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POISSON-COMMUTATIVE SUBALGEBRAS OF $\mathcal{S}(\mathfrak{g})$ ASSOCIATED WITH INVOLUTIONS

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ABSTRACT. The symmetric algebra $\mathcal{S}(\mathfrak{g})$ of a reductive Lie algebra $\mathfrak{g}$ is equipped with the standard Poisson structure, i.e., the Lie–Poisson bracket. Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{g})$ attract a great deal of attention, because of their relationship to integrable systems and, more recently, to geometric representation theory. The transcendence degree of a Poisson-commutative subalgebra $\mathcal{C} \subset \mathcal{S}(\mathfrak{g})$ is bounded by the "magic number" $\boldsymbol{b}(\mathfrak{g})$ of $\mathfrak{g}$. The "argument shift method" of Mishchenko–Fomenko was basically the only known source of $\mathcal{C}$ with $\mathrm{tr.deg}\,\mathcal{C} = \boldsymbol{b}(\mathfrak{g})$. We introduce an essentially different construction related to symmetric decompositions $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Poisson-commutative subalgebras $\mathcal{Z}, \tilde{\mathcal{Z}} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ of the maximal possible transcendence degree are presented. If the $\mathbb{Z}_2$-contraction $\mathfrak{g}_0 \ltimes \mathfrak{g}_1^{\mathsf{ab}}$ has a polynomial ring of symmetric invariants, then $\tilde{\mathcal{Z}}$ is a polynomial maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$, and its free generators are explicitly described.

CONTENTS

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INTRODUCTION

The ground field $\Bbbk$ is algebraically closed and of characteristic 0. A commutative associative $\Bbbk$-algebra $\mathcal{A}$ is a *Poisson algebra* if there is an additional anticommutative bilinear operation $\{\ ,\ \}\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ called a *Poisson bracket* such that

$$\{a,bc\}=\{a,b\}c+b\{a,c\}, \qquad \text{(the Leibniz rule)}$$
$$\{a,\{b,c\}\}+\{b,\{c,a\}\}+\{c,\{a,b\}\}=0 \quad \text{(the Jacobi identity)}$$

for all $a,b,c\in\mathcal{A}$. A subalgebra $\mathcal{C}\subset\mathcal{A}$ is *Poisson-commutative* if $\{\mathcal{C},\mathcal{C}\}=0$. The *Poisson centre* $\mathcal{ZA}$ of $\mathcal{A}$ is defined by the condition $\mathcal{ZA}=\{z\in\mathcal{A}\mid\{z,a\}=0\ \forall a\in\mathcal{A}\}$.

Usually, Poisson algebras occur as algebras of functions on varieties (manifolds), and we are only interested in the case, where such a variety is an affine n-space $\mathbb{A}^n$ and hence $\mathcal{A}=\Bbbk[\mathbb{A}^n]$ is a polynomial ring in n variables. Two Poisson brackets on $\mathbb{A}^n$ are said to be *compatible*, if all their linear combinations are again Poisson brackets.

There is a general method for constructing a "large" Poisson-commutative subalgebra of $\mathcal{A}$ associated with a pair of compatible brackets, see e.g. [BB02]. Let $\{\ ,\ \}'$ and $\{\ ,\ \}''$ be compatible Poisson brackets on $\mathbb{A}^n$. This yields a two parameter family of Poisson brackets $a\{\ ,\ \}'+b\{\ ,\ \}''$, $a,b\in\Bbbk$. As we are only interested in the corresponding Poisson centres, it is convenient to organise this, up to scaling, in a 1-parameter family $\{\ ,\ \}_t=\{\ ,\ \}'+t\{\ ,\ \}''$, $t\in\mathbb{P}=\Bbbk\cup\{\infty\}$, where $t=\infty$ corresponds to the bracket $\{\ ,\ \}''$. The *central rank* $\mathsf{rkc}\{\ ,\ \}$ of a Poisson bracket $\{\ ,\ \}$ is defined as the codimension of a symplectic leaf in general position, see Definition 1. For almost all $t\in\mathbb{P}$, $\mathsf{rkc}\{\ ,\ \}_t$ has one and the same (minimal) value, and we set $\mathbb{P}_{\mathsf{reg}}=\{t\in\mathbb{P}\mid \mathsf{rkc}\{\ ,\ \}_t \text{ is minimal}\}$, $\mathbb{P}_{\mathsf{sing}}=\mathbb{P}\setminus\mathbb{P}_{\mathsf{reg}}$. Let $\mathcal{Z}_t$ denote the centre of $(\mathcal{A},\{\ ,\ \}_t)$. The key fact is that the algebra $\mathcal{Z}$ generated by $\{\mathcal{Z}_t\mid t\in\mathbb{P}_{\mathsf{reg}}\}$ is Poisson-commutative w.r.t. to any bracket in the family. In many cases, this construction provides a Poisson-commutative subalgebra of $\mathcal{A}$ of maximal transcendence degree. We demonstrate this with a well-known important example.

Example 0.1. For any finite-dimensional Lie algebra $\mathfrak{q}$, the dual space $\mathfrak{q}^*$ has a Poisson structure. Here $\Bbbk[\mathfrak{q}^*]\cong\mathcal{S}(\mathfrak{q})$ and the Lie–Poisson bracket $\{\ ,\ \}_{\mathsf{LP}}$ is defined by $\{\xi,\eta\}_{\mathsf{LP}}=[\xi,\eta]$ for $\xi,\eta\in\mathfrak{q}$. The Poisson centre of $(\mathcal{S}(\mathfrak{q}),\{\ ,\ \}_{\mathsf{LP}})$ coincides with the ring $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ of symmetric $\mathfrak{q}$-invariants. The celebrated "argument shift method", which goes back to Mishchenko–Fomenko [MF78], provides large Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{q})$ starting from the Poisson centre $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Given $\gamma\in\mathfrak{q}^*$, the γ-shift of argument produces the *Mishchenko–Fomenko subalgebra* $\mathcal{A}_\gamma$. Namely, for $F\in\mathcal{S}(\mathfrak{q})=\Bbbk[\mathfrak{q}^*]$, let $\partial_\gamma F$ be the directional derivative of F with respect to γ, i.e.,

$$\partial_\gamma F(x)=\frac{d}{dt}F(x+t\gamma)\Big|_{t=0}.$$

Then $\mathcal{A}_\gamma$ is generated by all $\partial_\gamma^k F$ with $k \geqslant 0$ for all $F \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. The core of this method is that for any $\gamma \in \mathfrak{q}^*$ there is the Poisson bracket $\{\ ,\ \}_\gamma$ on $\mathfrak{q}^*$ such that $\{\xi, \eta\}_\gamma = \gamma([\xi, \eta])$ for $\xi, \eta \in \mathfrak{q}$, and that this new bracket is compatible with $\{\ ,\ \}_{\mathsf{LP}}$. One can prove that $\mathsf{rkc}\{\ ,\ \}_t$ takes one and the same value for all $\{\ ,\ \}_t = \{\ ,\ \}_{\mathsf{LP}} + t\{\ ,\ \}_\gamma$ with $t \in \Bbbk$, i.e., $\Bbbk \subset \mathbb{P}_{\mathsf{reg}}$, and $\mathcal{A}_\gamma$ is generated by all the corresponding centres $\mathcal{Z}_t$, $t \in \Bbbk$. (Actually, $\mathbb{P}_{\mathsf{reg}} = \mathbb{P}$ if and only if γ is regular in $\mathfrak{q}^*$.) The importance of these subalgebras and their quantum counterparts is explained e.g. in [FFR, Vi91]. If $\mathfrak{q}$ is reductive and γ is regular, then $\mathcal{A}_\gamma$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q})$ [PY08].

Our main object is a certain 1-parameter family of Poisson brackets on the dual of a semisimple Lie algebra $\mathfrak{g}$. Let σ be an involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the corresponding $\mathbb{Z}_2$-*grading* (or *symmetric decomposition*). We also say that $(\mathfrak{g}, \mathfrak{g}_0)$ is a *symmetric pair*. Without loss of generality, we may assume that the pair $(\mathfrak{g}, \sigma)$ is *indecomposable*, i.e., $\mathfrak{g}$ has no proper σ-stable ideals. Then either $\mathfrak{g}$ is simple or $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is simple and σ is a permutation. Our family of Poisson brackets is related to the decomposition:

$$\{\ ,\ \}_{\mathsf{LP}} = \{\ ,\ \}_{0,0} + \{\ ,\ \}_{0,1} + \{\ ,\ \}_{1,1},$$

where $\{\ ,\ \}_{i,j} = [\ ,\ ]_{i,j} \colon \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ for $i, j \in \mathbb{Z}_2 \simeq \{0, 1\}$, see Section 2 for details. Using this, we consider the 1-parameter family of Poisson brackets on $\mathfrak{g}^*$:

$$\{\ ,\ \}_t = \{\ ,\ \}_{0,0} + \{\ ,\ \}_{0,1} + t\{\ ,\ \}_{1,1}, \tag{0·1}$$

where $t \in \mathbb{P}$ and $\{\ ,\ \}_\infty = \{\ ,\ \}_{1,1}$. Each element of this family is a Poisson bracket and here $\mathbb{P}_{\mathsf{reg}} = \Bbbk$ unless $\mathfrak{g} = \mathfrak{sl}_2$. For $\mathfrak{sl}_2$, one has $\mathbb{P}_{\mathsf{reg}} = \mathbb{P}$, and this case has to be considered separately. Nevertheless, the final result can be stated uniformly, for all simple $\mathfrak{g}$, see below.

Let $\mathcal{Z}_t$ ($t \in \mathbb{P}$) denote the centre of $(\mathcal{S}(\mathfrak{g}), \{\ ,\ \}_t)$ and $\mathcal{Z}$ the subalgebra of $\mathcal{S}(\mathfrak{g})$ generated by all $\mathcal{Z}_t$ with $t \in \mathbb{P}_{\mathsf{reg}}$. Then $\{\mathcal{Z}, \mathcal{Z}\}_{\mathsf{LP}} = 0$. Moreover, $\{\mathfrak{g}_0, \mathcal{Z}\}_{\mathsf{LP}} = 0$, i.e., $\mathcal{Z}$ is a Poisson commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$. By [MY, Prop. 1.1], we have

$$\mathrm{tr.deg}\,\mathcal{C} \leqslant \frac{1}{2}(\dim \mathfrak{g}_1 + \mathsf{rk}\,\mathfrak{g} + \mathsf{rk}\,\mathfrak{g}_0)$$

for any Poisson-commutative subalgebra $\mathcal{C} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$. We prove that this upper bound is attained for $\mathcal{Z}$, see Theorem 2.7.

The computation of $\mathrm{tr.deg}\,\mathcal{Z}$ is completely general and is valid for any σ. However, this is not the case with more subtle properties. Our goal is to realise whether $\mathcal{Z}$ is polynomial and is maximal Poisson commutative in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$. For $t = 0$ in Eq. (0·1), one obtains the Lie–Poisson bracket of the Lie algebra $\mathfrak{g}_{(0)} := \mathfrak{g}_0 \ltimes \mathfrak{g}_1^{\mathsf{ab}}$. The symmetric invariants of $\mathfrak{g}_{(0)}$ have intensively been studied in [P07′, Y14, Y17]. The output is that there are four "bad" involutions of a simple $\mathfrak{g}$ in which $\mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{g}_{(0)}}$ is not polynomial. These cases are related to $\mathfrak{g}$ of type $\mathcal{E}_n$. In all other cases, $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ has a *good generating system* (= g.g.s.) for $(\mathfrak{g}, \mathfrak{g}_0)$, say

$H_1, \dots, H_l$ ($l = \mathsf{rk}\,\mathfrak g$), and a set of free generators of $\mathcal S(\mathfrak g_{(0)})^{\mathfrak g_{(0)}}$ is then obtained from the H_i's via a simple procedure, see Section 3 for details.

In the rest of the introduction, we assume that σ is "good" and $\mathfrak g \ne \mathfrak{sl}_2$. In particular, there is a g.g.s. for $(\mathfrak g, \mathfrak g_0)$. This is of vital importance for us, because we then prove that $\mathcal Z$ is freely generated by the nonzero bi-homogeneous components of all H_i's and is therefore polynomial, see Theorems 3.3 and 3.6. Let $r_0 \colon \mathcal S(\mathfrak g)^{\mathfrak g} \to \mathcal S(\mathfrak g_0)^{\mathfrak g_0}$ be the restriction homomorphism related to the embedding $\mathfrak g_0^* \hookrightarrow \mathfrak g^* = \mathfrak g_0^* \oplus \mathfrak g_1^*$. Furthermore,

- $\mathcal Z$ is a maximal Poisson commutative subalgebra of $\mathcal S(\mathfrak g)^{\mathfrak g_0}$ if and only if r_0 is onto, see Theorem 4.5.
- In general, let $\tilde{\mathcal Z}$ be the subalgebra of $\mathcal S(\mathfrak g)$ generated by $\mathcal Z$ and $\mathcal S(\mathfrak g_0)^{\mathfrak g_0}$. (Hence $\tilde{\mathcal Z} = \mathcal Z$ if and only if r_0 is onto.) We prove that $\tilde{\mathcal Z}$ is still polynomial and that it is a maximal Poisson commutative subalgebra of $\mathcal S(\mathfrak g)^{\mathfrak g_0}$, see Theorem 4.12. This statement also embraces the $\mathfrak{sl}_2$-case, because then $\mathcal Z = \tilde{\mathcal Z}$ is polynomial, etc.

In Section 5, we present a Poisson interpretation of the Kostant regularity criterion for $\mathfrak g$ [K63, Theorem 9] and give new related formulas arising from $\mathbb Z_2$-gradings and compatible Poisson structures. As a by-product, we describe $\mathcal Z_\infty$ for all σ.

Section 6.1 contains a discussion of possible quantisations of $\mathcal Z$ and $\tilde{\mathcal Z}$, i.e., their lifting to the enveloping algebra $\mathcal U(\mathfrak g)$. We conjecture that quantum analogues of these algebras may have applications in representation theory, and more explicitly, in the branching problem $\mathfrak g \downarrow \mathfrak g_0$. In Section 6.2, it is explained how to construct a polynomial maximal Poisson-commutative subalgebra of $\mathcal S(\mathfrak g)$ related to a chain of symmetric subalgebras

$$\mathfrak g = \mathfrak g^{(0)} \supset \mathfrak g^{(1)} \supset \mathfrak g^{(2)} \supset \dots \supset \mathfrak g^{(m)}$$

with $[\mathfrak g^{(m)}, \mathfrak g^{(m)}] = 0$.

In the Appendix, we gather auxiliary results on the kernels of a 1-parameter family of skew-symmetric bilinear forms on a vector space.

We refer to [DZ05] for generalities on Poisson varieties, Poisson tensors, symplectic leaves, etc.

1. Preliminaries on the coadjoint representation

Let Q be a connected affine algebraic group with Lie algebra $\mathfrak q$. The symmetric algebra $\mathcal S(\mathfrak q)$ over $\Bbbk$ is identified with the graded algebra of polynomial functions on $\mathfrak q^*$, and we also write $\Bbbk[\mathfrak q^*]$ for it.

Let $\mathfrak q^\xi$ denote the stabiliser in $\mathfrak q$ of $\xi \in \mathfrak q^*$. The *index of* $\mathfrak q$, $\mathrm{ind}\,\mathfrak q$, is the minimal codimension of Q-orbits in $\mathfrak q^*$. Equivalently, $\mathrm{ind}\,\mathfrak q = \min_{\xi\in\mathfrak q^*} \dim \mathfrak q^\xi$. By Rosenlicht's theorem [VP89, 2.3], one also has $\mathrm{ind}\,\mathfrak q = \mathrm{tr.deg}\,\Bbbk(\mathfrak q^*)^Q$. The "magic number" associated with $\mathfrak q$ is $\boldsymbol b(\mathfrak q) = (\dim \mathfrak q + \mathrm{ind}\,\mathfrak q)/2$. Since the coadjoint orbits are even-dimensional, the magic number is an integer. If $\mathfrak{q}$ is reductive, then $\operatorname{ind}\mathfrak{q} = \mathsf{rk}\,\mathfrak{q}$ and $\boldsymbol{b}(\mathfrak{q})$ equals the dimension of a Borel subalgebra. The Lie–Poisson bracket on $\Bbbk[\mathfrak{q}^*]$ is defined on the elements of degree 1 (i.e., on $\mathfrak{q}$) by $\{x, y\}_{\mathsf{LP}} := [x, y]$. The *Poisson centre* of $\mathcal{S}(\mathfrak{q})$ is

$$\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \{H \in \mathcal{S}(\mathfrak{q}) \mid \{H, x\}_{\mathsf{LP}} = 0 \ \ \forall x \in \mathfrak{q}\}.$$

As Q is connected, we have $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathcal{S}(\mathfrak{q})^{Q} = \Bbbk[\mathfrak{q}^*]^{Q}$. The set of *$Q$-regular* elements of $\mathfrak{q}^*$ is

$$\mathfrak{q}^*_{\mathsf{reg}} = \{\eta \in \mathfrak{q}^* \mid \dim \mathfrak{q}^{\eta} = \operatorname{ind}\mathfrak{q}\}. \tag{1·1}$$

Set $\mathfrak{q}^*_{\mathsf{sing}} = \mathfrak{q}^* \setminus \mathfrak{q}^*_{\mathsf{reg}}$. We say that $\mathfrak{q}$ has the *codim*–n property if $\operatorname{codim}\mathfrak{q}^*_{\mathsf{sing}} \geqslant n$. By [K63], the semisimple algebras $\mathfrak{g}$ have the *codim*–3 property.

1.1. **The Poisson tensor.** Let Ω^i be the $\mathcal{A}$-module of differential i-forms on $\mathbb{A}^n$. Then $\Omega = \bigoplus_{i=0}^{n} \Omega^i$ is the $\mathcal{A}$-algebra of regular differential forms on $\mathbb{A}^n$. Likewise, $\mathcal{W} = \bigoplus_{i=0}^{n} \mathcal{W}^i$ is the graded skew-symmetric algebra of polyvector fields generated by the $\mathcal{A}$-module $\mathcal{W}^1$ of polynomial vector fields on $\mathbb{A}^n$. Both algebras are free $\mathcal{A}$-modules. If $\mathbb{A}^n$ has a Poisson structure $\{\ ,\ \}$, then π is the corresponding *Poisson tensor (bivector)*. That is, $\pi \in \operatorname{Hom}_{\mathcal{A}}(\Omega^2, \mathcal{A})$ is defined by the equality $\pi(\boldsymbol{d}f \wedge \boldsymbol{d}g) = \{f, g\}$ for $f, g \in \mathcal{A}$. Then $\pi(x)$, $x \in \mathbb{A}^n$, defines a skew-symmetric bilinear form on $T^*_x(\mathbb{A}^n) \simeq (\mathbb{A}^n)^*$. Formally, if $v = \boldsymbol{d}_x f$ and $u = \boldsymbol{d}_x g$ for $f, g \in \mathcal{A}$, then $\pi(x)(v, u) = \pi(\boldsymbol{d}f \wedge \boldsymbol{d}g)(x) = \{f, g\}(x)$.

Definition 1. The *central rank* of a Poisson bracket $\{\ ,\ \}$ on $\mathbb{A}^n$, denoted $\mathsf{rkc}\{\ ,\ \}$, is the minimal codimension of the symplectic leaves in $\mathbb{A}^n$.

It is easily seen that if π is the corresponding Poisson tensor, then

$$\mathsf{rkc}\{\ ,\ \} = \min_{x \in \mathbb{A}^n} \dim \ker \pi(x) = n - \max_{x \in \mathbb{A}^n} \mathsf{rk}\,\pi(x).$$

Example. For a Lie algebra $\mathfrak{q}$ and the dual space $\mathfrak{q}^*$ equipped with the Lie–Poisson bracket $\{\ ,\ \}_{\mathsf{LP}}$, the symplectic leaves are the coadjoint Q-orbits. Hence $\mathsf{rkc}\{\ ,\ \}_{\mathsf{LP}} = \operatorname{ind}\mathfrak{q}$.

In view of the duality between differential 1-forms and vector fields, we may regard π as an element of $\mathcal{W}^2$. Let $[[\ ,\ ]] : \mathcal{W}^i \times \mathcal{W}^j \to \mathcal{W}^{i+j-1}$ be the Schouten bracket. The Jacobi identity for π is equivalent to that $[[\pi, \pi]] = 0$, see e.g. [DZ05, Chapter 1.8].

Lemma 1.1. *Two Poisson brackets $\{\ ,\ \}'$ and $\{\ ,\ \}''$ are compatible if and only if a sole linear combination, non-proportional to either of the initial brackets, is a Poisson bracket.*

Proof. In place of Poisson brackets, we may consider the corresponding Poisson tensors. Given two tensors π' and π'', consider $R = a\pi' + b\pi''$ with $a, b \in \Bbbk^{\times}$. Then R is a Poisson tensor if and only if $[[R, R]] = 0$. In view of the fact that $[[\pi', \pi'']] = [[\pi'', \pi']]$, this reduces to the condition $[[\pi', \pi'']] = 0$ regardless of nonzero a, b. □

1.2. **Contractions and compatibility.** Let $\mathfrak q = \mathfrak h \oplus V$ be a vector space decomposition, where $\mathfrak h$ is a subalgebra. For any $s \in \Bbbk^\times$, define a linear map $\varphi_s : \mathfrak q \to \mathfrak q$ by setting $\varphi_s|_{\mathfrak h} = \mathsf{id}$, $\varphi_s|_V = s{\cdot}\mathsf{id}$. Then $\varphi_s\varphi_{s'} = \varphi_{ss'}$ and $\varphi_s^{-1} = \varphi_{s^{-1}}$, i.e., this yields a one-parameter subgroup of $\mathrm{GL}(\mathfrak q)$. The invertible map φ_s defines a new (isomorphic to the initial) Lie algebra structure $[\ ,\ ]_{(s)}$ on the same vector space $\mathfrak q$ by the formula

$$[x, y]_{(s)} = \varphi_s^{-1}([\varphi_s(x), \varphi_s(y)]). \tag{1·2}$$

The corresponding Poisson bracket is $\{\ ,\ \}_s$. We naturally extend φ_s to an automorphism of $\mathcal S(\mathfrak q)$. Then the centre of the Poisson algebra $(\mathcal S(\mathfrak q), \{\ ,\ \}_s)$ equals $\varphi_s^{-1}(\mathcal S(\mathfrak q)^{\mathfrak q})$.

The condition $[\mathfrak h, \mathfrak h] \subset \mathfrak h$ implies that there is the limit of the brackets $[\ ,\ ]_{(s)}$ as s tends to zero. The limit bracket is denoted by $[\ ,\ ]_{(0)}$ and the corresponding Lie algebra is the semi-direct product $\mathfrak h \ltimes V^{\mathsf{ab}}$, where $[V^{\mathsf{ab}}, V^{\mathsf{ab}}]_{(0)} = 0$. The algebra $\mathfrak h \ltimes V^{\mathsf{ab}}$ is called an *Inönü-Wigner* or *one-parameter contraction* of $\mathfrak q$, see e.g. [PY12, Y14].

Having a family of Poisson brackets $\{\ ,\ \}_s$ on $\mathfrak q^*$ associated with the maps φ_s, it is natural to ask whether these brackets are compatible.

Lemma 1.2. *As above, let $\mathfrak q = \mathfrak h \oplus V$, where $\mathfrak h \subset \mathfrak q$ is a subalgebra. Let $s, s' \in \Bbbk$.*

- (i) *If $(\mathfrak q, \mathfrak h)$ is a symmetric pair, i.e., $[\mathfrak h, V] \subset V$ and $[V, V] \subset \mathfrak h$, then $\{\ ,\ \}_s = \{\ ,\ \}_{-s}$ and $\{\ ,\ \}_s + \{\ ,\ \}_{s'} = 2\{\ ,\ \}_{\tilde s}$ with $2\tilde s^2 = s^2 + (s')^2$.*
- (ii) *If $[V, V] \subset V$, i.e., V is a subalgebra, too, then $\{\ ,\ \}_s + \{\ ,\ \}_{s'} = 2\{\ ,\ \}_{\tilde s}$ with $2\tilde s = s + s'$.*

Proof. All statements are verified by an easy direct computation. □

In this article, we are interested in case (i) of Lemma 1.2 under the assumption that $\mathfrak q$ is semisimple.

2. Constructing a Poisson-commutative subalgebra $\mathcal Z$

Let $\mathfrak g$ be a $\mathbb Z_2$-graded semisimple Lie algebra and σ the corresponding involution of $\mathfrak g$, i.e., $\mathfrak g = \mathfrak g_0 \oplus \mathfrak g_1$ and $\sigma(x) = (-1)^j x$ for $x \in \mathfrak g_j$. Occasionally, we will need the related connected algebraic groups G and G_0, i.e., $\mathfrak g = \mathsf{Lie}\,(G)$ and $\mathfrak g_0 = \mathsf{Lie}\,(G_0)$. We may assume that $G_0 \subset G$. Under the presence of σ, the Lie–Poisson bracket is being decomposed as follows:

$$\{\ ,\ \}_{\mathsf{LP}} = \{\ ,\ \}_{0,0} + \{\ ,\ \}_{0,1} + \{\ ,\ \}_{1,1}.$$

More precisely, if $x = x_0 + x_1 \in \mathfrak g$, then $\{x, y\}_{0,0} = [x_0, y_0]$, $\{x, y\}_{0,1} = [x_0, y_1] + [x_1, y_0]$, and $\{x, y\}_{1,1} = [x_1, y_1]$. Using this decomposition, we introduce a 1-parameter family of Poisson brackets on $\mathfrak g^*$:

$$\{\ ,\ \}_t = \{\ ,\ \}_{0,0} + \{\ ,\ \}_{0,1} + t\{\ ,\ \}_{1,1},$$

where $t \in \mathbb{P} = \Bbbk \cup \{\infty\}$ and $\{\ ,\ \}_\infty = \{\ ,\ \}_{1,1}$. It is easily seen that $\{\ ,\ \}_t$ with $t \in \Bbbk^\times$ is given by the map φ_s, where $s^2 = t$ (see Section 1.2), and it follows from Lemmas 1.1 and 1.2 that all these brackets are compatible. Hence

$$\{\ ,\ \}_t = \{\ ,\ \}_0 + t\{\ ,\ \}_\infty, \quad t \in \mathbb{P},$$

in accordance with the general method outlined in the introduction, Note that $\{\ ,\ \}_{\mathsf{LP}} = \{\ ,\ \}_0 + \{\ ,\ \}_\infty$. Write $\mathfrak{g}_{(t)}$ for the Lie algebra corresponding to $\{\ ,\ \}_t$. Of course, we merely write $\mathfrak{g}$ in place of $\mathfrak{g}_{(1)}$. All Lie algebras $\mathfrak{g}_{(t)}$ have the same underlying vector space $\mathfrak{g}$.

Convention. We identify $\mathfrak{g}, \mathfrak{g}_0$, and $\mathfrak{g}_1$ with their duals via the Killing form on $\mathfrak{g}$. Hence $\mathfrak{g}_0^* \oplus \mathfrak{g}_1^* \simeq \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We regard $\mathfrak{g}^*$ as the dual of any algebra $\mathfrak{g}_{(t)}$ and sometimes omit the subscript '(t)' in $\mathfrak{g}^*_{(t)}$. However, if $\xi \in \mathfrak{g}^*$, then the stabiliser of ξ in the Lie algebra $\mathfrak{g}_{(t)}$ (i.e., with respect to the coadjoint representation of $\mathfrak{g}_{(t)}$) is denoted by $\mathfrak{g}^\xi_{(t)}$.

Let π_t be the Poisson tensor for $\{\ ,\ \}_t$ and $\pi_t(\xi)$ the skew-symmetric bilinear form on $\mathfrak{g} \simeq T^*_\xi(\mathfrak{g}^*)$ corresponding to $\xi \in \mathfrak{g}^*$, cf. Section 1.1. A down-to-earth description is that $\pi_t(\xi)(x_1, x_2) = \{x_1, x_2\}_{(t)}(\xi)$. Set $\mathsf{rk}\,\pi_t = \max_{\xi\in\mathfrak{g}^*} \mathsf{rk}\,\pi_t(\xi)$.

Lemma 2.1. *We have* $\operatorname{ind}\mathfrak{g}_{(t)} = \mathsf{rkc}\{\ ,\ \}_t = \begin{cases} \mathsf{rk}\,\mathfrak{g}, & t \neq \infty; \\ \dim\mathfrak{g}_0 + \mathsf{rk}\,\mathfrak{g} - \mathsf{rk}\,\mathfrak{g}_0, & t = \infty. \end{cases}$

Proof. We know that $\mathsf{rkc}\{\ ,\ \}_{\mathsf{LP}} = \mathsf{rkc}\{\ ,\ \}_1 = \mathsf{rk}\,\mathfrak{g}$, if $\mathfrak{g}$ is semisimple.

1) If $t \neq 0, \infty$, then the existence of φ_s with $s^2 = t$ implies that $\{\ ,\ \}_t$ is isomorphic to $\{\ ,\ \}_1$. For $t = 0$, one obtains the Poisson bracket of the semi-direct product ($\mathbb{Z}_2$-contraction) $\mathfrak{g}_{(0)} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1^{\mathsf{ab}}$, and it is proved in [P07, Cor. 9.4] that $\operatorname{ind}(\mathfrak{g}_0 \ltimes \mathfrak{g}_1^{\mathsf{ab}}) = \mathsf{rk}\,\mathfrak{g}$.

2) By definition, $\mathsf{rkc}\{\ ,\ \}_\infty = \operatorname{ind}\mathfrak{g}_{(\infty)} = \min_{\xi\in\mathfrak{g}^*} \dim\mathfrak{g}^\xi_{(\infty)}$. Here $\{\ ,\ \}_\infty$ represents the degenerated Lie algebra structure on the vector space $\mathfrak{g}$ such that $[x_0 + x_1, y_0 + y_1]_\infty = [x_1, y_1] \in \mathfrak{g}_0$. One easily verifies that if $\xi = \xi_0 + \xi_1 \in \mathfrak{g}^*$, then $\mathfrak{g}^\xi_{(\infty)} = \mathfrak{g}_0 \oplus \mathfrak{g}_1^{\xi_0}$. Therefore,

$$\operatorname{ind}\mathfrak{g}_{(\infty)} = \dim\mathfrak{g}_0 + \min_{\xi_0\in\mathfrak{g}_0^*} \dim\mathfrak{g}_1^{\xi_0} = \dim\mathfrak{g} - \max_{\xi_0\in\mathfrak{g}_0} \dim[\mathfrak{g}_1, \xi_0].$$

In the last step, we use the fact that upon the identification of $\mathfrak{g}_0^*$ and $\mathfrak{g}_0$, the coadjoint action of $\mathfrak{g}_1 \subset \mathfrak{g}_{(\infty)}$ on $\mathfrak{g}_0^* \subset \mathfrak{g}^*_{(\infty)}$ becomes the usual bracket in $\mathfrak{g}$.

By a well-known property of $\mathbb{Z}_2$-gradings, $\mathfrak{g}_0$ always contains a regular semisimple element of $\mathfrak{g}$. If $\xi_0 \in \mathfrak{g}_0$ is regular semisimple in $\mathfrak{g}$ and hence in $\mathfrak{g}_0$, then $[\mathfrak{g}, \xi_0] = [\mathfrak{g}_0, \xi_0] \oplus [\mathfrak{g}_1, \xi_0]$, $\dim[\mathfrak{g}, \xi_0] = \dim\mathfrak{g} - \mathsf{rk}\,\mathfrak{g}$, and $\dim[\mathfrak{g}_0, \xi_0] = \dim\mathfrak{g}_0 - \mathsf{rk}\,\mathfrak{g}_0$. Hence

$$\max_{\xi_0\in\mathfrak{g}_0} \dim[\mathfrak{g}_1, \xi_0] = \dim\mathfrak{g}_1 + \mathsf{rk}\,\mathfrak{g}_0 - \mathsf{rk}\,\mathfrak{g},$$

and we are done. □

It follows from Lemma 2.1 that $t=\infty$ is regular in $\mathbb{P}$ if and only if $\dim\mathfrak{g}_0=\mathsf{rk}\,\mathfrak{g}_0$, i.e., $\mathfrak{g}_0$ is Abelian. For the indecomposable pairs, this happens if and only if $\mathfrak{g}=\mathfrak{sl}_2$. For this reason, it is necessary to handle the $\mathfrak{sl}_2$-case separately.

Example 2.2. Let $\mathfrak{g}=\mathfrak{sl}_2$ with a standard basis $\{e,h,f\}$ such that $[h,e]=2e,\ [h,f]=-2f,\ [e,f]=h$. Then $\mathcal{S}(\mathfrak{sl}_2)^{\mathfrak{sl}_2}=\Bbbk[h^2+4ef]$. For the unique (up to conjugation) non-trivial σ, one has $\mathfrak{g}_0=\Bbbk h$ and $e,f\in\mathfrak{g}_1$. Then $\mathcal{Z}_t$ ($t\ne 0,\infty$) is generated by $h^2+t^{-1}ef$. An easy calculation shows that $\mathcal{Z}_0=\Bbbk[ef]$ and $\mathcal{Z}_\infty=\Bbbk[h]$. Here $\mathbb{P}_{\mathsf{reg}}=\mathbb{P}$, hence $\mathcal{Z}$ is generated by all $\mathcal{Z}_t$ with $t\in\mathbb{P}$ and $\mathcal{Z}=\Bbbk[h,ef]$. This is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ and it lies in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.

Unless otherwise explicitly stated, we assume below that $\mathfrak{g}\ne\mathfrak{sl}_2$. We then obtain a 1-parameter family of compatible Poisson brackets on $\mathfrak{g}^*$, with generic central rank being equal to $\mathsf{rk}\,\mathfrak{g}$ and $\mathbb{P}_{\mathsf{sing}}=\{\infty\}$, where the central rank jumps up to $\dim\mathfrak{g}_0+\mathsf{rk}\,\mathfrak{g}-\mathsf{rk}\,\mathfrak{g}_0$. Hence $\mathbb{P}_{\mathsf{reg}}=\mathbb{P}\setminus\{\infty\}=\Bbbk$. For each Lie algebra $\mathfrak{g}_{(t)}$, there is the related singular set $\mathfrak{g}^*_{(t),\mathsf{sing}}=\mathfrak{g}^*\setminus\mathfrak{g}^*_{(t),\mathsf{reg}}$, cf. Eq. (1·1). Then, clearly,

$$\mathfrak{g}^*_{(t),\mathsf{sing}}=\{\xi\in\mathfrak{g}^*\mid \mathsf{rk}\,\pi_t(\xi)<\mathsf{rk}\,\pi_t\},$$

which is the union of the symplectic $\mathfrak{g}_{(t)}$-leaves in $\mathfrak{g}^*$ having a non-maximal dimension. For aesthetic reasons, we write $\mathfrak{g}^*_{\infty,\mathsf{sing}}$ instead of $\mathfrak{g}^*_{(\infty),\mathsf{sing}}$.

Let $\mathcal{Z}_t$ denote the centre of the Poisson algebra $(\mathcal{S}(\mathfrak{g}),\{\ ,\ \}_t)$. Then $\mathcal{Z}_1=\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. For $\xi\in\mathfrak{g}^*$, let $d_\xi F$ denote the differential of $F\in\mathcal{S}(\mathfrak{g})$ at ξ. It is standard that for any $H\in\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, $d_\xi H\in\mathfrak{z}(\mathfrak{g}^\xi)$, where $\mathfrak{z}(\mathfrak{g}^\xi)$ is the centre of $\mathfrak{g}^\xi$.

Let $\{H_1,\dots,H_l\}$ be a set of homogeneous algebraically independent generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. By the *Kostant regularity criterion* for $\mathfrak{g}$ [K63, Theorem 9],

$$\langle d_\xi H_j\mid 1\leqslant j\leqslant l\rangle_{\Bbbk}=\mathfrak{g}^\xi \ \text{ if and only if }\ \xi\in\mathfrak{g}^*_{\mathsf{reg}}. \tag{2·1}$$

(Recall that $\mathfrak{g}^\xi=\mathfrak{z}(\mathfrak{g}^\xi)$ if and only if $\xi\in\mathfrak{g}^*_{\mathsf{reg}}$ [P03, Thm. 3.3].) Set $d_\xi\mathcal{Z}_t=\langle d_\xi F\mid F\in\mathcal{Z}_t\rangle_{\Bbbk}$. Then $d_\xi\mathcal{Z}_t\subset\ker\pi_t(\xi)$ for each t. The regularity criterion obviously holds for any $t\ne 0,\infty$. That is,

$$\text{if }\ t\ne 0,\infty,\ \text{ then }\ \xi\notin\mathfrak{g}^*_{(t),\mathsf{sing}}\ \Leftrightarrow\ d_\xi\mathcal{Z}_t=\ker\pi_t(\xi)\Leftrightarrow\ \dim\ker\pi_t(\xi)=\mathsf{rk}\,\mathfrak{g}. \tag{2·2}$$

A certain analogue of this statement holds for $t=0$, i.e., for $\mathfrak{g}_{(0)}$ and $d_x\mathcal{Z}_0$, but only for involutions σ such that $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ has a g.g.s. for $(\mathfrak{g},\mathfrak{g}_0)$, see [Y14].

The centres $\mathcal{Z}_t$ ($t\in\Bbbk$) generate a Poisson-commutative subalgebra with respect to any bracket $\{\ ,\ \}_t$, $t\in\mathbb{P}$, cf. Corollary A.2. Write $\mathcal{Z}=\mathsf{alg}\langle\mathcal{Z}_t\rangle_{t\in\Bbbk}$ for this subalgebra. Note that $d_\xi\mathcal{Z}$ is the linear span of $d_\xi\mathcal{Z}_t$ with $t\ne\infty$. There is a method for estimating the dimension of such subspaces, see Appendix A.

Lemma 2.3. *Suppose that $\xi\in\mathfrak{g}^*$ satisfy the properties:*

(1) $\dim\ker\pi_t(\xi) = \mathsf{rk}\,\mathfrak g$ *for all* $t \ne \infty$;

(2) *the rank of the skew-symmetric form* $\pi_0(\xi)|_{\ker\pi_\infty(\xi)}$ *equals* $\dim\ker\pi_\infty(\xi) - \mathsf{rk}\,\mathfrak g$.

Then $\dim d_\xi\mathcal Z = \mathsf{rk}\,\mathfrak g + \frac12\mathsf{rk}\,\pi_\infty(\xi)$ *and* $\dim\big(d_\xi\mathcal Z\cap\ker\pi_\infty(\xi)\big) = \mathsf{rk}\,\mathfrak g$.

Proof. By definition, $d_\xi\mathcal Z \subset \sum_{t\ne\infty}\ker\pi_t(\xi)$. Then Eq. (2·2) and hypothesis **(1)** on ξ imply that $d_\xi\mathcal Z \supset \sum_{t\ne 0,\infty}\ker\pi_t(\xi)$. Observe that we have a 2-dimensional vector space of skew-symmetric bilinear forms $a{\cdot}\pi_t(\xi)$ on $\mathfrak g \simeq T^*_\xi\mathfrak g^*$, where $a\in\Bbbk$, $t\in\mathbb P$. Moreover, $\mathsf{rk}\,\pi_t(\xi) = \dim\mathfrak g - \mathsf{rk}\,\mathfrak g$ for each $t\ne\infty$. By Lemma A.1, we have $\sum_{t\ne 0,\infty}\ker\pi_t(\xi) = \sum_{t\ne\infty}\ker\pi_t(\xi)$. Now the desired equalities follow from Theorem A.4. □

It is not clear yet whether such elements $\xi\in\mathfrak g^*$ actually exist! However, we will immediately see that there are plenty of them.

Proposition 2.4. *The hypotheses of Lemma 2.3 hold for generic* $\xi\in\mathfrak g^*$ *and therefore*

$$\mathrm{tr.deg}\,\mathcal Z = \frac12\mathsf{rk}\,\pi_\infty + \mathsf{rk}\,\mathfrak g = \frac12(\dim\mathfrak g - \mathsf{rkc}\{\ ,\ \}_\infty) + \mathsf{rk}\,\mathfrak g.$$

Proof. The first task is to prove that a generic point $\xi = \xi_0+\xi_1\in\mathfrak g^*$ satisfies condition **(1)** in Lemma 2.3.

One can safely assume that ξ is regular for $\{\ ,\ \}_0$ and $\{\ ,\ \}_\infty$. Next, we are lucky that $\xi_0+\xi_1\in\mathfrak g^*_{\mathsf{sing}} = \mathfrak g^*_{(1),\mathsf{sing}}$ if and only if $\xi_0 + s^{-1}\xi_1\in\mathfrak g^*_{(s^2),\mathsf{sing}}$. Therefore,

$$\bigcup_{t\ne 0,\infty}\mathfrak g^*_{(t),\mathsf{sing}} = \{\xi_0+t\xi_1 \mid \xi_0+\xi_1\in\mathfrak g^*_{\mathsf{sing}},\ t\ne 0,\infty\}. \tag{2·3}$$

Since $\operatorname{codim}\mathfrak g^*_{(t),\mathsf{sing}} = 3$ for each $t\in\Bbbk^\times$, the closure of $\bigcup_{t\ne0,\infty}\mathfrak g^*_{(t),\mathsf{sing}}$ is a proper subset of $\mathfrak g^*$. Hence the condition $\dim\ker\pi_t(\xi) = \mathsf{rk}\,\mathfrak g$ ($t\ne\infty$) holds for ξ in a dense open subset.

The next step is to check condition **(2)**, i.e., compute the rank of the restriction of $\pi_0(\xi)$ to $\ker\pi_\infty(\xi)$. Write $\xi = \xi_0+\xi_1$, where $\xi_i\in\mathfrak g_i^*$. We can safely assume that ξ_0 is regular in $\mathfrak g$ and hence also in $\mathfrak g_0$.

• For the inner involutions, one has $\mathsf{rk}\,\mathfrak g = \mathsf{rk}\,\mathfrak g_0$. Here $\ker\pi_\infty(\xi) = \mathfrak g_0$ and the rank in question is $\dim\mathfrak g_0 - \mathsf{rk}\,\mathfrak g$, as required in Lemma 2.3**(2)**.

• Suppose that σ is outer. Then $\ker\pi_\infty(\xi) = \mathfrak g_0\oplus\mathfrak g_1^{\xi_0}$ with $\dim\mathfrak g_1^{\xi_0} = \mathsf{rk}\,\mathfrak g - \mathsf{rk}\,\mathfrak g_0$. The rank of the form $\pi_0(\xi_0)$ on this kernel is equal to

$$\dim\ker\pi_\infty(\xi_0) - \mathsf{rk}\,\mathfrak g_0 - \dim\mathfrak g_1^{\xi_0} = \dim\ker\pi_\infty(\xi_0) - \mathsf{rk}\,\mathfrak g.$$

For a generic ξ, where ξ_1 is generic as well, the value in question cannot be smaller than $\dim\ker\pi_\infty(\xi_0) - \mathsf{rk}\,\mathfrak g$. On the other hand, it cannot be larger by Lemma A.3. That is, we have obtained the required value again!

Now, it follows from Lemma 2.3 that

$$\mathrm{tr.deg}\,\mathcal Z = \max_{\xi\in\mathfrak g^*}\dim d_\xi\mathcal Z = \frac12(\dim\mathfrak g - \mathsf{rkc}\{\ ,\ \}_\infty) + \mathsf{rk}\,\mathfrak g.$$ □

Combining Lemma 2.1 and Proposition 2.4, we obtain

$$\text{tr.deg}\,\mathcal{Z} = \frac{1}{2}(\dim \mathfrak{g}_1 + \mathsf{rk}\,\mathfrak{g} + \mathsf{rk}\,\mathfrak{g}_0). \tag{2·4}$$

Lemma 2.5 ([MY, Prop. 1.1]). *If $\mathcal{A} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ and $\{\mathcal{A}, \mathcal{A}\}_{\mathsf{LP}} = 0$, then*

$$\text{tr.deg}\,\mathcal{A} \leqslant \boldsymbol{b}(\mathfrak{g}) - \boldsymbol{b}(\mathfrak{g}_0) + \text{ind}\,\mathfrak{g}_0.$$

Note that in our situation, $\boldsymbol{b}(\mathfrak{g}) - \boldsymbol{b}(\mathfrak{g}_0) + \text{ind}\,\mathfrak{g}_0 = \frac{1}{2}(\dim \mathfrak{g}_1 + \mathsf{rk}\,\mathfrak{g} + \mathsf{rk}\,\mathfrak{g}_0)$.

Lemma 2.6. *We have $\mathcal{Z} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.*

Proof. For all Poisson brackets $\{\ ,\ \}_t$ with $t \neq \infty$, the commutators $[x_0, y]$ are the same as in $\mathfrak{g}$. Hence $\mathcal{Z}_t \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ for each $t \neq \infty$. □

A posteriori, this lemma is true for $\mathfrak{g} = \mathfrak{sl}_2$ as well, cf. Example 2.2. Combining previous formulae, together with computations for $\mathfrak{sl}_2$, we obtain the next general assertion.

Theorem 2.7. *For any $\mathfrak{g}$ and any σ, the algebra $\mathcal{Z} = \mathsf{alg}\langle \mathcal{Z}_t \rangle_{t \in \mathbb{P}_{\mathsf{reg}}}$ is a Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ of the maximal possible transcendence degree, which is given by Eq.* (2·4).

In Section 3, we provide an explicit set of generators of $\mathcal{Z}$, if $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ has a good generating system for $(\mathfrak{g}, \mathfrak{g}_0)$. From this, we deduce that $\mathcal{Z}$ is a polynomial algebra. Although $\mathcal{Z}$ has the maximal transcendence degree among the Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$, it is not always maximal. In Section 4, we construct the extended algebra $\tilde{\mathcal{Z}}$ such that $\mathcal{Z} \subset \tilde{\mathcal{Z}} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ and show that $\tilde{\mathcal{Z}}$ is maximal and still polynomial.

3. The algebra $\mathcal{Z}$ is polynomial whenever σ is good

Let $\{H_1, \dots, H_l\}$, $l = \mathsf{rk}\,\mathfrak{g}$, be a set of homogeneous algebraically independent generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Set $d_i = \deg H_i$. Then $\sum_{i=1}^{l} d_i = \boldsymbol{b}(\mathfrak{g})$. Associated with the vector space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, one has the bi-homogeneous decomposition of each H_j:

$$H_j = \sum_{i=0}^{d_j} (H_j)_{(i, d_j - i)},$$

where $(H_j)_{(i,d_j-i)} \in \mathcal{S}^i(\mathfrak{g}_0) \otimes \mathcal{S}^{d_j-i}(\mathfrak{g}_1) \subset \mathcal{S}^{d_j}(\mathfrak{g})$. Let $H_j^\bullet$ be the nonzero bi-homogeneous component of H_j with maximal $\mathfrak{g}_1$-degree. Then $\deg_{\mathfrak{g}_1} H_j = \deg_{\mathfrak{g}_1} H_j^\bullet$ and we set $d_j^\bullet = \deg_{\mathfrak{g}_1} H_j^\bullet$.

Definition 2. Let us say that $H_1, \dots, H_l$ is a *good generating system* in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ (g.g.s. for short) for $(\mathfrak{g}, \mathfrak{g}_0)$ or for σ, if $H_1^\bullet, \dots, H_l^\bullet$ are algebraically independent.

If the pair $(\mathfrak g,\mathfrak g_0)$ is indecomposable, which we always tacitly assume, then there is no g.g.s. for four involutions related to $\mathfrak g$ of type $\mathcal E_n$ [P07′, Remark 4.3] and a g.g.s. exists in all other cases, see [Y14]. The importance of g.g.s. is clearly seen in the following fundamental result.

Theorem 3.1 ([Y14, Theorem 3.8]). *Let $H_1,\dots,H_l$ be an arbitrary set of homogeneous algebraically independent generators of $\mathcal S(\mathfrak g)^{\mathfrak g}$. Then*

(i) $\sum_{j=1}^{l}\deg_{\mathfrak g_1}H_j\geqslant\dim\mathfrak g_1$;
(ii) *$H_1,\dots,H_l$ is a g.g.s. if and only if* $\sum_{j=1}^{l}\deg_{\mathfrak g_1}H_j=\dim\mathfrak g_1$;
(iii) *if $H_1,\dots,H_l$ is a g.g.s., then $\mathcal S(\mathfrak g_{(0)})^{\mathfrak g_{(0)}}=\Bbbk[H_1^\bullet,\dots,H_l^\bullet]$ is a polynomial algebra.*

Recall that $\mathfrak g_{(0)}=\mathfrak g_0\ltimes\mathfrak g_1^{\mathsf{ab}}$ is a $\mathbb Z_2$-contraction of $\mathfrak g$ and $\operatorname{ind}\mathfrak g_{(0)}=\operatorname{ind}\mathfrak g$. We continue to assume that $\mathfrak g\ne\mathfrak{sl}_2$, hence $\mathbb P_{\mathsf{reg}}=\Bbbk$ and $\mathcal Z=\mathsf{alg}\langle\mathcal Z_t\rangle_{t\in\Bbbk}$.

Theorem 3.2. *Suppose that $\{H_i\}$ is a g.g.s. for σ. Then the algebra $\mathcal Z$ is generated by*

$$\{(H_j)_{(i,d_j-i)}\mid j=1,\dots,l\ \&\ i=0,1,\dots,d_j\},\tag{3·1}$$

i.e., by all bi-homogeneous components of $H_1,\dots,H_l$.

Proof. To begin with, $\mathcal Z(\{\ ,\ \}_1)=\mathcal Z(\mathcal S(\mathfrak g))=\Bbbk[H_1,\dots,H_l]$. By the definition of $\{\ ,\ \}_t$, we have $\mathcal Z(\{\ ,\ \}_t)=\varphi_s^{-1}(\mathcal Z(\mathcal S(\mathfrak g)))$ for $t\ne0,\infty$, where $s^2=t$ and

$$\varphi_s(H_j)=(H_j)_{(d_j,0)}+s(H_j)_{(d_j-1,1)}+s^2(H_j)_{(d_j-2,2)}+\dots$$

Using the Vandermonde determinant, we deduce from this that all $(H_j)_{(i,d_j-i)}$ belong to $\mathcal Z$ and the algebra generated by them contains $\mathcal Z_t$ with $t\in\Bbbk\setminus\{0\}$. Moreover, the specific bi-homogeneous components $H_1^\bullet,\dots,H_l^\bullet$ generate $\mathcal Z_0$, since $H_1,\dots,H_l$ is a g.g.s. Therefore, the polynomials (3·1) generate the whole of $\mathcal Z$. □

However, not every $i\in\{0,1,\dots,d_j\}$ provides a nonzero bi-homogeneous component of H_j. Let us make this precise. Since the case of inner involutions is technically easier, we consider it first.

Theorem 3.3. *Suppose that $\sigma\in\mathsf{Aut}(\mathfrak g)$ is inner, and let $H_1,\dots,H_l$ be a g.g.s. in $\mathcal S(\mathfrak g)^{\mathfrak g}$ with $d_j^\bullet=\deg_{\mathfrak g_1}H_j$. Then*

(i) *all $d_j^\bullet$, $j=1,\dots,l$, are even;*
(ii) *$(H_j)_{(i,d_j-i)}\ne0$ if and only if d_j-i is even and $0\leqslant d_j-i\leqslant d_j^\bullet$;*
(iii) *the polynomials $\{(H_j)_{(i,d_j-i)}\mid j=1,\dots,l;\ \&\ d_j-i=0,2,\dots,d_j^\bullet\}$ freely generate $\mathcal Z$.*

Proof. (1) Since σ is inner, $\sigma(H_j)=H_j$ for all j. On the other hand, $\sigma|_{\mathfrak g_0}=\mathsf{id}$, $\sigma|_{\mathfrak g_1}=-\mathsf{id}$, and hence $\sigma((H_j)_{(i,d_j-i)})=(-1)^{d_j-i}(H_j)_{(i,d_j-i)}$. This yields (i) and one implication in (ii).

(2) In view of part (1), the number of non-zero bi-homogeneous components of H_j is at most $(d_j^\bullet/2)+1$. Hence the total number of nonzero bi-homogeneous components of all H_j is at most $\sum_{j=1}^{l}(d_j^\bullet/2)+1=(\dim\mathfrak g_1/2)+\mathsf{rk}\,\mathfrak g$.

As σ is inner, one also has $\mathsf{rk}\mathfrak g=\mathsf{rk}\mathfrak g_0$. Therefore, $\mathrm{tr.deg}\,\mathcal Z=(\dim\mathfrak g_1/2)+\mathsf{rk}\,\mathfrak g$, see Eq. (2·4). Because the bi-homogeneous components of all H_j generate $\mathcal Z$ (Theorem 3.2), we see that all $(H_j)_{(i,d_j-i)}$ with $d_j-i=0,2,\dots,d_j^\bullet$ are nonzero and algebraically independent. Thus, they freely generate $\mathcal Z$. □

With extra technical details, Theorem 3.3 extends to the outer involutions as well. Let σ be an arbitrary involution of $\mathfrak g$. It is easily seen that a set of homogeneous generators of $\mathcal S(\mathfrak g)^{\mathfrak g}$ can be chosen so that each H_j is an eigenvector of σ, i.e., $\sigma(H_j)=\varepsilon_jH_j=\pm H_j$. Moreover, the set of pairs $\{(d_j,\varepsilon_j)\mid j=1,\dots,l\}$ does not depend on the set of generators, cf. [S74, Lemma 6.1]. However, we need a set of free generators that both is a g.g.s. and consists of σ-eigenvectors.

Lemma 3.4. *If there is a* g.g.s. *for* $(\mathfrak g,\mathfrak g_0)$, *then there is also a* g.g.s. *that consists of eigenvectors of* σ.

Proof. Let $H_1,\dots,H_l$ be a g.g.s., hence $\sum_{j=1}^{l}\deg_{\mathfrak g_1}H_j=\dim\mathfrak g_1$ in view of Theorem 3.1.

Let $\mathcal A_+$ be the ideal in $\mathcal S(\mathfrak g)^{\mathfrak g}$ generated by all homogeneous invariants of positive degree. Then $\boldsymbol{\mathcal A}:=\mathcal A_+/\mathcal A_+^2$ is a finite-dimensional $\Bbbk$-vector space. If $H\in\mathcal A_+$, then $\bar H:=H+\mathcal A_+^2\in\boldsymbol{\mathcal A}$. As is well-known, $F_1,\dots,F_m$ is a generating system for $\mathcal S(\mathfrak g)^{\mathfrak g}$ if and only if the $\Bbbk$-linear span of $\bar F_1,\dots,\bar F_m$ is the whole of $\boldsymbol{\mathcal A}$. In our situation, $\dim_\Bbbk\boldsymbol{\mathcal A}=l$ and $\boldsymbol{\mathcal A}=\langle\bar H_1,\dots,\bar H_l\rangle$.

If H_i is not a σ-eigenvector, i.e., $\sigma(H_i)\ne\pm H_i$, then we consider the generating set

$$H_1,\dots,H_{i-1},\frac{H_i+\sigma(H_i)}{2},\frac{H_i-\sigma(H_i)}{2},H_{i+1},\dots,H_l$$

for $\mathcal S(\mathfrak g)^{\mathfrak g}$ that includes $l+1$ polynomials. Since $\bar H_,\dots,\bar H_{i-1},\bar H_{i+1},\dots,\bar H_l$ are linearly independent in $\boldsymbol{\mathcal A}$, we obtain a better generating set by replacing H_i with one of the functions $H_i^{(+)}=\frac{H_i+\sigma(H_i)}{2}$ or $H_i^{(-)}=\frac{H_i-\sigma(H_i)}{2}$. Let us demonstrate that there is actually only one suitable replacement for H_i, and this yields again a g.g.s. Recall that $d_j^\bullet=\deg_{\mathfrak g_1}H_j^\bullet=\deg_{\mathfrak g_1}H_j$.

(a) Suppose that $d_j^\bullet$ is even. Then $\sigma(H_i^\bullet)=H_i^\bullet$ and $H_i^\bullet$ cancel out in $H_i^{(-)}$. Therefore, $\deg_{\mathfrak g_1}H_i^{(-)}<\deg_{\mathfrak g_1}H_i$ and the sum of $\mathfrak g_1$-degrees for $H_1,\dots,H_{i-1},H_i^{(-)},H_{i+1},\dots,H_l$ is less than $\dim\mathfrak g_1$. By Theorem 3.1, this means that the choice of $H_i^{(-)}$ in place of H_i does not provide a generating system, and the only right choice is to take $H_i^{(+)}$. Moreover, $H_i^\bullet=(H_i^{(+)})^\bullet$ and hence $H_1,\dots,H_{i-1},H_i^{(+)},H_{i+1},\dots,H_l$ is a g.g.s.

(b) If $d_j^\bullet$ is odd, then we end up with the g.g.s. $H_1,\dots,H_{i-1},H_i^{(-)},H_{i+1},\dots,H_l$.

The procedure reduces the number of generators that are not σ-eigenvectors, and we eventually obtain a g.g.s. that consists of σ-eigenvectors. □

Without loss of generality, we can assume that $H_1, \dots, H_l$ is a g.g.s. and $\sigma(H_j) = \pm H_j$.

Lemma 3.5. *For any involution $\sigma \in \mathsf{Aut}(\mathfrak g)$, we have*

(1) $\sigma(H_j) = H_j$ *if and only if $d_j^\bullet$ is even;*

(2) $\mathsf{rk}\,\mathfrak g_0 = \#\{j \mid \sigma(H_j) = H_j\}$.

Proof. (1) The proof is similar to that of Theorem 3.3(i).
(2) This follows from results of T. Springer on regular elements of finite reflection groups [S74, Corollary 6.5]. To this end, one has to consider the Weyl group corresponding to a Cartan subalgebra $\mathfrak t = \mathfrak t_0 \oplus \mathfrak t_1 \subset \mathfrak g_0 \oplus \mathfrak g_1$ such that $\mathfrak t_0$ is a Cartan in $\mathfrak g_0$. □

Now, we can state and prove the main result of this section.

Theorem 3.6. *Let σ be an involution of $\mathfrak g$ such that $\mathcal S(\mathfrak g)^{\mathfrak g}$ has a g.g.s. Then $\mathcal Z$ is a polynomial algebra that is freely generated by the bi-homogeneous components of all $\{H_j\}$. More precisely, if $\sigma(H_j) = H_j$, then $d_j^\bullet$ is even and the nonzero bi-homogeneous components of H_j are $(H_j)_{(i,d_j-i)}$ with $d_j - i = 0, 2, \dots, d_j^\bullet$; if $\sigma(H_j) = -H_j$, then $d_j^\bullet$ is odd and the nonzero bi-homogeneous components of H_j are $(H_j)_{(i,d_j-i)}$ with $d_j - i = 1, 3, \dots, d_j^\bullet$.*

Proof. By Lemma 3.5, we may order the basic invariants $\{H_j\}$ such that

$$d_j^\bullet \text{ is } \begin{cases} \text{even} & i \leqslant k := \mathsf{rk}\,\mathfrak g_0; \\ \text{odd} & i \geqslant k+1. \end{cases}$$

Clearly, if $d_j^\bullet$ is even, then $\varepsilon_j = 1$ and H_j has at most $(d_j^\bullet/2)+1$ nonzero bi-homogeneous components, while if $d_j^\bullet$ is odd, then $\varepsilon_j = -1$ and H_j has at most $(d_j^\bullet+1)/2$ nonzero bi-homogeneous components. Hence the total number of all nonzero bi-homogeneous components is at most

$$\sum_{j=1}^{k}\left(\frac{d_j^\bullet}{2}+1\right) + \sum_{j=k+1}^{l}\frac{d_j^\bullet+1}{2} = \sum_{j=1}^{l}\frac{d_j^\bullet}{2} + k + \frac{l-k}{2} = \frac{\dim\mathfrak g_1 + \mathsf{rk}\,\mathfrak g + \mathsf{rk}\,\mathfrak g_0}{2} = \mathrm{tr.deg}\,\mathcal Z.$$

Therefore, all admissible bi-homogeneous components must be nonzero and algebraically independent. □

Remark 3.7. If there is no g.g.s. for $(\mathfrak g, \mathfrak g_0)$, then $\sum_j \deg_{\mathfrak g_1} H_j > \dim\mathfrak g_1$ for any set of basic invariants. Hence the number of the bi-homogeneous components of $\{H_j\}$ is bigger than $\mathrm{tr.deg}\,\mathcal Z$ and these generators of $\mathcal Z$ are algebraically dependent. Moreover, the algebra $\mathcal Z_0 = \mathcal Z(\mathcal S(\mathfrak g_0 \ltimes \mathfrak g_1^{\mathsf{ab}}))$, which is contained in $\mathcal Z$, is not polynomial [Y17, Section 6], and also $H_1^\bullet, \dots, H_l^\bullet$ are algebraically dependent, cf. Theorem 3.1. Thus, we cannot say anything good about $\mathcal Z$ in the four "bad" cases.

Remark 3.8. Recall from the introduction the map $r_0 : \mathcal{S}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. If σ is inner, then $\mathfrak{g}_0$ contains a Cartan subalgebra of $\mathfrak{g}$ and r_0 is injective. Hence $(H_j)_{(d_j,0)} = r_0(H_j) \neq 0$ for all j, which also follows from Theorem 3.3. Clearly, $r_0(\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}) \subset \mathcal{Z}$ for any σ. More precisely, $r_0(\mathcal{S}(\mathfrak{g})^{\mathfrak{g}})$ is freely generated by the $r_0(H_j) = (H_j)_{(d_j,0)}$ such that $\sigma(H_j) = H_j$ (i.e., $d_j^\bullet$ is even). However, for the inner (and some outer) involutions, $r_0(\mathcal{S}(\mathfrak{g})^{\mathfrak{g}})$ is a proper subalgebra of $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. And this is the reason, why $\mathcal{Z}$ appears to be not always a maximal commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.

4. THE EXTENDED ALGEBRA $\tilde{\mathcal{Z}}$ IS POLYNOMIAL AND MAXIMAL POISSON-COMMUTATIVE

In this section, we assume that $\mathfrak{g} \neq \mathfrak{sl}_2$, $(\mathfrak{g}, \mathfrak{g}_0)$ is indecomposable, and there is a g.g.s. for $(\mathfrak{g}, \mathfrak{g}_0)$. We write $\mathfrak{z}(\mathfrak{q})$ for the centre of a Lie algebra $\mathfrak{q}$. An open subset of $\mathfrak{g}^*$ is said to be *big*, if its complement does not contain divisors.

There is an extraordinary powerful tool for proving maximality of certain subalgebras.

Theorem 4.1 ([PPY, Theorem 1.1])**.** *Let $F_1, \dots, F_r \in \mathcal{S}(\mathfrak{g})$ be homogeneous algebraically independent polynomials such that their differentials $\{\textsf{d}F_i\}$ are linearly independent on a big open subset of $\mathfrak{g}^*$. Then $\Bbbk[F_1, \dots, F_r]$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{g})$, i.e., if $H \in \mathcal{S}(\mathfrak{g})$ is algebraic over the field $\Bbbk(F_1, \dots, F_r)$, then $H \in \Bbbk[F_1, \dots, F_r]$.*

In order to apply this theorem to $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$, we need some properties of divisors in $\mathfrak{g}^*$.

Lemma 4.2. *Let $D \subset \mathfrak{g}^*$ be an irreducible divisor. Then there is a non-empty open subset $U \subset D$ such that, for each $\xi \in U$, we have*

(i) *$\xi \notin \mathfrak{g}^*_{(t),\mathsf{sing}}$, if $t \neq \infty$;*
(ii) *if $\xi = \xi_0 + \xi_1$ with $\xi_i \in \mathfrak{g}_i^*$, then $\xi_0 \in (\mathfrak{g}_0^*)_{\mathsf{reg}}$.*

Proof. (i) The Lie algebra $\mathfrak{g}_{(0)} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1^{\mathsf{ab}}$ has the *codim*–2 property, see [P07′, Theorem 3.3]. Hence $\operatorname{codim} \mathfrak{g}^*_{(0),\mathsf{sing}} \geqslant 2$. Recall that $\dim \mathfrak{g}^*_{\mathsf{sing}} = \dim \mathfrak{g} - 3$. Therefore, the union of the singular subsets $\mathfrak{g}^*_{(t),\mathsf{sing}}$, $t \in \Bbbk^\times$, is a subset of codimension 2, as follows from Eq. (2·3). Hence there is a non-empty open subset of D such that $\mathsf{rk}\,\pi_t(\xi) = \mathsf{rk}\,\pi_t$ for each $\xi \in D$ and $t \neq \infty$.

(ii) Since $\mathfrak{g}_0$ is reductive, we also have $\dim(\mathfrak{g}_0^*)_{\mathsf{sing}} \leqslant \dim \mathfrak{g}_0 - 3$. □

Lemma 4.3. *Suppose that the differentials $\{\textsf{d}(H_j)_{(i,d_j-i)}\}$ are linearly dependent on an irreducible divisor $D \subset \mathfrak{g}^*$. Then $D \subset \mathfrak{g}^*_{\infty,\mathsf{sing}}$.*

Proof. Combining Lemmas 2.3 and 4.2, we see that if the differentials of the $(H_j)_{(i,d_j-i)}$'s are linearly dependent at a generic point $\xi \in D$, then

– either $\mathsf{rk}\,\pi_\infty(\xi) < \mathsf{rk}\,\pi_\infty$,

– or $\mathsf{rk}\,\pi_\infty(\xi) = \mathsf{rk}\,\pi_\infty$, but the restriction of $\pi_0(\xi)$ to $\ker\pi_\infty(\xi)$ does not have the prescribed (maximal possible) rank.

In the first case, we have $\xi \in \mathfrak{g}^*_{\infty,\mathsf{sing}}$ by the very definition. Let us show that the second possibility does not realise. Write $\xi = \xi_0 + \xi_1$. By Lemma 4.2(ii), we may assume that $\xi_0 \in \mathfrak{g}_{0,\mathsf{reg}}$. Since $\mathsf{rk}\,\pi_\infty(\xi) = \mathsf{rk}\,\pi_\infty$, we also have $\xi_0 \in \mathfrak{g}_{\mathsf{reg}}$. As in the proof of Proposition 2.4, the rank of $\pi_0(\xi_0)$ on $\ker\pi_\infty(\xi)$ equals $\dim\ker\pi_\infty(\xi_0) - \mathsf{rk}\,\mathfrak{g}$. And again the same holds for the restriction of $\pi_0(\xi)$. □

We also need the following simple but useful observation on $\mathfrak{g}^*_{\infty,\mathsf{sing}}$.

Lemma 4.4. *The subvariety $\mathfrak{g}^*_{\infty,\mathsf{sing}}$ is of the form $X_0 \times \mathfrak{g}_1^*$, where $X_0 \subset \mathfrak{g}_0^*$ is a conical subvariety. Moreover, $X_0 \cap \mathfrak{g}^*_{\mathsf{reg}} = \varnothing$.*

Proof. Let $\xi = \xi_0 + \xi_1 \in \mathfrak{g}^*$. Since $\mathfrak{g}^{\xi}_{(\infty)} = \mathfrak{g}_0 \oplus \mathfrak{g}_1^{\xi_0}$, the value $\mathsf{rk}\,\pi_\infty(\xi)$ depends only on $\xi_0 = \xi|_{\mathfrak{g}_0}$. Therefore, $\mathfrak{g}^*_{\infty,\mathsf{sing}} = X_0 \times \mathfrak{g}_1^*$, where $X_0 = \mathfrak{g}^*_{\infty,\mathsf{sing}} \cap \mathfrak{g}_0^*$.

It follows from the proof of Lemma 2.1 that $\min_{\xi_0\in\mathfrak{g}_0}\dim\mathfrak{g}_1^{\xi_0} = \mathsf{rk}\,\mathfrak{g} - \mathsf{rk}\,\mathfrak{g}_0$, and $\xi \in \mathfrak{g}^*_{\infty,\mathsf{sing}}$ if and only if $\dim\mathfrak{g}_1^{\xi_0} > \mathsf{rk}\,\mathfrak{g} - \mathsf{rk}\,\mathfrak{g}_0$. But, if $\xi_0 \in \mathfrak{g}^*_{\mathsf{reg}}$, then $\dim\mathfrak{g}_0^{\xi_0} = \mathsf{rk}\,\mathfrak{g}_0$ and $\dim\mathfrak{g}_1^{\xi_0} = \mathsf{rk}\,\mathfrak{g} - \mathsf{rk}\,\mathfrak{g}_0$. □

A particularly nice situation occurs if $r_0 : \mathcal{S}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ is onto. This condition is rather restrictive. If σ is inner, then $\boldsymbol{b}(\mathfrak{g}) = \boldsymbol{b}(\mathfrak{g}_0) + (\dim\mathfrak{g}_1)/2$. And since $\sum_{j=1}^{l} d_j = \boldsymbol{b}(\mathfrak{g})$, the nonzero polynomials $\{(H_j)_{(d_j,0)}\}_{j=1}^{l}$ cannot form a generating system in $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. Hence r_0 cannot be onto for the inner σ. Another observation is that $\mathfrak{g}_0$ has to be simple. This leads to the following list of suitable symmetric pairs:

$$(4{\cdot}1) \qquad (\mathfrak{h}\oplus\mathfrak{h},\mathfrak{h}),\ (\mathfrak{sl}_n,\mathfrak{so}_n),\ (\mathfrak{sl}_{2n},\mathfrak{sp}_{2n}),\ (\mathfrak{so}_{2n},\mathfrak{so}_{2n-1}),\ (\mathcal{E}_6,\mathfrak{sp}_8),\ (\mathcal{E}_6,\mathcal{F}_4)$$

Among them the map r_0 is onto for $(\mathfrak{h}\oplus\mathfrak{h},\mathfrak{h})$, $(\mathfrak{sl}_{2n+1},\mathfrak{so}_{2n+1})$, $(\mathfrak{sl}_{2n},\mathfrak{sp}_{2n})$, $(\mathfrak{so}_{2n},\mathfrak{so}_{2n-1})$, and $(\mathcal{E}_6,\mathcal{F}_4)$. But, the pair $(\mathcal{E}_6,\mathcal{F}_4)$ is not needed, because it does not have a g.g.s.

Theorem 4.5. (1) *If the restriction homomorphism $r_0 : \mathcal{S}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ is onto, then $\mathfrak{g}^*_{\infty,\mathsf{sing}}$ does not contain divisors and $\mathcal{Z}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.*

(2) *Conversely, if $\mathcal{Z}$ is maximal Poisson-commutative, then r_0 is onto.*

Proof. (1) The list of suitable symmetric pairs is quite short. For each item in the list, $\mathfrak{g}_0$ contains a nilpotent element that is regular in $\mathfrak{g}$. This implies that **every** fibre of the quotient morphism $\mathfrak{g}_0 \to \mathfrak{g}_0/\!\!/G_0$ contains a regular element of $\mathfrak{g}$ and hence $(\mathfrak{g}_0)^*_{\mathsf{reg}} \subset \mathfrak{g}^*_{\mathsf{reg}}$. Thus, $\dim(\mathfrak{g}^*_{\mathsf{sing}}\cap\mathfrak{g}_0) \leqslant \dim\mathfrak{g}_0 - 3$. Since $\mathsf{rk}\,\pi_\infty(\xi) = \mathsf{rk}\,\pi_\infty$ for each $\xi \in \mathfrak{g}_0^*\cap\mathfrak{g}^*_{\mathsf{reg}}$ (Lemma 4.4), the subset $\mathfrak{g}^*_{\infty,\mathsf{sing}}$ does not contain divisors. Therefore, the differentials $d(H_j)_{(i,d_j-i)}$ are linearly independent on a big open subset, in view of Lemma 4.3. Then, by Theorem 4.1, $\mathcal{Z}$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{g})$. Since it is a Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ of the maximal possible transcendence degree, it is also maximal.

(2) If r_0 is not onto, then the algebra generated by $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ and $\mathcal{Z}$ is Poisson-commutative, is contained in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$, and properly contains $\mathcal{Z}$. □

Remark 4.6. (1) Consider the following four conditions:

(a) the restriction homomorphism $r_0\colon \mathcal{S}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ is onto;
(b) $\mathfrak{g}_0$ contains a regular nilpotent element of $\mathfrak{g}$;
(c) $\mathfrak{g}^*_{\infty,\mathsf{sing}}$ does not contain divisors;
(d) $\mathcal{Z}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.

In the proof of Theorem 4.5(1), we have seen that (a) ⇒ (b) ⇒ (c) ⇒ (d), whereas part (2) of Theorem 4.5 states that (d) ⇒ (a). Thus, all these conditions are equivalent. One can also give a direct proof for (b) ⇒ (a) that does not invoke $\mathfrak{g}_{(\infty)}$ and $\mathcal{Z}$. However, the implication (a) ⇒ (b) is obtained case-by-case as yet.

(2) There is a g.g.s. for $(\mathfrak{g}, \mathfrak{g}_0)$ if and only if the restriction homomorphism $r_1\colon \mathcal{S}(\mathfrak{g})^{\mathfrak{g}} \to \Bbbk[\mathfrak{g}_1^*]^{\mathfrak{g}_0}$ is onto [P07′, Y14]. Therefore, $\mathcal{Z}$ is a polynomial maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ whenever both r_0 and r_1 are onto.

Our ultimate goal is to prove that, in general, $\tilde{\mathcal{Z}} = \mathsf{alg}\langle \mathcal{Z}, \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}\rangle$ is a polynomial maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$. Unfortunately, the proof requires many technical preparations, if $\mathfrak{g}^*_{\infty,\mathsf{sing}}$ contains divisors (i.e., r_0 is not onto).

Lemma 4.7. *Suppose that* $\dim \mathfrak{g}^*_{\infty,\mathsf{sing}} = n-1$, *and let* $D \subset \mathfrak{g}^*_{\infty,\mathsf{sing}}$ *be an irreducible component of dimension* $n-1$. *Then*

(i) $D = D_0 \times \mathfrak{g}_1^*$, *where* D_0 *is a* G_0*-stable conical divisor in* $\mathfrak{g}_0^*$, *and* D_0 *does not contain regular elements of* $\mathfrak{g}$;
(ii) *generic elements of* D_0 *are semisimple, regular in* $\mathfrak{g}_0 \simeq \mathfrak{g}_0^*$, *and subregular in* $\mathfrak{g}$;
(iii) $\mathsf{rk}\, \pi_\infty(\xi) = \mathsf{rk}\, \pi_\infty - 2$ *for generic point* $\xi \in D$.

Proof. (i) This follows from Lemma 4.4.

(ii) If σ is <u>inner</u>, then $\mathfrak{g}_0$ contains a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and $\mathfrak{t} \cap D_0$ is a W_0-stable divisor in $\mathfrak{t}$, where W_0 is the Weyl group of $(\mathfrak{g}_0, \mathfrak{t})$. It is easily seen that any such divisor contains a subregular element of $\mathfrak{g}$.

The case of an <u>outer</u> σ is more involved. We use an argument, which is also valid for the inner case. If $\mathfrak{t}_0 \subset \mathfrak{g}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, then a generic element $\nu \in D_0 \cap \mathfrak{t}_0$ is either regular or subregular in $\mathfrak{g}_0$. Consider these two possibilities in turn.

(a) Suppose first that ν is regular in $\mathfrak{g}_0$. Then $\mathfrak{g}_0^\nu = \mathfrak{t}_0$ and therefore $\mathfrak{g}^\nu$ is a sum of a toral subalgebra and several copies, say k, of $\mathfrak{sl}_2$. Let $\mathfrak{s}_i$ be the i-th copy of $\mathfrak{sl}_2$. Every such $\mathfrak{s}_i$ is determined by a root β_i of $\mathfrak{g}$. That is,

$$\mathfrak{s}_i = \mathfrak{g}_{-\beta_i} \oplus (\mathfrak{s}_i)^\sigma \oplus \mathfrak{g}_{\beta_i}.$$

Moreover, the one-dimensional subspace $(\mathfrak{s}_i)^\sigma$ is generated by the coroot $\beta_i^\vee$. It is also clear that $(\mathfrak{s}_i)^\sigma \subset \mathfrak{t}_0$ and $\beta_i^\vee$ is orthogonal to ν. Assume that $k \geqslant 2$. Then ν is orthogonal to at least two different coroots. Since the number of relevant pairs $\{\beta_i, \beta_j\}$ is finite, we obtain that $D_0 \cap \mathfrak{t}_0$ lies in a finite union of subspaces of $\mathfrak{t}_0$ of codimension $\geqslant$2. A contradiction! Hence $k \leqslant 1$. If $k = 0$, then ν is regular in $\mathfrak{g}$, which is impossible, see (i). Thus, $k = 1$ and ν is subregular in $\mathfrak{g}$.

(b) Suppose now that D_0 does not contain regular semisimple elements of $\mathfrak{g}_0$. Our goal is to prove that this case does not occur.

Here $\mathfrak{t}_0 \cap D_0$ is a union of reflection hyperplanes of W_0. Let $\mathfrak{z}_0$ be one of these hyperplanes and $\nu \in \mathfrak{z}_0$ generic. Then $\mathfrak{g}_0^\nu = \mathfrak{s} \oplus \mathfrak{z}_0$, where $\mathfrak{s} \simeq \mathfrak{sl}_2$ and $\mathfrak{z}_0$ is the centre of $\mathfrak{g}_0^\nu$. Here $[\mathfrak{z}_0, \mathfrak{g}^\nu] = 0$, since $\nu \in \mathfrak{z}_0$ is generic. Write $\mathfrak{g}^\nu = \mathfrak{h} \oplus \mathfrak{z}(\mathfrak{g}^\nu)$, where $\mathfrak{h} = [\mathfrak{g}^\nu, \mathfrak{g}^\nu]$ is semisimple. Then the symmetric pair $(\mathfrak{g}^\nu, \mathfrak{g}_0^\nu)$ decomposes as

$$(\mathfrak{g}^\nu, \mathfrak{g}_0^\nu) = (\mathfrak{h}, \mathfrak{s}) \oplus (\mathfrak{z}(\mathfrak{g}^\nu), \mathfrak{z}_0).$$

The only possibilities for the symmetric pair $(\mathfrak{h}, \mathfrak{s})$ are:

$$(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \mathfrak{sl}_2),\ (\mathfrak{sl}_3, \mathfrak{so}_3 \simeq \mathfrak{sl}_2),\ (\mathfrak{sl}_2, \mathfrak{sl}_2). \tag{4·2}$$

For $\mathfrak{s} = [\mathfrak{g}_0^\nu, \mathfrak{g}_0^\nu]$, the intersection $D_0 \cap (\Bbbk\nu \oplus \mathfrak{s})$ is a conical divisor of $\Bbbk\nu \oplus \mathfrak{s}$ that contains ν. If $\eta \in \mathfrak{s}$ is non-zero semisimple, then $\nu + \eta \in (\mathfrak{g}_0)_{\mathsf{reg}}$ is semisimple. Hence $\nu + \eta \notin D_0$. Therefore, $D_0 \cap (\Bbbk\nu \oplus \mathfrak{s})$ has to contain a sum $\nu + e$, where $e \in \mathfrak{s}$ is regular nilpotent. For all pairs in (4·2), e is also regular in $\mathfrak{h}$. Hence e is a regular element of $\mathfrak{g}^\nu$. Thereby $\nu + e$ is a regular element of $\mathfrak{g}$. However, this contradicts part (i).

Therefore, case (b) does not materialise and, according to (a), D_0 contains a semisimple element ν that is regular in $\mathfrak{g}_0$ and subregular in $\mathfrak{g}$. Since $D_0 \cap \mathfrak{g}_{\mathsf{reg}} = \varnothing$, subregular semisimple elements of $\mathfrak{g}$ are dense in D_0.

(iii) Since ν is regular in $\mathfrak{g}_0$ and subregular in $\mathfrak{g}$, we have $\dim \mathfrak{g}_0^\nu = \mathsf{rk}\, \mathfrak{g}_0$ and $\dim \mathfrak{g}_1^\nu = \mathsf{rk}\, \mathfrak{g} + 2 - \mathsf{rk}\, \mathfrak{g}_0$. The latter precisely means that $\mathsf{rk}\, \pi_\infty(\nu) = \mathsf{rk}\, \pi_\infty - 2$ for ν in a non-empty open subset of D_0. This completes the proof. □

Example 4.8. Let $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{sl}_{2n}, \mathfrak{so}_{2n})$. Then $D_0 \subset \mathfrak{g}_0$ is the zero set of the Pfaffian. If $\mathfrak{g}_0$ consists of skew-symmetric matrices with respect to the antidiagonal, then

$$x = \mathrm{diag}(a_1, \ldots, a_{n-1}, 0, 0, -a_{n-1}, \ldots, -a_1) \in D_0$$

is subregular whenever all a_i are nonzero and $a_i \neq \pm a_j$ for $i \neq j$.

Recall that $\{H_i\}$ is a g.g.s. for σ such that $\sigma(H_i) = \varepsilon_i H_i = \pm H_i$ for each i. As before, $d_i = \deg H_i$ and $l = \mathsf{rk}\, \mathfrak{g}$. Until the end of this section, we assume that $d_1 \leqslant \cdots \leqslant d_l$. If $\mathfrak{g}$ is simple, then there is a unique basic invariant of degree d_l, i.e., $d_{l-1} < d_l$.

Lemma 4.9. *If $\mathfrak g$ is simple and $x \in \mathfrak g$ is subregular, then the differentials $\{\boldsymbol{d}_x H_i \mid i < l\}$ are linearly independent. Moreover, $\sigma(H_l) = H_l$ unless $(\mathfrak g, \mathfrak g_0) = (\mathfrak{sl}_{2k+1}, \mathfrak{so}_{2k+1})$, where $l = 2k$ and $d_l = 2k+1$.*

Proof. Let $e \in \mathfrak g$ be a subregular nilpotent element. Then $\boldsymbol{d}_e H_l = 0$ [V68, Corollary 2] and $\{\boldsymbol{d}_e H_i \mid i < l\}$ are linearly independent [Sl80, Chapter 8.2]. If x is subregular and non-nilpotent, then the theory of *associated cones* developed in [BK79, § 3] shows that $Ge \subset \overline{\Bbbk^\times(Gx)}$. This implies that $\boldsymbol{d}_x H_i$ with $i < l$ are linearly independent, too.

The equality $\sigma(H_l) = H_l$ is obvious for the inner involutions. If σ is outer, then going through the list of outer involutions, one checks that $\sigma(H_l) = -H_l$ if and only if $\mathfrak g = \mathfrak{sl}_{2k+1}$ and $l = 2k$. Here necessary $\mathfrak g_0 = \mathfrak{so}_{2k+1}$. □

We need below some formulae for the differential and partial derivatives of a homogeneous polynomial $F \in \mathcal S(\mathfrak g) = \Bbbk[\mathfrak g^*]$. If $x \in \mathfrak g^*$ and $d = \deg F$, then $\partial_x^{d-1} F$ is a linear form on $\mathfrak g^*$, i.e., an element of $\mathfrak g$. In fact, one has

$$(d-1)!\, \boldsymbol{d}_x F = \partial_x^{d-1} F. \tag{4·3}$$

By linearity, it suffices to check this for a monomial of degree d. Furthermore, for the operator $\partial^k_{x+sx'} : \mathcal S^m(\mathfrak g) \to \mathcal S^{m-k}(\mathfrak g)$ with $x, x' \in \mathfrak g^*$ and $s \in \Bbbk$, there is the following expansion:

$$\partial^k_{x+sx'} = \partial^k_x + \binom{k}{1} s \partial_{x'} \partial_x^{k-1} + \dots + \binom{k}{i} s^i \partial^i_{x'} \partial_x^{k-i} + \dots + s^k \partial^k_{x'}\, . \tag{4·4}$$

Lemma 4.10. *Suppose that the restriction homomorphism r_0 is not onto (equivalently, $\mathfrak g^*_{\infty,\mathsf{sing}}$ contains divisors). Then*

- (i) *there is $x \in \mathfrak g_0^* \simeq \mathfrak g_0$ such that x is semisimple, regular in $\mathfrak g_0$, and subregular in $\mathfrak g$ (i.e., $\dim \mathfrak g^x = \mathsf{rk}\, \mathfrak g + 2$). Moreover, for a generic $x' \in \mathfrak g_1^* \simeq \mathfrak g_1$, we have $y := x + x' \in \mathfrak g_{\mathsf{reg}}$;*
- (ii) *$\lim_{t\to\infty} \langle \boldsymbol{d}_y F \mid F \in \mathcal Z_t\rangle_{\Bbbk} = \lim_{s\to 0} \varphi_s(\mathfrak g^{x+sx'}) (=: \mathbb V)$;*
- (iii) *$\dim(\mathbb V/\mathbb V \cap \mathfrak g_0) = \mathsf{rk}\, \mathfrak g - \mathsf{rk}\, \mathfrak g_0 + 1$.*

Proof. (i) The existence of such an x follows from Lemma 4.7. Then $\mathfrak g_0^x = \mathfrak t_0$ and if x' is a generic element of $\mathfrak g_1^x$, then y is regular in $\mathfrak g$. Hence $x + x' \in \mathfrak g_{\mathsf{reg}}$ for almost all $x' \in \mathfrak g_1$.

(ii) By the definition of $\{\ ,\ \}_t$, we have $\mathcal Z_t = \varphi_s^{-1}(\mathcal S(\mathfrak g)^{\mathfrak g})$ if $t \ne 0, \infty$ and $s^2 = t$. Let $\varphi_s^* \colon \mathfrak g^* \to \mathfrak g^*$ be the dual map, i.e., $\varphi_s^*|_{\mathfrak g_0^*} = \mathsf{id}$, $\varphi_s^*|_{\mathfrak g_1^*} = s^{-1}{\cdot}\mathsf{id}$. For any $F \in \mathcal S(\mathfrak g)$, we have $\varphi_s(\boldsymbol{d}_y F) = \boldsymbol{d}_{\varphi_s^*(y)} \varphi_s(F)$. In particular,

$$\boldsymbol{d}_y \varphi_s^{-1}(H_i) = \varphi_s^{-1}(\boldsymbol{d}_{\varphi_s^*(y)} H_i),$$

where $\varphi_s^*(y) = x + s^{-1}x'$. If s tends to ∞, then s^{-1} tends to 0. It remains to notice, that for almost all s, the element $x + sx'$ is regular and then $\mathfrak g^{x+sx'}$ is the linear span of $\{\boldsymbol{d}_{x+sx'} H_j\}_{j=1}^l$, see Eq. (2·1).

(iii) The hypothesis that r_0 is not onto excludes the pairs $(\mathfrak h\oplus\mathfrak h,\mathfrak h)$ and $(\mathfrak{sl}_{2k+1},\mathfrak{so}_{2k+1})$. Hence $\mathfrak g$ is simple and, by Lemma 4.9, $d_xH_1,\dots,d_xH_{l-1}$ are linearly independent, d_xH_l is a linear combination of d_xH_j with $j<l$, and $\sigma(H_l)=H_l$. Since x is semisimple and subregular, $\mathfrak g^x=\mathfrak z(\mathfrak g^x)\oplus\mathfrak{sl}_2$ and $\dim\mathfrak z(\mathfrak g^x)=l-1$. Hence $\mathfrak z(\mathfrak g^x)=\langle d_xH_i\mid i<l\rangle_{\Bbbk}$.

Take $j<l$ and set $m_j=d_j-1$. Then by Eq. (4·3) and by Eq. (4·4) with $k=m_j$, we have

$$(m_j)!\,d_{x+sx'}H_j=\partial^{m_j}_{x+sx'}H_j=\sum_{i=0}^{m_j}\binom{m_j}{i}s^i\partial^i_{x'}\partial_x^{m_j-i}H_j,$$
$$(m_j)!\,\sigma(d_{x+sx'}H_j)=\partial^{m_j}_{x-sx'}\sigma(H_j)=\sum_{i=0}^{m_j}\binom{m_j}{i}(-s)^i\partial^i_{x'}\partial_x^{m_j-i}\sigma(H_j).$$

It follows that $\partial^i_{x'}\partial_x^{m_j-i}H_j\in\mathfrak g_0$ if and only if either i is even and $\sigma(H_j)=H_j$ or i is odd and $\sigma(H_j)=-H_j$. Therefore,

- if $\sigma(H_j)=H_j$, then $\lim_{s\to0}\varphi_s(d_{x+sx'}H_j)=d_xH_j\in\mathfrak g_0$; while
- if $\sigma(H_j)=-H_j$, then $d_xH_j\in\mathfrak g_1$ and

$$(m_j)!\,\varphi_s(d_{x+sx'}H_j)=s(\partial_x^{m_j}H_j+m_j\partial_{x'}\partial_x^{m_j-1}H_j)+(\text{terms of degree}\geqslant 2\text{ w.r.t. }s)$$
$$=s\big((m_j)!\,d_xH_j+m_j\partial_{x'}\partial_x^{m_j-1}H_j\big)+\dots$$

Thus, if $\sigma(H_j)=-H_j$, then

$$\lim_{s\to0}\langle\varphi_s(d_{x+sx'}H_j)\rangle_{\Bbbk}=\langle d_xH_j+\frac{1}{(m_j-1)!}\cdot\partial_{x'}\partial_x^{d_j-2}H_j\rangle_{\Bbbk}\,. \qquad (4{\cdot}5)$$

Note that here $\partial_{x'}\partial_x^{d_j-2}H_j\ \in\mathfrak g_0$. Write $\mathfrak z(\mathfrak g^x)=\mathfrak z(\mathfrak g^x)_0\oplus\mathfrak z(\mathfrak g^x)_1$, where $\mathfrak z(\mathfrak g^x)_i=\mathfrak z(\mathfrak g^x)\cap\mathfrak g_i$. Then $\mathfrak z(\mathfrak g^x)_1=\langle d_xH_j\mid\sigma(H_j)=-H_j\rangle_{\Bbbk}$ and $\mathfrak z(\mathfrak g^x)_0=\langle d_xH_j\mid\sigma(H_j)=H_j,\ j\ne l\rangle_{\Bbbk}$. Hence $\dim\mathfrak z(\mathfrak g^x)_0=\mathrm{rk}\,\mathfrak g_0-1$ and $\dim\mathfrak z(\mathfrak g^x)_1=\mathrm{rk}\,\mathfrak g-\mathrm{rk}\,\mathfrak g_0$.

Let $\mathbf p_1$ denote the projection $\mathfrak g\to\mathfrak g_1$ along $\mathfrak g_0$. Then $\mathbf p_1(\mathbb V)=\mathbb V/(\mathbb V\cap\mathfrak g_0)$ and our goal is to compute $\dim\mathbf p_1(\mathbb V)$. By Eq. (4·5), we have $\mathfrak z(\mathfrak g^x)_1\subset\mathbf p_1(\mathbb V)$.

For our further argument, some properties of $d_xH_l\in\mathfrak g_0^x$ are needed. It would be nice to have $d_xH_l=0$ for x as in (i). Since this is not always the case, we need a trick.

Let $\tilde{\mathfrak g}=\mathfrak g\oplus\mathfrak c$ be the central extension of $\mathfrak g$, where $\dim\mathfrak c=1$. We extend the $\mathbb Z_2$-grading to $\tilde{\mathfrak g}$ so that $\mathfrak c\subset\tilde{\mathfrak g}_0$ and φ_s to $\tilde{\mathfrak g}$ by letting $\varphi_s|_{\mathfrak c}=\mathrm{id}$. Take non-zero $z\in\mathfrak c$ and $\gamma\in\mathfrak c^*$. Note that $\tilde{\mathfrak g}^{y+\gamma}=\tilde{\mathfrak g}^y$ for any $y\in\mathfrak g^*$. Therefore $\mathbb V\oplus\mathfrak c=\lim\limits_{s\to0}\varphi_s(\tilde{\mathfrak g}^{x+\gamma+sx'})$. Set $\boldsymbol\zeta=x+\gamma$. Then $\boldsymbol\zeta\in\tilde{\mathfrak g}^*$ is still subregular and $z(\boldsymbol\zeta)\ne0$. Clearly, there is a linear combination

$$\mathbf H_l=H_l+c_{l-1}z^{d_l-d_{l-1}}H_{l-1}+\dots+c_jz^{d_l-d_j}H_j+\dots+c_0z^{d_l}$$

with $c_i\in\Bbbk$ such that $\partial^{d_l-1}_{\boldsymbol\zeta}\mathbf H_l=d_{\boldsymbol\zeta}\mathbf H_l=0$. Note that $z,H_1,\dots,H_{l-1},\mathbf H_l$ freely generate $\mathcal Z\mathcal S(\tilde{\mathfrak g})$.

Let $\mathcal{A}_{\boldsymbol{\zeta}}$ be the Mishchenko–Fomenko subalgebra of $\mathcal{S}(\tilde{\mathfrak{g}})$ associated with $\boldsymbol{\zeta}$. By definition, $\mathcal{A}_{\boldsymbol{\zeta}}$ is generated by

$$\{z,\ \partial^k_{\boldsymbol{\zeta}} H_j\ (j<l,\ 0\leqslant k\leqslant m_j),\ \partial^k_{\boldsymbol{\zeta}}\mathbf{H}_l\ (0\leqslant k\leqslant m_l-1)\}. \tag{4·6}$$

As the total number of these generators is $\boldsymbol{b}(\mathfrak{g})$ and $\operatorname{tr.deg}\mathcal{A}_{\boldsymbol{\zeta}}=\boldsymbol{b}(\tilde{\mathfrak{g}})-1=\boldsymbol{b}(\mathfrak{g})$ [MY, Lemma 2.1], we see that $\mathcal{A}_{\boldsymbol{\zeta}}$ is freely generated by them. Note that the set in (4·6) contains a basis for the l-dimensional space $\mathfrak{z}(\mathfrak{g}^x)\oplus\mathfrak{c}=\mathfrak{z}(\tilde{\mathfrak{g}}^{\boldsymbol{\zeta}})$. Therefore, $F=\partial^{m_l-1}_{\boldsymbol{\zeta}}\mathbf{H}_l$ does not lie in $\mathcal{S}^2\big(\mathfrak{z}(\mathfrak{g}^x)\oplus\mathfrak{c}\big)$. Since $\partial^{m_l}_{\boldsymbol{\zeta}}\mathbf{H}_l=0$, the polynomial F is a $\tilde{\mathfrak{g}}^{\boldsymbol{\zeta}}$-invariant in $\mathcal{S}^2(\tilde{\mathfrak{g}}^{\boldsymbol{\zeta}})$ [MY, Lemma 1.5]. It is clear that $\sigma(F)=F$ and therefore $F\in\mathcal{S}^2(\tilde{\mathfrak{g}}_0)\oplus\mathcal{S}^2(\mathfrak{g}_1)$. Now $\tilde{\mathfrak{g}}^{\boldsymbol{\zeta}}=\mathfrak{c}\oplus\mathfrak{g}^x=\mathfrak{c}\oplus\mathfrak{z}(\mathfrak{g}^x)\oplus\mathfrak{sl}_2$. There is a standard basis $\{e,h,f\}$ of this $\mathfrak{sl}_2$ such that $e,f\in\mathfrak{g}_1$ (cf. Example 2.2) and $F\in(4ef+h^2)+\mathcal{S}^2(\mathfrak{z}(\mathfrak{g}^x)\oplus\mathfrak{c})$.

If $x'\in\mathfrak{g}_1^*$ is generic enough, then $\partial_{x'}F=\eta+\xi$, where $\eta\in\mathfrak{z}(\mathfrak{g}^x)_1$ and ξ is a non-zero element in $\langle e,f\rangle_{\Bbbk}\subset\mathfrak{g}_1$. Note that in this case $(m_l-1)!\,\boldsymbol{d}_{\boldsymbol{\zeta}+sx'}\mathbf{H}_l$ lies in $s\partial_{x'}F+s^2\tilde{\mathfrak{g}}$. Further,

$$(m_l-1)!\varphi_s(\boldsymbol{d}_{\boldsymbol{\zeta}+sx'}\mathbf{H}_l)=s^2(\eta+\xi)+\frac{m_l-1}{2}s^2\partial^2_{x'}\partial^{d_l-3}_{\boldsymbol{\zeta}}\mathbf{H}_l+(\text{terms of degree}\geqslant 3\text{ w.r.t. }s).$$

Here $\partial^2_{x'}\partial^{d_l-3}_{\boldsymbol{\zeta}}\mathbf{H}_l\in\mathfrak{g}_0$. Hence $\mathbf{p}_1(\mathbb{V})=\mathfrak{z}(\mathfrak{g}^x)_1+\Bbbk(\eta+\xi)=\mathfrak{z}(\mathfrak{g}^x)_1\oplus\Bbbk\xi$. The desired equality $\dim\mathbf{p}_1(\mathbb{V})=\mathsf{rk}\,\mathfrak{g}-\mathsf{rk}\,\mathfrak{g}_0+1$ follows. □

Lemma 4.11. *Let $y=x+x'$ be as in Lemma 4.10 with x' generic. Then the rank of the restriction of $\pi_0(y)$ to $\ker\pi_\infty(y)=\mathfrak{g}_0\oplus\mathfrak{g}_1^x$ is equal to $\dim(\mathfrak{g}_0\oplus\mathfrak{g}_1^x)-\mathsf{rk}\,\mathfrak{g}$.*

Proof. Set $U=\ker\pi_\infty(x)=\mathfrak{g}_0\oplus\mathfrak{g}_1^x$. Consider the maximal torus $\mathfrak{t}=\mathfrak{g}_0^x+\Bbbk^{l-1}$, where $\Bbbk^{l-1}$ is the centre of $\mathfrak{g}^x$. The intersection $\mathfrak{t}_0=\mathfrak{g}_0\cap\mathfrak{t}=\mathfrak{g}_0^x$ is a Cartan subalgebra of $\mathfrak{g}_0$. Further, $\mathfrak{g}_0=\mathfrak{t}_0\oplus\mathfrak{m}$, where $\mathfrak{m}$ is the $\mathfrak{t}_0$-stable complement of $\mathfrak{t}_0$ in $\mathfrak{g}_0$. The torus $\mathfrak{t}$ defines a finer decomposition of U, namely

$$U=\mathfrak{m}\oplus\mathfrak{t}_0\oplus(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha})\oplus\mathfrak{z}$$

where $\mathfrak{g}_{\pm\alpha}$ are root spaces and $\mathfrak{z}\simeq\Bbbk^{l-\mathsf{rk}\,\mathfrak{g}_0}$.

Choose a very particular x', namely as $x'=\xi_\alpha-\xi_{-\alpha}$ with non-zero root vectors $\xi_\alpha\in\mathfrak{g}_\alpha$, $\xi_{-\alpha}\in\mathfrak{g}_{-\alpha}$ under the usual identification $\mathfrak{g}_1\simeq\mathfrak{g}_1^*$. Then the matrix of $(\pi_0(y)|_U)$ with respect to a basis for U adapted to the above finer decomposition has a block form with easy to understand blocks (see Fig. 1):

- $\pi_0(y)$ is non-degenerate on $\mathfrak{m}$;
- $\pi_0(y)(\mathfrak{m},\mathfrak{t}_0\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha})=0$;
- $\pi_0(y)(\mathfrak{t}_0,\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha})\neq 0$.

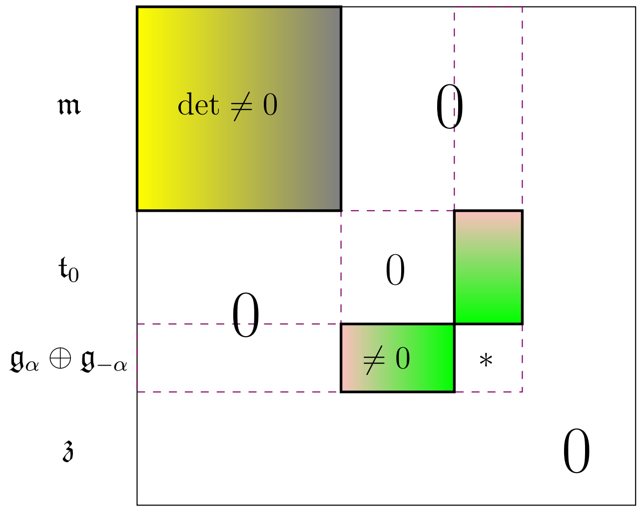

Fig. 1. The block structure of $\pi_0(y)|_U$

This is enough to see that the rank of $\pi_0(y)$ on $\mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is at least $\dim \mathfrak{g}_0 - \mathsf{rk}\, \mathfrak{g}_0 + 2$. Hence $\mathsf{rk}\,(\pi_0(y)|_U) \geqslant \dim U - \mathsf{rk}\, \mathfrak{g}$. This happens for one, not exactly generic x', however, the generic value cannot be smaller and it also cannot be larger by Lemma A.3. □

The algebra $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ is contained in the Poisson centre of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$. Let $\tilde{\mathcal{Z}}$ be the (Poisson-commutative) subalgebra of $\mathcal{S}(\mathfrak{g})$ generated by $\mathcal{Z}$ and $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. If $H_1, \dots, H_l$ is a g.g.s. for $(\mathfrak{g}, \mathfrak{g}_0)$ such that $\sigma(H_i) = \pm H_i$ for each i, then $\tilde{\mathcal{Z}}$ is freely generated by $(H_j)_{(i,d_j-i)}$ with $i \neq d_j$ and a set of basic invariants $\tilde{H}_1, \dots, \tilde{H}_{\mathsf{rk}\,\mathfrak{g}_0} \in \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. In other words, a set of basic invariants of $\tilde{\mathcal{Z}}$ is obtained from that of $\mathcal{Z}$ if one replaces the generators of $r_0(\mathcal{S}(\mathfrak{g})^{\mathfrak{g}})$ with the free generators of $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. [Recall that $(H_j)_{(d_j,0)} \neq 0$ if and only if $\varepsilon_j = 1$ and there are $\mathsf{rk}\,\mathfrak{g}_0$ such indices j, see Remark 3.8.]

Theorem 4.12. (i) *The differentials of the algebraically independent generators of $\tilde{\mathcal{Z}}$, chosen among $\{(H_j)_{(i,d_j-i)}\}$ and $\{\tilde{H}_j\}$, as above, are linearly independent on a big open subset of $\mathfrak{g}^*$.*

(ii) *The algebra $\tilde{\mathcal{Z}}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.*

Proof. (i) Assume that the differentials of the chosen algebraically independent generators of $\tilde{\mathcal{Z}}$ are linearly dependent at each point y of an irreducible divisor $D \subset \mathfrak{g}^*$. Since $\mathcal{Z} \subset \tilde{\mathcal{Z}}$, the same holds for $d(H_j)_{(i,d_j-i)}$. Then $D \subset \mathfrak{g}^*_{\infty,\mathsf{sing}}$ by Lemma 4.3 and $D = D_0 \times \mathfrak{g}_1^*$ by Lemma 4.7(i). Let $y = x + x'$ be a generic element of D.

Recall that $d_y\mathcal{Z}$ stands for the linear span of d_yF with $F \in \mathcal{Z}$. We have

$$d_y\mathcal{Z} = \sum_{t \neq \infty} d_y\mathcal{Z}_t.$$

According to Lemma 4.2, $y \in \mathfrak{g}^*_{(t),\mathsf{reg}}$ for each $t \neq \infty$. Hence $d_y\mathcal{Z}_t = \ker \pi_t(y)$ whenever $t \neq \infty$. By Lemma 4.7(iii), $\mathsf{rk}\,\pi_\infty(y) = \mathsf{rk}\,\pi_\infty - 2$. Combining Lemmas 4.7(ii), 4.10(i), and 4.11, we see that the rank of the restriction of $\pi_0(y)$ to $\ker\pi_\infty(y)$ is equal to $\dim\ker\pi_\infty(y) - \mathsf{rk}\,\mathfrak g$. Now Theorem A.4 applies and asserts that

$$\dim(d_y\mathcal Z/d_y\mathcal Z\cap\ker\pi_\infty(y)) = \frac12\mathsf{rk}\,\pi_\infty(y) = \frac12\mathsf{rk}\,\pi_\infty - 1. \tag{4·7}$$

By construction, $\mathbb V\subset d_y\mathcal Z\cap\ker\pi_\infty(y)$. Recall that $\ker\pi_\infty(y)=\mathfrak g_0\oplus\mathfrak g_1^x$. In view of (4·7) and Lemma 4.10(iii), we have

$$\dim(d_y\mathcal Z/d_y\mathcal Z\cap\mathfrak g_0)\geqslant\frac12\mathsf{rk}\,\pi_\infty - 1 + (\mathsf{rk}\,\mathfrak g-\mathsf{rk}\,\mathfrak g_0+1) = \frac12\mathsf{rk}\,\pi_\infty+\mathsf{rk}\,\mathfrak g-\mathsf{rk}\,\mathfrak g_0.$$

The differentials $\{d_x\tilde H_j \mid j=1,\dots,\mathsf{rk}\,\mathfrak g_0\}$ are linearly independent and lie in $\mathfrak g_0$. Hence

$$\dim d_y\mathcal Z\geqslant\frac12\mathsf{rk}\,\pi_\infty+\mathsf{rk}\,\mathfrak g-\mathsf{rk}\,\mathfrak g_0+\mathsf{rk}\,\mathfrak g_0=\mathrm{tr.deg}\,\mathcal Z.$$

Now we see that the differentials of all the generators of $\tilde{\mathcal Z}$ are linearly independent at y. A contradiction!

Part (ii) follows from (i) and Theorem 4.1. □

Remark. In the jargon of completely integrable systems, which is used e.g. in [MF78, B91], Eq. (4·7) means that the restriction of $\mathcal Z$ to the symplectic leaf of $\{\ ,\ \}_\infty$ at y is a "complete family in involution".

5. Fancy identities for Poisson tensors

In this section, the existence of a g.g.s. is of no importance, any indecomposable symmetric pair $(\mathfrak g,\mathfrak g_0)$ is admitted.

Let ω be the standard n-form on $\mathfrak g^*$, where $n=\dim\mathfrak g$, and let π be the Poisson tensor (bivector) of the Lie–Poisson bracket on $\mathfrak g^*$, see Section 1.1. Having a basis $\{e_1,\dots,e_n\}$ for $\mathfrak g$, one can write

$$\pi=\sum_{i<j}[e_i,e_j]\otimes\partial_i\wedge\partial_j,\ \text{ where }\ \partial_i=\partial_{e_i}.$$

For simplicity, we identify $\mathcal W^1$ with $\mathcal S(\mathfrak g)\otimes\mathfrak g^*$ and ∂_i with e_i^*, where e_i^* are the elements of the dual basis $\{e_1^*,\dots,e_n^*\}\subset\mathfrak g^*$. We also identify de_i with e_i and therefore Ω^1 with $\mathcal S(\mathfrak g)\otimes\mathfrak g$.

For any $k>0$, set

$$\bigwedge^k\pi=\underbrace{\pi\wedge\pi\wedge\ldots\wedge\pi}_{k\text{ factors}}$$

and regard it as an element of $\mathcal S^k(\mathfrak g)\otimes\bigwedge^{2k}\mathfrak g^*$. Then $\bigwedge^{(n-l)/2}\pi\neq0$ and all higher exterior powers of π are zero. There is a formula describing $\bigwedge^{(n-l)/2}\pi$ in terms of the Poisson centre of $\mathcal S(\mathfrak g)$. Applying the map φ_s^{-1}, one obtains a similar formula for $\varphi_s^{-1}(\pi)$, which is the Poisson tensor of $\{\ ,\ \}_s$, in terms of the Poisson centre of $(\mathcal S(\mathfrak g),\{\ ,\ \}_s)$. The main idea of [Y14] was to consider the minimal s-components of both sides. Here we consider the maximal s-components and obtain interesting new identities.

By definition, $dF \in \Omega^1$ for each $F \in \mathcal{S}(\mathfrak{g})$. Take $H_1, \dots, H_l \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Then

$$dH_1 \wedge \ldots \wedge dH_l \in \mathcal{S}(\mathfrak{g}) \otimes \bigwedge^l \mathfrak{g}.$$

At the same time, $\bigwedge^{(n-l)/2} \pi \in \mathcal{S}(\mathfrak{g}) \otimes \bigwedge^{n-l} \mathfrak{g}^*$. The volume form ω defines a non-degenerate pairing between $\bigwedge^l \mathfrak{g}$ and $\bigwedge^{n-l} \mathfrak{g}$. If $u \in \bigwedge^l \mathfrak{g}$ and $v \in \bigwedge^{n-l} \mathfrak{g}$, then $u \wedge v = c\omega$ with $c \in \Bbbk$. We write this as $\dfrac{u \wedge v}{\omega} = c$ and let $\dfrac{u}{\omega}$ be the element of $(\bigwedge^{n-l} \mathfrak{g})^*$ such that $\dfrac{u}{\omega}(v) = \dfrac{u \wedge v}{\omega}$. For any $\mathbf{u} \in \mathcal{S}(\mathfrak{g}) \otimes \bigwedge^l \mathfrak{g}$, we let $\dfrac{\mathbf{u}}{\omega}$ be the corresponding element of

$$\mathcal{S}(\mathfrak{g}) \otimes \left(\bigwedge^{n-l} \mathfrak{g}\right)^* \cong \mathcal{S}(\mathfrak{g}) \otimes \bigwedge^{n-l} \mathfrak{g}^*.$$

There is a Poisson interpretation of the Kostant regularity criterion [K63, Theorem 9], see also Eq. (2·1), the so-called *Kostant identity* (see [Y14]):

$$\frac{dH_1 \wedge \cdots \wedge dH_l}{\omega} = \bigwedge^{(n-l)/2} \pi.$$

The identity holds if the basic invariants are normalised correctly. It still holds if we apply φ_s^{-1} to both sides.

Suppose that σ is outer and $\sigma(H_j) = -H_j$. Then

$$d(H_j)_{(d_j-1,1)} \in \underbrace{\mathcal{S}^{d_j-1}(\mathfrak{g}_0) \otimes \bigwedge^1 \mathfrak{g}_1}_{I} \oplus \underbrace{\mathfrak{g}_1 \mathcal{S}^{d_j-2}(\mathfrak{g}_0) \otimes \bigwedge^1 \mathfrak{g}_0}_{II}.$$

Let $dH_j^{[1]}$ stand for the component of the first type. This is a 1-form on $\mathfrak{g}^*$. Suppose that $\sigma(H_i) = H_i$ for $i \leqslant k$ and $\sigma(H_i) = -H_i$ for $i > k$. Then $k = \mathsf{rk}\, \mathfrak{g}_0$ here, cf. Lemma 3.5.

Let $\pi_{\mathfrak{g}_0}$ denote the Poisson tensor of $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is reductive, $\bigwedge^{(\dim \mathfrak{g}_0 - \mathsf{rk}\, \mathfrak{g}_0)/2} \pi_{\mathfrak{g}_0}$ is non-zero on the big open subset $(\mathfrak{g}_0^*)_{\mathsf{reg}}$.

Proposition 5.1. *If σ is an inner involution, then*

$$\frac{d(H_1)_{(d_1,0)} \wedge \cdots \wedge d(H_l)_{(d_l,0)}}{\omega} = \bigwedge^{(\dim \mathfrak{g}_1)/2} \pi_\infty \otimes \bigwedge^{(\dim \mathfrak{g}_0 - l)/2} \pi_{\mathfrak{g}_0}. \tag{5·1}$$

If σ is an outer involution, then

$$\frac{d(H_1)_{(d_1,0)} \wedge \cdots \wedge d(H_k)_{(d_k,0)} \otimes dH_{k+1}^{[1]} \wedge \cdots \wedge dH_l^{[1]}}{\omega} = \\ = \bigwedge^{(\dim \mathfrak{g}_1 - l + k)/2} \pi_\infty \otimes \bigwedge^{(\dim \mathfrak{g}_0 - k)/2} \pi_{\mathfrak{g}_0}. \tag{5·2}$$

Proof. The product $dH_1 \wedge \cdots \wedge dH_l$ is an l-form on $\mathfrak{g}^*$ with polynomial coefficients. Among these coefficients, we are interested in those that have the maximal possible degree in $\mathfrak{g}_0$. It is not difficult to see that the degree in question is equal to $\boldsymbol{b}(\mathfrak{g}) - l = (n-l)/2$ and that the corresponding l-form is either $d(H_1)_{(d_1,0)} \wedge \cdots \wedge d(H_l)_{(d_l,0)}$ in the inner case or

$$d(H_1)_{(d_1,0)} \wedge \cdots \wedge d(H_k)_{(d_k,0)} \otimes dH_{k+1}^{[1]} \wedge \cdots \wedge dH_l^{[1]}$$

in the outer case. For the first one, we have

$$\frac{d(H_1)_{(d_1,0)}\wedge\cdots\wedge d(H_l)_{(d_l,0)}}{\omega}\in\mathcal{S}^{(n-l)/2}(\mathfrak{g}_0)\otimes\bigwedge^{\dim\mathfrak{g}_0-l}\mathfrak{g}_0^*\otimes\bigwedge^{\dim\mathfrak{g}_1}\mathfrak{g}_1^*.$$

In case of an outer involution σ, the $(n-l)$-vector belongs to

$$\mathcal{S}^{(n-l)/2}(\mathfrak{g}_0)\otimes\bigwedge^{\dim\mathfrak{g}_0-k}\mathfrak{g}_0^*\otimes\bigwedge^{\dim\mathfrak{g}_1-l+k}\mathfrak{g}_1^*.$$

The right hand side of the Kostant identity is a polyvector with polynomial coefficients of degree $\boldsymbol{b}(\mathfrak{g})-l$. If $\xi\otimes(x\wedge y)$ is a summand of π and $\xi\in\mathfrak{g}_0$, then either $x,y\in\mathfrak{g}_1^*$ or $x,y\in\mathfrak{g}_0^*$. This justifies the right hand sides of (5·1) and (5·2). □

If σ is inner, then $\{(H_i)_{(d_i,0)}\}$ are algebraically independent. Hence also the right hand side of (5·1) is nonzero. In particular, $\bigwedge^{\dim\mathfrak{g}_1/2}\pi_\infty\neq 0$ in complete accordance with Lemma 2.1. If σ is outer, then $\bigwedge^{(\dim\mathfrak{g}_1-l+k)/2}\pi_\infty\neq 0$ by Lemma 2.1. It is also clear that $\bigwedge^{(\dim\mathfrak{g}_0-k)/2}\pi_{\mathfrak{g}_0}\neq 0$. Therefore the left hand side of (5·2) is nonzero, too.

Suppose that σ is inner. Then $\bigwedge^{(\dim\mathfrak{g}_1)/2}\pi_\infty=F{\cdot}x_1\wedge\ldots\wedge x_{\dim\mathfrak{g}_1}$, where $F\in\mathcal{S}^{\dim\mathfrak{g}_1}(\mathfrak{g}_0)$ and $\{x_j\}$ is a basis for $\mathfrak{g}_1^*$. The zero set of F is exactly $\mathfrak{g}^*_{\infty,\mathsf{sing}}$. Under the identifications $\mathfrak{g}_0\simeq\mathfrak{g}_0^*$, we have that $F(\xi_0)=\det(\mathrm{ad}\,(\xi_0)|_{\mathfrak{g}_1})$ for $\xi_0\in\mathfrak{g}_0$.

Let $\{\tilde H_1,\ldots,\tilde H_l\}$ be a set of suitably normalised basic $\mathfrak{g}_0$-invariants in $\mathcal{S}(\mathfrak{g}_0)$. Then they satisfy the Kostant identity with $\bigwedge^{(\dim\mathfrak{g}_0-l)/2}\pi_{\mathfrak{g}_0}$ on the right hand side. In other words, if ω_0 is the volume form on $\mathfrak{g}_0^*$, then

$$\frac{d\tilde H_1\wedge\cdots\wedge d\tilde H_l}{\omega_0}=\bigwedge^{(\dim\mathfrak{g}_0-l)/2}\pi_{\mathfrak{g}_0}.$$

Plugging this identity into (5·1), we obtain the following statement.

Corollary 5.2. *Keep the assumption that σ is inner and regard $(H_j)_{(d_j,0)}$ as an element of $\mathcal{S}(\mathfrak{g}_0)$. Then*

$$d(H_1)_{(d_1,0)}\wedge\ldots\wedge d(H_l)_{(d_l,0)}=F{\cdot}d\tilde H_1\wedge\ldots\wedge d\tilde H_l,$$

where F is the same as above. Hence the differentials $\{d(H_i)_{(d_i,0)}\}$ are linearly dependent exactly on the subset $\mathfrak{g}^*_{\infty,\mathsf{sing}}\cup(\mathfrak{g}_0^*)_{\mathsf{sing}}$. □

Proposition 5.3. *Let σ be an outer involution. Then $(H_j)_{(d_j-1,1)}$, where $k<j\leqslant l$, together with a basis $\{\xi_1,\ldots,\xi_{\dim\mathfrak{g}_0}\}$ of $\mathfrak{g}_0$ freely generate $\mathcal{Z}_\infty$. Further, there is $Q\in\mathcal{S}(\mathfrak{g}_0)$ such that*

$$Q{\cdot}\frac{\xi_1\wedge\ldots\wedge\xi_{\dim\mathfrak{g}_0}\wedge dH^{[1]}_{k+1}\wedge\ldots\wedge dH^{[1]}_l}{\omega}=\bigwedge^{(\dim\mathfrak{g}_1-l+k)/2}\pi_\infty.$$

If Q is regarded as a function on $\mathfrak{g}^*$, then its zero locus is the maximal divisor of $\mathfrak{g}^*$ contained in $\mathfrak{g}^*_{\infty,\mathsf{sing}}$.

Proof. Set $P_0 = \bigwedge_{i=1}^{\dim \mathfrak{g}_0} \xi_i$, $P_1 = \bigwedge_{j=k+1}^{l} dH_j^{[1]}$, and $P = P_0 \wedge P_1$. By the construction of $H_j^{[1]}$, we have also $P = P_0 \wedge (\bigwedge_{j=k+1}^{l} d(H_j)_{(d_j-1,1)})$.

Take $x \in \mathfrak{g}_0^*$. If $\sigma(H_j) = -H_j$, then $d_x H_j = dH_j^{[1]}(x) = d_x(H_j)_{(d_j-1,1)} \in \mathfrak{g}_1$. If $y = x + x'$ with $x \in \mathfrak{g}_0^*$, $x' \in \mathfrak{g}_1^*$, then $P(y) = P_0 \wedge P_1(x)$. We wish to show that $P(y) \neq 0$ on a big open subset of $\mathfrak{g}^*$. This is equivalent to the **claim** that $P_1(x) \neq 0$ on a big open subset of $\mathfrak{g}_0^*$.

Assume that P_1 is zero on an irreducible divisor $X \subset \mathfrak{g}_0^*$. By Lemma 4.2(ii), $x \in (\mathfrak{g}_0^*)_{\mathsf{reg}}$ for a generic $x \in X$. If $x \in \mathfrak{g}_0^*$ is regular in $\mathfrak{g}$, then the elements $d_x H_i$ with $1 \leqslant i \leqslant l$ are linearly independent, see Eq. (2·1), and $P_1(x) \neq 0$. Thus, $\dim \mathfrak{g}^x \geqslant l + 2$ for all $x \in X$ and $X \times \mathfrak{g}_1 \subset \mathfrak{g}^*_{\infty,\mathsf{sing}}$. This settles the claim for the cases, where r_0 is surjective and $\mathfrak{g}^*_{\infty,\mathsf{sing}}$ does not contain divisors.

Suppose that $\dim \mathfrak{g}^*_{\infty,\mathsf{sing}} = n - 1$. Let $x \in X$ be generic. By Lemma 4.7, $\dim \mathfrak{g}^x = l + 2$. Lemma 4.9 states that the elements $d_x H_j$ with $\sigma(H_j) = -H_j$ are linearly independent. Thereby $P_1(x) \neq 0$. The claim is settled.

By Theorem 4.1, the subalgebra of $\mathcal{S}(\mathfrak{g})$ generated by $(H_j)_{(d_j-1,1)}$ with $k < j \leqslant l$ and ξ_i with $1 \leqslant i \leqslant \dim \mathfrak{g}_0$ is algebraically closed. Since it lies inside $\mathcal{Z}_\infty$ and has the same transcendence degree, $\dim \mathfrak{g}_0 + (l - k)$, it coincides with $\mathcal{Z}_\infty$.

Since P is non-zero on a big open subset, we have

$$Q \cdot \frac{\xi_1 \wedge \ldots \wedge \xi_{\dim \mathfrak{g}_0} \wedge dH_{k+1}^{[1]} \wedge \ldots \wedge dH_l^{[1]}}{\omega} = \bigwedge^{(\dim \mathfrak{g}_1 - l + k)/2} \pi_\infty$$

for some $Q \in \mathcal{S}(\mathfrak{g})$, see e.g. [Y14, Section 2]. Since all the coefficients in the right hand side are elements of $\mathcal{S}(\mathfrak{g}_0)$, we have $Q \in \mathcal{S}(\mathfrak{g}_0)$ as well. □

Remark 5.4. If σ is inner, then $\mathrm{tr.deg}\, \mathcal{Z}_\infty = \dim \mathfrak{g}_0$ and it is easily seen that $\mathcal{Z}_\infty = \mathcal{S}(\mathfrak{g}_0)$ as subalgebra of $\mathcal{S}(\mathfrak{g}_{(\infty)})$. In particular, $\mathcal{Z}_\infty$ is always a polynomial algebra.

Combining Proposition 5.3 with Eq. (5·2) and the Kostant identity for $\mathfrak{g}_0$, we obtain the following assertion.

Corollary 5.5. *Let $\tilde{H}_1, \ldots, \tilde{H}_k$ be properly normalised basic $\mathfrak{g}_0$-invariants in $\mathcal{S}(\mathfrak{g}_0)$. Then*

$$d(H_1)_{(d_1,0)} \wedge \ldots \wedge d(H_k)_{(d_k,0)} = Q \cdot d\tilde{H}_1 \wedge \ldots \wedge d\tilde{H}_k$$

in $\mathcal{S}(\mathfrak{g}_0) \otimes \bigwedge^k \mathfrak{g}_0$ with the same Q as in Proposition 5.3. The differentials $d(H_1)_{(d_1,0)}, \ldots, d(H_k)_{(d_k,0)}$ are linearly dependent exactly on the union of $(\mathfrak{g}_0^*)_{\mathsf{sing}}$ with the zero set of Q. □

Note that Q is the Pfaffian in the setting of Example 4.8.

6. Further developments and possible applications

We believe that this paper is the beginning of a long exciting journey. Several applications of our construction are already available and are presented below. Goals further ahead are stated as conjectures.

6.1. **Quantum perspectives.** Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Given a Poisson-commutative subalgebra $\mathcal{C} \subset \mathcal{S}(\mathfrak{g})$, it is natural to ask whether there exists a commutative subalgebra $\widehat{\mathcal{C}} \subset \mathcal{U}(\mathfrak{g})$ such that $\mathrm{gr}(\widehat{\mathcal{C}}) = \mathcal{C}$. This question was posed by Vinberg for the Mishchenko–Fomenko subalgebras [Vi91], and it is known nowadays as *Vinberg's problem*. For the semisimple $\mathfrak{g}$, the first conceptual solution was obtained in [R06]. The rôle of the symmetrisation map $\varpi : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ in that quantisation for the classical $\mathfrak{g}$ is explained in [MY].

Conjecture 6.1. *Suppose that there is a* g.g.s. *for σ. Let $\widehat{\mathcal{Z}}$ be the subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by $\varpi((H_j)_{(i,d_j-i)})$ with $1 \leqslant i \leqslant l$, $0 \leqslant i \leqslant d_i$. Then $\widehat{\mathcal{Z}}$ is commutative and $\mathrm{gr}(\widehat{\mathcal{Z}}) = \mathcal{Z}$.*

For the symmetric pairs $(\mathfrak{gl}_{n+m}, \mathfrak{gl}_n \oplus \mathfrak{gl}_m)$, $(\mathfrak{sp}_{2(n+m)}, \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m})$, and $(\mathfrak{so}_{n+m}, \mathfrak{so}_n \oplus \mathfrak{so}_m)$, there might be a connection between $\widehat{\mathcal{Z}}$ and commutative subalgebras of Yangians or twisted Yangians.

The Yangian $Y(\mathfrak{gl}_m)$ is a deformation of the enveloping algebra $\mathcal{U}(\mathfrak{gl}_m[z])$ of the current algebra $\mathfrak{gl}_m[z]$ given by explicit generators and relations. Then $\mathcal{U}(\mathfrak{gl}_m)$ is a subalgebra of $Y(\mathfrak{gl}_m)$. The facts on Yangians, which are used below, can be found in [M07], see in particular Chapter 8 therein. The most relevant for us is the *centraliser construction* of Olshanski [O91] and Molev–Olshanski [MO00]. For any n, there is an almost surjective map

$$\Psi_n \colon Y(\mathfrak{gl}_m) \to \mathcal{U}(\mathfrak{gl}_{n+m})^{\mathfrak{gl}_n},$$

where the words "almost surjective" mean that $\mathcal{U}(\mathfrak{gl}_{n+m})^{\mathfrak{gl}_n}$ is generated by the image of $Y(\mathfrak{gl}_m)$ and $\mathcal{U}(\mathfrak{gl}_n)^{\mathfrak{gl}_n}$. It is known that, for a fixed m, $\bigcap\limits_{n\geqslant 1} \ker \Psi_n = 0$.

Question 6.2. *Is there a commutative subalgebra $\mathcal{B} \subset Y(\mathfrak{gl}_m)$ such that $\mathrm{gr}(\Psi_n(\mathcal{B}))$ together with $\mathcal{Z}\mathcal{S}(\mathfrak{g}_0)$ generate $\tilde{\mathcal{Z}} \subset \mathcal{S}(\mathfrak{gl}_m \oplus \mathfrak{gl}_n)$?*

Let $Y(\mathfrak{sp}_{2m}) \subset Y(\mathfrak{gl}_{2m})$ be the twisted Yangian in the sense of G. Olshanski. Here $\mathcal{U}(\mathfrak{sp}_{2m}) \subset Y(\mathfrak{sp}_{2m})$ and there is again an almost surjective map

$$\Psi_n \colon Y(\mathfrak{sp}_{2m}) \to \mathcal{U}(\mathfrak{sp}_{2n+2m})^{\mathfrak{sp}_{2n}}.$$

Then one can pose an analogous question. A similar situation occurs for $Y(\mathfrak{so}_m) \subset Y(\mathfrak{gl}_m)$ and $\mathcal{U}(\mathfrak{so}_{n+m})$ with n even.

Any natural quantisation of $\mathcal{Z}$ has to provide a commutative subalgebra $\widehat{\mathcal{Z}} \subset \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$. By adding $\mathcal{U}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ one obtains the related quantisation $\widehat{\tilde{\mathcal{Z}}}$ of $\tilde{\mathcal{Z}}$. Let V be a finite-dimensional simple $\mathfrak{g}$-module. Then $\widehat{\tilde{\mathcal{Z}}}$ acts on the subspace $V^{\mathfrak{n}_0} \subset V$ of the highest weight vectors of $\mathfrak{g}_0$.

Conjecture 6.3. *Let $\widehat{\tilde{\mathcal{Z}}} \subset \mathcal{U}(\mathfrak{g})$ be the subalgebra generated by $\varpi((H_j)_{(i,d_j-i)})$ with $1 \leqslant i \leqslant l$, $0 \leqslant i \leqslant d_i$ and by $\mathcal{U}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. Then $\widehat{\tilde{\mathcal{Z}}}$ acts on $V^{\mathfrak{n}_0}$ diagonalisably and with a simple spectrum.*

If Conjecture 6.3 is true, then the action of $\widehat{\tilde{\mathcal{Z}}}$ produces a solution of the branching problem $\mathfrak{g} \downarrow \mathfrak{g}_0$. There are two renowned examples, where both conjectures are true.

Example 6.4 (The Gelfand–Tsetlin construction [GT50, GT50′])**.** Let $(\mathfrak{g}, \mathfrak{g}_0)$ be one of the symmetric pairs $(\mathfrak{sl}_{n+1}, \mathfrak{gl}_n)$, $(\mathfrak{so}_{n+1}, \mathfrak{so}_n)$. Then each H_i has at most two nonzero bi-homogeneous components. To be more precise, the Pfaffian in the case of $\mathfrak{g} = \mathfrak{so}_{2l}$ has one nonzero component, and all the other generators have exactly two. It follows that $\tilde{\mathcal{Z}}$ is generated by $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ and $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. The quantum analogue $\widehat{\tilde{\mathcal{Z}}}$ is generated by $\mathcal{U}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ and $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$.

For each irreducible finite-dimensional representation V of $\mathfrak{g}$, the restriction to $\mathfrak{g}_0$ is multiplicity free. Hence the action of $\widehat{\tilde{\mathcal{Z}}}$ on $V^{\mathfrak{n}_0}$ has a simple spectrum.

6.2. **Classical applications.** Let us return to the Poisson side of the story.

Suppose that there is a g.g.s. for σ. Although $\tilde{\mathcal{Z}}$ (or $\mathcal{Z}$) is not a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})$, it can be included into such a subalgebra in many natural ways. Let $\mathcal{C} = \Bbbk[F_1, \dots, F_{\boldsymbol{b}(\mathfrak{g}_0)}]$ be a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g}_0)$. Then necessary $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0} \subset \mathcal{C}$. Suppose further that the F_i's are homogeneous and their differentials are linearly independent on a big open subset of $\mathfrak{g}_0^*$. For instance, one can take $\mathcal{C} = \mathcal{A}_\gamma$ with $\gamma \in (\mathfrak{g}_0)^*_{\mathsf{reg}}$, see [PY08]. An easy calculation shows that $\mathsf{alg}\langle\tilde{\mathcal{Z}}, \mathcal{C}\rangle = \mathsf{alg}\langle\mathcal{Z}, \mathcal{C}\rangle$ has $\boldsymbol{b}(\mathfrak{g})$ generators. Indeed, $\tilde{\mathcal{Z}}$ (or $\mathcal{Z}$) has $\frac{1}{2}(\dim \mathfrak{g}_1 + \mathsf{rk}\,\mathfrak{g} + \mathsf{rk}\,\mathfrak{g}_0)$ free generators. Then we replace the generators sitting in $\mathcal{S}(\mathfrak{g}_0)$ (there are $\mathsf{rk}\,\mathfrak{g}_0$ of them) with the whole bunch of generators of $\mathcal{C}$. In this way, we obtain

$$\frac{1}{2}(\dim \mathfrak{g}_1 + \mathsf{rk}\,\mathfrak{g} + \mathsf{rk}\,\mathfrak{g}_0) - \mathsf{rk}\,\mathfrak{g}_0 + \boldsymbol{b}(\mathfrak{g}_0) = \boldsymbol{b}(\mathfrak{g})$$

generators $\{F_i, \boldsymbol{h}_j \mid 1 \leqslant i \leqslant \boldsymbol{b}(\mathfrak{g}_0), 1 \leqslant j \leqslant \boldsymbol{b}(\mathfrak{g}) - \boldsymbol{b}(\mathfrak{g}_0)\}$. Furthermore, the differentials $\{dF_i, d\boldsymbol{h}_j\}$ are linearly independent at $x \in \mathfrak{g}^*$ if and only if $\dim(\boldsymbol{d}_x\tilde{\mathcal{Z}} + \boldsymbol{d}_x\mathcal{C}) = \boldsymbol{b}(\mathfrak{g})$. Write $x = x_0 + x_1$ with $x_i \in \mathfrak{g}_i$ and suppose that $x_0 \in (\mathfrak{g}_0^*)_{\mathsf{reg}}$. Then

$$(\boldsymbol{d}_x\tilde{\mathcal{Z}} \cap \boldsymbol{d}_x\mathcal{C}) \subset \mathfrak{g}_0, \ \ \pi(x)(\mathfrak{g}_0, \boldsymbol{d}_x\tilde{\mathcal{Z}}) = 0, \ \text{ and hence } \ \boldsymbol{d}_x\tilde{\mathcal{Z}} \cap \boldsymbol{d}_x\mathcal{C} = \mathfrak{g}_0^{x_0}.$$

If in addition $\dim \boldsymbol{d}_x\tilde{\mathcal{Z}} = \operatorname{tr.deg} \mathcal{Z}$ and $\dim \boldsymbol{d}_{x_0}\mathcal{C} = \boldsymbol{b}(\mathfrak{g}_0)$, then $\dim(\boldsymbol{d}_x\tilde{\mathcal{Z}} + \boldsymbol{d}_x\mathcal{C}) = \boldsymbol{b}(\mathfrak{g})$. In view of Theorem 4.12(i), we can conclude that the differentials $\{dF_i, d\boldsymbol{h}_j\}$ are linearly independent on a big open subset of $\mathfrak{g}^*$. Thus, Theorem 4.1 applies and assures that $\mathsf{alg}\langle\tilde{\mathcal{Z}},\mathcal{C}\rangle$ is a maximal Poisson-commutative subalgebra of $\mathfrak{g}$.

Arguing inductively, one can produce a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ from a chain of symmetric subalgebras

$$\mathfrak{g}=\mathfrak{g}^{(0)}\supset\mathfrak{g}^{(1)}\supset\mathfrak{g}^{(2)}\supset\ldots\supset\mathfrak{g}^{(m)},$$

where $\mathfrak{g}^{(m)}$ is Abelian and each symmetric pair $(\mathfrak{g}^{(i)},\mathfrak{g}^{(i+1)})$ has a g.g.s.

Remark. (i) For any simple Lie algebra $\mathfrak{g}$, there is an involution σ that has a g.g.s. [P07′, Sect. 6]. Therefore our construction of a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ related to a chain of symmetric subalgebras works for any simple $\mathfrak{g}$.

(ii) In [Vi91, § 6], limits of Mishchenko–Fomenko subalgebras were introduced. The Poisson counterpart of the Gelfand–Tsetlin subalgebra of $\mathcal{U}(\mathfrak{sl}_{n+1})$ related to the chain

$$\mathfrak{sl}_{n+1}\supset\mathfrak{gl}_n\supset\mathfrak{gl}_{n-1}\supset\ldots\supset\mathfrak{gl}_2\supset\mathfrak{gl}_1,$$

appears as one of these limit subalgebras, see also Example 6.4. The key point of Vinberg's construction is that the Poincaré series of any limit subalgebra is the same as that of $\mathcal{A}_\gamma$ with $\gamma\in\mathfrak{g}^*_{\mathsf{reg}}$. With a few exceptions, our approach produces Poisson-commutative subalgebras with different Poincaré series. This can be illustrated by the chain

$$\mathfrak{so}_5\supset\mathfrak{so}_4\supset\mathfrak{so}_2\oplus\mathfrak{so}_2.$$

Here the degrees of the generators of the related maximal Poisson-commutative subalgebra are $(4,2,2,2,1,1)$ opposite to $(4,3,2,1,2,1)$ in the case of $\mathcal{A}_\gamma$.

Another feature is that $\mathcal{Z}$ can be used for constructing a Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g}_0)$. Let $(\mathfrak{g},\mathfrak{g}_0)$ be an arbitrary symmetric pair. If there is a g.g.s. for $(\mathfrak{g},\mathfrak{g}_0)$, then we are able to consider both algebras, $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$. For $\eta\in\mathfrak{g}_1^*$, let $\mathcal{Z}_\eta$, $\tilde{\mathcal{Z}}_\eta$ denote the restrictions of $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ to $\mathfrak{g}_0^*+\eta$. By choosing η as the origin, we identify $\mathfrak{g}_0^*+\eta$ with $\mathfrak{g}_0^*$. Then $\mathcal{Z}_\eta$ and $\tilde{\mathcal{Z}}_\eta$ are homogeneous subalgebras of $\mathcal{S}(\mathfrak{g}_0)$. Moreover, they Poisson-commute with $\mathfrak{g}_0^\eta$.

Lemma 6.5. *The subalgebras $\mathcal{Z}_\eta$ and $\tilde{\mathcal{Z}}_\eta$ are Poisson-commutative.*

Proof. Take $H,F\in\mathcal{Z}$ or $H,F\in\tilde{\mathcal{Z}}$ and $x\in\mathfrak{g}_0^*$. Let $\mathbf{h}$ and $\mathbf{f}$ be the restrictions of H,F to $\mathfrak{g}_0^*+\eta$. Then $d_{x+\eta}H=d_x\mathbf{h}+\xi_1$, $d_{x+\eta}F=d_x\mathbf{f}+\nu_1$, where $\xi_1,\nu_1\in\mathfrak{g}_1$. Set $\xi_0=d_x\mathbf{h}$, $\nu_0=d_x\mathbf{f}$. Our goal is to show that $x([\xi_0,\nu_0])=0$.

Since H and F commute w.r.t. any bracket $\{\ ,\ \}_t$ with $t\in\mathbb{P}$, we have in particular $x([\xi_1,\nu_1])=0$, as well as $(x+\eta)([\xi_0+\xi_1,\nu_0+\nu_1])=0$. Both are also $\mathfrak{g}_0$-invariants. Therefore

$$(x+\eta)([\xi_0,\nu_0+\nu_1])=0,\qquad 0=(x+\eta)([\nu_0,\xi_0+\xi_1])=x([\nu_0,\xi_0])+\eta([\nu_0,\xi_1]).$$

Now $0=(x+\eta)([\xi_1,\nu_0+\nu_1])=\eta([\xi_1,\nu_0])$ and it is clear that $x([\xi_0,\nu_0])=0$. □

Remark 6.6. Let $(\mathfrak g, \mathfrak g_0) = (\mathfrak{sl}_n, \mathfrak{so}_n)$. The corresponding involution σ is of *maximal rank* and any set of generators $H_1, \dots, H_l \in \mathcal S(\mathfrak g)^{\mathfrak g}$ is a g.g.s. for σ. The related Poisson-commutative subalgebra $\mathcal Z$ appeared, in a way, in work of Manakov [M76]. He stated that the restriction of $\mathcal Z$ to $\mathfrak g_0 + \eta$ with $\eta \in \mathfrak g_1$ is a Poisson-commutative subalgebra of $\mathcal S(\mathfrak g_0)$ of the maximal possible transcendence degree, which is $\boldsymbol b(\mathfrak g_0)$. Below we present a connection between his results and ours. We are grateful to E.B. Vinberg for bringing our attention to the fact that Manakov's construction involves an involution.

Let $\mathfrak c_1 \subset \mathfrak g_1$ be a Cartan subspace. If $\eta \in \mathfrak c_1$ is generic, then $\mathfrak l := \mathfrak g_0^\eta$ is reductive and it is also the centraliser of $\mathfrak c_1$ in $\mathfrak g_0$. There are well-known equalities: $\dim \mathfrak g_1 - \dim \mathfrak g_0 = \dim \mathfrak l - \dim \mathfrak c_1$ and $\mathrm{rk}\, \mathfrak l = \mathrm{rk}\, \mathfrak g - \dim \mathfrak c_1$.

Theorem 6.7. *For almost all $\eta \in \mathfrak c_1$, we have*

(i) $\mathrm{tr.deg}\, \mathcal Z_\eta = \boldsymbol b(\mathfrak g_0) - \boldsymbol b(\mathfrak l) + \mathrm{rk}\, \mathfrak l$;
(ii) *if there is a* g.g.s. *for σ, then $\tilde{\mathcal Z}_\eta$ is a maximal Poisson-commutative subalgebra of $\mathcal S(\mathfrak g_0)^{\mathfrak l}$. Besides, if $\mathfrak l$ is Abelian, then $\tilde{\mathcal Z}_\eta$ is a maximal Poisson-commutative subalgebra of $\mathcal S(\mathfrak g_0)$.*

Proof. Suppose that η is generic enough. Then

- $\dim \boldsymbol d_y\mathcal Z = \frac12(\dim \mathfrak g_1 + \mathrm{rk}\, \mathfrak g + \mathrm{rk}\, \mathfrak g_0)$ for y in a dense open subset of $\mathfrak g_0 + \eta$, and
- $\dim \boldsymbol d_y\tilde{\mathcal Z} = \frac12(\dim \mathfrak g_1 + \mathrm{rk}\, \mathfrak g + \mathrm{rk}\, \mathfrak g_0)$ for y in a **big** open subset of $\mathfrak g_0 + \eta$.

Note that the subspaces $\boldsymbol d_y\mathcal Z$ and $\boldsymbol d_y\tilde{\mathcal Z}$ are orthogonal to $\mathfrak g_0$ w.r.t. the bilinear form $\pi(y) = y([\ ,\ ])$. Hence for both of them, the intersection with $\mathfrak g_1$ has dimension at most $\dim \mathfrak c_1$. It is easily seen that actually $\dim(\boldsymbol d_y\tilde{\mathcal Z} \cap \mathfrak g_1) = \dim \mathfrak c_1$. Furthermore,

$$\boldsymbol d_y\mathcal Z_\eta \simeq \boldsymbol d_y\mathcal Z/(\boldsymbol d_y\mathcal Z \cap \mathfrak g_1)$$

and the same formula holds for $\tilde{\mathcal Z}$. Therefore

$$\begin{aligned}\mathrm{tr.deg}\, \mathcal Z_\eta \geqslant \frac12(\dim \mathfrak g_1 + \mathrm{rk}\, \mathfrak g + \mathrm{rk}\, \mathfrak g_0) - \dim \mathfrak c_1 &= \frac12(\dim \mathfrak g_1 - \dim \mathfrak c_1 + \mathrm{rk}\, \mathfrak g - \dim \mathfrak c_1 + \mathrm{rk}\, \mathfrak g_0)\\ &= \frac12(\dim \mathfrak g_0 - \dim \mathfrak l + \mathrm{rk}\, \mathfrak l + \mathrm{rk}\, \mathfrak g_0) = \boldsymbol b(\mathfrak g_0) - \boldsymbol b(\mathfrak l) + \mathrm{rk}\, \mathfrak l.\end{aligned}$$

Since $\mathcal Z_\eta \subset \mathcal S(\mathfrak g_0)^{\mathfrak l}$ and $\mathrm{rk}\, \mathfrak l = \mathrm{ind}\, \mathfrak l$, the transcendence degree of $\mathcal Z_\eta$ cannot be larger than $\boldsymbol b(\mathfrak g_0) - \boldsymbol b(\mathfrak l) + \mathrm{rk}\, \mathfrak l$ by [MY, Prop. 1.1]. Because $\tilde{\mathcal Z}$ in an algebraic extension of $\mathcal Z$, we also have $\mathrm{tr.deg}\, \tilde{\mathcal Z}_\eta = \mathrm{tr.deg}\, \mathcal Z_\eta$.

The difference $\mathrm{tr.deg}\, \tilde{\mathcal Z} - \mathrm{tr.deg}\, \tilde{\mathcal Z}_\eta$ is equal to $\dim \mathfrak c_1$. We consider the algebra $\tilde{\mathcal Z}$ only if there is a g.g.s for σ. In that case the map r_1 is surjective and therefore for certain members H_i of the g.g.s. we have $H_i^\bullet \in \mathcal S(\mathfrak g_1)$ [P07′]. The number of such element is equal to $\dim \mathfrak c_1$, and they restrict to constants on $\mathfrak g_0^* + \eta$.

We see that $\tilde{\mathcal Z}_\eta$ is freely generated by $\tilde H_1, \dots, \tilde H_{\mathrm{rk}\, \mathfrak g_0} \in \mathcal S(\mathfrak g_0)^{\mathfrak g_0}$ and the restrictions to $\eta + \mathfrak g_0$ of $(H_j)_{(i, d_j - i)}$ with $0 < i < d_j$. Moreover, the differentials of these generators are linearly independent on a big open subset. According to Theorem 4.1, $\tilde{\mathcal Z}_\eta$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{g}_0)$. By a standard argument, it is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{l}}$.

Suppose that $\mathfrak{l}$ is Abelian. Then $\dim\mathfrak{l} = \mathsf{rk}\,\mathfrak{l}$ and $\tilde{\mathcal{Z}}_\eta$ is a Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g}_0)$ of the maximal possible transcendence degree. Here $\tilde{\mathcal{Z}}_\eta$ is maximal in $\mathcal{S}(\mathfrak{g}_0)$. □

The statements of Theorem 6.7 are not entirely satisfactory. It would be nice to have an explicit description of η such that the results hold. In the original setting of Manakov, $\mathfrak{l}$ is trivial and the equality $\mathrm{tr.deg}\,\mathcal{Z}_\eta = \boldsymbol{b}(\mathfrak{g}_0)$ holds for each regular $\eta \in \mathfrak{c}_1$, see [GDI]. But a more precise assertion requires a further analysis of $\mathfrak{g}^*_{(t),\mathsf{sing}}$ and we prefer to postpone it.

Appendix A. On pencils of skew-symmetric forms

Here we gather some general facts concerning skew-symmetric bilinear forms. Let $\mathcal{P}$ be a two-dimensional vector space of (possibly degenerate) skew-symmetric bilinear forms on a finite-dimensional vector space V. Set $m = \max_{A\in\mathcal{P}} \mathsf{rk}\,A$, and let $\mathcal{P}_{\mathsf{reg}} \subset \mathcal{P}$ be the set of all forms of rank m. Then $\mathcal{P}_{\mathsf{reg}}$ is a conical open subset of $\mathcal{P}$. For each $A \in \mathcal{P}$, let $\ker A \subset V$ be the kernel of A. Our object of interest is the subspace $L := \sum_{A\in\mathcal{P}_{\mathsf{reg}}} \ker A$.

Lemma A.1 ([PY08, Appendix]). *If Ω is a non-empty open subset of $\mathcal{P}_{\mathsf{reg}}$, then $\sum_{A\in\Omega} \ker A = L$.*

Corollary A.2. *For all $A, B \in \mathcal{P} \setminus \{0\}$, we have $A(\ker B, L) = 0$ and therefore $A(L, L) = 0$.*

Proof. Clearly, the equality $A(\ker B, L) = 0$ holds if B is a scalar multiple of A. If not, then we consider $L_b := \ker(A + bB)$ for $b \in \Bbbk$. Here

$$A(\ker B, L_b) = (A + bB)(\ker B, L_b) - bB(\ker B, L_b) = 0.$$

By Lemma A.1, there is an open subset $\mathfrak{O} \subset \Bbbk$ such that L is spanned by $\{L_b \mid b \in \mathfrak{O}\}$. Hence

$$A(\ker B, L) = A(\ker B, \textstyle\sum_{b\in\mathfrak{O}} L_b) = 0.$$ □

Suppose that $C \in \mathcal{P}\setminus\mathcal{P}_{\mathsf{reg}}$. Then $U = \ker C$ may not be a subspace of L. Take $A \in \mathcal{P}\setminus\{0\}$ that is not proportional to C and restrict it to U. The resulting skew-symmetric form on U does not change if we replace A with any $A + bC$, where $b \in \Bbbk$.

Lemma A.3. *Let C, A, and U be as above. Then $\mathsf{rk}\,(A|_U) \leqslant \dim U - (\dim V - m)$.*

Proof. By Corollary A.2, we have $A(U, L) = 0$. Set $r = \dim V - m$. Because $\mathcal{P}$ is irreducible, $\overline{\mathcal{P}_{\mathsf{reg}}} = \mathcal{P}$ and there is a curve $\tau\colon \Bbbk^\times \to \mathcal{P}_{\mathsf{reg}}$ such that $\lim_{t\to 0}\tau(t) = C$. Hence

$$\lim_{t\to 0}(\ker\tau(t)) \subset \ker C,$$

where the limit is taken in the Grassmannian of the r-dimensional subspaces of V. Set $U_0 := \lim_{t\to 0}(\ker \tau(t))$. If $t \neq 0$, then $\ker \tau(t) \subset L$ and $A(\ker \tau(t), U) = 0$. Hence also $A(U_0, U) = 0$ and $U_0 \subset \ker(A|_U)$. It remains to notice that $\dim U_0 = r$. □

Remark. Lemma A.3 implies Vinberg's inequality: if $\mathfrak{q}$ is Lie algebra, then $\operatorname{ind}\mathfrak{q}^\gamma \geqslant \operatorname{ind}\mathfrak{q}$ for any $\gamma \in \mathfrak{q}^*$, see [P03, Cor. 1.7].

Theorem A.4. *Suppose that $\mathcal{P} \setminus \mathcal{P}_{\mathsf{reg}} = \Bbbk C$ with $C \neq 0$ and $U = \ker C$. Keep the notation of Lemma A.3 and suppose further that $\mathsf{rk}\,(A|_U) = \dim U - \dim V + m$. Then $\dim(L \cap U) = \dim V - m$ and $\dim L = (\dim V - m) + \frac{1}{2}(\dim V - \dim U)$.*

Proof. Let $B \in \mathcal{P}_{\mathsf{reg}}$ be non-proportional to A. Given $A, B \in \mathcal{P}_{\mathsf{reg}}$, there is the so-called *Jordan–Kronecker canonical form* of A and B, see [T91]. Namely, $V = V_1 \oplus \cdots \oplus V_d$, where $A(V_i, V_j) = 0 = B(V_i, V_j)$ for $i \neq j$, and accordingly, $A = \sum A_i$ and $B = \sum B_i$. There are two possibilities for (A_i, B_i), one obtains either a *Kronecker* or a *Jordan block* here, see figures below. Assume that $\dim V_i > 0$ for each i.

	A_i	B_i
A Jordan block ($\lambda_i \in \Bbbk$) :	$\begin{pmatrix} & \mathcal{J}(\lambda_i) \\ -\mathcal{J}^{\top}(\lambda_i) & \end{pmatrix}$	$\begin{pmatrix} & -\mathrm{I} \\ \mathrm{I} & \end{pmatrix}$,
a Kronecker block :	$\left(\begin{array}{c\|c} & \begin{matrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ 0 & \ddots & \\ & \ddots & -1 \\ & & 0 \end{matrix} & \end{array}\right)$	$\left(\begin{array}{c\|c} & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} \\ \hline \begin{matrix} 0 & & \\ -1 & \ddots & \\ & \ddots & 0 \\ & & -1 \end{matrix} & \end{array}\right)$,

where $\mathcal{J}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$. In general, there can occur "Jordan blocks with $\lambda_i = \infty$", but this is not the case here, since $B \in \mathcal{P}$ is assumed to be regular.

Note that if V_i gives rise to a Jordan block, then $\dim V_i$ is even and both A_i and B_i are non-degenerate on V_i. For a Kronecker block, $\dim V_i = 2k_i + 1$, $\mathsf{rk}\,A_i = 2k_i = \mathsf{rk}\,B_i$ and the same holds for every non-zero linear combination of A_i and B_i.

There is a unique $\lambda \in \Bbbk \setminus \{0\}$ such that $C = A + \lambda B$. This λ can be determined as the root of the equation $\det(A_i + \lambda B_i) = 0$ for any Jordan block (A_i, B_i). This readily follows from the uniqueness of the singular line $\Bbbk C \subset \mathcal{P}$. On the other hand, the above matrices show that the root corresponding to (A_i, B_i) is λ_i. Therefore, all λ_i's are equal and coincide with λ.

Let us assume that V_i defines a Kronecker block if and only if $1 \leqslant i \leqslant d'$. Then necessarily $d' = \dim V - m$. Let $\ker(A_i + bB_i) \subset V_i$ be the kernel of the bilinear form $A_i + bB_i$. Then

$$L = \bigoplus_{i=1}^{d'} \sum_{b\colon A+bB \in \mathcal{P}_{\mathsf{reg}}} \ker(A_i + bB_i) =: \bigoplus_{i=1}^{d'} L_i.$$

It follows from the above matrix form of a Kronecker block that $\dim L_i = k_i + 1$, cf. also [PY08, Appendix].

Set $C_i = A_i + \lambda B_i$ for each $i \in \{1, 2, \dots, d\}$. It is a bilinear form on V_i.

- If $i \leqslant d'$, then $\dim \ker C_i = 1$. Therefore $\ker C_i \subset L_i$ and $\dim(\ker C \cap L) = d'$.
- If $i > d'$, then $\dim \ker C_i = 2$.

Hence $\dim U = 2(d - d') + d' = 2d - d'$. Since $U = \bigoplus_{i=1}^{d} \ker C_i$ and the spaces $\{\ker C_i\}$ are pairwise orthogonal w.r.t. any form in $\mathcal{P}$, we have $A(\ker C_j, U) = 0$ for $j \leqslant d'$. Hence the condition $\mathsf{rk}\,(A|_U) = \dim U - \dim V + m$ implies that A_i is non-degenerate on $\ker C_i$ for any $i > d'$. The explicit matrix form of a Jordan block shows that $\ker C_i$ is spanned by two middle basis vectors of V_i. Therefore, A_i is non-degenerate on $\ker C_i$ if and only if $\dim V_i = 2$, and hence $C_i = 0$.

Summing up, we obtain

$$\dim L = \sum_{i=1}^{d'} (k_i + 1) = d' + \sum_{i=1}^{d'} \frac{1}{2} \mathsf{rk}\, C_i = d' + \frac{1}{2} \mathsf{rk}\, C = (\dim V - m) + \frac{1}{2}(\dim V - \dim U).$$

This completes the proof. □

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