{C}))_V \simeq \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}_{/V}), \mathcal{C}) =: \mathrm{Mack}_{\mathcal{T}_{/V}}(\mathcal{C})$$

Example 2.1.26. A key instance of this construction is the $\mathcal{T}$*–category of* $\mathcal{T}$*–spectra*, defined as $\underline{\mathrm{Sp}}_{\mathcal{T}} := \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Sp}))$. By the preceding remark, we see that $\underline{\mathrm{Sp}}_{\mathcal{T}}$ is fibrewise given by spectral Mackey functors $\mathrm{Mack}_{\mathcal{T}_{/V}}(\mathrm{Sp})$.

Construction 2.1.27 (Forgetful functor)**.** Suppose $\mathcal{T}$ has a final object T (this is merely a technical convenience which is mostly innocuous since for a fixed $V \in \mathcal{T}$, we always have a final object upon basechanging to $\mathcal{T}_{/V}$; see [Nar16, Def. 5.9] for the general case). Being a $\mathcal{T}$–commutative monoid object in a $\mathcal{T}$–category with finite indexed products is a structure, and one may functorially construct the forgetful functor as follows: the inclusion $\underline{*} \to \underline{\mathrm{Fin}}_*$ of the $\mathcal{T}$–object T yields by precomposition a transformation of functors

$$\Big(\underline{\mathrm{CMon}}_{\mathcal{T}}(-) \Rightarrow \mathrm{id} \simeq \underline{\mathrm{Fun}}(\underline{*}, -)\Big)\colon \mathrm{Cat}_{\mathcal{T},\underline{\prod}} \longrightarrow \mathrm{Cat}_{\mathcal{T},\underline{\prod}}$$

which we call the forgetful functor fgt. Here, $\mathrm{Cat}_{\mathcal{T},\underline{\prod}}$ is the non–full subcategory of $\mathrm{Cat}_{\mathcal{T}}$ whose objects are $\mathcal{T}$–categories with finite indexed products and morphisms which preserve these.

Proposition 2.1.28 ($\mathcal{T}$–semiadditivisation, [Nar16, Prop. 5.11])**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$-category with finite $\mathcal{T}$-products. Then the forgetful functor $\underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to \underline{\mathcal{C}}$ is an equivalence if and only if $\underline{\mathcal{C}}$ was $\mathcal{T}$-semiadditive.*

Notation 2.1.29. For $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$, we write $\underline{\mathrm{PSh}}(\underline{\mathcal{C}})$ and $\underline{\mathrm{PSh}}^{\mathrm{st}}(\underline{\mathcal{C}})$ for the presheaf categories $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{S}})$ and $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})$, respectively.

Moreover, Nardin in his thesis [Nar17] has also introduced the notion of $\mathcal{T}$–symmetric monoidal structures, upon which one may extract $\mathcal{T}$–commutative algebra objects. In the case when $\mathcal{T} = \mathcal{O}_G$, such commutative algebra objects encode precisely the multiplicative norms of [GM97; HHR16]. Much like the unparametrised notion from [Lur17], the notion of $\mathcal{T}$–operads was defined in [Nar17] as certain fibrations over $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ and $\mathcal{T}$–symmetric monoidal categories are then the $\mathcal{T}$–operads such that this fibration is $\mathcal{T}$–cocartesian. The $\mathcal{T}$–commutative algebras $\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\otimes})$ of a $\mathcal{T}$–symmetric monoidal category $\underline{\mathcal{C}}^{\otimes}$ is then defined to be the $\mathcal{T}$–category of $\underline{\mathrm{Fin}}_{*\mathcal{T}}$–sections of $\underline{\mathcal{C}}^{\otimes}$ which are maps of $\mathcal{T}$–operads. We refer the reader to [NS22, §2] for a more recent and fully–fledged development of this theory.

Notation 2.1.30. As in usual symmetric monoidal structures which in particular supply us with a "multiplication" $-\otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, we also have an indexed version of such maps. In more detail, for each map $f\colon U \to V$ in $\mathrm{Fin}_{/V}$, we also have an indexed multiplication map

$$f_{\otimes}\colon f_* f^* \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{C}}$$

associated to a $\mathcal{T}$–symmetric monoidal structure $\underline{\mathcal{C}}^{\otimes}$ on $\underline{\mathcal{C}}$. Here we have implicitly basechanged the $\mathcal{T}$–category $\underline{\mathcal{C}}$ to a $\mathcal{T}_{/V}$–category, which we also write $\underline{\mathcal{C}}$. This will be a convenience employed throughout the document to lighten our notational burdens.

Recollections 2.1.31 (Distributivity)**.** Another notion that will be important in our work is that of *distributivity* which generalises the idea of tensor products which are bicontinuous into the parametrised setting. This was first defined by Nardin in his thesis [Nar17], but see also [NS22, Def. 3.2.3] where the theory is further developed. A $\mathcal{T}$–symmetric monoidal structure $\underline{\mathcal{C}}^{\otimes}$ on a $\mathcal{T}$–category with all $\mathcal{T}$–colimits is said to be $\mathcal{T}$*–distributive* if the following holds: for any map $f\colon U \to V$ in $\mathrm{Fin}_{/V}$ and a $\mathcal{T}_{/U}$–colimit diagram $\partial\colon \underline{K}^{\triangleright} \to f^*\underline{\mathcal{C}}$, the diagram

$$f_{\otimes}\partial\colon (f_*\underline{K})^{\triangleright} \xrightarrow{\mathrm{can}} f_*(\underline{K}^{\triangleright}) \xrightarrow{f_*\partial} f_* f^* \underline{\mathcal{C}} \xrightarrow{f_{\otimes}} \underline{\mathcal{C}}$$

is a $\mathcal{T}_{/V}$–colimit diagram in $\underline{\mathcal{C}}$.

Crucial to our work will be two results by Nardin–Shah which we collect here as:

Theorem 2.1.32 ([NS22, Thm. 2.3.9, Prop. 2.8.7])**.** *Let $\underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T}}$ be the $\mathcal{T}$–category of $\mathcal{T}$–symmetric monoidal categories and $\mathcal{T}$–symmetric monoidal functors. Let $\underline{\mathcal{C}}^{\otimes} \in \mathrm{Cat}^{\otimes}_{\mathcal{T}}$.*

1. *There are equivalences $\underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T}} \simeq \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathrm{Cat}})$ and $\mathrm{Cat}^{\otimes}_{\mathcal{T}} \simeq \mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat})$.*
2. *There is a $\mathcal{T}$–functor $\underline{\mathrm{Env}}$ from $\mathcal{T}$–operads to $\mathcal{T}$–symmetric monoidal categories which participates in a natural equivalence $\mathrm{Fun}^{\otimes}_{\mathcal{T}}(\underline{\mathrm{Env}}(\underline{\mathrm{Fin}}_*), \underline{\mathcal{C}}^{\otimes}) \simeq \mathrm{CAlg}_{\mathcal{T}}(\underline{\mathcal{C}}^{\otimes})$, where $\mathrm{Fun}^{\otimes}_{\mathcal{T}}$ is the category of $\mathcal{T}$–symmetric monoidal functors.*

In particular, when $\mathcal{T} = \mathcal{O}_G$, we see that the category G–symmetric monoidal categories is given by $\mathrm{Mack}_G(\mathrm{Cat}) \simeq \mathrm{CMon}_G(\underline{\mathrm{Cat}}_G)$ (this equivalence is an immediate consequence of [Nar16, Thm. 6.5] and Construction 2.1.5), analogous to the fact that the category of symmetric monoidal categories may equivalently be described as $\mathrm{CMon}(\mathrm{Cat})$.

Recollections 2.1.33 (Pointwise symmetric monoidal structures)**.** Let $\underline{J} \in \mathrm{Cat}_{\mathcal{T}}$ and $\underline{\mathcal{D}}^{\otimes} \in \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathrm{Cat}})$. In the same way that one can equip the pointwise symmetric monoidal structure on the functor category $\mathrm{Fun}(I, \mathcal{C})$ for an arbitrary $I \in \mathrm{Cat}$ and $\mathcal{C}^{\otimes} \in \mathrm{CMon}(\mathrm{Cat})$, we can also construct a $\mathcal{T}$–symmetric monoidal structure $\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{D}}^{\otimes})$ on the $\mathcal{T}$–functor category $\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{D}})$ thanks to [NS22, §3.3]. As in the unparametrised case, this construction enjoys the following cotensor universal property: for any $\underline{\mathcal{C}}^{\otimes}$, there is a natural equivalence

$$\underline{\mathrm{Map}}_{\underline{\mathrm{CMon}}(\underline{\mathrm{Cat}})}(\underline{\mathcal{C}}^{\otimes}, \underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{D}}^{\otimes})) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}}(\underline{J}, \underline{\mathrm{Fun}}^{\otimes}(\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes}))$$

where $\underline{\mathrm{Fun}}^{\otimes}(\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes})$ is the $\mathcal{T}$–category of $\mathcal{T}$–symmetric monoidal functors. We refer the reader to [NS22, §2, §3.3], especially [NS22, Thm. 3.3.3], for more details on this.

The concept of parametrised presentability will be an indispensable component in our treatment of parametrised algebraic K–theory later, and so we recall some results from [Hil22b, §6]. This notion was first defined in [Nar17, §1.4] and further developed in [Hil22b]. A pleasant feature of the theory there is that parametrised Ind–completions and parametrised accessibility are fibrewise notions (c.f. [Hil22b, §3.5, §5.2]). For example, the κ–Ind–completion functor $\underline{\mathrm{Ind}}_{\kappa} \colon \mathrm{Cat}_{\mathcal{T}} \to \widehat{\mathrm{Cat}}_{\mathcal{T}}$ is given just by applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to the usual functor $\mathrm{Ind}_{\kappa} \colon \mathrm{Cat} \to \widehat{\mathrm{Cat}}$. Consequently, so are the notions of parametrised compactness and parametrised idempotent–completeness (c.f. [Hil22b, §5.1, §5.3]). For instance, an object X in $\underline{\mathcal{C}}$ is said to be parametrised κ–compact if it is so fibrewise, and we showed in [Hil22b, Prop. 5.1.4] that this definition can equivalently be characterised by saying that the parametrised mapping space functor $\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(X, -)$ preserves fibrewise (here fibrewise is in the sense of Recollection 2.1.11) κ–filtered colimits. Moreover, the main theorem [Hil22b, Thm. 6.1.2] given there was a "Lurie–Simpson" style characterisation of parametrised presentability, one characterisation of which we summarise as:

Theorem 2.1.34 (Omnibus presentability, [Hil22b, Thm. 6.1.2, Prop. 6.3.3])**.** *Let $\mathcal{T}$ be an orbital category and $\underline{\mathcal{C}}$ a $\mathcal{T}$-category. Then $\underline{\mathcal{C}}$ is $\mathcal{T}$-presentable if and only if $\underline{\mathcal{C}}$ satisfies the left Beck-Chevalley condition (cf. Terminology 2.1.19) and there is a regular cardinal κ such that the straightening $\underline{\mathcal{C}} : \mathcal{T}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}$ factors through $\underline{\mathcal{C}} \colon \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Pr}_{L,\kappa}$. Moreover, $\mathcal{T}$–presentable categories are also $\mathcal{T}$–complete.*

Theorem 2.1.35 (Parametrised adjoint functor theorem, [Hil22b, Thm. 4.2.1])**.** *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$-functor between $\mathcal{T}$-presentable categories.*

1. *If F preserves $\mathcal{T}$-colimits, then F admits a $\mathcal{T}$-right adjoint.*
2. *If F preserves $\mathcal{T}$-limits and is $\mathcal{T}$-accessible, then F admits a $\mathcal{T}$-left adjoint.*

Notation 2.1.36. For a fixed regular cardinal κ, we write $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa} \subset \underline{\widehat{\mathrm{Cat}}}$ for the non–full $\mathcal{T}$–subcategory of κ–accessible parametrised presentable categories and morphisms the left adjoint $\mathcal{T}$–functors which preserve κ–compact objects; we write $\underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}} \subset \underline{\widehat{\mathrm{Cat}}}$ for the non–full subcategory of κ–accessible parametrised presentable categories and morphisms the right adjoint $\mathcal{T}$–functors which preserve κ–filtered colimits. Furhtermore, we also write $\underline{\mathrm{Cat}}^{\mathrm{idem}(\kappa)}_{\mathcal{T}} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ for the non–full $\mathcal{T}$–subcategory of small parametrised-idempotent-complete $\mathcal{T}$–categories which are parametrised-κ-cocomplete and morphisms the functors which preserve κ-small parametrised colimits.

Proposition 2.1.37 ([Hil22b, Thm. 4.5.3])**.** *Let κ be a regular cardinal. Then we have an equivalence of $\mathcal{T}$–categories $(-)^{\underline{\kappa}} : \underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa} \rightleftarrows \underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{idem}(\kappa)} : \underline{\mathrm{Ind}}_{\kappa}$ where $(-)^{\underline{\kappa}}$ and $\underline{\mathrm{Ind}}_{\kappa}$ denote taking fibrewise κ–compact objects and fibrewise κ–Ind–completion, respectively.*

The following object – whose categorical properties we shall work out in §2.5 – will be one of the main players in this paper as it will be the domain of our parametrised algebraic K–theory functor.

Notation 2.1.38 ($\mathcal{T}$–perfect–stable categories)**.** We will use the notation $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}(\kappa)} := \underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{st},\mathrm{idem}(\kappa)} \subseteq \underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{idem}(\kappa)}$ for the *$\mathcal{T}$–perfect-stable categories* which are parametrised κ–cocomplete, where the word perfect is standard terminology for being idempotent-complete. When $\kappa = \omega$, we will often use the abbreviation $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}} := \underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}(\omega)}$. By Proposition 2.1.37, we have $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\omega} \simeq \underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}}$ implemented by taking fibrewise compact objects and fibrewise Ind–completion. Moreover, by Proposition 2.1.14, we see that the faithful inclusion $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ factors through $\underline{\mathrm{Cofree}}(\mathrm{Cat}^{\mathrm{perf}}) \subset \underline{\mathrm{Cofree}}(\mathrm{Cat}) = \underline{\mathrm{Cat}}_{\mathcal{T}}$.

2.2 Miscellaneous preliminaries

We consign to this subsection various elementary miscellany about parametrised categories that we shall need for main body of this article. As such, this subsection may be safely skipped on first reading, to be returned to as needed.

Our first task is to record an "internal" version of the parametrised straightening–unstraightening equivalence, building upon the "external" version given in [BDG+16b, Prop. 8.3]. A similar, and much more general statement in the setting of categories parametrised over ∞–topoi, has already appeared as [Mar22, Thm. 6.3.1]. We only include the proof of Theorem 2.2.2 in our setup for the reader's convenience.

To this end, first recall the notion of $\mathcal{T}$–(co)cartesian fibrations from [Sha23, Def. 7.1]. For $\underline{S} \in \mathrm{Cat}_{\mathcal{T}}$, we will write $\mathrm{coCart}_{\mathcal{T}}(\underline{S})$ for the category of $\mathcal{T}$–cocartesian fibrations over $\underline{S}$ and morphisms the maps of $\mathcal{T}$–cocartesian fibrations.

Observation 2.2.1. In [Sha23, Rem. 7.4] it is stated that a $\mathcal{T}$–functor $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a $\mathcal{T}$–cocartesian fibration if and only if $\int F\colon \int \underline{\mathcal{C}} \to \int \underline{\mathcal{D}}$ is a cocartesian fibration in the usual sense. For our purposes in the next theorem, we would need a slightly more refined information in the form of a description of what the cocartesian lifts look like under these equivalent conditions, which we provide here. Write $p\colon \int \underline{\mathcal{C}} \to \mathcal{T}^{\mathrm{op}}$ and $q\colon \int \underline{\mathcal{D}} \to \mathcal{T}^{\mathrm{op}}$ for the structure cocartesian fibrations and let $f\colon x \to y$ be a map in $\int \underline{\mathcal{D}}$ lying over $c := q(f)\colon V \to W$ in $\mathcal{T}^{\mathrm{op}}$. We may then obtain a unique factorisation

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[dr, "f"'] & c^*x \arrow[d, "\overline{f}"] \\ & y \end{tikzcd}$$

where $c^*x \in \mathcal{D}_W$, $u\colon x \to c^*x$ is the q–cocartesian morphism over c, and $q(\overline{f}) \simeq \mathrm{id}_W$. The claim now is that: (i) the F_W–cocartesian lift of $\overline{f}$ is already a $\int F$–cocartesian lift; (ii) any $\int F$–cocartesian lift of $f\colon x \to y$ is given by the composite of the $\int F$–cocartesian lift of u and the F_W–cocartesian lift of $\overline{f}$. Part (i) is gotten by the proof of [Sha23, Lem. 7.5] – but now using also that the fibrewise cocartesian lifts are preserved under the pushforward functors associated to morphisms in $\mathcal{T}^{\mathrm{op}}$ by the definition of $\mathcal{T}$–cocartesian fibrations. Combining (i) with the dual of [Lan21, Lem. 3.1.7] then yields part (ii) of the claim. Under this concrete elaboration of [Sha23, Rmk. 7.4] and since compositions of cocartesian fibrations are cocartesian fibrations by [Lan21, Lem. 3.1.4], we see that the datum of a map of cocartesian fibrations over $\int \underline{S}$ gives precisely the datum of a $\mathcal{T}$–map of $\mathcal{T}$–cocartesian fibrations over $\underline{S}$ upon postcomposing with

$\int \underline{S} \to \mathcal{T}^{\mathrm{op}}$. Here we have also used that $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a map of $\mathcal{T}$–categories (i.e. $\int F$ is a map of $\mathcal{T}$–cocartesian fibrations) if and only if the $\int F$–cocartesian lift of all such u coming from a map in $\mathcal{T}^{\mathrm{op}}$ as above is a $\mathcal{T}$–cocartesian lift, again by an application of [Lan21, Lem. 3.1.4].

Theorem 2.2.2 (Parametrised straightening/unstraightening)**.** *Let* $\underline{S} \in \mathrm{Cat}_{\mathcal{T}}$. *There is a natural equivalence* $\mathrm{Fun}_{\mathcal{T}}(\underline{S}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \simeq \mathrm{coCart}(\int \underline{S}) \simeq \mathrm{coCart}_{\mathcal{T}}(\underline{S})$.

Proof. We already know the first equivalence by [BDG+16b, Prop. 8.3]. To see the second equivalence, we write $p\colon \int \underline{S} \to \mathcal{T}^{\mathrm{op}}$ for the structure map. This map induces a functor $\mathrm{Cat}_{/\int \underline{S}} \to \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}}$ which in turn factors through an adjunction

$$p\colon \mathrm{Cat}_{/\int \underline{S}} \rightleftharpoons (\mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}})_{/p} : \mathrm{fgt}$$

which is an equivalence, where fgt forgets the map to $\mathcal{T}^{\mathrm{op}}$. That this is an equivalence is standard, and can for example be seen by computing explicitly that p induces equivalences on the mapping spaces, and that it is clearly seen to be essentially surjective. Next, recall that we had non–full subcategories $\mathrm{coCart}(\int \underline{S}) \subset \mathrm{Cat}_{/\int \underline{S}}$ and $\mathrm{Cat}_{\mathcal{T}} \simeq \mathrm{coCart}(\mathcal{T}^{\mathrm{op}}) \subset \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}}$. In particular, by the usual straightening–unstraightening equivalence, we have the non–full subcategories $\mathrm{coCart}_{\mathcal{T}}(\underline{S}) \subset (\mathrm{Cat}_{\mathcal{T}})_{/\underline{S}} \subset (\mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}})_{/p}$. Now consider the solid diagram

$$\begin{tikzcd}
\mathrm{Cat}_{/\int \underline{S}} \arrow[r, shift left, "p"] \arrow[r, phantom, "\perp\simeq"] & (\mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}})_{/p} \arrow[l, shift left, "\mathrm{fgt}"] \\
\mathrm{coCart}(\int \underline{S}) \arrow[u, hook] \arrow[r, dashed, shift left] & \mathrm{coCart}_{\mathcal{T}}(\underline{S}) \arrow[u, hook] \arrow[l, dashed, shift left]
\end{tikzcd}$$

Our goal is to show that we have the dashed factorisations giving inverse equivalences. Since both non–full subcategories contain all equivalences, it would suffice to show that p and fgt admit such factorisations since then the natural equivalences $p \circ \mathrm{fgt} \simeq \mathrm{id}$ and $\mathrm{fgt} \circ p \simeq \mathrm{id}$ are also contained in the non–full subcategories. Since factoring through subcategories is a property of a functor that can be checked on objects and morphisms, the desired factorisations are now easy consequences of the concrete description of the cocartesian edges from Observation 2.2.1. This completes the proof of the theorem. ☐

Construction 2.2.3. We have so far only dealt with cocartesian unstraightening. As in the unparametrised situation, we also have the cartesian version of Theorem 2.2.2. To describe this, first note by an easy inspection of [Sha23, Def. 7.1] that a $\mathcal{T}$–map $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a $\mathcal{T}$–cocartesian fibration if and only if $F^{\underline{\mathrm{op}}}\colon \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathcal{D}}^{\underline{\mathrm{op}}}$ is a $\mathcal{T}$–cartesian fibration, i.e. for a fixed $\underline{S} \in \mathrm{Cat}_{\mathcal{T}}$, the $\mathcal{T}$–functor $(-)^{\underline{\mathrm{op}}}\colon \mathrm{coCart}_{\mathcal{T}}(\underline{S}) \to \mathrm{Cart}_{\mathcal{T}}(\underline{S}^{\underline{\mathrm{op}}})$ is an equivalence. Hence, by the parametrised *cartesian* straightening–unstraightening equivalence, we will mean the composite

$$\mathrm{Fun}_{\mathcal{T}}(\underline{S}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \xrightarrow[\simeq]{(-)^{\underline{\mathrm{op}}}} \mathrm{Fun}_{\mathcal{T}}(\underline{S}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \simeq \mathrm{coCart}_{\mathcal{T}}(\underline{S}) \xrightarrow[\simeq]{(-)^{\underline{\mathrm{op}}}} \mathrm{Cart}_{\mathcal{T}}(\underline{S}^{\underline{\mathrm{op}}})$$

where the middle equivalence is by Theorem 2.2.2.

Notation 2.2.4. We write $\underline{\mathrm{RFun}}_{\mathcal{T}}$ (resp. $\underline{\mathrm{LFun}}_{\mathcal{T}}$) for the $\mathcal{T}$-full subcategories of $\underline{\mathrm{Fun}}_{\mathcal{T}}$ consisting of $\mathcal{T}$-right adjoint functors (resp. $\mathcal{T}$-left adjoint functors); we write $\underline{\mathrm{Fun}}^{R}_{\mathcal{T}}$ (resp. $\underline{\mathrm{Fun}}^{L}_{\mathcal{T}}$) for the $\mathcal{T}$-full subcategories of $\underline{\mathrm{Fun}}_{\mathcal{T}}$ consisting of strongly $\mathcal{T}$-limit-preserving functors (resp. strongly $\mathcal{T}$-colimit-preserving functors).

Proposition 2.2.5 ([Hil22b, Prop. 2.5.10])**.** *Let* $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$. *Then* $\underline{\mathrm{LFun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{RFun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})^{\underline{\mathrm{op}}}$.

Next, we now record several facts about Beck–Chevalley (i.e. adjointed) squares.

Lemma 2.2.6. *Suppose we have $\mathcal{T}$–adjunctions $L\colon \underline{\mathcal{C}} \rightleftharpoons \underline{\mathcal{D}} : R$ and $\widetilde{L}\colon \widetilde{\underline{\mathcal{C}}} \rightleftharpoons \widetilde{\underline{\mathcal{D}}} : \widetilde{R}$ with adjunction (co)units η, ε and $\widetilde{\eta}, \widetilde{\varepsilon}$ respectively, together with a right adjointable square*

$$\begin{tikzcd}
\underline{\mathcal{C}} \arrow[r, "\varphi"] \arrow[d, "L"'] & \widetilde{\underline{\mathcal{C}}} \arrow[d, "\widetilde{L}"'] \\
\underline{\mathcal{D}} \arrow[r, "\psi"'] \arrow[u, dashed, bend right, "R"'] & \widetilde{\underline{\mathcal{D}}} \arrow[u, dashed, bend right, "\widetilde{R}"']
\end{tikzcd}$$

For every $X \in \underline{\mathcal{C}}$, we then have a natural identification of $\varphi\eta_X : \varphi X \to \varphi RLX$ with $\widetilde{\eta}_{\varphi X} : \varphi X \to \widetilde{R}\widetilde{L}\varphi X$ via the identification $\varphi RLX \simeq \widetilde{R}\widetilde{L}\varphi X$ coming from adjointability. Similarly, for $A \in \underline{\mathcal{D}}$, we have a natural identification of $\psi\varepsilon_A \colon \psi LRA \to \psi A$ with $\widetilde{\varepsilon}_{\psi A} \colon \widetilde{L}\widetilde{R}\psi A \to \psi A$.

Proof. Recall that adjointability means the canonical Beck–Chevalley transformation

$$\varphi RY \xrightarrow{\widetilde{\eta}_{\varphi RY}} \widetilde{R}\widetilde{L}\varphi RY \simeq_{\widetilde{R}\sigma_R} \widetilde{R}\psi LRY \xrightarrow{\widetilde{R}\psi\varepsilon_Y} \widetilde{R}\psi Y$$

is an equivalence, where σ is the datum of the commutation $\psi L \simeq \widetilde{L}\varphi$. Hence, to prove the proposition, it would suffice to show that the diagram

$$\begin{tikzcd}
\varphi X \arrow[rrr, "\widetilde{\eta}_{\varphi X}"] \arrow[d, "\varphi\eta_X"'] & & & \widetilde{R}\widetilde{L}\varphi X \arrow[d, "\simeq"', "\widetilde{R}\sigma"] \\
\varphi RLX \arrow[r, "\widetilde{\eta}_{\varphi RLX}"] & \widetilde{R}\widetilde{L}\varphi RLX \arrow[r, "\widetilde{R}\sigma_{RLX}", "\simeq"'] & \widetilde{R}\psi LRLX \arrow[r, "\widetilde{R}\psi\varepsilon_{LX}"] & \widetilde{R}\psi LX
\end{tikzcd} \tag{7}$$

commutes. To this end, just observe that the bottom left composite participates in the following commuting diagram (where the bottom composite is an equivalence by adjointability)

$$\begin{tikzcd}
\varphi X \arrow[r, "\widetilde{\eta}_{\varphi X}"] \arrow[d, "\varphi\eta_X"'] & \widetilde{R}\widetilde{L}\varphi X \arrow[r, "\widetilde{R}\sigma_X", "\simeq"'] \arrow[d, "\widetilde{R}\widetilde{L}\varphi\eta_X"'] & \widetilde{R}\psi LX \arrow[d, "\widetilde{R}\psi L\eta_X"'] \arrow[dr, equal] & \\
\varphi RLX \arrow[r, "\widetilde{\eta}_{\varphi RLX}"] & \widetilde{R}\widetilde{L}\varphi RLX \arrow[r, "\widetilde{R}\sigma_{RLX}", "\simeq"'] & \widetilde{R}\psi LRLX \arrow[r, "\widetilde{R}\psi\varepsilon_{LX}"] & \widetilde{R}\psi LX
\end{tikzcd}$$

yielding the desired commutation (7). The case of counits is similar, using instead the commuting diagram

$$\begin{tikzcd}
& \widetilde{L}\varphi RA \arrow[r, "\sigma_{RA}", "\simeq"'] & \psi LRA \arrow[r, "\psi\varepsilon_A"] & \psi A \\
\widetilde{L}\varphi RA \arrow[ur, equal] \arrow[r, "\widetilde{L}\widetilde{\eta}_{\varphi RA}"] & \widetilde{L}\widetilde{R}\widetilde{L}\varphi RA \arrow[u, "\widetilde{\varepsilon}_{\widetilde{L}\varphi RA}"'] \arrow[r, "\widetilde{L}\widetilde{R}\sigma_{RA}", "\simeq"'] & \widetilde{L}\widetilde{R}\psi LRA \arrow[u, "\widetilde{\varepsilon}_{\psi LRA}"'] \arrow[r, "\widetilde{L}\widetilde{R}\psi\varepsilon_A"] & \widetilde{L}\widetilde{R}\psi A \arrow[u, "\widetilde{\varepsilon}_{\psi A}"']
\end{tikzcd}$$

where the bottom composite is the Beck–Chevalley equivalence. ☐

Proposition 2.2.7. *Fix a category J and suppose we have two objects $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Fun}(J^{\mathrm{op}}, \mathrm{Cat})$ together with a morphism $R\colon \underline{\mathcal{D}} \to \underline{\mathcal{C}}$. Suppose moreover that the map R is fibrewise a right adjoint and that for each morphism $f\colon i \to j$ in J, the Beck–Chevalley transformation*

$$\begin{tikzcd}
\mathcal{C}_i \arrow[r, "f^*"] & \mathcal{C}_j \\
\mathcal{D}_i \arrow[u, "L_i"] \arrow[r, "f^*"'] \arrow[ur, phantom, "\Leftarrow"] & \mathcal{D}_j \arrow[u, "L_j"']
\end{tikzcd}$$

is an equivalence, where $L_i \dashv R_i, L_j \dashv R_j$. Then the left adjoints assemble to a morphism $L\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ in $\mathrm{Fun}(J^{\mathrm{op}}, \mathrm{Cat})$ which furthermore induces an adjunction in Cat

$$\lim_{J^{\mathrm{op}}} L\colon \lim_{J^{\mathrm{op}}} \underline{\mathcal{C}} \rightleftharpoons \lim_{J^{\mathrm{op}}} \underline{\mathcal{D}} : \lim_{J^{\mathrm{op}}} R$$

Proof. For the first part, we will use Lurie's theory of relative adjunctions. Write $\int \underline{\mathcal{C}}, \int \underline{\mathcal{D}} \to J^{\mathrm{op}}$ for the respective cocartesian unstraightening, so that we have a map of J^{op}–cocartesian fibrations $R\colon \int \underline{\mathcal{D}} \to \int \underline{\mathcal{C}}$. There are two conditions in [Lur17, Prop. 7.3.2.11] to check in order to obtain a left adjoint $L\colon \int \underline{\mathcal{C}} \to \int \underline{\mathcal{D}}$ to R relative to J^{op}. Condition (1) there is immediate from our fibrewise adjunction hypothesis. A straightforward unwinding of condition (2) there states that we need to check the following: for every morphism $f\colon i \to j$, the map $f^* \xrightarrow{f^*\eta} f^* R_i L_i \simeq R_j f^* L_i$ adjoints to an equivalence $L_j f^* \xrightarrow{\simeq} f^* L_i$. But this adjointed map is precisely the Beck–Chevalley transformation, and so by hypothesis, is an equivalence. It is then an easy check to see that the relative left adjoint L is automatically a map of J^{op}–cocartesian fibrations (see for example [Hil22b, Prop. 2.2.5 (2)]), whence a map $L\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ in $\mathrm{Fun}(J^{\mathrm{op}}, \mathrm{Cat}) \simeq \mathrm{coCart}(J^{\mathrm{op}})$ as wanted. Finally, by the formula for limits in Cat in terms of cocartesian sections of the cocartesian unstraightening (cf. the dual of [Lur09, Cor. 3.3.3.2] recorded for example in [HW21, Prop. I.36]), we obtain the adjunction

$$\lim_{J^{\mathrm{op}}} L\colon \lim_{J^{\mathrm{op}}} \underline{\mathcal{C}} \simeq \Gamma_{\mathrm{cocart}}\Big(\int \underline{\mathcal{C}}\Big) \rightleftharpoons \Gamma_{\mathrm{cocart}}\Big(\int \underline{\mathcal{D}}\Big) \simeq \lim_{J^{\mathrm{op}}} \underline{\mathcal{D}} : \lim_{J^{\mathrm{op}}} R$$

as claimed. □

Fact 2.2.8 (Adjoints of equivariant functors). It is a standard categorical fact that the G–equivariant structure of a functor admitting an adjoint induces a G–equivariant structure on the adjoint and on the adjunction. Since we have not been able to find it anywhere in the literature, we will record a proof of this here which we learnt from Maxime Ramzi. Let $L\colon \mathcal{C} \to \mathcal{D}$ be a G–equivariant functor whose underlying functor admits a right adjoint. In particular, L can be encoded as a morphism in $\mathrm{Fun}(BG, \mathrm{Cat})$ and so upon unstraightening, we have a map $L\colon \int \mathcal{C} \to \int \mathcal{D}$ of cocartesian fibrations over BG. Now since BG was an ∞–groupoid, we get by [Lan21, Lem. 3.1.6] that all cocartesian morphisms in $\int \mathcal{C}$ and $\int \mathcal{D}$ are equivalences. In particular, any map between them over BG is a map of cocartesian fibrations. Hence, by the dual of [Lur17, Prop. 7.3.2.11], the fibrewise right adjoint coming from the underlying adjunction induces a relative right adjoint $R\colon \int \mathcal{D} \to \int \mathcal{C}$ which is automatically a map of cocartesian fibrations over BG by the previous sentence. All in all, we have obtained a G–equivariant structure on the right adjoint as well as on the unit and counit maps, as wanted.

The next pair of results provide abstract colimit decomposition results that will be crucial to our cubical theory in §3.

Lemma 2.2.9. *Suppose we are given a map*

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, "\gamma^*"] & \underline{\mathcal{E}} & \underline{\mathcal{D}} \arrow[l, "\beta^*"'] \\ \underline{\mathcal{A}} \arrow[u, "p^*"] \arrow[r, equal] & \underline{\mathcal{A}} \arrow[u, "r^*"] \arrow[r, equal] & \underline{\mathcal{A}} \arrow[u, "q^*"'] \end{tikzcd}$$

of diagrams in $\mathrm{Fun}(\Lambda^2_0, \mathrm{Cat}_{\mathcal{T}})$, where $\gamma^*, \beta^*, p^*, q^*, r^*$ admit right adjoints $\gamma_*, \beta_*, p_*, q_*, r_*$ respectively. If $\underline{\mathcal{A}}$ has pullbacks, then the functor $p^* \times_{r^*} q^* : \underline{\mathcal{A}} \to \underline{\mathcal{C}} \times_{\underline{\mathcal{E}}} \underline{\mathcal{D}}$ has a right adjoint given by the functor φ sending $(c, d) \in \underline{\mathcal{C}} \times_{\underline{\mathcal{E}}} \underline{\mathcal{D}}$ to the pullback in $\underline{\mathcal{A}}$

$$\begin{tikzcd} \varphi(c,d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & q_* d \arrow[d, "q_*\eta^\beta_d"] \\ p_* c \arrow[r, "p_*\eta^\gamma_c"] & r_*\gamma^* c \simeq r_*\beta^* d \end{tikzcd}$$

Similarly, if $\gamma^*, \beta^*, p^*, q^*, r^*$ admit instead left adjoints $\gamma_!, \beta_!, p_!, q_!, r_!$ respectively and $\underline{\mathcal{A}}$ admits pushouts, then the functor $p^* \times_{r^*} q^*$ admits a left adjoint given by the functor ψ sending $(c,d) \in \underline{\mathcal{C}} \times_{\underline{\mathcal{E}}} \underline{\mathcal{D}}$ to the pushout in $\underline{\mathcal{A}}$

$$\begin{tikzcd}
r_!\gamma^*c \simeq r_!\beta^*d \arrow[r, "q_!\varepsilon^\beta_d"] \arrow[d, "p_!\varepsilon^\gamma_c"'] \arrow[dr, phantom, "\ulcorner"] & q_!d \arrow[d] \\
p_!c \arrow[r] & \psi(c,d)
\end{tikzcd}$$

Proof. The putative right adjoint φ is clearly a $\mathcal{T}$–functor, and is fibrewise a right adjoint by [HY17, Thm. 5.5]. Hence, by Proposition 2.1.13, we get that φ is indeed the $\mathcal{T}$–right adjoint of $p^* \times_{r^*} q^*$. The statement for the left adjoint follows by passing to the opposite categories. $\square$

Corollary 2.2.10. *Suppose we have a pushout diagram in* $\mathrm{Cat}_{\mathcal{T}}$

$$\begin{tikzcd}
\underline{\mathcal{B}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \underline{\mathcal{D}} \arrow[d] \\
\underline{\mathcal{C}} \arrow[r] & \underline{\mathcal{P}}
\end{tikzcd}$$

and let $\underline{\mathcal{A}} \in \mathrm{Cat}_{\mathcal{T}}$ have all $\mathcal{T}$–colimits. Then for any $\partial \in \underline{\mathrm{Fun}}(\underline{\mathcal{P}}, \underline{\mathcal{A}})$, we have the following pushout diagram in $\underline{\mathcal{A}}$

$$\begin{tikzcd}
\underline{\mathrm{colim}}_{\underline{\mathcal{B}}}\partial \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \underline{\mathrm{colim}}_{\underline{\mathcal{D}}}\partial \arrow[d] \\
\underline{\mathrm{colim}}_{\underline{\mathcal{C}}}\partial \arrow[r] & \underline{\mathrm{colim}}_{\underline{\mathcal{P}}}\partial
\end{tikzcd}$$

where we have suppressed the restriction functors. A similar statement holds for limits, with all colimits in sight replaced with limits.

Proof. This is an immediate consequence of Lemma 2.2.9, using that $\underline{\mathrm{Fun}}(\underline{\mathcal{P}}, \underline{\mathcal{A}}) \simeq \underline{\mathrm{Fun}}(\underline{\mathcal{C}}, \underline{\mathcal{A}}) \times_{\underline{\mathrm{Fun}}(\underline{\mathcal{B}}, \underline{\mathcal{A}})} \underline{\mathrm{Fun}}(\underline{\mathcal{D}}, \underline{\mathcal{A}})$ by definition of $\underline{\mathcal{P}}$ as a pushout. $\square$

Next, recall the notion of $\mathcal{T}$–faithful functors from [Hil22b, Def. 3.4.4], i.e. a $\mathcal{T}$–functor which induces fibrewise inclusions of components on mapping spaces.

Lemma 2.2.11. *Let $\underline{\mathcal{C}} \subset \underline{\mathcal{D}}$ be a $\mathcal{T}$–faithful inclusion where $\underline{\mathcal{D}}$ is closed under limits of shape $\underline{J}$. Then $\underline{\mathcal{C}}$ inherits $\underline{J}$–shaped limits from $\underline{\mathcal{D}}$ if the following conditions hold:*

1. *for any object $\partial \in \underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{C}}) \subset \underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{D}})$, the limit of ∂ has the property of lying in $\underline{\mathcal{C}}$ and the adjunction counit $\underline{\mathrm{const}}_{\underline{J}}\underline{\lim}_{\underline{J}}\partial \to \partial$ lies in $\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{C}})$,*

2. *for any $C \in \underline{\mathcal{C}}$ equipped with a morphism $\underline{\mathrm{const}}_{\underline{J}}C \to \partial$ in $\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{C}})$, the induced morphism $C \to \underline{\lim}_{\underline{J}}\partial$ lies in $\underline{\mathcal{C}} \subset \underline{\mathcal{D}}$.*

A similar statement holds also for colimits by passing to the opposite categories.

Proof. Suppose we are given such an object $\partial \in \underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{C}})$ satisfying (1) and (2). We need to argue that the natural map $\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-, \underline{\lim}_{\underline{J}}\partial) \to \underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{C}})}(\underline{\mathrm{const}}_{\underline{J}}-, \partial)$ induced by the morphism $\underline{\mathrm{const}}_{\underline{J}}\underline{\lim}_{\underline{J}}\partial \to \partial$, which is in $\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{C}})$ by condition (1), is an equivalence. To this end, consider the commuting diagram

$$\begin{tikzcd}
\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-, \underline{\lim}_{\underline{J}}\partial) \arrow[r] \arrow[d, hook] & \underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{C}})}(\underline{\mathrm{const}}_{\underline{J}}-, \partial) \arrow[d, hook] \\
\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(-, \underline{\lim}_{\underline{J}}\partial) \arrow[r, "\simeq"] & \underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{D}})}(\underline{\mathrm{const}}_{\underline{J}}-, \partial)
\end{tikzcd}$$

where the bottom horizontal is an equivalence by definition of $\underline{\lim}_{\underline{I}}\partial$. Since the vertical maps are inclusion of subcomponents, so is the top horizontal map. Hence, we are left to showing that the top horizontal map is π_0–surjective, which is precisely supplied by condition (2). $\square$

We include the proof of the following standard observation for the reader's convenience as well as to establish what we mean by "reflecting (co)limits".

Lemma 2.2.12. *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}, \underline{I}, \in \mathrm{Cat}_{\mathcal{T}}$ such that $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ admits $\underline{I}$–shaped (co)limits. Let $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a conservative functor preserving $\underline{I}$–shaped (co)limits. Then F reflects $\underline{I}$–shaped (co)limits.*

Proof. Without loss of generality we deal with the case of limits. Suppose we have a coned $\underline{I}$–shaped diagram $\partial\colon \underline{I}^{\triangleleft} \to \underline{\mathcal{C}}$ such that $F\partial\colon \underline{I}^{\triangleleft} \to \underline{\mathcal{D}}$ is a limit diagram. This means that if we write ∞ for the cone point, then we have a map $\partial(\infty) \to \underline{\lim}_{\underline{I}}\partial$ in $\underline{\mathcal{C}}$ such that $F\partial(\infty) \to F\underline{\lim}_{\underline{I}}\partial \simeq \underline{\lim}_{\underline{I}}F\partial$ is an equivalence in $\underline{\mathcal{D}}$. Since F was conservative, we get that $\partial(\infty) \to \underline{\lim}_{\underline{I}}\partial$ was already an equivalence, as was to be shown. $\square$

We will deduce the parametrised analogue of [MP87, Lem 1.7.ii] from the unparametrised version proven in [CDH+]. We will need some terminology for this.

Terminology 2.2.13. Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$-cocomplete category and S be a set of objects in $\underline{\mathcal{C}}$. We say that it is *jointly conservative* if S induces a jointly conservative set of objects in each fibre of $\underline{\mathcal{C}}$, i.e. for every $V \in \mathcal{T}$ and writing S_V for the set of objects of $\mathcal{C}_V$ in the set S, the functor $\prod_{x\in S_V} \mathrm{Map}_{\mathcal{C}_V}(x,-) : \mathcal{C}_V \to \prod_{x\in S_V} \mathcal{S}$ is conservative. We say that it is a *set of parametrised generators of* $\underline{\mathcal{C}}$ if the smallest $\mathcal{T}$-cocomplete subcategory of $\underline{\mathcal{C}}$ containing S is $\underline{\mathcal{C}}$ itself. That is, every parametrised object in $\underline{\mathcal{C}}$ can be written as a parametrised colimit of objects in S.

Now recall the notion of parametrised compactness from the paragraph before Theorem 2.1.34.

Proposition 2.2.14 (Parametrised Makkai-Pitts)**.** *Let κ be a regular cardinal and $\underline{\mathcal{C}}$ a $\mathcal{T}$-cocomplete category. Let $S \subseteq \underline{\mathcal{C}}$ be a jointly conservative set of parametrised-κ-compact objects. Then S is a set of parametrised-κ-compact generators. In particular, $\underline{\mathcal{C}}$ is parametrised-κ-compactly generated.*

Proof. We want to show that for every $V \in \mathcal{T}$, any $\mathcal{T}_{/V}$–object in $\underline{\mathcal{C}}_{\underline{V}}$ is a $\mathcal{T}_{/V}$–colimit of objects in S. By hypothesis, $\prod_{x\in S_V} \mathrm{Map}_{\mathcal{C}_V}(x,-)\colon \mathcal{C}_V \to \prod_{x\in S_V} \mathcal{S}$ is jointly conservative. Hence, by [CDH+, Prop 1.1.2], every object in $\mathcal{C}_V$ is a κ-small colimit of objects in S_V. $\square$

Next, we supply the expected anti–equivalence of presentable categories.

Proposition 2.2.15. *There is a canonical equivalence of $\mathcal{T}$–categories $\underline{\mathrm{Pr}}_L \simeq \underline{\mathrm{Pr}}_R^{\mathrm{op}}$.*

Proof. This is just the proof of [Lur09, Prop. 5.5.3.3] written in our setting, except that we do not need to invoke the adjoint functor Theorem 2.1.35 since $\underline{\mathrm{Pr}}_L$ and $\underline{\mathrm{Pr}}_R$ were defined with morphisms being left and right adjoints respectively. Write $\underline{\widehat{\mathrm{Cat}}}_{\mathcal{T}}$ and $\underline{\mathrm{CAT}}_{\mathcal{T}}$ for the $\mathcal{T}$–categories of large and huge $\mathcal{T}$–categories, respectively. In particular, we have that $\underline{\mathrm{Pr}}_L, \underline{\mathrm{Pr}}_R \subset \underline{\widehat{\mathrm{Cat}}}_{\mathcal{T}} \in \underline{\mathrm{CAT}}_{\mathcal{T}}$. Now, fix a $\underline{S} \in \underline{\mathrm{CAT}}_{\mathcal{T}}$. Under the cocartesian unstraightening equivalence of Theorem 2.2.2, it is easy to see that elements in $\pi_0 \mathrm{Map}_{\mathrm{CAT}_{\mathcal{T}}}(\underline{S}, \underline{\widehat{\mathrm{Cat}}}_{\mathcal{T}})$ that lie in $\pi_0 \mathrm{Map}_{\mathrm{CAT}_{\mathcal{T}}}(\underline{S}, \underline{\mathrm{Pr}}_L) \subset \pi_0 \mathrm{Map}_{\mathrm{CAT}_{\mathcal{T}}}(\underline{S}, \underline{\widehat{\mathrm{Cat}}}_{\mathcal{T}})$ are precisely those $\mathcal{T}$–cocartesian fibrations $\underline{\mathcal{P}} \to \underline{S}$ which are also $\mathcal{T}$–cartesian and whose parametrised fibres are $\mathcal{T}$–presentable. Similarly, under the cartesian unstraightening from Construction 2.2.3, the subset $\pi_0 \mathrm{Map}_{\mathrm{CAT}_{\mathcal{T}}}(\underline{S}^{\mathrm{op}}, \underline{\mathrm{Pr}}_R) \subset \pi_0 \mathrm{Map}_{\mathrm{CAT}_{\mathcal{T}}}(\underline{S}^{\mathrm{op}}, \underline{\widehat{\mathrm{Cat}}}_{\mathcal{T}})$ is precisely described as those $\mathcal{T}$–cartesian fibrations $\underline{\mathcal{E}} \to \underline{S}$ which are also $\mathcal{T}$–cocartesian and whose parametrised fibres are $\mathcal{T}$–presentable. Therefore, we obtain bijections

$$\pi_0 \mathrm{Map}_{\mathrm{CAT}_{\mathcal{T}}}(\underline{S}, \underline{\mathrm{Pr}}_L) \cong \pi_0 \mathrm{Map}_{\mathrm{CAT}_{\mathcal{T}}}(\underline{S}^{\mathrm{op}}, \underline{\mathrm{Pr}}_R) \cong \pi_0 \mathrm{Map}_{\mathrm{CAT}_{\mathcal{T}}}(\underline{S}, \underline{\mathrm{Pr}}_R^{\mathrm{op}})$$

natural in $\underline{S} \in \underline{\mathrm{CAT}}_{\mathcal{T}}$. Since $\underline{\mathrm{CAT}}_{\mathcal{T}}$ admits pushouts, we may replace $\underline{S}$ with $\Sigma^n \underline{S}$ in the natural bijection of sets to upgrade it to an equivalence of mapping spaces, whence the equivalence $\underline{\mathrm{Pr}}_L \simeq \underline{\mathrm{Pr}}_R^{\mathrm{op}}$ as wanted. $\square$

Recall Notation 2.2.4. In [Nar17, §3.4], Nardin constructed a $\mathcal{T}$–symmetric monoidal structure on $\underline{\mathrm{Pr}}_L$ generalising Lurie's tensor product for presentable categories with the tensor unit given by the $\mathcal{T}$–category $\underline{\mathcal{S}}_{\mathcal{T}}$ of spaces. The following was then stated as [Nar17, Ex. 3.26] without proof, and we have supplied a proof in [Hil22b, Prop. 6.7.5].

Proposition 2.2.16 (Formula for presentable $\mathcal{T}$–tensors)**.** *Let $\mathcal{T}$ be an atomic orbital category, and let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}$–presentable categories. Then $\underline{\mathcal{C}} \otimes \underline{\mathcal{D}} \simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{D}})$.*

Notation 2.2.17. Write $\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}} \subseteq \underline{\mathcal{S}}_{*\mathcal{T}}$ for the smallest full $\mathcal{T}$–subcategory containing the zero object and finite parametrised colimits (i.e. closed under finite fibrewise colimits and finite indexed coproducts). Nardin proved in [Nar16, Thm. 7.4] that, much as in the unparametrised setting, the functor of $\mathcal{T}$–stabilisation on a $\mathcal{T}$–category $\underline{\mathcal{C}}$ admitting finite indexed products may be computed as $\underline{\mathrm{Sp}}(\underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{fin}}(\underline{\mathcal{S}}^{\mathrm{fin}}, \underline{\mathcal{C}})$

Proposition 2.2.18. *For $\underline{\mathcal{C}}$ a $\mathcal{T}$–presentable category, we have that $\underline{\mathrm{Sp}}_{\mathcal{T}}(\mathcal{C}) \simeq \underline{\mathcal{C}} \otimes \underline{\mathrm{Sp}}_{\mathcal{T}}$.*

Proof. Consider the sequence of equivalences

$$\begin{aligned}
\underline{\mathcal{C}} \otimes \underline{\mathrm{Sp}}_{\mathcal{T}} &\simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{\mathcal{T}}) \\
&\simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}})) \\
&\simeq \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}}, \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})) \\
&\simeq \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}}, \underline{\mathcal{C}} \otimes \underline{\mathcal{S}}_{\mathcal{T}}) \\
&\simeq \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{C}})
\end{aligned}$$

where the first equivalence is by Proposition 2.2.16. We have also used Nardin's formula for $\mathcal{T}$–stabilisation from [Nar16, Thm. 7.4] in the second and fifth equivalences. $\square$

Proposition 2.2.19 (Parametrised stabilisation is smashing, "[GGN15, Thm. 4.6]")**.** *The association $\underline{\mathcal{C}} \mapsto \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{C}})$ refines to a $\mathcal{T}$–symmetric monoidal localisation $\underline{\mathrm{Sp}}_{\mathcal{T}} \otimes - : \underline{\mathrm{Pr}}_{\mathcal{T},L} \longrightarrow \underline{\mathrm{Pr}}_{\mathcal{T},L}$ with essential image the $\mathcal{T}$–full subcategory of $\mathcal{T}$–presentable-stable categories $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L}$.*

Proof. That $\underline{\mathrm{Sp}}_{\mathcal{T}}(-) \simeq \underline{\mathrm{Sp}}_{\mathcal{T}} \otimes (-)$ is the proposition above, which also gives the required essential image. That the functor is a $\mathcal{T}$–symmetric monoidal localisation is by the $\mathcal{T}$–idempotence of $\underline{\mathrm{Sp}}_{\mathcal{T}}$ from [Nar17, Cor. 3.28]. $\square$

Corollary 2.2.20. *For $f : U \to W$ a map in $\mathrm{Fin}_{\mathcal{T}}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}_{/U}}$, there is a natural equivalence $f_{\otimes}\underline{\mathrm{PSh}}_U(\underline{\mathcal{C}}) \simeq \underline{\mathrm{PSh}}_W(f_*\underline{\mathcal{C}})$ and $f_{\otimes}\underline{\mathrm{PSh}}^{\mathrm{st}}_U(\underline{\mathcal{C}}) \simeq \underline{\mathrm{PSh}}^{\mathrm{st}}_W(f_*\underline{\mathcal{C}})$.*

Proof. Since $\underline{\mathrm{Sp}} \otimes -$ is a smashing localisation from Proposition 2.2.19, it suffices just to prove the case of presheaves in spaces. Let $\underline{\mathcal{D}} \in \underline{\mathrm{Pr}}_{\mathcal{T}_{/W}}$. By [Nar17, Prop. 3.19], the restriction map $\underline{\mathrm{Fun}}^L_W\big(f_{\otimes}\underline{\mathrm{PSh}}_U(\underline{\mathcal{C}}), \underline{\mathcal{D}}\big) \longrightarrow \underline{\mathrm{Fun}}_W\big(f_*\underline{\mathcal{C}}, \underline{\mathcal{D}}\big)$ is an equivalence. But then the target is naturally equivalent to $\underline{\mathrm{Fun}}^L_W\big(\underline{\mathrm{PSh}}_W(f_*\underline{\mathcal{C}}), \underline{\mathcal{D}}\big)$ by [Sha23, Thm. 11.5] and so we are done. $\square$

Observation 2.2.21 ($\mathcal{T}$–exactness on $\mathcal{T}$–stables). Write $\underline{\mathrm{Fun}}^{\mathrm{lex}}, \underline{\mathrm{Fun}}^{\mathrm{rex}}, \underline{\mathrm{Fun}}^{\mathrm{ex}} \subseteq \underline{\mathrm{Fun}}$ for the full subcategories of functors which preserve finite $\mathcal{T}$–limits, finite $\mathcal{T}$–colimits, and finite $\mathcal{T}$–(co)limits, respectively. If $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ are $\mathcal{T}$–stable, then note that the two $\mathcal{T}$–full subcategories $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \supseteq \underline{\mathrm{Fun}}^{\mathrm{rex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ agree. To wit, both imply that they are fibrewise right and left exact (since these are fibrewise stable after all); moreover, preserving finite $\mathcal{T}$–coproducts and preserving finite $\mathcal{T}$–products are equivalent since $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ were $\mathcal{T}$–semiadditive. Hence in this case we have $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) = \underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) = \underline{\mathrm{Fun}}^{\mathrm{rex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$.

Lemma 2.2.22. *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ have finite $\mathcal{T}$–limits and $\underline{\mathcal{A}}$ admit finite $\mathcal{T}$–colimits. Then we have a canonical equivalence* $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})) \simeq \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$.

Proof. Note that we have the identification $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})) \simeq \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$ since $\mathcal{T}$–limits of functor categories are computed in the target by [Hil22b, Prop. 3.1.12]. To see that we have the desired equivalence, consider the diagram

$$\begin{tikzcd}
\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})) \arrow[r, dashed, shift left] \arrow[d, hook] & \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \arrow[l, dashed, shift left] \arrow[d, hook] \\
\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})) \arrow[r, shift left, "\simeq"'] & \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \arrow[l, shift left]
\end{tikzcd}$$

That the bottom arrows restrict to the dashed arrows is because again by [Hil22b, Prop. 3.1.12], $\mathcal{T}$–limits in both $\underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})$ and $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ are computed in $\underline{\mathcal{D}}$. ☐

Corollary 2.2.23 (Internal hom object of $\mathcal{T}$–perfects)**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. Then the $\mathcal{T}$–full subcategory $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ on the $\mathcal{T}$–exact functors is also small $\mathcal{T}$–idempotent-complete-stable, that is, $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is again an object of $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$.*

Proof. That it is small is clear. To see that it is $\mathcal{T}$–stable, just note

$$\begin{aligned}
\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) &\simeq \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \\
&\simeq \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}}, \underline{\mathcal{D}})) \\
&\simeq \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}}, \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \\
&\simeq \underline{\mathrm{Fun}}^{\mathrm{lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}}, \underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))
\end{aligned}$$

where the first and last equivalences are by Observation 2.2.21, the second is by [Nar16, Thm. 7.4], and the third by Lemma 2.2.22. Hence, by [Nar16, Thm. 7.4] again, we see that $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is $\mathcal{T}$–stable. For $\mathcal{T}$–idempotent-completeness, note that $\mathcal{T}$–colimits of $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{rex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ are computed in $\underline{\mathcal{D}}$, and since being $\mathcal{T}$–idempotent-complete is just the condition of admitting certain fibrewise $\mathcal{T}$–colimits, this point is clear too. ☐

Proposition 2.2.24. *Let $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}(\kappa)}_{\mathcal{T}}$. Then $\underline{\mathrm{Fun}}^{\mathrm{ex}(\kappa)}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\kappa}_{\mathcal{T}}, \underline{\mathcal{C}}) \simeq \underline{\mathcal{C}}$, where $\underline{\mathrm{Fun}}^{\mathrm{ex}(\kappa)}_{\mathcal{T}}$ denotes the functors which preserves κ–finite (co)limits.*

Proof. Recall we had equivalence $\underline{\mathrm{Cat}}^{\mathrm{perf}(\kappa)}_{\mathcal{T}} \simeq \underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa}$ from Proposition 2.1.37 so that $(\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}})^{\kappa} \simeq \underline{\mathcal{C}}$. Writing $\underline{\mathrm{Fun}}^{\underline{\kappa}} \subseteq \underline{\mathrm{Fun}}$ for the $\mathcal{T}$–full subcategory of parametrised functors preserving parametrised κ–compact objects, consider

$$\begin{aligned}
\underline{\mathrm{Fun}}^{\mathrm{ex}(\kappa)}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\kappa}_{\mathcal{T}}, \underline{\mathcal{C}}) &\simeq \underline{\mathrm{Fun}}^{\mathrm{rex}(\kappa)}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\kappa}_{\mathcal{T}}, (\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}})^{\kappa}) \\
&\simeq \underline{\mathrm{Fun}}^{L,\kappa}_{\mathcal{T}}(\underline{\mathrm{Sp}}_{\mathcal{T}}, \underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}) \\
&\simeq \underline{\mathrm{Fun}}^{L,\kappa}_{\mathcal{T}}(\underline{\mathcal{S}}_{\mathcal{T}}, \underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}) \\
&\simeq (\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}})^{\kappa} \simeq \underline{\mathcal{C}}
\end{aligned}$$

where the second equivalence is by [Hil22b, Prop. 3.5.4]; the third equivalence is by Proposition 2.2.19; the fourth equivalence is by the universal property of $\mathcal{T}$–presheaves of [Sha23, Thm. 11.5]. ☐

2.3 Symmetric monoidality and localisations

Convention: From now on for the rest of the article, our base category $\mathcal{T}$ will always be assumed to be atomic orbital.

The main aim of this subsection is to give a multiplicative enhancement of the presentable Dwyer–Kan localisations from [Hil22b, §6.3].

Terminology 2.3.1. Let S be a collection of morphisms in a $\mathcal{T}$–category $\underline{\mathcal{C}}$. For any $U = U_1 \sqcup \cdots \sqcup U_n \in \mathrm{Fin}_{\mathcal{T}}$ where $U_i \in \mathcal{T}$, we will write S_U for the collection of morphisms inside S in $\mathcal{C}_U \simeq \mathcal{C}_{U_1} \times \cdots \times \mathcal{C}_{U_n}$. We will say that S is a *$\mathcal{T}$–collection* if for any morphism $f\colon U \to V$ in $\mathrm{Fin}_{\mathcal{T}}$ and any morphism $\varphi\colon A \to B$ in S_V (in general, this is a tuple of morphisms as is clear from our definition of S_V in the preceding sentence), the morphism $f^*\varphi\colon f^*A \to f^*B$ is a morphism in S_U.

Terminology 2.3.2. We say that a $\mathcal{T}$–collection of morphisms S in a $\mathcal{T}$–symmetric monoidal category $\underline{\mathcal{C}}^{\otimes}$ is *$\underline{\otimes}$–multiplicatively closed* if for any $V \in \mathcal{T}$, any morphism $p\colon U \to V$ in $\mathrm{Fin}_{\mathcal{T}}$, and any morphism $\varphi\colon A \to B$ in S_U, the morphism $p_{\otimes}\varphi\colon p_{\otimes}A \to p_{\otimes}B$ in $\mathcal{C}_V$ lies in S_V.

Notation 2.3.3. We recall the clarifying distinction between *Dwyer-Kan localisations* and *Bousfield localisations* due to [Hin16], which we have also adopted in [Hil22b]. By $\mathcal{T}$–Dwyer–Kan localisations, we will mean the following: let $\underline{\mathcal{C}}$ be a $\mathcal{T}$-category and S a $\mathcal{T}$–collection of morphisms in $\underline{\mathcal{C}}$. Suppose now that a $\mathcal{T}$-category $S^{-1}\underline{\mathcal{C}}$ exists and is equipped with a map $\mathrm{DK} : \underline{\mathcal{C}} \to S^{-1}\underline{\mathcal{C}}$ inducing the equivalence

$$\mathrm{DK}^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(S^{-1}\underline{\mathcal{C}}, \underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

for all $\mathcal{T}$-categories $\underline{\mathcal{D}}$, where $\underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is the $\mathcal{T}$-full subcategory of parametrised functors sending morphisms in S to equivalences. If such a $\mathcal{T}$-category exists, then it must necessarily be unique, and this is then defined to be the *$\mathcal{T}$–Dwyer–Kan localisation of $\underline{\mathcal{C}}$ with respect to S*.

By *$\mathcal{T}$–Bousfield localisations*, we mean a $\mathcal{T}$-adjunction $L : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : i$ where the $\mathcal{T}$-right adjoint i is $\mathcal{T}$-fully faithful. Writing Z for the morphisms in $\underline{\mathcal{C}}$ that get sent to equivalences under L, we may then view $\underline{\mathcal{D}}$ as precisely the $\mathcal{T}$–full subcategory of *Z–local objects*, i.e. those $X \in \underline{\mathcal{C}}$ such that for any morphism $\varphi\colon A \to B$ in Z, the induced map $\varphi^*\colon \underline{\mathrm{Map}}(B, X) \to \underline{\mathrm{Map}}(A, X)$ is an equivalence. In [Hil22b, Prop. 6.3.2] we showed that, much like in the unparametrised setting, a $\mathcal{T}$-Bousfield localisation $L\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is in particular a $\mathcal{T}$-Dwyer-Kan localisation with respect to this collection Z.

We now record the following proposition, which has appeared also as [QS22, Lem. 5.27]. While in all likelihood our proof is perhaps only cosmetically distinct from theirs, we think that it is slightly simpler to verify without having to "open the blackbox" of parametrised operads, so to speak.

Proposition 2.3.4 (Dwyer-Kan symmetric monoidality)**.** *Let $\underline{\mathcal{C}}^{\otimes}$ be a $\mathcal{T}$–symmetric monoidal category and S a $\mathcal{T}$–collection of morphisms in $\underline{\mathcal{C}}$ which is $\underline{\otimes}$–multiplicatively closed. Then:*

1. *the Dwyer–Kan localisation $S^{-1}\underline{\mathcal{C}}$ attains a unique $\mathcal{T}$–symmetric monoidal structure such that the canonical map $\mathrm{DK}\colon \underline{\mathcal{C}} \to S^{-1}\underline{\mathcal{C}}$ enhances to a $\mathcal{T}$–symmetric monoidal functor,*
2. *For any $\mathcal{T}$–symmetric monoidal category $\underline{\mathcal{D}}^{\otimes}$, the induced functor $\mathrm{DK}^* : \underline{\mathrm{Fun}}_{\mathcal{T}}^{\otimes}(S^{-1}\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}^{\otimes, S^{-1}}(\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes})$ is an equivalence.*

Proof. The proof will proceed by bootstrapping from the construction and proof of [Lur17, Prop. 4.1.7.4]. Recall from [Lur17, Cons. 4.1.7.1] that we have a category WCat whose objects are pairs $(\mathcal{C}, W)$ where $\mathcal{C}$ is a category and W is a collection of morphisms in $\mathcal{C}$ stable under composition and contains all equivalences in $\mathcal{C}$, and morphisms $f : (\mathcal{C}, W) \to (\mathcal{C}', W')$ are functors $f : \mathcal{C} \to \mathcal{C}'$ such that $f(W) \subseteq W'$. By [Lur17, Prop. 4.1.7.2] we have a Bousfield localisation

$$\mathrm{WCat} \underset{}{\overset{I}{\rightleftarrows}} \mathrm{Cat} \tag{8}$$

where both functors preserve finite products and the functor I sends $(\mathcal{C}, W)$ to the Dwyer-Kan localisation $W^{-1}\mathcal{C}$. Applying Construction 2.1.5 to this adjunction we get the $\mathcal{T}$–Bousfield localisation $\underline{I}_{\mathcal{T}} : \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{WCat}) \rightleftarrows \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Cat}) : \underline{\mathrm{incl}}_{\mathcal{T}}$. Moreover, since both functors in (8) preserve finite products, the functor $\underline{I}_{\mathcal{T}}$ preserves indexed products. Hence, applying the forgetful functor from Construction 2.1.27, we even have a commuting square of $\mathcal{T}$–Bousfield localisations

$$\begin{tikzcd}
\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{WCat}) \arrow[r, shift left, "\underline{I}_{\mathcal{T}}"] \arrow[d, "\mathrm{fgt}"'] & \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}) \arrow[l, shift left, "\underline{\mathrm{incl}}"] \arrow[d, "\mathrm{fgt}"] \\
\underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{WCat}) \arrow[r, shift left, "\underline{I}_{\mathcal{T}}"] & \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Cat}) \arrow[l, shift left, "\underline{\mathrm{incl}}"]
\end{tikzcd} \tag{9}$$

It is straightforward to see that our hypotheses on the pair $(\underline{\mathcal{C}}, S) \in \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{WCat})$ ensures that it lifts to an object in $\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{WCat})$. Furthermore, recall that $\mathcal{T}$–symmetric monoidal categories are equivalently $\mathcal{T}$–commutative monoids in $\underline{\mathrm{Cat}}_{\mathcal{T}}$ by Theorem 2.1.32 (1).

Given these, part (1) may now be obtained exactly by the argument in [GGN15, Lem. 3.6] (which also saw an immediate parametrised adaptation in [Hil22b, Lem. 4.2.3]). For part (2), we would like to argue that for any $\mathcal{T}$–category $\underline{\mathcal{E}}$, the map

$$\mathrm{DK}^* : \underline{\mathrm{Map}}_{\mathrm{Cat}_{\mathcal{T}}}\big(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}^{\otimes}_{\mathcal{T}}(S^{-1}\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes})\big) \longrightarrow \underline{\mathrm{Map}}_{\mathrm{Cat}_{\mathcal{T}}}\big(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}^{\otimes, S^{-1}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes})\big)$$

is an equivalence. But by the universal property of the pointwise $\mathcal{T}$–symmetric monoidal structure from Recollection 2.1.33, we may rewrite the domain and codomain respectively as $\underline{\mathrm{Map}}^{\otimes}_{\mathcal{T}}(S^{-1}\underline{\mathcal{C}}^{\otimes}, \underline{\mathrm{Fun}}(\underline{\mathcal{E}}, \underline{\mathcal{D}}^{\otimes}))$ and $\underline{\mathrm{Map}}^{\otimes, S^{-1}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\otimes}, \underline{\mathrm{Fun}}(\underline{\mathcal{E}}, \underline{\mathcal{D}}^{\otimes}))$. Under these identifications, the map displayed above is then an equivalence by the top $\mathcal{T}$–adjunction in (9). ◻

Remark 2.3.5. For our multiplicative considerations shortly, the following notations and observations will be important. For a $\mathcal{T}$–collection of morphisms S in a $\mathcal{T}$–presentable category $\underline{\mathcal{C}}$, we write $S_{\underline{\amalg}} \supseteq S$ for the closure of S under finite indexed coproducts of morphisms in S. Similarly, we denote by $S_{\underline{\mathrm{colim}}} \supseteq S_{\underline{\amalg}} \supseteq S$ the closure of S under *all* parametrised colimits valued in S. These are easily seen to be, again, $\mathcal{T}$–collections of morphisms. We write $S^{-1}_{\underline{\mathrm{colim}}}\underline{\mathcal{C}}, S^{-1}_{\underline{\amalg}}\underline{\mathcal{C}} \subseteq \underline{\mathcal{C}}$ for the $\mathcal{T}$–full subcategories of $S_{\underline{\mathrm{colim}}}$– and $S_{\underline{\amalg}}$–local objects in $\underline{\mathcal{C}}$ respectively (this is consistent with Notation 2.3.3 by virtue of [Hil22b, Thm. 6.3.7]). Since $S_{\underline{\mathrm{colim}}} \supseteq S_{\underline{\amalg}}$, we have the inclusion $S^{-1}_{\underline{\mathrm{colim}}}\underline{\mathcal{C}} \subseteq S^{-1}_{\underline{\amalg}}\underline{\mathcal{C}}$. To see that this inclusion is an equivalence, let $X \in S^{-1}_{\underline{\amalg}}\underline{\mathcal{C}}$ and let $\partial \colon \underline{J} \to \underline{\mathcal{C}}^{\Delta^1}$ be a diagram taking values in S. We need to show that X is local against the morphism $\underline{\mathrm{colim}}_{\underline{J}} \colon \underline{\mathrm{colim}}_{\underline{J}}\partial_0 \to \underline{\mathrm{colim}}_{\underline{J}}\partial_1$ in $S_{\underline{\mathrm{colim}}}$. But this is clearly implied by the commutation $\underline{\mathrm{Map}}(\underline{\mathrm{colim}}_{\underline{J}}\partial_1, X) \simeq \underline{\lim}_{\underline{J}^{\mathrm{op}}}\underline{\mathrm{Map}}(\partial_1, X)$ and the analogous ones for $\underline{\mathrm{colim}}_{\underline{J}}\partial_0$ and $\underline{\mathrm{colim}}_{\underline{J}}\partial$.

Lemma 2.3.6. *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–presentable category equipped with a $\mathcal{T}$–distributive–symmetric–monoidal structure $\underline{\mathcal{C}}^{\underline{\otimes}}$. Let S be a $\mathcal{T}$–collection of morphisms in $\underline{\mathcal{C}}$ and $S_{\underline{\mathrm{colim}}} \supseteq S$ its closure from Remark 2.3.5. If S is $\underline{\otimes}$–multiplicatively closed, then so is $S_{\underline{\mathrm{colim}}}$.*

Proof. Let $p \colon W \to U$ be a morphism in $\mathrm{Fin}_{\mathcal{T}}$ and suppose we have a diagram of morphisms $\partial \colon \underline{J} \to p^*\underline{\mathcal{C}}^{\Delta^1}$ in S. Our goal is to show that the morphism $p_{\otimes}\underline{\mathrm{colim}}_{\underline{J}}\partial$ is still contained in $S_{\underline{\mathrm{colim}}}$. By $\mathcal{T}$–distributivity, we have the equivalence

$$p_{\otimes}\underline{\mathrm{colim}}_{\underline{J}}\partial \simeq \underline{\mathrm{colim}}\big(p_*\underline{J} \xrightarrow{p_*\partial} p_*p^*\underline{\mathcal{C}}^{\Delta^1} \xrightarrow{p_{\otimes}} \underline{\mathcal{C}}^{\Delta^1}\big)$$

where we have endowed $\underline{\mathcal{C}}^{\Delta^1}$ with the pointwise $\mathcal{T}$–symmetric monoidal structure from [NS22, §3.3], which is again $\mathcal{T}$–distributive. Since S is $\underline{\otimes}$–multiplicatively closed, the right hand side is in $S_{\underline{\mathrm{colim}}}$, and so $p_{\otimes}\underline{\mathrm{colim}}_{\underline{J}}\partial$ is too, as required. □

Proposition 2.3.7. *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–presentable category equipped with a $\mathcal{T}$–distributive–symmetric–monoidal structure $\underline{\mathcal{C}}^{\underline{\otimes}}$. Let S be a $\mathcal{T}$–collection of morphisms in $\underline{\mathcal{C}}$.*

1. *There is a $\mathcal{T}$–presentable category $L_S\underline{\mathcal{C}}$ participating in a $\mathcal{T}$–Bousfield localisation*
$$L\colon \underline{\mathcal{C}} \rightleftarrows L_S\underline{\mathcal{C}} : i$$
satisfying the following universal property: for any other $\mathcal{T}$–presentable category $\underline{\mathcal{D}}$, the map L induces the equivalence $L^*\colon \underline{\mathrm{Fun}}^L(L_S\underline{\mathcal{C}}, \underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{L,S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$.

2. *If S was furthermore $\underline{\otimes}$–multiplicatively closed, then there is a canonical enhancement $L_S\underline{\mathcal{C}}^{\underline{\otimes}}$ of $L_S\underline{\mathcal{C}}$ to the structure of a $\mathcal{T}$–symmetric monoidal category. This is uniquely characterised by the following universal property: for any $\mathcal{T}$–cocomplete $\mathcal{T}$–symmetric monoidal category $\underline{\mathcal{D}}^{\underline{\otimes}}$, this functor induces the equivalence $L^*\colon \underline{\mathrm{Fun}}^{\underline{\otimes},L}(L_S\underline{\mathcal{C}}, \underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\underline{\otimes},L,S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$.*

Proof. Part (1) is an immediate consequence of [Hil22b, Thm. 6.3.7], using $L_S\underline{\mathcal{C}} := S_{\underline{\amalg}}^{-1}\underline{\mathcal{C}}$. For part (2), we recall that $S_{\underline{\mathrm{colim}}}^{-1}\underline{\mathcal{C}} \simeq S_{\underline{\amalg}}^{-1}\underline{\mathcal{C}}$ from Remark 2.3.5. Now, Lemma 2.3.6 ensures that $S_{\underline{\mathrm{colim}}}$ is $\underline{\otimes}$–multiplicatively closed, and so by Proposition 2.3.4 (1) we obtain the first sentence of (2). The final sentence is now an immediate combination of the universal property of part (1) and Proposition 2.3.4 (2). □

2.4 Borel equivariant theory

In this subsection, we specialise for the moment to the case of $\mathcal{T} = \mathcal{O}_G$. We write $b\colon BG \hookrightarrow \mathcal{O}_G$ for the fully faithful inclusion of the free transitive G–set G/e. This subsection pertains to working out some far–reaching consequences of constructions associated to this one map just from adjunction considerations. The philosophy here is to seriously reckon with the fact that the process of forgetting "genuine" G–structures to "Borel" G–structures is one that penetrates and appears on multiple categorical levels. In this sense, the "categorification–decategorification" approach we present here is very much in line with the so–called "metacosm–macrocosm–microcosm" trichotomy of [AMR22]. Our end point will be the full version of Theorem A in the form of Theorem 2.4.10 and our starting point is the Bousfield localisation

$$b^*\colon \widehat{\mathrm{Cat}}_G = \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \widehat{\mathrm{Cat}}) \rightleftarrows \mathrm{Fun}(BG, \widehat{\mathrm{Cat}}) : b_* \tag{10}$$

The G–category of small G–categories $\underline{\mathrm{Cat}}_G$ is then an object of $\widehat{\mathrm{Cat}}_G$. Due to its special role throughout, we denote $b_*b^*\underline{\mathrm{Cat}}_G \in \widehat{\mathrm{Cat}}_G$ by $\underline{\mathrm{Bor}}(\mathrm{Cat})$ for *Borel G–categories*. By the right Kan extension formula, we see that under the embedding b_* from (10), $\underline{\mathrm{Bor}}(\mathrm{Cat})$ is the G–category whose value at G/H is given by $\mathrm{Cat}^{hH} \simeq \mathrm{Fun}(BH, \mathrm{Cat})$. Importantly, the adjunction unit $\underline{\mathrm{Cat}}_G \to \underline{\mathrm{Bor}}(\mathrm{Cat})$ can be checked easily to be given precisely *again* by the map b^*. By right Kan extension, the G–functor $b^*\colon \underline{\mathrm{Cat}}_G \to \underline{\mathrm{Bor}}(\mathrm{Cat})$ admits fibrewise fully faithful right adjoints b_*. Over every fibre, objects in $\underline{\mathrm{Bor}}(\mathrm{Cat})$ may then be viewed via b_* precisely as those objects in $\underline{\mathrm{Cat}}$ which are *Borel local*, i.e. those $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}$ with the property that if $\varphi\colon \underline{I} \to \underline{J}$ is a map in $\underline{\mathrm{Cat}}$ such that $b^*\varphi$ is an equivalence (such maps are also called Borel equivalences), then $\varphi^*\colon \mathrm{Map}_{\underline{\mathrm{Cat}}}(\underline{J}, \underline{\mathcal{C}}) \to \mathrm{Map}_{\underline{\mathrm{Cat}}}(\underline{I}, \underline{\mathcal{C}})$ is an equivalence. Similarly, b^* also admits fibrewise fully faithful left adjoints $b_!$ by left Kan extension. Under this inclusion, $\underline{\mathrm{Bor}}(\mathrm{Cat})$ may be viewed as the *Borel colocal* objects, i.e. those $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}$ such that $\varphi_*\colon \mathrm{Map}_{\underline{\mathrm{Cat}}}(\underline{\mathcal{C}}, \underline{I}) \to \mathrm{Map}_{\underline{\mathrm{Cat}}}(\underline{\mathcal{C}}, \underline{J})$ is an equivalence for any Borel equivalence $\varphi\colon \underline{I} \to \underline{J}$. The first observation to be made is that these can be assembled to a G–Bousfield (co)localisation by the following:

Proposition 2.4.1. *The G–functor b^* participates in the G–Bousfield (co)localisation*

$$\begin{tikzcd} \underline{\mathrm{Cat}}_G \arrow[r, "b^*"] & \underline{\mathrm{Bor}}(\mathrm{Cat}) \arrow[l, bend right, "b_!"'] \arrow[l, bend left, "b_*"] \end{tikzcd}$$

Proof. We would like to use Proposition 2.1.13 and its dual to say that the fibrewise fully faithful right/left adjoints assemble to a G–functor. For the case of b_*, without loss of generality, we show that for every $H \leq G$, the adjointed square

$$\begin{tikzcd} \mathrm{Cat}_G \arrow[d, "\mathrm{Res}^G_H"'] \arrow[r, phantom, shift right=4, "\Rightarrow"] & \mathrm{Fun}(BG, \mathrm{Cat}) \arrow[l, hook', "b_*"'] \arrow[d, "\mathrm{Res}^G_H"] \\ \mathrm{Cat}_H & \mathrm{Fun}(BH, \mathrm{Cat}) \arrow[l, hook', "b_*"'] \end{tikzcd}$$

commutes. To this end, we will use the Borel local description given above and we show the following: if $\underline{\mathcal{C}} \in \mathrm{Cat}_G$ is Borel local, then $\mathrm{Res}^G_H \underline{\mathcal{C}} \in \mathrm{Cat}_H$ is also Borel local. So let $\varphi \colon \underline{I} \to \underline{J}$ be a Borel equivalence in Cat_H. Since the functor $\mathrm{fgt} \colon \mathrm{Fun}(BH, \mathrm{Cat}) \to \mathrm{Cat}$ is conservative, this is the same as requiring that $\mathrm{Res}^H_e \varphi$ is an equivalence. The key point now is that because of this, and because $\mathrm{Res}^G_e \coprod_{G/H} \simeq \coprod_{g\in G/H} \mathrm{Res}^H_e$, we see that $\coprod_{G/H} \varphi \colon \coprod_{G/H} \underline{I} \to \coprod_{G/H} \underline{J}$ is still a Borel equivalence. Hence, by the computation

$$\mathrm{Map}_{\mathrm{Cat}_H}(\underline{J}, \mathrm{Res}^G_H \underline{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Cat}_G}(\coprod_{G/H} \underline{J}, \underline{\mathcal{C}}) \xrightarrow[\simeq]{\varphi^*} \mathrm{Map}_{\mathrm{Cat}_G}(\coprod_{G/H} \underline{I}, \underline{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Cat}_H}(\underline{I}, \mathrm{Res}^G_H \underline{\mathcal{C}})$$

we see that $\mathrm{Res}^G_H \underline{\mathcal{C}}$ is still Borel local, as claimed. The case of $b_!$ is similar, using now instead that $\mathrm{Res}^G_e \prod_{G/H} \simeq \prod_{g\in G/H} \mathrm{Res}^H_e$, so that $\prod_{G/H} \varphi$ is still a Borel equivalence. □

In the rest of the article, by the word *Borelification* we will mean either the functor b^* or b_*b^* in all its various incarnations at various categorical levels.

Observation 2.4.2. A point that we will be using several times to prove properties relating to b^* is that it is the unique dashed lift

$$\begin{tikzcd} & \mathrm{Fun}(BG, \mathrm{Cat}) \arrow[d, "\mathrm{fgt}"] \\ \mathrm{Cat}_G \arrow[ur, dashed, "b^*"] \arrow[r, "\mathrm{Res}^G_e"'] & \mathrm{Cat} \end{tikzcd}$$

associated to the G–equivariant functor Res^G_e (with trivial G–equivariance everywhere).

Proposition 2.4.3 (Omnibus basics of Borelification)**.** *Let $\underline{J} \in \mathrm{Cat}_G$, $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}_G$ and $\mathcal{D} \in \underline{\mathrm{Bor}}(\mathrm{Cat})$.*

1. *There is a natural equivalence $b^*\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{C}}) \simeq \mathrm{Fun}(b^*\underline{J}, b^*\underline{\mathcal{C}})$ of objects in $\underline{\mathrm{Bor}}(\mathrm{Cat})$,*
2. *the adjunction unit $\underline{\mathrm{Fun}}(\underline{J}, b_*\mathcal{D}) \to b_*b^*\underline{\mathrm{Fun}}(\underline{J}, b_*\mathcal{D})$ is an equivalence,*
3. *if $\mathcal{D}$ has finite (co)products, then $b_*\mathcal{D} \in \mathrm{Cat}_G$ is a G–category with finite indexed (co)products. In the case of admitting products, we furthermore have a natural equivalence $\underline{\mathrm{CMon}}_G(b_*\mathcal{D}) \simeq b_* \mathrm{CMon}(\mathcal{D})$,*
4. *if $\mathcal{D}$ is moreover semiadditive as an object in* Cat*, then $b_*\mathcal{D} \in \mathrm{Cat}_G$ is a G–semiadditive G–category.*

Proof. For part (1), note that we have both commuting squares

$$\begin{tikzcd} \mathrm{Cat}_G \arrow[r, "\mathrm{Res}^G_e"] \arrow[d, "\underline{\mathrm{Fun}}(\underline{J}{,}-)"'] & \mathrm{Cat} \arrow[d, "\mathrm{Fun}(\mathrm{Res}^G_e\,\underline{J}{,}-)"] \\ \mathrm{Cat}_G \arrow[r, "\mathrm{Res}^G_e"] & \mathrm{Cat} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Fun}(BG,\mathrm{Cat}) \arrow[r, "\mathrm{Fun}(b^*\underline{J}{,}-)"] \arrow[d, "\mathrm{fgt}"] & \mathrm{Fun}(BG,\mathrm{Cat}) \arrow[d, "\mathrm{fgt}"] \\ \mathrm{Cat} \arrow[r, "\mathrm{Fun}(\mathrm{Res}^G_e\,\underline{J}{,}-)"] & \mathrm{Cat} \end{tikzcd}$$

Here, all categories and functors in the left square are endowed with the trivial G–equivariant structure. Hence, by Observation 2.4.2, the bottom left composite of the left square lifts to the functor $b^*\underline{\mathrm{Fun}}(\underline{J},-)\colon \mathrm{Cat}_G \to \mathrm{Fun}(BG,\mathrm{Cat})$. On the other hand, the same observation together with the right commuting square yields that the unique lift to $\mathrm{Fun}(BG,\mathrm{Cat})$ of the top right composite of the left square is given by $\mathrm{Fun}(b^*\underline{J}, b^*-)\colon \mathrm{Cat}_G \to \mathrm{Fun}(BG,\mathrm{Cat})$. This gives part (1).

For part (2), we just need to show that $\underline{\mathrm{Fun}}(\underline{J}, b_*\mathcal{D})$ is Borel local, and this is an immediate consequence of the definition and unwinding adjunctions, using also the fact that b^* preserves products (for example because it has a left adjoint $b_!$).

For point (3), we deal with the case of products since that of coproducts will be dual. Let $H \leq G$ and $w\colon G/H \to G/G$ be the unique equivariant map. This then induces a G–equivariant map $w^*\colon \mathcal{D} \simeq \mathrm{Fun}(G/G,\mathcal{D}) \to \mathrm{Fun}(G/H,\mathcal{D}) \simeq \prod_{G/H}\mathrm{Res}^G_H\mathcal{D}$. Since $\mathcal{D}$ has finite products and since w^* was G–equivariant, this map admits a right adjoint w_* which furthermore can be endowed with a canonical G–equivariant structure with G–equivariant (co)units by Fact 2.2.8. In particular, this induces, upon applying $(-)^{hG}$, the adjunction

$$(b_*\mathcal{D})^G = \mathcal{D}^{hG} \underset{w_*}{\overset{w^*}{\rightleftarrows}} (\textstyle\prod_{G/H}\mathrm{Res}^G_H\mathcal{D})^{hG} \simeq \mathcal{D}^{hH} = (w_*w^*b_*\mathcal{D})^G$$

To see that this can be assembled to a G–adjunction $w^*\colon b_*\mathcal{D} \rightleftharpoons w_*w^*b_*\mathcal{D} : w_*$, we need to show by Proposition 2.1.13 that w_* commutes with restrictions under the appropriate Beck–Chevalley transformation. The key to this is the double coset decomposition of finite G–sets. More precisely, let $K \leq G$ and $f\colon G/K \to G/G$. Then by the double coset decomposition we have the left pullback diagram of finite G–sets

$$\begin{tikzcd} \coprod_{g\in K\backslash G/H} G/K^g\cap H \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G/H \arrow[d, "w"] \\ G/K \arrow[r, "f"] & G/G \end{tikzcd} \qquad \begin{tikzcd} \prod_{g\in K\backslash G/H}\prod_{G/K^g\cap H}\mathrm{Res}^G_{G/K^g\cap H}\mathcal{D} \arrow[d, "w_*"'] & \prod_{G/H}\mathrm{Res}^G_H\mathcal{D} \arrow[l, "f^*"'] \arrow[d, "w_*"] \\ \prod_{G/K}\mathrm{Res}^G_K\mathcal{D} \arrow[ur, phantom, "\Leftarrow"] & \mathcal{D} \arrow[l, "f^*"] \end{tikzcd}$$

Applying $\mathrm{Fun}(-,\mathcal{D})$, we obtain the adjointed square on the right of G–equivariant maps and transformation, again by Fact 2.2.8. This commutes by [Lur17, Lem. 6.1.6.3]. Finally, applying $(-)^{hG}$ to the right square gives us the commutation of the Beck–Chevalley square in the aforementioned criterion, as required.

For the second part of (3), since $\mathrm{Res}^G_e\,\underline{\mathrm{CMon}}_G(b_*\mathcal{D}) \simeq \mathrm{CMon}(\mathcal{D}) \in \mathrm{Cat}$, we see by the same argument as for part (1) that $b^*\underline{\mathrm{CMon}}_G(b_*\mathcal{D}) \simeq \mathrm{CMon}(\mathcal{D}) \in \mathrm{Fun}(BG,\mathrm{Cat})$. Hence, we need only to argue that $\underline{\mathrm{CMon}}_G(b_*\mathcal{D})$ is Borel local. So let $\varphi\colon \underline{I} \to \underline{J}$ be a Borel equivalence. Recall from Definition 2.1.24 that $\underline{\mathrm{CMon}}_G(b_*\mathcal{D}) \coloneqq \underline{\mathrm{Fun}}^{\mathrm{sadd}}(\underline{\mathrm{Fin}}_*, b_*\mathcal{D})$. Now, simply consider the natural equivalences

$$\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathrm{Fun}}^{\mathrm{sadd}}(\underline{\mathrm{Fin}}_*, b_*\mathcal{D})) \simeq \underline{\mathrm{Fun}}^{\mathrm{sadd}}(\underline{\mathrm{Fin}}_*, \underline{\mathrm{Fun}}(\underline{J}, b_*\mathcal{D})) \simeq \underline{\mathrm{Fun}}^{\mathrm{sadd}}(\underline{\mathrm{Fin}}_*, b_*\,\mathrm{Fun}(b^*\underline{J},\mathcal{D}))$$

to conclude, where the second equivalence is by points (1) and (2).

Lastly, point (4) is simply because, as explained in the proof of part (3), the indexed (co)product for $b_*\mathcal{D}$ are induced by the left and right adjoint of the G–equivariant restriction functor $\mathcal{D} \to \prod_{G/H}\mathrm{Res}^G_H\mathcal{D}$. Hence, forgetting the G–equivariant structure on the G–semiadditivity norm map (6) gives the usual semiadditivity norm map associated to the set of size $|G/H|$, which is an equivalence by hypothesis. $\square$

Proposition 2.4.4. *Applying the transformation* $\mathrm{fgt}\colon \underline{\mathrm{CMon}}_G(-) \to \mathrm{id}(-)$ *from Construction 2.1.27 to the* $b^* \dashv b_*$ *G–Bousfield localisation of Proposition 2.4.1 yields the commuting square of G–Bousfield localisations (i.e. both the* (b^*, fgt) *and* (b_*, fgt) *squares commute)*

$$\begin{tikzcd} \underline{\mathrm{CMon}}_G(\underline{\mathrm{Cat}}) \arrow[r, shift left, "b^*"] \arrow[d, "\mathrm{fgt}"'] & \underline{\mathrm{Bor}}(\mathrm{CMon}(\mathrm{Cat})) \arrow[l, shift left, "b_*"] \arrow[d, "\mathrm{fgt}"] \\ \underline{\mathrm{Cat}}_G \arrow[r, shift left, "b^*"] & \underline{\mathrm{Bor}}(\mathrm{Cat}) \arrow[l, shift left, "b_*"] \end{tikzcd} \tag{11}$$

Proof. By $b_! \dashv b^* \dashv b_*$ from Proposition 2.4.1, both b^* and b_* preserve finite indexed products. Since $\underline{\mathrm{CMon}}_G(-)$ is functorial on finite indexed product–preserving functors and since the fgt transformation is implemented by precomposition from Construction 2.1.27, we immediately obtain a commuting square of G–Bousfield localisations. All that is left to argue is that $\underline{\mathrm{CMon}}_G(\underline{\mathrm{Bor}}(\mathrm{Cat})) \simeq \underline{\mathrm{Bor}}(\mathrm{CMon}(\mathrm{Cat}))$: this is precisely supplied by Proposition 2.4.3 (3). $\square$

Remark 2.4.5. Recall that by definition, $\mathrm{Mack}_G(\mathrm{Cat}) \coloneqq \mathrm{Fun}^{\times}(\mathrm{Span}(G), \mathrm{Cat})$. It is then standard that we have $\mathrm{Mack}_G(\mathrm{Cat}) \simeq \mathrm{Mack}_G(\mathrm{CMon}(\mathrm{Cat}))$: to wit, since $\mathrm{Span}(G)$ is semiadditive by [Bar17, Prop. 4.3, Ex. B], we may immediately deduce the fact for example from [HW21, Thm. II.19]. Evaluating the top horizontal G–adjunction from (11) at G/G and using that $\mathrm{CMon}_G(\underline{\mathrm{Cat}}) \simeq \mathrm{Mack}_G(\mathrm{Cat})$ from [Nar16, Thm. 6.5] yields the Bousfield localisation

$$b^*\colon\ \mathrm{Mack}_G(\mathrm{Cat}) \simeq \mathrm{Mack}_G(\mathrm{CMon}(\mathrm{Cat})) \rightleftarrows \mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat})) : b_*$$

This then recovers [BGS20, Cons. 8.1], and one may indeed view Proposition 2.4.4 as a parametrised enhancement of the cited result.

Observation 2.4.6. Suppose we are given an object $\mathcal{D}^{\otimes} \in \mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat}))$. Then by Theorem 2.1.32 (1), we know that $b_*\mathcal{D}^{\otimes} \in \mathrm{CMon}_G(\underline{\mathrm{Cat}}) \simeq \mathrm{Mack}_G(\mathrm{Cat})$ may now be viewed as a G–symmetric monoidal category. We explain now a concrete description of the multiplicative norm functors on $b_*\mathcal{D}^{\otimes}$. First of all, because $\mathrm{CMon}(\mathrm{Cat})$ is semiadditive, we get that $\underline{\mathrm{Bor}}(\mathrm{CMon}(\mathrm{Cat}))$ is G–semiadditive by Proposition 2.4.3 (4). In particular, this means that for any G–object $\mathcal{D}^{\otimes} \in \underline{\mathrm{Bor}}(\mathrm{CMon}(\mathrm{Cat}))$ and for any $H \leq G$, we have the counit

$$\prod_{G/H} \mathrm{Res}^G_H \mathcal{D}^{\otimes} \xrightarrow{\varepsilon_{\mathcal{D}}} \mathcal{D}^{\otimes}$$

from the adjunction $\prod_{G/H} \simeq \coprod_{G/H}\colon\ \mathrm{Fun}(BH, \mathrm{CMon}(\mathrm{Cat})) \rightleftharpoons \mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat})) : \mathrm{Res}^G_H$. As indicated in the proof of Proposition 2.4.3 (4), upon forgetting the G–actions, the adjunction counit is given by the multiplication map $\otimes\colon \prod_{|G/H|} \mathcal{D} \to \mathcal{D}$. Now note that an object in $\prod_{G/H} \mathrm{Res}^G_H \mathcal{D}$ admits an explicit description given by a tuple $(X_g)_{g \in G/H}$ and so an object in $(\prod_{G/H} \mathrm{Res}^G_H \mathcal{D})^{hG} \simeq \mathcal{D}^{hH}$ can be described also as the tuple $(gX)_{g\in G/H}$ where $X \in \mathcal{D}^{hH}$. Since $\varepsilon_{\mathcal{D}}$ is G–equivariant, we then get the map

$$(\prod_{G/H} \mathrm{Res}^G_H \mathcal{D})^{hG} \simeq \mathcal{D}^{hH} \xrightarrow{\varepsilon_{\mathcal{D}}^{hG}} \mathcal{D}^{hG} \quad :: \quad X \mapsto \bigotimes_{g \in G/H} gX$$

Now, by virtue of the commuting square of G–Bousfield localisations from Proposition 2.4.4, we get from Lemma 2.2.6 an identification of $b_*\varepsilon_{\mathcal{D}}$ with the counit in $\underline{\mathrm{CMon}}_G(\underline{\mathrm{Cat}})$

$$\prod_{G/H} \mathrm{Res}^G_H b_*\mathcal{D}^{\otimes} \xrightarrow{\varepsilon_{b_*\mathcal{D}}} b_*\mathcal{D}^{\otimes}$$

Since this adjunction counit *is* the map encoding the multiplicative norm of an object in $\mathrm{CMon}_G(\underline{\mathrm{Cat}}) \simeq \mathrm{Cat}^{\otimes}_G$, all in all, we see that the G–symmetric monoidal category $b_*\mathcal{D}^{\otimes}$ has G/H–norms given by the formula $X \mapsto \bigotimes_{g\in G/H} gX$.

Because pointwise symmetric monoidal structures are merely cotensors instead of being internal hom objects, the next two lemmas will require a proof separate from, but very much in the spirit of, that of Proposition 2.4.3.

Lemma 2.4.7. *Let $\underline{J} \in \mathrm{Cat}_G$ and $\underline{\mathcal{D}}^{\otimes} \in \underline{\mathrm{CMon}}_G(\underline{\mathrm{Cat}})$. Then we have an equivalence $b^*\underline{\mathrm{Fun}}(\underline{J}, \underline{\mathcal{D}}^{\otimes}) \simeq \mathrm{Fun}(b^*\underline{J}, b^*\underline{\mathcal{D}}^{\otimes})$ in $\mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat}))$.*

Proof. By [Sha23, Prop. 9.7], we see that the pairing construction of the pointwise G–symmetric monoidal structure from [NS22, §3.3] is compatible with restrictions (for an example argument, see the proof of [NS22, Prop. 3.2.2]). Hence, we have the left commuting square

$$\begin{tikzcd} \mathrm{CMon}_G(\underline{\mathrm{Cat}}) \arrow[r, "\mathrm{Res}^G_e"] \arrow[d, "\underline{\mathrm{Fun}}(\underline{J},-)"'] & \mathrm{CMon}(\mathrm{Cat}) \arrow[d, "\mathrm{Fun}(\mathrm{Res}^G_e \underline{J},-)"] \\ \mathrm{CMon}_G(\underline{\mathrm{Cat}}) \arrow[r, "\mathrm{Res}^G_e"] & \mathrm{CMon}(\mathrm{Cat}) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CMon}(\mathrm{Cat})^{BG} \arrow[r, "\mathrm{Fun}(b^*\underline{J},-)"] \arrow[d, "\mathrm{fgt}"] & \mathrm{CMon}(\mathrm{Cat})^{BG} \arrow[d, "\mathrm{fgt}"] \\ \mathrm{CMon}(\mathrm{Cat}) \arrow[r, "\mathrm{Fun}(\mathrm{Res}^G_e \underline{J},-)"] & \mathrm{CMon}(\mathrm{Cat}) \end{tikzcd}$$

Furthermore, the category $\mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat}))$ is also cotensored over $\mathrm{Fun}(BG, \mathrm{Cat})$ under the pointwise symmetric monoidal structure and so we also have the right commuting square in the diagram above. Now all maps in sight in the left commuting square are moreover G–equivariant with the trivial G–equivariant structures on all the categories. Hence, the bottom composite of the left square lifts uniquely to the map $b^*\underline{\mathrm{Fun}}(\underline{J}, -)\colon \mathrm{CMon}_G(\underline{\mathrm{Cat}}) \to \mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat}))$ whereas the top composite of the left square lifts uniquely to a map $\mathrm{Fun}(b^*\underline{J}, b^*-)\colon \mathrm{CMon}_G(\underline{\mathrm{Cat}}) \to \mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat}))$ (here we have also used the existence of the right commuting square). Hence, all in all, we have obtained an equivalence of functors $b^*\underline{\mathrm{Fun}}(\underline{J}, -) \simeq \mathrm{Fun}(b^*\underline{J}, b^*-)$ as required. ☐

Recall our notation of $s\colon * \hookrightarrow \mathcal{O}_G$ being the inclusion of the final object G/G from Observation 2.1.10.

Lemma 2.4.8. *Let $\mathcal{D}^{\otimes} \in \mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat}))$ and $I \in \mathrm{Cat}$. Then there is a natural map $\underline{\mathrm{Fun}}(s_!I, b_*\mathcal{D}^{\otimes}) \to b_*\,\mathrm{Fun}(I, \mathcal{D}^{\otimes})$ in $\mathrm{CMon}_G(\underline{\mathrm{Cat}})$ which is an equivalence.*

Proof. Since $b^*s_! \simeq \mathrm{triv}_G\colon \mathrm{Cat} \to \mathrm{Fun}(BG, \mathrm{Cat})$ and $b^*b_* \simeq \mathrm{id}$, by Lemma 2.4.7 we have an equivalence $b^*\underline{\mathrm{Fun}}(s_!I, b_*\mathcal{D}^{\otimes}) \simeq \mathrm{Fun}(I, \mathcal{D}^{\otimes})$. Adjointing this over we obtain the claimed map $\underline{\mathrm{Fun}}(s_!I, b_*\mathcal{D}^{\otimes}) \to b_*\,\mathrm{Fun}(I, \mathcal{D}^{\otimes})$ in $\mathrm{CMon}_G(\underline{\mathrm{Cat}})$. Since the composite $\mathrm{CMon}_G(\underline{\mathrm{Cat}}) \to \mathrm{Cat}_G \xrightarrow{\prod_{H\leq G}(-)^H} \prod_{H\leq G}\mathrm{Cat}$ is conservative, it is enough to show that the said map is an equivalence upon passing to fixed points for all subgroups of G. Without loss of generality, in order to keep notations to a minimum, we may just show it for G–fixed points since the other fixed points can be dealt with similarly after restriction. In this case, we need to show that the map in Cat

$$s^*\underline{\mathrm{Fun}}(s_!I, b_*\mathcal{D}) \longrightarrow s^*b_*\,\mathrm{Fun}(I, \mathcal{D})$$

is an equivalence. The target is given by $s^*b_*\,\mathrm{Fun}(I, \mathcal{D}) \simeq \mathrm{Fun}(I, \mathcal{D})^{hG}$, whereas the source is, via Observation 2.1.10, given by $s^*\underline{\mathrm{Fun}}(s_!I, b_*\mathcal{D}) \simeq \mathrm{Fun}(I, s^*b_*\mathcal{D}) \simeq \mathrm{Fun}(I, \mathcal{D}^{hG})$, and it can be checked easily that the map is also the canonical one implementing the limit exchange equivalence $\mathrm{Fun}(I, \mathcal{D}^{hG}) \xrightarrow{\simeq} \mathrm{Fun}(I, \mathcal{D})^{hG}$. This concludes the proof of the lemma. ☐

Proposition 2.4.9. *Let $\underline{\mathcal{C}}^{\otimes} \in \mathrm{CMon}_G(\underline{\mathrm{Cat}})$ and $\mathcal{D}^{\otimes} \in \mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat}))$. There is a natural equivalence of categories $\mathrm{Fun}^{\otimes}_{\underline{G}}(\underline{\mathcal{C}}^{\otimes}, b_*\mathcal{D}^{\otimes}) \simeq \mathrm{Fun}^{\otimes}(b^*\underline{\mathcal{C}}^{\otimes}, \mathcal{D}^{\otimes})^{hG}$.*

Proof. Fix $I \in \mathrm{Cat}$. Consider the following sequence of natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Cat}}\left(I, s^*\underline{\mathrm{Fun}}^{\otimes}(\underline{\mathcal{C}}^{\otimes}, b_*\mathcal{D}^{\otimes})\right) &\simeq \mathrm{Map}_{\mathrm{Cat}_G}\left(s_! I, \underline{\mathrm{Fun}}^{\otimes}(\underline{\mathcal{C}}^{\otimes}, b_*\mathcal{D}^{\otimes})\right)\\
&\simeq \mathrm{Map}_{\mathrm{CMon}_G(\underline{\mathrm{Cat}})}\left(\underline{\mathcal{C}}^{\otimes}, \underline{\mathrm{Fun}}(s_! I, b_*\mathcal{D}^{\otimes})\right)\\
&\simeq \mathrm{Map}_{\mathrm{CMon}_G(\underline{\mathrm{Cat}})}\left(\underline{\mathcal{C}}^{\otimes}, b_*\,\mathrm{Fun}(I, \mathcal{D}^{\otimes})\right)\\
&\simeq \mathrm{Map}_{\mathrm{CMon}(\mathrm{Cat})^{BG}}\left(b^*\underline{\mathcal{C}}^{\otimes}, \mathrm{Fun}(I, \mathcal{D}^{\otimes})\right)\\
&\simeq \mathrm{Map}_{\mathrm{CMon}(\mathrm{Cat})}\left(b^*\underline{\mathcal{C}}^{\otimes}, \mathrm{Fun}(I, \mathcal{D}^{\otimes})\right)^{hG}\\
&\simeq \mathrm{Map}_{\mathrm{Cat}}\left(I, \mathrm{Fun}^{\otimes}(b^*\underline{\mathcal{C}}^{\otimes}, \mathcal{D}^{\otimes})^{hG}\right)
\end{aligned}$$

where the second and sixth equivalence are by virtue of the universal property from Recollection 2.1.33, the third by Lemma 2.4.8, and the fifth by how the mapping space in $\mathrm{CMon}(\mathrm{Cat})^{BG}$ is computed. Since by definition, $s^*\underline{\mathrm{Fun}}^{\otimes}(\underline{\mathcal{C}}^{\otimes}, b_*\mathcal{D}^{\otimes}) = \mathrm{Fun}^{\otimes}_{G}(\underline{\mathcal{C}}^{\otimes}, b_*\mathcal{D}^{\otimes})$, we obtain the desired equivalence. □

We now distil all that we have done in this subsection into the following principle which establishes an abstract but very important link between G–categories and their underlying category with G–action. We will use the notation $\underline{\mathrm{Bor}}$ for the functor $b_* \colon \mathrm{Bor}(\mathrm{Cat}) \hookrightarrow \underline{\mathrm{Cat}}_G$ for intuitive appeal. We thank Asaf Horev for discussions leading to it, especially in teaching us the trick of using symmetric monoidal envelopes.

Theorem 2.4.10 (Monoidal Borelification principle)**.** *Let $\underline{\mathcal{C}}^{\otimes} \in \mathrm{CMon}_G(\underline{\mathrm{Cat}}) \simeq \mathrm{Mack}_G(\mathrm{Cat})$ be a G–symmetric monoidal category and $\mathcal{D}^{\otimes} \in \mathrm{Fun}(BG, \mathrm{CMon}(\mathrm{Cat}))$ be a symmetric monoidal category with a G–action. Then:*

1. *The G–category $\underline{\mathrm{Bor}}(\mathcal{D}) \in \mathrm{Cat}_G$ canonically refines to a G–symmetric monoidal category $\underline{\mathrm{Bor}}(\mathcal{D}^{\otimes}) \in \mathrm{CMon}_G(\underline{\mathrm{Cat}})$. The multiplicative norm map $\mathrm{N}^G_H \colon \mathcal{D}^{hH} \to \mathcal{D}^{hG}$ can be concretely described as follows: for $X \in \mathcal{D}^{hH}$ a H–object in $\underline{\mathrm{Bor}}(\mathcal{D})$, the G–object $\mathrm{N}^G_H X \in \mathcal{D}^{hG}$ is given by $\bigotimes_{g\in G/H} gX$,*
2. *Writing $\mathcal{C}_e \in \mathrm{Fun}(BG, \mathrm{Cat})$ for the value of $\underline{\mathcal{C}} \in \mathrm{Cat}_G$ at G/e, the adjunction unit $\underline{\mathcal{C}} \to \underline{\mathrm{Bor}}(\mathcal{C}_e)$ of Proposition 2.4.1 canonically refines to a G–symmetric monoidal functor $\underline{\mathcal{C}}^{\otimes} \to \underline{\mathrm{Bor}}(\mathcal{C}_e^{\otimes})$.*
3. *There is a natural equivalence $\mathrm{CAlg}_G\big(\underline{\mathrm{Bor}}(\mathcal{D}^{\otimes})\big) \simeq \mathrm{CAlg}(\mathcal{D}^{\otimes})^{hG}$.*

Proof. Part (1) is by the fact that the (b_*, fgt)–square in Proposition 2.4.4 commutes, and the description of the norm is by Observation 2.4.6. Part (2) is by the fact that we have a commuting square of G–Bousfield localisations from Proposition 2.4.4. And finally, for part (3), we just compute:

$$\begin{aligned}
\mathrm{CAlg}_G(\underline{\mathrm{Bor}}(\mathcal{D}^{\otimes})) &\simeq \mathrm{Fun}^{\otimes}_{G}(\underline{\mathrm{Env}}(\underline{\mathrm{Fin}}_*), \underline{\mathrm{Bor}}(\mathcal{D}^{\otimes}))\\
&\simeq \mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathrm{Fin}_*), \mathcal{D}^{\otimes})^{hG}\\
&\simeq \mathrm{CAlg}(\mathcal{D}^{\otimes})^{hG}
\end{aligned}$$

as required, where the first and last equivalences are by Theorem 2.1.32 (2) and the second equivalence is by Proposition 2.4.9. □

Remark 2.4.11. The structure of G–commutative algebra objects and morphisms thereof are often tricky to construct. Part (3) of the theorem guarantees us, however, that at least when the G–category involved is Borel, such algebras and their morphisms are nothing but algebras and morphisms in the underlying category equipped with a G–action. By further applying suitable G–lax symmetric monoidal functors on these Borel G–commutative algebras, we may construct out of them many interesting non–Borel examples of G–commutative algebras. We refer the reader to §4.4 for an illustration of this strategy.

Remark 2.4.12. While we have not pursued it here so as not to obfuscate the general exposition and since we will not be needing it for our purposes, we believe that the notion of Borel objects and its attendant monoidality theory above can be developed more generally for any atomic orbital base category $\mathcal{T}$ equipped with a full subcategory $b\colon \mathcal{B} \hookrightarrow \mathcal{T}$ which is a *sieve*, i.e. if we are given a morphism $C \to X$ in $\mathcal{T}$ where $X \in \mathcal{B}$, then $C \in \mathcal{B}$ too. The main point is that this will allow the proof of Proposition 2.4.1 to go through, from which much else should follow via careful analyses of all the notions involved.

Remark 2.4.13. The theorem above is a slight expansion and strengthening of [Hil22a, Thm. 3.3.4, Prop. 3.3.6] from the author's thesis. Since then, an article [Yan23] of Lucy Yang's has appeared that gave a concrete description of C_p–$\mathbb{E}_\infty$–algebras which in particular also yields Theorem 2.4.10 (3) in the special of $G = C_p$.

2.5 Perfect–stable categories and Mackey functors

In this subsection, we will work out some basic categorical properties of the $\mathcal{T}$–category $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ of $\mathcal{T}$–perfect–stable categories. This will be the domain of the $\mathcal{T}$–parametrised K–theory functor we consider in the sequel. The highlight here is that, as expected, $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ will be shown to be a $\mathcal{T}$–semiadditive–presentable $\mathcal{T}$–category (c.f. Corollary 2.5.8). Moreover, we will show in Theorem 2.5.11 how this internally defined $\mathcal{T}$–category relates to $\mathcal{T}$–Mackey functors valued in $\mathrm{Cat}^{\mathrm{perf}}$. Apart from providing a psychological reassurance that $\mathcal{T}$–perfect–stable categories are a reasonable notion, the aforementioned relationship will also allow us to transport results about split Verdier sequences on $\mathrm{Cat}^{\mathrm{perf}}$ to our setting as well as relate our notion of $\mathcal{T}$–parametrised algebraic K–theory with, for instance, the one considered in [BGS20; CMN+20].

To begin, let us first record some formalities on $\mathcal{T}$–semiadditivity.

Lemma 2.5.1. *Suppose $\underline{\mathcal{C}}$ is $\mathcal{T}$–pointed with finite indexed (co)products. Let $f\colon U \to V$ be in $\underline{\mathrm{Fin}}_{/V}$ and $Y \in \underline{\mathcal{C}}$. We then have an identification*

$$f_*f^*\eta_Y \simeq \eta_{f_*f^*Y}\colon f_*f^*Y \longrightarrow f_*f^*f_*f^*Y$$

Proof. This is an immediate consequence of Lemma 2.2.6, using the adjointable square

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, "f_*f^*"] \arrow[d, "f^*"'] & \underline{\mathcal{C}} \arrow[d, "f^*"'] \\ f_*f^*\underline{\mathcal{C}} \arrow[r, "f_*f^*"] \arrow[u, bend right, dashed, "f_*"'] & f_*f^*\underline{\mathcal{C}} \arrow[u, bend right, dashed, "f_*"'] \end{tikzcd}$$

coming from the fact that both the restriction functor $f^*\colon \underline{\mathcal{C}} \to f_*f^*\underline{\mathcal{C}}$ and indexed product functor $f_*\colon f_*f^*\underline{\mathcal{C}} \to \underline{\mathcal{C}}$ preserve indexed products. ☐

Observation 2.5.2. Suppose $\underline{\mathcal{C}}$ is $\mathcal{T}$–pointed with finite indexed products. Let $f\colon U \to V$ be in $\underline{\mathrm{Fin}}_{/V}$ and let $X \in f^*\underline{\mathcal{C}}$. Observe that X is a retract of f^*f_*X since, by the pullback decomposition from (5), we have $f^*f_*X \simeq X \times \overline{c}_*c^*X$. Hence, using the map $* \to \overline{c}_*c^*X \to *$, we can get a retraction

$$X \simeq X \times * \longrightarrow f^*f_*X \simeq X \times \overline{c}_*c^*X \xrightarrow{\pi} X \simeq X \times *$$

Corollary 2.5.3. *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–pointed category admitting finite indexed products. Let $f\colon U \to V$ be in $\underline{\mathrm{Fin}}_{/V}$ and let $X \in f^*\underline{\mathcal{C}}$. Then the composite*

$$f_*X \xrightarrow{\eta_{f_*X}} f_*f^*f_*X \xrightarrow{f_*\pi} f_*X$$

is an equivalence.

Proof. Since retractions of equivalences are equivalences, by Observation 2.5.2 which gives that X is a retract of f^*f_*X, it suffices to show the equivalence when $X = f^*Y$ for some $Y \in \underline{\mathcal{C}}$. But then, in this case, this composite is identified with

$$f_*f^*Y \xrightarrow{f_*f^*\eta_Y} f_*f^*f_*f^*Y \xrightarrow{f_*\pi} f_*f^*Y \tag{12}$$

by Lemma 2.5.1. Now by a simple unwinding of adjunctions, we see via the decomposition from (5) that $f^*\eta_Y : f^*Y \to f^*f_*f^*Y \simeq f^*Y \times \overline{c}_*c^*f^*Y$ has the effect of the identity map on the f^*Y component in $f^*f_*f^*Y$. Therefore, we see indeed that the composite (12) is indeed an equivalence. ∎

With these generalities explained, we now begin our categorical study of $\underline{\mathrm{Cat}}^{\mathrm{perf}}$ in earnest.

Proposition 2.5.4. *The $\mathcal{T}$–categories $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa}$ and $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ are $\mathcal{T}$–semiadditive, where the $\mathcal{T}$–products are created in $\widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$. In particular, we have that $\underline{\mathrm{Cat}}^{\mathrm{perf}}$ is $\mathcal{T}$–semiadditive and the faithful inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}} \subset \underline{\mathrm{Cat}}$ is closed under finite $\mathcal{T}$–products.*

Proof. We only show that $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ is $\mathcal{T}$–semiadditive. This would then imply that the $\mathcal{T}$–full subcategory $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa}$ is too, since $\mathcal{T}$–presentable-stables are closed under $\mathcal{T}$–products.

First, we show that $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ is ordinary semiadditive. Recall from Theorem 2.1.34 that we may view $\mathrm{Pr}_{\mathcal{T},L,\kappa}$ as a non–full subcategory of $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Pr}_{L,\kappa})$ consisting of those objects for which all the restriction functors have left adjoints and satisfy the left Beck–Chevalley condition, and the morphisms are those which satsify the left Beck–Chevalley condition. Since we already know that $\mathrm{Pr}_{L,\kappa}$ is semiadditive (cf. for instance [HL13, Ex. 4.3.11]), it suffices now to argue that $\mathrm{Pr}_{\mathcal{T},L,\kappa} \subset \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Pr}_{L,\kappa})$ creates finite (co)products by applying Lemma 2.2.11. Let $\underline{\mathcal{C}}_1, \underline{\mathcal{C}}_2 \in \mathrm{Pr}_{\mathcal{T},L,\kappa}$. The product and coproduct $\underline{\mathcal{C}}_1 \times \underline{\mathcal{C}}_2$ clearly still satisfies Beck–Chevalley, and the projection maps $\underline{\mathcal{C}}_1 \times \underline{\mathcal{C}}_2 \to \underline{\mathcal{C}}_i$ and inclusion maps $\underline{\mathcal{C}}_i \to \underline{\mathcal{C}}_1 \times \underline{\mathcal{C}}_2$ (defined using that $\underline{\mathcal{C}}_i$ had initial objects) also clearly satisfies Beck–Chevalley. These give condition (1), and to see condition (2), suppose we are given $h_i : \underline{\mathcal{D}} \to \underline{\mathcal{C}}_i$ and $f_i : \underline{\mathcal{C}}_i \to \underline{\mathcal{E}}$ all satisfying Beck–Chevalley. Then it is similarly easy to see that the maps $h_1 \times h_2 : \underline{\mathcal{D}} \to \underline{\mathcal{C}}_1 \times \underline{\mathcal{C}}_2$ and $f_1 \sqcup f_2 : \underline{\mathcal{C}}_1 \times \underline{\mathcal{C}}_2 \to \underline{\mathcal{E}}$ also satisfy Beck–Chevalley, whence condition (2) as wanted.

Next, we show that there is a canonical adjunction datum witnessing that $f_* \dashv f^*$ for every $f : W \to V$ in $\mathrm{Fin}_{\mathcal{T}}$. For this, simply observe the natural equivalences

$$\underline{\mathrm{Map}}_{\underline{\mathrm{Pr}}_{L,\kappa}}(f_*\underline{\mathcal{D}}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{Pr}}_{R,\kappa\text{-filt}}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) \simeq \underline{\mathrm{Map}}_{f_*f^*\underline{\mathrm{Pr}}_{R,\kappa\text{-filt}}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}}) \simeq \underline{\mathrm{Map}}_{f_*f^*\underline{\mathrm{Pr}}_{L,\kappa}}(\underline{\mathcal{D}}, f^*\underline{\mathcal{C}})$$

where the first and last equivalences is by Proposition 2.2.15, and the middle equivalence is by [Hil22b, Prop. 6.6.2]. Now write η, ε as the adjunction (co)unit for $f^* \dashv f_*$ in $\underline{\mathrm{Pr}}_{R,\kappa\text{-filt}}$ and $\widetilde{\eta}, \widetilde{\varepsilon}$ for the $f_* \dashv f^*$ (co)unit in $\underline{\mathrm{Pr}}_{L,\kappa}$. Tracing through the identifications above, we obtain that $\widetilde{\eta} \dashv \varepsilon$ and $\widetilde{\varepsilon} \dashv \eta$. Under these notations, the Beck–Chevalley equivalences $f^*f_* \simeq \mathrm{id} \sqcup \overline{c}_*c^*$ and $f^*f_* \simeq \mathrm{id} \times \overline{c}_*c^*$ are then implemented by

$$\widetilde{\mathrm{BC}} : \mathrm{id} \sqcup \overline{c}_*c^* \xrightarrow{(\mathrm{id} \sqcup \overline{c}_*c^*)\widetilde{\eta}^f} (\mathrm{id} \sqcup \overline{c}_*c^*)f^*f_* \simeq (\mathrm{id} \sqcup \overline{c}_*\overline{c}^*)f^*f_* \xrightarrow{\widetilde{\varepsilon}^{\overline{c}}_{f^*f_*}} f^*f_*$$

$$\mathrm{BC} : f^*f_* \xrightarrow{\eta^{\overline{c}}_{f^*f_*}} (\mathrm{id} \times \overline{c}_*\overline{c}^*)f^*f_* \simeq (\mathrm{id} \times \overline{c}_*c^*)f^*f_* \xrightarrow{(\mathrm{id} \times \overline{c}_*c^*)\varepsilon^f} \mathrm{id} \times \overline{c}_*c^*$$

Now since $\widetilde{\mathrm{BC}}$ was an equivalence, an inverse is given by the right adjoint, which may in turn be seen easily to be given by BC. Therefore, the composite $\mathrm{BC} \circ \widetilde{\mathrm{BC}} : \mathrm{id} \sqcup \overline{c}_*c^* \to \mathrm{id} \times \overline{c}_*c^*$ *is* the semiadditive equivalence $\sqcup \simeq \times$. In particular, we have the equivalence of maps

$$(\mathrm{id} \sqcup \emptyset) \simeq (\mathrm{id} \times \emptyset) \circ \mathrm{BC} \circ \widetilde{\mathrm{BC}} : \mathrm{id} \sqcup \overline{c}_*c^* \longrightarrow \mathrm{id} \tag{13}$$

Finally, to see that $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ is $\mathcal{T}$–semiadditive via $f_* \dashv f^*$, we need to show that the composite

$$f_* \xrightarrow{\eta_{f_*}} f_*f^*f_* \xleftarrow[\simeq]{f_*\widetilde{\mathrm{BC}}} f_*(\mathrm{id} \sqcup \overline{c}_*c^*) \xrightarrow{f_*(\mathrm{id} \sqcup \emptyset)} f_*$$

is an equivalence by definition of $\mathcal{T}$–semiadditivity. By (13), this composite is equivalent to the one of the form in Corollary 2.5.3, which is an equivalence. This completes the proof that $\underline{\mathrm{Pr}}_{L,\kappa}$ is $\mathcal{T}$–semiadditive. The last statement about $\underline{\mathrm{Cat}}^{\mathrm{perf}}$ follows immediately from Notation 2.1.38. $\square$

Lemma 2.5.5. *The inclusion $\underline{\mathrm{Pr}}_{\mathcal{T},R,\mathrm{st},\kappa\text{-filt}} \subset \underline{\widehat{\mathrm{Cat}}}$ is closed under arbitrary parametrised limits. In particular, $\underline{\mathrm{Pr}}_{\mathcal{T},L,\mathrm{st},\kappa}$ is $\mathcal{T}$–cocomplete.*

Proof. By the equivalence $\underline{\mathrm{Pr}}_{L,\mathrm{st},\kappa} \simeq \underline{\mathrm{Pr}}^{\mathrm{op}}_{R,\mathrm{st},\kappa\text{-filt}}$ from Proposition 2.2.15 and the $\mathcal{T}$–semiadditivity from Proposition 2.5.4, we know that $\underline{\mathrm{Pr}}_{R,\mathrm{st},\kappa\text{-filt}} \subset \underline{\widehat{\mathrm{Cat}}}$ is closed under finite indexed products. Hence, since arbitrary parametrised limits can be decomposed into arbitrary fibrewise limit and finite indexed products, we are left to argue in the case of arbitrary fibrewise limits. This can in turn be split up into showing the case of arbitrary products and pullbacks. We will only treat the case of pullbacks since that of arbitrary products is simpler.

We would like to apply Lemma 2.2.11. Since fibrewise limits in $\widehat{\mathrm{Cat}}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \widehat{\mathrm{Cat}})$ are computed pointwise and since we already know from the unparametrised case that $\mathrm{Pr}_{L,\mathrm{st},\kappa}, \mathrm{Pr}_{R,\mathrm{st},\kappa\text{-filt}} \subset \widehat{\mathrm{Cat}}$ are closed under limits, we know that $\widehat{\mathrm{Cat}}_{\mathcal{T}}$–pullbacks $\underline{\mathcal{C}}_1 \times_{\underline{\mathcal{C}}_3} \underline{\mathcal{C}}_2$ of objects in $\mathrm{Pr}_{R,\mathrm{st},\kappa\text{-filt}}$ are still in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Pr}_{L,\mathrm{st},\kappa})$. Using Proposition 2.2.7, we can easily check that $\underline{\mathcal{C}}_1 \times_{\underline{\mathcal{C}}_3} \underline{\mathcal{C}}_2$ still satisfies the left Beck–Chevalley conditions, and so by Theorem 2.1.34, we get that $\underline{\mathcal{C}}_1 \times_{\underline{\mathcal{C}}_3} \underline{\mathcal{C}}_2$ is an object in $\mathrm{Pr}_{\mathcal{T},R,\mathrm{st},\kappa\text{-filt}}$. Moreover, the projection maps $\underline{\mathcal{C}}_1 \times_{\underline{\mathcal{C}}_3} \underline{\mathcal{C}}_2 \to \underline{\mathcal{C}}_i$ for $i \in \{1,2,3\}$ are also easily seen to preserve fibrewise (co)limits and indexed products, whence condition (1) of Lemma 2.2.11. Since these projection maps preserves said (co)limits, this means that such (co)limits are computed pointwise in $\underline{\mathcal{C}}_1 \times_{\underline{\mathcal{C}}_3} \underline{\mathcal{C}}_2$ and hence condition (2) of the lemma is easily seen to be satisfied also, as required. The last statement is then an immediate consequence of the first statement and the equivalence $\underline{\mathrm{Pr}}_{L,\kappa} \simeq \underline{\mathrm{Pr}}^{\mathrm{op}}_{R,\kappa\text{-filt}}$ from Proposition 2.2.15. $\square$

Proposition 2.5.6. *The faithful inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}} \subset \underline{\mathrm{Cat}}$ is closed under arbitrary parametrised limits.*

Proof. Since parametrised limits can be decomposed into fibrewise limits and arbitrary indexed products by [Sha23, §12], by Proposition 2.5.4 we are left to show that the inclusion is closed under fibrewise limits. Concretely, since fibrewise limits are computed fibrewise by the dual of [Sha23, Cor. 5.9], we need to show that the faithful inclusion $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}} \subset \mathrm{Cat}_{\mathcal{T}}$ is closed under arbitrary limits.

To do so, we first show that limits of $\mathcal{T}$–perfect stable categories along $\mathcal{T}$–exact functors are again $\mathcal{T}$–perfect stable. Let $\underline{\mathcal{C}} \colon I \to \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$ be a diagram. Because limits in $\mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ are computed pointwise and since we know this statement in the unparametrised situation, we know already that $\lim_{a\in I} \underline{\mathcal{C}}_a$ is fibrewise perfect stable. Hence, we are reduced to showing that $\lim_{a\in I} \underline{\mathcal{C}}_a$ is $\mathcal{T}$–semiadditive. Without loss of generality, assume $\mathcal{T}$ has a final object T and let $f \colon V \to T$ be a map in $\mathcal{T}$. Since each $\underline{\mathcal{C}}_a$ is $\mathcal{T}$–semiadditive, we know that for all $a \in I$, the semiadditivity norm map in

$$\begin{tikzcd} \underline{\mathcal{C}}_a \arrow[r, phantom, "\simeq\Downarrow"] & f_*f^*\underline{\mathcal{C}}_a \arrow[l, bend right, "f_!"'] \arrow[l, bend left, "f_*"] \end{tikzcd}$$

is an equivalence. But since fibrewise limits along $\mathcal{T}$–exact functors preserve the indexed (co)product adjunctions ($\mathcal{T}$–exact functors satisfy the Beck–Chevalley condition in Proposition 2.2.7), zero objects, and ordinary finite biproducts (ie. the zero object and ordinary finite biproducts in $\lim_{a\in I} \underline{\mathcal{C}}_a$ are given by the ones in each $\underline{\mathcal{C}}_a$), we see that applying $\lim_{a\in I}$ to the diagram above yields the semiadditivity norm map equivalence

$$\begin{tikzcd} \lim_{a\in I}\underline{\mathcal{C}}_a \arrow[r, phantom, "\simeq\Downarrow"] & f_*f^*\lim_{a\in I}\underline{\mathcal{C}}_a \arrow[l, bend right, "f_!"'] \arrow[l, bend left, "f_*"] \end{tikzcd}$$

whence the $\mathcal{T}$–semiadditivity of $\lim_{a\in I}\underline{\mathcal{C}}_a$ as claimed.

To see that this has the correct universal property in $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$, we need to check the two conditions in Lemma 2.2.11. For condition (2), we need to argue that if we have a transformation of I–shaped diagrams $\varphi\colon \mathrm{const}_I\,\underline{\mathcal{D}} \to \underline{\mathcal{C}}$ in $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$, then the map $\varphi\colon \underline{\mathcal{D}} \to \lim_{a\in I}\underline{\mathcal{C}}_a$ induced by the universal property in $\mathrm{Cat}_{\mathcal{T}}$ is already $\mathcal{T}$–exact. As before, since this statement holds in the unparametrised setting, we are left with showing that the induced functor preserves finite $\mathcal{T}$–biproducts. To see this, letting $f\colon V \to T$ as in the preceding paragraph, we need to show that the left square in

$$\begin{tikzcd} f_*f^*\underline{\mathcal{D}} \arrow[r, "f_*"] \arrow[d, "f_*f^*\varphi"'] & \underline{\mathcal{D}} \arrow[d, "\varphi"] \\ f_*f^*\lim_{a\in I}\underline{\mathcal{C}}_a \arrow[r, "f_*"] & \lim_{a\in I}\underline{\mathcal{C}}_a \end{tikzcd} \qquad \begin{tikzcd} f_*f^*\underline{\mathcal{D}} \arrow[r, "f_*"] \arrow[d, "f_*f^*\varphi"'] & \underline{\mathcal{D}} \arrow[d, "\varphi"] \\ f_*f^*\underline{\mathcal{C}}_a \arrow[r, "f_*"] & \underline{\mathcal{C}}_a \end{tikzcd}$$

commutes. This is simply because for each $a \in I$, the right square commutes by assumption that everything in sight is $\mathcal{T}$–exact, from which we can conclude that the left square also commutes by applying $\lim_{a\in I}$ to the bottom horizontal map in the right square. Finally, for condition (1), by virtue of the previous paragraph, we are left to argue that the adjunction counit is $\mathcal{T}$–exact. Since arbitrary limits can be decomposed in terms of arbitrary products and pullbacks and the case of products is simple to see, we will only argue in the case of pullbacks. In this case, we need to argue that the projection maps $\underline{\mathcal{C}}_1 \times_{\underline{\mathcal{C}}_3} \underline{\mathcal{C}}_2 \to \underline{\mathcal{C}}_i$ preserve finite indexed products for $i \in \{1,2,3\}$. But this is clear since $\mathcal{T}$–biproducts in $\underline{\mathcal{C}}_1 \times_{\underline{\mathcal{C}}_3} \underline{\mathcal{C}}_2$ are created pointwise. □

Next, we mimic the techniques of [CDH+, §1.1] to prove:

Proposition 2.5.7. *The set $\{\underline{\mathrm{Sp}}^{\underline{\omega}}, \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{Sp}}^{\underline{\omega}})\}$ consists of ω–compact objects and is jointly conservative on $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. Thus, $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ is κ-compactly generated for all regular cardinals κ.*

Proof. Since compactness and joint conservativity are checked fibrewise, we show that $\{\underline{\mathrm{Sp}}^{\underline{\omega}}, \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{Sp}}^{\underline{\omega}})\}$ are ω–compact and jointly conservative on $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}_{/V}}$ for an arbitrary $V \in \mathcal{T}$. We claim that $\underline{\mathrm{Sp}}^{\underline{\omega}}$ and $\underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{Sp}}^{\underline{\omega}})$ corepresent the functors $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}_{/V}} \to \underline{\mathcal{S}}_{\mathcal{T}_{/V}}$

$$\underline{\mathcal{C}} \mapsto \underline{\mathcal{C}}^{\simeq} \quad \text{ and } \quad \underline{\mathcal{C}} \mapsto \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathcal{C}})^{\simeq} \tag{14}$$

respectively. We only show this for the second one since the first is easier:

$$\begin{aligned} \underline{\mathrm{Map}}_{\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}_{/V}}}\left(\underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{Sp}}^{\underline{\omega}}), \underline{\mathcal{C}}\right) &\simeq \underline{\mathrm{Map}}_{\mathrm{Pr}_{L,\mathrm{st},\omega}}\left(\underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{Sp}}), \underline{\mathrm{Ind}}_{\underline{\omega}}\underline{\mathcal{C}}\right) \\ &\simeq \underline{\mathrm{Fun}}^{L,\underline{\omega}}(\underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{Sp}}), \underline{\mathrm{Ind}}_{\underline{\omega}}\underline{\mathcal{C}})^{\simeq} \\ &\simeq \underline{\mathrm{Fun}}^{R,\underline{\omega}\text{-filt}}\left(\underline{\mathrm{Ind}}_{\underline{\omega}}\underline{\mathcal{C}}, \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{Sp}})\right)^{\simeq} \\ &\simeq \underline{\mathrm{Fun}}\Big(\Delta^1, \underline{\mathrm{Fun}}^{R,\underline{\omega}\text{-filt}}(\underline{\mathrm{Ind}}_{\underline{\omega}}\underline{\mathcal{C}}, \underline{\mathrm{Sp}})\Big)^{\simeq} \\ &\simeq \underline{\mathrm{Fun}}\Big(\Delta^1, \underline{\mathrm{Fun}}^{L,\underline{\omega}}(\underline{\mathrm{Sp}}, \underline{\mathrm{Ind}}_{\underline{\omega}}\underline{\mathcal{C}})\Big)^{\simeq} \\ &\simeq \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathcal{C}})^{\simeq} \end{aligned}$$

where the first equivalence is by Proposition 2.1.37; the third and fifth are by Proposition 2.2.5 and Theorem 2.1.35; the fourth by Notation 2.1.9; and the last is by Proposition 2.2.24. Since the two corepresented functors preserve ω–filtered colimits, we thus see that $\underline{\mathrm{Sp}}^{\underline{\omega}}$ and $\underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{Sp}}^{\underline{\omega}})$ are ω–compact objects. To see that the two functors of (14) are jointly conservative, suppose $\varphi : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a functor such that

$$\varphi : \underline{\mathcal{C}}^{\simeq} \xrightarrow{\simeq} \underline{\mathcal{D}}^{\simeq} \qquad \text{and} \qquad \varphi : \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathcal{C}})^{\simeq} \xrightarrow{\simeq} \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathcal{D}})^{\simeq}$$

are equivalences of $\mathcal{T}_{/V}$–spaces. In particular, the first equivalence implies that φ is $\mathcal{T}_{/V}$–essentially surjective. On the other hand, the fibre over $[W \to V]$ of $\underline{\mathrm{Fun}}(\Delta^1, \underline{\mathcal{C}})$ is $\mathrm{Fun}(\Delta^1, \mathcal{C}_W)$ and so the second equivalence together with the the formula for unparametrised mapping spaces as pullbacks $\mathrm{Fun}(\Delta^1, \mathcal{C}_W) \times_{\mathcal{C}_W^{\times 2}} \{*\}$ gives us that $\varphi : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is $\mathcal{T}_{/V}$–fully faithful. Therefore, φ is an equivalence, as wanted, and so by Proposition 2.2.14, we may conclude the proof. ◻

We may now summarise the preceding results in the following package.

Corollary 2.5.8. *The $\mathcal{T}$–category $\underline{\mathrm{Cat}}^{\mathrm{perf}}$ is $\mathcal{T}$–semiadditive–presentable. The parametrised limits in it are computed in $\underline{\mathrm{Cat}}$ and the parametrised colimits may be computed by the formula $\underline{\mathrm{colim}}_{\underline{J}}\partial \simeq (\underline{\lim}_{\underline{J}^{\mathrm{op}}}\widetilde{\underline{\mathrm{Ind}}\partial})^{\underline{\omega}}$, where $\widetilde{\underline{\mathrm{Ind}}}\partial \colon \underline{J}^{\mathrm{op}} \to \underline{\mathrm{Pr}}_{R,\kappa\text{-filt}} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ is the diagram obtained by passing to right adjoints.*

Proof. $\mathcal{T}$–semiadditivity is by Proposition 2.5.4. To see $\mathcal{T}$–presentability, by Lemma 2.5.5, we know that $\underline{\mathrm{Cat}}^{\mathrm{perf}} \simeq \underline{\mathrm{Pr}}_{L,\mathrm{st},\omega}$ is $\mathcal{T}$–cocomplete. This, together with Proposition 2.5.7 and [Hil22b, Thm. 6.1.2 (6)], then implies that $\underline{\mathrm{Cat}}^{\mathrm{perf}}$ is $\mathcal{T}$–presentable. That parametrised limits in $\underline{\mathrm{Cat}}^{\mathrm{perf}}$ are created in $\underline{\mathrm{Cat}}$ was shown in Proposition 2.5.6. Finally, the formula for parametrised colimits is an immediate consequence of Lemma 2.5.5 and the equivalences

$$\underline{\mathrm{Cat}}^{\mathrm{perf}} \underset{(-)^{\underline{\omega}}}{\overset{\mathrm{Ind}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \underline{\mathrm{Pr}}_{L,\mathrm{st},\omega} \xrightarrow{\ \simeq\ } \underline{\mathrm{Pr}}^{\mathrm{op}}_{R,\mathrm{st},\omega\text{-filt}}$$

where the second equivalence is by Proposition 2.2.15. ◻

Our next goal is to articulate the relationship between $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$ and $\mathcal{T}$–Mackey functors valued in $\mathrm{Cat}^{\mathrm{perf}}$. The basic ingredient will be the following evaluation functors.

Construction 2.5.9 (The evaluation functor)**.** Let $V \in \mathcal{T}$. Writing $s \colon * \hookrightarrow \mathcal{T}_{/V}$ for the inclusion of the final object, consider the solid part of the diagram

$$\begin{tikzcd} \mathrm{Cat}^{\mathrm{perf}} \arrow[r, hook] & \mathrm{Cat} \\ \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}_{/V}} \arrow[r, hook] \arrow[u, dashed, "s^*"] & \mathrm{Cat}_{\mathcal{T}_{/V}} \arrow[u, "s^*"'] \end{tikzcd} \tag{15}$$

where the hooked arrows are faithful functors. The functor s^* here implements the evaluation at $V \in \mathcal{T}$. Since objects in $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$ are in particular fibrewise idempotent–complete and stable, and morphisms are in particular fibrewise exact, we obtain the dashed lift as shown. Observe that the top horizontal inclusion and the solid s^* functors preserve all limits. Observe also that, on objects, $s_* \colon \mathrm{Cat} \to \mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ is concretely given by the functor which sends $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}} \subset \mathrm{Cat}$ to the object $\underline{\mathcal{C}} \coloneqq s_*\mathcal{C} \in \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ given by $\mathcal{C}$ at the final object in $\mathcal{T}$ and $*$ everywhere else.

Lemma 2.5.10. *For every $V \in \mathcal{T}$, the composite functor $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}} \xrightarrow{\mathrm{Res}} \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}_{/V}} \xrightarrow{s^*} \mathrm{Cat}^{\mathrm{perf}}$, where s^* is as constructed above, preserves arbitrary limits and colimits.*

Proof. Since the restriction functor is given by the global section of the $\mathcal{T}$–functor $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \xrightarrow{p^*} \underline{\mathrm{Fun}}(\underline{V}, \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})$ where $p\colon \underline{V} \to \underline{*}$ is the unique $\mathcal{T}$–functor to the final $\mathcal{T}$–category, it clearly preserves all limits and colimits. Hence, we are left to studying s^* for a fixed $V \in \mathcal{T}$. Thus without loss of generality, instead of writing $\mathcal{T}_{/V}$ everywhere, we just assume that $\mathcal{T}$ has a final object.

For the case of limits, by Proposition 2.5.6 the faithful inclusion $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}} \subset \mathrm{Cat}_{\mathcal{T}}$ is closed under limits. Thus, we see that all the solid arrows in (15) preserves limits, and so the dashed lift s^* also preserves limits as wanted.

For the case of colimits, we claim that the right adjoint s_* to the solid s^* functor in (15) restricts to a right adjoint $s_*\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$. To see this, note that the concrete description of the functor s_* from Construction 2.5.9 clearly yields a fibrewise stable and $\mathcal{T}$–semiadditive category (since all the proper restrictions are zero from Observation 2.1.18). It is similarly easy to see that morphisms in $\mathrm{Cat}^{\mathrm{perf}}$ (ie. exact functors) get sent to morphisms in $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$ (ie. fibrewise exact and preserve $\mathcal{T}$–products, the latter of which are all zero by the argument above). Hence, all in all, we obtain the restricted adjunction $s^*\colon \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}} \rightleftharpoons \mathrm{Cat}^{\mathrm{perf}} : s_*$, which implies that s^* preserves colimits, as desired. ∎

Given these, we are now ready to phrase the embedding of $\mathcal{T}$–perfect–stable categories into $\mathcal{T}$–Mackey functors valued in perfect–stable categories.

Theorem 2.5.11. *We have a conservative $\mathcal{T}$–faithful inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subset \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$. Moreover, this inclusion preserves and reflects parametrised (co)limits.*

Proof. We first construct the said $\mathcal{T}$–faithful functor. By definition we have the following solid non-full $\mathcal{T}$–faithful inclusions

$$\begin{tikzcd} \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \arrow[r, hook] \arrow[dr, hook, dashed] & \underline{\mathrm{Cat}}_{\mathcal{T}} = \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Cat}) \\ & \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \arrow[u, hook] \end{tikzcd}$$

which preserve finite $\mathcal{T}$–products: the top horizontal inclusion by Proposition 2.5.4 and the vertical inclusion since $\mathrm{Cat}^{\mathrm{perf}} \subset \mathrm{Cat}$ preserves limits. By Notation 2.1.38, we in fact have the dashed factorisation which must, by the preceding points, also preserve finite $\mathcal{T}$–products. Now by definition $\underline{\mathrm{CMon}}_{\mathcal{T}}(-) := \underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, -) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, -)$ and so applying $\underline{\mathrm{CMon}}_{\mathcal{T}}(-)$ and invoking [Hil22b, Cor. 3.4.6] we get a $\mathcal{T}$–faithful inclusion

$$\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subset \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$$

where we can dispense with the $\mathcal{T}$–semiadditivisation of the source by virtue of Proposition 2.1.28 and Proposition 2.5.4.

Now to see that it is conservative, simply note that we have the commuting triangle of categories

$$\begin{tikzcd} \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}} \arrow[d, hook] \arrow[drr, "\prod_{V\in\mathcal{T}} \mathrm{ev}_V"] & & \\ \mathrm{CMon}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \simeq \mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \arrow[rr, "\prod_{V\in\mathcal{T}} \mathrm{ev}_V"'] & & \prod_{V\in\mathcal{T}} \mathrm{Cat}^{\mathrm{perf}} \end{tikzcd} \tag{16}$$

where the diagonal map is conservative. Therefore, the vertical map must be conservative as well.

For the final statement, first note that by construction, the inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subset \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ preserves finite indexed products, and so also finite indexed (co)products by $\mathcal{T}$–semiadditivity of both source and target. Hence, by Lemma 2.2.12 the inclusion also reflects these. To deal with the fibrewise (co)limits, since all the restriction functors in sight preserve (co)limits, it suffices by [Sha23, Cor. 5.9] to argue that $\mathrm{Cat}^{\underline{\mathrm{perf}}}_{\mathcal{T}} \to \mathrm{CMon}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \simeq \mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ preserves and reflects arbitrary limits and colimits. Since (co)limits in $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ are computed pointwise by [Bar17, Cor. 6.7.1], the bottom horizontal evaluation map in (16) is conservative and preserves (co)limits. On the other hand, the diagonal functor in (16) preserves arbitrary (co)limits by Lemma 2.5.10. Combining these, we get that the vertical functor in (16) preserves arbitrary (co)limits in addition to being conservative from the previous paragraph, and hence by Lemma 2.2.12 it also reflects (co)limits, as wanted. $\square$

Remark 2.5.12. We now give an intuitive description of objects in $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \simeq \mathrm{CMon}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ which lie in $\mathrm{Cat}^{\underline{\mathrm{perf}}}_{\mathcal{T}}$. Let $\underline{\mathcal{C}} \in \mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ and let $f\colon W \to V$ be a map in $\mathcal{T}$. Write $f_{\#}\colon \mathcal{C}_W \to \mathcal{C}_V$ for the associated transfer map. From the pullback

$$\begin{tikzcd} W \coprod C \arrow[r, "\mathrm{id}\sqcup c"] \arrow[d, "\mathrm{id}\sqcup\overline{c}"'] \arrow[dr, phantom, "\lrcorner"] & W \arrow[d, "f"] \\ W \arrow[r, "f"] & V \end{tikzcd}$$

we obtain by the datum of a Mackey functor a decomposition $f^*f_{\#} \simeq \mathrm{id} \oplus \overline{c}_{\#}c^*$. Hence, by inclusion and projection, we obtain the following transformations

$$u\colon \mathrm{id} \Longrightarrow f^*f_{\#} \qquad \text{and} \qquad c\colon f^*f_{\#} \Longrightarrow \mathrm{id}$$

By unwinding the definitions in the proof of the theorem, we then see that $\underline{\mathcal{C}}$ lies in the non–full subcategory $\mathrm{Cat}^{\underline{\mathrm{perf}}}_{\mathcal{T}}$ if u exhibits $f_{\#} \dashv f^*$ and c exhibits $f^* \dashv f_{\#}$. These conditions *can* however be enforced provide we work with the $(\infty,2)$*–category* of spans. As such, we expect that an $(\infty,2)$-categorical version of spans and of Mackey functors should precisely yield $\mathrm{Cat}^{\underline{\mathrm{perf}}}_{\mathcal{T}}$.

For the purposes of our motivic analyses later, we will also record here the following:

Proposition 2.5.13. *Fix $\underline{\mathcal{C}} \in \mathrm{Pr}_{\mathcal{T},L,\kappa}$. The functor $\underline{\mathrm{Fun}}(-,\underline{\mathcal{C}})\colon \underline{\mathrm{Cat}}_{\mathcal{T}} \to \underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ taken along left Kan extensions preserve finite parametrised colimits.*

Proof. Since every finite parametrised colimit can be decomposed as the finite indexed coproducts and finite fibrewise colimits, we will split up the proof into these two cases. For the indexed coproducts, we just note that for a fixed $f\colon U \to V$ in $\underline{\mathrm{Fin}}_{/V}$, we have $\underline{\mathrm{Fun}}(f_!\underline{I},\underline{\mathcal{C}}) \simeq f_*f^*\underline{\mathrm{Fun}}(\underline{I},\underline{\mathcal{C}})$, and so since Proposition 2.5.4 gives that $f_*f^*\underline{\mathrm{Fun}}(\underline{I},\underline{\mathcal{C}})$ is also the indexed coproduct in $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$, we are done in this case. Since we have argued for an arbitrary $U \in \underline{\mathrm{Fin}}_{\mathcal{T}}$, this also covers the case of ordinary finite coproducts by setting $U = \coprod_{i=1}^n V$.

Hence, we are left with showing the case of pushouts. Suppose we have a pushout diagram

$$\begin{tikzcd} \underline{I} \arrow[r, "i"] \arrow[d, "k"'] & \underline{J} \arrow[d, "p"] \\ \underline{K} \arrow[r, "q"'] \arrow[ur, phantom, "\ulcorner"] & \underline{P} \end{tikzcd}$$

By Lemma 2.5.5, we have an equivalence $\underline{\mathrm{Fun}}(\underline{P},\underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}(\underline{J},\underline{\mathcal{C}}) \times_{\underline{\mathrm{Fun}}(\underline{I},\underline{\mathcal{C}})} \underline{\mathrm{Fun}}(\underline{K},\underline{\mathcal{C}})$ in $\underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}}$ using the restriction maps p^*, q^*, i^*, k^*, and so upon passing to left adjoints under $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa} \simeq \underline{\mathrm{Pr}}^{\mathrm{op}}_{\mathcal{T},R,\kappa\text{-filt}}$, we obtain the desired result. $\square$

3 The theory of parametrised cubes

In this section, we lay down the theory of parametrised cubes associated to atomic orbital base categories $\mathcal{T}$. As we shall see in Construction 3.1.5, this hypothesis on $\mathcal{T}$ will be exploited in an essential way to identify the "singletons" in a parametrised cube, which will in turn play a key role in our treatment of C_2–pushouts in the setting of equivariant K–theory in §4.3. The key insight is that the inclusions of the initial and final objects in Δ^1 will allow us to encode the idea of a "subset" in a purely abstract and categorical manner (where 1 means being in a "subset" and 0 means the converse). The advantage of this point of view is at least two–fold in that it not only lets one speak of cubes in a very general setting but also allows many proofs to be carried out using concise adjunction manipulations.

As far as we are aware, this use case of atomic orbitality of base categories is new and might represent a third expression – alongside that of $\mathcal{T}$–semiadditivity and $\mathcal{T}$–symmetric monoidality for which it was first designed by [BDG+16a] – of the algebraic richness of the atomic orbitality hypothesis. As alluded to in the introduction, the rather general and abstract treatment of cubes here will serve as the foundations for a theory of parametrised functor calculus, a subject that will be treated in forthcoming work.

We now outline the contents of this section: in §3.1, we will introduce the basic definitions and constructions of cubes; we will then provide a general utility (co)limit decomposition result Proposition 3.2.2 and record the interaction of parametrised tensor powers with cofibres as Proposition 3.2.8 in §3.2; lastly, we will specialise the general theory to the equivariant setting where we look at G/H–cubes when $|G/H| = 2$ in preparation for our K–theoretic applications in §4.3.

3.1 Basic notions

Definition 3.1.1. Let $w\colon W \to T$ be a map in $\underline{\mathrm{Fin}}_{\mathcal{T}}$ and let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category. We write $\underline{\Delta}^1 \in \mathrm{Cat}_T = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}_{/T}, \mathrm{Cat})$ for the constant $\mathcal{T}_{/T}$–category with value Δ^1. By *the parametrised w–cube* we will mean the $\mathcal{T}_{/T}$–category $w_*w^*\underline{\Delta}^1$ and by *a parametrised w–cube in $\underline{\mathcal{C}}$* we will mean a $\mathcal{T}$–functor $Q\colon w_*w^*\underline{\Delta}^1 \to \underline{\mathcal{C}}$.

Proposition 3.1.2. *The parametrised cubes $w_*w^*\underline{\Delta}^1$ are all parametrised posets, i.e. they belong to the full subcategory* $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Poset}) \subseteq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$. *In fact, these are fibrewise given by cubes of various dimensions.*

Proof. This is just because the inclusion $\mathrm{Poset} \subseteq \mathrm{Cat}_\infty$ preserves all limits (with left adjoint $\tau_{\leq -1}$) and hence in particular preserves products. On the other hand, we know that for any $\mathcal{C} \in \mathrm{Cat}_\infty$ admitting finite products, $\underline{\mathrm{Cofree}}(\mathcal{C})$ also have all indexed products coming from the products on $\mathcal{C}$. Therefore all in all, $\underline{\mathrm{Cofree}}(\mathrm{Poset}) \subseteq \underline{\mathrm{Cofree}}(\mathrm{Cat}_\infty)$ admits (and is closed under) indexed products. In particular, since $\underline{\Delta}^1$ is a parametrised poset, so is $w_*w^*\underline{\Delta}^1$ as required. The last statement is simply because these indexed products are computed as various products in Poset of Δ^1, which are cubes. ◻

Construction 3.1.3. Upon applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$, the join–slice adjunction (cf. for instance [Lan21, Cor. 1.4.17]) induces the parametrised join–slice adjunction

$$(-)^{\underline{\triangleleft}}\colon \mathrm{Cat}_{\mathcal{T}} \rightleftharpoons \mathrm{Cat}_{\mathcal{T},\underline{*}/} : (-)_{p/}$$

where $(-)_{p/}$ is the slice construction on a category equipped with a choice of object, i.e. $(\underline{\mathcal{D}}, d)_{/p} \coloneqq \underline{*} \times_{\underline{\mathcal{D}}} \underline{\mathcal{D}}^{\Delta^1}$ where the map $\underline{\mathcal{D}}^{\Delta^1} \to \underline{\mathcal{D}}$ is the source map. In particular, if $\underline{\mathcal{D}}$ has an initial object $*$, then any functor $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ will induce an extension $\underline{\mathcal{C}}^{\underline{\triangleleft}} \to \underline{\mathcal{D}}$.

We thank Sil Linskens for pointing out the need for the strict initial object assumption in the following result.

Proposition 3.1.4. *Let $\underline{\mathcal{D}}$ be a $\mathcal{T}$–category with a strict initial object (i.e. any morphism to the initial object in $\underline{\mathcal{D}}$ must be an equivalence) and $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be an arbitrary fully faithful $\mathcal{T}$–functor whose essential image does not contain the initial object of $\underline{\mathcal{D}}$. Then the extension $F^{\trianglelefteq}\colon \underline{\mathcal{C}}^{\trianglelefteq} \to \underline{\mathcal{D}}$ is also fully faithful.*

Proof. Since fully faithfulness is a fibrewise notion, we can just prove this fibrewise and hence reduce to proving it in the unparametrised case. Since we already know that the composite $\mathcal{C} \to \mathcal{C}^{\triangleleft} \to \mathcal{D}$ is fully faithful and since, on the space of objects, it is standard that $(\mathcal{C}^{\triangleleft})^{\simeq} \simeq \mathcal{C}^{\simeq} \sqcup \{\emptyset_{\mathcal{C}}\}$ (see for example [CMN+20, Prop. A.4]), we are left to argue that the induced map of mapping spaces is an equivalence when one of the objects is $\emptyset_{\mathcal{C}}$ the initial object in $\mathcal{C}^{\triangleleft}$. To this end, first note that $\emptyset_{\mathcal{C}}$ does not admit any map from any $X \in \mathcal{C}$: this is because we have a functor $\mathcal{C}^{\triangleleft} \to *^{\triangleleft} \simeq \Delta^1$, and so any such map would give a map from 1 to 0 in Δ^1, which does not exist. Thus the object $\emptyset_{\mathcal{C}} \in \underline{\mathcal{C}}^{\trianglelefteq}$ is also a strict initial object. Hence, we are left to argue that $\mathrm{Map}_{\mathcal{C}^{\triangleleft}}(\emptyset_{\mathcal{C}}, X) \to \mathrm{Map}_{\mathcal{D}}(F^{\triangleleft}(\emptyset_{\mathcal{C}}), F^{\triangleleft}(X))$ is an equivalence. But this is true because by the universal property, we must have that $F^{\triangleleft}(\emptyset_{\mathcal{C}}) \simeq \emptyset_{\mathcal{D}}$, and so both sides are contractible. □

The following is the key construction in the theory of parametrised cubes.

Construction 3.1.5 (Singleton inclusion)**.** Let $w\colon W \to T$ be a map in $\underline{\mathrm{Fin}}_{\mathcal{T}}$. We would like to construct a map

$$\psi_w \colon w_! w^* \underline{*} \longrightarrow w_* w^* \underline{\Delta}^1$$

which generalises the inclusion of the singletons in Goodwillie's definition of strong cocartesianness. First note by atomic orbitality that we have the pullback in $\mathrm{Fin}_{\mathcal{T}}$

$$\begin{tikzcd} W \coprod C \arrow[r, "\mathrm{id} \coprod c"] \arrow[d, "\mathrm{id} \coprod \overline{c}"'] \arrow[dr, phantom, "\lrcorner"] & W \arrow[d, "w"] \\ W \arrow[r, "w"'] & T \end{tikzcd} \tag{17}$$

where C is some object in $\mathrm{Fin}_{\mathcal{T}}$. In particular, we have the decomposition

$$w^* w_* w^* \underline{\Delta}^1 \simeq \mathrm{id}_* \mathrm{id}^* w^* \underline{\Delta}^1 \times \overline{c}_* c^* w^* \underline{\Delta}^1 \simeq w^* \underline{\Delta}^1 \times \overline{c}_* c^* w^* \underline{\Delta}^1$$

Now by adjunction, to construct ψ_w, it would suffice to construct its adjoint $\overline{\psi}_w \colon w^* \underline{*} \to w^* w_* w^* \underline{\Delta}^1$. By the decomposition above, this is equivalent to constructing maps

$$w^* \underline{*} \to w^* \underline{\Delta}^1 \qquad \big(w^* \underline{*} \to \overline{c}_* c^* w^* \underline{\Delta}^1\big) \Leftrightarrow \big(\overline{c}^* w^* \underline{*} \simeq c^* w^* \underline{*} \to c^* w^* \underline{\Delta}^1\big)$$

To this end, we declare the first map to be the inclusion of the target and the second map to be the inclusion of the source. This yields the map ψ_w which one should think of the inclusion of the singletons in a cube. We will see that this is always fully faithful in Corollary 3.1.8. Therefore, by Construction 3.1.3, the map ψ_w constructed above induces a map

$$\varphi_w \colon (w_! w^* \underline{*})^{\trianglelefteq} \longrightarrow w_* w^* \underline{\Delta}^1$$

This map φ_w should be thought of generalising the subsets of size at most 1 in a cube, as the following important example will illustrate.

Example 3.1.6. In the special case when $T \in \mathcal{T}$ is the final object, $W = \coprod_{j=1}^n T$, and $w\colon W \to T$ is the fold map, we will see how the above construction yields the usual inclusion of singletons in the n–cube $\prod_{j=1}^n \Delta^1 = \mathrm{Pos}([n])$. To wit, since $T \times T \simeq T$ by finality, we have a decomposition

$$(\coprod_{j=1}^{n} T) \times (\coprod_{j=1}^{n} T) \simeq \coprod_{\substack{(a,b)\in \mathbb{Z}/n\times\mathbb{Z}/n, \\ a-b\equiv_n 0}} T \sqcup \coprod_{j=1}^{n-1} \coprod_{\substack{(a,b)\in \mathbb{Z}/n\times\mathbb{Z}/n, \\ a-b\equiv_n j}} T$$

where the term $\coprod_{\substack{(a,b)\in\mathbb{Z}/n\times\mathbb{Z}/n,\\ a-b\equiv_n 0}} T$ should be thought of as the diagonal tuples in $(\coprod^n T)\times(\coprod^n T)$. Hence, the pullback (17) in this case looks like

$$\begin{tikzcd}
\coprod\limits_{\substack{(a,b)\in\mathbb{Z}/n\times\mathbb{Z}/n,\\ a-b\equiv_n 0}} T \sqcup \coprod\limits_{j=1}^{n-1} \coprod\limits_{\substack{(a,b)\in\mathbb{Z}/n\times\mathbb{Z}/n,\\ a-b\equiv_n j}} T \arrow[r, "\mathrm{id}\sqcup c"] \arrow[d, "\mathrm{id}\sqcup\overline{c}"'] & \coprod_{j=1}^n T \arrow[d, "w"] \\
\coprod_{j=1}^n T \arrow[r, "w"'] & T
\end{tikzcd}$$

In this case, the map $\psi\colon \coprod_{j=1}^n \underline{*} \simeq w_!\underline{*} \to \prod_{j=1}^n \underline{\Delta}^1 \simeq w_*\underline{\Delta}^1$ constructed in Construction 3.1.5 comes from specifying the map $w^*\underline{*} \to w^*w_*\underline{\Delta}^1 \simeq \underline{\Delta}^1 \times \overline{c}_*c^*\underline{\Delta}^1$ describable as

$$(*,\ldots,*) \longrightarrow (\Delta^1 \times \prod_{1\leq j\leq n, j\neq 1} \Delta^1, \Delta^1 \times \prod_{1\leq j\leq n, j\neq 2} \Delta^1, \ldots, \Delta^1 \times \prod_{1\leq j\leq n, j\neq n} \Delta^1)$$

choosing 1 in the first copy of Δ^1 and 0's in $\prod_{1\leq j\leq n, j\neq k}\Delta^1$ for all k. Equivalently, the k–th summand in $w_!\underline{*} \simeq \coprod_{j=1}^n *$ picks out 1 in the k–th copy of Δ^1 in $\prod_{1\leq j\leq n}\Delta^1$ and 0's in the other copies of Δ^1. Hence, the k–th summand in $w_!\underline{*}$ sits as the k–th singleton in $\prod_{j=1}^n \Delta^1 \simeq \mathrm{Pos}([n])$, as claimed.

We now record the following fundamental observation which will serve both as a basic principle for our proofs as well as as an indication that our notion of singletons is "correct" from the parametrised point of view, in that it is a notion that is stable under basechange along arbitrary morphisms in $\mathcal{T}$.

Remark 3.1.7 (Stability of singleton inclusions under basechange). Let $b\colon B\to T$ be a map in $\mathcal{T}$. Recall first that we have an adjunction from Recollection 2.1.20

$$b_!\colon \mathrm{Fin}_B \rightleftharpoons \mathrm{Fin}_T : b^*$$

where $b_!$ is given by postcomposing with $b\colon B\to T$ and b^* is pullback along b. In particular, applying b^* to the pullback (17) gives us a pullback

$$\begin{tikzcd}
b^*W \coprod b^*C \arrow[r, "\mathrm{id}\coprod a"] \arrow[d, "\mathrm{id}\coprod\overline{a}"'] \arrow[dr, phantom, "\lrcorner"] & b^*W \arrow[d, "\overline{w}"] \\
b^*W \arrow[r, "\overline{w}"'] & B
\end{tikzcd}$$

We would like to show now that $b^*\psi_w \simeq \psi_{\overline{w}}$, so that singleton inclusions are stable under basechange.

By pasting pullback squares, we also see that we have the pullback

$$\begin{tikzcd}
b^*C \arrow[r, "\overline{z}"] \arrow[d, "\overline{a}"'] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "c"] \\
b^*W \arrow[r, "z"'] & W
\end{tikzcd} \tag{18}$$

Now recall that the adjoint of ψ_w was a map defined as the composite

$$w^*\underline{*} \hookrightarrow w^*\underline{*} \amalg \overline{c}_!c^*w^*\underline{*} \xrightarrow{w^*\psi_w \simeq \big((1,0)\ (0,0)\big)} w^*\underline{\Delta}^1 \times \overline{c}_*c^*w^*\underline{\Delta}^1$$

Here, the map $\big((1,0)\;(0,0)\big)$ is matrix notation for the map induced by the four maps

$$w^*\underline{*} \xrightarrow{1} w^*\underline{\Delta}^1 \qquad w^*\underline{*} \xrightarrow{0} \overline{c}_*c^*w^*\underline{\Delta}^1 \qquad \overline{c}_!c^*w^*\underline{*} \xrightarrow{0} w^*\underline{\Delta}^1 \qquad \overline{c}_!c^*w^*\underline{*} \xrightarrow{0} \overline{c}_*c^*w^*\underline{\Delta}^1$$

We claim now that $b^*\psi_w \colon \overline{w}_!\overline{w}\underline{*} \to \overline{w}_*\overline{w}^*\underline{\Delta}^1$ is defined similarly, so that $b^*\psi_w \simeq \psi_{\overline{w}}$ as wanted. Adjoining over $b^*\psi_w$ gives the map

$$\overline{w}^*\underline{*} \hookrightarrow \overline{w}^*\underline{*} \amalg \overline{a}_!a^*\underline{*} \xrightarrow{\overline{w}^*b^*\psi_w} \overline{w}^*\underline{\Delta}^1 \times \overline{a}_*a^*\underline{\Delta}^1 \tag{19}$$

But then $\overline{w}^*b^* \simeq z^*w^*$, and so together with the pullback (18), the map $\overline{w}^*b^*\psi_w$ from (19) can be analysed as

$$z^*w^*\underline{*} \amalg \overline{a}_!a^*\underline{*} \xrightarrow{z^*w^*\psi_w \simeq ((1,0)\;(0,0))} z^*w^*\underline{\Delta}^1 \times \overline{a}_*a^*\overline{w}^*\underline{\Delta}^1$$

which is the desired form of map.

Corollary 3.1.8. *The map $\varphi_w \colon (w_!w^*\underline{*})^{\trianglelefteq} \to w_*w^*\underline{\Delta}^1$ is fully faithful.*

Proof. By Proposition 3.1.4, it suffices to show that the map $\psi_w \colon w_!w^*\underline{*} \to w_*w^*\underline{\Delta}^1$ is fully faithful. Now, since fully faithfulness is a fibrewise notion, it suffices to check the statement in each fibre, and since these maps are stable under basechange by Remark 3.1.7, we may without loss of generality argue in the fibre over $T \in \mathcal{T}$. In case $w \colon W \to T$ is not an equivalence, then we know by Observation 2.1.18 that $\mathrm{ev}_T(w_!\underline{*}) \simeq \emptyset$, and so the map ψ_w is vacuously fully faithful in the fibre over $T \in \mathcal{T}$. On the other hand, if $w \colon W \to T$ were an equivalence, then the map $\psi_w \colon w_!\underline{*} \to w_*\underline{\Delta}^1$ is just the inclusion of 0 in Δ^1, which is also fully faithful. Hence, ψ is fully faithful in either case, as was to be shown. $\square$

Example 3.1.9 (G–cubes and C_2–pushouts). Let G be a finite group, $H \leq G$ a subgroup, and consider the case of $\mathcal{T} = \mathcal{O}_G$. The G–category $w_*\underline{\Delta}^1 = \prod_{G/H}\underline{\Delta}^1$ should then be thought of as the $|G/H|$–dimensional cube equipped with the G–action dictated by the one on G/H. For example, one would expect such a diagram to have G–fixed points only the initial and the final object, and the underlying diagram to be a $|G/H|$–cube. And indeed, we do have that $\mathrm{ev}_{G/G}(w_*w^*\underline{\Delta}^1) \simeq \Delta^1$ and $\mathrm{ev}_{G/e}(w_*w^*\underline{\Delta}^1) \simeq \prod_{|G/H|}\Delta^1$, as expected. In the rest of the article, we will be especially interested in the case when $|G/H| = 2$. In this case, a G–diagram indexed by $(w_!w^*\underline{*})^{\trianglelefteq} \subseteq w_*w^*\underline{\Delta}^1$ may be schematically represented as the datum

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[dl, dash, dashed] \\
B &
\end{tikzcd}$$

where A is a G–object, B is a H–object, the maps f are maps of H–objects $\mathrm{Res}^G_H A \xrightarrow{f} B$, and the G–action on G/H swaps the two copies of f's. We will call the G–colimits of such diagrams C_2*–pushouts* for reasons that we hope are clear given this schematic representation. In the next section, we will give a formula to compute such G–colimits in terms of ordinary pushouts and indexed coproducts.

3.2 Cubical decompositions

We would like now to apply the general nonsense Corollary 2.2.10 to the cubical setting, the highlights of which are the decomposition Propositions 3.2.2 and 3.2.8.

Observation 3.2.1. Let $\underline{I} \in \mathrm{Cat}_{\mathcal{T}}$. Then there is a pushout in $\mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$

$$\begin{tikzcd} \underline{I} \arrow[r, "c"] \arrow[d, "t"'] & \underline{*} \arrow[d, "\ell"] \\ \underline{I} \times \underline{\Delta}^1 \arrow[r, "\pi"] & \underline{I}^{\triangleright} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

To see this, since all constructions and operations in sight in the square above are done fibrewise, we may reduce to showing the statement in the unparametrised case in Cat. Here, we know that we have the join–slice adjunction (cf. for instance [Lan21, Cor. 1.4.17])

$$(-)^{\triangleright} \colon \mathrm{Cat} \rightleftharpoons \mathrm{Cat}_{*/} : (-)_{/p}$$

where $(-)_{/p}$ is the slice construction on a category equipped with a choice of object, i.e. $(\mathcal{D}, d)_{/p} \coloneqq * \times_{\mathcal{D}} \mathcal{D}^{\Delta^1}$ where $\mathcal{D}^{\Delta^1} \to \mathcal{D}$ is the target projection. Hence, we just need to show that the pushout construction $P(I) \coloneqq (I \times \Delta^1) \cup_I *$ is also left adjoint to $(-)/p$. For this, let $(\mathcal{D}, d) \in \mathrm{Cat}_{*/}$ and simply consider

$$\begin{aligned} \mathrm{Map}_{\mathrm{Cat}_{*/}}(P(I), (\mathcal{D}, d)) &\simeq \mathrm{Map}_{\mathrm{Cat}}(P(I), \mathcal{D}) \times_{\mathrm{Map}_{\mathrm{Cat}}(*, \mathcal{D})} \{d\} \\ &\simeq \mathrm{Map}_{\mathrm{Cat}}(I \times \Delta^1, \mathcal{D}) \times_{\mathrm{Map}_{\mathrm{Cat}}(I, \mathcal{D})} \{d\} \\ &\simeq \mathrm{Map}_{\mathrm{Cat}}(I, \mathcal{D}^{\Delta^1} \times_{\mathcal{D}} *) = \mathrm{Map}_{\mathrm{Cat}}(I, (\mathcal{D}, d)_{/p}) \end{aligned}$$

as required.

Using the notations from the pushout in the observation above, we may now extract the following important decomposition result for parametrised (co)limits that we will be especially interested in.

Proposition 3.2.2. *Let $i \colon \underline{I} \hookrightarrow \underline{I}^{\triangleright}$ be the inclusion and let the morphism* can: $i^* \to c^*\ell^*$ *in $\underline{\mathrm{Fun}}(\underline{I}, \underline{\mathcal{C}})$ be the canonical structure map. The limit functor $\underline{\lim}_{\underline{I}^{\triangleright}} \colon \underline{\mathrm{Fun}}(\underline{I}^{\triangleright}, \underline{\mathcal{C}}) \to \underline{\mathcal{C}}$ can be computed as follows: letting $\partial \in \underline{\mathrm{Fun}}(\underline{I}^{\triangleright}, \underline{\mathcal{C}})$, we have a pullback in $\underline{\mathcal{C}}$*

$$\begin{tikzcd} \underline{\lim}_{\underline{I}^{\triangleright}} \partial \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \ell^* \partial \arrow[d, "\eta"] \\ c_* i^* \partial \arrow[r, "c_* \mathrm{can}"] & c_* c^* \ell^* \partial \end{tikzcd}$$

Similarly, for any $\partial \in \underline{\mathrm{Fun}}(\underline{I}^{\triangleleft}, \underline{\mathcal{C}})$, we obtain the pushout decomposition

$$\begin{tikzcd} c_! c^* \ell^* \partial \arrow[r, "\varepsilon"] \arrow[d, "c_! \mathrm{can}"'] & \ell^* \partial \arrow[d] \\ c_! i^* \partial \arrow[r] & \underline{\mathrm{colim}}_{\underline{I}^{\triangleleft}} \partial \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Proof. We will first need to collect a few elementary formal observations:

- Since the inclusion of the source $s \colon \underline{I} \hookrightarrow \underline{I} \times \underline{\Delta}^1$ has a right adjoint f (because $0 \in \Delta^1$ is an initial object), we get that $f^* \dashv s^*$ and so $s^* \simeq f_*$.
- Hence, if we write $v \colon \underline{I} \times \underline{\Delta}^1 \to \underline{*}$ for the unique map, then since $v \simeq cf$, we get $v_* \simeq c_* f_* \simeq c_* s^*$.
- Moreover, since $ft \simeq \mathrm{id}$, we also get that $s^* t_* \simeq f_* t_* \simeq \mathrm{id}$.
- Next, since $\pi s \simeq i$, we get $s^* \pi^* \simeq i^*$.

- Under these identifications, the canonical map in the statement of the result can be obtained by applying s^* to $\eta^t_{\pi^*}\colon \pi^* \to t_*t^*\pi^*$ since $s^*\pi^* \simeq i^*$ and $s^*t_*t^*\pi^* \simeq \mathrm{id}\circ c^*\ell^* \simeq c^*\ell^*$. It is a straightforward check to see that this is the canonical structure map.

Now, by Observation 3.2.1, we have a pullback description of $\underline{\mathrm{Fun}}(\underline{I}^{\triangleright},\underline{\mathcal{C}})$ as

$$\begin{tikzcd} \underline{\mathrm{Fun}}(\underline{I}^{\triangleright},\underline{\mathcal{C}}) \arrow[r, "\ell^*"] \arrow[d, "\pi^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathcal{C}} \arrow[d, "c^*"] \\ \underline{\mathrm{Fun}}(\underline{I},\underline{\mathcal{C}})^{\underline{\Delta}^1} \arrow[r, "t^*"] & \underline{\mathrm{Fun}}(\underline{I},\underline{\mathcal{C}}) \end{tikzcd} \tag{20}$$

Hence, by Lemma 2.2.9, we then get the pullback diagram

$$\begin{tikzcd} \underline{\lim}_{\underline{I}^{\triangleright}}\partial \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \ell^*\partial \arrow[d, "\eta^c_{\ell^*\partial}"] \\ v_*\pi^*\partial \arrow[r, "v_*\eta^t_{\pi^*\partial}"] & c_*t^*\pi^*\partial \simeq c_*c^*\ell^*\partial \end{tikzcd}$$

The identifications above now allow us to conclude that this pullback is of the form in the statement of the result. As usual, the the case of colimits can now be deduced be deduced by passing to opposite categories. □

Construction 3.2.3. We view the inclusion $\Lambda^2_0 \subseteq (\Delta^1)^{\times 2}$ as

$$\begin{tikzcd} 00 \arrow[r] \arrow[d] & 10 & & 00 \arrow[r] \arrow[d] & 10 \arrow[d] \\ 01 & & \hookrightarrow & 01 \arrow[r] & 11 \end{tikzcd}$$

In this way, for any $n \geq 1$, we may describe the objects in the full subcategory $\prod_{i=1}^n \Lambda^2_0 \subseteq \prod_{i=1}^n (\Delta^1)^{\times 2}$ as those tuples $(a_i)_{1\leq i\leq n}$ where $a_i \in \{00, 10, 01\}$. We want to decompose $\prod_{i=1}^n \Lambda^2_0$ in terms of subcategories that we will now specify:

- Let $M_n \subseteq \prod_{i=1}^n \Lambda^2_0$ be the full subcategory $(\prod_{i=1}^n \Delta^1) \times \{0\}$. Concretely in terms of the tuples description above, this is the full subcategory consisting of those tuples such that $a_i \in \{00, 10\}$.
- Let $B_n \subseteq \prod_{i=1}^n \Lambda^2_0$ be the full subcategory $\prod_{i=1}^n \Lambda^2_0 \backslash \{(10, \ldots, 10)\}$, i.e. of those tuples where some of the a_i's must either be 00 or 01.
- Let $F_n \subseteq B_n$ be the full subcategory of those tuples where $a_i \in \{01, 10\}$, that is, where none of the a_i's are 00 and at least one of them is 01.
- Let $J_n \coloneqq T_n \cap B_n$.

Similarly, since for any map $f\colon U \to V$ in $\mathcal{T}$, $f_*\underline{\Lambda}^2_0$ is fibrewise just products of multiple copies of Λ^2_0, we may similarly define in a fibrewise fashion the $\mathcal{T}$–full subcategories $\underline{J}_f, \underline{F}_f, \underline{B}_f, \underline{M}_f \subseteq f_*\underline{\Lambda}^2_0$. Here, $\underline{J}_f, \underline{B}_f, \underline{M}_f$ are clearly $\mathcal{T}$–subcategories, and to see that $\underline{F}_f$ is too, we need to argue that it is closed under restrictions. More precisely, we need to show that if $A \xrightarrow{a} W \xrightarrow{w} V$ are maps in $\mathcal{T}$, then the map $(\underline{F}_f)_W \subseteq (f_*\underline{\Lambda}^2_0)_W \xrightarrow{a^*} (f_*\underline{\Lambda}^2_0)_A$ factors through $(\underline{F}_f)_A$. For this, let us set up some notations and consider the pullback diagram in $\mathrm{Fin}_{\mathcal{T}}$

$$\begin{tikzcd} \coprod_{i,j} Y_{ij} \arrow[r, "\sqcup q_{ij}"] \arrow[d, "\sqcup p_{ij}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_i Z_i \arrow[r, "\sqcup v_i"] \arrow[d, "\sqcup u_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "f"] \\ A \arrow[r, "a"] & W \arrow[r, "w"] & V \end{tikzcd}$$

We then get $(f_*\underline{\Lambda}^2_0)_W \simeq (w^*f_*\underline{\Lambda}^2_0)_W \simeq (\prod_i u_{i*}\underline{\Lambda}^2_0)_W \simeq \prod_i \Lambda^2_0$ and $(a^*w^*f_*\underline{\Lambda}^2_0)_A \simeq (\prod_i\prod_j p_{ij*}\underline{\Lambda}^2_0)_A \simeq \prod_i\prod_j \Lambda^2_0$. Under these identifications, the map $(f_*\underline{\Lambda}^2_0)_W \xrightarrow{a^*} (f_*\underline{\Lambda}^2_0)_A$ is then simply the product $\prod_i \Delta\colon \prod_i \Lambda^2_0 \to \prod_i\prod_j \Lambda^2_0$ of the diagonal maps. From the tuples definition of F_n above, it is then clear that $(\underline{F}_f)_W \subseteq \prod_i \Lambda^2_0$ is sent to $(\underline{F}_f)_A \subseteq \prod_i\prod_j \Lambda^2_0$ as was to be argued.

Observation 3.2.4. The reason we will be interested in the subcategories above is that we have a union of posets $\prod_{i=1}^n \Lambda^2_0 = M_n \cup_{J_n} B_n$ and so we have a pushout in Cat

$$\begin{tikzcd} J_n \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & M_n \arrow[d, hook] \\ B_n \arrow[r, hook] & \prod_{i=1}^n \Lambda^2_0 \end{tikzcd}$$

One way to see this is that fully faithful inclusions of posets induce monomorphisms of their associated simplicial sets under the nerve functor, and so such inclusions are in particular cofibrations in the Joyal model structure (cf. [Lur09, Thm. 2.2.5.1]), whence the strict pushout being a homotopy pushout in Cat. Hence, since pushouts in $\mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ are computed pointwise, we also have a pushout in $\mathrm{Cat}_{\mathcal{T}}$

$$\begin{tikzcd} \underline{J}_f \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \underline{M}_f \arrow[d, hook] \\ \underline{B}_f \arrow[r, hook] & f_*\underline{\Lambda}^2_0 \end{tikzcd}$$

Lemma 3.2.5. *The inclusion $F_n \subseteq B_n$ is cofinal.*

Proof. To apply Quillen's Theorem A (e.g. [Lan21, Thm. 4.4.20]), we need to show that for every tuple $\underline{a} \in B_n$, the category $(F_n)_{\underline{a}/}$ is weakly contractible. We will show this by showing that $(F_n)_{\underline{a}/}$ has an initial object. To this end, consider the tuple $\underline{\widetilde{a}}$ defined as follows:

$$\widetilde{a}_i = \begin{cases} 01 & \text{if } a_i \in \{00, 01\} \\ 10 & \text{if } a_i = 10 \end{cases}$$

The map $00 \to 01$ then supplies us with a map $\eta\colon \underline{a} \to \underline{\widetilde{a}}$. Since $\underline{a} \in B_n$, some of the a_i's are either 00 or 01, and so $\underline{\widetilde{a}}$ indeed lies in F_n. Hence, we have an object $(\eta\colon \underline{a} \to \underline{\widetilde{a}})$ in $(F_n)_{\underline{a}/}$.

Now since we are dealing with a poset, to show that $\underline{\widetilde{a}}$ is initial in $(F_n)_{\underline{a}/}$ and since $B_n \subseteq \prod_n \Lambda^2_0$ is fully faithful, it suffices to show that $\mathrm{Map}_{\prod_n \Lambda^2_0}(\underline{\widetilde{a}}, \underline{f}) \simeq *$ for all $(\underline{a} \to \underline{f}) \in (F_n)_{\underline{a}/}$. For this, consider the map

$$\mathrm{Map}_{\prod_n \Lambda^2_0}\left(\underline{\widetilde{a}}, \underline{f}\right) \xrightarrow{\eta^*} \mathrm{Map}_{\prod_n \Lambda^2_0}\left(\underline{a}, \underline{f}\right) \simeq * \tag{21}$$

where the last term is contractible by definition of $(\underline{a} \to \underline{f}) \in (F_n)_{\underline{a}/}$. Now noting that $\mathrm{Map}_{\prod_n \Lambda^2_0}(\underline{x}, \underline{z}) \simeq \prod_i \mathrm{Map}_{\Lambda^2_0}(x_i, z_i)$, the map (21) is easily seen to be an equivalence since $00 \to 01$ induces an equivalence $\mathrm{Map}_{\Lambda^2_0}(01, 01) \to \mathrm{Map}_{\Lambda^2_0}(00, 01)$. $\square$

Corollary 3.2.6. *The inclusion of $\mathcal{T}$–categories $\underline{F}_f \subseteq \underline{B}_f$ is $\mathcal{T}$–cofinal, and so restriction along this inclusion induces an equivalence $\underline{\mathrm{colim}}_{\underline{B}_f} \simeq \underline{\mathrm{colim}}_{\underline{F}_f}$.*

Proof. We may deduce this statement from the unparametrised Lemma 3.2.5 immediately since $\mathcal{T}$–cofinality may be checked fibrewise by [Sha23, Thm. 6.7]. $\square$

Notation 3.2.7. Let $\underline{\mathcal{C}}^{\otimes}$ be a $\mathcal{T}$–symmetric monoidal category, $f\colon U \to V$ a map in $\mathcal{T}$, and $(A \to B) = \varphi\colon \underline{\Delta}^1 \to f^*\underline{\mathcal{C}}$ a morphism in $f^*\underline{\mathcal{C}}$. It will be convenient to use the notation $\underline{\mathrm{colim}}_{\underline{J}_f} f_{\otimes}(A \to B)$ for the $\underline{J}_f$–shaped colimit of the functor $\underline{J}_f \subseteq f_*\underline{\Delta}^1 \xrightarrow{f_*\varphi} f_*f^*\underline{\mathcal{C}} \xrightarrow{f_{\otimes}} \underline{\mathcal{C}}$.

We now come to the main proposition of this subsection:

Proposition 3.2.8. *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–pointed category with all $\mathcal{T}$–colimits equipped with a $\mathcal{T}$–distributive symmetric monoidal structure $\underline{\mathcal{C}}^{\otimes}$. Let $f\colon U \to V$ be a map in $\mathcal{T}$. If we have a cofibre sequence $A \to B \to C$ in $f^*\underline{\mathcal{C}}$, then applying $f_{\otimes}$ yields a cofibre sequence in $\underline{\mathcal{C}}$*

$$\underline{\mathrm{colim}}_{\underline{J}_f} f_{\otimes}(A \to B) \longrightarrow f_{\otimes}B \longrightarrow f_{\otimes}C.$$

Proof. We write $\partial\colon \underline{\Lambda}^2_0 \to f^*\underline{\mathcal{C}}$ for the diagram $(0 \leftarrow A \to B)$ for the diagram defining the cofibre C. Since $f_{\otimes}$ was assumed to be distributive, we have a colimit diagram

$$\delta^{\trianglerighteq}\colon (f_*\underline{\Lambda}^2_0)^{\trianglerighteq} \longrightarrow f_*\big((\underline{\Lambda}^2_0)^{\trianglerighteq}\big) \xrightarrow{f_*(\partial^{\trianglerighteq})} f_*f^*\underline{\mathcal{C}} \xrightarrow{f_{\otimes}} \underline{\mathcal{C}}$$

and so we obtain that $f_{\otimes}C$ is the colimit of the diagram $\delta\colon f_*\underline{\Lambda}^2_0 \to \underline{\mathcal{C}}$. Now applying the colimit decomposition Corollary 2.2.10 on the pushout from Observation 3.2.4, we obtain the pushout diagram in $\underline{\mathcal{C}}$ (where we have suppressed the restriction functors for readability)

$$\begin{tikzcd} \underline{\mathrm{colim}}_{\underline{J}_f}\delta \arrow[r] \arrow[d] & \underline{\mathrm{colim}}_{\underline{M}_f}\delta \arrow[d] \\ \underline{\mathrm{colim}}_{\underline{B}_f}\delta \arrow[r] \arrow[ur, phantom, "\ulcorner"] & f_{\otimes}C \end{tikzcd}$$

But then $\underline{M}_f$ has a final object and the evaluation at the final object of the functor $\underline{M}_f \subseteq f_*\underline{\Lambda}^2_0 \xrightarrow{\delta} \underline{\mathcal{C}}$ is $f_{\otimes}B$, hence we get $\underline{\mathrm{colim}}_{\underline{M}_f}\delta \simeq f_{\otimes}B$. Next, by distributivity, tensoring with the zero object yields the zero object, and so since by definition $\underline{F}_f$ contains at least one 01 in the tuples description from Construction 3.2.3, we see that the restricted functor $\underline{F}_f \subseteq f_*\underline{\Lambda}^2_0 \xrightarrow{\delta} \underline{\mathcal{C}}$ has constant value the zero object. Therefore, all in all, by Corollary 3.2.6, we have that $\underline{\mathrm{colim}}_{\underline{B}_f}\delta \simeq \underline{\mathrm{colim}}_{\underline{F}_f}\delta \simeq 0$. Finally, by definition of our notation, $\underline{\mathrm{colim}}_{\underline{J}_f} f_{\otimes}(A \to B) = \underline{\mathrm{colim}}_{\underline{J}_f}\delta$. Combining all of these gives the claimed cofibre sequence. □

3.3 Special case of index 2 quotients in the equivariant setting

In preparation for our application to the algebraic K–theory of 2–groups in the next section, we will be considering the special case of $\mathcal{T} = \mathcal{O}_G$ and where all parametrised colimits in sight come from an index 2 subgroup $H \leq G$. As we shall see, the special facts in the following observation conspire to make this situation particularly simple.

Observation 3.3.1. Let us collect all the elementary group theory we will need here. Let $H \leq G$ be an index 2 subgroup and $K \leq G$ be another subgroup.

1. If the inclusion $H \cap K \leq K$ is proper, then it is also an index 2 inclusion. This is because, fixing any $x \in K\backslash H$, we know by $|G/H| = 2$ that for any other $k \in K$, $k = xh$ for some $h \in H$. But then $h = x^{-1}k \in K$ and so $h \in H \cap K$, whence $|K/H \cap K| = 2$.

2. By the usual double coset decomposition, we know that $G/K \times G/H \cong \coprod_{g \in K\backslash G/H} G/H^g \cap K = \coprod_{g \in K\backslash G/H} G/H \cap K$ where we have used that $H \leq G$ was a normal subgroup since $|G/H| = 2$. But then, again by $|G/H| = 2$, there are only two possibilities for what the set $K\backslash G/H$ can be, whence

$$G/K \times G/H = \begin{cases} (G/K)^{\amalg 2} & \text{if } K \leq H \\ G/K \cap H & \text{if } K \not\leq H \end{cases}$$

Proposition 3.3.2. *Let $H \leq G$ have index 2 and let $w\colon G/H \to G/G$ be the unique map. In this case, the map $\varphi\colon (w_!w^*\underline{*})^{\trianglelefteq} \hookrightarrow w_*w^*\underline{\Delta}^1$ from Construction 3.1.5 induces via Proposition 3.1.4 an equivalence $\overline{\varphi}\colon ((w_!w^*\underline{*})^{\trianglelefteq})^{\trianglerighteq} \to w_*w^*\underline{\Delta}^1$. Equivalently, in the notation of Construction 3.2.3, the map φ induces an equivalence $\varphi\colon (w_!w^*\underline{*})^{\trianglelefteq} \xrightarrow{\simeq} \underline{J}_w$.*

Proof. Equivalences can be checked fibrewise, and so fixing a subgroup $K \leq G$ and writing $r\colon G/K \to G/G$ for the unique map, we will prove by induction on the order of G that

$$r^*\overline{\varphi}\colon r^*((w_!w^*\underline{*})^{\trianglelefteq})^{\trianglerighteq} \simeq ((r^*w_!w^*\underline{*})^{\trianglelefteq})^{\trianglerighteq} \longrightarrow r^*w_*w^*\underline{\Delta}^1 \tag{22}$$

is an equivalence. As the base case, the statement is vacuously true when $|G| = 1$.

First suppose that $K \lneq G$. By the dichotomy in Observation 3.3.1 (2), we have the two cases of pullbacks

$$\begin{tikzcd} G/K \coprod G/K \arrow[r, "i\sqcup gi"] \arrow[d, "\mathrm{id}\sqcup\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & G/H \arrow[d, "w"] \\ G/K \arrow[r, "r"'] & G/G \end{tikzcd} \qquad \begin{tikzcd} G/H\cap K \arrow[r, "j"] \arrow[d, "\ell"'] \arrow[dr, phantom, "\lrcorner", very near start] & G/H \arrow[d, "w"] \\ G/K \arrow[r, "r"'] & G/G \end{tikzcd}$$

according as $K \leq H$ or $K \not\leq H$. In the case $K \leq H$, we see that $r^*\psi\colon r^*w_!w^*\underline{*} \to r^*w_*w^*\underline{\Delta}^1$ is identified with the ordinary singleton inclusion

$$(i^* \sqcup i^*g^*)w^*\underline{*} \simeq r^*\underline{*} \sqcup r^*\underline{*} \longrightarrow r^*\underline{\Delta}^1 \times r^*\underline{\Delta}^1 \simeq (i^* \times i^*g^*)w^*\underline{\Delta}$$

since the map ψ was stable under basechange by Remark 3.1.7, and the map $r^*\underline{*} \sqcup r^*\underline{*} \to r^*\underline{\Delta}^1 \times r^*\underline{\Delta}^1$ is the ordinary singleton inclusion by Example 3.1.6. Therefore, in this case we indeed have that (22) is an equivalence. As for the case $K \not\leq H$, writing $z\colon G/H\cap K \to G/G$ for the unique map, we have the identification of $r^*\psi$ with

$$\psi\colon \ell_!z^*\underline{*} \longrightarrow \ell_*z^*\underline{\Delta}^1$$

again by stability of singleton inclusions with basechange Remark 3.1.7. But then by Observation 3.3.1 (1) we know that $H \cap K \leq K$ was an index 2 inclusion and $|K| < |G|$, and so by the inductive hypothesis, (22) is also an equivalence, finishing the proof for this case.

Finally, for the case when $K = G$, we write $i\colon \{G/G\} \hookrightarrow \mathcal{O}_G^{\mathrm{op}}$ for the inclusion. We would like to argue that applying i^* (which is the evaluation at G/G) to $\overline{\varphi}$ gives an equivalence in Cat. Now, we know that $i^*w_*w^*\underline{\Delta}^1 \simeq \Delta^1$. On the other hand, $i^*w_!\underline{*} \simeq \emptyset$ by Observation 2.1.18. Therefore, we see now that $i^*\overline{\varphi}$ is identified with the map

$$(\emptyset^{\triangleleft})^{\triangleright} \longrightarrow \Delta^1$$

which is clearly an equivalence. This completes the inductive step and hence the proof of the proposition. □

4 Noncommutative motives and equivariant algebraic K–theory

In the final section, we put together all the general theory developed above to treat the parametrised version of algebraic K–theory. As with ordinary algebraic K–theory, the key notion is that of split Verdier sequences. Building upon the theory of $\mathcal{T}$–perfect-stable categories from §2.5, we shall introduce these sequences in the parametrised context in §4.1 and deduce their properties from the unparametrised context via Theorem 2.5.11. Using these sequences, we construct two variants of parametrised noncommutative motives $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw}}$ and

$\underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$, called *pointwise* and *normed* motives, respectively. The former will be the one with a desirable universal property and which corepresents K–theory, whereas the latter will be the one that may be refined multiplicatively for formal reasons (c.f. Theorems 4.2.11 and 4.2.15 resp. Proposition 4.2.17). Next, we shall employ the theory of parametrised cubes laid down in §3 to show that these two types of motives coincide in the equivariant case when G is a 2–group, i.e. $|G| = 2^n$ for some n. The strategy is to show that additive functors satisfy descent against certain C_2–pushouts by transforming these diagrams into ordinary pushouts for which descent is known to be true. By a dévissage argument using the solvability of p–groups (which may be viewed as an equivariant replacement of "currying"), we then bootstrap the C_2–square descent to show in Theorem 4.3.11 that the two motives agree for arbitrary 2–groups. Finally, we combine this with Theorem 2.4.10 to treat the case of Swan K–theory in §4.4.

4.1 Split Verdier sequences and additive functors

In our setting, the notion of (split) Verdier sequences, so central in giving a universal characterisation of algebraic K–theory, will simply be a direct adaptation of those of [CDH+21] in light of Theorem 2.5.11.

Definition 4.1.1. A sequence $\underline{\mathcal{C}} \xrightarrow{i} \underline{\mathcal{D}} \xrightarrow{p} \underline{\mathcal{E}}$ in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ with vanishing composite is called a *Verdier sequence* if it is both a fibre and cofibre sequence. It is moreover said to be a *split* Verdier sequence if it can be completed to $\mathcal{T}$-adjunctions

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, "i"] & \underline{\mathcal{D}} \arrow[l, bend right, "q"'] \arrow[l, bend left, "r"] \arrow[r, "p"] & \underline{\mathcal{E}} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "j"] \end{tikzcd} \tag{23}$$

where an arrow stacked above another denotes being a left adjoint. If we only have the left adjoints q and ℓ (resp. only right adjoints r and j), then we say that the Verdier sequence is left–split (resp. right–split).

Remark 4.1.2. Since $\mathrm{Cat}^{\mathrm{perf}}$ is semiadditive, we get that (co)limits in $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ are computed pointwise by [Bar17, Cor. 6.7.1]. On the other hand, [CDH+21, §$A.1, A.2$] give us very good control of the fibre and cofibre sequences in $\mathrm{Cat}^{\mathrm{perf}}$ in terms of (split) Verdier sequences. Hence, in conjunction with the creation of fibre and cofibre sequences under the inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subset \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ from Theorem 2.5.11, we will have a good control of the parametrised (split) Verdier sequences as defined above. The following is a word–for–word adaptation of [CDH+21, Lem. A.2.5] in our setting.

Proposition 4.1.3. *Suppose we have a sequence $\underline{\mathcal{C}} \xrightarrow{i} \underline{\mathcal{D}} \xrightarrow{p} \underline{\mathcal{E}}$ in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ with vanishing composite. Then the following conditions are equivalent:*

1. *the given sequence is a fibre sequence, and p admits a fully faithful left (resp. right) adjoint ℓ,*

2. *the given sequence is a cofibre sequence, and i is fully faithful and admits a left (resp. right) adjoint q.*

Furthermore, if (1) and (2) hold, then both the original sequence and the left (resp. right) sequence $\underline{\mathcal{E}} \xrightarrow{\ell} \underline{\mathcal{D}} \xrightarrow{q} \underline{\mathcal{C}}$ are Verdier sequences.

Proof. Since the inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subset \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ preserves and reflects fibres and cofibres by Theorem 2.5.11, and since these are pointwise in $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ by the remark above, we can check the Verdierness of these sequences by checking fibrewise. This immediately reduces the result to [CDH+21, Lem. A.2.5]. $\square$

Corollary 4.1.4. *Suppose we have a cofibre sequence $\underline{\mathcal{C}} \xrightarrow{i} \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ such that i was fully faithful. Then it is already a Verdier sequence.*

Proof. Again, by appealing to Theorem 2.5.11, we may prove this statement in $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$, where it is known by [CDH+21, Cor. A.1.10]. ◻

Lemma 4.1.5. *Let $f\colon W \to V$ be in $\mathcal{T}$. Then a split Verdier sequence*

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, hook, "i"] & \underline{\mathcal{D}} \arrow[l, bend right, "q"'] \arrow[l, bend left, "r"] \arrow[r, two heads, "p"] & \underline{\mathcal{E}} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "j"] \end{tikzcd}$$

in $\big(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}\big)_W$ gives rise to one in $\big(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}\big)_V$.

$$\begin{tikzcd} f_*\underline{\mathcal{C}} \arrow[r, hook, "i"] & f_*\underline{\mathcal{D}} \arrow[l, bend right, "q"'] \arrow[l, bend left, "r"] \arrow[r, two heads, "p"] & f_*\underline{\mathcal{E}} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "j"] \end{tikzcd}$$

Proof. We saw in Proposition 2.5.4 that $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ is $\mathcal{T}$-semiadditive, and so $f_! \simeq f_*$. Hence f_* preserves (co)fibre sequences and we have bifibre sequences in the three directions above. Furthermore, [Hil22b, Lem. 4.3.2] says that the desired three layers of sequences are all adjoints of each other, and hence they form a split Verdier sequence. ◻

Fact 4.1.6 (Split Verdier classification). Suppose

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, "i"] & \underline{\mathcal{D}} \arrow[l, bend right, "q"'] \arrow[l, bend left, "r"] \arrow[r, "p"] & \underline{\mathcal{E}} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "j"] \end{tikzcd}$$

is a split Verdier sequence. This can then be recovered as the pullback

$$\begin{tikzcd} \underline{\mathcal{D}} \arrow[r, "q \to qjp"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathcal{C}}^{\Delta^1} \arrow[d, "\mathrm{tgt}"] \\ \underline{\mathcal{E}} \arrow[r, "qj"] & \underline{\mathcal{C}} \end{tikzcd}$$

whose vertical fibres are then $\underline{\mathcal{C}}$. This can be deduced from the analogous nonparametrised result in $\mathrm{Cat}^{\mathrm{perf}}$, recorded for instance in [CDH+21, Prop. A.2.12], since the inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subset \mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ preserves and reflects (co)limits by Theorem 2.5.11, and since (co)limits in $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ are computed pointwise by Remark 4.1.2.

We learnt of the following observation from the forthcoming [CDH+, Lem. 1.1.4], which was written in the more structured setting of Poincaré categories. We include the argument for $\mathrm{Cat}^{\mathrm{perf}}$ here for the reader's convenience.

Lemma 4.1.7. *There is an adjunction $L : \mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{perf}}) \rightleftarrows \mathrm{Cat}^{\mathrm{perf}} : R$ where $L(\mathcal{C} \xrightarrow{f} \mathcal{D}) \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{\Delta^1}$ and $R(\mathcal{E}) \simeq (\mathcal{E}^{\Delta^1} \to \mathcal{E})$, where both $\mathcal{D}^{\Delta^1} \to \mathcal{D}$ and $\mathcal{E}^{\Delta^1} \to \mathcal{E}$ are the target maps. Moreover, the right adjoint R preserves all colimits.*

Proof. To see that we have such an adjunction, we would like to construct an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{perf}})}(\mathcal{C} \xrightarrow{f} \mathcal{D}, \mathcal{E}^{\Delta^1} \xrightarrow{\mathrm{tgt}} \mathcal{E}) \simeq \mathrm{Map}_{\mathrm{Cat}^{\mathrm{perf}}}(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{\Delta^1}, \mathcal{E}) \tag{24}$$

natural in all the variables. For this, consider the split Verdier sequence $\mathcal{D} \hookrightarrow \mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{\Delta^1} \twoheadrightarrow \mathcal{C}$ classified by $\mathcal{C} \xrightarrow{f} \mathcal{D}$ via Fact 4.1.6. Since the functor $\mathrm{Fun}^{\mathrm{ex}}(-, \mathcal{E})$ preserves split Verdier sequences, we obtain another split Verdier sequence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E}) \hookrightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{\Delta^1}, \mathcal{E}) \twoheadrightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E})$$

By the classification Fact 4.1.6 yet again, we then obtain an equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{\Delta^1}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E}) \times_{\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})^{\Delta^1}$$

Applying core groupoids $(-)^{\simeq}$ to this yields the equivalence (24), which is clearly natural in all the inputs, as wanted. Finally, to see that the right adjoint preserves colimits, we just need to argue that the functor $(-)^{\Delta^1}$ commutes with colimits. Using that $\mathrm{Cat}^{\mathrm{perf}} \simeq \mathrm{Pr}_{L,\mathrm{st},\omega} \simeq \mathrm{Pr}^{\mathrm{op}}_{R,\mathrm{st},\omega\text{-filt}}$ and that the faithful inclusion $\mathrm{Pr}_{R,\mathrm{st},\omega\text{-filt}} \subset \widehat{\mathrm{Cat}}$ creates limits, we obtain

$$\mathrm{Ind}(\operatorname*{colim}_{j\in J} \mathcal{E}_j)^{\Delta^1} \simeq (\operatorname*{colim}_{j\in J}^{\mathrm{Pr}_L} \mathrm{Ind}\mathcal{E}_j)^{\Delta^1} \simeq (\lim_{j\in J^{\mathrm{op}}}^{\mathrm{Pr}_R} \mathrm{Ind}\mathcal{E}_j)^{\Delta^1} \simeq \lim_{j\in J^{\mathrm{op}}}^{\mathrm{Pr}_R} (\mathrm{Ind}\mathcal{E}_j)^{\Delta^1} \simeq \operatorname*{colim}_{j\in J}^{\mathrm{Pr}_L} \mathrm{Ind}(\mathcal{E}_j)^{\Delta^1}$$

where the outer equivalences are by [Lur09, Prop. 5.3.5.15], whence the desired conclusion. □

Construction 4.1.8. Since $\mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{perf}})$ and $\mathrm{Cat}^{\mathrm{perf}}$ are semiadditive, the left adjoint L from Lemma 4.1.7 preserves finite products. Hence we can apply $\underline{\mathrm{CMon}}_{\mathcal{T}}$ to obtain a $\mathcal{T}$-adjunction $\underline{L}_{\mathcal{T}} : \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})) \rightleftarrows \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) : \underline{R}_{\mathcal{T}}$ where the $\mathcal{T}$-right adjoint preserves all fibrewise colimits. Furthermore, both adjoints clearly restrict to $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subset \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ so that we get the $\mathcal{T}$–adjunction

$$\underline{L}_{\mathcal{T}} : \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}) \rightleftarrows \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} : \underline{R}_{\mathcal{T}}$$

By the preservation and reflection of fibrewise (co)limits from Theorem 2.5.11, the right adjoint here also preserves all fibrewise colimits, and so in particular, κ–filtered colimits. Thus, $\underline{L}_{\mathcal{T}}$ preserves κ-compact objects for all regular cardinals κ. Thus, if $(\underline{\mathcal{C}} \xrightarrow{f} \underline{\mathcal{D}})$ is a $\mathcal{T}$-exact functor between κ-compact $\mathcal{T}$-perfect stable categories, then $\underline{\mathcal{C}} \times_{\underline{\mathcal{D}}} \underline{\mathcal{D}}^{\Delta^1}$ is κ-compact too.

Corollary 4.1.9. *For any regular cardinal κ there is a small set S_κ of split Verdier sequences on κ-compact $\mathcal{T}$-perfect-stable categories such that any split Verdier sequence in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ can be written as a fibrewise κ-filtered colimit of sequences in S_κ.*

Proof. First of all, for any regular cardinal κ, since $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$ is κ–compactly generated by Proposition 2.5.7, we have by [Lur09, Prop. 5.3.5.15] that $\mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}) \simeq \mathrm{Ind}_\kappa \mathrm{Fun}(\Delta^1, (\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}})^\kappa)$. On the other hand, by κ–compact generation, we also have that finite limits in $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$ commute with κ–filtered colimits: this is because equivalences can be checked by mapping out of a set of κ–compact objects by virtue of κ–compact generation, and mapping out of these commutes with all κ–filtered colimits and all limits; hence, we can reduce this commutation statement to the category of spaces where it is true.

All in all, combining the statements from the previous paragraph with the fact that the functor $(\underline{\mathcal{E}} \xrightarrow{f} \underline{\mathcal{C}}) \mapsto \underline{\mathcal{E}} \times_{\underline{\mathcal{C}}} \underline{\mathcal{C}}^{\Delta^1}$ preserves κ–compactness by Construction 4.1.8, we obtain that if we are given a split Verdier sequence in $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$ as in Definition 4.1.1, then we may write $\underline{\mathcal{E}} \xrightarrow{qj} \underline{\mathcal{C}}$ as a κ–filtered colimit $\mathrm{colim}_{a\in I}(\underline{\mathcal{E}}_a \xrightarrow{(qj)_a} \underline{\mathcal{C}}_a)$ where $\underline{\mathcal{C}}_a, \underline{\mathcal{E}}_a \in (\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}})^\kappa$. This, moreover, yields that the map of pullback squares

$$\begin{tikzcd} \underline{\mathcal{D}}_a \arrow[r] \arrow[d, "p_a"'] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathcal{C}}_a^{\Delta^1} \arrow[d, "\mathrm{tgt}"] \\ \underline{\mathcal{E}}_a \arrow[r, "(qj)_a"] & \underline{\mathcal{C}}_a \end{tikzcd} \longrightarrow \begin{tikzcd} \underline{\mathcal{D}} \arrow[r, "q\to qjp"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathcal{C}}^{\Delta^1} \arrow[d, "\mathrm{tgt}"] \\ \underline{\mathcal{E}} \arrow[r, "qj"] & \underline{\mathcal{C}} \end{tikzcd}$$

is an equivalence upon applying $\mathrm{colim}_{a\in I}$, where $\underline{\mathcal{D}}_a$ is also in $(\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}})^{\kappa}$. In this way, we have written the given split Verdier sequence as a κ–filtered colimit of split Verdier sequences of κ–compact objects $\underline{\mathcal{C}}_a \hookrightarrow \underline{\mathcal{D}}_a \twoheadrightarrow \underline{\mathcal{E}}_a$, as wanted. ☐

Before moving on to the next subsection where we construct (two variants of) parametrised noncommutative motives, let us now declare what we mean by an *additive $\mathcal{T}$–functor* out of $\underline{\mathrm{Cat}}^{\mathrm{perf}}$ and collect some standard consequences of the notion.

Definition 4.1.10. Let $\underline{\mathcal{A}}$ be a $\mathcal{T}$–stable category. A $\mathcal{T}$-functor $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathcal{A}}$ is said to be *additive* if it sends split Verdier sequences to fibre sequences and preserves the final objects. We write $\underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{A}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{A}})$ for the $\mathcal{T}$-full subcategory of such.

Observation 4.1.11 (Waldhausen sequences). Write $s\colon \Delta^0 \to \Delta^1$ and $t\colon \Delta^0 \to \Delta^1$ for the inclusion of the source and the target, respectively. One of Waldhausen's many key original insights, translated in the parametrised setup, is the following: for $\underline{\mathcal{C}} \in \mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$, we have the split Verdier sequence

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, "s_*"] & \underline{\mathcal{C}}^{\Delta^1} \arrow[l, bend right, "s^*"'] \arrow[l, bend left, "\mathrm{fib}"] \arrow[r, two heads, "t^*"] & \underline{\mathcal{C}} \arrow[l, bend right, "t_!"'] \arrow[l, bend left, "t_*"] \end{tikzcd} \tag{25}$$

where c takes X to $X \xrightarrow{=} X$. This means that for any additive $F\colon \underline{\mathrm{Cat}}^{\mathrm{perf}} \to \underline{\mathcal{A}}$, we have the split fibre sequence

$$\begin{tikzcd} F(\underline{\mathcal{C}}) \arrow[r, "F(s_*)"] & F(\underline{\mathcal{C}}^{\Delta^1}) \arrow[l, bend left, "F(\mathrm{fib})"] \arrow[r, "F(t^*)"] & F(\underline{\mathcal{C}}) \arrow[l, bend left, "F(t_*)"] \end{tikzcd}$$

which yields the equivalence

$$F(\mathrm{fib}) \times F(t^*)\colon F(\underline{\mathcal{C}}^{\Delta^1}) \xleftrightarrow{\simeq} F(\underline{\mathcal{C}}) \times F(\underline{\mathcal{C}}) : F(s_*) + F(t_*)$$

This is of foundational importance as we will see in the next basic observation which may be seen as "trickling down" the additivity property through a level of decategorification. As in the nonparametrised situation, it is a straightforward matter to see using Proposition 2.2.16 that all such sequences are obtained by applying $\underline{\mathcal{C}} \otimes -$ to the version of (25) for $\underline{\mathrm{Sp}}^{\omega}$.

Lemma 4.1.12 (Waldhausen's trick, [CDH+21, Rmk. 2.7.6 (ii)])**.** *Let $F\colon \underline{\mathrm{Cat}}^{\mathrm{perf}} \to \underline{\mathcal{A}}$ be an additive invariant. If we have a cofibre sequence $\alpha \Rightarrow \beta \Rightarrow \gamma$ of maps $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ in $\mathrm{Cat}^{\mathrm{perf}}_{\mathcal{T}}$, then we have an equivalence of morphisms $F\beta \simeq F\alpha \oplus F\gamma\colon F\underline{\mathcal{C}} \to F\underline{\mathcal{D}}$.*

Proof. The key for these kinds of statements is that both natural transformations

$$(\beta \Rightarrow \gamma), (\alpha \oplus \gamma \Rightarrow \gamma)\colon \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{D}}^{\Delta^1}$$

have the same fibres, i.e. α. Hence, applying F and postcomposing further with the equivalence $F(\mathrm{fib}) \times F(t^*)\colon F(\underline{\mathcal{D}}^{\Delta^1}) \xrightarrow{\simeq} F(\underline{\mathcal{D}}) \times F(\underline{\mathcal{D}})$ from Observation 4.1.11 yields that the two morphisms

$$F(\beta \Rightarrow \gamma),\ F(\alpha \oplus \gamma \Rightarrow \gamma)\colon F(\underline{\mathcal{C}}) \longrightarrow F(\underline{\mathcal{D}}^{\Delta^1})$$

are equivalent. Finally, postcomposing now these two equivalent morphisms with $F(s^*)\colon F(\underline{\mathcal{D}}^{\Delta^1}) \to F(\underline{\mathcal{D}})$ shows that we have an equivalence of morphisms

$$F(\beta) \simeq F(\alpha \oplus \gamma) \simeq F(\alpha) \oplus F(\gamma)\colon F(\underline{\mathcal{C}}) \longrightarrow F(\underline{\mathcal{D}})$$

as was to be shown. ☐

Remark 4.1.13. In fact, the proof of the second part of the lemma above shows that any additive functor already sends left–split or right–split Verdier sequences to fibre sequences.

We end this subsection by recording how stable valued additive functors interact with certain pushout diagrams. This will be needed in our analysis of equivariant algebraic K–theory when G is a 2–group in §4.3.

Definition 4.1.14. A square in $\underline{\mathrm{Cat}^{\mathrm{perf}}}$

$$\begin{tikzcd} \underline{\mathcal{A}} \arrow[r] \arrow[d, hook, shift right] & \underline{\mathcal{B}} \arrow[d, hook, shift right] \\ \underline{\mathcal{C}} \arrow[r] \arrow[u, dashed, shift right] \arrow[ur, phantom, "\ulcorner"] & \underline{\mathcal{P}} \arrow[u, dashed, shift right] \end{tikzcd}$$

is said to be a *right-split Verdier pushout* if it is a pushout diagram and the vertical arrows are right-split Verdier inclusions.

The following lemma gives the source of right-split Verdier pushouts that will concern us.

Lemma 4.1.15. *Suppose we have the solid diagram in* $\underline{\mathrm{Cat}^{\mathrm{perf}}}$

$$\begin{tikzcd} \underline{\mathcal{A}} \arrow[r] \arrow[d, hook, shift right] & \underline{\mathcal{B}} \arrow[l, dashed, bend left] \\ \underline{\mathcal{C}} \arrow[u, dashed, shift right] & \end{tikzcd}$$

where the dashed arrows are the respective right adjoints. Then the pushout in $\underline{\mathrm{Cat}^{\mathrm{perf}}}$ *of the solid diagram is a right-split Verdier pushout.*

Proof. We work in the presentable setting by virtue of the equivalence $\underline{\mathrm{Cat}^{\mathrm{perf}}} \simeq \underline{\mathrm{Pr}}_{L,\mathrm{st},\omega}$ from Proposition 2.1.37. Now recall that colimits in $\underline{\mathrm{Pr}}_{L,\mathrm{st},\omega}$ are computed as limits in $\underline{\mathrm{Pr}}_{R,\mathrm{st},\omega\text{-filt}}$ by Proposition 2.2.15. And so we get the solid pushout in $\underline{\mathrm{Pr}}_{L,\mathrm{st},\omega}$ and dashed pullback square in $\underline{\mathrm{Pr}}_{R,\mathrm{st},\omega\text{-filt}}$

$$\begin{tikzcd} \underline{\mathrm{Ind}}(\underline{\mathcal{A}}) \arrow[r] \arrow[d, hook, shift right] & \underline{\mathrm{Ind}}(\underline{\mathcal{B}}) \arrow[l, dashed, bend left] \arrow[d, shift right] \\ \underline{\mathrm{Ind}}(\underline{\mathcal{C}}) \arrow[r] \arrow[u, dashed, shift right] \arrow[ur, phantom, "\ulcorner"] & \underline{\mathrm{Ind}}(\underline{\mathcal{P}}) \arrow[u, dashed, shift right] \arrow[l, dashed, bend left] \end{tikzcd}$$

Now since limits in both $\underline{\mathrm{Pr}}_{L,\mathrm{st},\omega}$ and $\underline{\mathrm{Pr}}_{R,\mathrm{st},\omega\text{-filt}}$ are computed underlying by Lemma 2.5.5 and Proposition 2.5.6, and since the top and left dashed maps are themselves compact–preserving left adjoints by our hypothesis, we see that the bottom and right dashed maps are also compact–preserving left adjoints (i.e. the dashed square is also a pullback in $\underline{\mathrm{Pr}}_{L,\mathrm{st},\omega}$). Moreover, since sections pull back to sections and since $\underline{\mathrm{Ind}}(\underline{\mathcal{A}}) \rightarrow \underline{\mathrm{Ind}}(\underline{\mathcal{C}})$ is a section of $\underline{\mathrm{Ind}}(\underline{\mathcal{C}}) \rightarrow \underline{\mathrm{Ind}}(\underline{\mathcal{A}})$, we see that $\underline{\mathrm{Ind}}(\underline{\mathcal{B}}) \rightarrow \underline{\mathrm{Ind}}(\underline{\mathcal{P}})$ is a section of $\underline{\mathrm{Ind}}(\underline{\mathcal{P}}) \rightarrow \underline{\mathrm{Ind}}(\underline{\mathcal{B}})$. This pair being adjoint to each other then automatically implies that $\underline{\mathrm{Ind}}(\underline{\mathcal{B}}) \rightarrow \underline{\mathrm{Ind}}(\underline{\mathcal{P}})$ is fully faithful. Since all maps in sight preserve compact objects, we may apply $(-)^{\underline{\omega}}$ to the solid diagram to get a right–split Verdier pushout, as wanted. $\square$

The following result is where our stability hypothesis comes in.

Lemma 4.1.16. *Let* $F \colon \underline{\mathrm{Cat}^{\mathrm{perf}}} \rightarrow \underline{\mathcal{M}}$ *be an additive* $\mathcal{T}$*–functor where* $\underline{\mathcal{M}}$ *is* $\mathcal{T}$*–stable. If we have a right-split Verdier pushout as in Definition 4.1.14, then*

$$\begin{tikzcd}
F(\underline{\mathcal{A}}) \arrow[r] \arrow[d] & F(\underline{\mathcal{B}}) \arrow[d] \\
F(\underline{\mathcal{C}}) \arrow[r] & F(\underline{\mathcal{P}})
\end{tikzcd}$$

is a pushout in $\underline{\mathcal{M}}$.

Proof. First we extend the diagram with $\underline{\mathcal{E}} := \mathrm{cofib}(\underline{\mathcal{B}} \hookrightarrow \underline{\mathcal{P}})$ to obtain

$$\begin{tikzcd}
\underline{\mathcal{A}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \underline{\mathcal{B}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\
\underline{\mathcal{C}} \arrow[r] \arrow[u, dashed, bend right] & \underline{\mathcal{P}} \arrow[r] \arrow[u, dashed, bend right] & \underline{\mathcal{E}}
\end{tikzcd}$$

Since taking cofibres of right-split Verdier inclusions give right-split Verdier sequences by Proposition 4.1.3, we get right-split Verdier sequences

$$\begin{tikzcd}
\underline{\mathcal{A}} \arrow[r, hook] & \underline{\mathcal{C}} \arrow[r, two heads] \arrow[l, dashed, bend left] & \underline{\mathcal{E}} \arrow[l, dashed, bend left]
\end{tikzcd} \qquad \begin{tikzcd}
\underline{\mathcal{B}} \arrow[r, hook] & \underline{\mathcal{P}} \arrow[r, two heads] \arrow[l, dashed, bend left] & \underline{\mathcal{E}} \arrow[l, dashed, bend left]
\end{tikzcd}$$

Hence, by Remark 4.1.13, the maps $F(\underline{\mathcal{C}})/F(\underline{\mathcal{A}}) \to F(\underline{\mathcal{E}})$ and $F(\underline{\mathcal{P}})/F(\underline{\mathcal{B}}) \to F(\underline{\mathcal{E}})$ are equivalences. Now consider the horizontal maps of vertical cofibre sequences

$$\begin{tikzcd}
F(\underline{\mathcal{A}}) \arrow[r] \arrow[d] & F(\underline{\mathcal{B}}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
F(\underline{\mathcal{C}}) \arrow[r] \arrow[d] & F(\underline{\mathcal{P}}) \arrow[r] \arrow[d] & F(\underline{\mathcal{E}}) \arrow[d, equal] \\
F(\underline{\mathcal{C}})/F(\underline{\mathcal{A}}) \arrow[r, dashed] \arrow[rr, bend right, "\simeq"'] & F(\underline{\mathcal{P}})/F(\underline{\mathcal{B}}) \arrow[r, "\simeq"] & F(\underline{\mathcal{E}})
\end{tikzcd}$$

where the equivalences are by the previous sentence. Hence, the dashed map is an equivalence too. On the other hand, we have this map of cofibre sequences

$$\begin{tikzcd}
F(\underline{\mathcal{B}}) \arrow[r] \arrow[d, equal] & F(\underline{\mathcal{B}}) \coprod_{F(\underline{\mathcal{A}})} F(\underline{\mathcal{C}}) \arrow[r] \arrow[d] & F(\underline{\mathcal{C}})/F(\underline{\mathcal{A}}) \arrow[d, "\simeq"] \\
F(\underline{\mathcal{B}}) \arrow[r] & F(\underline{\mathcal{P}}) \arrow[r] & F(\underline{\mathcal{P}})/F(\underline{\mathcal{B}})
\end{tikzcd}$$

Since we are working stably, this implies that the middle vertical is an equivalence, as was to be shown. ◻

4.2 Two variants of noncommutative motives

In this section, we follow closely the methods of [CDH+] in constructing the noncommutative motives in our setting. Essentially all the proofs of this section are straightforward parametrised modifications of their arguments and we are grateful to them for sharing a draft of their upcoming work which made our motivic approach possible.

Notation 4.2.1. Let κ be a regular cardinal. We write $\widetilde{\underline{\mathrm{Cat}}}{}^{\mathrm{perf},\kappa}_{\mathcal{T}}$ for the smallest $\mathcal{T}$-symmetric monoidal subcategory of $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ containing $(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}}$. In particular, since $(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}}$ is small by Proposition 2.5.7, $\widetilde{\underline{\mathrm{Cat}}}{}^{\mathrm{perf},\kappa}_{\mathcal{T}}$ is also small. We need this slight enlargement for the technical reason that we do not know a priori that $(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}}$ inherits the $\mathcal{T}$-symmetric monoidal structure of $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}}$ since it is not clear that the multiplicative norms preserve parametrised-κ-compact objects.

Definition 4.2.2. Let κ be a regular cardinal. Let $\mathcal{R}_{\mathrm{pw},\kappa}$ be the collection of diagrams in $\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa})$ consisting of:

- the diagram $\underline{\mathrm{const}}_{\mathcal{T}}(\emptyset)^{\trianglerighteq} = \underline{*} \to \widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa}$ picking the zero category (ie. the initial object),
- all split Verdier sequences in $(\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}})^{\underline{\kappa}} \subseteq \widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa}$.

Let $\mathcal{R}_{\mathrm{nm},\kappa}$ be the closure of $\mathcal{R}_{\mathrm{pw},\kappa}$ under $f_{\otimes}$ for arbitrary maps $f: U \to V$ in $\underline{\mathrm{Fin}}_{\mathcal{T}}$.

Now, using the construction and notation from Proposition 2.3.7, we may define the following intermediate notions of noncommutative motives.

Definition 4.2.3. Let κ be a regular cardinal. We define:

- *unstable pointwise κ-motives* $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\kappa}$ to be $L_{\mathcal{R}_{\mathrm{pw},\kappa}}\underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa})$,
- *unstable normed κ-motives* $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{nm},\mathrm{un},\kappa}$ to be $L_{\mathcal{R}_{\mathrm{nm},\kappa}}\underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa})$.

Remark 4.2.4. Note that $\mathcal{R}_{\mathrm{pw},\kappa}$ and $\mathcal{R}_{\mathrm{nm},\kappa}$ are small since $\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa}$ is, and so $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\kappa}$ and $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{nm},\mathrm{un},\kappa}$ are $\mathcal{T}$-presentable.

Remark 4.2.5. For the purposes of capturing the notion of additivity, we may without loss of generality let $\mathcal{R}_{\mathrm{pw}}$ be the tensor ideal generated by the map $\underline{\mathrm{Sp}}^{\omega,\Delta^1}/\underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}$ associated to the Waldhausen sequence for $\underline{\mathrm{Sp}}^{\omega}$. To see this, we argue now that a functor $F\colon \underline{\mathrm{Cat}}^{\mathrm{perf}} \to \underline{\mathcal{A}}$ which sends sequences of the form (25) to fibre sequences already sends any split Verdier sequence (see (23) for notation) to a fibre sequence. To wit, given such a split Verdier sequence, it is easy to see that we have a cofibre sequence $iq \Rightarrow \mathrm{id}_{\underline{\mathcal{D}}} \Rightarrow jp$ in $\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}},\underline{\mathcal{D}})$. Hence, by Lemma 4.1.12, we get a splitting

$$\mathrm{id}_{F\underline{\mathcal{D}}} \simeq F(i)\circ F(q) \oplus F(j)\circ F(p)\colon F\underline{\mathcal{D}} \to F\underline{\mathcal{D}}$$

which implies that the sequence $F\underline{\mathcal{C}} \to F\underline{\mathcal{D}} \to F\underline{\mathcal{E}}$ is a fibre sequence in $\underline{\mathcal{A}}$, as wanted. Therefore, it is enough to require that $\mathcal{R}_{\mathrm{pw}}$ consists only of the Waldhausen sequences (25), which by the last sentence in Observation 4.1.11, is a tensor ideal generated by the Waldhausen sequence for $\underline{\mathrm{Sp}}^{\omega}$. This completes the proof of the lemma.

Notation 4.2.6. Write $j^{\kappa}_{\mathrm{un}}\colon \widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa} \xhookrightarrow{y^{\kappa}_{\mathrm{un}}} \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa}) \to \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\kappa}$ for the canonical composition. Since split Verdier sequences were already cofibre sequences in $\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa}$ by definition, we get from [Hil22b, Thm. 6.4.2] that this functor is $\mathcal{T}$-fully faithful.

Recall now the notion of additive $\mathcal{T}$–functors from Definition 4.1.10.

Proposition 4.2.7. *For every $\mathcal{T}$-cocomplete category $\underline{\mathcal{E}}$, $(j^{\kappa}_{\mathrm{un}})^* : \underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\kappa},\underline{\mathcal{E}}) \to \underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa},\underline{\mathcal{E}})$ is an equivalence.*

Proof. This is immediate by Proposition 2.3.7 and the universal property of presheaves [Sha23, Thm. 11.5]. □

Construction 4.2.8 (The big unstable pointwise motives)**.** Let $\kappa \leq \kappa'$ be two regular cardinals. Then the composition $\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa} \subseteq \widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\kappa'} \hookrightarrow \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\kappa'}$ preserves initial objects and sends split Verdier sequences to cofibre sequences. Hence by Proposition 4.2.7 we obtain a strongly $\mathcal{T}$-colimit-preserving functor $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\kappa} \to \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\kappa'}$. This is $\mathcal{T}$-fully faithful since it sends compact-generators to compact objects and is $\mathcal{T}$-fully faithful on these. Similar considerations also apply when we replace motives with presheaves. From these, and since $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \simeq \mathrm{colim}_\kappa \widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}$, we then define

$$\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}} := \underset{\kappa}{\mathrm{colim}}\, \underline{\mathrm{NMot}}^{\mathrm{pw,un},\kappa}_{\mathcal{T}} \qquad \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}) := \underset{\kappa}{\mathrm{colim}}\, \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}})$$

Applying colim_κ on all functors in sight give us a $\mathcal{T}$-fully faithful functor

$$j_{\mathrm{un}} \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \xhookrightarrow{y_{\mathrm{un}}} \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}) \longrightarrow \underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}$$

Since the poset of regular cardinals is a large category and each of $\underline{\mathrm{NMot}}^{\mathrm{pw,un},\kappa}_{\mathcal{T}}$ is large, we deduce that $\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}$ is a large $\mathcal{T}$-presentable category since large unions of large sets is large. We refer to [CDH+, §1.2] for a more thorough discussion of set-theoretic considerations.

Proposition 4.2.9 ("[CDH+, Prop. 1.2.6]")**.** *For a $\mathcal{T}$–(co)complete category $\underline{\mathcal{E}}$, $(j_{\mathrm{un}})^* : \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}, \underline{\mathcal{E}}) \to \underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}})$ is an equivalence.*

Proof. By Corollary 4.1.9 we have $\underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}}) \simeq \lim_\kappa \underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf},\kappa}_{\mathcal{T}}, \underline{\mathcal{E}})$ by restricting from the tautological equivalence $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}}) \simeq \lim_\kappa \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf},\kappa}_{\mathcal{T}}, \underline{\mathcal{E}})$. But we also have the tautological equivalence $\underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}, \underline{\mathcal{E}}) \simeq \lim_\kappa \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{NMot}}^{\mathrm{pw,un},\kappa}_{\mathcal{T}}, \underline{\mathcal{E}})$. Therefore we can apply Proposition 4.2.7 to conclude. ☐

Construction 4.2.10 (Big stable motives)**.** Define the $\mathcal{T}$-presentable-stable category of *parametrised noncommutative motives* to be $\underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}} := \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}})$. This yields

$$\mathcal{Z}_{\mathrm{pw}} \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \xhookrightarrow{j_{\mathrm{un}}} \underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}} \xrightarrow{\mathrm{can}} \underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}}$$

Since $\mathcal{T}$–stabilisation is a left adjoint in $\underline{\mathrm{Pr}}_{\mathcal{T},L}$, we also have $\underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}} \simeq \mathrm{colim}_\kappa \underline{\mathrm{NMot}}^{\mathrm{pw},\kappa}_{\mathcal{T}}$ where $\underline{\mathrm{NMot}}^{\mathrm{pw},\kappa}_{\mathcal{T}} := \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathrm{NMot}}^{\mathrm{pw,un},\kappa}_{\mathcal{T}})$. We then obtain commuting composites

$$\begin{tikzcd}
 & & \underline{\mathrm{NMot}}^{\mathrm{pw,un},\kappa}_{\mathcal{T}} \arrow[drr, "\mathrm{can}"] & & \\
\mathcal{Z}_{\mathrm{pw},\kappa} \colon \widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}} \arrow[urr, hook, "j_{\mathrm{un}}"] \arrow[rr, "\mathcal{U}_\kappa"'] & & \underline{\mathrm{PSh}}^{\mathrm{st}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}) \arrow[rr, "\lambda_\kappa"'] & & \underline{\mathrm{NMot}}^{\mathrm{pw},\kappa}_{\mathcal{T}}
\end{tikzcd}$$

where $\mathcal{U}_\kappa$ is the composite

$$\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}) \xrightarrow{\mathrm{can}} \underline{\mathrm{PSh}}^{\mathrm{st}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}) := \underline{\mathrm{Sp}} \otimes \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}})$$

We will use this second description to handle monoidal matters later.

Theorem 4.2.11 (Universal property of pointwise stable motives)**.** *For every $\mathcal{T}$-presentable-stable category $\underline{\mathcal{E}}$, the precomposition $\mathcal{Z}^*_{\mathrm{pw}} \colon \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}}, \underline{\mathcal{E}}) \to \underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}})$ is an equivalence.*

Proof. This is an immediate consequence of Proposition 4.2.9 and Proposition 2.2.19. ☐

Construction 4.2.12 (Pointwise connective algebraic K-theory)**.** Recall from [BGT13] that connective algebraic K-theory is given by the finite product-preserving functor

$$\mathcal{K} : \mathrm{Cat}^{\mathrm{perf}} \xrightarrow{\mathrm{Q}_\bullet} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{perf}}) \xrightarrow{(-)^{\simeq}} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{colim}} \mathcal{S}$$

where $\underline{\mathrm{Q}}_n \underline{\mathcal{C}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{TwAr}(\Delta^n), \underline{\mathcal{C}})$, Quillen's Q–construction. Since $\mathrm{CMon}(\mathcal{S}) \to \mathcal{S}$ preserves sifted colimits by [Lur17, §3.2.3], it in particular preserves geometric realisations. Hence the geometric realisation used above to define $\mathcal{K}$ acquires a canonical commutative monoid structure because we have the factorisation

$$\begin{tikzcd} \mathrm{Cat}^{\mathrm{perf}} \arrow[r, "(-)^{\simeq}"] \arrow[dr, dashed] & \mathcal{S} \\ & \mathrm{CMon}(\mathcal{S}) \arrow[u] \end{tikzcd}$$

Thus we can apply the $\mathcal{T}$-cofree Construction 2.1.5 and $\mathcal{T}$-semiadditivise to get

$$\underline{\mathcal{K}}^{\mathrm{pw}}_{\mathcal{T}} : \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \longrightarrow \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathcal{S})$$

which we call the *pointwise parametrised algebraic K-theory space*. Levelwise, this looks like

$$\begin{aligned} \mathrm{Mack}_{\mathcal{T}}(\mathcal{K}) :\ \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{T}), \mathrm{Cat}^{\mathrm{perf}}) &\xrightarrow{\mathrm{Q}_\bullet} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{T}), \mathrm{Cat}^{\mathrm{perf}})) \\ &\xrightarrow{(-)^{\simeq}} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{T}), \mathrm{CMon}(\mathcal{S}))) \\ &\xrightarrow{\mathrm{colim}} \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{T}), \mathrm{CMon}(\mathcal{S})) \end{aligned}$$

We will have use of this description soon in analysing motivic suspensions. Note also that $\underline{\mathcal{K}}^{\mathrm{pw}}_{\mathcal{T}}$ is an additive theory by the usual unparametrised additivity theorem and since we define split Verdier sequences in $\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ as those that are pointwise split Verdier in the usual sense. Moreover, one can deloop the algebraic K-theory space $\mathcal{K}$ to get an *algebraic K-theory spectrum* $\mathrm{K} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ which is the spectrum associated to the prespectrum whose n-th term is $\mathrm{colim}_{\bullet \in (\mathbf{\Delta}^{\mathrm{op}})^n}(\mathrm{Q}_\bullet \mathcal{C})^{\simeq}$ (cf. [BGT13, §7.2] or Waldhausen's original treatment [Wal85] for more details using the equivalent $\mathrm{S}_\bullet$–construction), and we write $\underline{\mathrm{K}}^{\mathrm{pw}}_{\mathcal{T}}$ for the analogous pointwise K–theory spectrum. By construction, this fits into the commuting diagram

$$\begin{tikzcd} \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \arrow[r, hook] \arrow[dr, "\underline{\mathrm{K}}^{\mathrm{pw}}_{\mathcal{T}}"'] & \underline{\mathrm{Mack}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \arrow[d, "\underline{\mathrm{Mack}}_{\mathcal{T}}(\mathrm{K})"] \\ & \underline{\mathrm{Mack}}_{\mathcal{T}}(\mathrm{Sp}) \simeq \underline{\mathrm{Sp}}_{\mathcal{T}} \end{tikzcd}$$

Lemma 4.2.13. *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. Then $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathrm{Q}}_n \underline{\mathcal{C}}) \simeq \underline{\mathrm{Q}}_n \underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})$.*

Proof. Since $\underline{\mathrm{Q}}_n \underline{\mathcal{C}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{TwAr}(\Delta^n), \underline{\mathcal{C}})$, we get $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathrm{Q}}_n \underline{\mathcal{C}}) \simeq \underline{\mathrm{Q}}_n \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})$ from Notation 2.1.9 (1). But then, both $\underline{\mathrm{Q}}_n \underline{\mathcal{C}}$ and $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})$ inherit $\mathcal{T}$-(co)limits from $\underline{\mathcal{C}}$ (the former by Notation 2.1.9 (2)), and so clearly we obtain the statement required. ☐

Lemma 4.2.14 (Motivic suspension, "[BGT13, §7.3], [CDH+, Prop. 1.2.9]")**.** *Let $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. Then $\mathrm{colim}_{\bullet \in \mathbf{\Delta}^{\mathrm{op}}} y_{\mathrm{un}} \underline{\mathrm{Q}}_\bullet(\underline{\mathcal{C}})$ is already motivically local and moreover,*

$$\underset{\bullet \in \mathbf{\Delta}^{\mathrm{op}}}{\mathrm{colim}}\, j_{\mathrm{un}} \underline{\mathrm{Q}}_\bullet(\underline{\mathcal{C}}) \simeq \Sigma j_{\mathrm{un}}(\underline{\mathcal{C}}) \in \underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}$$

Proof. To see the first part, let $\underline{\mathcal{D}} \in \underline{\mathrm{Cat}^{\mathrm{perf}}}$. Then note that

$$\begin{aligned} \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}}(y_{\mathrm{un}}\underline{\mathcal{D}}, \underset{\bullet \in \mathbf{\Delta}^{\mathrm{op}}}{\mathrm{colim}}\, y_{\mathrm{un}} \underline{\mathrm{Q}}_\bullet(\underline{\mathcal{C}})) &\simeq \underset{\bullet \in \mathbf{\Delta}^{\mathrm{op}}}{\mathrm{colim}}\, \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}}(y_{\mathrm{un}}\underline{\mathcal{D}}, y_{\mathrm{un}} \underline{\mathrm{Q}}_\bullet(\underline{\mathcal{C}})) \\ &\simeq \underset{\bullet \in \mathbf{\Delta}^{\mathrm{op}}}{\mathrm{colim}}\, \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}, \underline{\mathrm{Q}}_\bullet(\underline{\mathcal{C}}))^{\simeq} \\ &\simeq \underset{\bullet \in \mathbf{\Delta}^{\mathrm{op}}}{\mathrm{colim}}\, \big(\underline{\mathrm{Q}}_\bullet \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})\big)^{\simeq} =: \underline{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})) \end{aligned}$$

and hence, since $\underline{\mathrm{Fun}}^{\mathrm{ex}}(-, \underline{\mathcal{C}})$ preserves split Verdier sequences and since $\underline{\mathcal{K}}_{\mathcal{T}}$ is additive, we obtain that indeed $\mathrm{colim}_{\bullet \in \mathbf{\Delta}^{\mathrm{op}}} y_{\mathrm{un}} \underline{\mathrm{Q}}_\bullet(\underline{\mathcal{C}})$ is motivically local as claimed.

For the second part, recall we have the simplicial split Verdier sequence

$$\underline{\mathcal{C}} \to \underline{\mathrm{Déc}}_\bullet \underline{\mathcal{C}} \to \underline{\mathrm{Q}}_\bullet \underline{\mathcal{C}}$$

where we have adopted the terminology décalage from [CDH+21, Lem. 2.4.7]. The construction $\underline{\mathrm{Déc}}_\bullet\underline{\mathcal{C}}$ is also called the simplicial path object in [BGT13, Proof of Prop. 7.17]. Now since $j_{\mathrm{un}} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}$ sends split Verdier sequences to cofibre sequences by definition of unstable motives, and cofibre sequences are stable under colimits, we can apply j_{un} to the simplicial split Verdier sequence and take geometric realisation in $\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}$ to get a cofibre sequence in $\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}$

$$j_{\mathrm{un}}(\underline{\mathcal{C}}) \to \operatorname*{colim}_{n\in\mathbf{\Delta}^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Déc}}_\bullet\underline{\mathcal{C}} \to \operatorname*{colim}_{\bullet\in\mathbf{\Delta}^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\underline{\mathcal{C}}$$

But then we know that the middle term is always augmented over 0 and so is zero, hence the last term is a suspension of the first term, as required. $\square$

Theorem 4.2.15 (Motivic corepresentability of pointwise K-theory, "[CDH+, Prop. 2.1.5]")**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. Then there is a natural equivalence*

$$\underline{\mathrm{map}}_{\underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}}}(\mathcal{Z}_{\mathrm{pw}}\underline{\mathcal{C}}, \mathcal{Z}_{\mathrm{pw}}\underline{\mathcal{D}}) \simeq \underline{\mathrm{K}}^{\mathrm{pw}}_{\mathcal{T}}(\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$$

In particular, $\underline{\mathrm{K}}^{\mathrm{pw}}_{\mathcal{T}}$ is corepresented by $\mathcal{Z}_{\mathrm{pw}}\big(\underline{\mathrm{Sp}}^{\omega}\big)$ by Proposition 2.2.24.

Proof. Firstly, note that in $\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}$, $\Sigma^n j_{\mathrm{un}}\mathcal{D} \simeq \operatorname{colim}_{\bullet\in(\mathbf{\Delta}^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\mathcal{D}$ since

$$\begin{aligned}\Sigma^n j_{\mathrm{un}}\mathcal{D} \simeq \Sigma^{n-1}\big(\operatorname*{colim}_{\bullet\in\mathbf{\Delta}^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\mathcal{D}\big) &\simeq \operatorname*{colim}_{\bullet\in\mathbf{\Delta}^{\mathrm{op}}} \big(\Sigma^{n-1} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\mathcal{D}\big)\\ &\simeq \operatorname*{colim}_{\bullet\in\mathbf{\Delta}^{\mathrm{op}}} \big(\Sigma^{n-2}\big(\operatorname*{colim}_{\bullet\in\mathbf{\Delta}^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\mathcal{D}\big)\big)\end{aligned}$$

and so on by Lemma 4.2.14. Now, the left hand parametrised spectrum in the theorem statement is the one associated to the prespectrum whose n-th term, for $n \geq 1$, is

$$\begin{aligned}\underline{\mathrm{Map}}_{\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}}(j_{\mathrm{un}}\underline{\mathcal{C}}, \Sigma^n j_{\mathrm{un}}\underline{\mathcal{D}}) &\simeq \underline{\mathrm{Map}}_{\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}}(j_{\mathrm{un}}\underline{\mathcal{C}}, \operatorname*{colim}_{\bullet\in(\mathbf{\Delta}^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\underline{\mathcal{D}})\\ &\simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}}(y_{\mathrm{un}}\underline{\mathcal{C}}, \operatorname*{colim}_{\bullet\in(\mathbf{\Delta}^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\underline{\mathcal{D}})\\ &\simeq \operatorname*{colim}_{\bullet\in(\mathbf{\Delta}^{\mathrm{op}})^n} \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}}(y_{\mathrm{un}}\underline{\mathcal{C}}, j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\underline{\mathcal{D}})\\ &\simeq \operatorname*{colim}_{\bullet\in(\mathbf{\Delta}^{\mathrm{op}})^n} \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}}(\underline{\mathcal{C}}, \underline{\mathrm{Q}}_\bullet\underline{\mathcal{D}})\\ &\simeq \operatorname*{colim}_{\bullet\in(\mathbf{\Delta}^{\mathrm{op}})^n} \big(\underline{\mathrm{Q}}_\bullet\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})\big)^{\simeq}\\ &\simeq \Omega^\infty\Sigma^n\underline{\mathrm{K}}^{\mathrm{pw}}_{\mathcal{T}}\big(\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})\big)\end{aligned}$$

where the second equivalence is since for $n \geq 1$, $\operatorname{colim}_{\bullet\in(\mathbf{\Delta}^{\mathrm{op}})^n} y_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\mathcal{D}$ is already in $\underline{\mathrm{NMot}}^{\mathrm{pw,un}}_{\mathcal{T}}$ by Lemma 4.2.14; the fourth since j_{un} is $\mathcal{T}$-fully faithful; the fifth by Lemma 4.2.13; and the last by definition of $\underline{\mathrm{K}}^{\mathrm{pw}}_{\mathcal{T}}$. Hence both parametrised spectra in the statement have equivalent associated spectra, giving the desired conclusion. $\square$

As in Construction 4.2.8, we can construct $\underline{\mathrm{NMot}}^{\mathrm{nm,un}}_{\mathcal{T}}$, and we denote the canonical maps by $k_{\mathrm{un}} : \underline{\mathrm{Cat}}^{\mathrm{perf}} \to \underline{\mathrm{NMot}}^{\mathrm{nm,un}}_{\mathcal{T}}$ and $\mathcal{Z}_{\mathrm{nm}} : \underline{\mathrm{Cat}}^{\mathrm{perf}} \to \underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$. By definition of $\mathcal{R}_{\mathrm{nm}}$, the functors k_{un} and $\mathcal{Z}_{\mathrm{nm}}$ are in particular additive.

Proposition 4.2.16 ("[CDH+, Prop. 1.2.11]")**.** *There is a $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{NMot}}^{\mathrm{nm,un}}_{\mathcal{T}}$ such that $k_{\mathrm{un}} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \longrightarrow \underline{\mathrm{NMot}}^{\mathrm{nm,un}}_{\mathcal{T}}$ refines canonically to a $\mathcal{T}$-symmetric monoidal functor.*

Proof. We first argue for the case of small motives. From [NS22, Prop. 6.0.12], the Yoneda embedding $\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}})$ refines to a $\mathcal{T}$-symmetric monoidal functor. Furthermore, since the $\mathcal{T}$–cartesian symmetric monoidal on $\underline{\mathcal{S}}_{\mathcal{T}}$ is $\mathcal{T}$–distributive by [NS22, Prop. 3.2.5], we see by [NS22, Thm. 3.2.6] that the $\mathcal{T}$–Day convolution symmetric monoidal structure on $\underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}})$ is $\mathcal{T}$–distributive. Hence, by Proposition 2.3.7 and by construction of $\mathcal{R}_{\mathrm{nm}}$, $\underline{\mathrm{NMot}}^{\mathrm{nm,un},\kappa}_{\mathcal{T}}$ attains a canonical $\mathcal{T}$–symmetric monoidal structure which affords a refinement of k_{un} to a $\mathcal{T}$–symmetric monoidal functor. This completes the case of small motives.

Now for the case of the big motives, applying again [NS22, Cor. 6.0.12], we get that the $\mathcal{T}$-symmetric monoidal inclusion $\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}} \subseteq \widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa'}_{\mathcal{T}}$ induces a $\mathcal{T}$-symmetric monoidal refinement of $\underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}) \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa'}_{\mathcal{T}})$. On the other hand, Proposition 2.3.7 (2) implies that this induces a $\mathcal{T}$-symmetric monoidal refinement of $\underline{\mathrm{NMot}}^{\mathrm{nm,un},\kappa}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{nm,un},\kappa'}_{\mathcal{T}}$. Thus since filtered colimits of $\mathcal{T}$-symmetric monoidal categories are formed underlying by the straightforward parametrised analogue of [Lur17, §3.2.3], we obtain a canonical $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{NMot}}^{\mathrm{nm,un}}_{\mathcal{T}}$ together with a $\mathcal{T}$-symmetric monoidal refinement of $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{nm,un}}_{\mathcal{T}}$. □

Proposition 4.2.17 (Monoidality of normed motives)**.** *The $\mathcal{T}$-functor $\mathcal{Z}_{\mathrm{nm}} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$ canonically refines to a $\mathcal{T}$-symmetric monoidal functor.*

Proof. We already know that $k_{\mathrm{un}} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{nm,un}}_{\mathcal{T}}$ is canonically a $\mathcal{T}$-symmetric monoidal functor by Proposition 4.2.16. Moreover, by [Hil22b, Lem. 4.2.3] and Proposition 2.2.19, $\underline{\mathrm{NMot}}^{\mathrm{nm,un}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$ also refines uniquely to a $\mathcal{T}$-symmetric monoidal functor. □

Unlike in the pointwise situation where algebraic K–theory is a construction and its corepresentability in motives is a result, we now *define* the *normed* parametrised algebraic K–theory to be that which is corepresented by the unit in normed motives.

Definition 4.2.18. The *normed parametrised algebraic K–theory spectrum* $\underline{\mathrm{K}}^{\mathrm{nm}}_{\mathcal{T}}$ is defined as

$$\underline{\mathrm{K}}^{\mathrm{nm}}_{\mathcal{T}} \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \xrightarrow{\mathcal{Z}_{\mathrm{nm}}} \underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}} \xrightarrow{\underline{\mathrm{map}}(\mathbb{1},-)} \underline{\mathrm{Sp}}_{\mathcal{T}}$$

Observation 4.2.19. Let us now highlight some points based on all our considerations so far:

1. The $\mathcal{T}$–functor $\underline{\mathrm{K}}^{\mathrm{nm}}_{\mathcal{T}} \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{Sp}}_{\mathcal{T}}$ canonically refines to a $\mathcal{T}$–lax symmetric monoidal functor because $\underline{\mathrm{map}}(\mathbb{1},-)$ canonically refines to such.

2. Since $\mathcal{Z}_{\mathrm{nm}} \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$ is additive, by Theorem 4.2.11, we obtain a canonical comparison map
$$\Psi \colon \underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}} \longrightarrow \underline{\mathrm{NMot}}^{\mathrm{nm}}_{\mathcal{T}}$$
which also yields a transformation of additive functors
$$\big(\underline{\mathrm{K}}^{\mathrm{pw}}_{\mathcal{T}} \Rightarrow \underline{\mathrm{K}}^{\mathrm{nm}}_{\mathcal{T}}\big) \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \longrightarrow \underline{\mathrm{Sp}}_{\mathcal{T}}$$
We do not know in general if these comparison maps are equivalences. However, we are able to show that it *is* so in the case of equivariant algebraic K–theory for G a 2-group, and this is the content of the rest of the article.

3. It might be tempting to try to prove directly that the map $\underline{\mathrm{K}}^{\mathrm{pw}}_{\mathcal{T}} \Rightarrow \underline{\mathrm{K}}^{\mathrm{nm}}_{\mathcal{T}}$ is an equivalence by proving Theorem 4.2.15 in the case of normed motives. When one traces through this strategy, the key hiccup is in proving Lemma 4.2.14 that the Q–construction is already motivically local where we have used crucially that the parametrised algebraic K–theory

space functor $\underline{\mathcal{K}}_{\mathcal{T}}$ is additive, i.e. that it interacts well with maps in $\mathcal{R}_{\mathrm{pw}}$. What is needed in the normed setting is of course that it interacts well with maps in $\mathcal{R}_{\mathrm{nm}} \supseteq \mathcal{R}_{\mathrm{pw}}$. Indeed, as mentioned above, this is what we will show in the equivariant case for 2–groups in §4.3.

4.3 Pointwise K–theory is normed for 2-groups

In this subsection, we specialise the considerations of §4.2 to the case of $\mathcal{T} = \mathcal{O}_G$ where G is a finite group, giving G–equivariant algebraic K–theory. The end goal is to show Theorem 4.3.11, which says that $\underline{\mathrm{K}}_G$ refines to the structure of a normed ring G–spectrum when G is a 2–group. In other words, we will show that the comparison map

$$\Psi\colon \underline{\mathrm{NMot}}_G^{\mathrm{pw}} \to \underline{\mathrm{NMot}}_G^{\mathrm{nm}} \qquad \text{and hence also} \qquad \underline{\mathrm{K}}_G^{\mathrm{pw}} \Longrightarrow \underline{\mathrm{K}}_G^{\mathrm{nm}}$$

from Observation 4.2.19 (2) are equivalences. First, recall the notations $\mathcal{U}$ and λ from Construction 4.2.10. Let us also take stock of the theory developed in §3 in our current situation. While we have opted to use the more efficient stars and shrieks notation there, it would be beneficial now to switch to the more conventional notation in equivariant homotopy theory. We do this mainly because it is easier to keep track of the groups involved and because C_2–pushouts have an especially suggestive notation.

Notation 4.3.1. For a subgroup $H \leq G$ and $w\colon G/H \to G/G$ the unique map, we will from now on denote $w^*, w_!, w_*$ by $\mathrm{Res}^G_H, \mathrm{Ind}^G_H, \prod_{G/H}$ respectively. In the presence of G–symmetric monoidal structures, we will write N^G_H for $w_\otimes$. When $|G/H| = 2$, we will write the colimit of the C_2–pushouts discussed in Example 3.1.9 by $B\underline{\amalg}_A B$. In particular, by Proposition 3.2.2 in the case when $\underline{I} = w_!\underline{*}$, we may express $B\underline{\amalg}_A B$ as the *ordinary* pushout

$$\begin{tikzcd} \mathrm{Ind}^G_H \mathrm{Res}^G_H A \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{Ind}^G_H f"'] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d] \\ \mathrm{Ind}^G_H B \arrow[r] & B\underline{\amalg}_A B \end{tikzcd}$$

Finally, given a cofibre sequence $A \to B \to C$ of H–objects in a pointed G–category with a G–distributive symmetric monoidal structure, a straightforward combination of Proposition 3.2.8 and Proposition 3.3.2 gives us a cofibre sequence of G–objects

$$A \otimes B\underline{\amalg}_{\mathrm{N}^G_H A} B \otimes A \longrightarrow \mathrm{N}^G_H B \longrightarrow \mathrm{N}^G_H C$$

We aim to prove that $\mathcal{R}_{\mathrm{pw}}$ is $\underline{\otimes}$–multiplicatively closed, i.e. if $H \leq G$ and we have a split Verdier in $\mathrm{Cat}^{\mathrm{perf}}_H$

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, hook] & \underline{\mathcal{D}} \arrow[r, two heads] \arrow[l, dashed, bend right] \arrow[l, dashed, bend left] & \underline{\mathcal{E}} \arrow[l, dashed, bend right] \arrow[l, dashed, bend left] \end{tikzcd}$$

then $\mathrm{N}^G_H(\mathcal{U}(\underline{\mathcal{D}})/\mathcal{U}(\underline{\mathcal{C}})) \to \mathcal{U}(\mathrm{N}^G_H\underline{\mathcal{E}})$ induced by the λ-equivalence $\mathcal{U}(\underline{\mathcal{D}})/\mathcal{U}(\underline{\mathcal{C}}) \to \mathcal{U}(\underline{\mathcal{E}})$ is itself a λ-equivalence. If we can show this, then we would have shown that the inclusion $\mathcal{R}_{\mathrm{pw},\kappa} \subseteq \mathcal{R}_{\mathrm{nm},\kappa}$ (cf. §4) is an identification, and so the comparison map $\Psi : \underline{\mathrm{NMot}}_G^{\mathrm{pw}} \to \underline{\mathrm{NMot}}_G^{\mathrm{nm}}$ from Observation 4.2.19 is an equivalence. Since size issues will not play a role in our discussions here, we will suppress any mention of κ.

Corollary 4.3.2. *Let $H \leq G$ with $|G/H| = 2$. Suppose we have a pushout*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in a G–distributive symmetric monoidal G-stable category $\underline{\mathcal{C}}$. Then we have the pushout

$$\begin{tikzcd} A\otimes B\underline{\amalg}_{\mathrm{N}^G_H A} B\otimes A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \mathrm{N}^G_H B \arrow[d] \\ X\otimes Y\underline{\amalg}_{\mathrm{N}^G_H X} Y\otimes X \arrow[r] & \mathrm{N}^G_H Y \end{tikzcd}$$

Proof. Writing C for $\mathrm{cofib}(A\to B)\simeq \mathrm{cofib}(X\to Y)$, we get from the G/H-distributivity of N^G_H together with Notation 4.3.1 that we have the map of cofibre sequences

$$\begin{tikzcd} A\otimes B\underline{\amalg}_{\mathrm{N}^G_H A} B\otimes A \arrow[r] \arrow[d] & \mathrm{N}^G_H B \arrow[r] \arrow[d] & \mathrm{N}^G_H C \arrow[d, equal] \\ X\otimes Y\underline{\amalg}_{\mathrm{N}^G_H X} Y\otimes X \arrow[r] & \mathrm{N}^G_H Y \arrow[r] & \mathrm{N}^G_H C \end{tikzcd}$$

and so since $\underline{\mathcal{C}}$ was stable, the left square is a fibrewise pushout. ☐

Lemma 4.3.3. *Suppose $H\leq G$ with $|G/H|=2$, and $\underline{\mathcal{A}}\xrightarrow{i}\underline{\mathcal{B}}$ is a split Verdier inclusion in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_H$. Then the canonical map*

$$\mathcal{Z}(\underline{\mathcal{A}}\otimes\underline{\mathcal{B}})\underline{\amalg}_{\mathcal{Z}(\mathrm{N}^G_H\underline{\mathcal{A}})}\mathcal{Z}(\underline{\mathcal{B}}\otimes\underline{\mathcal{A}}) \longrightarrow \mathcal{Z}(\underline{\mathcal{A}}\otimes\underline{\mathcal{B}}\underline{\amalg}_{\mathrm{N}^G_H\underline{\mathcal{A}}}\underline{\mathcal{B}}\otimes\underline{\mathcal{A}})$$

is an equivalence in $\underline{\mathrm{NMot}}_G$.

Proof. By Notation 4.3.1 we have the pushout

$$\begin{tikzcd} \mathrm{Ind}^G_H\mathrm{Res}^G_H\mathrm{N}^G_H\underline{\mathcal{A}} \arrow[r, "\varepsilon"] \arrow[d, shift right] \arrow[dr, phantom, "\ulcorner", very near end] & \mathrm{N}^G_H\underline{\mathcal{A}} \arrow[d, shift right] \\ \mathrm{Ind}^G_H(\underline{\mathcal{A}}\otimes\underline{\mathcal{B}}) \arrow[r] \arrow[u, dashed, shift right] & \underline{\mathcal{A}}\otimes\underline{\mathcal{B}}\underline{\amalg}_{\mathrm{N}^G_H\underline{\mathcal{A}}}\underline{\mathcal{B}}\otimes\underline{\mathcal{A}} \arrow[u, dashed, shift right] \end{tikzcd}$$

which is moreover a right-split Verdier pushout by Lemma 4.1.15. Hence by Lemma 4.1.16 we obtain the pushout square

$$\begin{tikzcd} \mathrm{Ind}^G_H\mathrm{Res}^G_H\mathcal{Z}(\mathrm{N}^G_H\underline{\mathcal{A}}) \arrow[r, "\varepsilon"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \mathcal{Z}(\mathrm{N}^G_H\underline{\mathcal{A}}) \arrow[d] \\ \mathrm{Ind}^G_H\mathcal{Z}(\underline{\mathcal{A}}\otimes\underline{\mathcal{B}}) \arrow[r] & \mathcal{Z}(\underline{\mathcal{A}}\otimes\underline{\mathcal{B}}\underline{\amalg}_{\mathrm{N}^G_H A}\underline{\mathcal{B}}\otimes\underline{\mathcal{A}}) \end{tikzcd}$$

from which, using Notation 4.3.1, we may conclude as desired. ☐

Next, recall the notion of saturation closure from [Hil22b, Def. 6.3.5].

Observation 4.3.4. Let $\underline{\mathcal{C}}$ be equipped with a G–distributive symmetric monoidal structure. Suppose that S were a tensor ideal, i.e. if for any $Z\in\underline{\mathcal{C}}$ and $f\colon A\to B$ in S, we have that $\mathrm{id}_Z\otimes f$ is also in S. Then by a standard argument, we know that $\overline{S}$ is also a tensor ideal. To wit, consider the collection $U\subseteq\overline{S}$ of morphisms f such that $\mathrm{id}_Z\otimes f$ is again in $\overline{S}$ for any $Z\in\underline{\mathcal{C}}$. By hypothesis on S, we know that $S\subseteq U$. Moreover, since $Z\otimes -$ commutes with all colimits by G–distributivity of the symmetric monoidal structure, it is easy to see that the three axioms in [Hil22b, Def. 6.3.5] are satisfied so that U is a G–strongly saturated collection containing S, whence $U=\overline{S}$ by minimality of $\overline{S}$.

Lemma 4.3.5. *Let $H \leq G$ with $|G/H| = 2$ and $\underline{\mathcal{C}}$ a G–distributive symmetric monoidal G-stable category. Suppose S is a collection of morphisms in $\underline{\mathcal{C}}$ which is a tensor ideal and $\overline{S}$ its G-strong saturation. If N_H^G sends morphisms in S to morphisms in $\overline{S}$, then N_H^G also preserves all morphisms in the saturation $\overline{S}$.*

Proof. Write $U \subseteq \overline{S}$ for the collection of morphisms which get sent to a morphism in $\overline{S}$ by N_H^G. By hypothesis, $S \subseteq U$. We claim that U is G–strongly saturated. The 2-out-of-3 property is clear, and so we only have to check the first two axioms. To see axiom (2), we need to show that if $\partial : \underline{J} \to \underline{\mathrm{Fun}}_H(\Delta^1, \underline{\mathcal{C}})$ is a diagram that is pointwise in the full subcategory $\underline{\mathrm{Fun}}_H^U(\Delta^1, \underline{\mathcal{C}})$, then $\mathrm{N}_H^G \underline{\mathrm{colim}}_J \partial \in \mathrm{Fun}_G^{\overline{S}}(\Delta^1, \underline{\mathcal{C}})$. For this, recall by G/H-distributivity that $\mathrm{N}_H^G \underline{\mathrm{colim}}_{\underline{J}} \partial$ is computed as the cone point of the G-colimit diagram

$$\Big(\prod_{G/H} \underline{J}\Big)^{\triangleright} \to \prod_{G/H} (\underline{J}^{\triangleright}) \xrightarrow{\prod_{G/H} \partial} \prod_{G/H} \underline{\mathrm{Fun}}_H(\Delta^1, \underline{\mathcal{C}}) \xrightarrow{\mathrm{N}_H^G} \underline{\mathrm{Fun}}_G(\Delta^1, \underline{\mathcal{C}})$$

Now the hypothesis on ∂ ensures that, when restricted to $\prod_{G/H} \underline{J}$, this composite lands in $\underline{\mathrm{Fun}}_G^{\overline{S}}(\Delta^1, \underline{\mathcal{C}}) \subseteq \underline{\mathrm{Fun}}_G(\Delta^1, \underline{\mathcal{C}})$ and since by definition $\underline{\mathrm{Fun}}_G^{\overline{S}}(\Delta^1, \underline{\mathcal{C}})$ is closed under G-colimits, we obtain that the cone point $\mathrm{N}_H^G \underline{\mathrm{colim}}_J \partial$ is indeed in $\underline{\mathrm{Fun}}_G^{\overline{S}}(\Delta^1, \underline{\mathcal{C}})$ as required.

Finally, to see axiom (3), suppose we have a pushout of H–objects in $\underline{\mathcal{C}}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where the left vertical is in U (and so, by definition of saturation, the right vertical is in $\overline{S}$). Then by Corollary 4.3.2 we obtain the pushout square

$$\begin{tikzcd} A \otimes B \underline{\amalg}_{\mathrm{N}_H^G A} B \otimes A \arrow[r] \arrow[d] & \mathrm{N}_H^G B \arrow[d] \\ X \otimes Y \underline{\amalg}_{\mathrm{N}_H^G X} Y \otimes X \arrow[r] & \mathrm{N}_H^G Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{26}$$

Hence if we can show that the left vertical map is in $\overline{S}$, then by definition, the right vertical map will be in $\overline{S}$ too. For this, by Notation 4.3.1 we have

$$\begin{tikzcd} \mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{N}_H^G A \arrow[r, "\varepsilon"] \arrow[d] & \mathrm{N}_H^G A \arrow[d] \\ \mathrm{Ind}_H^G (A \otimes B) \arrow[r] & A \otimes B \underline{\amalg}_{\mathrm{N}_H^G A} A \otimes B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

and similarly for $X \otimes Y \underline{\amalg}_{\mathrm{N}_H^G X} X \otimes Y$. Since the respective maps on the upper three terms between the ones for the pair (A, B) and the ones for the pair (X, Y) are all in $\overline{S}$ by hypothesis (the bottom left uses that $\overline{S}$ is a tensor ideal by Observation 4.3.4), so is the induced map $A \otimes B \underline{\amalg}_{\mathrm{N}_H^G A} A \otimes B \to X \otimes Y \underline{\amalg}_{\mathrm{N}_H^G X} X \otimes Y$ by axiom (2) of [Hil22b, Def. 6.3.5]. Therefore, the pushout (26) gives that $\mathrm{N}_H^G B \to \mathrm{N}_H^G Y$ is also in $\overline{S}$ as required. □

Lemma 4.3.6. *Let $H \leq G$ with $|G/H| = 2$ and $\underline{\mathcal{C}}$ a G–distributive symmetric monoidal G-stable category. Let T be a collection of morphisms and $S \supseteq T$ the smallest tensor ideal containing T. If N_H^G sends morphisms in T to morphisms in $\overline{S}$, then N_H^G also preserves all morphisms in $\overline{S}$.*

Proof. By Lemma 4.3.5, it suffices to show that N_H^G sends morphisms in S to $\overline{S}$. So let $U \subseteq S$ be the collection of morphisms which get sent to $\overline{S}$ under N_H^G. By hypothesis, this contains T, and it is easy to check that it is also a tensor ideal because $\overline{S}$ is again a tensor ideal by Observation 4.3.4. Hence by minimality we see that $U = S$ as required. $\square$

To state the next result, observe that by Proposition 2.5.13 we have the identification $\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_!}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1} \simeq \underline{\mathrm{Fun}}_G(\Delta^1\underline{\amalg}_{\Delta^0}\Delta^1, \underline{\mathrm{Sp}}_G)$. Furthermore, by Corollary 2.2.20, we also have the identification $\bigotimes_{G/H}\underline{\mathrm{Fun}}_H(\Delta^1, \underline{\mathrm{Sp}}_H) \simeq \underline{\mathrm{Fun}}_G(\prod_{G/H}\Delta^1, \underline{\mathrm{Sp}}_G)$.

Lemma 4.3.7. *Let $s : \Delta^0 \hookrightarrow \Delta^1$ be the source inclusion, $H \leq G$ with $|G/H| = 2$, and $\varphi : \Delta^1\underline{\amalg}_{\Delta^0}\Delta^1 \hookrightarrow \prod_{G/H}\Delta^1$ the inclusion from Proposition 3.3.2. Then the functor*

$$\underline{\mathrm{Fun}}_G(\Delta^1\underline{\amalg}_{\Delta^0}\Delta^1, \underline{\mathrm{Sp}}_G) \to \underline{\mathrm{Fun}}_G(\prod_{G/H}\Delta^1, \underline{\mathrm{Sp}}_G)$$

induced by $\bigotimes_{G/H}(\underline{\mathrm{Sp}}_H \xrightarrow{s_!} \underline{\mathrm{Fun}}_H(\Delta^1, \underline{\mathrm{Sp}}_H))$ is given by left Kan extension along the inclusion φ, and so in particular preserves ω–compact objects and is G-fully faithful since φ is G-fully faithful.

Proof. Write $\partial\colon \underline{\Delta}^1 \to \underline{\mathrm{Cat}}_H$ for the map classifying $\underline{\Delta}^0 \xrightarrow{s} \underline{\Delta}^1$. By Corollary 2.2.20, we obtain the commuting square in

$$\begin{tikzcd}
\prod_{G/H}\underline{\Delta}^1 \arrow[r, "\prod_{G/H}\partial"] & \prod_{G/H}\underline{\mathrm{Cat}}_H \arrow[r, "\prod_{G/H}\underline{\mathrm{Fun}}(-{,}\underline{\mathrm{Sp}})"] \arrow[d, "\times"'] & \prod_{G/H}\underline{\mathrm{Pr}}_{H,L,\mathrm{st}} \arrow[d, "\otimes"] \\
& \underline{\mathrm{Cat}}_G \arrow[r, "\underline{\mathrm{Fun}}(-{,}\underline{\mathrm{Sp}})"] & \underline{\mathrm{Pr}}_{G,L,\mathrm{st}}
\end{tikzcd}$$

Note, importantly, that it is easy to see that the essential image of these compositions factor through $\underline{\mathrm{Pr}}_{G,L,\mathrm{st},\omega} \subset \underline{\mathrm{Pr}}_{G,L,\mathrm{st}}$. By the commutativity of this diagram and the identification $\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_!}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1} \simeq \underline{\mathrm{Fun}}_G(\Delta^1\underline{\amalg}_{\Delta^0}\Delta^1, \underline{\mathrm{Sp}}_G)$ from Proposition 2.5.13, applying $\underline{\mathrm{Fun}}(-, \underline{\mathrm{Sp}})$ to the diagram in $\underline{\mathrm{Cat}}_G$

$$\begin{tikzcd}
\underline{\Delta}^0 \arrow[r, hook, "s"] \arrow[d, hook, "s"'] & \underline{\Delta}^1 \arrow[d] \arrow[ddr, bend left, "s\times\mathrm{id}"] & \\
\underline{\Delta}^1 \arrow[r] \arrow[drr, bend right, "\mathrm{id}\times s"'] & \underline{\Delta}^1\underline{\amalg}_{\underline{\Delta}^0}\underline{\Delta}^1 \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[dr, hook, dashed, "\varphi"] & \\
& & \prod_{G/H}\underline{\Delta}^1
\end{tikzcd}$$

where the outer diagram is encoded by $\prod_{G/H}\underline{\Delta}^1$–shaped diagram $\prod_{G/H}\partial$ then yields the desired statement. $\square$

For the next result, recall the notation from Observation 4.1.11 as well as [Hil22b, Prop. 6.3.6] which says that strong saturations are the same as motivic equivalences in the current setting.

Proposition 4.3.8. *Let $H \leq G$ be a subgroup of index 2. Then N_H^G sends the morphism $t^* : \mathcal{U}((\underline{\mathrm{Sp}}_H^{\omega})^{\Delta^1})/\mathcal{U}(\underline{\mathrm{Sp}}_H^{\omega}) \to \mathcal{U}(\underline{\mathrm{Sp}}_H^{\omega})$ in $\mathcal{R}_{\mathrm{pw}}$ to a morphism in $\overline{\mathcal{R}}_{\mathrm{pw}}$.*

Proof. Note the following commutative square, which we learnt from Achim Krause.

$$\begin{tikzcd}
\underline{\mathrm{Sp}}_H \arrow[r, "s_!"] \arrow[d, equal] & \underline{\mathrm{Sp}}_H^{\Delta^1} \arrow[d, "\mathrm{cofib}"', "\simeq"] \\
\underline{\mathrm{Sp}}_H \arrow[r, "s_*"] & \underline{\mathrm{Sp}}_H^{\Delta^1}
\end{tikzcd}$$

Hence applying N_H^G to the whole square, we get in turn the diagram

$$\begin{tikzcd}
\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_!}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1} \simeq \underline{\mathrm{Fun}}_G(\Delta^1\underline{\amalg}_{\Delta^0}\Delta^1, \underline{\mathrm{Sp}}_G) \arrow[r, hook, "\varphi_!"] \arrow[d, "\simeq"'] & \underline{\mathrm{Fun}}_G(\prod_{G/H}\Delta^1, \underline{\mathrm{Sp}}_G) \simeq \mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1}) \arrow[d, "\simeq"] \\
\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1} \arrow[r] & \mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1})
\end{tikzcd}$$

where the G/H-pushout on the top left is with respect to the $s_!$ diagram and the bottom left is with respect to the s_* diagram. Since, by Lemma 4.3.7, the top arrow is $\varphi_!$ which preserves ω–compact objects and is G-fully faithful, so is the bottom arrow. Therefore, together with the G/H-distributivity of N_H^G, we obtain from Notation 4.3.1 the following solid cofibre sequence in $\mathrm{Pr}_{G,L,\mathrm{st},\omega} \simeq \underline{\mathrm{Cat}}^{\mathrm{perf}}_G$

$$\begin{tikzcd}
\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1} \arrow[r, hook] & \mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1}) \arrow[r, two heads, "(t\times t)^*"] & \mathrm{N}_H^G\underline{\mathrm{Sp}}_H \simeq \underline{\mathrm{Sp}}_G \arrow[l, dashed, bend right, "(t\times t)_!"'] \arrow[l, dashed, bend left]
\end{tikzcd}$$

which is moreover Verdier by Corollary 4.1.4. Note that this is then automatically split by Proposition 4.1.3 since the right hand Verdier projection admits the dashed adjoints by [Hil22b, Lem. 4.4.3], where everything in sight preserves ω–compact objects. Hence, upon applying $(-)^{\underline{\omega}}$ and by definition of the motivic localisation, the diagonal map in

$$\begin{tikzcd}
\dfrac{\mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1}))}{\mathcal{U}(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1})\underline{\amalg}^{s_*}_{\mathcal{U}(\underline{\mathrm{Sp}}_G^{\underline{\omega}})}\mathcal{U}(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1})} \arrow[r] \arrow[d] & \mathcal{U}(\mathrm{N}_H^G\underline{\mathrm{Sp}}_H^{\underline{\omega}}) \\
\dfrac{\mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1}))}{\mathcal{U}(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1}\underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G^{\underline{\omega}}}\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1})} \arrow[ur]
\end{tikzcd}$$

is a morphism in $\mathcal{R}_{\mathrm{pw}}$. So to show that the top horizontal map is in $\overline{\mathcal{R}}_{\mathrm{pw}}$, it will suffice to show that the left vertical map is in $\overline{\mathcal{R}}_{\mathrm{pw}}$: this is merely the observation that we have, by definition, a map of cofibre sequences in $\underline{\mathrm{PSh}}^{\mathrm{st}}(\underline{\mathrm{Cat}}^{\mathrm{perf}})$

$$\begin{tikzcd}
\mathcal{U}(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1})\underline{\amalg}^{s_*}_{\mathcal{U}(\underline{\mathrm{Sp}}_G^{\underline{\omega}})}\mathcal{U}(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1}) \arrow[r] \arrow[d] & \mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1})) \arrow[r] \arrow[d, equal] & \dfrac{\mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1}))}{\mathcal{U}(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1})\underline{\amalg}^{s_*}_{\mathcal{U}(\underline{\mathrm{Sp}}_G^{\underline{\omega}})}\mathcal{U}(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1})} \arrow[d] \\
\mathcal{U}(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1}\underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G^{\underline{\omega}}}\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1}) \arrow[r] & \mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1})) \arrow[r] & \dfrac{\mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1}))}{\mathcal{U}(\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1}\underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G^{\underline{\omega}}}\underline{\mathrm{Sp}}_H^{\underline{\omega},\Delta^1})}
\end{tikzcd}$$

and the left vertical is in $\overline{\mathcal{R}}_{\mathrm{pw}}$ by Lemma 4.3.3, and hence the right vertical is in $\overline{\mathcal{R}}_{\mathrm{pw}}$ too. □

Lemma 4.3.9. *Let $H \leq G$ with $|G/H| = 2$. Then N_H^G preserves morphisms in $\overline{\mathcal{R}}_{\mathrm{pw}}$.*

Proof. By Remark 4.2.5, we could have replaced $\mathcal{R}_{\mathrm{pw}}$ with the tensor ideal generated by the map $t^*\colon \mathcal{U}(\underline{\mathrm{Sp}}^{\omega,\Delta^1})/\mathcal{U}(\underline{\mathrm{Sp}}^{\omega}) \to \mathcal{U}(\underline{\mathrm{Sp}}^{\omega})$ coming from the Waldhausen split Verdier sequence. By Lemma 4.3.6, it suffices to show that N_H^G sends this morphism to one $\overline{\mathcal{R}}_{\mathrm{pw}}$. This input is in turn supplied by Proposition 4.3.8. □

The final ingredient to the main theorem is the following standard fact in group theory.

Proposition 4.3.10. *Let p be a prime, G be a p-group, and $H \leq G$ a subgroup. There is a normal series $H = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_k = G$ such that the quotients $N_m/N_{m-1} \cong C_p$ for all m.*

Proof. If $H \leq G$ is itself already normal, then this is immediate since we can just obtain this from the C_p-solvability of the p-group G/H. Suppose $H \leq G$ is a proper subgroup. We claim that we have the proper inclusion $H \lneq \mathrm{N}_H G$ into the normaliser: given this, we can now induct by taking successive normalisers and applying the statement in the case of $H \leq G$ being normal. To see the claim, consider the left action of H on the left H-cosets of G. Since H fixes the coset H, this action has a point with singleton orbit, and so since everything in sight are p-groups, we get from the orbit-stabiliser theorem that there is another H-coset gH, for some $g \in G\backslash H$, fixed by the left H–action. This means that for all $h \in H$, we get that $hgH = gH$, so that $g \in G\backslash H$ is a normaliser of H which is not in H, as asserted. □

Theorem 4.3.11. *Let G be a 2-group. The inclusion $\mathcal{R}_{\mathrm{pw},\kappa} \subseteq \mathcal{R}_{\mathrm{nm},\kappa}$ is an identification, and hence the comparison $\Psi : \underline{\mathrm{NMot}}_G^{\mathrm{pw}} \to \underline{\mathrm{NMot}}_G^{\mathrm{nm}}$ from Observation 4.2.19 is an equivalence.*

Proof. Let $H \leq G$ be a subgroup. We need to show that N_H^G preserves λ-equivalences. Equivalently, by [Hil22b, Prop. 6.3.6], we need to show N_H^G preserves morphisms in $\overline{\mathcal{R}}_{\mathrm{pw}}$, the G-strong saturation of $\mathcal{R}_{\mathrm{pw}} = \langle \underline{\mathcal{C}} \xrightarrow{s_*} \underline{\mathcal{C}}^{\Delta^1} \xrightarrow{t^*} \underline{\mathcal{C}} \rangle$. By Proposition 4.3.10, let $H = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_k = G$ be a C_2-normal series. Since $\mathrm{N}_H^G \simeq \mathrm{N}_{N_{k-1}}^{N_k} \circ \cdots \circ \mathrm{N}_{N_0}^{N_1}$, it would suffice to show that $\mathrm{N}_{N_{m-1}}^{N_m}$ preserves morphisms in $\overline{\mathcal{R}}_{\mathrm{pw}}$. But then $N_{m-1} \triangleleft N_m$ is a normal inclusion of index 2, and so this assertion is true by Lemma 4.3.9. □

In view of Observation 4.2.19, the following is now an immediate consequence of the theorem.

Corollary 4.3.12. *Let G be a 2–group. Then $\underline{\mathrm{K}}_G^{\mathrm{pw}} \Rightarrow \underline{\mathrm{K}}_G^{\mathrm{nm}} : \underline{\mathrm{Cat}}_G^{\mathrm{perf}} \longrightarrow \underline{\mathrm{Sp}}_G$ is an equivalence. In particular, $\underline{\mathrm{K}}_G^{\mathrm{pw}}$ refines to the a G–lax symmetric monoidal structure and induces*

$$\underline{\mathrm{K}}_G : \mathrm{CAlg}_G(\underline{\mathrm{Cat}}_G^{\mathrm{perf}}) \longrightarrow \mathrm{CAlg}_G(\underline{\mathrm{Sp}}_G)$$

4.4 Borel equivariant algebraic K–theory

Having performed a general analysis of normed equivariant algebraic K–theory, we record here a large source of examples via Theorem 2.4.10 coming from categories with G–actions.

Proposition 4.4.1. *Let G be a finite group. The functor $\mathrm{ev}_{G/e} : \underline{\mathrm{Cat}}_G^{\mathrm{perf}} \to \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{perf}})$ canonically refines to a G–symmetric monoidal functor $\mathrm{ev}_{G/e} : (\underline{\mathrm{Cat}}_G^{\mathrm{perf}})^{\underline{\otimes}} \to \underline{\mathrm{Bor}}((\mathrm{Cat}^{\mathrm{perf}})^{\otimes})$. Moreover, it admits a G–fully faithful right adjoint $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{perf}}) \hookrightarrow \underline{\mathrm{Cat}}_G^{\mathrm{perf}}$.*

Proof. By Theorem 2.4.10 (2), we are left to show that $\mathrm{ev}_{G/e}$ is the unit of the Bousfield localisation in the $\widehat{\mathrm{Cat}}$ version of Proposition 2.4.1. As noted there, this is fibrewise induced by taking homotopy fixed points in the target of the H–equivariant map $\mathrm{Res} : \mathrm{Cat}_H^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{perf}}$ to yield

$$\mathrm{ev} : \mathrm{Cat}_H^{\mathrm{perf}} \to (\mathrm{Cat}^{\mathrm{perf}})^{hH} \simeq \mathrm{Fun}(BH, \mathrm{Cat}^{\mathrm{perf}})$$

as desired. We now immediately obtain that the G–right adjoint is as claimed because fibrewise the adjunction is given by the dashed lift

$$\begin{tikzcd}
\mathrm{Cat}_H^{\mathrm{perf}} \arrow[r, shift left, "\mathrm{ev}_{H/e}"] \arrow[d, hook] & \mathrm{Fun}(BH, \mathrm{Cat}^{\mathrm{perf}}) \arrow[l, shift left, dashed, hook'] \arrow[dl, shift left, hook'] \\
\mathrm{Mack}_H(\mathrm{Cat}^{\mathrm{perf}}) \arrow[ur, shift left, "\mathrm{ev}_{H/e}"] &
\end{tikzcd}$$

for which the diagonal Bousfield localisation is given for instance by [BGS20, §8] (compare also with Proposition 2.4.4), and the vertical inclusion is by Theorem 2.5.11. $\square$

Corollary 4.4.2. *Let G be a 2–group. Then the G–functor*

$$\underline{\mathrm{K}}_G : \underline{\mathrm{Bor}}_G(\mathrm{Cat}^{\mathrm{perf}}) \hookrightarrow \underline{\mathrm{Cat}}_G^{\mathrm{perf}} \subset \underline{\mathrm{Mack}}_G(\mathrm{Cat}^{\mathrm{perf}}) \xrightarrow{\underline{\mathrm{Mack}}_G(\mathrm{K})} \underline{\mathrm{Sp}}_G \tag{27}$$

canonically refines to a G–lax symmetric monoidal functor $\underline{\mathrm{K}}_G\colon \underline{\mathrm{Bor}}((\mathrm{Cat}^{\mathrm{perf}})^{\otimes}) \longrightarrow \underline{\mathrm{Sp}}_G^{\otimes}$. In particular, we obtain a functor $\mathrm{K}_G\colon\ \mathrm{Fun}\left(BG, \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})\right)^{\simeq} \longrightarrow \mathrm{CAlg}_G\left(\underline{\mathrm{Sp}}_G^{\otimes}\right)^{\simeq}$.

Proof. That the functor $\underline{\mathrm{Cat}}_G^{\mathrm{perf}} \to \underline{\mathrm{Sp}}_G$ refines to a G–lax symmetric monoidal functor is by Corollary 4.3.12. That the inclusion $\underline{\mathrm{Bor}}_G(\mathrm{Cat}^{\mathrm{perf}}) \hookrightarrow \underline{\mathrm{Cat}}_G^{\mathrm{perf}}$ refines to a G–lax symmetric monoidal functor is because it is right adjoint to $\mathrm{ev}_{G/e}$, which in turn enhances to a G–symmetric monoidal functor by Proposition 4.4.1. Finally, applying CAlg_G to this composite and Theorem 2.4.10 (3) gives the last statement. $\square$

Example 4.4.3. We collect here two important sources of examples, showing that normed equivariant algebraic K–theory are in ample supply.

1. Since $\underline{\mathrm{Sp}}_G^{\omega} \in (\underline{\mathrm{Cat}}_G^{\mathrm{perf}})^{\otimes}$ is the unit object, it is a G–commutative algebra, and hence $\underline{\mathrm{K}}_G(\underline{\mathrm{Sp}}_G^{\omega})$ canonically refines a G–normed ring spectrum. In light of [BH21, Prop. 7.6] - the connection to which we do not make precise in our work - we expect that *any* G–normed ring spectrum will give rise to a G–normed ring K–theory spectrum. This would then specialise to the case above by considering the G–normed ring spectrum $\mathbb{S}_G$.

2. By the last statement in Corollary 4.4.2, any symmetric monoidal perfect–stable $\mathcal{C}^{\otimes}$ equipped with a symmetric monoidal G–action gives rise to a collection of spectra $\left\{\mathrm{K}(\mathcal{C}^{hH})\right\}_{H\leq G}$ which canonically assemble to a G–normed ring spectrum.

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