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On classical tensor categories attached to the irreducible representations of the general linear supergroups $GL(n|n)$

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Abstract

We study the quotient of $\mathcal{T}_n = Rep(GL(n|n))$ by the tensor ideal of negligible morphisms. If we consider the full subcategory $\mathcal{T}_n^+$ of $\mathcal{T}_n$ of indecomposable summands in iterated tensor products of irreducible representations up to parity shifts, its quotient is a semisimple tannakian category $Rep(H_n)$ where H_n is a pro-reductive algebraic group. We determine the H_n and the groups H_λ corresponding to the tannakian subcategory in $Rep(H_n)$ generated by an irreducible representation $L(\lambda)$. This gives structural information about the tensor category $Rep(GL(n|n))$, including the decomposition law of a tensor product of irreducible representations up to summands of superdimension zero. Some results are conditional on a hypothesis on 2-torsion in $\pi_0(H_n)$.

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Contents

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1 Introduction

1.1 Semisimple quotients

The categories of finite dimensional representations $\mathcal{T}_{m|n}$ of the general linear supergroups $GL(m|n)$ over an algebraically closed field k of characteristic zero are abelian tensor categories, where representations in $\mathcal{T}_{m|n}$ are always understood to be algebraic. However, contrary to the classical case of the general linear groups $GL(n)$ these categories are not semisimple. Whereas the tensor product $V \otimes V$, $V \simeq k^{m|n}$, is completely reducible, this is no longer true for the tensor product $\mathbb{A} = V \otimes V^\vee$. Indeed $\mathbb{A}$ defines the indecomposable adjoint representation of $GL(n|n)$ in the case $m = n$, hence admits a trivial one dimensional subrepresentation defined by the center and a trivial one dimensional quotient representation defined by the supertrace. In contrast to the classical case the supertrace is trivial on the center, and $\mathbb{A}$ is indecomposable with three irreducible Jordan–Hoelder factors $1, S^1, 1$ with the superdimensions $1, -2, 1$ respectively defined by the filtration $\mathfrak{z} \subseteq \mathfrak{sl}(n|n) \subseteq \mathfrak{gl}(n|n)$, where $\mathfrak{z}$ denotes the center of $\mathfrak{gl}(n|n)$.

Although the irreducible representations of $GL(m|n)$ can be classified by highest weights similarly to the classical case, this implies that the tensor product of irreducible representations is in general far from being completely reducible. In fact Weyl's unitary trick fails in the superlinear setting. While the structure of $\mathcal{T}_{m|n}$ as an abelian category is now well understood [14], its monoidal structure remains mysterious.

The perspective of this article is that, in order to restore parts of the classical picture, two finite dimensional representations M and M' of $GL(m|n)$ should not be distinguished, if there exists an isomorphism

$$M \oplus N \cong M' \oplus N'$$

where N and N' are negligible modules. Here we use the notion that a finite dimensional module is said to be negligible if it is a direct sum of indecomposable modules whose superdimensions are zero. A typical example of a negligible module is the indecomposable adjoint representation $\mathbb{A}$ in the case $m = n$. To define this precisely we divide our category $\mathcal{T}_{m|n}$ by the tensor ideal $\mathcal{N}$ [2] of negligible morphisms. The quotient is a semisimple abelian tensor category. By a fundamental result of Deligne it is equivalent to the representation category of a pro-reductive supergroup G^{red} [34].

Taking the quotient of a non-semisimple tensor category by objects of categorial dimension 0 has been studied in a number of different cases. A well-known example is the quotient of the category of tilting modules by the negligible modules (of quantum dimension 0) in the representation category of the Lusztig quantum group $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple Lie algebra over k [1, 6]. The modular categories so obtained have been studied extensively in their applications to the 3-manifold invariants of Reshetikhin–Turaev. In [40] Jannsen proved that the category of numerical motives as defined via algebraic correspondences modulo numerical equivalence is an abelian semisimple category. It was noted by André and Kahn [2] that taking numerical equivalence amounts to taking the quotient by the negligible morphisms. Jannsen's theorem has been generalized to a categorical setting by [2]. In particular they study quotients of tannakian categories by the ideal of negligible morphisms. Recently Etingof and Ostrik [28] studied semisimplifactions with an emphasis on finite tensor categories.

A general study of $Rep(G)/\mathcal{N}$, where G is a supergroup scheme, was initiated in [34] where in particular the reductive group G^{red} given by $Rep(G^{red}) \simeq Rep(GL(m|1))/\mathcal{N}$ was determined. This example is rather special since $Rep(GL(m|1))$ has tame representation type. One can always assume $m \geq n$. Note for $m \geq n \geq 2$ the problem of classifying irreducible representations of G^{red} is wild [34]. Therefore one should not study the entire quotient $\mathcal{T}_{m|n}/\mathcal{N}$, but rather pass to a suitably small tensor subcategory in $\mathcal{T}_{m|n}$. In our case we consider for this the Karoubi envelope of the irreducible objects $Rep(G)$, which for convenience are only considered up to suitable parity shift. The image $\overline{\mathcal{T}}_{m|n}$ of this subcategory in $Rep(G)/\mathcal{N}$ defines a tannakian category, and the aim of this paper is to determine its Tannaka group H_n in the cases $G = Gl(n|n)$. Indeed, as we show in [39], the computation of the corresponding Tannaka group $H_{m|n}$ in the case $G = Gl(m|n)$ can be reduced to the case $m = n$ besides an additional factor $GL(m-n)$ that appears in the Tannaka group $H_{m|n}$ which arises from the Tannakian group of the Tannakian subcategory generated by the covariant tensor representations. In this sense the major complications of the constructions arise in the case $m = n$. So, for simplicity, we restrict ourselves to the case $m = n$ and postpone the additional combinatorial arguments that are necessary for the general case $m > n$ to the paper [39].

1.2 The Tannaka category $\overline{\mathcal{T}}_n$

For convenience we now write $\mathcal{T}_n$ instead of $\mathcal{T}_{n,n}$. To define the karoubienne hull of the irreducible representations in $\mathcal{T}_n$, we work with the tensor subcategory $\mathcal{T}_n^+$ generated by the irreducible representations of nonnegative superdimension in the following sense. Up to parity shift the irreducible representations $L = L(\lambda)$ of $\mathcal{T}_n$ are parametrized up to isomorphy by their highest weights λ. We define an equivalence relation on the set of highest weights, such that λ and λ' are called equivalent if $L(\lambda)$ or its Tannaka dual $L(\lambda)^\vee$ is isomorphic to $Ber^r \otimes L(\lambda')$ for some power Ber^r, $r \in \mathbb{Z}$ of the Berezin determinant representation Ber in $\mathcal{T}_n$. Let $Y_0^+(n)$ denote the set of equivalence classes $\lambda/\sim$ of maximal atypical weights. The cases where $L(\lambda)^\vee \cong Ber^r \otimes L(\lambda)$ holds are called (SD)-cases if $\dim(L(\lambda)) > 1$. The remaining cases are called (NSD)-cases. Notice that an irreducible representation $L(\lambda)$ of $GL(n|n)$

can be replaced by a parity shift X_λ of $L(\lambda)$ so that the superdimension $\mathrm{sdim}(X_\lambda)$ becomes ≥ 0. For maximal atypical representations $L(\lambda) = [\lambda_1, \ldots, \lambda_n]$ this is well defined since the superdimension is not zero. But of course this is ambiguous for irreducible representations of $GL(n|n)$ with $\mathrm{sdim}(L) = 0$, i.e. for the irreducible representations that are not maximal atypical. But these representations are negligible in the sense above. Thus we may consider only objects that are retracts of iterated tensor products of direct sums of maximal atypical irreducible representations X_λ of $GL(n|n)$ satisfying $\mathrm{sdim}(X_\lambda) > 0$. The tensor category thus obtained will be baptized $\mathcal{T}_n^+$. The full tensor subcategory $\mathcal{T}_n^+$ of $\mathcal{T}_n$ has more amenable properties than the full category $\mathcal{T}_n = Rep(GL(n|n))$. To motivate this, let us compare it with the tensor category of finite dimensional algebraic representations $Rep(G)$ of an arbitrary algebraic group G over k. In this situation the tensor subcategory generated by irreducible representations is semisimple[1] and can be identified with the tensor category of the maximal reductive quotient of G. The tensor category $\mathcal{T}_n^+$ however is not a semisimple tensor category in general. To make it semisimple we proceed as follows:

Let $\overline{\mathcal{T}}_n$ denote the quotient category of $\mathcal{T}_n^+$ obtained by killing the negligible morphisms in the tensor ideal $\mathcal{N}$ and hence in particular all neglegible objects, i.e.

$$\overline{\mathcal{T}}_n \cong \mathcal{T}_n^+/\mathcal{N}.$$

In order to analyze these categories, we work inductively using the cohomological tensor functors $DS : \mathcal{T}_n \to \mathcal{T}_{n-1}$ of [36]. We show in Lemma 5.5 that DS induces a tensor functor $\eta_n : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$.

Theorem 1.1 *The categories $\overline{\mathcal{T}}_n$ are semisimple tannakian categories. A fibre functor ω is provided by the composite of functors η_m for $1 \leq m \leq n$. Their Tannaka groups H_n are projective limits of reductive algebraic groups over k such that there is an equivalence of tensor categories*

$$\overline{\mathcal{T}}_n \;\cong\; Rep(H_n).$$

The functor η_n induces a closed embedding of affine group schemes $H_{n-1} \hookrightarrow H_n$ over k such that $\eta_n : Rep(H_n) \to Rep(H_{n-1})$, defined by DS on objects, can be identified with the restriction functor for this group scheme embedding.

The Tannaka group H_n is subgroup of the product of Tannaka groups H_λ, where λ runs over the set of equivalence classes $\lambda/\sim$ of maximal atypical highest weights λ. By definition H_λ is the Tannaka group of the tannakian subcategory $\mathcal{T}_\lambda$ that is generated by the simple object X_λ in $\overline{\mathcal{T}}_n$. For the twisted Berezin $X_\lambda = B$ the group H_λ is isomorphic to the multiplicative group $\mathbb{G}_m$ whose characters correspond to the irreducible one dimensional representations in $\mathcal{T}_n^+$, the powers B^r of B. In general H_λ may be considered as an algebraic subgroup of the general linear group $GL(V_\lambda)$ of the finite dimensional k-vectorspace $V_\lambda = \omega(X_\lambda)$ defined by the fibre functor ω. Note that $\dim(V_\lambda) = \mathrm{sdim}(X_\lambda)$ and this value is bounded by $n!$. The restriction of the

[1] In [42, p.231, 1.22ff] it was forgotten to mention the important passage to the tensor subcategory generated by simple objects. The corresponding statement is false without it as kindly pointed out by Y. André.

determinant character of $GL(V_\lambda)$ will be denoted $\det_\lambda$. In Theorem 14.3 we show that $\det_\lambda$ is a power $B^{\ell(\lambda)}$ of the character defined by B.

Theorem 1.2 *The character* $\det_\lambda$ *of the group* H_λ *is represented by the image of* $\det(X_\lambda) = \Lambda^{\mathrm{sdim}(X_\lambda)}(X_\lambda)$ *in the Tannaka category* $\overline{\mathcal{T}}_n$. *In* $\mathcal{T}_n^+$ *one has*

$$\det(X_\lambda) \;\cong\; B^{\ell(\lambda)} \,\oplus\, N$$

for some negligible object N *with the integer*

$$\ell(\lambda) \;=\; n^{-1}\,\mathrm{sdim}(X_\lambda)\,D(\lambda)$$

where $D(\lambda)$ *is explicitely described by the weight* λ *in Sect.* 13.

In the (SD)-cases the isomorphism $L(\lambda)^\vee \cong Ber^r \otimes L(\lambda)$ defines a nondegenerate pairing $X_\lambda \otimes X_\lambda \to B^r$. It induces a nondegenerate k-bilinear pairing on V_λ such that H_λ becomes a subgroup of its group of similitudes. In "Appendix C" we determine the parity of this pairing.

Theorem 1.3 *In the (SD)-cases the parity* $\varepsilon(X_\lambda)$ *of the invariant pairing* $\langle .,. \rangle$ *on* V_λ *defining* H_λ *is*

$$\varepsilon(X_\lambda) = \varepsilon(\lambda_{basic})$$

where $\varepsilon(\lambda_{basic}) = (-1)^{\sum_{i=1}^n \mu_i}$ *if* $L(\lambda_{basic}) \cong [\mu_1, \ldots, \mu_n]$ *is the irreducible basic representation associated to* $L(\lambda)$.

For each maximal atypical weight λ we define characters

$$\mu_\lambda : H_\lambda \to \mathbb{G}_m$$

of the Tannaka groups H_λ as follows: First suppose that X_λ is not isomorphic to a power of B. Then, in the (NSD)-cases μ_λ is defined to be $\det_\lambda$. In the (SD)-cases μ_λ is defined as the restriction of the similitude character to H_λ. The similitude character of H_λ is defined by an object in $\mathcal{T}_\lambda$ that is isomorphic to the image of B^r. To make these characters well defined notice the following: For the twisted Berezin the associated Tannaka group H_λ is isomorphic to the multiplicative group $\mathbb{G}_m$, whose characters correspond to the irreducible one-dimensional representations in $\mathcal{T}_n^+$, the powers B^r of B. Any tensor functor between tannakian categories (compatible with the fibre functors) induces a group homorphism between the Tannaka groups that is uniquely determined up to conjugacy by the functor. This observation, applied to the tannakian subcategories of $\mathcal{T}_\lambda$ that are generated by $\det(X_\lambda)$ resp. B^r, together with the fact that inner automorphisms of $\mathbb{G}_m$ are trivial, shows that all characters μ_λ are uniquely defined once an isomorphism μ_λ for $X_\lambda \cong B$ between the Tannaka group $H_B := H_\lambda$ and the multiplicative group $\mathbb{G}_m$ has been chosen. We fix such an isomorphism, denoted μ_B in the following.

This being said, a conjectural description of the structure of H_n can be given as follows:

Conjecture 1.4 (1) *H_n is the subgroup of the infinite product*

$$\prod_{\lambda/\sim\ \in Y_0^+(n)} H_\lambda$$

defined by the elements $h = (h_\lambda)_{\lambda/\sim}$ that satisfy $\mu_\lambda(h_\lambda) = \mu_B(h_B)$ for all λ.
(2) *For the (NSD)-cases the group H_λ, considered as a subgroup of $GL(V_\lambda)$, is equal to $GL(V_\lambda)$ if $\mu_\lambda \neq 1$ holds and is equal to the subgroup $SL(V_\lambda)$ otherwise. Recall, $\mu_\lambda = \det_\lambda$ holds in that case.*
(3) *Otherwise H_λ, considered as a subgroup of the group of similitudes of the pairing $\langle ., .\rangle$ on V_λ, is equal to the connected component of the similitude group if $\mu_\lambda \neq 1$ holds and is otherwise equal to the kernel of the similtude homomorphism μ_λ on this connected component.*

The parity shift of the twisted Berezin $B = \Pi^n(Ber)$ is an invertible object of the Tannaka category $\overline{\mathcal{T}}_n$, i.e. an object I of superdimension 1 such that $I \otimes I^\vee \cong \mathbf{1}$ holds. Conjecture 1.4 implies that the group $Pic(\overline{\mathcal{T}}_n)$ of isomorphism classes of invertible objects in $\overline{\mathcal{T}}_n$ is generated by the twisted Berezin B. For this notice that in 1.4 (1) the kernel of the projection from $H_n \subset \prod_{\lambda/\sim} H_\lambda$ in the product to the $L(\lambda) = B$-component is a connected semisimple profine groupscheme by 1.4 (2) and (3), hence only admits trivial characters. We also prove the converse.

Theorem 1.5 *A necessary and sufficient condition that Conjecture 1.4 holds is that $Pic(\overline{\mathcal{T}}_n)$ is generated by the twisted Berezin B.*

So the main obstacle for the proof Conjecture 1.4 turns out to be the structure of $Pic(\mathcal{T}_\lambda)$. Conjecture 1.4 is also equivalent to the assertion that exceptional (SD)-cases in the sense of the next theorem do not occur.

Theorem 1.6 *For all maximal atypical highest weights λ there is a homomorphism $\nu : Pic(\mathcal{T}_\lambda) \to \mathbb{Z}$ whose kernel is a two-torsion group μ_2^k of rank $k = k(\lambda)$ where $0 \leq k \leq 2$. In the (so called) regular case where $k(\lambda) = 0$, the group Tannaka group H_λ of $\mathcal{T}_\lambda$ is the one described in Conjecture 1.4.*

Corollary 1.7 *The Picard group $Pic(\overline{\mathcal{T}}_n)$ is a direct product of $\mathbb{Z}$ and a 2-power torsion group.*

1.3 The exceptional cases

The nonregular cases in the sense of the last Theorem 1.6 will be called exceptional cases. For these exceptional λ we show that there exists a subgroup $\tilde{H}_\lambda$ of H_λ of index two

$$0 \to \tilde{H}_\lambda \to H_\lambda \to \mu_2 \to 0.$$

The restriction of the irreducible representation V_λ of H_λ to the subgroup $\tilde{H}_\lambda$ decomposes into a direct sum $W_\lambda \oplus W_\lambda^\vee$ of two irreducible faithful nonisomorphic representations of $\tilde{H}_\lambda$ on orthogonal Lagrangian subspaces W_λ and $W_\lambda^\vee$ of the metric space $(V_\lambda, \langle ., . \rangle)$; see Theorem 11.4 and Sect. 11.7. In this way we can view $\tilde{H}_\lambda$ as a subgroup of $GL(W_\lambda)$, and we show that the following holds

$$SL(W_\lambda) \subseteq \tilde{H}_\lambda \subseteq GL(W_\lambda).$$

Finally let G_λ denote the derived group of the connected component H_λ^0 of H_λ, and let similar G_n denote the derived group of the connected component H_n^0. For the derived connected subgroup G_n of H_n we prove the following result.

Theorem 1.8 (1) *G_n is isomorphic to the infinite product*

$$\prod_{\lambda/\sim\, \in Y_0^+(n)} G_\lambda.$$

The groups G_λ are isomorphic to $SL(V_\lambda)$, $SO(V_\lambda)$, $Sp(V_\lambda)$ in the (NSD) resp. the even or odd regular (SD)-cases and they are isomorphic to $SL(W_\lambda)$, for a Lagrangian subspace W_λ of V_λ, in the exceptional (SD)-cases.

(2) *If λ is not an exceptional (SD)-case, the groups H_λ are as described in Conjecture 1.4. Furthermore the analog of Conjecture 1.4 holds for the Tannaka group generated by the simple objects X_λ for which which λ does not belong to an exceptional (SD)-case.*

In view of Theorem 1.8 our Conjecture 1.4 is hence equivalent to the conjectural nonexistence of exceptional (SD)-cases.

Reformulating these statements for the category of representations of $GL(n|n)$, what we have achieved is

- a (partial) description of the decomposition law of tensor products of irreducible representations into indecomposable modules up to negligible indecomposable summands; and
- a classification (in terms of the highest weights of H_λ and H_μ) of the indecomposable modules of non-vanishing superdimension in iterated tensor products of $L(\lambda)$ and $L(\mu)$.

To determine this decomposition it suffices to know the Clebsch–Gordan coefficients for the classical simple groups of type A, C, D. Notice that $\dim(V_\lambda)$ is always even in the (SD)-cases, hence simple groups of type B do not occur. Furthermore the superdimensions of the indecomposable summands are just the dimensions of the corresponding irreducible summands of the tensor products in $Rep_k(H_n)$. Without this, to work out any such decomposition is rather elaborate. For the case $n = 2$ see [37]. In fact the knowledge of the Jordan–Hölder factors usually gives too little information on the indecomposable objects itself. In the (NSD) and the odd (SD)-case it is enough for these two applications to know the connected derived group G_λ since the restriction of any irreducible representation of H_λ to G_λ stays irreducible. Therefore these results hold unconditionally in these cases. In the even (SD)-case we need the finer (but conjectural) results of Sect. 12 to see that H_λ is connected. We refer the reader to Example 9.7 and Sect. 15 for some examples.

1.4 Relation to physics

Part of the motivation for our computations of the Tannaka groups H_n comes from the real supergroups $G = SU(2, 2|N)$ which are covering groups of the super conformal groups $SO(2, 4|N)$. The complexification $\mathfrak{g}$ of $Lie(G)$ is isomorphic to the complex Lie superalgebras $\mathfrak{sl}(4|N)$. The finite dimensional representations of $\mathfrak{g}$ are hence related to those of the Lie superalgebras $\mathfrak{gl}(n|n)$ for $n \leq 4$. Since the complexification $\mathfrak{g}$ defines complex supervector fields on four dimensional Minkowski superspace M and compactifications of it, these Lie superalgebras play a role in string theory and the AdS/CFT correspondence.

For supersymmetric fields ψ on M with values in a finite dimensional representation L of $\mathfrak{g}$, the Feynman integrals of conformal theories are computed from tensor contractions and superintegration. These can be considered as contractions between tensor products of fields. The computations will require the analysis of higher tensor products $L^{\otimes r} \otimes (L^\vee)^{\otimes s}$ and generalizations of Fierz rules. The results will strongly depend on the rules of the underlying tensor categories $\mathcal{T}_n$. Since it seems reasonable to consider not only fields of superdimension zero, but besides the constant representation also such with values in maximal atypical representations L of $\mathfrak{g}$, our study of tensor categories may be a little step into this direction. To look at examples for $n \leq 4$ we now replace the groups H_λ by their compact inner forms H_λ^c and $\mathbb{G}_m$ by $U(1)$, to make things look more familiar to physicists. Indeed notice that the tensor categories $Rep_k(H_\lambda)$ and $Rep_{\mathbb{C}}(H_\lambda^c)$ are equivalent.

For $\mathfrak{gl}(n|n)$ and $n \leq 4$ there only exist finitely many isomorphism classes of quotient groups H_λ of the tannakian groups H_n of $\overline{\mathcal{T}} = Rep(H_n)$. Besides $U(1)$ that corresponds to the twisted Berezin, the smallest such groups are $SU(2)$ and $SU(3)$ related to representations denoted $L = S^1$ and $L = S^2$. If we pass to $\mathfrak{sl}(n|n)$, for $n \leq 3$ these are the only groups except for $Sp^c(6)$ in the case $n = 3$. For the more involved discussion of the case $n = 4$ we refer to Sect. 15 and Example 15.1. One may ask whether the appearence of the groups $U(1)$, $SU(2)$, $SU(3)$ here is a mere accident, or whether there does exist some connection with the symmetry groups arising in the standard model of elementary particle physics? Looking for relations between internal symmetries and supersymmetry has a long history going back to the Coleman-Mandula theorem [17], so this may be of interest. A possible relation could be the following:

If in such a theory, for some mysterious physical reasons, the tensor product contributions to the Feynman integrals from direct summands of $L^{\otimes r} \otimes (L^\vee)^{\otimes s}$ of superdimension zero would be relatively small in a certain energy range due to supersymmetry cancellations, then to first order they would be negligible. Hence a physical observer might come up with the impression that the underlying rules of symmetry are imposed by the invariant theory of the quotient tensor category $\overline{\mathcal{T}}_n = Rep(H_n)$ instead of $\mathcal{T}_n$, i.e. the tensor categories that are obtained by ignoring negligible indecomposable summands of superdimension zero. Thus H_n would appear as an internal symmetry group of the theory in an approximate sense. Of course, any such speculation is highly tentative for various reasons: Computations along such lines will probably be extremely involved. Fields with values in maximal atypical representations V may produce ghosts in the associated infinite dimensional representations of $\mathfrak{g}$. In other words, such field theories may a priori not be superunitary and it is unclear whether the passage to the cohomology groups for operators like DS or the Dirac operator H_D [36], breaking the conformal symmetry, would suffice to get rid of ghosts.

1.5 Structure of the article

Our main tool are the cohomological tensor functors $DS : \mathcal{T}_n \to \mathcal{T}_{n-1}$ of [36]. In the main theorem of [36, Theorem 16.1] we calculate $DS(L(\lambda))$. In particular $DS(L(\lambda))$ is semisimple and multiplicity free. We show in Lemma 5.5 that DS induces a tensor functor $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$ and by Lemma 5.11 one can construct a tensor functor on the quotient categories

$$\eta : \mathcal{T}_n^+/\mathcal{N} \to \mathcal{T}_{n-1}^+/\mathcal{N}.$$

This seemingly minor observation is one of the crucial points of the proof since it allows us to determine the groups H_n and G_n inductively. We also stress that it is not clear whether DS naturally induces a functor between the quotients $\mathcal{T}_n^+/\mathcal{N}$ and $\mathcal{T}_{n-1}^+/\mathcal{N}$ on the level of morphisms. In fact, if one enlarges $\mathcal{T}^+$ to the larger category $\mathcal{T}^{ev}$ of Sect. 14, DS does not preserve negligible morphisms. The DS functor however agrees with the functor η on objects. The quotient $\mathcal{T}_n^+/\mathcal{N}$ is equivalent to the representation category $Rep(H_n)$ of finite-dimensional representations of a pro-reductive group. By the deep and powerful Theorem 5.15 of Deligne the induced DS functor determines an embedding of algebraic groups $H_{n-1} \hookrightarrow H_n$ and the functor DS is the restriction functor with respect to this embedding.

Hence the main theorem of [36] tells us the branching laws for the representation V_λ with respect to the embedding $H_{n-1} \hookrightarrow H_n$. Our strategy is to determine the groups H_n or G_n inductively using the functor DS. For $n = 2$ we need the explicit results of [37] to give us the fusion rule between two irreducible representations and we describe the corresponding Tannaka group in Lemma 9.2. Starting from the special case $n = 2$ we can proceed by induction on n. For this we use the embedding $H_{n-1} \hookrightarrow H_n$ along with the known branching laws and the classification of small representations due to Andreev et al. [5] which allows to determine inductively the connected derived groups $G_n = (H_n^0)_{der}$ for $n \geq 3$; see Sect. 10. The passage to the connected derived group means that we have to deal with the possible decomposition of V_λ when restricted to G_n. In order to determine G_n we first determine the connected derived groups G_λ corresponding to the tensor subcategory generated by the image of $L(\lambda)$ in $\overline{\mathcal{T}}_n$ in Theorem 6.2. Roughly speaking the strategy of the proof is the following: We use the inductively known situation for G_{n-1} to show that for sufficiently large n the rank and the dimension of G_λ is large compared to the dimension of V_λ, i.e. V_λ or any of its irreducible constituents in the restriction to G_λ should be *small* in the sense of [5]. We refer to Sect. 10 for more details on the proof.

The final sections are devoted to the determination of H_λ and $Rep(H_n)$. We determine the groups H_λ in Sect. 11. We split this determination into three cases: NSD, regular SD and exceptional SD. The crucial tool here is the determination of the determinant $det(X_\lambda)$. This determinant is computed in the later Sects. 13 and 14.

In Sect. 12 we conjecture a stronger structure theorem, namely that there are no exceptional SD-cases. We describe various conditions which are equivalent to this statement. We end the article with some low-rank cases and a discussion of cases of potential physical relevance.

We have outsourced a large number of technical (but necessary) results to the "Appendices A, C and D" as to not distract the reader too much from the structure of the arguments. "Appendix E" discusses some evidence for our conjectures.

1.6 Outlook

For the general $\mathcal{T}^+_{m|n}$-case (where $m \geq n$) recall that every maximal atypical block in $\mathcal{T}_{m|n}$ is equivalent to the principal block of $\mathcal{T}_{n|n}$. We fix the standard block equivalence due to Serganova and denote the image of an irreducible representation $L(\lambda)$ under this equivalence by $L(\lambda^0)$.

Theorem 1.9 *Suppose that* $\mathrm{sdim}(L(\lambda)) > 0$. *Then* $H_\lambda \cong GL(m-n) \times H_{\lambda^0}$ *and* $L(\lambda)$ *corresponds to the representation* $L_\Gamma \boxtimes V_{\lambda^0}$ *of* H_λ. *Here* L_Γ *is an irreducible representation of* $GL(m-n)$ *which only depends on the block* Γ *(the* core *of* Γ *as defined by Serganova).*

We prove this in [39]. The problem of determining the semisimplification of $Rep(G)$ can be asked for any basic supergroup. We expect that the strategy employed in this paper (induction on the rank via the DS functor) serves as a blueprint for other cases, but one can certainly not expect uniform proofs or results. In fact the semisimplicity of DS is now known in the OSp-case [32], but fails for the $P(n)$-case [26]. Note also that even in the OSp-case no exact analog of the structure theorem can hold since the supergroup $OSp(1|2n)$ will appear as a Tannaka group.

About the Arxiv version

The present version is identical to the version posted on the Arxiv except that the latter contains further appendices with sample calculations. In particular the theorem numbering is identical.

2 The superlinear groups

Let k be an algebraically closed field of characteristic zero. We adopt the notations of [36]. With $GL(m|n)$ we denote the general linear supergroup and by $\mathfrak{g} = \mathfrak{gl}(m|n)$ its Lie superalgebra. A representation ρ of $GL(m|n)$ is a representation of $\mathfrak{g}$ such that its restriction to $\mathfrak{g}_{\bar{0}}$ comes from an algebraic representation of $G_{\bar{0}} = GL(m) \times GL(n)$. We denote by $\mathcal{T} = \mathcal{T}_{m|n}$ the category of all finite dimensional representations with parity preserving morphisms.

2.1 The category $\mathcal{R}$

Fix the morphism $\varepsilon : \mathbb{Z}/2\mathbb{Z} \to G_{\overline{0}} = GL(n) \times GL(n)$ which maps -1 to the element $diag(E_n, -E_n) \in GL(n) \times GL(n)$ denoted ϵ_n. Notice that $Ad(\epsilon_n)$ induces the parity morphism on the Lie superalgebra $\mathfrak{gl}(n|n)$ of G. We define the abelian subcategory $\mathcal{R} = sRep(G, \varepsilon)$ of $\mathcal{T}$ as the full subcategory of all objects (V, ρ) in $\mathcal{T}$ with the property $p_V = \rho(\epsilon_n)$; here p_V denotes the parity morphism of V and ρ denotes the underlying homomorphism $\rho : GL(n) \times GL(n) \to GL(V)$ of algebraic groups over k. The subcategory $\mathcal{R}$ is stable under the dualities $^\vee$ and *. For $G = GL(n|n)$ we usually write $\mathcal{T}_n$ instead of $\mathcal{T}$, and $\mathcal{R}_n$ instead of $\mathcal{R}$. The irreducible representations in $\mathcal{R}_n$ are parametrized by their highest weight with respect to the Borel subalgebra of upper triangular matrices. A weight $\lambda = (\lambda_1, \ldots, \lambda_n \mid \lambda_{n+1}, \cdots, \lambda_{2n})$ of an irreducible representation in $\mathcal{R}_n$ satisfies $\lambda_1 \geq \ldots \lambda_n$, $\lambda_{n+1} \geq \ldots \lambda_{2n}$ with integer entries. The Berezin determinant of the supergroup $G = G_n$ defines a one dimensional representation Ber. Its weight is is given by $\lambda_i = 1$ and $\lambda_{n+i} = -1$ for $i = 1, \ldots, n$. For each representation $M \in \mathcal{R}_n$ we also have its parity shifted version $\Pi(M)$ in $\mathcal{T}_n$. Since we only consider parity preserving morphisms, these two are not isomorphic. In particular the irreducible representations in $\mathcal{T}_n$ are given by the $\{L(\lambda), \Pi L(\lambda) \mid \lambda \in X^+\}$. The whole category $\mathcal{T}_n$ decomposes as $\mathcal{T}_n = \mathcal{R}_n \oplus \Pi\mathcal{R}_n$ [11, Corollary 4.44]. For maximal atypical λ exactly one of $L(\lambda), \Pi L(\lambda))$ has positive superdimension. We call this irreducible module X_λ and $B = \Pi^n(Ber)$ in $\mathcal{T}_n^+$, for $Ber = [1, \ldots, 1]$, the twisted Berezin.

2.2 Kac objects

We put $\mathfrak{p}_\pm = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(\pm 1)}$ for the usual $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$. We consider a simple $\mathfrak{g}_{(0)}$-module as a $\mathfrak{p}_\pm$-module in which $\mathfrak{g}_{(1)}$ respectively $\mathfrak{g}_{(-1)}$ acts trivially. We then define the Kac module $V(\lambda)$ and the anti-Kac module $V'(\lambda)$ via

$$V(\lambda) = Ind_{\mathfrak{p}_+}^{\mathfrak{g}} L_0(\lambda) \ \ V'(\lambda) = Ind_{\mathfrak{p}_-}^{\mathfrak{g}} L_0(\lambda)$$

where $L_0(\lambda)$ is the simple $\mathfrak{g}_{(0)}$-module with highest weight λ. The Kac modules are universal highest weight modules. $V(\lambda)$ has a unique maximal submodule $I(\lambda)$ and $L(\lambda) = V(\lambda)/I(\lambda)$ [41, Proposition 2.4]. We denote by $\mathcal{C}^+$ the tensor ideal of modules with a filtration by Kac modules in $\mathcal{R}_n$ and by $\mathcal{C}^-$ the tensor ideal of modules with a filtration by anti-Kac modules in $\mathcal{R}_n$.

2.3 Equivalence classes of weights

Two irreducible representations M, N in $\mathcal{T}$ are said to be equivalent $M \sim N$, if either $M \cong Ber^r \otimes N$ or $M^\vee \cong Ber^r \otimes N$ holds for some $r \in \mathbb{Z}$. This obviously defines an equivalence relation on the set of isomorphism classes of irreducible representations of T. A self-equivalence of M is given by an isomorphism $f : M \cong Ber^r \otimes M$ (which implies $r = 0$ and f to be a scalar multiple of the identity) respectively an isomorphism $f : M^\vee \cong Ber^r \otimes M$. If it exists, such an isomorphism uniquely determines r and is unique up to a scalar and we say M is of type (SD). Otherwise we say M is of type (NSD). The isomorphism f can be viewed as a nondegenerate G-equivariant bilinear form

$$M \otimes M \to Ber^r$$

which is either symmetric or alternating. So we distinguish between the cases ($SD_\pm$).

3 Weight and cup diagrams

3.1 Weight diagrams and cups

Consider a weight

$$\lambda = (\lambda_1, \ldots, \lambda_n | \lambda_{n+1}, \ldots, \lambda_{2n}).$$

Then $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_{n+1} \geq \cdots \geq \lambda_{2n}$ are integers, and every $\lambda \in \mathbb{Z}^{2n}$ satisfying these inequalities occurs as the highest weight of an irreducible representation $L(\lambda)$. The set of highest weights will be denoted by $X^+ = X^+(n)$. Following [14] to each highest weight $\lambda \in X^+(n)$ we associate two subsets of cardinality n of the numberline $\mathbb{Z}$

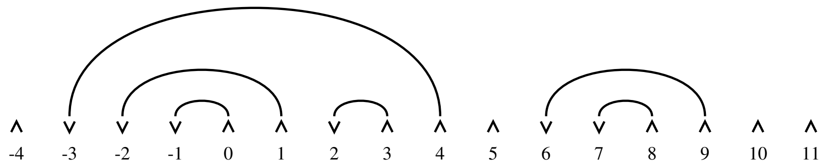
$$\begin{aligned} I_\times(\lambda) &= \{\lambda_1, \lambda_2 - 1, \ldots, \lambda_n - n + 1\} \\ I_\circ(\lambda) &= \{1 - n - \lambda_{n+1}, 2 - n - \lambda_{n+2}, \ldots, -\lambda_{2n}\}. \end{aligned}$$

We now define a labeling of the numberline $\mathbb{Z}$. The integers in $I_\times(\lambda) \cap I_\circ(\lambda)$ are labeled by $\vee$, the remaining ones in $I_\times(\lambda)$ resp. $I_\circ(\lambda)$ are labeled by $\times$ respectively $\circ$. All other integers are labeled by $\wedge$. This labeling of the numberline uniquely characterizes the weight vector λ. If the label $\vee$ occurs r times in the labeling, then $r = atyp(\lambda)$ is called the *degree of atypicality* of λ. Notice $0 \leq r \leq n$, and for $r = n$ the weight λ is called *maximal atypical*. A weight is maximally atypical if and only if $\lambda_i = -\lambda_{2n-i+1}$ for $i = 1, \ldots, n$ in which case we write

$$L(\lambda) = [\lambda_1, \ldots, \lambda_n].$$

To each weight diagram we associate a cup diagram as in [13, 36]. The outer cups in a cup diagram define the sectors of the weight as in [36]. We number the sectors from left to right $S_1, S_2, \ldots, S_k$.

Example 3.1 Consider the (maximal atypical) irreducible representation [7, 7, 4, 2, 2, 2] of $GL(6|6)$. Its associated weight and cup diagram have two sectors:



The cup diagram of λ

3.2 Important invariants

The segment and sector structure of a weight diagram is completely encoded by the positions of the ∨'s. Hence any finite subset of $\mathbb{Z}$ defines a unique weight diagram in a given block. We associate to a maximal atypical highest weight the following invariants:

- the type (SD) resp. (NSD),
- the number $k = k(\lambda)$ of sectors of λ,
- the sectors $S_\nu = (I_\nu, K_\nu)$ from left to right (for $\nu = 1, \ldots, k$),
- the ranks $r_\nu = r(S_\nu)$, so that $\# I_\nu = 2r_\nu$,
- the distances d_ν between the sectors (for $\nu = 1, \ldots, k-1$),
- and the total shift factor $d_0 = \lambda_n + n - 1$.

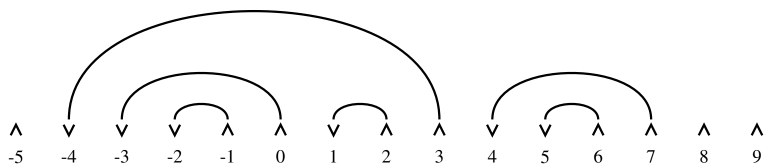
If convenient, k sometimes may also denote the number of segments, but hopefully no confusion will arise from this.

A maximally atypical weight $[\lambda]$ is called basic if $(\lambda_1, \ldots, \lambda_n)$ defines a decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$ with the property $n - i \geq \lambda_i$ for all $i = 1, \ldots, n$. The total number of such *basic weights* in $X^+(n)$ is the Catalan number C_n. Reflecting the graph of such a sequence $[\lambda]$ at the diagonal, one obtains another basic weight $[\lambda]^*$. By [36, Lemma 21.4] a basic weight λ is of type (SD) if and only if $[\lambda]^* = [\lambda]$ holds. To every maximal atypical highest weight λ is attached a unique maximal atypical highest weight λ_{basic}

$$\lambda \mapsto \lambda_{basic}$$

having the same invariants as λ, except that $d_1 = \cdots = d_{k-1} = 0$ holds for λ_{basic} and the leftmost ∨ is at the vertex $-n+1$.

Example 3.2 In Example 3.1 the weight $[7, 7, 4, 2, 2, 2]$ is of type (NSD). It has two $k = 2$ sectors of rank $r_1 = 4$ and $r_2 = 2$ with shift factor $d_0 = 7$ and $d_1 = 1$. Its associated basic weight is



The cup diagram of λ_{basic}

4 Cohomological tensor functors

4.1 The Duflo–Serganova functor

We attach to every irreducible representation a sign. If $L(\lambda)$ is maximally atypical in $\mathcal{R}_n$ we put $\varepsilon(L(\lambda)) = (-1)^{p(\lambda)}$ for the parity $p(\lambda) = \sum_{i=1}^n \lambda_i$. For the general case see [36]. Now for ε define the full subcategories $\mathcal{R}_n(\varepsilon)$. These consists of all objects in $\mathcal{R}_n$ whose irreducible constituents L have sign $\varepsilon(L) = \varepsilon$. Then by [36, Corollary 15.1] the categories $\mathcal{R}_n(\varepsilon)$ are semisimple categories.

Note that $\mathrm{sdim}(X) \geq 0$ holds for all irreducible objects $X \in \mathcal{R}_n(\varepsilon)$ in case $\varepsilon(X) = 1$ and also for all irreducible objects $X \in \Pi\mathcal{R}_n(\varepsilon)$ in case $\varepsilon(X) = -1$. For each irreducible representation $L(\lambda)$ with $sdim(X_\lambda) \neq 0$ let

$$X_\lambda = \Pi^{p(\lambda)}(L(\lambda))$$

denote the parity shift of $L(\lambda)$ that satisfies $sdim(X_\lambda) \geq 0$. In the case of the Berezin representation $Ber = [1, \dots, 1]$ we also write B for this parity shift. Notice, $B = Ber$ if n is even and $B = \Pi(Ber)$ if n is odd.

We recall some constructions from the article [36]. Fix the following element $x \in \mathfrak{g}_1$,

$$x = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \text{ for } y = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & & \dots & \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since x is an odd element with $[x, x] = 0$, we get

$$2 \cdot \rho(x)^2 = [\rho(x), \rho(x)] = \rho([x, x]) = 0$$

for any representation (V, ρ) of $GL(n|n)$ in $\mathcal{R}_n$. Notice $d = \rho(x)$ supercommutes with $\rho(GL(n-1|n-1))$. Then we define the cohomological tensor functor DS as

$$DS = DS_{n,n-1} : \mathcal{T}_n \to \mathcal{T}_{n-1}$$

via $DS_{n,n-1}(V, \rho) = V_x := Kern(\rho(x))/Im(\rho(x))$.

In fact $DS(V)$ has a natural $\mathbb{Z}$-grading and decomposes into a direct sum of $GL(n-1|n-1)$-modules

$$DS(V, \rho) \ = \ \bigoplus_{\ell \in \mathbb{Z}} \Pi^\ell(H^\ell(V))$$

for certain cohomology groups $H^\ell(V)$. If we want to emphasize the $\mathbb{Z}$-grading, we also write this in the form

$$DS(V, \rho) \ = \ \bigoplus_{\ell \in \mathbb{Z}} H^\ell(V)[-\ell].$$

Theorem 4.1 [36, Theorem 16.1] *Suppose $L(\lambda) \in \mathcal{R}_n$ is an irreducible atypical representation, so that λ corresponds to a cup diagram*

$$\bigcup_{j=1}^{r} [a_j, b_j]$$

with r sectors $[a_j, b_j]$ for $j = 1, \ldots, r$. Then

$$DS(L(\lambda)) \cong \bigoplus_{i=1}^{r} \Pi^{n_i} L(\lambda_i)$$

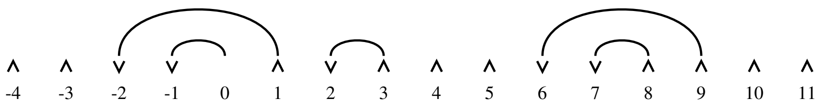
is the direct sum of irreducible atypical representations $L(\lambda_i)$ in $\mathcal{R}_{n-1}$ with shift $n_i \equiv p(\lambda) - p(\lambda_i)$ modulo 2. The representation $L(\lambda_i)$ is uniquely defined by the property that its cup diagram is

$$[a_i + 1, b_i - 1] \quad \cup \bigcup_{j=1, j \neq i}^{r} [a_j, b_j]$$

the union of the sectors $[a_j, b_j]$ for $1 \leq j \neq i \leq r$ and (the sectors occuring in) the segment $[a_i + 1, b_i - 1]$.

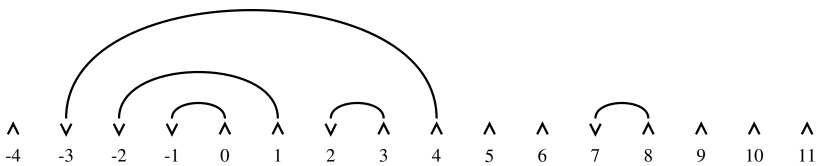
In particular $DS(L(\lambda))$ is semisimple and multiplicity free.

Example 4.2 Consider again the (maximal atypical) irreducible representation [7, 7, 4, 2, 2, 2] of $GL(6|6)$ of Example 3.1. The parity is $\varepsilon(\lambda) = 1$. Applying DS gives 2 irreducible representations. The representation $[\lambda_1] = [7, 7, 4, 2, 2]$ is associated to the derivative of the first sector



The cup diagram of $L(\lambda_1)$

Then the parity is $\varepsilon(\lambda_1) = 1 = \varepsilon(\lambda)$. The second irreducible representation is $\Pi[7, 3, 1, 1, 1]$ (note the parity shift since $\varepsilon(\lambda_2) \neq \varepsilon(\lambda)$) with cup diagram



The cup diagram of $L(\lambda_2)$

All in all $DS[7, 7, 4, 2, 2, 2] \cong [7, 7, 4, 2, 2] \oplus \Pi[7, 3, 1, 1, 1]$.

4.2 The Hilbert polynomial

Similarly to DS we can define the tensor functors $DS_{n,n-m} : \mathcal{T}_n \to T_{n-m}$ by replacing the x in the definition of DS by an x with m 1's on the antidiagonal. These functors admit again a $\mathbb{Z}$-grading. In particular we can consider the functor $DS_{n,0} : \mathcal{T}_n \to T_0 = svec_k$ with its decomposition $DS_{n,0}(X) = \bigoplus_{\ell\in\mathbb{Z}} D^{\ell}_{n,0}(X)[-\ell]$ for objects X in $\mathcal{T}_n$ and objects $D^{\ell}_{n,0}(X)$ in $svec_k$ where $D^{\ell}_{n,0}(X)[-\ell]$ is the object $\Pi^{\ell} D^{\ell}_{n,0}(X)$ concentrated in degree ℓ with respect to the $\mathbb{Z}$-gradation of $DS_{n,0}(X)$. For $X \in \mathcal{T}_n$ we define the Laurent polynomial

$$\omega(X, t) = \sum_{\ell\in\mathbb{Z}} \mathrm{sdim}(DS^{\ell}_{n,0}(X)) \cdot t^{\ell}$$

as the Hilbert polynomial of the graded module $DS^{\bullet}_{n,0}(X) = \bigoplus_{\ell\in\mathbb{Z}} DS^{\ell}_{n,0}(X)$. Since $\mathrm{sdim}(W[-\ell]) = (-1)^{\ell}\,\mathrm{sdim}(W)$ and $X = \bigoplus DS^{\ell}_{n,0}(X)[-\ell]$ holds, the formula

$$\mathrm{sdim}(X) = \omega(X, -1)$$

follows. For $X = Ber^i_n$

$$\omega\big(Ber^i_n, t\big) \;=\; t^{ni}.$$

For more details we refer the reader to [36, section 25].

4.3 The Dirac functor

In [36, Section 5] we also consider the Dirac operator

$$D = \partial + \overline{\partial}.$$

Here $\overline{x} = x^T$ denotes the supertranspose of x and $\overline{\partial} = i\rho(\overline{x})$. Let $H = diag(0_{n-1}, 1, 1, 0_{n-1})$ and $M := V^H$. Then we show that

$$H_D(V) = Kern(D : M \to M)/Im(D : M \to M)$$

defines a symmetric monoidal functor $\mathcal{T}_n \to \mathcal{T}_{n-s}$ where s is the rank of x. It follows from Lemma 5.8 that H_D agrees with DS on the subcategory $\mathcal{T}^+_n$.

5 Tannakian arguments

5.1 The category $\mathcal{T}^+_n$

Let $\mathcal{T}^+_n$ denote the Karoubian envelope of the simple nonnegative representations, i.e the full subcategory of $\mathcal{T}_n$, whose objects consist of all retracts of iterated tensor products of irreducible representations in $\mathcal{T}_n$ that are not maximal atypical and of maximal atypical irreducible representations X_λ in $\mathcal{R}_n(+1) \oplus \Pi\mathcal{R}_n(-1)$, defined as at the begining of Sect. 4.1. Obviously $\mathcal{T}_n^+$ is a symmetric monoidal idempotent complete k-linear category closed under the $*$-involution. It contains all irreducible objects of $\mathcal{T}_n$ up to a parity shift. It contains the standard representation V and its dual $V^\vee$, and hence contains all mixed tensors [35]. Furthermore all objects X in $\mathcal{T}_n^+$ satisfy condition $\mathtt{T}$ (see section 6 in [36]) and $\mathcal{T}_n^+$ is rigid. For this it suffices for irreducible $X \in \mathcal{T}_n^+$ that $X^\vee \in \mathcal{T}_n^+$. This is obvious since $X^\vee$ is irreducible with $\mathrm{sdim}(X^\vee) = \mathrm{sdim}(X) \geq 0$, and hence $X^\vee \in \mathcal{T}_n^+$.

5.2 Conventions on tensor categories

We use the same definition of a tensor category as is used in [27] except that we do not require the category to be abelian. Tensor functors are additive as in [27] but need not be exact. Our definition of a tensor functor therefore agrees with the one used in [22].

5.3 The ideal of negligible morphisms

An ideal in a k-linear category $\mathcal{A}$ is for any two objects X, Y the specification of a k-submodule $\mathcal{I}(X, Y)$ of $Hom_{\mathcal{A}}(X, Y)$, such that $g\mathcal{I}(X', Y)f \subseteq \mathcal{I}(X, Y')$ holds for all pairs of morphisms $f \in Hom_{\mathcal{A}}(X, X')$, $g \in Hom_{\mathcal{A}}(Y, Y')$. Let $\mathcal{I}$ be an ideal in $\mathcal{A}$. By definition $\mathcal{A}/\mathcal{I}$ is the category with the same objects as $\mathcal{A}$ and with

$$Hom_{\mathcal{A}/\mathcal{I}}(X, Y) = Hom_{\mathcal{A}}(X, Y)/\mathcal{I}(X, Y).$$

An ideal in a tensor category is a tensor ideal if it is stable under $\mathbf{1}_C \otimes -$ and $- \otimes \mathbf{1}_C$ for all $C \in \mathcal{A}$. Let Tr be the trace. For any two objects A, B we define $\mathcal{N}(A, B) \subset Hom(A, B)$ by

$$\mathcal{N}(A, B) = \{f \in Hom(A, B) \mid \forall g \in Hom(B, A),\ Tr(g \circ f) = 0\}.$$

The collection of all $\mathcal{N}(A, B)$ defines a tensor ideal $\mathcal{N}$ of $\mathcal{A}$ [2].

Let $\mathcal{A}$ be a super tannakian category. An indecomposable object will be called *negligible*, if its image in $\mathcal{A}/\mathcal{N}$ is the zero object. By [34] an object is negligible if and only if its categorial dimension is zero.

Example 5.1 An irreducible representation has superdimension zero if and only if it is not maximal atypical, see Sect. 3. The standard representation $V \simeq k^{n|n}$ has superdimension zero and therefore also the indecomposable adjoint representation $\mathbb{A} = V \otimes V^\vee$.

Any super tannakian category is equivalent (over an algebraically closed field) to the representation category of a supergroup scheme by [20]. In that case the categorial dimension is the superdimension of a module. If $\mathcal{A}$ is a super tannakian category over k, the quotient of $\mathcal{A}$ by the ideal $\mathcal{N}$ of negligible morphisms is again a super tannakian category by [2, 34]. More generally, for any pseudo-abelian full subcategory $\tilde{\mathcal{A}}$ in $\mathcal{A}$ closed under tensor products, duals and containing the identity element the following holds:

Lemma 5.2 *The quotient category $\tilde{\mathcal{A}}/\mathcal{N}_{\tilde{\mathcal{A}}}$ is a semisimple super tannakian category.*

Proof The quotient is a k-linear semisimple rigid tensor category by [3, Theorem 1 a)]. The quotient is idempotent complete by lifting of idempotents (or see [2, 2.3.4 b)] and by [2, 2.1.2] a k-linear pseudoabelian category is abelian. The Schur finiteness [20, 34] is inherited from $\mathcal{A}$ to $\tilde{\mathcal{A}}/\mathcal{N}$. □

This in particular applies to the situation where $\tilde{\mathcal{A}}$ is the full subcategory of objects which are retracts of iterated tensor products of a fixed set of objects in $\mathcal{A}$. In particular for $\tilde{\mathcal{A}} = \mathcal{T}_n^+$ and $\mathcal{A} = \mathcal{T}_n$ this implies

Corollary 5.3 *The tensor functor $\mathcal{T}_n^+ \to \mathcal{T}_n^+/\mathcal{N}$ maps $\mathcal{T}_n^+$ to a semisimple super tannakian category $\overline{\mathcal{T}}_n := \mathcal{T}_n^+/\mathcal{N}$.*

Proposition 5.4 *The category $\overline{\mathcal{T}}_n$ is a tannakian category, i.e. there exists a pro-reductive algebraic k-groups H_n such that the category $\overline{\mathcal{T}}_n$ is equivalent as a tensor category to the category $Rep_k(H_n)$ of finite dimensional k-representations of H_n*

$$\overline{\mathcal{T}}_n \sim Rep_k(H_n).$$

Proof By a result of Deligne [21, Theorem 7.1] it suffices to show that for all objects X in $\mathcal{T}_n^+$ we have $\mathrm{sdim}(X) \geq 0$. We prove this by induction on n. Suppose we know this assertion for $\mathcal{T}_{n-1}$ already. Then all objects of $\mathcal{T}_{n-1}^+$ have superdimension ≥ 0 (for the induction start $n = 0$ our assertion is obvious). Since the tensor functor $DS : \mathcal{T}_n \to \mathcal{T}_{n-1}$ preserves superdimensions, it suffices for the induction step that DS maps $\mathcal{T}_n^+$ to $\mathcal{T}_{n-1}^+$. □

Lemma 5.5 *The functors $DS_{n,n-m} : \mathcal{T}_n \to \mathcal{T}_{n-m}$ and $\omega_{n,n-m} : \mathcal{T}_n \to \mathcal{T}_{n-m}$ restrict to functors from $\mathcal{T}_n^+$ to $\mathcal{T}_{n-m}^+$. In particular*

$$DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+.$$

Proof Since $DS_{n,n-m}$ and ω_{n-m} preserve tensor products and idempotents, it suffices by the definition of $\mathcal{T}_n^+$ that $DS_{n-m}(X), \omega_{n-m}(X) \in \mathcal{T}_{n-m}^+$ holds for all irreducible objects X in $\mathcal{T}_n^+$. Now Theorem 4.1 implies $DS(X) \in \mathcal{T}_{n-1}^+$ since any irreducible representation X maps to a semisimple representation $DS(X)$ and for maximal atypical $X \in \mathcal{T}_{n-1}^+$ all summands of $DS(X)$ are in $\mathcal{T}_{n-1}^+$. This proves the claim for $DS(X)$, X irreducible. But then also for $DS_{n,n-m}(X)$, X irreducible, since then again $DS_{n,n-m}(X)$ is semisimple by proposition 8.1 in [36]. The same then also holds for $\omega_{n,n-m}(X) = H_{\overline{\partial}}(DS_{n-m}(X))$ by loc.cit. □

Corollary 5.6 *Under DS negligible objects in $\mathcal{T}_n^+$ map to negligible objects in $\mathcal{T}_{n-1}^+$.*

Proof We have shown $\mathrm{sdim}(Y) \geq 0$ for all objects Y in $\mathcal{T}_{n-1}^+$. Therefore $\mathrm{sdim}(DS(X)) = \mathrm{sdim}(X) = 0$ implies $sdim(Y_i) = 0$ for all indecomposable summands Y_i of $Y = DS(X)$, since $sdim(Y_i) \geq 0$. □

Remark 5.7 Since irreducible objects L satisfy condition $\mathtt{T}$ in the sense that $\overline{\partial}$ is trivial on $DS_{n,n-m}(L)$ [36, proposition 8.5], and since condition $\mathtt{T}$ is inherited by tensor products and retracts, all objects in $\mathcal{T}_n^+$ satisfy condition $\mathtt{T}$. Hence [36, proposition 8.5] implies the following lemma.

Lemma 5.8 *On the category $\mathcal{T}_n^+$ the functor $H_D(.)$ is naturally equivalent to the functor $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$. Similarly the functors $\omega_{n,n-m}(.) : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$ are naturally equivalent to $DS_{n,n-m}(.)$.*

Corollary 5.9 *$DS(X) = 0$ in $\mathcal{T}_{n-1}^+$ if and only if X is a projective object in $\mathcal{T}_n$.*

Proof Any negligible maximal atypical object in $\mathcal{T}_n^+$ (i.e. a negligible object in the principal block) maps under DS to a negligible maximal atypical object in $\mathcal{T}_{n-1}^+$. Furthermore $DS(X) = 0$ for X in $\mathcal{T}_n^+$ implies that X is an anti-Kac object. If $X \neq 0$, then X^* is a Kac object in $\mathcal{T}_n^+$. Hence $H_D(X^*) = 0$. Since $X^* \in \mathcal{T}_n^+$ satisfies condition $\mathtt{T}$, this implies $DS(X^*) = 0$ and hence X^* is a Kac and anti-Kac object. The corollary follows since $\mathcal{C}^+ \cap \mathcal{C}^- = Proj$. □

Corollary 5.10 *If $X \in \mathcal{T}_n^+$ and X is a Kac or anti-Kac object, then $X \in Proj$.*

Even though negligible objects map to negligible objects by Corollary 5.6, it is highly non-trivial negligible morphisms map to negligible morphism and that we therefore get an inducted functor between the quotient categories.

Lemma 5.11 *The functor $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$ gives rise to a k-linear exact tensor functor between the quotient categories*

$$\eta : \overline{\mathcal{T}}_n \to \overline{\mathcal{T}}_{n-1}.$$

In particular the iterated tensor functor $\omega = \eta \circ \ldots \eta : \overline{\mathcal{T}}_n \to vec_k$ defines a fibre functor for $\overline{\mathcal{T}}_n$.

Proof We define the ideal $\mathcal{I}^0$ via

$$\mathcal{I}^0(X, Y) = \{f : X \to Y \mid f \text{ factorizes over a negligible object.}\}$$

Obviously $\mathcal{I}^0$ is a tensor ideal for $\mathcal{T}_n^+$. As for any tensor ideal $\mathcal{I}^0 \subset \mathcal{N}$ the quotient $\mathcal{T}_n^+/\mathcal{I}^0 =: \mathcal{A}_n^+$ becomes a rigid tensor category and $\mathcal{T}_n^+ \to \mathcal{T}_n^+/\mathcal{I}^0 = \mathcal{A}_n^+$ a tensor functor. Under this tensor functor an indecomposable object X in $\mathcal{T}_n^+$ maps to zero in the quotient $\mathcal{A}_n^+$ if and only if $\mathrm{sdim}(X) = 0$. Furthermore, since the tensor functor DS maps negligible objects of $\mathcal{T}_n^+$ to negligible objects of $\mathcal{T}_{n-1}^+$, the functor DS induces a k-linear tensor functor $DS' : \mathcal{A}_n^+ \to \mathcal{A}_{n-1}^+$. The category $\mathcal{A}_n^+$ is pseudoabelian since we have idempotent lifting in the sense of [44, Theorem 5.2] due to the finite dimensionality of the Hom spaces. By the definition of $\mathcal{A}_n^+$ and $\mathcal{T}_n^+$, the dimension of each object in $\mathcal{A}_n^+$ is a natural number and, contrary to $\mathcal{T}_n^+$, it does not contain any nonzero object that maps to an element isomorphic to zero under the quotient functor $\mathcal{A}_n^+ \to \mathcal{A}_n^+/\mathcal{N}$. Therefore $\mathcal{A}_n^+$ satisfies conditions d) and g) in [2, Theorem 8.2.4]. By [2, Theorem 8.2.4 (i),(ii)] this implies that $\mathcal{N}(\mathcal{A}_n^+)$ equals the radical $\mathcal{R}(\mathcal{A}_n^+)$ of $\mathcal{A}_n^+$; note that $\mathcal{N}(\mathcal{A}_n^+) = \mathcal{N}(\mathcal{T}_n^+)/\mathcal{I}^0$ and that $\mathcal{N}(A, A)$ is a nilpotent ideal in $End(A)$ for any A in $\mathcal{A}_n^+$ by assertion b) of [2, Theorem 8.2.4 (i),(ii)]. Since $\mathcal{N}$ always is a tensor ideal, $\mathcal{R}(\mathcal{A}_n^+)$ in particular is a tensor ideal. This allows to apply [2, Theorem 13.2.1] to construct a monoidal section $s_n : \mathcal{A}_n^+/\mathcal{N}(\mathcal{A}_n^+) \to \mathcal{A}_n^+$ for the tensor functor $\pi_n : \mathcal{A}_n^+ \to \mathcal{A}_n^+/\mathcal{N}(\mathcal{A}_n^+)$. The composite tensor functor

$$\eta := \pi_{n-1} \circ DS' \circ s_n$$

defines a k-linear tensor functor

$$\eta : \overline{\mathcal{T}}_n \to \overline{\mathcal{T}}_{n-1}.$$

Since DS' is additive and $\overline{\mathcal{T}}_n$ is semisimple, η is additive and hence exact. □

Remark 5.12 (1) The k-linear tensor functor $\pi_{n-1} \circ DS' : \mathcal{A}_n^+ \to \overline{\mathcal{T}}_{n-1}$ defines the tensor ideal $\mathcal{K}_n$ of $\mathcal{A}_n^+$ of morphisms annihilated by $\pi_{n-1} \circ DS'$. Obviously $\mathcal{K}_n \subseteq \mathcal{N}$.
(2) Let S be the image of a simple object in $\mathcal{A}_n^+$. Since $\mathcal{N}(\mathcal{A}_n^+) = \mathcal{R}(\mathcal{A}_n^+)$, some given morphism $f \in Hom_{\mathcal{A}_n^+}(S, A)$ is in $\mathcal{N}(\mathcal{A}_n^+)(S, A)$ if and only if for all $g \in Hom_{\mathcal{A}_n^+}(S, A)$ the composite $g \circ f$ is zero [2, Lemma 1.4.9] (note that the endomorphisms of S in $\mathcal{A}_n^+$ are in $k \cdot id$, hence [2, Lemma 1.4.9] can be applied.).
(3) By [2, Theorem 13.2.1] the section s_n is unique up to isomorphism. Therefore the functor η so constructed is unique up to isomorphism.

Remark 5.13 We do not know whether $DS(\mathcal{N}(\mathcal{T}_n^+)) \subseteq \mathcal{N}(\mathcal{T}_{n-1}^+)$ holds. If this were true for all n, then also $DS_{n,n-i}(\mathcal{N}(\mathcal{T}_n^+)) \subseteq \mathcal{N}(\mathcal{T}_{n-i}^+)$ would hold. We consider this a fundamental question in the theory. For $n = 1$ observe that $\mathcal{A}_1^+ = \mathcal{T}_1^+/\mathcal{N}$. Indeed $\mathcal{T}_1^+$ has only one proper tensor ideal $\mathcal{N} = \mathcal{I}^0$ as can be easily seen by looking at the maximal atypical objects Ber^i and $P(Ber^j)$ in $\mathcal{T}_1^+$. The tensor ideal $\mathcal{I}^0$ could be different from $\mathcal{N}$ for $n \geq 2$. With respect to the partial ordering on the set of tensor ideals given by inclusion, $\mathcal{I}^0$ is the minimal element in the fibre of the decategorification map of the thick ideal of indecomposable objects of superdimension 0 [19, Theorem 4.1.3]. The negligible morphisms are the largest tensor ideal in this fibre.

Example 5.14 Note that it is important here to work in $\mathcal{T}_n^+$ since for example $DS(K(\mathbf{1}))$ (where $K(\mathbf{1})$ is the Kac-module of the trivial representation) splits into a direct sum of maximal atypical irreducible modules (see [36, Section 10]). Hence the identity morphism of $K(\mathbf{1})$ does not map to a negligible morphism. There are even counter examples in the smaller category $\mathcal{T}^{ev}$ of Sect. 14. In $\mathcal{T}_1$ consider the indecomposable ZigZag module (see [34]) with socle Ber^{-1} and Ber and top $\mathbf{1}$. The inclusion of Ber^{-1} induces an isomorphism when taking DS-cohomology. On the other hand the inclusion is negligible. Note that Ber^{-1} is odd. The ZigZag module can be obtained as $\mathbf{1}[1]$ in the stable category $\mathcal{K}$. Since $\mathbf{1}$ is even, $\mathbf{1}[1]$ is odd. Hence their parity shifts define even objects in $\mathcal{T}_1^{ev}$.

5.4 *DS* as a restriction functor

Recall from [21, Theorem 8.17] the following fundamental theorem on k-linear tensor categories: Suppose $\mathcal{A}_1, \mathcal{A}_2$ are k-linear abelian rigid symmetric monoidal tensor categories with $k \cong End_{\mathcal{A}_i}(\mathbf{1})$ as in loc. cit. Assume that all objects of $\mathcal{A}_i$ have finite length and all Hom-groups have finite k-dimension. Assume that k is a perfect field so that $\mathcal{A}_1 \otimes \mathcal{A}_2$ is again k-linear abelian rigid symmetric monoidal tensor categories with $k \cong End_{\mathcal{A}_i}(\mathbf{1})$ as in [21, 8.1]. Suppose

$$\eta : \mathcal{A}_1 \to \mathcal{A}_2$$

is an *exact tensor functor*. Then η is faithful [22, Proposition 1.19].

Theorem 5.15 [21, Theorem 8.17] *Under the assumptions above there exists a morphism*

$$\pi(\mathcal{A}_2) \to \eta(\pi(\mathcal{A}_1))$$

as in [21, 8.15.2] *such that η induces a tensor equivalence between the category $\mathcal{A}_1$ and the tensor category of objects in $\mathcal{A}_2$ equipped with an action of $\eta(\pi(\mathcal{A}_1))$, so that the natural action of $\pi(\mathcal{A}_2))$ is obtained via the morphism $\pi(\mathcal{A}_2) \to \eta(\pi(\mathcal{A}_1))$.*

Suppose $\omega : \mathcal{A}_2 \to Vec_k$ is fiber functor of $\mathcal{A}_2$, i.e. ω is an exact faithful tensor functor. Then $\mathcal{A}_2$ is a Tannakian category and $\mathcal{A}_2 \cong Rep_k(H)$ as a tensor category. If $\mathcal{A}_2 = Rep_k(H)$ is a Tannakian category for some affine group H over k, then $\pi(\mathcal{A}_2) = H$ by [21, Example 8.14 (ii)]. More precisely, an $\mathcal{A}_2$-group is the same as an affine k-group equipped with an H-action, and here H acts on itself by conjugation. The forgetful functor ω of $Rep_k(G)$ to Vec_k is a fiber functor. By applying this fiber functor we obtain a fiber functor $\omega \circ \eta : \mathcal{A}_1 \to Vec_k$ for the tensor category $\mathcal{A}_1$. In particular $\mathcal{A}_1$ becomes a Tannakian category with Tannaka group $H' = \omega \circ \eta(\pi(\mathcal{A}_1))$. Furthermore, by applying η to the morphism $\pi(\mathcal{A}_2) \to \eta(\pi(\mathcal{A}_2))$ in $\mathcal{A}_2$, we get a morphism $\omega(\pi(\mathcal{A}_2)) \to (\omega \circ \eta)(\pi(\mathcal{A}_1))$ in the category of k-vectorspaces, which defines a group homomorphism

$$f : H' \to H$$

of affine k-groups inducing a pullback functor

$$Rep(H') \to Rep(H)$$

that gives back the functor $\eta : \mathcal{A}_1 \to \mathcal{A}_2$ via the equivalences $\mathcal{A}_1 = Rep_k(H')$ and $\mathcal{A}_2 = Rep_k(H)$ obtained from the fiber functors.

Lemma 5.16 [22, Proposition 2.21(b)] *The morphism $f : H' \to H$ thus obtained is a closed immersion if and only if every object Y of $\mathcal{A}_2$ is isomorphic to a subquotient of an object of the form $\eta(X)$, $X \in \mathcal{A}_1$.*

The statements above will now be applied for the tensor functor

$$\eta : \mathcal{A}_1 \to \mathcal{A}_2$$

obtained from DS between the quotient categories $\mathcal{A}_1 = \mathcal{T}_n^+/\mathcal{N}$ and $\mathcal{A}_2 = \mathcal{T}_{n-1}^+/\mathcal{N}$. Notice that the assumptions above on k and $\mathcal{A}_i$ are satisfied so that $\mathcal{A}_2$ is a tannakian category with fiber functor ω giving an equivalence of tensor categories $\mathcal{A}_2 = Rep_k(H_{n-1})$. Obviously η induces an *exact tensor functor* between the quotient categories, since DS is additive, maps negligible objects of $\mathcal{T}_n^+$ into negligible objects of $\mathcal{T}_{n-1}^+$ and since the categories $\mathcal{A}_i$ are semisimple. As in our case k is algebraically closed, we know that up to an isomorphism the group H_n only depends on $\mathcal{A}_1$ but not on the choice of a fiber functor. As explained above, this defines a homomorphism of affine k-groups (uniquely defined up to conjugacy) $f : H_{n-1} \longrightarrow H_n$.

Theorem 5.17 *The homomorphism $f : H_{n-1} \to H_n$ is injective and the functor $\eta : Rep_k(H_n) \to Rep_k(H_{n-1})$ induced by $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$ can be identified with the restriction functor for the homomorphism f.*

Proof By Lemma 5.16 it suffices that every indecomposable Y in $\mathcal{T}_{n-1}^+$ with $\mathrm{sdim}(Y) > 0$ is a subobject of an object $DS(X)$, $X \in \mathcal{T}_n^+$. By assumption Y is a retract of a tensor product of irreducible modules $L_i \in \mathcal{T}_{n-1}^+$. So it suffices that each L_i is a subobject of some object $DS(X_i)$, $X_i \in \mathcal{T}_n^+$. We can assume that Y is not negligible and irreducible, hence maximal atypical and $Y = \Pi^r L(\lambda)$ for some r. Then $L(\lambda) = [\lambda] = [\lambda_1, \ldots, \lambda_{n-1}]$. By a twist with Berezin we may assume that $\lambda_{n-1} \geq 0$. Then we define $[\tilde{\lambda}] = [\lambda_1, \ldots, \lambda_{n-1}, 0]$ so that for $X = \Pi^r L(\tilde{\lambda})$ we get by Theorem 4.1 and [36, Lemma 10.2] the assertion $DS(X) = Y \oplus$ other summands. Notice that by construction $X = \Pi^r L(\tilde{\lambda})$ is in $\mathcal{T}_n^+$. But this proves our claim. □

In other words, the description of the functor DS on irreducible objects in $\mathcal{T}_n$ given by Theorem 4.1 can be interpreted as branching rules for the inclusion

$$f : H_{n-1} \hookrightarrow H_n.$$

We will show later how this fact gives information on the groups H_n.

5.5 Enriched morphism

Using the $\mathbb{Z}$-grading of DS (see Sect. 4.1), we can define an extra structure on the tower of Tannaka groups. This extra structure will not be used in the later determination of the Tannaka groups. The collection of cohomology functors $H^i : \mathcal{R}_n \to \mathcal{R}_{n-1}$ for $i \in \mathbb{Z}$ defines a tensor functor

$$H^\bullet : \mathcal{R}_n \to Gr^\bullet(\mathcal{R}_{n-1})$$

to the category of $\mathbb{Z}$-graded objects in $\mathcal{R}_{n-1}$. Using the parity shift functor Π, this functor can be extended to a tensor functor

$$H^\bullet : \mathcal{T}_n^+ \to Gr^\bullet\big(\mathcal{T}_{n-1}^+\big)$$

which induces a corresponding tensor functor on the level of the quotient categories

$$H^\bullet : \overline{\mathcal{T}}_n = \mathcal{T}_n^+/\mathcal{N} \to Gr^\bullet\big(\mathcal{T}_{n-1}^+/\mathcal{N}\big)\big) = Gr^\bullet(\overline{\mathcal{T}}_{n-1}).$$

Using the language of tannakian categories this induces an 'enriched' group homomorphism

$$f^\bullet :\ H_{n-1} \times \mathbb{G}_m \to H_n.$$

Its restriction to the subgroup $1 \times H_{n-1}$ is the homomorphism f from above.

5.6 The involution τ

Note that the category $\mathcal{T}_n^+$ is closed under $\vee$ and $*$ and hence is equipped with the tensor equivalence $\tau : X \mapsto (X^\vee)^*$. This tensor equivalence induces a tensor equivalence of $\overline{\mathcal{T}}_n = \mathcal{T}_n^+/\mathcal{N}$ and hence an automorphism $\tau = \tau_n$ (unique up to conjugacy) of the group H_n. Since all objects of $\mathcal{T}_n^+$ satisfy property $\mathtt{T}$ [36, Section 6], the involution $*$ commutes with DS. Since this also holds for the Tannaka duality, we get a compatibility

$$(H_{n-1}, \tau_{n-1}) \hookrightarrow (H_n, \tau_n).$$

5.7 Characteristic polynomial

By iteration the morphisms $f^\bullet$ successively define homomorphisms $H_{n-i} \times (\mathbb{G}_m)^i \to H_n$ and therefore we get a homomorphism in the case $i = n$

$$h : (\mathbb{G}_m)^n \to H_n.$$

This allows to define a characteristic polynomial, defined by the restriction $h^*(V_X)$ of the representation $V_X = \omega(X)$ of H to the torus $(\mathbb{G}_m)^n$

$$h_X(t) = \sum_\chi \dim(h^*(V_X)_\chi) \cdot t^\chi$$

where χ runs over the characters $\chi = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n = \mathbb{X}^*((\mathbb{G}_m)^n)$. It is easy to see that $\omega(X, t) = h_X(t, \ldots, t)$ (see Sect. 4.2).

6 The structure of the derived connected groups G_n

6.1 Setup and notations

The Tannaka group generated by the object $X_\lambda = \Pi^{p(\lambda)} L(\lambda)$ for $p(\lambda) = \sum_{i=1}^n \lambda_i$ will be denoted H_λ and we define

$$G_\lambda := \left(H_\lambda^0\right)_{der} \subseteq H_\lambda^0 \subseteq H_\lambda.$$

Finally define $V_\lambda \in Rep(H_\lambda)$ as the irreducible finite dimensional faithful representation (or the underlying vector space) of H_λ corresponding to X_λ, i.e. the representation $\omega(X_\lambda)$ for the fibre functor ω defined in Sect. 5.

(SD)-Types. Now assume for $L(\lambda)$ that

$$\varphi : L(\lambda)^\vee \cong L(\lambda) \otimes Ber^{-r}$$

holds for some $r \in \mathbb{Z}$ and some isomorphism φ in $\mathcal{T}_\lambda$. The evaluation morphism $eval : L(\lambda)^\vee \otimes L(\lambda) \to \mathbf{1}$ and the isomorphism $L(\lambda)^\vee \cong L(\lambda) \otimes Ber^{-r}$ gives rise to a nondegenerate pairing

$$L(\lambda) \otimes L(\lambda) \to Ber^r$$

As a nondegenerate pairing of the simple object $L(\lambda)$ it is even either or odd, where the parity $\varepsilon_\lambda \in \{\pm 1\}$ is given by $\varphi^\vee = \varepsilon_\lambda \cdot \varphi$ (see "Appendix C", where this sign will be computed). If we replace L_λ by the parity shift $X_\lambda = \Pi^{p(\lambda)}(L_\lambda)$, our pairing on $L(\lambda)$ induces a pairing $X_\lambda \otimes X_\lambda \to Ber^r$ on X_λ. For this notice that $\Pi^2(L \otimes L) \cong \Pi(L) \otimes \Pi(L)$ holds and the underlying vectorspaces of $L(\lambda)$ and X_λ coincide. Notice that nr is even by Lemma C.7 so that $B^r = Ber^r$ always holds. The resulting pairing

$$X_\lambda \otimes X_\lambda \to B^r$$

will be denoted $< \cdot, \cdot >$. Since a parity shift switches symmetric and antisymmetric pairings [46], the parity $\varepsilon(X_\lambda)$ of the induced pairing $< \cdot, \cdot >$ for X_λ is

$$\varepsilon(X_\lambda) = (-1)^{p(\lambda)} \varepsilon_\lambda.$$

This sign $\varepsilon(X_\lambda)$ will only depend on λ_{basic} by Lemma 6.1. Since $< \cdot, \cdot >$ is uniquely defined up to a nonvanishing constant, we will fix the pairing once for all.

The nondegenerate pairing $< \cdot, \cdot >$ on X_λ induces a nondegenerate pairing on V_λ of the same parity $\varepsilon(X_\lambda)$, and will also be denoted $< \cdot, \cdot >$ by abuse of notation. Indeed, since $< \cdot, \cdot >$ is defined in terms of Tannaka duality, the evaluation morphism $eval$ and the isomorphism φ, this follows by functoriality.

The Tannaka group H_λ of $\mathcal{T}_\lambda = <X_\lambda>$ acts faithfully on V_λ such that $<hv_1, hv_2> = \mu(h) \cdot <v_1, v_2>$ holds for all $h \in H_\lambda$ and the similitude character $\mu : GSp(V_\lambda) \to \mathbb{G}_m$ of the pairing $< \cdot, \cdot >$ on V_λ.

For a vector space V_λ over an algebraically closed field with a nondegenerate symmetric or antisymmetric pairing $\langle ., . \rangle$ let

$$G(V_\lambda, \langle ., . \rangle) = \{g \in GL(V_\lambda) | <gv, gw> = \mu(g) <v, w>, \forall v, w \in V_\lambda\}$$

be the similitude group with its similitude character $\mu : G(V_\lambda, \langle ., . \rangle) \to k^*$. The sign character $sgn : G(V_\lambda, \langle ., . \rangle) \to \mu_2$ is defined by

$$sgn(g) = \frac{det(g)}{\mu(g)^m}.$$

Notice that $\dim(V_\lambda) = 2m$ by Lemma D.4 will always be even unless $X(\lambda)$ has dimension one and hence is a power of B. In the symmetric resp. antisymmetric case this above similtude group defines the orthogonal similitude group $GO(V_\lambda)$ resp. the symplectic similitude group $GSp(V_\lambda)$.

In the GSp-case sgn is trivial and the kernel of μ is the connected symplectic group $Sp(V_\lambda)$. In the GO-case the kernel of μ is the orthogonal group $O(V_\lambda)$, and the kernel of sgn on $O(V_\lambda)$ is the connected group $SO(V_\lambda)$. The kernel of sgn on $GO(V_\lambda)$ is the connected subgroup $GSO(V_\lambda)$.

The Tannaka group H_λ of the Tannaka category $\mathcal{T}_\lambda = <X_\lambda>$ generated by X_λ acts faithfully on $V_\lambda = \omega(X_\lambda)$ such that $<hv_1, hv_2> = \mu(h) \cdot <v_1, v_2>$ holds for all $h \in H_\lambda$. Hence H_λ is a subgroup of $G\big(V_\lambda, \langle ., . \rangle\big)$.

A priori bounds. We distinguish two cases: Either X_λ is a weakly selfdual object (SD), i.e. $X_\lambda^\vee \cong B^r \otimes X_\lambda$ for some r; or alternatively X_λ is not weakly selfdual (NSD). Summarizing we obtain the following bounds for the groups H_λ. We have $H_\lambda \subseteq GL(V_\lambda)$ in the case (NSD) and $H_\lambda \subseteq GO(V_\lambda)$ resp. $H_\lambda \subseteq GSp(V_\lambda)$ for even resp. odd $\varepsilon(X_\lambda)$ in the (SD)-cases. If X_λ is properly self dual in the sense $X_\lambda^\vee \cong X_\lambda$, these simlitude groups can be replaced by their subgroups $O(V_\lambda)$ resp. $Sp(V_\lambda)$.

6.2 The structure of G_λ

Recall that two maximal atypical weights $\lambda, \ \mu$ are equivalent $\lambda \sim \mu$ if there exists $r \in \mathbb{Z}$ such that $L(\lambda) \cong Ber^r \otimes L(\mu)$ or $L(\lambda)^\vee \cong Ber^r \otimes L(\mu)$ holds. Another way to express this is to consider the restriction of the representations $L(\lambda)$ and $L(\mu)$ to the Lie superalgebra $\mathfrak{sl}(n|n)$. These restrictions remain irreducible and $\lambda \sim \mu$ holds if and only if $L(\lambda) \cong L(\mu)$ or $L(\lambda) \cong L(\mu)^\vee$ as representations of $\mathfrak{sl}(n|n)$. Let $X^+(n)$ be the set of dominant weights and let $Y^+(n)$ be the set of equivalence classes of dominant weights. Similarly let $X_0^+(n)$ denote the class of maximal atypical dominant weights and $Y_0^+(n)$ the set of corresponding equivalence classes. If we write $\lambda \in Y_0^+(n)$, we mean that $\lambda \in X_0^+(n)$ is some representative of the class in $Y_0^+(n)$ defined by λ.

Let λ be of SD-type. Then there exists $r \in \mathbb{Z}$ such that $L(\lambda) \cong Ber^r \otimes L(\lambda)^\vee$. Hence there exists an equivariant nondegenerate pairing

$$X_\lambda \times X_\lambda \longrightarrow B^r.$$

This pairing is either symmetric (even) or antisymmetric (odd). We then say X_λ is even and put $\varepsilon(X_\lambda) = 1$, or odd and $\varepsilon(X_\lambda) = -1$. The next lemma is proven in "Appendix C".

Lemma 6.1 *For all irreducible objects X_λ of SD-type in $\mathcal{T}_n^+$ we have*

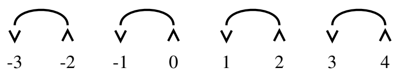
$$\varepsilon(X_\lambda) = \varepsilon(X_{\lambda_{basic}}) = (-1)^{p(\lambda_{basic})}.$$

Theorem 6.2 *$G_\lambda = SL(V_\lambda)$ if X_λ is (NSD). If X_λ is (SD) and $V_\lambda|_{G_{\lambda'}}$ is irreducible, then $G_\lambda = SO(V_\lambda)$ respectively $G_\lambda = Sp(V_\lambda)$ according to whether X_λ is even respectively odd. If X_λ is (SD) and $V_\lambda|_{G_{\lambda'}}$ is not irreducible, then $G_\lambda \cong SL(W)$ for $V_\lambda|_{G_{\lambda'}} \cong W \oplus W^\vee$.*

This theorem is proven in Sects. 7–10. Many examples can be found in Sect. 9. We conjecture that a stronger version is true: V_λ should always stay irreducible. We refer to Sect. 11 for a discussion of this case.

Remark 6.3 The (NSD) case is the generic case for $n \geq 4$. Since $SL(V_\lambda) \cong G_\lambda \subset GL(V_\lambda)$, all representations of H_λ stay irreducible upon restriction to G_λ. Hence the derived group sees already the entire tensor product decomposition into indecomposable representations up to superdimension zero. The same remark is true for a selfdual weight of symplectic type. In the orthogonal case we could have a decomposition of an irreducible representation of H_λ into two irreducible representations of G_λ since $O(V_\lambda)$ and $GO(V_\lambda)$ have two connected components.

Example 6.4 The smallest case for which V_λ could decompose when restricted to G_λ is the case $[\lambda] = [3, 2, 1, 0] \in \mathcal{T}_4^+$ with sector structure



The cup diagram of $L(\lambda)$

Then $DS[3, 2, 1, 0]$ decomposes into four irreducible representations

$$L_1 = [3, 2, 1], L_2 = [3, 2, -1], L_3 = [3, 0, -1], L_4 = [1, 0, -1].$$

Since $L_1 = Ber^2 L_4$ and $L_2 \cong L_3^\vee$ we have two equivalence classes

$$\{L_1, L_4\}, \{L_2, L_3\}.$$

In fact $V_{\lambda_1} \cong V_{\lambda_4} \cong st(SO(6))$ and $V_{\lambda_2} \cong st(SL(6))$, $V_{\lambda_3} \cong st(SL(6))^\vee$.

If $V_{[3,2,1,0]}$ does not decompose under restriction to $G_{[3,2,1,0]}$, then $G_\lambda \cong SO(24)$ and $V_\lambda \cong st(SO(24))$. If it decomposes $V_\lambda = W \oplus W^\vee$, then $G_\lambda \cong SL(12)$ and $W \cong st(SL(12))$. Since $W \not\sim W^\vee$ this implies that the embedding $SO(6) \times SL(6) \to SL(12)$ gives the branching rules

$$W \mapsto st(SL(6)) \oplus st(SO(6))$$
$$W^\vee \mapsto st(SL(6))^\vee \oplus st(SO(6)).$$

6.3 The structure theorem on G_n

We now determine G_n.

Lemma 6.5 *Suppose a tannakian category $\mathcal{R}$ with Tannaka group H is $\otimes$-generated as a tannakian category by the union of two subsets V' and V''. Let H' and H'' be the Tannaka groups of the tannakian subcategories generated by V' respectively V''. Then there exists an embedding $H \hookrightarrow H' \times H''$ so that the composition with the projections is surjective.*

Proof For arbitrary V', V'' (not necessarily finite) we get $\mathcal{T}(V') \hookrightarrow T(V' \cup V'')$, $\mathcal{T}(V'') \hookrightarrow \mathcal{T}(V \cup V")$ for the tensor categories $\mathcal{T}$ generated by V', $V"$, $V' \cup V"$ respectively. This gives natural epimorphisms $\pi' : H \to H'$ and $\pi'' : H \to H''$ which induce a morphism $i : H \to H' \times H''$ so that the composition with the projections are π' and π''. It remains to show that i is injective. The morphism i is injective since $g \in H$ is trivial if it acts trivially on all generators in $V' \cup V''$ of $\mathcal{T}(V' \cup V'')$. □

We remark that the inclusion $H \hookrightarrow H' \times H''$ induces an inclusion $H^0 \hookrightarrow (H')^0 \times (H'')^0$ of the Zariski connected components and hence an inclusion of the corresponding adjoint groups $H^0_{ad} := (H^0)_{ad}$

$$H^0_{ad} \hookrightarrow (H')^0_{ad} \times (H'')^0_{ad}$$

and, abbreviating $H^0_{der} \hookrightarrow (H')^0_{der} \times (H'')^0_{der}$, similarly for the derived groups $G := H^0_{der} := (H^0)_{der}$

$$G \hookrightarrow G' \times G''.$$

We also need the following variant of Goursat's lemma.

Lemma 6.6 *Suppose H is a connected reductive subgroup of the product $A \times B$ of two semisimple affine algebraic k-groups A and B, so that the projections to A and B are surjective. Then*

(1) *If A and B are connected simple k-groups, then either $H_{ad} = A_{ad} \times B_{ad}$ or $H_{ad} \cong A_{ad} \cong B_{ad}$.*
(2) *$H \cong A \times B$, if A and B are of adjoint type without common factor.*
(3) *If A and B are connected, $H \cong A \times B$ if and only if $H_{ad} \cong A_{ad} \times B_{ad}$.*
(4) *Suppose A is a connected semisimple group and B is a connected simple group. Let H be a proper subgroup H of $A \times B$, that surjects onto A and B for the projections. Then there exists a simple normal subgroup C of A, such that the image H/C of H in $(A/C) \times B$ is a proper subgroup of $(A/C) \times B$, if A is not a simple group.*

Proof (1)–(3) are obvious. Part (4) can be reduced to the case of adjoint groups by part (3). So we may assume that B and A are groups of adjoint type. We now use the following fact. Any semisimple A group of adjoint type is isomorphic to the product

$\prod_{i=1}^{r} A_i$ of its simple subgroups A_i. Its factors are the normal simple subgroups of A. These factors and hence this product decomposition is unique up to a permutation of the factors. Any nontrivial algebraic homomorphism of A to a simple group B is obtained as projection of A onto some factor A_i of the product decomposition composed with an injective homomorphism $A_i \to B$. Since $H \subseteq A \times B$ projects onto the first factor A and B is simple, and since H is a proper subgroup of the connected semisimple group $A \times B$, the kernel of the projection $p_A : H \to A$ is a finite normal and hence central subgroup of H. It injects into the center of B, hence is trivial. Thus $p_A : H \to A$ is an isomorphism so that H defines the graph of a group homomorphism $A \to B$. Since A is of adjoint type and therefore a product of simple groups $A \cong \prod_{i=1}^{r} A_i$, the kernel of the homomorphism $A \to B$ must be of the form $\prod_{i \neq j} A_i$. Unless A is simple, for $C = A_i$ and any $i \neq j$ assertion (4) becomes obvious. □

Corollary 6.7 *Let λ and μ be two maximal atypical weights and denote by $G_{\lambda,\mu}$ the connected derived group of the Tannaka group $H_{\lambda,\mu}$ corresponding to the subcategory in $\overline{\mathcal{T}}_n$ generated by $L(\lambda)$ and $L(\mu)$. If λ is not equivalent to μ,*

$$G_{\lambda,\mu} \cong G_\lambda \times G_\mu .$$

Proof If G_λ and G_μ are not isomorphic, Lemma 6.6 implies the claim. Otherwise $G_{\lambda,\mu} \cong G_\lambda \cong G_\mu$ (special case of Lemma 6.6.1). We assume by induction that the statement holds for smaller n. By Theorem B.3 there exists either $L(\lambda_i)$ which is not equivalent to any $L(\mu_j)$ or there exists $L(\mu_j)$ which is not equivalent to any $L(\lambda_i)$—a contradiction since then the branching of V_λ and V_μ to G_{n-1} would not be the same. For $n = 2$ we give an adhoc argument in Sect. 9. □

Theorem 6.8 Structure Theorem for G_n. *The connected derived group G_n of the Tannaka group H_n of the category $\mathcal{T}_n^+$ is isomorphic to the product*

$$G_n \cong \prod_{\lambda \in Y_0^+(n)} G_\lambda .$$

Proof This follows essentially from Theorem 6.2, where the structure of the individual groups G_λ was determined. Using Lemma 6.6, one reduces the statement of the theorem to a situation that involves only two inequivalent weights λ and μ: By part (3) of Lemma 6.6 we may replace the derived groups by the adjoint groups. Then the assertion follows from part (4) of the lemma by induction on the number of factors reducing the assertion to the case of two groups G_λ, G_μ dealt with in Corollary 6.7. □

Example 6.9 Consider the tensor product of two inequivalent representations $L(\lambda)$ and $L(\mu)$ of non-vanishing superdimension. Then

$$L(\lambda) \otimes L(\mu) = I \mod \mathcal{N}$$

for an indecomposable representation I. Indeed $L(\lambda)$ and $L(\mu)$ correspond to representations of the derived connected Tannaka groups G_λ and G_μ. Since G_λ and G_μ

are disjoint groups in G_n, tensoring with $L(\lambda)$ and $L(\mu)$ corresponds to taking the external tensor product of these representations.

7 Proof of the structure theorem: overview

We now determine G_λ inductively using the k-linear exact tensor functor between the quotient categories of the representation categories

$$\eta : \overline{\mathcal{T}}_n \to \overline{\mathcal{T}}_{n-1}$$

constructed in Lemma 5.11 with the help of $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$. By the main theorem on DS (Theorem 4.1), the restriction of $V_\lambda = \omega(X_\lambda)$ to the subgroup H_{n-1} is a multiplicity free representation. We assume by induction that Theorems 6.2 and 6.8 hold for H_{n-1} and G_{n-1}.

We have inclusions

$$G_{\lambda'} \hookrightarrow G_\lambda \hookrightarrow H_\lambda^0 \hookrightarrow H_\lambda$$

where $G_{\lambda'}$ denotes the image of the natural map $(H_{n-1}^0)_{der} \to G_\lambda = (H_\lambda^0)_{der}$. The restriction of V_λ to $G_{\lambda'}$ decomposes

$$V_\lambda \cong \bigoplus_{i=1}^{k} V_{\lambda_i}$$

where the V_{λ_i} are the irreducible representations in the category $\overline{\mathcal{T}}_{n-1}^+$ corresponding to the irreducible constituents $L(\lambda_i), i = 1, \ldots, k$, of $DS(L(\lambda))$. By induction we obtain

$$G_{\lambda'} \cong \prod_{\lambda_i/\sim} G_{\lambda_i}$$

where the G_{λ_i} are described in Theorem 6.2.

In a first step we discuss the situation in the $n = 2$ and the $n = 3$ case as well as the Tannaka groups G_λ for $L(\lambda) = Ber^r \otimes [i, 0, \ldots, 0]$, $r, i \in \mathbb{Z}$. The $n = 2$-case is needed for the start of the inductive determination of G_n. In this case we can use the known tensor product decomposition between irreducible modules in $\mathcal{T}_2$ to determine G_2 and H_2. In order to get a clear induction scheme in the proof of the structure theorem, we need to rule out certain exceptional cases which can only occur for $n \leq 3$ and for the modules $Ber^r \otimes [i, 0, \ldots, 0]$. This will allow us to assume $n \geq 4$ in Sect. 10.

In the next step we show that G_λ is simple. By induction all the V_{λ_i} are standard representations for simple groups of type A, B, C, D or $V_{\lambda_i}|_{G_{\lambda_i}} = W \oplus W^\vee$ for $G_{\lambda_i} \cong SL(W)$. The representation V_λ decomposes under restriction to G_λ in the form $W_1 \oplus \ldots \oplus W_s$ (we later show that s is at most 2). If we restrict these W_ν to $G_{\lambda'}$, they are meager representation of $G_{\lambda'}$ in the sense of Definition 10.2. The crucial Lemma 10.3 shows then that G_λ is simple. This allows us to use the classification of small representations due to Andreev–Elashvili–Vinberg.

Our aim is then to show that the dimension of the subgroup $G_{\lambda'}$ is large compared to the dimension of V_λ (given by the superdimension formula for $L(\lambda)$ in [36]) as in Lemma 8.1 or Corollary 8.2. A large rank and a large dimension of $G_{\lambda'}$ implies that the rank and the dimension of G_λ must be large, forcing V_λ to be a small representation of G_λ in the sense of Lemma 8.1 and Corollary 8.2. If we additionally know that G_λ is simple and that also $r(G_\lambda) \geq \frac{1}{2}(\dim(V_\lambda) - 1)$, Corollary 8.2 will immediately imply that G_λ is of type $SL(V_\lambda)$, $SO(V_\lambda)$ or $Sp(V_\lambda)$. However the strong rank estimate will not always hold and we will be in the less restrictive situation of Lemma 8.1.

Here the (NSD) and the (SD) case differ considerably. In the (NSD) case each irreducible representation V_{λ_i} (corresponding to $L(\lambda_i)$ in $DS(L(\lambda))$) gives a distinct direct factor in the product $G_{\lambda'} \cong \prod_{\lambda_i/\sim} G_{\lambda_i}$ since all irreducible representations of $DS(L(\lambda))$ are inequivalent in the (NSD) case by Lemma A.4. The dimension estimate for G_λ so obtained then implies that V_λ is a small representation. In the (SD) case however two representations V_{λ_i}, V_{λ_j} will contribute the same direct factor $G_{\lambda_i} \simeq G_{\lambda_j}$ if $\lambda_i \sim \lambda_j$. This decreases the dimension and rank estimate of the subgroup $G_{\lambda'}$ in G_λ and therefore of G_λ.

To finish the proof we need to understand the restriction of V_λ to G_λ. The group of connected components acts transitively on the irreducible constituents $V_\lambda = W_1 \oplus \cdots \oplus W_s$ of the restriction to H_λ^0 and G_λ. Using that the decomposition of V_λ to H_{n-1} is multiplicity free in a weak sense (obtained from an analysis of the derivatives of $L(\lambda)$ in "Appendix A"), we show finally in Sect. 10.3, using Clifford–Mackey theory, that V_λ can decompose into at most $s = 2$ irreducible representations of G_λ.

8 Small representations

Our aim is to understand the Tannaka groups associated to an irreducible representation by means of the restriction functor $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$. We have a formula for the superdimension of an irreducible representation [36] and we know inductively the ranks and dimensions of the groups arising for $k < n$. This gives strong restrictions about the groups in the $\mathcal{T}_n^+$-case due to the following list of small representations.

List of small representations. For a simple connected algebraic group H and a nontrivial irreducible representation V of H the following holds [5]

Lemma 8.1 $\dim(V) = \dim(H)$ *implies that* V *is isomorphic to the adjoint representation of* H. *Furthermore, except for a finite number of exceptional cases,* $\dim(V) < \dim(H)$ *implies that* V *belongs to the regular cases*

R.1 $V \cong st$, $S^2(st)$, $\Lambda^2(st)$ *or their duals in the* A_r*-case,*
R.2 $V = st$ *(the standard representation) in the* B_r, D_r*-case,*
R.3 $V \cong st$ *in the* C_r*-case,*
R.4 $V \hookrightarrow \Lambda^2(st)$ *in the* C_r*-case*

where the list of exceptional cases is

E.1 $\dim(V) = 20, 35, 56$ *for* $V = \Lambda^3(st)$ *and* A_r *in the cases* $r = 5, 6, 7$.
E.2 $\dim(V) = 4, 8, 16, 32, 64$ *for the spin representations of* B_r *in the cases* $r = 2, 3, 4, 5, 6$.
E.3 $\dim(V) = 8, 8, 16, 16, 32, 32, 64, 64$ *for the two spin representations of* D_r *in the cases* $r = 4, 5, 6, 7$.
E.4 $\dim(V) = 27, 27$ *for* E_6 *with* $\dim(E_6) = 78$ *(standard representation and its dual).*
E.5 $\dim(V) = 56$ *for* E_7 *with* $\dim(E_7) = 133$.
E.6 $\dim(V) = 7$ *for* G_2 *with* $\dim(G_2) = 14$.
E.7 $\dim(V) = 26$ *for* F_4 *with* $\dim(F_4) = 52$.

In particular $\dim(V) \geq r + 2$ *holds, except for* $G = A_r$ *in the cases* $V \cong st$ *or* $V \cong st^\vee$.

Corollary 8.2 *Let* V *be an irreducible representation of a simple connected group* H *such that* $4 \leq \dim(V) < \dim(H)$ *and*

$$2r(H) \geq \dim(V) - 1$$

holds. Then H *is of type* A_r, B_r, C_r, D_r *and* $V = st$ *the standard representation of this group of dimension* $r + 1,\ 2r + 1,\ 2r,\ 2r$ *for* $r \geq 3,\ 2,\ 2,\ 2$ *respectively, or* $H = D_4$ *and* V *is one of the two 8-dimensional spin representations.*

From the classification in Lemma 8.1 one also obtains

Lemma 8.3 *For a simple connected grous* H *with an irreducible root system of rank* r *we have* $\dim(H) \geq r(2r - 1)$ *except for* $H \cong SL(n)$ *with* $\dim(H) = r(r + 2)$. *Furthermore* $r \leq \dim(V)$ *holds for any nontrivial irreducible representation* V *of* H.

9 The cases $n = 2, 3$ and the S^i-case

In the next sections we determine the group G_n and the groups G_λ. Since we will determine these groups inductively starting from $n = 2$, we need to start with this case. We also discuss the $n = 3$ case separately since we have to rule out some exceptional low rank examples in the classification of [5] in Sect. 8.

Warm-up. Suppose $n = 1$. Then H_1 is the multiplicative group $\mathbb{G}_m$. Indeed the irreducible representations of it correspond to the irreducible modules $\Pi^i Ber^i$ for $i \in \mathbb{Z}$.

9.1 The case $n = 2$

For $S^i = L([i, 0])$ and $i \geq 1$ let denote

$$X_i := \Pi^i([i, 0]).$$

Then $X_i^\vee \cong B^{1-i} \otimes X_i$, hence $X_1^\vee \cong X_1$. We use from [37] the fusion rule

$$[i,0] \otimes [j,0] \;=\; \text{indecomposable} \;\oplus\; \delta_i^j \cdot Ber^{i-1} \oplus \text{negligible}$$

for $1 \leq i \leq j$ together with $Ber^r \otimes [i,0] \cong [r+i, r]$ for all $r \in \mathbb{Z}$.

Lemma 9.1 *If H_{X_i} denotes the Tannaka group of X_i, then*

$$H_{X_i} \simeq \begin{cases} SL(2) & i = 1 \\ GL(2) & i \geq 2. \end{cases}$$

Proof Since $H_1 \hookrightarrow H_2 \twoheadrightarrow H_{X_i}$ can be computed from DS we see that H_1 injects into $H = H_{X_i}$ and the two dimensional irreducible representation $V = V_{X_i}$ of H_{X_i} attached to X_i decomposes into

$$V|_{H_1} \;=\; det^{-1} \oplus det^i.$$

corresponding to $DS(X_i) \;=\; Ber^{-1} \oplus Ber^i$. If $H^0_{X_i} \cong \mathbb{G}_m$, the finite group $\pi_0(H)$ acts on H^0. By Mackey's theorem the stabilizer of the character Ber^{-1} has index two in H_{X_i} and acts by a character on V. Since the only automorphisms of $\mathbb{G}_m$ are the identity and the inversion, this would imply $i = 1$. Hence $V \otimes V$ would restrict to $\mathbb{G}_m$ with at least three irreducible constituents $det^{-2} \oplus det^2$ (corresponding to $Ber^{-2} \oplus Ber^2$) and a two dimensional module W with an action of $\pi_0(H)$ such that a subgroup of index two acts by a character. But $X_1^\vee \cong X_1$ implies that V is self dual, and hence W contains the trivial representation. This contradicts the fusion rule from above. Hence $H^0 \neq \mathbb{G}_m$ and the same argument as above shows that H^0 can not be a torus. Hence the rank r of each irreducible component of the Dynkin diagram of $(H^0_{der})_{sc}$ is $r \geq 1$ and hence $\dim(H) \geq 3$. By Lemma 8.3 we know $r \leq \dim(V) = 2$ and accordingly $\dim(H) = 3$ by Lemma 8.1. Therefore $(H^0_{der}) = SL(2)$ and $V|_{H^0_{der}}$ is the irreducible standard representation. Since H acts faithful on V

$$SL(2) \subseteq H \subseteq GL(2).$$

Now we use $V^\vee \cong Ber^{i-1} \otimes V$, which implies $H = GL(2)$ for $i > 1$. Indeed $\Lambda^2(V)$ is the character Ber^{i-1} by the fusion rules above. For $i = 1$ the isomorphism $V^\vee \cong V$ implies that $det(V)$ is trivial on H, hence

$$H = SL(2)$$

in the case $i = 1$. □

9.2 The H_2-case

We discuss the Tannaka group generated by all irreducible representations. First consider the Tannaka group H of $\langle X_i, X_j \rangle_\otimes$ for some pair $j > i$. The derived groups of the Tannaka groups H' resp. H'' of $\langle X_i \rangle_\otimes$ and $\langle X_j \rangle_\otimes$ are $SL(2)$.

We claim that $H_{der} \cong H'_{der} \times H''_{der}$. If this were not the case, then $H_{der} \cong SL(2)$ (special case of Lemma 6.6.1). But then the tensor product $X_i \otimes X_j$ considered as a representation of H corresponds to the tensor product of two standard representation and hence is a reducible representation with two irreducible factors. However this contradicts the fusion rules stated above. This implies $H_{der} \cong SL(2) \times SL(2)$ and hence $H_{ad} \cong H'_{ad} \times H''_{ad}$.

Now consider the Tannaka group H of $\langle X_{i_1}, \ldots, X_{i_k} \rangle_\otimes$ for $k > 2$. We claim that H is connected and that it is the product

$$H_{der} \cong \prod_{\nu=1}^{k} H_{der}(X_{i_\nu})$$

of the derived Tannaka groups of the $\langle X_{i_\nu} \rangle_\otimes$. This is an immediate consequence of Lemma 6.6

The fusion rule $S^i \otimes S^i \cong Ber^{i-1} \oplus$ indecomposable $\oplus$ negligible implies $\Lambda^2(X_i) \cong B^{i-1} \oplus$negligible. In particular the image of B^{i-1} is contained in $Rep(H_{X_i})$ and generates a subgroup of form $\mathbb{G}_m$. So the Tannaka group H_2 of the category $\mathcal{T}_2^+/\mathcal{N}$ sits in an exact sequence

$$0 \to \lim_k \prod_{\nu=0}^{k-1} SL(2) \to H_2 \to \mathbb{G}_m \to 0.$$

The derived group of H_2 is the projective limit of groups $SL(2)$ with a copy for each irreducible object $X_{\nu+1}$ for $\nu = 0, 1, 2, 3, \ldots$. The structure of the extension is now easily recovered from the following decription:

Lemma 9.2 *$H_2 \subset \prod_{\nu=0}^{\infty} GL(2)$ is the subgroup defined by all elements $g = \prod_{\nu=0}^{\infty} g_\nu$ in the product with the property $det(g_\nu) = det(g_1)^\nu$. The automorphism τ_2 is inner.*

We usually write $GL(2)_\nu$ for the ν-th factor of the product $\prod_{\nu=0}^{\infty} GL(2)$. Using the description of the last lemma, the torus $H_1 \cong \mathbb{G}_m$ embeds into H_2 as follows

$$H_1 \ni t \mapsto \prod_{\nu=0}^{\infty} diag(t^{\nu+1}, t^{-1}) \in H_2 \subset \prod_{\nu=0}^{\infty} GL(2)_\nu.$$

Defining $det(g) = det(g_1)$ for $g = \prod_{\nu=0}^{\infty} g_\nu$ in H_2, the representation of the quotient group $\mathbb{G}_m$ of H_2 defined by the Berezin determinant $Ber \in \mathcal{T}_2$, corresponds to the character $det(g)$ of the group H_2.

We continue with two special cases: The S^i-case for any n, and the case G_3.

9.3 The S^i-case

Consider the modules $X_i = \Pi^i([i, 0, 0])$ in $\mathcal{T}_3^+$. They have super dimension 3 for $i \geq 2$. Let H (or sometimes H_{X_i}) denote the associated Tannaka group and V the associated irreducible representation of H.

Lemma 9.3 *We have $H_{X_1} = SL(2)$ and $G_{X_i} \simeq SL(3)$ for any $i \geq 2$ and $H_{X_i} \simeq GL(3)$ for any $i \geq 3$.*

Proof The natural map $H_2 \to H_3 \to H$ allows to consider V as a representation of H_2, and as such we get

$$V|_{H_2} \cong \det^{-1} \oplus X_i$$

for $i \geq 2$ (here X_i on the right is the irreducible 2-dimensional standard representation of $GL(2)_{i-1}$, restricted to H_2). Hence $\dim(A) \geq 3$ for at least one simple factor A of H^0 and every irreducible summand W of $V|_A$ has dimension $\leq \dim(A)$. By Lemma 8.1 therefore W either has dimension 3 and $A_{sc} = SL(3)$, $W = st$ or $W = st^\vee$, or $A_{sc} = SL(2)$ and $W = S^2(st)$. If H^0_{der} is not simple, we replace it by its simply connected cover and write $(H^0_{der})_{sc} = A_{sc} \times A'$ (where A' is a product of simple groups). The representation V is then an external tensor product

$$V = W \boxtimes W'$$

of irreducible representations W, W' of A_{sc} and A'. Since V is a faithful representation of H, the lift of V (again denoted V) to $(H^0_{der})_{sc}$ has finite kernel. Since it has finite kernel, $\dim(W) > 1$, $\dim(W') > 1$ holds. Hence $\dim(W) = 3$ implies $(H^0_{der}) = A$ and $V|_{H^0}$ and $V|_{H^0_{der}}$ remain irreducible by dimension reasons. If $A_{sc} = SL(2)$ and $W = S^2(st)$, the image of H_2 surjects onto H_{der}. This contradicts the fact that V is irreducible but $V|_{H_2}$ decomposes, and excludes the case $A_{sc} = SL(2)$. Hence

$$H^0_{der} \cong SL(3).$$

Since H acts faithfully on V, we also have $H \subseteq GL(V) = GL(3)$. The restriction of V to H_2 has determinant $det^{-1} \cdot \det(X_i) \cong det^{-1} det^{i-1} = det^{i-2}$. Hence

$$H \cong GL(3)$$

for all $i \geq 3$. □

For $j > i \geq 2$ let H denote the Tannaka group of $\langle X_i, X_j \rangle_\otimes$ and H', H'' the connected components of the Tannaka groups of $\langle X_i \rangle_\otimes$ resp. $\langle X_j \rangle_\otimes$. Then we claim

$$H^0_{der} \cong H'_{der} \times H''_{der}$$

since otherwise $H'_{der} \cong H''_{der}$ by Lemma 6.6.1. But this is impossible since then the morphisms $H_2 \to H_3 \to H$ would induce the same morphisms $(H_2)_{der} \to H_{der} \to H'_{der}$ and $(H_2)_{der} \to H_{der} \to H''_{der}$, which contradicts Theorem 4.1. Indeed the factor $SL(2)_{i-1}$ maps nontrivially to H'_{der} but trivially to H''_{der}. Since H acts faithfully on the representation associated to the object $X_i \oplus X_j$ on the other hand $H \subseteq GL(\omega(X_i)) \times GL(\omega(X_j))$.

The same arguments enable us to determine the connected derived groups for any $n \geq 3$:

Lemma 9.4 *The Tannaka group H of the modules $X_i = \Pi^i([i,0,\ldots,0])$ in $\mathcal{T}_n^+$ satisfies $H^0_{der} \cong SL(n)$ and $H \subseteq GL(n)$ for all $i \geq n-1$, and $H = GL(n)$ for all $i \geq n$. For $i < n-1$ we get $H^0_{der} \cong SL(\mathrm{sdim}(X_i))$.*

Proof Indeed we have in H^0_{der} a simple component A of semisimple rank $r \geq n-1$ by induction. Obviously A contains $SL(n-1)$ and cannot be of Dynkin type A_r unless $A = SL(n)$ by Lemma 8.1.

Notice that $\dim(A) \geq r(2r-1) \geq (n-1)(2n-3) > n$ or $\dim(A) \geq r(r+2) \geq (n-1)(2n) > n$, for $n \geq 3$ by Lemma 8.3. The restriction of V decomposes into irreducible summands $W, W', \ldots$ of dimension $\dim(W) \leq n$, and the dimension of all these representations is $\leq r$. So the possible representations are listed in Lemma 8.1. None of them has dimension $\leq r+1$ except for the case where A is of type A_r and $V \cong st$ or $V \cong st^\vee$. □

9.4 The $n = 3$-case

We analyse the remaining $n = 3$-cases.

Lemma 9.5 *The derived connected group $G_3 = (H_3)^0_{der}$ of H_3 is*

$$G_3 \;\cong\; \prod_\lambda G_\lambda$$

where λ runs over all $\lambda = [\lambda_1, \lambda_2, 0]$ with integers λ_1, λ_2 such that

$$0 \leq 2\lambda_2 \leq \lambda_1$$

and $G_\lambda \cong 1, SL(2), SL(3), Sp(6), SL(6)$ according to whether λ is $0, [1,0,0]$ or $[2+\nu, 0, 0]$, for $\nu \geq 0$, or $\lambda = [2\lambda_2, \lambda_2, 0]$, for $\lambda_2 > 0$, or $0 < 2\lambda_2 < \lambda_1$.

Remark 9.6 We discuss the general case in the next section assuming $n \geq 4$. The assumption $n \geq 4$ is only relevant because we want to have a uniform behaviour regarding derivatives. Essentially all the arguments regarding simplicity of G_λ and Clifford–Mackey theory apply to the $n = 3$ case at hand. In the proof we discuss $[2,1,0]$ in detail and sketch the key inputs for the other cases.

Proof Let us consider $X = X_\lambda$ for $L(\lambda) = [210]$. The associated irreducible representation Tannaka group $H = H_X$ admits an alternating pairing, hence H_X is contained in the symplectic group of this pairing

$$H_X \;\subseteq\; Sp(6).$$

We claim that H^0_{der} is simple. If not, we replace it by its simply connected cover and write it as a product

$$\left(H^0_{der}\right)_{sc} = G_1 \times G_2.$$

The faithful representation V_X of H_X has finite kernel when seen lifted to a representation of $(H^0_{der})_{sc}$. Therefore V_λ as a representation of $(H^0_{der})_{sc}$ is of the form $V_1 \boxtimes V_2$ with $\dim(V_i) > 1$. The representation V_λ restricts to the subgroup $SL(2) \times SL(2) = G_{\lambda'}$ as

$$V_\lambda|_{G_{\lambda'}} \cong 2 \cdot (st \boxtimes \mathbf{1}) \oplus (\mathbf{1} \boxtimes st).$$

This is easily seen using

$$DS(\Pi[2,1,0]) \cong \Pi[2,1] \oplus \Pi[2,-1] \oplus \Pi[0,-1].$$

Since $\Pi[2,1] \cong Ber^{-2} \otimes [0,-1]$ they both give a copy of the standard representation of the same $SL(2)$. Hence the restriction of V_λ to the first $SL(2)$-factor is of the form

$$V_\lambda|_{SL(2)} \cong 2 \cdot st \oplus 2 \cdot \mathbf{1}$$

and

$$V_\lambda|_{SL(2)} \cong st \oplus 4 \cdot \mathbf{1}$$

for the second $SL(2)$-factor. Now consider the restriction to any of the two $SL(2)$-factors

$$V|_{SL(2)} = V_1|_{SL(2)} \otimes V_2|_{SL(2)}.$$

Since $dim(V_1) = 2$ and $dim(V_2) = 3$, their restriction to $SL(2)$ is either st or $2 \cdot \mathbf{1}$ for V_1 and $st \oplus \mathbf{1}$ or $3 \cdot \mathbf{1}$ for V_2. The Clebsch–Gordan rule for $SL(2)$ shows that $V|_{SL(2)} = V_1|_{SL(2)} \otimes V_2|_{SL(2)}$ is not possible, hence H^0_{der} must be simple. The image of H_2 in H contains two copies of $SL(2)$. Since H^0_{der} is not $SL(2) \times SL(2)$, we get $\dim(H^0_{der}) \geq 7$ and the representation V is small. Since V_λ restricted to the subgroup $SL(2) \times SL(2)$ has 3 summands of dimension 2 each, the restriction to H^0_{der} can decompose into at most 3 summands: either V_λ stays irreducible, or decomposes in the form $W \oplus W^\vee$ or in the form $W_1 \oplus W_2 \oplus W_3$ with $\dim(W_i) = 2$. But the latter implies $W_i \cong st$ for the standard representation of $SL(2)$. This would mean $\dim(H^0_{der}) \leq 6$, a contradiction. The case $W \oplus W^\vee$ cannot happen either since the restriction of $W \oplus W^\vee$ to $SL(2) \times SL(2)$ would have an even number of summands. Therefore $V_\lambda|_{H^0_{der}}$ is irreducible. Since it is selfdual irreducible of dimension 6 and carries a symplectic pairing, we conclude from Lemma 8.1 or Lemma 8.2 that $H^0_{der} = Sp(6)$ and V is the standard representation. But then

$$H_X \;\cong\; Sp(6).$$

Similarly consider $X = \Pi(Ber^{1-b} \otimes [2b, b, 0])$ for $b > 1$. Then $X^\vee \cong X$. Then either $H \subseteq O(6)$ or $H \subseteq Sp(6)$ for $H = H_X$. The image of H_2 in H contains $SL(2)^2$. Hence $\dim(H^0_{der}) \geq 6$ and $r \geq 2$. Furthermore $H^0_{der} \ncong SL(3)$. If $r = 2$, then we get a contradiction by Mackey's lemma. Hence $r \geq 3$ and the restriction of the 6-dimensional representation $V = \omega(X)$ of H to H^0_{der} remains irreducible. By the upper bound obtained from duality therefore the semisimple rank is $r = 3$. Hence V is a small irreducible representation of H^0_{der} of dimension 6. Hence by Lemma 8.1 we get $H^0_{der} = SO(V)$ resp. $Sp(V)$, since $H^0_{der} \not\cong SL(3)$. In the second case then $H = Sp(6)$. In the first case it remains to determine whether $H = SO(6)$ or $H = O(6)$.

Finally the case $X = X_\lambda$ where $L(\lambda) = [a, b, 0]$ for $a > b > 0$ and $a \neq 2b$. In this case $X^\vee \not\cong Ber^\nu \otimes X$ for all $\nu \in \mathbb{Z}$. The image of H_2 in $H = H_X$ contains $SL(2)^3$, hence the restriction of $V = \omega(X)$ to H^0_{der} remains again irreducible and defines a small representation of dimension 6. This now implies $H^0_{der} = SL(6)$, since X is not weakly selfdual which excludes the cases $Sp(6)$ and $SO(6)$. On the other hand we know that $det(V)$ is nontrivial on the image of H_1, and hence

$$H_X \cong GL(6).$$

The structure of G_3 follws from Theorem 6.8. □

Example 9.7 For $\Pi[2, 1, 0]$ the associated Tannaka group is $H_X = Sp(6)$. Furthermore X corresponds to the standard representation of $Sp(6)$ and decomposes accordingly. Hence

$$X \otimes X = I_1 \oplus I_2 \oplus I_3 \mod \mathcal{N}$$

with the indecomposable representations $I_i \in \mathcal{R}_3$ corresponding to the irreducible $Sp(6)$ representations $L(2, 0, 0)$, $L(1, 1, 0)$ and $L(0, 0, 0)$. Now consider the tensor product $I_1 \otimes I_1$. For I_1 corresponding to $L(2, 0, 0)$ it decomposes as

$$I_1 \otimes I_1 = \bigoplus_{i=1}^{6} J_i \mod \mathcal{N}$$

with the 6 indecomposable representations J_i corresponding to the 6 irreducible $Sp(6)$-representations in the decomposition

$$L(2, 0, 0)^{\otimes 2} = L(4, 0, 0) \oplus L(3, 1, 0) \oplus L(2, 2, 0) \oplus L(2, 0, 0) \oplus L(1, 1, 0) \oplus \mathbf{1}.$$

In this way we obtain the tensor product decomposition up to superdimension 0 for any summand of nonvanishing superdimension in such an iterated tensor product. Furthermore these indecomposable summands are parametrized by the irreducible representation of $Sp(6)$. Although $n = 3$ and the weight $[2, 1, 0]$ are small, we found it hardly possible to achieve this result by a brute force calculation.

10 Tannakian induction: proof of the structure theorem

10.1 Restriction to the connected derived group

Recall that H_λ denotes the Tannaka group of the tensor category generated by X_λ and $V_\lambda = \omega(X_\lambda)$ is a faithful representation of H_λ. We have inclusions

$$G_{\lambda'} \hookrightarrow G_\lambda \hookrightarrow H_\lambda^0 \hookrightarrow H_\lambda$$

where $G_{\lambda'}$ denotes the image of the natural map $(H_{n-1}^0)_{der} \to G_\lambda = (H_\lambda^0)_{der}$. Similarly we denote by $H_{\lambda'}$ the image of H_n in H_λ. The restriction of V_λ to H_{n-1} (or $H_{\lambda'}$) decomposes

$$V_\lambda \cong \bigoplus_{i=1}^{k} V_{\lambda_i}$$

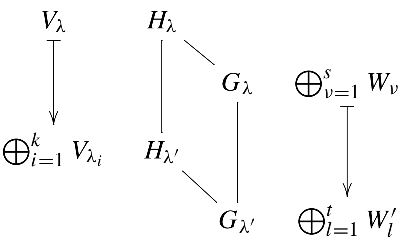
where V_{λ_i} are the irreducible representations in the category $Rep(H_{n-1})$ corresponding to the irreducible constituents $L(\lambda_i), i = 1, \ldots, k$ of $DS(L(\lambda))$. To describe $G_{\lambda'}$ we use the structure theorem for $\mathcal{T}_{n-1}^+$ (induction assumption). Therefore it suffices to group the highest weight λ_i for $i = 1, \ldots, k$ into equivalence classes. Using the structure theorem for the category $\mathcal{T}_{n-1}^+$ and Theorem 4.1, we then obtain

$$G_{\lambda'} \cong \prod_{\lambda_i/\sim} G_{\lambda_i}$$

Again using the structure theorem for G_{n-1}, each V_{λ_i} is either irreducible on G_{λ_i} or it decomposes in the form $W_i \oplus W_i^\vee$ and $G_{\lambda_i} \cong SL(W)$. The groups G_{λ_i} are independent in case (NSD). For (SD) the only dependencies between them come from the equalities $G_{\lambda_{k+1-i}} = G_{\lambda_i}$ for $i = 1, \ldots, k$ by "Appendix A". Using these strong conditions let us consider V_λ as a representation of H_λ^0. Since an irreducible representation of H_λ^0 is an irreducible representation of its derived group G_λ, the decomposition of V_λ into irreducible representation for the restriction to H_λ^0 resp. G_λ coincide. Let

$$V_\lambda = \bigoplus_{\nu=1}^{s} W_\nu$$

denote this decomposition. We then restrict each W_ν to $G_{\lambda'}$.



By induction each W'_l can be seen as the standard representation or its dual of a simple group of type A, B, C, D.

10.2 Meager representations

If we use by induction the structure theorem for G_{n-1}, we see that the representations W_i in $V_\lambda|_{G_\lambda}$ are meager in the sense below. We analyze in this section the implications of W_i to be meager.

Definition 10.1 A finite dimensional representation V of a reductive group H will be called small if $\dim(V) < \dim(H)$ holds.

Definition 10.2 A representation V of a semisimple connected group G will be called meager, if every irreducible constituent W of V factorizes over a simple quotient group of G and is isomorphic to the standard representation of this simple quotient group or isomorphic to the dual of the standard representation for a simple quotient group of Dynkin type A, B, C, D.

If a representation V of H is small resp. meager, any subrepresentation of V is small resp. meager.

We now relax the notation and write G instead of $(H^0_\lambda)_{der}$ and G' instead of $(H_{\lambda'})^0_{sc}$, the simply connected cover of $G_{\lambda'} = (H^0_{\lambda'})_{der}$. Then there exists a homomorphism $\varphi : G' \to G$ with finite kernel. We show later in Theorem 10.7 that except for some special cases ($n \leq 3$ or Berezin twists of S^i) the situation will be as in the assumptions of the next Lemma 10.3.

So suppose G' is a semisimple connected simply connected group and V is a faithful meager representation of G. Each irreducible constituent of V then factorizes over one of the projections $p_\mu : G' \to G'_\mu$ where $G' \cong \prod G'_\mu$. We then say that the corresponding constituent is of type μ.

Lemma 10.3 *Suppose V is an irreducible faithful representation of the semisimple connected group G of dimension ≥ 2. Suppose G' is a connected semisimple group and $\varphi : G' \to G$ is a homomorphism with finite kernel such that*

(1) *The restriction $\varphi^*(V)$ of V to G' is meager and for fixed μ every (nontrivial) irreducible constituents of type μ in the restriction of V to G' has multiplicity at most 2.*
(2) *If an irreducible constituent W' of $V|_{G'}$ occurs with multiplicity 2 for a type μ in $V|_{G'}$ (such a μ is called an exceptional type), then either*
   (i) *W' is the standard representation of $G_\mu \cong SL(2)$, or*
   (ii) *there is a unique type, say $\mu = \mu_2$, such that the restriction of V to G'_μ is equal to either $2W \oplus 2W^\vee$ as a representation of the quotient $SL(W)$ of G' for $\dim(W) \geq 3$ or equal to $W \oplus W^\vee$ for $\dim(W) = 2$ or*
   (iii) *there is a unique type, say $\mu = \mu_0$, with $G'_\mu \cong Sp(W')$ or $(G'_\mu)_{sc} = Spin(W)$ such that the standard representation st of G'_μ occurs twice.*
(3) *No irreducible constituent of the restriction of $V|_{G'}$ is a trivial representation of G'.*

(4) *The semisimple group G' has at most one simple factor isomorphic to $SL(2)$. The index, if it occurs, will be denoted μ_1.*

Under these assumptions G is a simple group or G' is a product of exceptional types in the sense of (2).

Remark 10.4 For the connection to our case see Theorem 10.7. The cases (2)(ii) and (2)(iii) can appear for weakly selfdual weights, see "Appendix A.5" for the possible λ. It is crucial here that we can assume $n \geq 4$.

Proof We may replace G and G' by their simply connected coverings without changing our assumptions, so that we can assume that G and $G' = \prod_\mu G'_\mu$ decompose into a product of simple groups. Then V is not faithful any more, but has finite kernel. The restriction of the meager representation V to G' decomposes into the sum $\bigoplus_\mu J_\mu$ of representations J_μ such that J_μ is trivial on $\prod_{\lambda \neq \mu} G'_\lambda$

$$V|_{G'} = \bigoplus_\mu J_\mu$$

hence J_μ can be considered as a representation of the factor G'_μ of G'. Furthermore J_μ is either an irreducible representation of G'_μ, or the direct sum $J_\mu \cong W \oplus W^\vee$ (as a representation of $G'_\mu \cong SL(W)$) by the assumption (1) and (2) or there exists a unique type μ of Dynkin type B, C, D where $J_\mu = st \oplus st$ for the standard representation st of this group G'_μ.

If the semisimple connected G is not simple, $G = G_1 \times G_2$ is a product of semisimple groups and the irreducible representation V is an external tensor product

$$V = V_1 \boxtimes V_2$$

of irreducible representations V_1, V_2 of G_1 resp. G_2. Since V has finite kernel and G is connected, $\dim(V_i) > 1$ holds. For each factor $G'_\mu \hookrightarrow G' = \prod_\mu G'_\mu$ consider the composed map

$$G'_\mu \to G_1 \times G_2.$$

This map has finite kernel.

We claim that there exists at least one index μ such that both compositions $G'_\mu \to G_i$ with the projections $G \to G_i$ $(i = 1, 2)$ are nontrivial except when G' has only exceptional types. To prove the claim, suppose $G'_\mu \to G_2$ would be the trivial map. Then the restriction of V to $G'_\mu \subseteq G'$ is $V|G'_\mu = \dim(V_2) \cdot V_1|_{G'_\mu}$. Hence $\dim(V_2) \leq 2$, since otherwise we get a contradiction to assumption (1) of the lemma. $V_1|_{G'_\mu}$ also contains at least one nontrivial irreducible constituent by assumption (3), and this constituent can occur by assumption (1) at most with multiplicity two in $V|_{G'}$. If then $\dim(V_2) = 2$, then there must exist a nontrivial irreducible constituent $I_\mu \subseteq V_1|_{G'_\mu}$ of G'_μ by assumption (3). Hence if $\dim(V_2) = 2$, $V|_{G'_\mu}$ contains $I_\mu \oplus I_\mu$ both of some type μ and we are in an exceptional type [see (2)].

We assume now that $\{\mu\}$ is not an exceptional type. We may therefore choose μ so that both $G'_\mu \to G_i$ are both nontrivial. Then

$$V|_{G'_\mu} = V_1|_{G'_\mu} \otimes V_2|_{G'_\mu}$$

is the tensor product of two nontrivial representations $V_1|_{G'_\mu}$ and $V_2|_{G'_\mu}$ of G'_μ. Since $V|_{G'}$ is a meager representation of G', all irreducible constituents of the restriction of $V|_{G'}$ to G'_μ are trivial representations of G'_μ except for at most two of them [see assumption (1)], which are standard representations up to duality. Since V_i are irreducible representations of G (recall $V \cong V_1 \boxtimes V_2$) and V has finite kernel, the restriction of V to G'_ν has finite kernel. Hence both of the representations $V_i|_{G'_\mu}$ have finite kernel, hence contain an irreducible nontrivial representation of G'_μ. Otherwise the restriction $V|_{G'_\mu}$ would be trivial contradicting that $G'_\mu \to G_i$ have finite kernel for both $i = 1, 2$ and V_i both have finite kernel on G_i. For every nontrivial irreducible representations $I_1 \subseteq V_1|_{G'_\mu}$ and $I_2 \subseteq V_2|_{G'_\mu}$ of G'_μ the representation

$$I_1 \otimes I_2$$

only contains trivial representations and standard representations st up to duality by assumption (2). Since the trivial representation occurs at most once in the tensor product of two irreducible representations, this implies $I_1 \otimes I_2 \subseteq J_\mu \oplus 1 \subseteq st \oplus st^\vee \oplus 1$ (note that μ is not exceptional). Hence $\dim(I_1)\dim(I_2) \leq 1 + 2 \cdot \dim(st) < 1 + 2 \cdot \dim(st) + \dim(st)^2$. Hence $\min(\dim(I_\nu)) < 1 + \dim(st)$. In particular, the corresponding representation with minimal dimension, say I_1, has dimension $\leq \dim(st)$ and hence I_1 is a small representation of G'_μ. Since it is small, it belongs to the list of Lemma 8.2. Therefore I_1 is the standard representation of G'_μ or its dual, unless the group G'_μ is of Dynkin type D_4 and I_1 is a spin representation. In the first case, considering highest weights it is clear that $st \otimes I_2 \subseteq st \oplus st^\vee \oplus 1$ is impossible. In the remaining orthogonal case G'_μ of Dynkin type D_4, the representation $I_1 \otimes I_2$ must have dimension $\geq 8^2$. But this contradicts $\dim(I_1)\dim(I_2) \leq 1 + 2 \cdot \dim(st) = 1 + 8 + 8 = 17$, and finally proves our assertion. □

Corollary 10.5 *In the situation of Lemma 10.3, the restriction of the representation V to the group G' is multiplicity free unless G' contains an exceptional type (in which case the irreducible constituent has multiplicity 2). If G' has at least one non-exceptional type, then the restriction contains at least one constituent with multiplicity 1.*

Proof If the restriction of V to G' contains an irreducible summand I of G' with multiplicity ≥ 2, then the restriction of I at least under one map $G'_\mu \to G$ contains a nontrivial constituent of G'_μ with multiplicity > 1. Hence the restriction of I contains J_μ by the assumption (1) and (2) of the main lemma such that $J_\mu \cong I_\mu \oplus I_\mu$ and we are in an exceptional type. □

Definition 10.6 Let $G,\ G'$ be semisimple connected groups and $\varphi : G' \to G$ a homomorphism with finite kernel. The restriction of the irreducible representation V of G to G' is called *weakly multiplicity-free* if at least one irreducible constituent has multiplicity 1.

10.3 Mackey–Clifford theory

Let H be a reductive group and H^0 its connected component. We assume that G is the connected derived group of H^0. Let V be a finite dimensional irreducible faithful representation of H and let

$$V|_{H^0} = W_1 \oplus \cdots \oplus W_s$$

be the decomposition of V into irreducible summands (W_ν, ρ_ν) after restriction to H^0. The restriction of each W_ν to G remains irreducible (this follows from Schur's lemma and the fact that the image of H^0 in $GL(W_\nu)$ is generated by the image of G in $GL(W_\nu)$ and the image of the connected component of the center of H^0, whose image is in the center of $GL(W)$). By Clifford theory [16] $\pi_0(H) = H/H^0$ acts on the isotypic components $m_\mu W_\nu$ permuting them transitively; i.e. $\rho_\nu(g) = \rho_1(hgh^{-1})$ for certain $h \in H$. Here we define the isotypic part of an irreducible W_ν to be the sum of all subrepresentations of $V|_{H^0}$ which are isomorphic to W_ν. Since $\pi_0(H)$ acts transitively on these isotypic components, the multiplicity $m = m_\mu$ of each isotypic part is the same. Let us write

$$V|_{H^0} = m \cdot (W_1 \oplus \cdots \oplus W_{\tilde{s}}).$$

Representations (W_ν, ρ_ν) from different isotypic parts are pairwise nonisomorphic representations of H^0 (in our application later this also remains true for the restriction to G by the G'-multiplicity arguments). But $\rho_1(h_1gh_1^{-1}) \cong \rho_1(h_2gh_2^{-1})$ as representations of $g \in H^0$ (or $g \in G$) holds if $h_1^{-1}h_2 \in H^0$ (resp. $h = h_1^{-1}h_2 \in H^0$). Therefore the automorphism $int_h : H^0 \to H^0$ acts trivially and the isotypic components mW_ν are permuted transitively by $Out(H^0) = Aut(H^0)/Inn(H^0)$. If a finite group acts transitvely on a set X, this implies that the cardinality of the set divides the order of the group. Therefore $\tilde{s} \leq |Out(H^0)|$. Furthermore $\dim W_i = \frac{1}{m}\frac{1}{\tilde{s}} \dim V$.

If $H = H_\lambda$ is the Tannaka group of an irreducible maximal atypical module $L(\lambda) \in \mathcal{T}_n^+$ and $V = V_\lambda = \omega(L(\lambda))$ is the associated irreducible representation of H and $W_1, \ldots, W_s$ are the irreducible constituents of the restriction of V to H^0, then the following theorem holds.

Theorem 10.7 *Suppose that $L(\lambda)$ is not a Berezin twist of S^i for some i or its dual, and suppose $n \geq 4$. Then for $G = (H_\lambda^0)_{der}$ and $G' = G_{\lambda'}$ the irreducible representations $W_1, \ldots, W_s$ of G satisfy the conditions of Lemma 10.3 and G' has at least one non-exceptional type μ. In particular G is a connected simple algebraic group and V is a weakly multiplicity free representation of H^0.*

Remark 10.8 See also "Appendix A.5" for an overview.

Proof The irreducibility and faithfulness is a tannakian consequence of the definitions. We claim that condition (1) and (2) follow from induction on n and the classification of similar and selfdual derivatives λ_i of λ in "Appendix A".

If we restrict V_{λ_i} to $G_{\lambda'}$, the induction assumption implies that the restriction is either irreducible (the regular case) or V_{λ_i} decomposes as $W \oplus W^\vee$ for the group

$SL(W)$ (the exceptional case). The exceptional case can only happen if λ_i is of type (SD).

If λ is (NSD), then equivalence classes of its derivatives consist of one element by Proposition A.4. At most one λ_i is of (SD) type. If λ is (SD), equivalence classes can consist of one or two elements by Corollary A.6. At most two derivatives λ_ν, λ_μ can be of type (SD) by Lemma A.9. In case λ_i is (NSD), the restriction of V_{λ_i} to $G_{\lambda'}$ remains irreducible.

Let us then assume that we are in the case where λ_ν is not equivalent to any other derivative. Then λ_ν belongs to one the three cases (1), (2), (3) treated in the proof of Lemma A.9. In the regular cases V_{λ_i} remains irreducible. If λ_i is exceptional, it decomposes as $W \oplus W^\vee$ for the group $SL(W)$. Then the restriction of W_ν to $G_{\lambda'}$ has multiplicity 1 unless $\dim(W) = 2$ or $\dim(W) = 1$ since $W^\vee \not\cong W$ for $\dim(W) \geq 3$. These two cases would lead to an irreducible constituent of multiplicity 2 (trivial representation or standard representation of $SL(2)$).

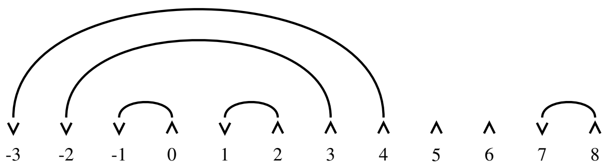
Assume therefore that $\lambda_\nu \sim \lambda_\mu$ for $\nu \neq \mu$. Then λ is of ladder type and $\{\nu, \mu\} = \{1, k\}$ such that $G_\nu = G_\mu$ is of (SD)-type and either symplectic or orthogonal regular or exceptional (see Lemma A.9). Here we use that $n \geq 4$.

In the exceptional case the standard representation W and its dual $W^\vee$ of $SL(W)$ appear with multiplicity 2 in the restriction of V to $G_{\lambda'}$. We don't have $W \cong W^\vee$ unless $\dim(W_\mu) = 2$ which is impossible for $n \geq 4$ (it would mean $\dim(V_{\lambda_1}) = 4$, but $\dim(V_{\lambda_1}) = (n-1)!$ since λ is of ladder type).

In the regular case (SD)-case we have $G_1 = G_k = SO(V_{\lambda_1})$ or $Sp(V_{\lambda_1})$. Both V_{λ_1} and V_{λ_k} remain irreducible after restriction, hence the multiplicity is again 2.

The uniqueness assertion about the types in (2)(ii) and (2)(iii) follows since case (iii) can only occur for $\lambda_\nu \sim \lambda_\mu$ of (SD) type, and there are at most two such derivatives. The multiplicity 2 assertion for (2)(ii) holds since at most one selfdual derivative can have dimension 4 or 2 (see proof of condition 3 and 4).

Condition (3) is seen as follows: The trivial representation of G' is attached to a derivative λ_μ of λ only if $L(\lambda)$ isomorphic to $S^i \otimes Ber^j$ for some $i \geq 1$ and some $j \in \mathbb{Z}$ by Lemma D.3. Concerning condition 4): A factor G'_μ of G' of rank 1 (i.e. with derived group $SL(2)$) is attached to some derivative λ_μ of λ only if $L(\lambda) = S^1$ or λ has only two sectors, one sector S of rank 1 and the other sector S' corresponds to S^1 on the level $n-1$. In other words $\partial SS'$ resp. $S'\partial S$ gives S^1 and the corresponding group $SL(2)$, but not the other derivative unless $n \leq 3$.



The first derivative is S^1 for $n = 4$.

Hence by our assumptions, the group G' has at most one simple factor $SL(2)$. If an irreducible constituent of the restriction of V to G' has multiplicity 2, it comes from a derivative of type (SD). Hence if all types of G' are exceptional, all derivatives of

$L(\lambda)$ would have to be selfdual. This can only happen for $n \leq 3$ by the analysis in "Appendix A". Hence Lemma 10.3 and Corollary 10.5 imply the last statement. □

Theorem 10.9 *The simple group G is of type A, B, C, D and $W_1|_G$ is either the standard representation of G or its dual.*

Proof We suppose that $L(\lambda)$ is not a Berezin twist of S^i for some i and suppose $n \geq 4$. We distinguish the cases NSD and SD. In the NSD-case we claim that we have

$$r(G_\lambda) \geq (\dim(V_\lambda) - 1)/2$$

and that for $n \geq 4$ and $\dim(V_\lambda) \geq 4$

$$\dim(G_\lambda) > \dim(V_\lambda)$$

holds (note that $\dim(V_\lambda) \leq 3$ for $n \geq 4$ implies $k = 1$ and $\dim(V_\lambda) = \dim(V_{\lambda_1})$). For all $i = 1, \ldots, k$ the superdimension formula of [51] [36, Section 16] implies by Lemma D.5 that

$$\dim(V_\lambda) \leq n \cdot \dim(V_{\lambda_i})/r_i$$

where $r_i = r(V_{\lambda_i}) \geq 1$ is the rank of λ_i. Obviously $\dim(G_{\lambda_i}) \leq \dim(G_\lambda)$.

Since we excluded the S^i-case, no V_{λ_i} has dimension 1 by Lemma D.3. At most one of the representations V_{λ_i} is selfdual by Lemma A.9. We make a case distinction on whether there exists one V_{λ_i} that splits in the form $W_i' \oplus (W_i')^\vee$ upon restriction to $G_{\lambda'}$ or not. In the latter case we know $r(G_{\lambda_i}) \geq \frac{1}{2}\dim(V_{\lambda_i})$ by Theorem 6.2 and the induction assumption. Now by Proposition A.4 and the assumption (NSD) all λ_i in the derivative of λ are inequivalent for $i \neq j$. Hence we get

$$r(G_\lambda) \geq \sum_i r(G_{\lambda_i}) \geq \sum_i \frac{1}{2}\dim(V_{\lambda_i}) \geq \frac{1}{2}(\dim(V_\lambda)).$$

Since $\dim(G_{\lambda_i}) \geq 3r(G_{\lambda_i})$, this implies $\dim(G_\lambda) \geq \frac{3}{2}(\dim(V_\lambda) - 1)$ and hence $\dim(G_\lambda) > \dim(V_\lambda)$ (note that we have at least one SL factor G_{λ_i} for which $r(G_{\lambda_i}) > \frac{1}{2}\dim(V_{\lambda_i})$). If V_λ splits $V_\lambda = W_1 \oplus \cdots \oplus W_s$ we may replace V_λ by any W_ν for an even better estimate. Therefore Lemma 8.2 implies that V_λ (or W_ν) is the standard representation or its dual of a simple group of type A, B, C, D. If V_λ stays irreducible, then we obtain $G_\lambda \cong SL(V_\lambda)$ since V_λ is not selfdual.

If V_{λ_i} splits, $G_{\lambda_i} \cong SL(W_i)$ for $V_\lambda \cong W_i \oplus W_i^\vee$ by induction assumption. If the dimension of V_{λ_i} is $2d_i$, we then have $r(G_{\lambda_i}) = d_i - 1$ and therefore have to replace the estimate $r(G_{\lambda_i}) \geq \frac{1}{2}\dim(V_{\lambda_i})$ by the estimate $r(G_{\lambda_i}) \geq \frac{1}{2}(\dim(V_{\lambda_i} - 2))$. Since V_{λ_i} can only decompose if it is of type SD, $L(\lambda)$ has more than one sector. All the other $k - 1 \geq 1$ derivatives $L(\lambda_i)$ are of type NSD and define inequivalent $SL(V_{\lambda_i})$. For each of these we obtain $r(G_{\lambda_j}) = \dim V_{\lambda_j} - 1$. Summing up we obtain

$$r(G_\lambda) \geq \sum_i r(G_{\lambda_i}) \geq \frac{1}{2}(\dim(V_{\lambda_i}) - 2) + \sum_{j \neq i} \dim(V_{\lambda_j}) - 1.$$

This implies again the necessary estimates to apply Lemma 8.2.

We now consider the SD-case. If V_λ decomposes

$$V_\lambda|_{G_\lambda} \cong W_1 \oplus \cdots \oplus W_s$$

then we can assume by reindexing that $\dim(W_i) = \frac{1}{s}\dim(V_\lambda)$. Note that $\dim(W_1) > 1$ follows from the induction assumption.

In the SD case we proceed as follows: We first show that V_λ or W_1 is small. Since we cannot prove the strong rank estimates for $r(G_\lambda)$ as in the NSD case, we work through the list of exceptional cases in Lemma 8.1.

The list of superdimensions in the n = 4 and n = 5 case in Sects. 15 and D.2 along with the induction assumption shows in these cases that V_λ is small. Therefore we can assume $n \geq 5$. We use the known formulas $\dim(SL(n)) = n^2 - 1$, $\dim SO(n) = \frac{n(n-1)}{2}$ and $\dim(Sp(2n)) = n(2n+1)$.

We recall from the analysis in Lemma A.9 that $L(\lambda)$ can only have more than one selfdual derivative if it is completely unnested, i.e. it has n sectors of cardinality 2. In this case it has 2 selfdual derivatives coming from the left and rightmost sectors and, if n is odd, another derivative coming from the middle sectors. If λ is not of this form, then the unique weakly selfdual derivative comes from the middle sector (of arbitrary rank).

We want to show $\dim(G_\lambda) > \dim(V_\lambda)$. By induction G_{λ_i} is either $SO(V_{\lambda_i})$, $Sp(V_{\lambda_i})$, $SL(V_{\lambda_i})$ or $SL(W_i)$ for $V_{\lambda_i} = W_i' \oplus (W_i')^\vee$. We estimate the dimension of G_{λ_i} via $\sum \dim(G_{\lambda_i})$. We claim that we can assume that we have more than one sector because otherwise $\dim(V_\lambda) = \dim(V_{\lambda_1})$ implies that V_λ is small using the induction assumption. If V_{λ_1} is an irreducible representation of $G_{\lambda'}$ the claim is clear by induction assumption. If it splits $V_{\lambda_1} = W_1' \oplus (W_1')^\vee$, then $\dim(V_{\lambda_1}) < \dim(SL(W_1'))$ provided $\mathrm{sdim}(L(\lambda_1)) \geq 3$. Now $\mathrm{sdim}(L(\lambda_1)) = 2$ can only happen for $L(\lambda_i) \cong Ber^r \otimes S^1$ for some r (and then V_{λ_1} is an irreducible representation of $G_{\lambda'}$). We therefore assume $k > 1$. The worst estimate for the dimension is obtained if all V_{λ_i} split as $W_i' \oplus (W_i')^\vee$ and therefore $G_{\lambda_i} \cong SL(W_i)$. This case can only happen if either $n = 2$ or $n = 3$. For $n \geq 4$ the lowest estimate for the dimension of G_λ occurs if λ is completely unnested with 2 selfdual derivatives coming from the left and right sector and we have $n/2$ equivalence classes of derivatives (or $\lfloor n/2 \rfloor + 1$ for odd n). The left and right sector then contribute a single $SL(W_1') = SL(W_k')$ and if n is even for all other derivatives $G_{\lambda_i} \cong SL(V_{\lambda_i})$ with $V_{\lambda_i} \sim V_{\lambda_{k-i}}$ and therefore same connected derived Tannaka group. If $n = 2l+1$ is odd the middle sector can contribute another derivative of type SD with Tannaka group $SL(W_{l+1}')$. The dimension estimate works as in the case above and we therefore ignore this case.

We show now that $\dim(G_\lambda) > \dim(V_\lambda)$ provided we have two SD derivatives coming from the left- and rightmost sector. Denote by d_i the dimension of V_{λ_i}. For $i = 1, k$ it is even $d_1 = 2d_1' = 2d_k'$ by Lemma D.4. We then obtain for the dimension of $G_{\lambda'}$

$$\dim(G_{\lambda'}) = \frac{1}{2}\big((d_1')^2 - 1 + (d_k')^2 - 1\big) + \frac{1}{2}\sum_{j \neq 1,k} d_j^2 - 1.$$

It is enough to show $2\dim V_{\lambda_i} < \dim G_{\lambda_i}$ for each i. The smallest possible superdimensions for a selfdual irreducible representation are $2, 4, 12, \ldots$. The dim $= 2$ case can only happen for $L(\lambda_i) \cong Ber^r \otimes S^1$ which is not possible by assumption. Hence $d_1' \geq 3$. This case occurs for $[2, 1, 0]$ for $n = 3$, $[2, 2, 0, 0]$ for $n = 4$ and all their counterparts for larger n by appending zeros to the weight (e.g. $[2, 1, 0, 0]$). These are not derivatives of a selfdual representation $L(\lambda)$ unless $L(\lambda)$ has one sector (which we excluded). Therefore we can assume $d_1' \geq 6$. Then

$$2\ \dim(V_{\lambda_1}) = 4d_1' < \left(d_1'\right)^2 - 1 = \dim(G_{\lambda_1}).$$

For the NSD derivatives we can exclude the case $d_i = 2$ since this only happens for $L(\lambda_i) \cong Ber^{\cdots} \otimes S^1$. For $d_i \geq 3$ we obtain $2d_i < d_i^2 - 1$, hence again $2\dim(V_{\lambda_i}) < \dim(G_{\lambda_i})$. Clearly this estimates also hold if we have more than $n/2$ equivalence classes of weights or if we have $SO(V_{\lambda_i})$ or $Sp(V_{\lambda_i})$ in case of $SL(W_i)$.

Hence $\dim(V_\lambda) < \dim(G_\lambda)$. If V_λ is an irreducible representation of G_λ, it is a small representation of G_λ and Lemma 8.1 applies. If it decomposes $V_\lambda \cong W_1 \oplus \cdots \oplus W_s$, then each W_ν is an irreducible small representation of G_λ.

Assume first that $V_\lambda \cong W_1 \oplus \cdots \oplus W_s$ with $s \geq 3$ and $\dim(W_1) \leq \frac{1}{s}\dim(V_\lambda)$. Again the smallest rank estimate for the subgroup $G_{\lambda'}$ occurs for $n \geq 4$ if λ is completely unnested with 2 selfdual derivatives coming from the left and right sector and we have $n/2$ equivalence classes of derivatives (we assume here n even. In the odd case we can have another derivative from the middle sector. The estimate below still holds). Then

$$\begin{aligned} r(G_\lambda) \geq r(G_{\lambda'}) &\geq \frac{1}{2}\left(d_1/2 - 1 + d_k/2 - 1 + \sum_{j \neq 1,k} d_j - 1\right) \\ &= \frac{1}{2}(\dim(V_\lambda) - k - d_1/2 - d_k/2). \end{aligned}$$

In the completely unnested case this equals

$$\frac{1}{2}(n! - n - (n-1)!).$$

We need $r(G_\lambda) \geq \frac{1}{2}(\dim(V_\lambda) - 1)$ to apply Lemma 8.2. We replace now V_λ by W_1 with $\dim(W_1) \leq 1/s\ \dim(V_\lambda)$. For $n \geq 4$ and $s \geq 2$ we obtain $(n!/s) - 1 \leq n! - n - (n-1)!$, hence Lemma 8.2 can be applied to the irreducible representation W_1.

If λ is not completely unnested, it can have at most one SD derivative coming from the middle sector for $k = 2l + 1$ odd. Then we obtain

$$\begin{aligned} r(G_\lambda) \geq r(G_{\lambda'}) &\geq \frac{1}{2}\left(d_{l+1}/2 - 1 + \sum_{j \neq l+1} d_j - 1\right) \\ &= \frac{1}{2}(\dim(V_\lambda) - k - d_{l+1}/2). \end{aligned}$$

As above we replace V_λ with W_1 with $\dim(W_1) \leq \frac{1}{s} V_\lambda$ and show $\dim(V_\lambda/s - 1) \leq \dim(V_\lambda) - k - d_{l+1}/2$. For $s = 2$ this is equivalent to $\dim(V_\lambda) \geq d_{l+1} + 2(k-1)$. This follows easily from $\dim(V_\lambda) = \dim(V_{\lambda_{l+1}}) \frac{n}{r_{l+1}}$ (Lemma D.5). For $s > 2$ the estimates are even stronger. The cases where the SD derivative occurs and contributes $SO(V_{\lambda_{l+1}})$ or $Sp(V_{\lambda_{l+1}})$, or the case in which no SD derivative occurs, can be treated the same way.

We can therefore assume that either (a) V_λ is an irreducible representation of G_λ or it splits in the form $V_\lambda = W \oplus W^\vee$. The analysis of small superdimensions in Sect. D.2 shows that the possible superdimensions of weakly selfdual irreducible representations less than 129 are

$$1,\ 2,\ 6,\ 12,\ 20,\ 24,\ 30,\ 42,\ 56,\ 70,\ 72,\ 80,\ 90,\ 110,\ 112.$$

Except for the numbers 20 and 56 none of the exceptional dimensions in Lemma 8.1 is equal to either the superdimension or half the superdimension of an irreducible weakly selfdual representation in $\mathcal{T}_n^+$. It is easy to exclude these two cases (see Sect. D.2) since in this case V_λ or W would be either a symmetric or alternating square of a standard representation (which would give a contradiction to the induction assumption) or the irreducible representation of minimal dimension of E_7 which is impossible by rank estimates. □

Theorem 10.10 *Either the restriction of V_λ to H_λ^0 and G_λ is irreducible, or $G \cong SL(W)$ and $V|_G \cong W \oplus W^\vee$ for a vectorspace W of dimension ≥ 3. If $V|_G \cong W \oplus W^\vee$, then*

$$V_\lambda \cong Ind_{H_1}^H(W)$$

for a subgroup H_1 of index 2 between H^0 and H. In particular V_λ is an irreducible representation of G_λ if $L(\lambda)$ is not weakly selfdual.

Proof As in the statement of Theorem 10.7 we can assume that $n \geq 4$ and that $L(\lambda)$ is not a Berezin twist of S^i (or its dual) since these cases were already treated in Sect. 9.

We claim that the representation $V|_{H^0} = W_1 \oplus \cdots \oplus W_s$ is multiplicity free. Since the restriction of V to G' is weakly multiplicity free, at least one irreducible constituent occurs only with multiplicity 1 for some (non-exceptional) μ. By Clifford theory the multiplicity of each isotypic part in the restriction of V to H^0 is the same (since π_0 acts transitively). If the multiplicity of each isotypic part would be bigger than 1, the restriction of V to G' could not be weakly multiplicity free. Therefore the multiplicity of each isotypic part is 1. Any W_ν restricted to G_λ is irreducible (restriction to the derived group). Since G_λ is a normal subgroup of H, H still operates transitively on the set $\{W_\nu|_{G_\lambda}\}$. Fix any $W_\nu|_{G_\lambda}$. Its H-orbit has s' elements where s' divides s and s/s' is the multiplicity of each $W_\nu|_{G_\lambda}$ in $V_\lambda|_{G_\lambda}$. Hence the argument from Clifford theory explained preceding Theorem 10.7 shows

$$s' \leq |Out(G)|.$$

But a nontrivial outer automorphism of G that does not fix the isomorphism class of the standard representation W_1 of G exists only for the groups G of the Dynkin type A_r for $r \geq 2$ (note that we can ignore the D_4-case since no weakly selfdual irreducible representation has superdimension 8). For the special linear groups $G = SL(\mathbb{C}^{r+1})$ the nontrivial representative in $Out(G)$ it is given by $g \mapsto g^{-t}$. The twist of the standard representation by this automorphism gives the isomorphism class of the dual standard representation $W_1^{\vee}$. This implies $s' = 1$ or $s' = 2$. If $s' = 2$, then $V_\lambda|_{G_\lambda} \cong W \oplus W^{\vee}$ where W is the standard representation of SL and $G_\lambda \cong SL(W)$. Since $V_\lambda|_{G_{\lambda'}}$ is weakly multiplicity free and $G_{\lambda'} \subset G_\lambda$, $V_\lambda|_{G_\lambda}$ is weakly multiplicity free as well. Accordingly $s/s' = 1$ and we also obtain $s = 1$ or 2. If $s = 2$, Clifford theory further implies that

$$V_\lambda \cong Ind_{H_1}^{H}(W)$$

for a subgroup H_1 of index 2 between H^0 and H. □

Remark 10.11 Since $W \oplus W^{\vee}$ is selfdual, this implies in particular that V_λ can only decompose if $L(\lambda)$ is weakly selfdual. If V_λ decomposes, its restriction to $G_{\lambda'}$ is of the form $\bigoplus_i W_i \oplus W_i^{\vee}$. This leads to some restrictions on SD weights λ such that V_λ decomposes in the form $W \oplus W^{\vee}$. Consider for an instance the weakly selfdual weight $[n-1, n-2, \ldots, 1, 0]$ for odd $n = 2l+1$. Then V_λ can only decompose if the irreducible representation V_{λ_l+1} associated to the middle derivative $L(\lambda_{l+1})$ decomposes upon restriction to $G_{\lambda'}$ in the form $W'_{l+1} \oplus (W'_{l+1})^{\vee}$.

11 The structure theorem on the full Tannaka groups

We discuss the full Tannaka groups H_λ in this section. To this end we analyze the invertible elements in $Rep(H_n)$, i.e. $Pic(H_n)$, or in down-to-earth terms the character group of H_n.

11.1 Invertible elements

For a rigid symmetric k-linear tensor category $\mathcal{C}$ an object I of $\mathcal{C}$ is called invertible if $I \otimes I^{\vee} \cong \mathbf{1}$ holds. The tensor product of two invertible objects of $\mathcal{C}$ is an invertible object of $\mathcal{C}$. Let $Pic(\mathcal{C})$ denote the set of isomorphism classes of invertible objects of $\mathcal{C}$. The tensor product canonically turns $(Pic(\mathcal{C}, \otimes))$ into an abelian group with unit object $\mathbf{1}$, the Picard group of $\mathcal{C}$.

Suppose that the categorial dimension dim is an integer ≥ 0 for all indecomposable objects of $\mathcal{C}$. An indecomposable object I of $\mathcal{C}$ is an invertible object in $\overline{\mathcal{C}} = \mathcal{C}/\mathcal{N}$ if and only if $\mathrm{sdim}(I) = 1$ holds. In fact $\dim(I) = 1$ implies $\dim(I^{\vee}) = 1$ and hence $\dim(I \otimes I^{\vee}) = 1$. Hence $I \otimes I^{\vee} \cong \mathbf{1} \oplus N$ for some negligible object N. Note that the evaluation morphisms $eval : I \otimes I^{\vee} \to \mathbf{1}$ splits since $\dim(I) \neq 0$.

11.2 $Pic(\overline{\mathcal{T}}_n)$ and its generators

Since $\overline{\mathcal{T}}_n \sim Rep_k(H_n)$, to determine the Picard group $Pic(\overline{\mathcal{T}}_n)$ is tantamount to determine the character group of H_n. It coincides with the character group of the factor commutator group H_n^{ab} of H_n. Hence $H_n^{ab} = H_n/G_n$ is determined by $Pic(\overline{\mathcal{T}}_n)$.

The Picard group $Pic(\overline{\mathcal{T}}_n)$ can be determined from the individual $Pic(H_\lambda)$, but it is preferable to choose different generators.

For a Tannakian category $\mathcal{T}$ over an algebraically closed field k of characteristic zero, generated by finitely many irreducible objects, let $H(\mathcal{T})$ denote the Tannaka group of $\mathcal{T}$. In our situation the Tannaka group H_n is a projective limit of certain algebraic Tannaka groups $H(\mathcal{T})$ as above, so that $Pic(H_n)$ correspondingly is an inductive limit

$$Pic(H_n) = \lim_{\rightarrow} Pic(H(\mathcal{T}))$$

of the Picard groups $Pic(H(\mathcal{T}))$. Let us consider generators of this inductive limit. Such generators are the Picard groups $Pic(H(\overline{\mathcal{T}}_\lambda))$, attached to the Tannakian categories $\overline{\mathcal{T}}_\lambda$ that are generated by X_λ and the normalized Berezin B. By definition, the Tannakian category $\overline{\mathcal{T}}_\lambda$ only depends on the equivalence class $\lambda/\sim$ of λ since $\overline{\mathcal{T}}_{\lambda'} = \overline{\mathcal{T}}_\lambda$ for $X_{\lambda'} = X_\lambda \otimes B$.

In the limit, the passage from H_λ to $H(\overline{\mathcal{T}}_\lambda)$ allows a slicker description of the structure of the projective limit H_n: Obviously there exists a canonical splitting

$$Pic(H(\overline{\mathcal{T}}_\lambda)) \cong Pic^0(H(\overline{\mathcal{T}}_\lambda)) \times \mathbb{Z}$$

compatible with the splitting $Pic(H_n) \cong Pic^0(H_n) \times \mathbb{Z}$ given in 13, in such a way that $Pic^0(H_n)$ is generated by the images of the groups $Pic^0(H(\overline{\mathcal{T}}_\lambda))$ for λ ranging over the equivalence classes $\lambda/\sim$.

11.3 $Pic(\overline{\mathcal{T}}_n)$ and the determinant

The elements of $Pic(\overline{\mathcal{T}}_n)$ are represented by indecomposable objects $I \in \mathcal{T}_n^+$ with the property

$$I \otimes I^\vee \cong \mathbf{1} \oplus \text{ negligible.}$$

Since $\text{sdim}(X_\lambda) \geq 0$, we can define $det(X_\lambda) = \Lambda^{\text{sdim}(X_\lambda)}(X_\lambda)$. Notice

$$\det(X_\lambda) \;=\; I_\lambda \;\oplus\; \text{negligible}$$

is the sum of a unique indecomposable module I_λ in $\mathcal{T}_n^+$ and a direct sum of negligible indecomposable modules in $\mathcal{T}_n^+$. Furthermore $I_\lambda^* \cong I_\lambda$ and $\text{sdim}(I_\lambda) = 1$ holds, and if X_λ is selfdual, then I_λ is selfdual. In particular, $det(X_\lambda)$ in $\mathcal{T}_n^+$ has superdimension one, hence its image defines an invertible object of the representation category $\overline{\mathcal{T}}_n \sim Rep_k(H_n)$. By abuse of notation we also write $det(X_\lambda) \in Rep_k(H_n)$.

11.4 The NSD-case

According to the structure theorem Theorem 10.10 in the (NSD)-cases the group H_λ satisfies

$$SL(V_\lambda) \subseteq H_\lambda \subseteq GL(V_\lambda) \dots$$

Hence to determine H_λ it suffices to show that the restriction of the determinant $\det : GL(V_\lambda) \to k^*$ to the H_λ is either trivial or surjective. In Theorem 14.3 we later show that $\det : H_\lambda \to \mathbb{G}_m$ a a representation is represented by a power $B^{\ell(\lambda)}$ of the twisted Berezin object B in $\mathcal{T}_n{}^+$. Hence $H_\lambda \cong SL(V_\lambda)$ if and only if the integer $\ell(\lambda)$ is zero, and $H_\lambda \cong GL(V_\lambda)$ holds otherwise. In particular H_λ^{ab} is the Tannaka group of the Tannaka category generated by $B^{\ell(\lambda)}$ in the (NSD)-cases.

11.5 The SD-cases

We distinguish between the exceptional cases where $V_\lambda|_{G_\lambda} \cong W \oplus W^\vee$ and the remaining cases where this does not happen. We call these the regular (SD)-cases since we conjecture that the exceptional (SD)-case do not occur.

In all (SD)-cases the group H_λ is a genuine subgroup of the similitude group $G(V_\lambda, \langle ., . \rangle)$. As such it inherits the similitude character μ resp. the determinant character det of $G(V_\lambda, \langle ., . \rangle)$ that are represented by B^r resp. $B^{\ell(\lambda)}$. Recall from Sect. 6 that $sign = \det / \mu^n$ is a character of order two of the similitude group $G(V_\lambda, \langle ., . \rangle)$. Since the restriction of $sign$ to H_λ is represented by $B^{\ell(\lambda)-r}$, and the latter is either trivial or non-torsion. This implies that $sign$ is trivial on H_λ, hence $H_\lambda \subset GSO(V_\lambda)$ resp. $H_\lambda \subset GSp(V_\lambda)$ holds according to the parity $\varepsilon(X_\lambda)$ of the pairing $\langle ., . \rangle$.

11.6 The regular SD-cases

For the *regular (SD)-cases* the group H_λ satisfies $H_\lambda \subseteq GO(V_\lambda)$ resp. $H_\lambda \subseteq GSp(V_\lambda)$ according to the parity of the pairing $\langle ., . \rangle$. This follows from 6 as well as the fact that the kernel of the characters det and the similtude character μ are the subgroup $SO(V_\lambda)$ resp. $Sp(V_\lambda)$. Both $\mu = B^r$ and $\det = B^{\ell(\lambda)}$ are represented by tensor powers powers of the twisted Berezin by theorem 14.3. Hence in the regular (SD)-cases Theorem 10.10 implies the following structure result: The group H_λ is isomorphic to $SO(V_\lambda)$ resp. $Sp(V_\lambda)$ if and only if $\ell(\lambda) = 0$, and it is isomorphic to $GSO(V_\lambda)$ resp. $GSp(V_\lambda)$ otherwise.

11.7 The exceptional (SD)-cases

For this exceptional situation we recall the following facts:

In the exceptional case we have shown that W is not isomorphic to $W^\vee$ as a representation of G_λ (in other words we have $m > 2$). By Schur's lemma this implies that the restriction of the pairing $< \cdot, \cdot >$ must be trivial on the subspaces $W \subset V_\lambda$ and $W^\vee \subset V_\lambda$, and that these two subspaces are orthogonal to each other for the nondegenerate pairing $\langle ., . \rangle$ on V_λ. Hence W and $W^\vee$ define an orthogonal pair of Lagrangian subspaces of V_λ. In the exceptional cases the representation of G_λ on the vectorspace

$$V_\lambda|_{G_\lambda} \cong W \oplus W^\vee$$

decomposes into two faithful nonisomorphic irreducible representations on the subspaces W and the subspace $W^\vee$ such that the image of G_λ in $GL(W)$ contains the perfect group $SL(W)$. Furthermore $H = H_\lambda$ (as well as $H = H(\mathcal{T}_\lambda)$) preserves the unordered pair $\{W, W^\vee\}$ of disjoint Lagrangian subspaces W and $W^\vee$ and the pairing on V_λ (up to a similitude factor). Since they fix the Lagrangian decomposition $V_\lambda = W \oplus W^\vee$ up to a permutation of the two subspaces, this induces a permutation character $\chi_\lambda : H_\lambda \to \mu_2$ and similarly for $H(\mathcal{T}_\lambda)$. In the exceptional cases, by definition there exist elements w in H that $\chi_\lambda(w) \neq 1$; let us fix such w. For exceptional λ we thus obtain an exact sequence

$$0 \to \tilde{H} \to H \to \mu_2 \to 0.$$

The kernel $\tilde{H}$ is a subgroup of the group $G(W)$ of similitudes of V_λ that individually preserve the subspaces W and $W^\vee$, and $\tilde{H}$ contains $SL(W)$. In terms of a basis of W and a dual basis of $W^\vee$ the elements of $G(W)$ are of the blockdiagonal form $g = diag(A, \lambda \cdot A^{-t})$ for $A \in GL(W)$ and $\lambda \in k^*$. In fact $g \mapsto (A, \lambda)$ induces a group isomorphism $G(W) \cong GL(W) \times \mathbb{G}_m$, such that the projection onto the second factor $\mathbb{G}_m$ induces the similitude character $\mu : G(W) \to \mathbb{G}_m$. The determinant $\det_W : GL(W) \to \mathbb{G}_m$ on the first factor induces an isomorphism

$$(\det_W, \mu) : \; G(W)/SL(W) \;\cong\; \mathbb{G}_m \times \mathbb{G}_m .$$

The action of w by conjugation on $G(W)$ preserves the subgroups $SL(W)$ and $\tilde{H}$ of $G(W)$ for either $H = H_\lambda$ or $H = \tilde{H}(\overline{\mathcal{T}}_\lambda)$ such that the induced action on $\mathbb{G}_m \times \mathbb{G}_m \cong G(W)/SL(W)$ is inversion on the first factor and the identity on the second factor. This follows from $\det_W(wgw^{-1}) = \det_W(g)^{-1}$ and $\mu(wgw^{-1}) = \mu(g)$. The algebraic group $H/SL(W)$, as a subgroup of $G(W)/SL(W)$, therefore is a closed subgroup of $\mathbb{G}_m \times \mathbb{G}_m$. Since $\tilde{H}/SL(W)$ is contained in the first factor $\mathbb{G}_m$, the quotient $Q = \tilde{H}/SL(W)$ is either $\mathbb{G}_m$ or a finite cyclic group. In the first case Q is the full commutator group of $H/SL(W)$. But in both cases, the factor commutator group H^{ab} as a diagonalizable group is isomorphic to a direct product of a finite torsion group and a torus of rank ≤ 1. Its rank is nonzero if and only if the similitude $\mu : H \to \mathbb{G}_m$ is nontrivial and thus surjective, i.e. for the cases $H = H(\overline{\mathcal{T}}_\lambda)$ resp. $r(\lambda) \neq 0$ for $H = H_\lambda$. Recall that the subgroup $Pic^0(H)$ of $Pic(H) = X^*(H^{ab})$ is the annihilator of a cocharacter $\mathbb{G}_m \to H$ that is induced by the embedding $i : \mathbb{G}_m \cong H_1 \to H_n$ composed with the surjection $H_n \twoheadrightarrow H$. Since $\mu \circ i$ is surjective unless $\mu : H \to \mathbb{G}_m$ is trivial, it easily follows that $Pic^0(H)$ is the torsion subgroup $Pic(H)_{tor}$ of $Pic(H)$, and this group $Pic^0(H)$ only depends on the equivalence $\lambda/\sim$ of λ and is the same for $H = H_\lambda$ and $H(\overline{\mathcal{T}}_\lambda)$. We make this more explicit in the next section.

To study the equivalence class $\lambda/\sim$, we consider normalised representatives λ. Twisting λ by the ath power of the Berezin B, in the pairing $\mu : X_\lambda \times X_\lambda \to B^r$ the character $\mu = B^r$ changes to $\mu \otimes B^{2a} = B^{r+2a}$. Hence one can choose a *normalised* representative λ in its class so that $r = r(\lambda)$ is one or zero. In the first case $r = 1$ the Tannakian category $\overline{\mathcal{T}}_\lambda$ is generated by X_λ, hence $H_\lambda = H(\overline{\mathcal{T}}_\lambda)$ contains the group Z of scalar homotheties of V_λ. In the second case $r = 0$, the similitude character μ of H_λ is trivial. Then it is easy to see that $H(\overline{\mathcal{T}}_\lambda) = Z \cdot H_\lambda$, and $Z \cap H_\lambda = \{\pm id_{V_\lambda}\}$. Hence $H(\overline{\mathcal{T}}_\lambda) = Z \cdot H_\lambda$ holds for any exceptional λ. Since therefore $H(\overline{\mathcal{T}}_\lambda)$ contains the group Z of all diagonal matrices, it is easy to see that we can modify w by an element of $\tilde{H}(\overline{\mathcal{T}}_\lambda)$ (without changing its conjugation action on S, but possibly the sign of $\chi_\lambda(w)$) such that $w^2 = id_{V_\lambda}$ holds. As a consequence, in the exceptional cases for $Q = Q_\lambda \subseteq \mathbb{G}_m$ this implies

$$H^{ab}(\overline{\mathcal{T}}_\lambda) \cong (Q/Q^2) \times \mu_2 \times \mathbb{G}_m.$$

Notice Q/Q^2 is trivial if Q is finite of odd order or $Q \cong \mathbb{G}_m$, and it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ otherwise.

For $H(\langle B\rangle) = \mathbb{G}_m$ the Tannakian subcategory $\langle B^r\rangle$ generated by B^r gives rise to an r-fold covering $H(\langle B\rangle) \to H(\langle B^r\rangle)$, i.e. $H(\langle B^r\rangle)$ is the quotient of $H(\langle B\rangle) \cong \mathbb{G}_m$ by the unique cyclic subgroup of order r. In a similar vein we can recover H_λ from $H(\overline{\mathcal{T}}_\lambda)$. This is seen as follows: The inclusion

$$H(\overline{\mathcal{T}}_\lambda) \hookrightarrow H_\lambda \times H(\langle B\rangle)$$

composed with both projections, defines surjective maps. Furthermore (h, t) for $h \in H_\lambda$ and $t \in k^*$ is in $H(\overline{\mathcal{T}}_\lambda)$ if and only if $t^r = \mu(h)$ holds, where $\mu = \mu_\lambda$ is the similitude character defined on H_λ and $r = r(\lambda)$ is defined by $\mu_\lambda = B^r$. Thus $H(\overline{\mathcal{T}}_\lambda)$ is a fibre product of H_λ and $\mathbb{G}_m$ by μ and the r-th power map, and H_λ is obtained from its r-fold covering group $H(\overline{\mathcal{T}}_\lambda)$ as the quotient by the cyclic subgroup of order r that is contained in the second factor $H(\langle B\rangle) \cong \mathbb{G}_m$.

From the discussion above of the exceptional case we immediately derive the next two Lemmas 11.1 and 11.2.

Lemma 11.1 *Suppose that the class $\lambda/\sim$ is exceptional. Then either for the normalized representative λ the Tannaka group H_λ has trivial similitude character (if r is even), or (understood nonexclusively) for the normalized representative λ the group $H(\overline{\mathcal{T}}_\lambda)$ is isomorphic as an algebraic group to $(GL(\dim(V_\lambda)) \cdot \mu_2) \times \mathbb{G}_m$, so it has two connected components and $Pic^0(H(\overline{\mathcal{T}}_\lambda))$ is isomorphic to μ_2.*

Lemma 11.2 *For exceptional classes $\lambda/\sim$ the group $Pic^0(H(\overline{\mathcal{T}}_\lambda)) = Pic^0(H_\lambda)$ is a nontrivial two-torsion group of rank ≤ 2.*

Remark 11.3 The object I realizing the surjective projection $\chi_\lambda : H_\lambda \to \mu_2 = \langle w\rangle$ corresponds to an element with the properties in Lemma E.1. Indeed, I appears as an indecomposable constituent of superdimension 1 in $L(\lambda) \otimes L(\lambda)^\vee$ that is not isomorphic to the trivial representation. This follows immediately from the description of $V_\lambda \cong Ind_{H_1}^H(W)$ as an induced representation.

We summarize the results of this section in the following theorem.

Theorem 11.4 *In the (NSD) resp. in the regular (SD)-cases the groups H_λ are isomorphic to $GL(V_\lambda)$, resp. to $Kern(sign : G(V_\lambda, \langle ., .\rangle) \to \mu_2)$, i.e. $GSO(V_\lambda)$ or $GSp(V_\lambda)$, if $\ell(\lambda) \neq 0$, and to $SL(V_\lambda)$, resp. to $SO(V_\lambda)$ or $Sp(V_\lambda)$ if $\ell(\lambda) \neq 0$. So in these cases H_λ is connected and $H_\lambda/G_\lambda \cong H_\lambda^{ab}$. In the exceptional (SD)-cases the groups H_λ are not connected and there exists a nontrivial homomorphism $\chi_\lambda : H_\lambda \to \mu_2$. The kernel $\tilde{H}_\lambda$ is a subgroup that contains $SL(W)$ such that $S = \tilde{H}_\lambda/SL(W)$ becomes an algebraic subgroup of $\mathbb{G}_m^2$ of dimension one or two, depending on whether $\ell(\lambda) = 0$ or $\ell(\lambda) \neq 0$ (or, equivalently, $r = 0$ or $r \neq 0$).*

11.8 The full Tannaka group

Now consider the full tannakian category $\bar{\mathcal{T}}_n$. Note that

$$H_n \hookrightarrow \prod_{\lambda/\sim} H_\lambda .$$

As in the introduction we fix an isomorphism μ_B between the Tannaka group $H_B := H_\lambda$ of the twisted Berezin and the multiplicative group $\mathbb{G}_m$. Recall further that μ_λ was given by $\det_\lambda$ in the (NSD)-case and the similitude character in the (SD)-case. Then H_n is a subgroup of the infinite fibre product defined by the elements $h = (h_\lambda)_{\lambda/\sim}$ in $\prod_{\lambda/\sim\, \in Y_0^+(n)} H_\lambda$ that satisfy $\mu_\lambda(h_\lambda) = \mu_B(h_B)$ for all λ. The induced fibre homomorphism $\mu : H_n \to \mathbb{G}_m$ defined by μ_B is surjective. Its kernel $\tilde{H}_n = Ker(\mu : H_n \to \mathbb{G}_m)$ contains the projective limit G_n of the derived groups of the connected component of the H_λ as a normal subgroup. For H_n^{ab} determined by $X^*(H_n^{ab}) = Pic(H_n) = Pic^0(H_n) \times \mathbb{Z}$ and $X^*(\tilde{H}_n^{ab}) \cong Pic^0(H_n)$, as explained above, our computations imply

Corollary 11.5 *H_n^{ab} is the factor commutator group of H_n/G_n, and the commutator subgroup M of H_n/G_n is a pro-diagonalizable group.*

Proof Although only left exact, projective limits preserve exactness if the so called Mittag-Leffler conditions are satisfied. For a projective limit of algebraic groups as in our case, these conditions hold. Thus the projective limit H_n contains the projective limit G_n as a normal subgroup, and the quotient group H_n/G_n admits the projective limit H_n^{ab} as quotient group such that the remaining kernel M in H_n/G_n is the projective limit of the diagonalizable algebraic groups $\prod_{\lambda/\sim \in I}(Q_\lambda)^2$ extended over all finite subsets I of the set of exceptional classes $\lambda/\sim$. Here the Q_λ are the diagonalizable groups from Sect. 11.7. □

Remark If we knew that the characters χ_λ of H_λ (attached to the exceptional classes $\lambda/\sim$) were linear independent on H_n, this would easily imply that $Pic^0(H_n)$ is a two torsion group. To prove linear independency of the χ_λ, by Schur's lemma amounts to show that every finite tensor product $\bigotimes_\lambda X_\lambda$ of simple objects attached to exceptional classes $\lambda/\sim$ is a simple object. However, in absence of such stronger results, we have to be content with the following weaker statement.

Corollary 11.6 *$Pic^0(H_n)$ is a 2-power torsion group (possibly infinite), and this group is nontrivial if and only if exceptional classes $\lambda/\sim$ exist.*

Proof By our previous computations it suffices to consider the contribution of the exceptional classes. Let λ_i, $i \in I$, denote a finite set I of exceptional weights λ_i. Let H_I denote the Tannaka group attached to the Tannakian category $\overline{\mathcal{T}}_I$ that is generated by B and the objects X_{λ_i} for $i \in I$. Notice $H_I \subset \prod_{i\in I} H(\overline{\mathcal{T}}_{\lambda_i})$, and $\prod_{i\in T} \mu_i$ defines a character μ_I on H_I, factorizing over the quotient group H_I^{ab}. To prove our assertion it is enough to show that the kernel of $\mu_I : H_I^{ab} \to \mathbb{G}_m$ is a finite torsion group annihilated by $2^{\#I}$. For each exceptional class $\lambda/\sim$, the group $H(\overline{\mathcal{T}}_\lambda)$ contains in its commutator group the perfect group $SL(W_{\lambda_i})$. Hence we may replace H_I by its image D in $\prod_{i\in I} D_i$, for $D_i = Kern(\mu_i : H_{\lambda_i} \to \mathbb{G}_m)/SL(W_{\lambda_i})$, as is easy to see. Each group D_i for $i \in I$ is a generalized dihedral group, i.e. a semidirect product

$$D_i = S_i \cdot \langle w_i \rangle, \quad w_i^2 = 1$$

for an algebraic group S_i of multiplicative type, such that the involution w_i acts on S_i by conjugation: $w_i s_i w_i = s_i^{-1}$. As a subgroup of $\prod_{i\in I} D_i$ the group D admits an exact sequence

$$0 \to S \to D \to \prod_{i\in I} \langle w_i \rangle$$

where S is a subgroup of $\prod_{i\in I} S_i$. The image W of D is a subvectorspace of $\prod_{i\in I} \langle w_i \rangle$, considered as a vector space over the prime field of characteristic 2. The latter may be identified with the power set of I, so that the elements of W may be considered as subsets $J \subseteq I$ of I. Let $\chi_J : I \to \mathbb{Z}$ thus denote the characteristic function of $J \subset I$ in this sense. Any $\mathbb{Z}$-valued function f on I can be considered as an endomorphism of $\prod_{i\in I} S_i$ via $f((s_i)_{i\in I}) = (f(i) \cdot s_i)_{i\in I}$. For $J \in W$ with representative $w \in D$, the commutator of w with elements of S induces an endomorphism of S. Considered as a subgroup of $\prod_{i\in I} S_i$, this is the endomorphism attached to the integer-valued function $f = 2\chi_J$ on I. By definition this endomorphism of $\prod_{i\in I} S_i$ preserves the subgroup S. Therefore any element of S in the image of the endomorphism $\sum_{J\in W} 2\chi_J$ of S is contained in the commutator group of D. By Lemma 11.7, the factor commutator group of D thus is a finite group annihilated by $2^{\dim(W)}$, so also by $2^{\#I}$. Notice, Lemma 11.7 can be applied since $\bigcup_{J\in W} = I$ is satisfied, as follows from the fact that all projections $H_I \mapsto H_{\lambda_i}$, $i \in I$ are surjective. □

Lemma 11.7 *If $\bigcup_{J\in W} J = I$ holds, then as endomorphism of S we obtain:*

$$\sum_{J\in W} 2\chi_J \; = \; 2^{\dim(W)} \cdot id_S.$$

Proof For a basis $J_1, \ldots, J_d$ of the $\mathbb{F}_2$-vectorspace W any $J \in W$ can be uniquely written in the form $J = \sum_{\nu=1}^{d} a_\nu J_\nu$ for $a_\nu \in \{0, 1\}$. For $i \in I$ fixed, we may reorder the basis such that $i \in J_1, \ldots, J_k$ and $i \notin J_{k+1}, \ldots, J_d$. Notice $k = k(i) \geq 1$ holds for all $i \in I$ by $\bigcup_{J\in W} J = I$. Hence $\chi_J(i) = 1$ if and only if $a_1 + \cdots + a_k$ is odd. Since $k \geq 1$, therefore there exist precisely $\frac{1}{2}2^k \cdot 2^{d-k} = 2^{\dim(W)-1}$ vectors J in W with $\chi_J(i) = 1$. The sum of the endomorphisms $2\chi_J$ of S, for $J \in W$, therefore gives $2^{\dim(W)}$ times the identity of S. □

12 A conjectural picture

12.1 Equivalent conjectures

We conjecture that all SD-cases are regular:

Conjecture 12.1 *$G_\lambda = SL(V_\lambda)$ resp. $G_\lambda = SO(V_\lambda)$ resp. $G_\lambda = Sp(V_\lambda)$ according to whether X_λ satisfies (NSD) respectively (SD) with either X_λ being even respectively odd.*

Lemma 12.2 *The following are equivalent:*

(1) *Conjecture 12.1 holds.*
(2) *The module $I = I_\lambda$ is trivial.*
(3) *Any invertible object I in $\overline{\mathcal{T}}_n$ is represented in $\mathcal{T}_n^+$ by a power of the Berezin determinant.*
(4) *Conjecture 12.8 holds, i.e. every special module is trivial.*

Proof By Theorem 11.4 the Picard group is generated by Berezin powers if there are no exceptional SD-cases, i.e. $I \cong \mathbf{1}$. □

We discuss a possible approach to proving $I \cong \mathbf{1}$ in "Appendix E".

12.2 An element of the Picard group of $\mathcal{T}_{2|2}/\mathcal{N}$

The assertion (3) from Lemma 12.2 cannot hold for $\mathcal{T}_n$ instead of $\mathcal{T}_n^+$. In the category $\mathcal{T}_2$ consider the indecomposable representation I, defined uniquely up to isomorphism by the nontrivial extension

$$0 \longrightarrow [0,-1] \longrightarrow I \longrightarrow \mathbf{1} \longrightarrow 0$$

This extension is realized in the Kac module of the trivial representation

$$K(\mathbf{1}) = \begin{pmatrix} \mathbf{1} \\ [0,-1] \\ Ber^{-2} \end{pmatrix}$$

Since $\mathrm{sdim}(K(\mathbf{1})) = 0$, I has superdimension -1, so its parity shift has $\mathrm{sdim}(\Pi I) = 1$. Clearly $I \otimes I^\vee \cong \mathbf{1} \oplus N$ with $\mathrm{sdim}(N) = 0$. We show that N is negligible. For this we can pass to the homotopy category $\mathcal{H}o\mathcal{T}$ attached to the lower parabolic $\mathfrak{p}^-$ [38]. The kernel of the homotopy functor $\pi : \mathcal{T}_2 \to \mathcal{H}o\mathcal{T}$ consists of the Kac objects, i.e.

those modules with a filtration by Kac-modules. The short exact sequence for $K(\mathbf{1})$ induces in the tensor triangulated category $\mathcal{H}oT$ an exact triangle

$$Ber^{-2} \longrightarrow K(\mathbf{1}) \longrightarrow I \longrightarrow Ber^{-2}[1].$$

Since $K(\mathbf{1})$ is zero in $\mathcal{H}oT$, this gives the identification $I \cong Ber^{-2}[1]$. In particular

$$I \otimes I^\vee \cong Ber^{-2}[1] \otimes (Ber^{-2})^\vee[1] \cong \mathbf{1}[2].$$

Now suppose N would not be negligible. Then $N \cong N_1 \oplus N_2$ with $\mathrm{sdim}(N_1) \neq 0$. In particular $\pi(N_1) \neq 0$, a contradiction since π is a symmetric monoidal functor and $\mathbf{1}[2]$ is indecomposable.

Corollary 12.3 *$I \otimes I^\vee = \mathbf{1} \oplus N$ and N is negligible. Therefore I and its parity shift ΠI define elements of the Picard group of $\mathcal{T}_2/\mathcal{N}$.*

12.3 An application

The conjectural structure theorem would have the following consequences.

Corollary 12.4 *For given $L = L(\lambda)$ in $\mathcal{T}_n$ and $r \in \mathbb{Z}$ there can exist at most one summand M in $L \otimes (Ber^r \otimes L^\vee)$ with the property $\mathrm{sdim}(M) = \pm 1$. If it exists then $M \cong Ber^r$.*

Proof of the corollary We can assume that L is maximal atypical. Then $\mathbf{1}$ is a direct summand of $L \otimes L^\vee$ and hence Ber^r is a direct summand of $L \otimes (Ber^r \otimes L^\vee)$. Hence it suffices to show that $\mathbf{1}$ is the unique summand M of $L \otimes L^\vee$ with $\mathrm{sdim}(M) = \pm 1$. Equivently it suffices to show that $V_\lambda \otimes V_\lambda^\vee$ contains no one-dimensional summand except $\mathbf{1}$. This now follows from conjecture 12.1 using the well known fact that $st \otimes st^\vee$ for the standard representation st of $SL(V)$, $SO(V)$, $Sp(V)$ contains only one summand of dimension 1. □

12.4 The Tannaka groups H_λ revisited

The following theorem is an immediate consequence of theorem 11.4.

Theorem 12.5 *Assuming Conjecture 12.1, the Tannaka groups H_λ of X_λ are the following:*

(1) *NSD non-basic: $H_\lambda = GL(V_\lambda)$.*
(2) *NSD basic: $H_\lambda = SL(V_\lambda)$.*
(3) *SD, proper selfdual, $\mathrm{sdim}(L(\lambda_{basic})) > 0$: $H_\lambda = SO(V_\lambda)$.*
(4) *SD, proper selfdual, $\mathrm{sdim}(L(\lambda_{basic})) < 0$: $H_\lambda = Sp(V_\lambda)$.*
(5) *SD, weakly selfdual, $\mathrm{sdim}(L(\lambda_{basic})) > 0$: $H_\lambda = GSO(V_\lambda)$.*
(6) *SD, weakly selfdual, $\mathrm{sdim}(L(\lambda_{basic})) < 0$: $H_\lambda = GSp(V_\lambda)$.*

In each case the representation V_λ of H_λ coming from X_λ corresponds to the standard representation. In the GL case the determinant comes from a (nontrivial) Berezin power.

Note that a basic representation of SD type always satisfies $L \cong L^\vee$. In the (SD) case $\ell(\lambda) = 0$ if and only if $L(\lambda) \simeq L(\lambda)^\vee$. In the (NSD)-case $\ell(\lambda) = 0$ if and only if λ is basic.

12.5 Special modules

We discuss a conjecture which would show $I \cong \mathbf{1}$.

Definition 12.6 An indecomposable module V in $\mathcal{T}_n^+$ with $\mathrm{sdim}(V) = 1$ will be called *special*, if $V^* \cong V$ and $H^0(V)$ contains $\mathbf{1}$ as a direct summand.

For special modules

$$DS(V) \cong \mathbf{1} \oplus N$$

holds for some negligible module N, since $\mathrm{sdim}(DS(V)) = \mathrm{sdim}(V)$. This also implies

$$H_D(V) = \mathbf{1} \oplus N.$$

Lemma 12.7 *Suppose $V \cong V^* \cong V^\vee$ and $DS(V) \cong \mathbf{1} \oplus N$ holds for some negligible module N. Then V is special.*

Proof The assumptions imply that there exists a unique integer ν for which $H^\nu(V)$ is not a negligible module. Since $H^\nu(V)^\vee \cong H^{-\nu}(V^\vee)$, the assumption $V \cong V^\vee$ implies $\nu = 0$. Hence $H^0(V) = \mathbf{1} \oplus N$ for some negligible N. □

Conjecture 12.8 *Up to a parity shift, any special module V in $\mathcal{T}_n^+$ is isomorphic to the trivial module* $\mathbf{1}$.

13 The Picard group of $\overline{\mathcal{T}}_n$

We study the determinant $\det(X_\lambda)$ in this section.

13.1 The invariant $\ell(\lambda)$

As one easily shows, for any object X of $\mathcal{T}_n$

$$\det(B^m \otimes X) = B^{m \cdot \mathrm{sdim}(X)} \otimes \det(X).$$

Hence to determine I_λ we may assume $\lambda_n = 0$. So let us assume this for the moment. Then, for a maximal atypical weight λ with the property $\lambda_n = 0$, let $S_1, \ldots, S_k$ denote its corresponding sectors, from left to right. If $i = 1, \dots, k-1$ let $d_i = dist(S_i, S_{i+1})$ denote the distances between these sectors and $r(S_i)$ denotes the rank of S_i, then $\sum_{i=1}^{k} r(S_i) = n$. Furthermore $d = \sum_{i=1}^{k} d_i = 0$ holds if and only if the weight λ is a basic weight. Recall, if we translate S_2 by shifting it d_1 times to the left, then shift S_3 translating it $d_1 + d_2$ to left and so on, we obtain a basic weight. This basic weight is called the *basic weight associated to* λ. The weighted total number of shifts necessary to obtain this associated basic weight by definition is the *integer*

$$\ell(\lambda) := \sum_{i=1}^{k} \mathrm{sdim}(X_{\lambda_i}) \cdot \left(\sum_{j<i} d_j \right)$$

where $L(\lambda_i) \in \mathcal{R}_{n-1}$ denote the irreducible representations associated to the derivatives $S_1 \dots \partial S_i \dots S_k$. By [51] [36, Section 16] $\mathrm{sdim}(X_{\lambda_i}) = \frac{r_i}{n} \cdot \mathrm{sdim}(X_\lambda)$ holds for $r_\nu = r(S_\nu)$, which allows to rewrite this in the form

$$\ell(\lambda) = n^{-1}\, \mathrm{sdim}(X_\lambda) D(\lambda)$$

where $D(\lambda)$ is the *total number of left moves* needed to shift the support of the plot λ into the support of the associated basic plot λ_{basic}, i.e. the integer

$$D(\lambda) \ = \ \sum_{\nu=1}^{k} r_\nu \cdot \left(\sum_{\mu<\nu} d_\mu \right).$$

Now, to remove our temporary assumption $\lambda_n = 0$ and hence to make the formulas above true unconditionally, we have to introduce the additional terms $d_0 = \lambda_n$ (for $\mu = 0$) in the formulas above. For further details on this see [36, Section 25]. We remark that in the following we also write $D(L)$ instead of $D(\lambda)$ for the irreducible representations $L = L(\lambda)$ and similarly $\ell(L)$ instead of $L(\lambda)$.

13.2 *Pic*^{0}

We return to indecomposable objects $I \in \mathcal{T}_n^+$ representing invertible objects of $\overline{\mathcal{T}}_n$.

Since $I \otimes I^\vee \cong \mathbf{1} \oplus$ negligible objects, we obtain

$$\omega(I, t)\omega(I^\vee, t) = \omega(I \otimes I^\vee, t) = 1.$$

Indeed, the functor ω annihilates negligible objects. For the Laurent polynamial $\omega(I, t)$ this now implies

$$\omega(I, t) = t^\nu$$

for some integer $\nu \in \mathbb{Z}$ which defines the degree $\nu(I) = \nu$. Obviously this degree $\nu(I)$ induces a homomorphism $Pic(\mathcal{R}_n) \to \mathbb{Z}$ of groups by $I \mapsto \nu = \nu(I) \in \mathbb{Z}$ and gives an exact sequence

$$0 \longrightarrow Pic^0(\overline{\mathcal{T}}_n) \longrightarrow Pic(\overline{\mathcal{T}}_n) \xrightarrow{\nu} \mathbb{Z}$$

with kernel $Pic^0(\overline{\mathcal{T}}_n)$. Clearly $\nu(B) = n$, hence the next lemma follows.

Lemma 13.1 *The intersection of $Pic^0(\overline{\mathcal{T}}_n)$ with the subgroup generated by the normalized Berezin B is trivial.*

Lemma 13.2 *For any irreducible object X in $\mathcal{T}_n^+$ the invertible element* $\det(X) \in \mathcal{T}_n$ *has the property*

$$\nu(\det(X)) = \mathrm{sdim}(X) \cdot D(X) = \ell(X) \cdot n.$$

In particular, the image of the homomorphism ν contains $n \cdot \mathbb{Z}$.

Proof Fix some $X = X_\lambda$. We can assume λ to be maximally atypical. The functor $\omega : \mathcal{T}_n^+ \to gr{-}vec_k$ is a tensor functor. Hence $\nu(\det(X)) = \nu(\det(\omega(X))$. Hence

$$\nu(\det(X)) = \sum_i i \cdot a_i \qquad (*)$$

for $\omega(X,t) = \sum_i a_i t^i$. By [36, Lemma 25.2] $\omega(X, t^{-1}) = t^{-2D(\lambda)}\omega(X,t)$ and hence $\omega(X_{basic}, t) = \omega(X_{basic}, t^{-1})$, the latter because of $D(X_{basic}) = 0$. So $a_i = a_{-i}$ holds for basic X, and formula (*) implies $\nu(\det(X_{basic})) = 0$. From $\omega(X,t) = t^{D(X)}\omega(X_{basic}, t)$ and $\mathrm{sdim}(X_{basic}) = \mathrm{sdim}(X)$, again by (*) we therefore obtain

$$\sum_i i\, a_i = \sum_i i\, a_{basic,i} + D(\lambda) \cdot \sum_i a_i = D(\lambda)\, \mathrm{sdim}(X).$$

Note that $X \in \mathcal{T}_n^+$ has superdimension ≥ 0, hence $\omega(X,1) = \sum a_i$ is the superdimension of X (not only up to a sign). □

Since $\omega(L(\lambda), t)t^{-D(\lambda)}$ is invariant under $t \mapsto t^{-1}$ for irreducible $L = L(\lambda)$, we also obtain

Corollary 13.3 $d\, log(\omega(L,t))|_{t=1} = D(L)$.

Corollary 13.4 *We have* $\det(X) \otimes B^{-\ell(X)} \in Pic^0(\overline{\mathcal{T}}_n^+)$, *i.e.*

$$\det(X) \in Pic^0(\overline{\mathcal{T}}_n) \times B^{\mathbb{Z}}$$

for irreducible $X \in \mathcal{T}_n^+$.

Example 13.5 For $GL(2|2)$ we obtained (up to parity shifts) in [37] the formula $S^i \otimes S^i = Ber^{i-1} \oplus M$ for some module M of superdimension 3. Since $\mathrm{sdim}(S^i) = 2$, $\det(S^i) = Ber^{i-1} \oplus$ negligible. Indeed for S^i we obtain $\ell([i,0]) = r_1 d_0 + r_2 d_1$ where r_i denotes the rank of the i-th sector. Clearly $r_1 = r_2 = 1$ and $d_0 = 0$ and $d_1 = i - 1$, hence $\ell([i,0]) = i - 1$.

14 The determinant of an irreducible representation

We now compute the determinant of irreducible representations. The computation uses the passage to the stable category with its triangulated structure. This determinant calculation determines in all regular cases (along with the results of Sect. 11) the full Picard group of H_λ.

14.1 The full even category $\mathcal{T}^{ev}$

Since a svectorspace is the sum of an even and odd subspace, we have $svec_k = vec_k \oplus \Pi(vec_k)$ as a decomposition of abelian categories. We say, an object X of $\mathcal{T}$ is even resp. odd if $\omega(X)$ is in vec_k resp. $\Pi(vec_k)$. In terms of the Hilbert polynomial $\omega(X,t)$ defined in Sect. 4.2 even means that all t-powers are even. Let $\mathcal{T}^{ev}$ and $\mathcal{T}^{odd}$ denote the corresponding full subcategories. In [36, Section 24] it is shown that simple objects in $\mathcal{T}$ are always even or odd, hence

$$\mathcal{T}^{+} \subset \mathcal{T}^{ev}.$$

Although D^n is not an exact functor, exact sequences in $\mathcal{T}$ become exact hexagons in $\mathcal{T}_0$. [Sometimes it is useful that in a certain sense we need not distinguish between D^{n-1} and D^n since $D : \overline{\mathcal{T}}_1 \to \mathcal{T}_0$ is faithful. This refines the notion of even/odd if the D^{n-1}-image is semisimple and even resp. odd.] Extensions of even (odd) objects in $\mathcal{T}$ are even (odd) objects in $\mathcal{T}$. Obviously $\mathcal{T}^{ev}$, as a full karoubian subcategory of $\mathcal{T}$, is closed under extensions, retracts and tensor products and Tannaka duals. In particular we have stability with respect to Schur functors.

We consider now the semisimplification of $\mathcal{T}^{ev}$ with the same method as in Sect. 5. Here we however use the Dirac tensor functor D (see Sect. 4.3) instead of DS (note they agree on $\mathcal{T}^{+}$). Iterated n times it factorizes over the additive quotient category $\mathcal{T}^{ev} \to \mathcal{A}$ defined by dividing through the ideal of all morphisms that factorize over null objects (objects whose indecomposable summands have superdimension zero). i.e. $D^n : \mathcal{T}_n^{ev} \to \mathcal{T}_0^{ev}$. We define

$$\omega = D^n \circ s$$

via a section s of the semisimplification functor $\nu : \mathcal{A} \to \overline{\mathcal{T}}^{ev}$, i.e. $\nu \circ s = id_{\overline{\mathcal{T}}}$ whose existence follows from [2]. The k-linear tensor functor ω is an exact functor since $\overline{\mathcal{T}}^{ev}$ is semisimple. Thus it defines a superfibre functor $\omega : \overline{\mathcal{T}}^{ev} \to svec_k$ with values in vec_k.

Since for indecomposable objects X of $\mathcal{T}^{ev}$ the space $\omega(X)$ has dimension $sdim(X) > 0$ (unless $sdim(X) = 0$ and X is negligible), the quotient $\overline{\mathcal{T}}^{ev}$ defines a Tannakian category. Hence, as a tensor category $\overline{\mathcal{T}}^{ev}$ is equivalent to $Rep_k(H^{ev}, \varepsilon)$ for some affine (pro)reductive group scheme H^{ev} over k.

Remark 14.1 All in all we attached to the category $\mathcal{T}$ three different semisimple super-tannakian categories:

$$\mathcal{T}^+/\mathcal{N},\ \mathcal{T}^{ev}/\mathcal{N},\ \mathcal{T}/\mathcal{N}.$$

The inclusion $\mathcal{T}^+ \subset \mathcal{T}^{ev}$ is strict. Already for $GL(1|1)$, $\mathcal{T}^{ev}$ contains ZigZag modules of length $2m+1$ for $m \in \mathbb{N}$ [34] which have nonvanishing superdimension. In fact for $GL(1|1)$ the quotients $\mathcal{T}^{ev}/\mathcal{N}$ and $\mathcal{T}/\mathcal{N}$ are isomorphic. The relationship betweeen $\mathcal{T}$ and $\mathcal{T}^{ev}$ and their corresponding semisimple quotients is unclear as it is not obvious how to find indecomposable objects of nonvanishing superdimension which are mixed (i.e. neither even nor odd).

14.2 The stable category

Recall that $\mathcal{T}_n = \mathcal{T}$ is a k-linear tensor category and as an abelian category it is a Frobenius category. Associated to a Frobenius category $\mathcal{T}$ one defines its stable category $\mathcal{K}$ as a quotient category [33]. For the quotient functor

$$\alpha : \mathcal{T} \to \mathcal{K}$$

the objects of $\mathcal{K}$ are those of $\mathcal{T}$, but morphisms are equivalence classes of morphisms in $\mathcal{T}$. Two morphisms become equivalent if their difference is a morphism that factorizes over a projective module in $\mathcal{T}$. $\mathcal{K}$ is a triangulated category with a suspension functor $S(X) = X[1]$ such that $Ext^i_{\mathcal{T}}(X, Y) \cong Hom_{\mathcal{K}}(X, Y[i])$ holds for all $i \geq 0$ and α is a tensor functor. The α-image of an exact sequence in $\mathcal{T}$ induces a distinguished triangle in $\mathcal{K}$. Any distinguished triangle in $\mathcal{K}$ is isomorphic as a triangle to the α-image of an exact sequence in $\mathcal{T}$ [33].

Let $\mathcal{K}^{ev}$, $\mathcal{K}^{odd}$ denote the corresponding full subcategories of $\mathcal{K}$ corresponding to $\mathcal{T}^{ev}$, $\mathcal{T}^{odd}$. Similarly to the $\mathcal{T}$-case under the Dirac functor D exact triangles in $\mathcal{K}$ become exact triangles in $svec_k$. In $\mathcal{K}$ the following holds: If $X \to Y \to Z \to$ is a distigiuished triangle and X and Z are in $\mathcal{K}^{ev}$, then also Y. This follows since $\omega = D^n$ induces an exact hexagon from each exact sequence in $\mathcal{T}$ as in [36, Lemma 2.1] via $\omega = \omega^+ \oplus \omega^-$.

14.3 Determinants

Recall $\mathrm{sdim}(X) \geq 0$ for $X \in \mathcal{T}^{ev}$. Hence $\det(X) = \Lambda^{\mathrm{sdim}(X)}(X)$ is defined so that $\Lambda^{\mathrm{sdim}(X)+1}(X)$ is negligible. Here $L = \det(X)$ by definition is $\mathbf{1}$ if $sdim(X) = 0$. The image of $\det(X)$ under the functor $\omega = D^n$ defines an invertible object in the Tannakian subcategory generated by $\omega(X)$ in $\overline{\mathcal{T}}^{ev}$.

For exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{T}^{ev}$ we have $\det(Y) \cong \det(X) \otimes \det(Z)$ in $\mathcal{T}^{ev}$. The analogous assertion holds for a distinguished triangle $0 \to X \to Y \to Z \to$ with $X, Y, Z \in \mathcal{K}^{ev}$, simply by lifting this to an exact sequence $0 \to X' \to Y' \to Z' \to 0$ in $\mathcal{T}^{ev}$. For this notice, if X, X' in $\mathcal{T}^{ev}$ become isomorphic in $\mathcal{K}^{ev}$, then $\det(X)$ and $\det(X')$ become isomorphic on $\mathcal{K}^{ev}$. In particular, their images in the Tannakian representation category $\overline{\mathcal{T}}^{ev}$ of the reductive group H^{ev} become isomorphic.

Now consider the following special situation, where I in $\mathcal{T}$ has a filtration of length 3 whose graded pieces are a submodule S, a middle layer M and the quotient $T = I/M$ on top.

Lemma 14.2 *Suppose S, T are in $\mathcal{T}^{odd}$ and I is in $\mathcal{T}^{ev}$. Then M, $S[1]$ and $T[-1]$ are in $\mathcal{K}^{ev}$ and the following holds in $\mathcal{K}$ up to negligible summands:*

$$\det(M) \cong \det(I) \otimes \det(S[1]) \otimes \det(T[-1]).$$

Proof If S is odd/even in the stable category, then $S[1]$ is even/odd in the stable category, and conversely. Indeed $S[1]$ is represented by a quotient P/S in the representation category for a suitable projective/injective module P. To the exact sequence $0 \to S \to P \to P/S \to 0$ the functor $\omega = D^n$ attaches an exact hexagon for $\omega = \omega^+ \oplus \omega^-$ which immediately implies $\omega^{\pm}(S) \cong \omega^{\mp}(P/S)$ in $\mathcal{T}$ and then $\mathcal{K}$. From this the above assertion follows. We now have two distinguished triangles $I \to Y \to S[1] \to$ for suitable Y and $T[-1] \to M \to Y \to$ in $\mathcal{K}$. Here $S[1]$, $T[-1]$ and I are in $\mathcal{K}^{ev}$ by our assumptions. Therefore Y and then also M are in $\mathcal{K}^{ev}$. All objects $S[1]$, $T[-1]$, M, I and Y are represented by objects U, V, M', I', and Y'' in $\mathcal{T}^{ev}$ so that there are exact sequences $0 \to I' \to Y' \to U \to 0$ and $0 \to V \to M' \to Y'' \to 0$ in $\mathcal{T}$. Since all the superdimensions are ≥ 0, we conclude $\det(Y') \cong \det(I') \otimes \det(U)$ and $\det(M') \cong \det(V) \otimes \det(Y'')$. Since $\det(Y') \cong \det(Y'')$, $\det(I') \cong \det(I)$ and $\det(M') \cong \det(M)$ hold in $\mathcal{K}^{ev}$, in the stable category $\mathcal{K}^{ev}$ we obtain $\det(M) \cong \det(I) \otimes \det(S[1]) \otimes \det(T[-1])$. □

A symbolic way of writing.

For $Y \in \mathcal{K}^{ev}$ define $Y\langle \pm 1 \rangle = \Pi(Y)[\pm 1]$ in $\mathcal{K}^{ev}$. Then $Y\langle \pm 1 \rangle \cong Y \otimes \mathbf{1}\langle \pm \rangle$. For the iterated tensor powers $(\mathbf{1}\langle \pm m \rangle)$ of $(\mathbf{1}\langle \pm \rangle)$ one has $\mathbf{1}\langle n_1 \rangle \otimes \mathbf{1}\langle n_2 \rangle \cong \mathbf{1}\langle n_1 + n_2 \rangle$ for all $n_1, n_2 \in \mathbb{Z}$. For $X \in \mathcal{K}^{odd}$ put $\det(X) := \det(\Pi(X))^{\vee} \in \mathcal{K}^{ev}$. Using this definition, up to negligible objects the formula in the lemma above becomes

$$\det(I) \cong \det(S) \otimes \det(M) \otimes \det(T)\langle \mathrm{sdim}(T) - \mathrm{sdim}(S) \rangle.$$

For this notice $\det(\Pi(X)\langle \pm 1 \rangle) \cong \det(\Pi(X))\langle \pm\, \mathrm{sdim}(X) \rangle$. This formula follows from [20, Proposition 1.11]

$$\det(X \otimes Y) \cong det(X)^{\mathrm{sdim}(Y)} \otimes \det(Y)^{\mathrm{sdim}(X)}$$

applied for $Y = \mathbf{1}$ using $\mathrm{sdim}(\mathbf{1}\langle 1 \rangle) = 1$ and $X\langle 1 \rangle \cong X \otimes \mathbf{1}\langle 1 \rangle$.

14.4 Calculation of determinants

Theorem 14.3 *For any maximal atypical weight λ defining X_λ in $\mathcal{T}_n^+$, for $\lambda_n = 0$ the module* $\det(X_\lambda)$ *satisfies*

$$\det(X_\lambda) \; = \; B^{\ell(\lambda)} \, \oplus \, negligible.$$

In particular, for $\lambda_n = 0$ we have $\det(X_\lambda) = \mathbf{1}$ *if (and only if) the maximal atypical weight weight λ is a basic weight.*

Proof We prove this claim in $\mathcal{K}$ by a kind of induction, using the method of [36]. This requires a certain ordering of the maximal atypical simple representations, described in the section on the algorithms I, II, and III in [51] [36, Section 20]. For that we define an order on the set of cup diagrams for a fixed block such that the representations with completely nested cup diagrams (in our case the Ber powers) are the minimal elements.

In [51] [36, Section 20] it is also shown that for every maximal atypical irreducible module X there exists a negligible indecomposable object I of Lowey length 3 in $\mathcal{T}$ such that the socle $S \cong A$ and cosocle $T \cong A$ are isomorphic simple objects A, and the middle layer $M = X \oplus M'$ is a direct sum of simple objects such that A and all simple summands of M' are smaller than X with respect to the ordering. More precisely, this indecomposable module is one of the translation functors $F_i L^{\times\circ}$ of [36, Section 18]. Furthermore, it was shown that all simple objects in M have the same parity [36, Section 20]. Without restriction of generality we may therefore assume that $M \in \mathcal{T}^{ev}$ and $A \in \mathcal{T}^{odd}$ holds. Hence Lemma 14.2 implies that $\omega(\det(X))$ is a power of $\omega(B)$, by induction on X with respect to the mentioned ordering.

Concerning the start of this induction: the claim holds for groundstates X in the sense of [36, 51]. In the present situation for $GL(n|n)$ these are the powers B^k of B for $k \leq 0$. Since the groundstates are the start of the induction above, the determinant is a Ber-power. The specific power $\ell(\lambda)$ follows then from Corollary 13.4. ◻

Remark 14.4 In [36] we showed that if i is chosen correctly, one can find for given maximal atypical L an irreducible module $L^{\times\circ}$ such that the translation functor $F_i(L^{\times\circ})$ satisfies the conditions of the proof. In particular it contains L in the middle layer such that all other composition factors of $F_i(L^{\times\circ})$ are of lower order. For L with more then one segment we can choose i and $L^{\times\circ}$ in such a way that all composition factors have one segment less then L. We can now apply the same procedure to all the composition factors of $F_i(L^{\times\circ})$ with more then one segment. Iterating this we finally end up with a finite number of indecomposable modules where all composition factors have weight diagrams with only one segment. This procedure is called Algorithm I. In Algorithm II we decrease the number of sectors in the same way. Iterating we finally relate L to a finite number of maximal atypical representations with only one sector. Hence after finitely many iterations we have reduced everything to irreducible modules with one segment and one sector. This sector might not be completely nested. In this case we can apply Algorithm II to the internal cup diagram having one segment enclosed by the outer cup. If we iterate this procedure we will finally end up in a collection of Berezin powers.

15 The conformal group and low rank cases

In view of the relation with the conformal group G of the Lorentz metric we discuss ceretain cases of rank ≤ 4. The complexified Lie algebra $Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$ of the conformal group is isomorphic to the complex Lie algebra $\mathfrak{sl}(4)$. So the Lie superalgebras $\mathfrak{gl}(4|N)$ are of potential interest as supersymmetry algebras of conformal field theories and the finite dimensional representations L of these Lie superalgebras may serve as targets of fields $\psi : M \to L$ on certain supermanifolds M related to Minkowski space such that $Lie(G)$ acts on M by supervector fields. A covering of the Poincare group can be embedded into G, and in particular the universal covering $SL(2,\mathbb{C})$ of the Lorentz group $SO(1,3)$. The restriction of the representation L to the Lie subalgebra $Lie(SL(2,\mathbb{C}))$ decomposes into irreducible representations of the complex Lie algebra $\mathfrak{sl}(2)$ and their highest weights defines the underlying classical spin values of the L-valued fields. For physical reasons it seems relevant that these spins s are contained in the set $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$. In other words, the highest weights should not exceed 5. We refer to this as the spin condition.

The structure of tensor products of irreducible representation of $GL(4|N)$ resp. $SL(4|N)$ is controlled by the number $m = min(N,4)$. For $m < 4$ the information is encoded in the tensor products of irreducible representations of the reduced group $GL(m|m) \times GL(4-m)$. For $m = 4$, this reduced group has to be replaced by $GL(4|4) \times GL(N-4)$. So these leads us to consider $GL(n|n)$ for $n = 4$ and 3. The case $n = 2$ was completely discussed in Sect. 9. In the following we therefore list some interesting candidates for irreducible superrepresentations L where the spin condition is satisfied. In fact there exist only finitely many isomorphism classes of irreducible representation where the above spin condition is satisfied. The most prominent example is given by $L = S^1$ where only spin $s = 0$ and $s = \frac{1}{2}$ shows up in the restriction to the Lie algebra of the Lorentz group of this representation of dimension $\dim(L) = n^2 - 2$. In the case $n = 4$ the largest and most interesting example we give is probably the irreducible representation $L = [3,2,1,0]$ of $\mathfrak{gl}(4|4)$ of dimension $\dim(L) = 11{,}163{,}160$. Here all spins s of the restriction are in $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ and all these numbers occur. As already explained, the Tannaka groups H_λ related to the irreducible representations $L = L(\lambda)$ may perhaps show up in such theories as hidden approximate symmetry groups. $L(\lambda) = [3,2,1,0]$ defines a symmetric (SD)-case. The underlying group H_λ should be the group $SO(24)$ if this case is regular (if not G_λ would be $SL(12)$, but we could not exclude this). This case is the only case of our example where we could not exclude exceptional (SD)-case.

One remark for this section. For the convenience of physicists we replace here the groups H_λ by their compact inner forms H_λ^c. So we write $U(1)$ instead of $\mathbb{G}_m$ and $SU(k)$ instead of $SL(k)$, $Sp^c(2k)$ of $Sp(2k)$ etc. In fact, the tensor categories $\mathcal{T}_\lambda$ of the complex algebraic groups H_λ are isomorphic to the tensor categories of their compact inner forms H_λ^c.

The expected behaviour of the groups H_λ was summarized in Theorem 12.5 (all of which is proven except for the exceptional SD-case!). Here we discuss the $GL(3|3)$ and $GL(4|4)$-case.

Example 15.1 The $GL(3|3)$-case. For $n = 3$ the structure theorem on the G_λ holds unconditionally and therefore also the results on the H_λ. Here is a list of the nontrivial basic representations and their Tannaka groups. We automatically consider the possible parity shifted representation with positive superdimension here.

(1) $[2,1,0]$, sdim $= 6$, $H_\lambda = Sp^c(6)$.
(2) $[1,1,0]$, sdim $= 3$, $H_\lambda = SU(3)$.
(3) $[2,0,0]$, sdim $= 3$, $H_\lambda = SU(3)$.

(4) $[1,0,0]$, sdim $=2$, $H_\lambda = SU(2)$.

Twisting any of these with a nontrivial Berezin power gives the GL, GSO or GSp version. The appearing groups exhaust all possible Tannaka groups arising from an $L(\lambda)$.

Example 15.2 The $GL(4|4)$-case. Here the structure theorem for G_λ (and therefore the determination of H_λ) holds unconditionally for basic weights except for the case where $L(\lambda)$ is weakly selfdual with $[\lambda] \neq [3,2,1,0]$ by the following lemma:

Lemma 15.3 *The basic representations of (SD) type*

$$[3,1,1,0],\ [2,1,0,0],\ [2,2,0,0]$$

are regular (i.e. $I \cong \mathbf{1}$*).*

Proof For $[2,2,0,0]$ this follows from "Appendix E" and example E.8. It is enough to verify that $DS([2,2,0,0])$ does not contain a summand $L(\lambda_i)$ with $(\lambda_i)_{basic} = [2,1,0]$. The irreducible representations $[3,1,1,0]$ and $[2,1,0,0]$ have $k=3$ sectors each. However V_λ can only decompose under the restriction to G_λ if k is even. Alternatively note that we have embedded subgroups $Sp(6)$ and $Sp(6) \times SL(3)$ in $G_{[2,1,0,0]}$ and $G_{[3,1,1,0]}$ respectively which implies that G_λ cannot be $SL(3)$ or $SL(6)$. □

For $n=4$ there are 14 maximal atypical basic irreducible representations in $\mathcal{R}_4$, the self dual representations

$$\mathbf{1} = [0,0,0,0],\ S^1 = [1,0,0,0],\ [2,1,0,0],\ [2,2,0,0],\ [3,1,1,0],\ [3,2,1,0]$$

of superdimension $1, -2, -6, 6, -12, 24$ and the representations

$$S^2 = [2,0,0,0],\ S^3 = [3,0,0,0],\ [3,1,0,0],\ [3,2,0,0]$$

of superdimension $3, -4, 8, -12$ and their duals

$$[1,1,0,0],\ [1,1,1,0],\ [2,1,1,0],\ [2,2,1,0].$$

Here is a list of the nontrivial basic representations and their Tannaka groups. We automatically consider the possible parity shifted representation with positive superdimension here. Note that the result for the first example $[3,2,1,0]$ assumes that $G_\lambda \cong SO(24)$ (a consequence of the conjectural structure Theorem 12.1).

(1) $[3,2,1,0]$, sdim $=24$, $H_\lambda = SO(24)$ (conjecturally).
(2) $[3,2,0,0]$, sdim $=12$, $H_\lambda = SU(12)$.
(3) $[3,1,1,0]$, sdim $=12$, $H_\lambda = Sp^c(12)$.
(4) $[3,1,0,0]$, sdim $=8$, $H_\lambda = SU(8)$.
(5) $[3,0,0,0]$, sdim $=4$, $H_\lambda = SU(4)$.
(6) $[2,2,1,0]$, sdim $=12$, $H_\lambda = SU(12)$.

(7) $[2, 2, 0, 0]$, $\text{sdim} = 6$, $H_\lambda = SO(6)$.
(8) $[2, 1, 1, 0]$, $\text{sdim} = 8$, $H_\lambda = SU(8)$.
(9) $[2, 1, 0, 0]$, $\text{sdim} = 6$, $H_\lambda = Sp^c(6)$.
(10) $[2, 0, 0, 0]$, $\text{sdim} = 3$, $H_\lambda = SU(3)$.
(11) $[1, 1, 1, 0]$, $\text{sdim} = 4$, $H_\lambda = SU(4)$.
(12) $[1, 1, 0, 0]$, $\text{sdim} = 3$, $H_\lambda = SU(3)$.
(13) $[1, 0, 0, 0]$, $\text{sdim} = 2$, $H_\lambda = SU(2)$.

In addition there is the normalised Berezin representation B, with $[1, 1, 1, 1]$ and $\text{sdim} = 1$ and in the notation above

$$H_\lambda = U(1).$$

Twisting any of the basic representations above with a nontrivial Berezin power gives the GL, GSO or GSp versions. For $n = 4$ the appearing groups exhaust all possible Tannaka groups arising from an $L(\lambda)$.

Theorem 4.1 implies the following branching rules (the lower index indicates the superdimensions up to a sign):

(1) $DS([3, 2, 1, 0]_{24}) \cong [3, 2, 1]_6 \oplus [1, 0, -1]_6 \oplus [3, 0, -1]_6 \oplus [3, 2, -1]_6$
(2) $DS([3, 2, 0, 0]_{12}) \cong [3, 2, 0]_6 \oplus [1, -1, -1]_3 \oplus [3, -1, -1]_3$
(3) $DS([3, 1, 1, 0]_{12}) \cong [3, 1, 1]_3 \oplus [3, 1, -1]_6 \oplus [0, 0, -1]_3$
(4) $DS([3, 1, 0, 0]_8) \cong [3, 1, 0]_6 \oplus [0, -1, -1]_2$
(5) $DS([3, 0, 0, 0]_4) \cong [3, 0, 0]_3 \oplus [-1, -1, -1]_1$
(6) $DS([2, 2, 0, 0]_6) \cong [2, 2, 0]_3 \oplus [2, -1, -1]_3$
(7) $DS([2, 1, 0, 0]_6) \cong [2, 1, 0]_6$
(8) $DS([2, 0, 0, 0]_3) \cong [2, 0, 0]_3$
(9) $DS([1, 0, 0, 0]_2) \cong [1, 0, 0]_2$
(10) $DS([1, 1, 1, 1]_1) \cong [1, 1, 1]_1$

and $DS([n, 0, 0, 0]_4) \cong [n, 0, 0]_3 \oplus [-1, -1, -1]_1$ for all $n \geq 4$. We also have to consider the dual representations in the cases (2), (4), (5) and (8). We remark that even while most of the derivatives are not basic, they also give examples for $n = 3$ of representations which satisfy the spin condition.

Example 15.4 Consider $L(\lambda) = [6, 6, 1, 1]$. It is weakly selfdual with the dual representation $[1, 1, -4, -4] = Ber^{-5} \otimes [6, 6, 1, 1,]$. Its superdimension is 6. Since $\ell(\lambda) \neq 0$ and its basic weight $[2, 2, 0, 0]$ carries an even pairing, the associated Tannakagroup is therefore $H_\lambda = GSO(V_\lambda) \simeq GSO(6)$. This does not depend on the conjecture $I \simeq \mathbf{1}$. Indeed $DS([6, 6, 1, 1])$ does not contain an irreducible summand $L(\lambda_i)$ with $(\lambda_i)_{basic} = [2, 1, 0]$ and one can argue as in Lemma 15.3.

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Appendix A. Equivalences and derivatives

Recall that two weights $\lambda,\ \mu$ are equivalent $\lambda \sim \mu$ if there exists $r \in \mathbb{Z}$ such that $L(\lambda) \cong Ber^r \otimes L(\mu)$ or $L(\lambda)^\vee \cong Ber^r \otimes L(\mu)$ holds. We denote the equivalence classes of maximal atypical weights by $Y_0^+(n)$. The embedding $H_{n-1} \to H_n$ induces an embedding $G_{n-1} \to G_n$. Since inductively $G_{n-1} = \prod_{\lambda \in Y_0^+(n)} G_\lambda$, we need to understand the equivalence classes of weights and their behaviour under DS.

A.1 Plots

We use the notion of plots from [36, Section 13] to describe weight diagrams and their sectors. A plot λ is a map

$$\lambda : \mathbb{Z} \to \{\boxplus, \boxminus\}$$

such that the cardinality r of the fiber $\lambda^{-1}(\boxplus)$ is finite. Then by definition $r = r(\lambda)$ is the degree and $\lambda^{-1}(\boxplus)$ is the support of λ. The fiber $\lambda^{-1}(\boxplus)$ corresponds to those vertices of the weight diagram which are labeled by a $\vee$. An interval $I = [a, b]$ of even cardinality $2r$ and a subset K of cardinality of rank r defines a plot λ of rank r with support K. We consider formal finite linear combinations $\sum_i n_i \cdot \lambda_i$ of plots with integer coefficients. This defines an abelian group $R = \bigoplus_{r=0}^{\infty} R_r$ (gradn uu7887u87u. ation by rank r). In [36] we defined a derivation on R called derivative. Any plot can be written as a product of prime plots and we use the formula $\partial(\prod_i \lambda_i) = \sum_i \partial\lambda_i \cdot \prod_{j \neq i} \lambda_j$ to reduce the definition to the case of a prime plot λ. For prime λ let (I, K) be its associated sector. Then $I = [a, b]$. Then for prime plots λ of rank n with sector (I, K) we define $\partial\lambda$ in R by $\partial\lambda = \partial(I, K),\ I = [a, b]$ with $\partial(I, K) = (I, K)' = (I', K')$ for $I' = [a+1, b-1]$ and $K' = I' \cap K$. The importance of ∂ is that it describes the effect of DS on irreducible representations according to Theorem 4.1: If $L(\lambda)$ has sector structure $S_1 \ldots S_k$, $L(\lambda_i)$ has sector structure $S_1 \ldots \partial S_i \ldots S_k$. For a segment (I, K) with $I = [a, b]$ put

$$\int (I, K) = ([a-1, b+1], K \cup \{a-1\})$$

increasing the rank by 1. We call this *integrating*. Observe that $([a-1, b+1], K \cup \{a-1\})$ always defines a sector.

A.2 Duality

If $L = L(\lambda)$ is an irreducible maximal atypical representation in $\mathcal{R}_n$, its weight λ is uniquely determined by its plot. Let $S_1 \ldots S_2 \ldots S_k$ denote the segments of this plot. Each segment S_ν has even cadinality $2r(S_\nu)$, and can be identified up to a translation with a unique basic weight of rank $r(S_\nu)$ and a partition in the sense of [36, Lemma 21.4]. For the rest of this section we denote the segment of rank $r(S_\nu)$ attached to the dual partition by S_ν^*, hoping that this will not be confused with the contravariant functor $*$. Using this notation, Tannaka duality maps the plot $S_1 \ldots S_2 \ldots S_k$ to the plot $S_k^* \ldots S_2^* \ldots S_1^*$ so that the distances d_i between S_i and S_{i+1} coincide with the distances between S_{i+1}^* and S_i^*. This follows from [36, proposition 21.5] and determines the Tannaka dual $L^\vee$ of L up to a Berezin twist.

If we identify the basic plots with rooted trees $S_i \leftrightarrow \mathcal{T}_i$, we can describe a weight by a *spaced* forest

$$\mathcal{F} = (d_0, \mathcal{T}_1, d_1, \mathcal{T}_2, \ldots, d_{k-1}, \mathcal{T}_k).$$

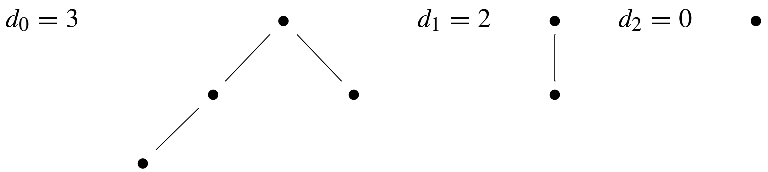
Lemma A.1 [36, Lemma 21.5] *The weight of the dual representation corresponds to the spaced forest*

$$\mathcal{F}^\vee = \left(d_0^*, \mathcal{T}_k^*, d_1^*, \mathcal{T}_{k-1}^*, d_2^*, \ldots, d_{k-1}^*, \mathcal{T}_1^*\right)$$

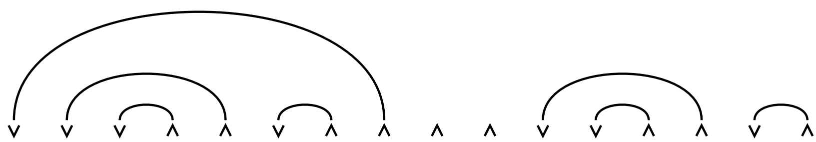
where $d_i^* := d_{k-i}$ *for* $i = 1, \ldots, k-1$ *and* $d_0^* = -d_0 - d_1 - \ldots - d_{k-1}$ *and* $\mathcal{T}_i^*$ *denotes the mirror image (along the root axis) of the planar tree* $\mathcal{T}_i$.

In the following we will use $S_1 \ldots S_i \ldots S_k$ to denote the sectors of λ since the effect of DS on $L(\lambda)$ can be described conveniently in this setup. It is however important to note that the description of the Tannaka dual does not require the S_i to be sectors (segments, i.e. unions of adjacent sectors, is enough). In particular if ∂S_i is not a sector, the dual of $S_1 \ldots \partial S_i \ldots S_k$ is still $S_k^* \ldots (\partial S_i)^* \ldots S_1^*$ (up to a shift).

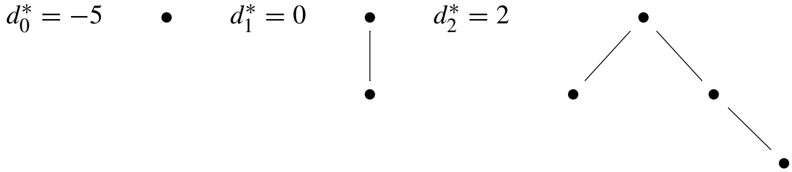
Example A.2 Consider the irreducible representation $[11, 9, 9, 5, 3, 3, 3]$ in $\mathcal{T}_7$. It can be either described by the spaced forest



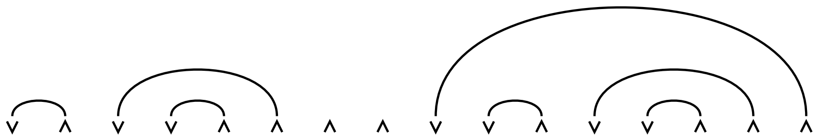
or by the cup diagram



The dual is the representation [1, 1, 0, 0, −4, −4, −5] with spaced forest



and cup diagram



The dual weight

We will switch between the language of sectors, cup diagrams and forests as is convenient.

A.3 Equivalent weights

Let λ be a maximal atypical highest weight in $X^+(n)$ with the sectors $S_1 \dots S_k$. The constituents λ_i (for $i = 1, \dots, k$) of the derivative have the sector-structure $S_1 \dots \partial S_i \dots S_k$. Recall that two irreducible representations M, N in $\mathcal{T}_n$ are equivalent $M \sim N$, if either $M \cong Ber^r \otimes N$ or $M^\vee \cong Ber^r \otimes N$ holds for some $r \in \mathbb{Z}$. Assume that λ_i and λ_j are equivalent for $i \neq j$. Then

$$S_1 \dots (\partial S_i) \dots S_j \dots S_k \ \sim \ S_1 \dots S_i \dots (\partial S_j) \dots S_k$$

define equivalent weights of $\mathcal{T}_{n-1}^+$. Passing from $L(\lambda)$ to $Ber^i \otimes L(\lambda)$ involves a shift of the vertices in the weight diagram by i. We refer to this as the translation case. Applying the duality functor $L(\lambda) \mapsto L(\lambda)^\vee$ is described in terms of the cup diagram as a kind of reflection. We refer to this as the reflection case.

Lemma A.3 *For a maximal atypical weight λ assume that there exists an equivalence $\lambda_i \sim \lambda_j$ for some $i \neq j$ between two constituents λ_i, λ_j of the derivative of λ. Then $S_\nu \equiv S^*_{k+1-\nu}$ holds for all $\nu = 1, \dots, k$ and $d_{k-\nu} = d_\nu$ holds for all $\nu = 1, \dots, k$.*

Proof (1) *Translation case* We first discuss whether this equivalence can be achieved by a translation and show that this implies

$$r(S_\nu) = 1 \text{ for all } \nu, i = 1 \text{ and } j = k.$$

To prove this we first exclude $1 < i, j$. Indeed then the starting sector is S_1 in both cases and a translation equivalence different from the identity is impossible. Now assume $i = 1$. Again an equivalence is not possible unless $\partial S_1 = \emptyset$, since otherwise S_1 and ∂S_1 would be starting sectors of different cardinality and hence they can not be identified by a translation. So the only possibility could be $i = 1$ and $r(S_1) = 1$ (so that $\partial S_1 = \emptyset$). The equivalence of $S_2 \ldots S_k$ with $S_1 \ldots \partial S_j \ldots S_k$ then implies $\partial S_j = \emptyset$, since both plots must have the same number of sectors. But then the only equivalence comes from a left shift by two. Hence it is not hard to see that this implies $r(S_\nu) = 1$ for all ν, $i = 1$ and $j = k$. Furthermore $d_1 = \cdots = d_k$ must hold. But then we see that this translation equivalence is also induced by a reflection equivalence.

(2) *Reflection case* Let us consider equivalences between $S_1 \ldots (\partial S_i) \ldots S_j \ldots S_k$ and $S_1 \ldots S_i \ldots (\partial S_j) \ldots S_k$ involving duality as in Sect. A.2.

The case $r(S_i) > 1$. Notice that $r(S_i) > 1$ is equivalent to $\partial S_i \neq \emptyset$. Furthermore notice that $r(S_i) > 1$ implies $r(S_j) > 1$, since equivalent plots need to have the same number of sectors. To proceed let us temporarily ignore the distances between the different sectors S_ν; we write $\equiv$ to indicate this. Then for all $\nu \neq i, j, k+1-i, k+1-j$ we get

$$S_\nu^* \equiv S_{k+1-\nu}$$

(equality up to a shift). Let us assume that ∂S_i is one sector (which implies the same for ∂S_j). The easy case now is $j = k + 1 - i$, where we get the further condition (*)

$$S_i^* \equiv S_{k+1-i} \text{ and hence } \partial S_i^* \equiv \partial S_{k+1-i}.$$

We also then conclude

$$d_\nu = d_{k-\nu} \quad \text{for all} \quad \nu = 1, \ldots, k.$$

If ∂S_i consists of several sectors dualizing still yields for $j = k+1-i$ that $\partial S_i \equiv \partial S_j^*$. This in turn implies $\mathrm{S}_i \equiv S_j^*$.

We now show that the more complicated looking case $i \neq j$ and $i \neq k + 1 - j$, where we also have $j \neq k + 1 - i$, can not occur. We again assume that ∂S_i and ∂S_j consist of one sector. In this case [36, proposition 20.1], implies, from comparing

$$\ldots \quad \partial S_i \quad \ldots \quad S_{k+1-j} \quad \ldots \quad S_j \quad \ldots \quad S_{k+1-i} \quad \ldots$$

and the reflection of

$$\ldots \quad S_i \quad \ldots \quad S_{k+1-j} \quad \ldots \quad \partial S_j \quad \ldots \quad S_{k+1-i} \quad \ldots$$

the following assertions

$$\partial S_i \equiv S_{k+1-i}^*, \quad \partial S_j \equiv S_{k+1-j}^*$$
$$S_i \equiv S_{k+1-i}^*, \quad S_j \equiv S_{k+1-j}^*.$$

However this is absurd, since it would imply $r(S_i) = r(S^*_{k+1-i}) = r(S_i) - 1$. If ∂S_i consists of $r > 1$ sectors, so does ∂S_j (since we assume that the weights are equivalent). The same matching $S_1 \equiv S^*_k, \dots$ as above of the $k-1$ other sectors forces again $\partial S_i \equiv \partial S^*_j$ and therefore $S_i \equiv S^*_j$.

So now $r(S_i) = 1$. Then $\partial S_i = \emptyset$ and hence also $\partial S_j = \emptyset$ since the cardinality of sectors of equivalent plots coincide. First assume $j = k+1-i$. In the case of a reflection symmetry this implies

$$S_\nu \equiv S^*_{k+1-\nu} \quad \text{for all} \quad \nu \neq i, k+1-i.$$

Furthermore it implies

$$d_{k-\nu} = d_\nu, \quad \nu = 1, \dots, k.$$

This follows by comparing

$$S_1 \quad \dots d \dots \quad \partial S_i \quad d_i \quad S_{i+1} \quad \dots \quad S_{k+1-i} \quad \dots d \dots \quad S_k$$

with the reflection of

$$S_1 \quad \dots d \dots \quad S_i \quad d_i \quad S_{i+1} \quad \dots S_{k-i} \quad \partial S_{k+1-i} \quad \dots d \dots \quad S_k.$$

Then $d_1 = d_{k+1-i}, \dots, d_{i-1} = d_{k-i+1}$, by a comparison of the lower left side and the upper right side, and then also $d_i = d_{k-i}$ and so on till $d_{k-i-1} = d_{i+1}$, but then also $d_{k-i} + d = d_i + d$ for $d = d_1 + \cdots + d_{i-1}$. Hence we conclude that $d_\nu = d_{k-\nu}$ holds for all $\nu = 1, \dots, k$. Similarly we see $S_\nu \equiv S^*_{k+1-\nu}$ for $\nu \neq i, k+1-i$. But taking into account $r(S_i) = r(S_{k+1-i})$ the assertion $S_\nu \equiv S^*_{k+1-\nu}$ also holds for $\nu = i, k+1-i$.

Finally we want to show that we have now covered all case. This means that again for $r(S_i) = 1$ the case $j \neq k+1-j$ is impossible. To show this we can assume $min(i, k+1-i) < min(j, k+1-j)$ by reverting the role of i and j and we can then assume $i < k+1-i$ by left-right reflection. Then we have to compare the reflection of

$$S_1 \quad \dots d \dots \quad \partial S_i S_{i+1} \quad \dots \quad S_{k+1-j} \quad \dots \quad S_j \quad \dots \quad S_{k+1-i} \quad \dots d \dots \quad S_k$$

with

$$S_1 \quad \dots d \dots \quad S_i S_{i+1} \quad \dots \quad S_{k+1-j} \quad \dots \quad \partial S_j \quad \dots \quad S_{k+1-i} \quad \dots d \dots \quad S_k.$$

We claim that an equivalence is not possible by a reflection! (We could easily reduce to the case where $i = 1$ by the way). In fact, by comparing the left side of the second plot with the right side of the first plot, then $S_i \equiv S^*_{k+1-i}$ and the distance $d = d_1 + \cdots + d_{i-1}$ between S_1 and S_i must be the same as the distance $d_{k+1-i} + \cdots + d_{k-1}$ between S_{k+1-i} and S_k. However, by comparing the left side of the first plot with the right side of the second plot, then $S_{i+1} \equiv S^*_{k+1-i}$ and the distance $d + 2 + d_i$ between

S_1 and S_{i+1} must be the same as the distance d between S_{k+1-i} and S_k. In fact this follows from the fact $\partial S_i = \emptyset$ and $\# S_i = 2r(S_i) = 2$. This implies $2 + d_i = 0$. A contradiction! □

From Lemma A.3 we easily get

Proposition A.4 *Suppose for the k irreducible constituents $L(\lambda_i)$ of $DS(\lambda)$ there are two different integers $i, j \in \{1, \dots, k\}$ such that $\lambda_i \sim \lambda_j$. Then there exists an integer r such that $L(\lambda)^\vee \cong Ber^r \otimes L(\lambda)$ holds. If conversely $L(\lambda)$ is weakly selfdual with sector structure $S_1 \dots S_i \dots S_k$, then $\lambda_i \sim \lambda_{k+1-i}$ for all i.*

Proof By the last lemma we conclude $S_\nu \equiv S^*_{k+1-\nu}$ and $d_{k-\nu} = d_\nu$ for all sectors S_ν, $\nu = 1, \dots, k$ of λ. By proposition [36, Proposition 20.1] or Sect. A.2 this implies $L(\lambda)^\vee \cong Ber^r \otimes L(\lambda)$ for some integer r. The converse statement is obvious from the description of the dual and the DS rule. □

Another conclusion of the considerations above is

Lemma A.5 *For fixed i between 1 and k the plot $S_1 \dots \partial S_i \dots S_k$ can only be equivalent to at most one of the plot $S_1 \dots S_i \dots (\partial S_j) \dots S_k$ for $j \neq i$.*

Corollary A.6 *Every equivalence class of the constituents λ_i of the derivative of λ can contain at most $s = 2$ representatives.*

A.4 Selfdual derivatives

We discuss the question under what circumstances a weight λ can have weakly selfdual derivatives. Before the characterize these, we discuss the special case of *ladder types* first.

Definition A.7 We call λ to be of ladder type if λ is an equidimensional union of sectors of minimal length (i.e. $r_i = 2$ for all $i = 1, \dots, k$ and all distances $d_1, \dots, d_{k-1}$ are the same).

Example A.8 Here is a weight of ladder type along with its derivatives.

A ladder type weight

The derivative λ_1 (SD)

The derivative λ_5 (SD)

The derivative λ_2 (NSD)

The derivative λ_4

The derivative λ_3

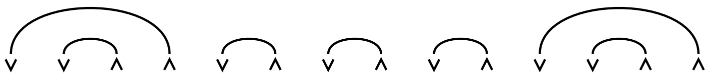
Here λ_1 and λ_5 are weakly selfdual (in fact one is a Ber^2-shift from the other) and λ_2 and λ_4 are dual to each other. The derivative λ_3 is weakly selfdual. Hence we have three derivatives of (SD)-type.

Obviously we have the following general observation about ladder type representations with at least two sectors:

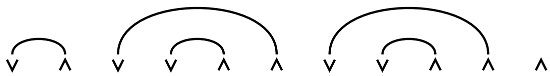
(1) If $k = 2m + 1$ is odd, then λ has three weakly selfdual derivatives: λ_1, λ_k and λ_{m+1}.
(2) If $k = 2m$ is even, λ has two weakly selfdual derivatives: λ_1 and λ_k.

Lemma A.9 *Suppose the maximal atypical weight λ has a weakly selfdual derivative λ_i for some $i = 1, \ldots, k$. Then λ_i is the unique weakly selfdual derivative except in the case where λ is weakly selfdual and has equidistant sectors all of cardinality two or $n = 3$.*

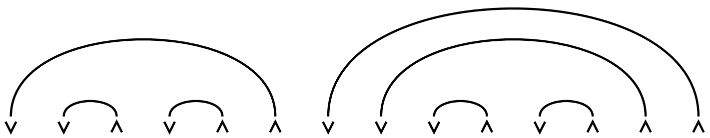
Example A.10 We show three examples of such weights. They correspond to the cases (1), (2) and (3) in the proof of the lemma.



A weight of type (1) with a selfdual derivative



A weight of type (2) with a selfdual derivative



A weight of type (3) with a selfdual derivative

Proof Suppose λ is a maximal atypical weight such that one of its derivatives λ_i is weakly selfdual. Let $S_1, \dots, S_k$ denote the sectors of λ. Then λ belongs to one of the following cases mutually exclusive cases:

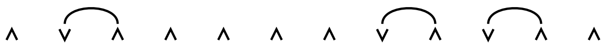
1. $k = 2m + 1$ is odd and $S^1 \dots \partial S_{m+1} \dots, S_k$ is weakly selfdual.
2. $S_1 \dots \partial S_\nu \dots, S_k$ is weakly selfdual such that $\partial S_\nu = \emptyset$ and not of type (1).
3. $S_1 \dots \partial S_\nu \dots S_k$ is weakly selfdual and we are in neither of the two cases above.

In the first case either λ is of ladder type (in which case it is weakly selfdual) or $\lambda_{m+1} = S^1, \dots, \partial S_{m+1} \dots, S_k$ is the unique selfdual derivative of λ. This immediately follows from Lemma A.3. Furthermore, if $S^1, \dots, \partial S_{m+1} \dots, S_k$ is weakly selfdual, $S^1, \dots, S_{m+1} \dots, S_k$ is weakly selfdual in this first case.

In the second case we change notation and we can suppose that

$$\lambda = S_1, S_2, \dots, S_{\nu-1}, [a, a+1], S_{\nu+1}, \dots, S_2^*, S_1^*$$

where we also allow the the sector $[a, a+1]$ to be at the left or right. If $n = 3$ with $k = 3$ sectors, λ_1 and λ_3 are always weakly selfdual (but λ is in general not weakly selfdual).



The derivatives λ_1 and λ_2 are weakly selfdual

Let us ignore this case now. We show that λ is of ladder type if there are two weakly selfdual derivatives in case (2). It is immediately clear that there does not exist a weakly selfdual derivative λ_j different from λ_i except if $[a, a+1]$ is the rightmost or leftmost sector of λ (since λ is not of type (1)). Without restriction of generality we may assume it is the leftmost sector of λ, i.e. $\lambda = S_0, S_1, S_2, \dots, S_2^*, S_1^*$ for $S_0 = [a, a+1]$ holds, i.e. $S_\nu^* = S_{k+1-\nu}$. If λ_j is obtained by $S_j \mapsto \partial S_j$, we distinguish two cases: The first is where $j = \min(j, k+1-j)$ and the second is where $j = \max(j, k+1-j)$. In the first case we obtain $S_0^* = S_k = S_1^*, S_1^* = S_{k-1} = S_2^*, \dots, S_{j-1}^* = S_j^*$ and it is immediately clear that $\partial S_j = \emptyset$. It is then clear, that there is a conflict with the symmetry of distances at j unless $j = k$. In this case $j = \max(j, k+1-j)$. So let us turn to this case now, where it follows in a similar way that the only possible case is $j = k$. Then we can easily show that all distances of λ are the same and all sectors have length two, hence λ is of ladder type and again λ is weakly selfdual.

In the third case by Lemma A.3 and its proof any translation equivalence between two derivatives implies $r(S_\nu) = 1$ for all ν, hence $\partial S_\nu = \emptyset$. Therefore λ_i is the unique weakly selfdual derivative of λ. □

Corollary A.11 *Suppose a maximal atypical weight λ that is not weakly selfdual admits a weakly selfdual derivative λ_i for some $i = 1, \dots, k$. Then λ_i is unique with this property and we are in case 3 above.*

A.5 Multiplicities

We reformulate some of the previous results in the language of Sect. 10. The same group G_μ can appear for different derivatives $\mu_1, \dots, \mu_k$ (G_μ is also called the type of

μ). Different derivatives lead to the same G_μ if they are equivalent. By Corollary A.6 the number of derivatives that lead to the same μ can be at most 2. We write this number as $m(G_\mu)$. Assuming $n \geq 4$, the results of this appendix can be summarized as follows. If $m(G_\mu) = 2$, then either

- $G_\mu = G_\nu = SL(W)$ and $W_\nu = W_\mu^\vee$ as a representation of $H_\mu = H_\nu$
- or λ is of ladder type and $\{\nu, \mu\} = \{1, k\}$ such that $G_\mu = G_\nu$ is of (SD)-type, either symplectic, orthogonal (this leads to case (2)(iii) in Lemma 10.3) or exceptional (this leads to case (2)(ii) in Lemma 10.3).

Corollary A.12 *If* $m(G_\mu) \neq 1$, *then the representation* $W_\mu|_{G_\mu} = W_\mu \oplus W_\mu^\vee \not\cong 2W_\mu$ *unless* $\dim(W_\mu) = 2$. *This can only happen for* $n = 3$.

Appendix B. Equivalences and separated weights

B.1 Equivalent and separated weights

Suppose that $\lambda, \tilde{\lambda}$ are maximal atypical weights in X_0^+ and $X = L(\lambda)$ and $Y = L(\tilde{\lambda})$. Recall the equivalence relation $\sim$ on classes of such representations and the relation $\equiv$ (meaning 'up to translation').

Definition B.1 We say that X is not separated from Y if

1. X is not equivalent to Y
2. For all derivatives $\partial_i X$ of X there exists an equivalent derivative $\partial_j Y$ of Y (where j depends on i).

Conversely we say for inequivalent weights X and Y that X is separated from Y if the second condition fails. This definition only depends on the equivalence class of X resp. Y, but Corollary B.3 shows that this is an asymmetric relation on the set of equivalence classes.

Theorem B.2 *If X is not separated from Y, then X and Y have at most two sectors and the possible cases are described in Sect. B.6.*

Corollary B.3 *Suppose X and Y are inequivalent of rank* > 2. *If X is not separated from Y, then Y is separated from X.*

Proof This can be checked easily for all the examples in Sect. B.6. □

The corollary is false for rank 2 since then S^i is not separated from S^j for all $j \neq i$.

B.2 Anchoring

If X is not separated from Y, we can replace X and Y by other representatives in their equivalence class such that

$$\partial_i X \equiv \partial_j Y \text{ for some pair } (i, j).$$

Note $i \neq j$ holds since X and Y are inequivalent. By duality we may therefore suppose $i < j$. *Hence we make this assumption once and for all.*

B.2.1 Sector structure

Let σ_i denote the segment of $\partial_i X$ obtained as the derivative of S_i; here we have to allow that σ_i is an 'empty segment' i.e. an empty interval of length 2 if S_i is of rank 1. Similarly denote the derivative of the j-th sector of Y by σ_j. If S_k denotes the last sector of Y, then the above assertion $\partial_i X \equiv \partial_j Y$ for $i < j$ defines a chain of sectors and segments $(*)$

$$\begin{aligned} X &= S_1\, d_1\, S_2\, \ldots d_{i-1}{-}1\, S_i\, d_i - 1 \ldots\, d_{j-1}\;\; \sigma_j \;\; d_j \;\; \ldots\; S_k \\ Y &= S_1\, d_1\, S_2\, \ldots\;\; d_{i-1} \;\; \sigma_i \;\; d_i \ldots\, d_{j-1}{-}1\, S_j\, d_j - 1 \;\; \ldots\; S_k \end{aligned}$$

where S_ν for $\nu \neq j$ are sectors of X and S_i, S_j are sectors such that

$$S_i = \int \sigma_i \;\; \text{(for X)} \;\; \text{and} \;\; S_j = \int \sigma_j \;\; \text{(for Y)}.$$

Obviously the i-th sector of Y is the first sector of the segment σ_i, and the j-th sector of X is the first sector of the segment σ_j.

B.2.2 Inductive and non-inductive cases

It is helpful to distinguish the following cases: The *non-inductive cases*

- where $i = 1$ is and $j = k$.
- where $i = 1$, but $1 < j < k$.

and the *inductive cases*,

- where $1 < i < j < k$.

B.2.3 Overview of the proof

The crucial Lemma B.4 shows $\partial_s X \equiv \partial_t(Y^\vee)$ for all $s \neq i$ and $t = t(s)$. This gives too many conditions to hold as soon as X has more than three sectors, thereby proving Theorem B.3. The case of at most 2 sectors is discussed in Sect. B.6 If X has at least three sectors, we partition X into three intervals $L\ M\ R$ (depending on whether we are in the inductive or non-inductive case) and check what happens if condition (**) holds with respect to ∂_s corresponding to a sector in L, M or R. Case distinctions are necessary depending on whether the derivative of the last sector is $\emptyset$ or not.

B.3 Condition $(**)$

Suppose that X has r sectors and Y r' sectors. If X is not separated from Y, then for all $s \in \{1, \ldots, r\} \setminus \{i\}$ either $\partial_s X \equiv \partial_m Y$, or $\partial_s X \equiv \partial_t Y^\vee$ (i.e. equal up to a translation) holds for some integers $m = m(s)$ resp. $t = t(s)$ in $\{1, \ldots, r'\}$.

Lemma B.4 *If X is not separated from Y, then for all $s \in \{1, \dots, r\}\backslash\{i\}$ there exists an integer $t \in \{1, \dots, r'\}$ such that*

$$(**) \qquad \partial_s X \equiv \partial_t (Y^\vee).$$

Proof *First case.* $\partial_s X \equiv \partial_m Y$ cannot hold for $s > i$. Indeed, by $(*)$ and $i < j$ the i-th sector of X is larger than the i-th sector of Y (since the latter is the left sector of σ_i).

Second case. If $s < i$, then by $(*)$ (the sectors of X and Y match until position i) the partner m of s has to be equal to s except possibly for the cases where either (i) $s = 1$ and $|S_1| = 1$ or (ii) $m = 1$ and $|S_1| = 1$. In case $s = m$, since σ_i is smaller than $S_i = \int \sigma_i$, we can argue as in the case $s > i$.

Case 2(i). Suppose $s = 1$ and $m \neq 1$. Since X and Y agree up to the i-th sector in case $m < i$ we get a contradiction by $(*)$. If $m \geq i$, then $\partial S_1 = \emptyset$ and $(*)$ forces $S_1 = S_2 = \cdots = S_{i-1} = S_i$. Since $|S_1| = 1$, this implies $|S_i| = 1$. But the cardinality of S_i is larger than 1 since $S_i = \int \sigma_i$, a contradiction, unless $\sigma_i = \emptyset$, i.e. $|S_i| = 1$. Condition (*) implies that S_{i+1} equals the i-th sector of Y.

Suppose first that $m = i$. But then the derivative of the i-th sector of Y is equal to $S_{i+1} \cap S_{i+2} \cap \cdots \cap S_{i+l}$ for some $l \geq 1$, a contradiction. Similarly, if $i < m < j$, then S_{i+l} equals the $(i + l - 1)$-th sector of Y, but $\partial_1 X \equiv \partial_m Y$ implies that the derivative of the m-th sector of Y is $S_{m+1} \cap$ other sectors, a contradiction. If $m > j$, then in order for $\partial_1 X \equiv \partial_m Y$ and $\partial_i X \equiv \partial_j Y$ to hold, the distances between S_i and S_{i+1} and the $(i-1)$-th and i-th sector in Y would have to be different in both cases.

Case 2(ii). Suppose $m = 1$ and $s > 1$. Since we also have $s < i$, we can argue similarly as for 2(i) and conclude that S_i is equal to the first sector of σ_i, a contradiction as seen before. □

Notation. Since σ_i as a segment may consist of several sectors, from now on we are changing the notation: Let denote $X_1, \dots, X_r$ the sectors of X and $Y_1, \dots, Y_{r'}$ the sectors of Y, in consequtive ordering from left to right.

Lemma B.5 *Suppose the number of sectors X_ν of X is $r \geq 2$. Suppose $\partial_s X \equiv \partial_u X$ holds for two different sectors X_s and X_u of X. Then X is of ladder type and $s = 1$ and $u = r$, or in reversed order $s = r$ and $u = 1$.*

Proof This is part 1 (translation case) of the proof of Lemma A.3. □

B.4 Non-inductive cases

All cases, where X has 1 or 2 sectors will be determined in Sect. B.6. Hence we suppose that X has at least 3 sectors. Instead of the more complicated formula $(*)$ we now use short symbolic diagrams.

We first consider the *non-inductive* case (where $i = 1$). Put $L = \sigma_1$ in both non-inductive cases and $R = \sigma_j$ respectively $R = S_j$ depending on whether $j = k$ or $j < k$. X can be partitioned into three intervals. Depending on whether $i = 1$ and $j = k$ (*left diagram*) or $j < k$ (*right diagram*) the decomposition $(*)$ will be written symbolically

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \int L & M & R \\ L & M & \int R \end{pmatrix} \text{ or } \begin{pmatrix} \int L & M & R \\ L & \int M & R \end{pmatrix},$$

Then $S_1 = \int L$ is the first sector of X. In the right diagram R is the last sector of X. In the left diagram $\int R$ is the last sector of Y. Notice

$$\begin{pmatrix} X \\ Y^\vee \end{pmatrix} = \begin{pmatrix} \int L & M & R \\ \int R^\vee & M^\vee & L^\vee \end{pmatrix} \text{ or } \begin{pmatrix} \int L & M & R \\ R^\vee & (\int M)^\vee & L^\vee \end{pmatrix},$$

For an interval I we write $I^\vee$ for the interval (of the same length) that is obtained by the reflection at its middle point. Note that this notation is in conflict with the previously used S^* for a sector S. It is convenient in this section to treat if on equal footing with $X^\vee$ (the dual of the irreducible representation X) and we hope that this notation doesn't lead to any confusion.

Proposition B.6 *Suppose X has $r \geq 3$ sectors and suppose that X is not separated from Y. Then for the first non-inductive case (left diagrams)*

$$L = R^\vee$$

follows, and for the second non-inductive case (right diagrams)

$$\int L = R^\vee.$$

Proof In the non-inductive cases, for $Z = Y^\vee$, the first sectors of X and Z are $X_1 = \int L$, $Z_1 = \int R^\vee$ (left diagram) resp. $X_1 = \int L$, $Z_1 = R^\vee$ (right diagram). Here it is important to recall that R is always a sector in the second non-inductive case (right diagram). X has at least three sectors by assumption. Hence in condition $(**)$ we have at least two different choices for $s \neq i = 1$. For at least one of these choices $t(s) \neq 1$ must hold. Otherwise Lemma B.5 would imply that X is of ladder type and the two choices s, u belong to $u = 1$ and $s = r$. But $s, u \neq i = 1$ would yield a contradiction. This shows the existence of s such that $t(s) \neq 1$. Hence, by construction we have $s > 1$ and $t(s) > 1$, and $\partial_s X = X_1 \ldots \partial(X_s) \ldots \equiv \partial_{t(s)} Z = Z_1 \ldots \partial(Z_{t(s)}) \ldots$ finally implies $X_1 \cong Z_1$. □

B.4.1 The special subcase of the non-inductive cases

For the right diagram of the non-inductive type R as a sector can not be empty. For the left type diagram, R as a segment could be empty. Then $X = (\int L M \emptyset)$ and $Y^\vee = (C M^\vee L^\vee)$ for a cap C of rank 1. Now there two cases: In the first, the partner $t = t(s)$ is different from 1 for all $s > 1$. Then $(**)$ implies $\int L(\partial_s M) \equiv C \partial_{t(s)} M^\vee$ for all $s > 1$. This implies $\int L = C$ and $L = \emptyset$, and hence $(\partial_s M) = \partial_{t(s)} M^\vee$ as true equalities for all $s > 1$. It is easy to see that this implies $t(s) = s$, and hence

$M = M^\vee$. Thus $Y^\vee = X$ and $X \sim Y$, a contradiction! Hence there exist $s > 1$ such that $t(s) = 1$ holds, and by Lemma B.5 there exists at least one s' such that $t(s') \neq 1$ holds. For s' condition $(**)$ implies $\int L^\vee = C$, whereas for s condition $(**)$ implies $\partial_s(X) = (C\partial_s(M)\emptyset) \equiv \partial_t(CM^\vee\emptyset) = M^\vee$. Therefore we obtain the identity

$$M^\vee = C\partial_s(M).$$

This case where $L = R^\vee$ is empty and $M^\vee = C\partial_s(M)$ holds, will be called the *special case*.

Lemma B.7 *In the special subcase of the non-inductive case X is of ladder type and $X \sim Y$.*

Notation. In the following distances will play a crucial role. We use the notation IaJ to denote a distance a between the intervals I and J.

Before the proof of the lemma we first prove an auxiliary technical lemma.

Lemma B.8 *Let C be a cap and let M be an interval such that $C\partial_s(M) \equiv M^\vee$ holds. Then $CM = CaIaC$ for selfdual $I = I^\vee$ and s is the last derivate.*

Proof The left 'critical' equation implies by duality $M = (\partial_s M)^\vee C = bIaC$ for $bIa := (\partial_s M)^\vee$ and minimal interval I. Then the left critical equation becomes $C\partial_s(bIaC) = CaI^\vee b$. If s is not the last derivative, then $C\partial_s(bIaC)$ has true length $2 + b + \ell(I) + a + 2$, whereas $CaI^\vee b$ has true length $2 + a + \ell(I)$. This leads to a contradiction and shows $CbI = CaI^\vee$. This implies $a = b$ and $I = I^\vee$. Hence $CM = CaIaC = CaI^\vee aC = CaIaC = M^\vee C$. ◻

Notice that the left critical equation $C\partial_s(M) \equiv M^\vee$ for M is equivalent to the right critical equation $\partial_s(N)C \equiv N^\vee$ for $N = M^\vee$.

We now proof Lemma B.7.

Proof In the *special cases* we have $X = CM$ and $Y = MC$ where C is as in Lemma B.8 and M is an interval with at least two sectors. Then $Y^\vee = CM^\vee$ and we had an equation of the form $C\partial_s M \equiv M^\vee$ for a nonempty interval C of rank 1. Concerning solutions of this equation, Lemma B.8 shows $M = aIaC$ for a selfdual interval $I = I^\vee$. Hence $X = CM = CaIaC$ is symmetric and $Y^\vee = CCaIa$ holds.

We assume now by induction that X and Y are of the form

$$X = C^r I C^r, \quad Y = C^{r+1} I C^{r-1}$$

for some selfdual interval $I = I^\vee$ and some r. We can assume $I \neq \emptyset$. Consider now $s = r + 1$ (i.e. the first derivative of I) with $t(s) \neq 1$. Then

$$\begin{aligned}\partial_s(C^r I C^r) &= C^r \partial_s(I) C^r = \partial_t(Y^\vee) \\ &= \partial_t(C^{r+1} I C^{r-1}) = C\partial_t(C^r I C^{r-1}).\end{aligned}$$

This implies $t \geq r + 1$, hence the last entry becomes $C^r \partial_t(CIC^{r-1})$. The comparison gives $\partial_{r+1}(I)C^r = \partial_t(CIC^{r-1})$. Put $I = aI_1J$ for the first sector I_1 of I. The left side $\partial_{r+1}(I)C^r$ is $(a+2)I_1C^r$ if $\partial(I_1) = \emptyset$ and $(a+1)\partial I_1 JC^r$ if $\partial(I_1) \neq \emptyset$. The right side $\partial_t(CIC^{r-1})$ is $(a+2)I_1JC^{r-1}$ if $t = r+1$ and $Ca\partial_t(I_1JC^{r-1})$ if $t > r+1$. Hence $t = r+1$ and $I_1 = C$. This implies $I = aCJ$ and by comparison of both sides $JC^r = CJC^{r-1}$ and then $JC = CJ$. This in turn implies $J = J_1C$ for some J_1 and therefore $I = aCJ_1C$. Since $I = I^\vee$ we get $a = 0$ and $J_1 = J_1^\vee$. Since r was arbitrary, this implies that X is of ladder type $X = CC\ldots CC$ and $Y^\vee = CC\ldots CC$ as well. □

Granting this, in the proof of the following proposition we can assume $R \neq \emptyset$ not only for the right type diagrams, but also for the left type diagrams of the non-inductive situation.

Proposition B.9 *Assume that X has at least 3 sectors and X is not separated from Y. Then the non-inductive cases do not occur.*

Proof *Left type of diagrams*. There Proposition B.6 gives $L = R^\vee$ and where we can suppose $R \neq \emptyset$ (since we already discussed the special cases). Thus $X = (\int L\, M\, L^\vee)$ and $Y^\vee = (\int L\, M^\vee\, L^\vee)$. Take $s = r$ to be the derivative with respext to the last sector of X. Then $1 < t(s) < k$ can not be occur, since the total length of $\partial_s X$ would then be smaller than the total length of $\partial_{t(s)}(Y^\vee)$. This is not possible by $(**)$. Hence $t(r) \in \{1, r\}$. If $t(r) = 1$ then $\int L = \partial \int L = L$. A contradiction. If $t(r) = r$, then $M = M^\vee$ and hence $X \sim Y$.

Right type of diagrams. Here Proposition B.6 gives $\int L = R^\vee$ and R is a sector and L are sectors. Thus $X = (\int L\, M\, \int L^\vee)$ and $Y^\vee = (\int L\, (\int M)^\vee L^\vee)$. Take $s = r$ to be the derivative with respect to the last sector of X. As before $1 < t(s) < k$ can not be occur, since the total length of $\partial_s X$ would then be smaller than the total length of $\partial_{t(s)}(Y^\vee)$. Hence $t(r) \in \{1, r\}$ and hence $t(r) = r$. Notice $M = S_2, \ldots, S_{r-1}$, Therefore $t(r) = r$ together with Lemma B.5 implies that $s \mapsto t(s)$ defines an injection $\{2, \ldots, r-1\} \to \{2, \ldots, r-1\}$ [since $(\int_j M)^\vee$ also has $r-2$ sectors and since $t(s) = 1$ is not possible by reasons of total length]. Hence $(**)$ implies $\int L^\vee = L^\vee$ by a comparison of the last sectors of the derivatives $\partial_s X$ and $\partial_{t(s)}(Y^\vee)$. A contradiction. □

B.5 The inductive case

The *inductive situation* $1 < i < j < k$ and $(*)$ leads to the following diagrams

$$\begin{pmatrix} X \\ Y^\vee \end{pmatrix} = \begin{pmatrix} L & \int_i M & R \\ R^\vee & (\int_j M)^\vee & L^\vee \end{pmatrix} = \begin{pmatrix} L & \tilde{X} & R \\ R^\vee & \tilde{Y}^\vee & L^\vee \end{pmatrix}$$

where $\tilde{X} = \int_i M$ and $\tilde{Y} = \int_j M$. Note that L denotes the first sector of X and R the last sector of X, such that M is the remaining middle interval defined by the decomposition $(*)$. We may as in the non-inductive case assume that X has at least 3 sectors.

Proposition B.10 *Suppose X has $r \geq 3$ sectors and is not separated from Y. Then for the inductive case we have an equality of sectors $L = R^\vee$.*

Proof Use Lemma B.5 to find $s > 1$ such that $t(s) \neq 1$ and then us $(**)$ as in the proof of Proposition B.6. □

By Proposition B.9 non-inductive cases do not contribute if X has more than 3 sectors. Hence the next proposition proves Theorem B.2.

Proposition B.11 *Suppose X has $r \geq 3$ sectors and is not separated from Y. Then the inductive case can not occur.*

Proof Consider condition $(**)$ for the last derivative ∂_k of X, i.e. $s = k$. This implies $\partial_t(Y^\vee) = \partial_s(X) = (L\tilde{X}\partial R)$. Since $R = L^\vee$ by Proposition B.10, $\partial(R)$ can not contain the sector $L^\vee$. Hence the only possibilities are case (1) where $t = t(k) = k$ or case (2) where $t(k) = 1$ [consider the total length] and $\partial R = \emptyset$ (then also $\partial L^\vee = \emptyset$ by $R = L^\vee$).

Case (1) *Suppose L and R do not have rank 1*. Then $t(k) = k$ and $(**)$ implies $\tilde{X} = \tilde{Y}^\vee$. Hence $Y^\vee = (R^\vee \tilde{Y}^\vee L^\vee)$ by Proposition B.10 is equal to $(L\tilde{X}R) = X$. This shows $X \sim Y^\vee$, hence a contradiction.

Case (2) *Now suppose R and L have rank 1*. Then in $(**)$ we can derive X with respect to $s = 1$ and $s = k$ since $i \neq 1, k$. But $t(s) \notin \{1, k\}$ for $s \in \{1, k\}$ is impossible since otherwise the total length (i.e. the distance from the beginning of the first sector to the end of the last sector) of $\partial_s X$ would be strictly smaller than that of $\partial_t(Y^\vee)$. This however contradicts $\partial_s X \equiv \partial_t(Y^\vee)$. Thus $t(\{1, k\}) \subseteq \{1, k\}$. Then $t(1) = k$ and $t(k) = 1$ since otherwise $\tilde{X} = \tilde{Y}^\vee$, and therefore $X = Y^\vee$ would contradict $X \nsim Y$. This implies $(\tilde{X}R) \equiv (L\tilde{Y}^\vee)$ and also $(L\tilde{X}) \equiv (\tilde{Y}^\vee R)$ for the sectors R, L of rank 1. Therefore the first and last sector of $\tilde{X}$ and $\tilde{Y}$ must have rank 1. If we can exclude this, our proof of Proposition B.11 is complete.

For this we first observe $t(s) \notin \{1, k\}$ for $s \notin \{1, k\}$. This is seen by arguing with the total length as before. We now apply induction with respect to the number of sectors. By the induction we can therefore either assume $\tilde{X} \sim \tilde{Y}$, or $\tilde{X}$ and $\tilde{Y}$ only have at most two sectors. In the latter case $\tilde{X}$ and $\tilde{Y}$ have two sectors of rank 1 by the last observation of case (2). The first case is impossible because we could assume $\tilde{X} \equiv \tilde{Y}^\vee$ [otherwise replace Y by $Y^\vee$]. Then $t(s) = s$ for all $s \notin \{1, k\}$. Furthermore $(**)$ would imply $\partial_s X = \partial_s Y^\vee$ for all $s \notin \{1, k\}$ (the equality sign comes from the fact that the sectors R and L rigidify the situation). This shows $\tilde{X} = \tilde{Y}^\vee$ and hence $X = Y^\vee$, a contradiction. Notice, this rigidified version of $(**)$ now implies $X = Y^\vee$ also in the last remaining case where $\tilde{X}$, $\tilde{Y}$ both consist of two rank 1 sectors. □

B.6 The examples where *X* has ≤ 2 sectors

Case where X has only one sector. Consider a sector X and $Y = (\partial(X)C)$ for a cap diagram C of rank 1 with an arbitrary distance between the segment $\partial(X)$ and C. Then X is not separated from Y. It is easy to see that these are the only cases.

Case of two sectors. If X has only two sectors, suppose X can not be separated from $Y \nsim X$. So without restriction of generality we can assume $\partial_1 X = (\partial X_1)X_2 \equiv \partial_2 Y = Y_1(\partial Y_2)$ as in condition $(*)$. If $\partial X_1 = \emptyset$, then also $\partial Y_2 = \emptyset$ and $X_2 = Y_1$. Hence $X = CR$ and $Y = RC$ for C of rank 1. We can assume $R \neq R^\vee$.

(1) Example: If R has rank 1, then we obtain examples where X can not be separated from X in the form

$$X = CaC \text{ and } Y = CbC \text{ for } a \neq b.$$

In this case X has rank 2. If R has rank > 1 no further examples arise: Indeed, $\partial_2(X) = C\partial R$ can not be $\partial_1(Y) = \partial(R)C$ nor $\partial_2(Y) = R$ nor $\partial_1(Y^\vee) = R^\vee$ nor $\partial_2(Y^\vee) = C\partial R^\vee$.

(2) It remains to discuss the case $\partial X_1 \neq \emptyset$. Then $\partial_1 X \equiv \partial_2 Y$ implies $\partial X_1 = Y_1$ and $X_2 = \partial Y_2$. Hence $X = \int L\, R$ and $Y = L \int R$. Hence we are in the left type non-inductive case with $M = \emptyset$ and $Y^\vee = \int R^\vee L^\vee$. But now show $R = \int L^\vee$ (in contrast to the case with more than 3 sectors discussed in Proposition B.6). In fact, condition $(**)$ for $s = 2$ implies $\int L\partial(R) \equiv \partial_t(\int R^\vee L^\vee)$. For $t = 1$ this implies $\int L = R^\vee$, for $t = 2$ it implies $L = R^\vee$ and hence $X = Y^\vee$ and hence $X \sim Y$. This gives the second examples $Y = (L \ \int\int L^\vee)$ with

$$\begin{pmatrix} X \\ Y^\vee \end{pmatrix} \sim \begin{pmatrix} \int L & \int L^\vee \\ \int\int L & L^\vee \end{pmatrix}$$

Obviously, X and Y thus defined for an arbitrary sector L have the property that X can not be separated from Y since $\partial_1 X = \partial_2 Y$ and $\partial_2 X = \partial_1 Y^\vee$.

So if X is not separated from Y, X is selfdual. For rank $n > 2$, however Y is not selfdual. Hence Proposition B.3 follows.

Appendix C. Pairings

Selfdual objects $L(\lambda)$ will give rise to groups of type $B,\ C, D$ according to Sect. 6. In order to distinguish between the orthogonal and the symplectic case we check whether these representations are even or odd in the sense defined below.

C.1 Strong selfduality

We say that an object M is strongly selfdual, if there exists an isomorphism $\rho : M \to M^\vee$ such that $\rho^\vee = \pm\rho$ holds and call it even or odd depending on the sign. Here $\rho^\vee : M \to M^\vee$ is the dual morphism of ρ. Here we use the canonical identification $M = (M^\vee)^\vee$, since a priori we only have $\rho^\vee : (M^\vee)^\vee \to M^\vee$. Note that any selfdual irreducible object is strongly selfdual in this sense. Slightly more general: If L is an invertible object in a tannakian category and $\rho : M \cong M^\vee \otimes L$, then $(\rho^\vee \otimes id_L) \circ (id_M \otimes coeval_L) = \pm\rho$. Furthermore any multiplicity one retract of a strongly selfdual object is strongly selfdual. Finally, if F is a tensor functor between rigid symmetric tensor categories, then $F(M)$ is strongly selfdual if M is strongly selfdual. We remark that we can define the similar notion of strong selfduality for $*$-duality.

By [46, (4.30)] a supersymmetric invariant bilinear form on a representation (V, ρ) in T defines a skew-supersymmetric invariant bilinear form on the representation $\Pi(V, \rho)$.

Suppose $L \cong L^\vee$ in $\mathcal{R}$ is a maximal atypical self dual representation. We consider now irreducible representations of the form $[\lambda] = [\lambda_1, \ldots, \lambda_{n-1}, 0]$. We call these positive. For general λ we can twist with an appropriate Berezin power to get this form. We will induct on the degree $\sum \lambda_i$, hence we start with the case S^1.

Lemma C.1 *S^1 is an even selfdual representation.*

Proof Obviously $S^1 \cong (S^1)^\vee$, and therefore there exists a nondegenerate super bilinear form

$$B : S^1 \otimes S^1 \to \mathbf{1} = k.$$

Note that the adjoint representation of G_n on $\mathbb{A} := \mathfrak{g}_n$ carries the nondegenerate invariant Killing form

$$K : \mathfrak{g}_n \otimes \mathfrak{g}_n \to \mathbf{1} = k.$$

This bilinear form is supersymmetric: $K(S(x \otimes y)) = K(x \otimes y)$ for the symmetry constraint $S : \mathfrak{g}_n \otimes \mathfrak{g}_n \cong \mathfrak{g}_n \otimes \mathfrak{g}_n$, or $K(x, y) = (-1)^{|x||y|} K(y, x)$. Let $\mathfrak{g}_n^0$ denote the kernel of the supertrace $\mathfrak{g}_n \to \mathbf{1}$. Then $S^1 = \mathfrak{g}_n^0/z$, where z is the center of G_n. The Killing form K restricts to a supersymmetric form on $\mathfrak{g}_n^0$ which becomes nondegenerate on $S^1 = \mathfrak{g}_n^0/z$. Hence S^1 carries a nondegenerate supersymmetric bilinear form. □

We now treat the general $[\lambda] = [\lambda_1, \ldots, \lambda_{n-1}, 0]$-case. Recall that the direct summands of $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$ are called mixed tensors. The maximal atypical mixed tensors are parametrized by partitions λ satisfying $k(\lambda) \leq n$ for an integer $k(\lambda)$ defined in [15, 6.17] [35, Section 4]. We furthermore recall from [35, Theorem 12.3]: For every such $[\lambda]$ the mixed tensor $R(\lambda)$ contains $[\lambda]$ with multiplicity 1 in the middle Loewy layer. $[\lambda]$ is the constituent of highest weight of $R(\lambda)$. If we define $deg\ [\lambda] = \sum_{i=1}^n \lambda_i$, then $[\lambda]$ has larger degree then all other constituents. We denote the degree of a partition by $|\lambda|$. We recall further: If λ and μ are two partitions of length $\leq n$, the tensor product $R(\lambda) \otimes R(\mu)$ splits in $\mathcal{R}_n$ as

$$R(\lambda) \otimes R(\mu) \quad \bigoplus_{|\nu|=|\lambda|+|\mu|, k(\nu)\leq n} \left(c_{\lambda\mu}^\nu\right)^2 R(\nu) \oplus \bigoplus_{|\nu|<|\lambda|+|\mu|, k(\nu)\leq n} d_{\lambda\mu}^\nu R(\nu)$$

for some coefficients $d_{\lambda\mu}^\nu \in \mathbb{N}$, the Littlewood–Richardson coefficients $c_{\lambda\mu}^\nu$ and the invariant $k(\lambda)$ [35, Lemma 14.4].

Proposition C.2 *Let $[\lambda]$ be positive of degree r. Then $R(\lambda)$ occurs as a direct summand with multiplicity 1 in a tensor product $\mathbb{A} \otimes R(\lambda_i)$ where $l(\lambda_i) \leq n$, $|\lambda_i| = r - 1$ and $R(\lambda_i)$ is a direct summand in $\mathbb{A}^{\otimes r-1}$. The constituent $[\lambda]$ occurs with multiplicity 1 as a composition factor in the tensor product $\mathbb{A} \otimes R(\lambda_i)$.*

Proof For λ, μ of length $\leq n$ we know that

$$R(\lambda) \otimes R(\mu) = \bigoplus_{|\nu|=|\lambda|+|\mu|, k(\nu)\leq n} \left(c^{\nu}_{\lambda\mu}\right)^2 R(\nu) \oplus \tilde{R}$$

where $\tilde{R}$ are the terms of lower degree. We apply this for $\lambda = \mu = (1)$ (i.e. $\mathbb{A} \otimes \mathbb{A}$) and then to tensor products of the form $R(\lambda) \otimes \mathbb{A}$. Since every summand in a tensor product of the standard representation of $SL(n)$ with any other irreducible module has multiplicity 1, $\mathbb{A}^{\otimes r}$ decomposes as the standard representation of $SL(n)$ modulo contributions of lower degree and contributions of length $l(\nu) > n$. Since every irreducible $SL(n)$-representation with highest weight λ of degree $deg(\lambda) = \sum \lambda_i = r$ occurs as a summand in $st^{\otimes r}$, every mixed tensor $R(\lambda)$ with $l(\lambda) \leq n$ and $deg(\lambda) = r$ occurs as a direct summand in $\mathbb{A}^{\otimes r}$. Hence there exists in $\mathbb{A}^{\otimes r-1}$ a mixed tensor $R(\lambda_i)$ of length $\leq n$ and degree $deg(\lambda_i) = r - 1$ with

$$\mathbb{A} \otimes R(\lambda_i) = R(\lambda) \oplus \bigoplus R(\nu_i)$$

and $\nu_i \neq \lambda$ for all i. $R(\lambda)$ contains the composition factor $[\lambda]$ with multiplicity 1 and no other mixed tensor in this decomposition contains $[\lambda]$. Indeed if $deg(\nu_i) < r$, its constituent of highest weight has degree $< r$. If $R(\nu_i)$ has degree r and $l(\nu_i) \leq n$, its constituent of highest weight is $[\nu_i] \neq [\lambda]$ and if $R(\nu_i)$ has degree r and $l(\nu_i) > n$, its constituent of highest weight has degree $< r$ by [35, Section 14]. □

This applies in particular to positive $[\lambda]$ which are (Tannaka) selfdual. Every such $[\lambda]$ occurs as a multiplicity 1 constituent in a multiplicity 1 summand in a tensor product $\mathbb{A} \otimes R(\lambda_i)$ for $|\lambda_i| = r - 1$ which in turn appears as a multiplicity 1 summand in a tensor product $\mathbb{A} \otimes R(\lambda_{i_2})$ with $|\lambda_{i_2}| = r - 2$ etc.

Corollary C.3 *The selfdual representation* $[\lambda] = [\lambda_1, \ldots, \lambda_{n_1}, 0]$ *is even. Its parity shift* $\Pi[\lambda]$ *is odd.*

Proof The parity is inherited to super tensor products (look at the even parts) and to multiplicity 1 summands. □

Since for basic $L(\lambda)$ a weakly selfdual weight λ is selfdual, we obtain the next corollary.

Corollary C.4 *If* λ *is basic of type (SD), its parity is even.*

C.2 Combinatorics of selfdual weights

If the representation $L(\lambda)$ is only selfdual up to a Berezin twist, the argument breaks down. If one simply restricts $L(\lambda)$ to $SL(n|n)$ to get a selfdual weight, we lose the multiplicity one assertions from above.

Definition C.5 The total height of a forest is defined recursively as follows:

$$ht(\mathcal{F}) = \sum_i ht(\mathcal{T}_i),$$
$$ht(\mathcal{T}) = \sum_{x \in \mathcal{T}} ht(x)$$

where $ht(x)$ is the distance to the root x_0 of the tree (so $ht(x_0) = 0$ etc).

Recall from Sect. A.2 that in the (SD)-case we have $r_1 = r_k, \ldots$ and $d_1 = d_k - 1, d_2 = d_{k-2}, \ldots$ for the k sectors of rank $r_1, \ldots, r_k$ and the distances $d_1, \ldots, d_{k-1}$. From this we get

$$D(\lambda) = \sum_{i=0}^{k} \left(\sum_{1 \le \mu < i} d_\mu \right) = \left(\sum_{i=1}^{[k/2]} d_i \right) \cdot n - d_{middle} \cdot \frac{n}{2}$$

using $r_1 + \ldots r_k = n$ where $d_{middle} = 0$ unless $2|k$.

Lemma C.6 *For weakly selfdual forests $\mathcal{F}_\lambda$ with k trees we have*

$$D(\lambda) = (d_0 + \cdots + d_{[k/2]})n - \frac{nd_{middle}}{2}$$

where d_{middle} is zero by definition if the number of trees is odd.

Lemma C.7 *For weakly selfdual $L = L(\lambda)$ with $L^\vee \cong L \otimes Ber^{-r}$ we have $r = 2(d_0 + \cdots + d_{[k/2]}) + d_{middle}$ and $rn = 2D(\lambda)$, so that rn is even.*

Proof The isomorphism $L^\vee \cong L \otimes Ber^{-r}$ implies

$$\det(L)^{-1} \cong \det(L) Ber^{-nrdim(V_\lambda)}.$$

We apply $\nu : Pic(\mathcal{T}_n^+) \to \mathbb{Z}$ and $\det(L)^\vee \cong det(L)^{-1}$ and use results of Sect. 13. Then $\nu(det(L)) = \mathrm{sdim}(L)D(L)$ and $\nu(Ber) = n$ give the result. □

Lemma C.8 *We have $p(\lambda) - p(\lambda_{basic}) = D(\lambda)$ and furthermore $p(\lambda_{basic}) = n(n-1)/2 - ht(\mathcal{F}(\lambda))$ for the height $ht(\mathcal{F}(\lambda))$ of the forest $\mathcal{F}(\lambda)$.*

Proof Induction on n using $ht(\partial\mathcal{T}) = ht(\mathcal{T}) - \#\mathcal{T} + 1$. For the first assertion see [36, Corollary 25.4]. □

We obtain the following corollary.

Corollary C.9 *If L is weakly selfdual and the number of trees of $\mathcal{F}(L)$ is odd, let $\partial_{middle}\mathcal{F}(L)$ denote the forest obtained by the middle derivative of $\mathcal{F}(L)$. The associated weight $\tilde{\lambda}$ is weakly selfdual such that $p(\tilde{\lambda}_{basic}) = p(\lambda_{basic})$ mod 2.*

Proof By Lemma C.8 and its proof the congruence is equivalent to $n - 1 - (r_{middle} - 1) = 0$ mod 2, hence follows from $n \equiv r_{middle}$ mod 2. □

C.3 Weakly selfdual cases

Let L be weakly selfdual as above, then there exists a nondegenerate pairing

$$L \times L \to Ber^r$$

which is either symmetric or antisymmetric. Let ε_λ denote this parity of the pairing. It induces a pairing on the Dirac cohomology $V = \omega(L)$ of the same parity. Since we work mainly in $\mathcal{T}_n^+$, notice that $B^r = Ber^r$ is always in $\mathcal{T}_n^+$ by Lemma C.7 because nr is even in the (SD)-cases. But the twist $X_\lambda = \Pi^{p(\lambda)} L(\lambda)$ shifts the parity of the induced pairing on $\varepsilon(X_\lambda)$ which has parity $(-1)^{p(\lambda)}\varepsilon_\lambda$.

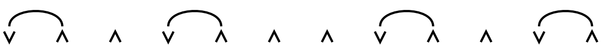
We now analyze the parity ε_λ. For this we make a case distinction into Case 1 (easier case) and Case 2 (more difficult case). Our proof is vaguely reminiscent of the classical proof in Bourbaki [10, Chapter IX, Section 7.2, Proposition 1] where one constructs a Lie subalgebra isomorphic to $\mathfrak{sl}(2)$ and considers the restriction of the bilinear form to a certain irreducible $\mathfrak{sl}(2)$-module. The sign can then be read off from the one for this restriction. In the difficult second case we reduce the computation of the sign to the restriction of the form to one direct summand in $DS_{n,2}(L(\lambda))$ (where we know the sign by [37]). Actually a similar argument could also be applied in the case 1 where n is odd, r is even or r is odd, n is even. However our previous approach in these cases is constructive, whereas we do not know an explicit description of the pairing in Case 2.

Case 1: *Let $\mathcal{F}(\lambda)$ be a weakly selfdual forest such that it either has an odd number of trees, or the middle distance d_{middle} is even.*

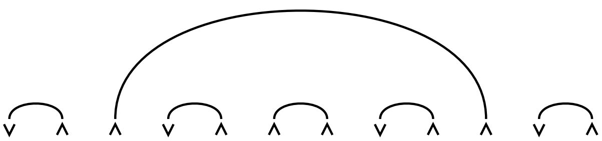
Then middle integration gives a chain of weakly selfdual weights

$$\lambda = \lambda^0, \lambda^1, \ldots, \lambda^s = \Lambda$$

such that Λ is basic selfdual up to a Berezin shift. Conversely we obtain λ by iterated middle derivatives from Λ.



The cup diagram of λ



The basic (up to Ber shift) weight Λ obtained from λ

Lemma C.10 *In the situation of our assumption the parity of the pairing $\varepsilon(X_\lambda)$ is equal to $(-1)^{p(\lambda_{basic})} = \varepsilon(X_{\lambda_{basic}})$.*

Proof By Corollary C.9 it suffices to show this for basic Λ selfdual up to a Berezin twist. This case has been settled in Corollary C.4. Indeed by Corollary C.9, for the middle derivative $\tilde{\lambda}$ of λ the parity ε_λ coincides with $\varepsilon_{\tilde{\lambda}}$ since $\tilde{\lambda}$ is an irreducible multiplicity one summand of $H^\bullet_D(L)$. □

C.4 Case 2

Now we turn to cases were the previous assumption is not satisfied. The integer d_{middle} then is odd. By Lemma C.6 then $D(L)$ is an odd multiple of $n/2$. Hence n must be even. By Lemma C.7 furthermore nr is not divisible by 4. Hence r must be odd.

Lemma C.11 *Under our new assumption n is even and r is odd.*

In the situation of the previous assumption notice that nr is always divisible by 4 and r is even. Hence $LBer^{-r/2}$ is selfdual and $Ber^{-r/2}$ was in $\mathcal{T}_n^+$. If we normalize now and replace L by the selfdual $\mathcal{L} := LBer^{-r/2}$, we need to be more careful. First we have to define an extended representation tensor category of $\mathcal{T}_n^+$ that contains $Ber^{1/2}$ (we sometimes write simply B instead of Ber). Second, the parity of $B^{1/2}$ and hence also of $Ber^{r/2}$ will be odd, so it needs a parity shift so that $\Pi B^{1/2}$ is in the extended tensor category $\mathcal{T}_n^{ext}$. To do this replace $GL(n|n)$ by the real supersubgroup generated by replacing $GL(n|n)_{\overline{0}} = GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ by the subgroup G generated by $diag(E,-E)$ and by the matrices $diag(A,D)$ in $GL(n,\mathbb{R}) \times GL(n,\mathbb{R})$ with $\det(A), \det(D) \in \mathbb{R}^*_{>0}$. Then $Ber^{1/2}$ is well-defined as a complex representation of the real super Liegroup G. The super parity decomposition of the space of a representation ρ is defined, as before, by the eigenvalues of $\rho(diag(E,-E))$. We define $\mathcal{T}_n^{ext}$ as the category of all finite dimensional representations on which $\rho(diag(E,-E))$ acts by the parity endomorphism as well as their parity shifts (analogously to $\mathcal{T}_n = \mathcal{R}_n \oplus \Pi\mathcal{R}_n$).

Hence the parity of $Ber^{1/2}$ becomes $(-1)^{nr} = -1$. Notice that $Lie(G) \otimes_\mathbb{R} \mathbb{C} = \mathfrak{gl}(n|n)$. Hence we can view our new category again as a tensor category $\mathcal{T}_n^{ext}$ of representations of $\mathfrak{gl}(n|n)$ containing $\mathcal{T}_n$.

Recall $\mathcal{L}^\vee \cong \mathcal{L}$. Let J, $J^2 = id$ be the antidiagonal unit matrix in $GL(2n)$, then $X \mapsto \phi(X) = X^J = JXJ$ defines an automorphism of the Lie superalgebra $\mathfrak{gl}(n|n)$ of order two. Notice that this automorphism exchanges the highest and lowest vectors v_+ and v_- of L. This follows from the fact that J interchanges the ideals $\mathfrak{p}_+$ and $\mathfrak{p}_-$ and the fact that irreducible modules are cyclic modules.

Lemma C.12 *For maximal atypical irreducible representations L the twisted representation $\rho^J_\mathcal{L}(X) := \rho_\mathcal{L}(X^J)$ is isomorphic to the dual representation given by $\rho_{\mathcal{L}^\vee}(X)$.*

Proof This uses $L^* \cong L$, for L^* defined on the representation space of $L^\vee$ by $X \mapsto \rho^\vee(-X^T)$ and the automorphism $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = X \mapsto -X^T = \left(\begin{smallmatrix} -A' & C' \\ -B' & -D' \end{smallmatrix}\right)$ of $\mathfrak{gl}(n|n)$. This automorphism, composed with conjugation by J, becomes the automorphism $X \mapsto \left(\begin{smallmatrix} -wD'w & -wB'w \\ wC'w & -wA'w \end{smallmatrix}\right)$ of $\mathfrak{gl}(n|n)$ for the antidiagonal unit matrix w in $GL(n)$. Clearly, if we twist a representation with this composed automorphism, this preserves highest weight vectors and their weights $(\lambda_1, \ldots, \lambda_n|\lambda_{n+1}, \ldots, \lambda_{2n})$ become

$(-\lambda_{2n}, \ldots, -\lambda_{n+1}|-\lambda_{\lambda_n}, \ldots, -\lambda_1)$. Recall that the highest weight of a maximal atypical irreducible representation is of the form $(\lambda_1, \ldots, \lambda_n \mid -\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1)$. Since $J^2 = id$, this implies $\mathcal{L}^\vee \cong (\mathcal{L}^\vee)^* \cong \mathcal{L}^J$ for maximal atypical irreducible objects and proves our claim. □

Hence for selfdual $\mathcal{L}$ we obtain from $\mathcal{L}^J \cong \mathcal{L}^\vee$ a composite isomorphism

$$\mathcal{L}^J \cong \mathcal{L}^\vee \cong \mathcal{L}.$$

Note that $\mathcal{L}^J$ and $\mathcal{L}$ share the same underlying representation space. By the isomorphism $\mathcal{L}^J \cong \mathcal{L}$ there exists an automorphism ϕ of this vectorspace such that $\rho^J_{\mathcal{L}}(X) = \phi \circ \rho_{\mathcal{L}}(X) \circ \phi$ holds. There is a unique choice for $\pm\phi$ if we normalize ϕ by a scalar such that $\phi^2 = id$ holds. Fix such ϕ (unique up to ± 1). Then this extends the representation $\mathcal{L}$ to a representation of the "semidirect product"

$$\mathfrak{gl}(n|n) \cdot \langle\phi\rangle$$

resp. $GL(n|n)\cdot\rangle\phi\rangle$ such that $(g_1, \phi^i)(g_2, \phi^j) = (g_1 g_2^{J^i}, \phi^{i+j})$. Notice that J acts on $H^\bullet_{\overline{D}}(L)$ (reversing the degrees) since J commutes with the Dirac operator $D = \partial + \overline{\partial}$ (see Sect. 4.3). The pairing $(.,.)_{\mathcal{L}} : \mathcal{L} \times \mathcal{L} \to \mathbf{1}$ descends to $H_D(\mathcal{L})$ since $(closed, exact)_{\mathcal{L}} = 0$. However this make sense only for $D = D_s : \mathcal{T}^>_n \to \mathcal{T}^>_{n-s}$ for even integers s, although it makes sense for the pairing of $L \times L \to B^r$ for all s. This fact will be important below. Furthermore, for the G-invariant pairing $(.,.)_{\mathcal{L}} : \mathcal{L} \times \mathcal{L} \to \mathbf{1}$ on $\mathcal{L}$, $(\phi(.), \phi(.))_{\mathcal{L}}$ is a G-invariant pairing as well. Again ϕ only makes sense for $D = D_s$ and even s. Hence by Schur's lemma $(\phi(v), \phi(w))_{\mathcal{L}} = \varepsilon_\phi \cdot (v, w)_{\mathcal{L}}$ for some constant ε_ϕ. Notice $\varepsilon^2_\phi = 1$ holds since $\phi^2 = 1$.

Lemma C.13 $\varepsilon_{\mathcal{L}} = -\varepsilon(X_\lambda)$.

Proof Since $\varepsilon_{\mathcal{L}} = (-1)^r \varepsilon(X_\lambda)$ and r now is odd, the assertion follows. □

C.5 Case 2 continued: derivatives

We consider the spaced forest of L (we prefer to write L instead of $\mathcal{L}$ since the spaces are the same, only the parity ε_L changes to $\varepsilon_{\mathcal{L}}$). This spaced forest $\mathcal{F}$ has the structure $\mathcal{F} = \mathcal{F}_L \ldots d_{middle} \ldots \mathcal{F}_L^{dual}$. Its first derivative (symbolically) is

$$\partial\mathcal{F} = \partial\mathcal{F}_L \ldots d_{middle} \ldots \mathcal{F}_L^{dual} \quad \cup \quad \mathcal{F} = \mathcal{F}_L \ldots d_{middle} \ldots \partial\mathcal{F}_L^{dual}.$$

We now use the following symbolic notation: The derivative $\partial\mathcal{F}_L$ is given by the union of the tree derivatives $\partial_{\mathcal{T}_i}\mathcal{F}_L$ for the trees $1, \ldots, k/2$ of the left subforest $\partial\mathcal{F}_L$ of $\partial\mathcal{F}(\lambda)$. We always focus on the derivative of the left middle tree $\mathcal{T}_{k/2}$ and its associated Tannakian quotient group of H_{n-1}. The irreducible subrepresentation W_L^{middle} in W_L associated to it has multiplicity one in V_L for all even $n \geq 4$. Its dual representation $(W_L^{middle})^\vee$ is associated to the right middle derivate $\partial_{\mathcal{T}_{k/2+1}}\mathcal{F}_L^\vee$. Since this dual representation is nonisomorphic, the restriction of the pairing $V_L \times V_L \to \mathbf{1}$ on V_L

must be trivial on W_L^{middle} and on its dual $(W_L^{middle})^\vee$. Hence both are Lagrangian subspaces of the nondegenerate pairing on $W_L^{middle} \oplus (W_L^{middle})^\vee$. The Lagrangian property will be substantial for the subsequent argument. So keep in mind, when we symbolically write $\partial\mathcal{F}_L$, we actually mean W_L^{middle} and by abuse of notation ignore all other contributions.

Consider the super fibre functor $\omega = H_D(L)$ for $D = D_n$ and $V_\lambda = V_L = \omega(L) = H_D(L)$. In V_λ) this gives an orthogonal decomposition $V_\lambda = W_L \oplus^\perp W_L^\vee$ where W_L is a Lagrangian subspace. This decomposition comes from considering $H_D(L)$ for $D = D_1$ as a representation in $\mathcal{T}_{n-1}$. It induces a corresponding decomposition of $\mathcal{T}_{n-1}$, since the pairing is inherited. Notice $V_L = H_D(\mathcal{L}) = H_D(L)$ as a vectorspace (not as a super vectorspace or representation). Hence this induces a decomposition of V_L, as a representation of H_{n-1} into orthogonal Lagrangian subspaces $V_{L,left}$ and $V_{L,right}$. As a representation of H_{n-1} we have $V_{L,left}^\vee \cong V_{L,right} \otimes B^r$ such that W_L^{middle} decomposes into an orthogonal direct sum of $W_L^{middle} \cap V_{L,left}$ and $W_L^{middle} \cap V_{L,right}$. Both nonisomorphic summands define irreducible representations of the middle type factor of the group G_{n-1} (the connected derived group).
Now we consider the second derivative, i.e the restriction of $V_L = V_\lambda = V_{\mathcal{L}}$ to the group H_{n-2}. The morphism $\phi = \phi_{n-2}$ is now defined and flips the two Lagrangian subspaces coming from the study of the first derivative. These two Lagrangians subspaces $V_{L,left}$ and $V_{L,right}$ remain orthogonal subspaces for the twisted pairing $(.,.)_{\mathcal{L}_{n-2}}$ and of course are invariant under the action of $H_{n-2} \subseteq H_{n-1}$. However now ϕ_{n-2} switches the two orthogonal Lagrangian subspaces.

Lemma C.14 *The involution $\phi = \phi_{n-2}$ defines an isomorphism of vector spaces*

$$\phi_{n-2} : V_{L,left} \to V_{L,right}$$

i.e. flips the two orthogonal Lagrangians.

Proof Indeed, ϕ_{n-1} is welldefined if we only consider the subgroups generated by $SL(n-1|n-1)$, and then G_{n-1} acts on the irreducible summands $W_L^{middle} \cap V_{L,left}$ and $W_L^{middle} \cap V_{L,right}$ nonisomorphically and these are irreducible representations of the middle type factor of the group G_{n-1}. If $SL(n-1|n-1)$ acts by π on $W_L^{middle} \cap V_{L,left} \subset H_{D_1}(L)$, then it acts on $\phi(W_L^{middle} \cap V_{L,left})$ by the dual representation, which is not isomorphic and irreducible and hence orthogonal to $W_L^{middle} \cap V_{L,left}$. By Schur's lemma the H_{n-1}-component is there therefore uniquely determined as the π^J-isotypic component. Since $\pi^J \cong \pi^\vee$, we obtain for the induced action of G_{n-1} on V_L that ϕ_{n-1} (defined on the level of $SL(n-1|n-1)$) switches the two irreducible summands $W_L^{middle} \cap V_{L,left}$ and $W_L^{middle} \cap V_{L,right}$ of G_{n-1} [The ladder type representations, where $H_{D_1}(L)$ may not be multiplicity free, do not cause a problem unless $n = 2$ since we consider only the middle derivative components]. □

C.6 Case 2: Second derivatives

Now let us consider the second derivatives. Besides $\partial^2\mathcal{F}_L \dots \mathcal{F}_L^{dual}$ and $\mathcal{F}_L \dots \partial^2\mathcal{F}_L^{dual}$ (again symbol writing) it produces spaced forests that appear with multiplicity two of type

$$\partial \mathcal{F}_L \ldots \partial \mathcal{F}_L^{dual}$$

(again symbolic writing with focus only on the second middle derivative). One of them, the right middle derivative of the left middle derivative of $\mathcal{F}_L$, defines an irreducible constituent subspace W_{n-2} of $V_{L,left}$, the other one, the left middle derivative of the right middle derivative of $\mathcal{F}_L$, defines an irreducible constituent subspace $W_{n-2}^{\vee}$ of $V_{\mathcal{L},right}$.

Example C.15

The cup diagram of λ

After deriving the second respectively third tree in this diagram, one obtains the cup diagrams

Derivative of the 2nd sector of λ

Derivative of the third sector of λ

Taking now either the derivative of the 2nd sector of the derivative of the 3rd sector or the derivative of the 3rd sector of the derivative of the 2nd sector gives the cup diagram

The cup diagram of $W_{n-2} \cong W_{n-2}^{\vee}$

Notice that $W_{n-2} \cong W_{n-2}^{\vee}$ as irreducible representations of G_{n-2} on $V_{\mathcal{L}} = H_{D_{n-2}}(\mathcal{L})$. Furthermore the isotypic component of this irreducible representation in $V_{\mathcal{L}}$ consist precisely of these two irreducible orthogonal constituents. By the last Lemma $\phi = \phi_{n-2}$ hence switches these two *Langragian subspaces* W_{n-2} and $W_{n-2}^{\vee}$ of the nondegenerate restriction of the pairing to $W_{n-2} \oplus W_{n-2}^{\vee}$.

Considered as a module under the semidirect product of G_{n-2} and ϕ_{n-2} this isotypic subspace remains stable and defines a representation of the semidirect product $G_{n-2}.\langle\phi_{n-2}\rangle$.

Lemma C.16 *As module of $G_{n-2}^* = G_{n-2}.\langle\phi_{n-2}\rangle$ the isotypic subspace $Y = W_{n-2} \oplus W_{n-2}^{\vee}$ contains each irreducible constituent with multiplicity 1.*

Proof We may assume that Y is not irreducible for G^*_{n-2}. Then $Y \cong W' \oplus W''$ with $Res(W') \cong Res(W'') \cong W_{n-2}$ as a representation of G_{n-2}. We have to show $W' \ncong W''$. Otherwise $Y \cong W' \oplus W' = 2W'$ and for every pair of constants $(\alpha, \beta) \neq 0$ and $0 \neq w = (w', 0) \in W$ the representation space $W_{\alpha,\beta}$ spanned by $(\alpha w', \beta w')$ is isomorphic to W' and hence invariant under ϕ_{n-2}. Since $Hom_{G_{n-2}}(Res(W'), Res(W)) = 2$, for suitable choice of $(\alpha, \beta) \neq 0$ the subspace $W_{\alpha,\beta}$ must coincide with the subspace W_{n-2} as a vectorspace and representation space of G_{n-2} (Schur's lemma). This would imply that also the subspace W_{n-2}, obtained for some special choice of (α, β), is stable under ϕ_{n-2}. Since $W_{n-2} \subset L_{L,left}$, ϕ_{n-2} flips into the orthogonal subspace $W^\vee_{n-2}$ of Y by Lemma C.14, this gives a contradiction and hence implies $W' \ncong W''$. □

Corollary C.17 *The parity of the pairing $(.,.)_{\mathcal{L}_n}$, or equivalently of $(.,.)_{\mathcal{L}_{n-2}}$, is therefore inherited to each of the non-isomorphic irreducible constituents (by multiplicity one).*

Corollary C.18 *In the situation where d_{middle} is odd, we have $\varepsilon_{W_{n-2}} = \varepsilon_{\mathcal{L}}$ for the irreducible module W_{n-2} of $GL(n-2|n-2)$ attached to the second middle derivative $\partial^2_{middle}\lambda$ of λ, i.e $W_{n-2} := \mathcal{L}''$ for the normalized representation $\mathcal{L}_{n-2} = \mathcal{L}''$ in $\mathcal{T}^{ext}_{n-2}$ that is attached to $L(\partial^2_{middle}\lambda)$ in $\mathcal{T}_{n-2}$.*

So for short, let $\varepsilon_{\mathcal{L}}$ and $\varepsilon_{\mathcal{L}_{n-2}}$ denote the parities of the parings $(.,.)_{\mathcal{L}}$ and of $(.,.)_{\mathcal{L}''}$, then we obtain from Corollary C.18

$$\varepsilon_{\mathcal{L}} = \varepsilon_{\mathcal{L}_{n-2}}.$$

Similarly let ε_λ and $\varepsilon_{\lambda_{n-2}}$ denote the parities of the pairings on $L(\lambda)$ and $L(\lambda_{n-2})$ for $\lambda_{n-2} = \partial^2_{middle}\lambda$. Then we conclude

$$\begin{aligned} \varepsilon_\lambda = \pm 1 \Longrightarrow \varepsilon_{\mathcal{L}} = \pm(-1)^{n/2} &\Longrightarrow \varepsilon_{\mathcal{L}_{n-2}} = \pm(-1)^{n/2} \\ &\Longrightarrow \varepsilon_{\lambda_{n-2}} = \pm(-1)^{n/2}(-1)^{(n-2)/2} = \mp 1 \ . \end{aligned}$$

The first shift comes from the normalization factor $(-1)^{nr/2} = (-1)^{n/2}$ that arises from the passage from L to $\mathcal{L}$ since r is odd and n is even (see [46]) by twisting with the invertible object $Ber^{-r/2}$ of parity $(-1)^{nr/2}$. The second assertion comes from Corollary C.18. The last assertion comes from the passage from $\mathcal{L}_{n-2}$ to $\tilde{\lambda} = \lambda_{n-2}$ again by twisting with a Berezin power.

C.7 Case 2: final step

We now reformulate this in terms of the parities for the objects $X_\lambda \in \mathcal{T}^+_n$ and $X_{\tilde{\lambda}} \in \mathcal{T}^+_{n-2}$. This passage produces an extra factor

$$(-1)^{p(\lambda)} = (-1)^{p(\lambda_{basic})+D(\lambda)}.$$

Now $D(\lambda) = nr/2$ is congruent to 1 resp. 0 mod 2 for $n = 2$ mod 4 resp. $n = 4$ mod 4. On the other hand $p(\lambda_{basic}) = n(n-1)/2 - ht(\mathcal{F}_\lambda) = n(n-1)/2$ mod 2 since $ht(\mathcal{F}_\lambda) = 2ht(\mathcal{F}_L)$ is even. For even n we have $n(n-1)/2 = 1$ mod 2 resp. 0 mod 2 if $n = 2$ mod 4 resp. $n = 4$ mod 4. Hence $2|p(\lambda)$ and

$$\varepsilon(X_\lambda) = \varepsilon_\lambda.$$

Finally we have $\varepsilon_{\lambda_{basic}} = (-1)^{p(\lambda_{basic})} = (-1)^{n(n-1)/2}$ from Lemma C.8. Similarly $\varepsilon_{\tilde{\lambda}_{basic}} = (-1)^{(n-2)(n-3)/2}$. Thus $\varepsilon_{\lambda_{basic}} = -\varepsilon_{\tilde{\lambda}_{basic}}$ implies

Corollary C.19 *The quotients $\varepsilon(X_\lambda)/\varepsilon(X_{\lambda_{basic}})$ and $\varepsilon(X_{\tilde{\lambda}})/\varepsilon(X_{\tilde{\lambda}_{basic}})$ coincide in the situation of our second assumption (r odd, n even).*

This being said, we immediately observe that the property $2|d_{middle}(\lambda)$ of λ is inherited by $\tilde{\lambda} = \lambda_{n-2}$. Also r remains the same. Hence by decending induction on even n we conclude

Theorem C.20 *For all irreducible objects X_λ in $\mathcal{T}_n^+$ we have*

$$\varepsilon(X_\lambda) = \varepsilon(X_{\lambda_{basic}})$$

and $\varepsilon(X_\lambda) = (-1)^{p(\lambda_{basic})}$.

Proof This holds for $n = 2$ by [37] (see Sect. 9), hence follows inductively from Corollary C.17 for all even n with odd r. The remaining cases, where d_{middle} is even or $\mathcal{F}_{middle} \neq \emptyset$, are covered by Lemma C.10. □

Appendix D. Technical lemmas on derivatives and superdimensions

D.1 Derivatives

Lemma D.1 *Suppose L is a simple module and suppose the trivial module $\mathbf{1}$ is a constituent in $H_D^0(L)$, then $L \cong \mathbf{1}$.*

Proof Suppose $H_D^0(L)$ contains $\mathbf{1}$ and suppose $L \ncong \mathbf{1}$. Then Theorem 4.1 implies that L has two sectors with sector structure $[-n+2, \ldots, 0, 1, \ldots, n-1]S_1$ and $r(S_1) = 1$, hence

$$L \cong Ber \otimes S^i$$

for some $i \geq n-1$, or has sector structure $S_2[-n+2, \ldots, 0, 1, \ldots, n-1]$ with $r(S_2) = 1$ and hence

$$L \cong (Ber \otimes S^i)^\vee$$

for some $i \geq n-1$. However $H_D^\nu(Ber \otimes S^i) \cong Ber \otimes H_D^{\nu-1}(S^i)$ vanishes unless $\nu - 1 = 0$ with $H_D^0(S^i) = S^i$ or $\nu - 1 = i - (n-1) \geq 0$, as follows from the next Lemma D.2. Hence this implies $H^0_D(Ber \otimes S^i) = 0$. Similarly then also $H^0_D((Ber \otimes S^i)^\vee) = 0$ holds by duality. This contradiction proves our claim. □

Lemma D.2 *Suppose $i \geq 1$. Then for S^i in $\mathcal{R}_n$ the cohomology is $H^\nu(S^i) = S^i$ for $\nu = 0$ and $H^\nu(S^i) = Ber^{-1}$ for $\nu = max(0, i-n+1)$, and $H^\nu(S^i)$ is zero otherwise.*

Proof An easy consequence of Theorem 4.1 and [36, Proposition 22.1]. □

The following lemma is an immediate consequence of Theorem 4.1 or Lemma D.1.

Lemma D.3 *$DS(L(\lambda))$ has a summand of superdimension 1 only if $L(\lambda) \cong Ber^r \otimes S^i$ for some r, i.*

Recall that an irreduble representation is weakly selfdual (or of type (SD)) if $L(\lambda)^\vee \cong Ber^r \otimes L(\lambda)$ for some $r \in \mathbb{Z}$.

Lemma D.4 *A (weakly) selfdual irreducible object $L = L(\lambda)$ with odd superdimension* sdim(L) *is a power of the Berezin determinant.*

Proof For (weakly) selfdual maximal atypical irreducible objects $L = L(\lambda)$ of odd dimension their plot has sectors $S_1, \ldots S_k$ from left to right of lengths say $2r_1, \ldots, 2r_k$ that must satisfy

$$r_{k+1-i} = r_i$$

and hence in particular $r_1 = r_k$. By [36, 51] the superdimension is divisible by the multinomial coefficient $n!/(\prod_i r_i!)$ for $n = \sum_i r_i$. Hence, in case $k \geq 2$, the superdimension is divisible by the integer $(r_1 + r_k)!/(r_1!r_k!)$, which is $(2r_1)!/(r_1)!(r_1)!$ and hence *even*. Therefore $\mathrm{sdim}(L) \notin 2\mathbb{Z}$ implies $k = 1$, i.e. the associated plot only has a single sector. For this sector, we may continue with the same argument using the recursion formula for the superdimension given in [36, 51]. □

Lemma D.5 *Let $L(\lambda)$ be a maximal atypical weight with k sectors of rank $r_1, \ldots, r_k$ and derivatives $L(\lambda_j)$, $j = 1, \ldots, k$. Then for all $j = 1, \ldots, k$*

$$\mathrm{sdim}(L(\lambda)) = \mathrm{sdim}(L(\lambda_j)) \cdot \frac{n}{r_j}.$$

Proof By the superdimension formula [36]

$$\mathrm{sdim}(L(\lambda)) = \binom{n}{r_1, \ldots, r_k} \cdot T(S_1, \ldots, S_k)$$

for a term $T(S_1, \ldots, S_k)$ that only depends on the sektors S_j such that

$$T(S_1, \ldots, S_k) = T(S_1, \ldots, \partial S_j, \ldots, S_k).$$

Since

$$\mathrm{sdim}(V_j) = \binom{n-1}{r_1, \ldots, r_j - 1, \ldots, r_k} T(S_1, \ldots, \partial S_i, \ldots, S_k)$$

this implies for all $j = 1, \ldots, k$

$$\mathrm{sdim}(L(\lambda)) = \mathrm{sdim}(L(\lambda_j)) \cdot \frac{n}{r_j}.$$

□

D.2 Small superdimensions

According to Lemma 8.1 a small representation belongs to one of four infinite families of regular cases or to a finite list of exceptional cases. The largest dimension occuring in the exceptional cases is 64 (the spin representations of D_7). Assume that V_λ restricted to G_λ splits as $V_\lambda = W_1 \oplus \ldots \oplus W_s$. We may assume $\dim(W_1) \leq \frac{1}{s}\dim(V_\lambda)$. The rank estimates in Sect. 10.3 show that W_1 belongs to the regular cases of Lemma 8.1 if $s \geq 3$. We therefore consider here the case where V_λ restricted to G_λ splits into at most two representations $V_\lambda = W \oplus W^\vee$. We want to rule out that W or $W^\vee$ is one of the exceptional cases. The dimension of W is $\dim(V_\lambda)/2$. Therefore we compute all superdimensions of irreducible weakly selfdual representations up to superdimension 128. Except for the numbers 20 and 56 none of the exceptional dimensions is equal to either the superdimension or half the superdimension of an irreducible weakly selfdual representation in $\mathcal{T}_n^+$.

Lemma D.6 *If* $[\lambda]$ *is a basic representation of* $\mathcal{T}_n^+$, *then* $[\lambda, 0]$ *is a basic representation of* $\mathcal{T}_{n+1}^+$ *of the same superdimension. Every basic representation of* $\mathcal{T}_{n+1}^+$ *with one sector is of this form.*

Therefore we can always assume that the irreducible representations have at least two sectors. Note also that a weakly selfdual representation cannot have an even number of sectors if n is odd. For a list of the basic representations in the case $n = 3$ and $n = 4$ we refer to the examples in Sect. 15.

D.2.1 Basic selfdual weights for $n = 5$

$[4, 3, 2, 1, 0]$,	sdim 120;	$[3, 3, 2, 0, 0]$,	sdim 30
$[4, 1, 1, 1, 0]$,	sdim 20;	$[1, 0, 0, 0, 0]$,	sdim 2
$[2, 1, 0, 0, 0]$,	sdim 6;	$[3, 2, 1, 0, 0]$,	sdim 24
$[2, 2, 0, 0, 0]$,	sdim 6;	$[3, 1, 1, 0, 0]$,	sdim 12

D.2.2 Basic selfdual weights for $n = 6$

By Lemma D.6 we can focus on the case of two or more sectors. These basic weights are listed below.

[5, 4, 3, 2, 1, 0],	sdim 720;	[3, 3, 3, 0, 0, 0],	sdim 20
[4, 3, 3, 1, 0, 0],	sdim 80;	[5, 1, 1, 1, 1, 0],	sdim 30
[4, 4, 2, 2, 0, 0],	sdim 90;	[5, 4, 2, 2, 1, 0],	sdim 360
[5, 3, 3, 1, 1, 0],	sdim 180;	[4, 3, 2, 2, 1, 0],	sdim 180

D.2.3 Basic selfdual weights for $n = 7$

By Lemma D.6 we can focus on the case of two or more sectors. Since n is odd, a weakly selfdual weight cannot have an even number of sectors. If the weight has ≥ 5 sectors, its superdimension exceeds 128. Therefore we list the basic SD weights with 3 sectors.

[4, 4, 4, 3, 0, 0, 0],	sdim 140;	[5, 5, 2, 2, 2, 0, 0],	sdim 210
[6, 1, 1, 1, 1, 1, 0],	sdim 30;	[6, 3, 3, 1, 1, 1, 0],	sdim 252
[6, 4, 3, 2, 1, 1, 0],	sdim 1008		

D.2.4 Basic selfdual weights for $n \geq 8$

If the weight has 2 sectors for $n \geq 8$, then the smallest possible superdimension is $\geq n!/((n/2)!(n/2)!)$. This equals the case $[\lambda] = [n/2, n/2, \dots, n/2, 0, 0, \dots, 0]$ (each $n/2$ times). For $n = 8$ the superdimension is then 70, for $n = 9$ it is already 252. All other weights with 2 sectors have superdimension > 128.

If the weight has 3 sectors and $n \geq 9$, the smallest superdimension is given by the hook weight $[n-1, 1, \dots, 1, 0]$ of superdimension $n(n-1)$. The next smallest superdimension is given by the irreducible representation $[n-1, 2, 1, \dots, 1, 0]$ of superdimension $2 \cdot n(n-1)$. For $n = 8$ these superdimensions are 56 and 112. For $n \geq 9$ the second case has superdimension larger than 128. In the first case the superdimensions are 72 ($n = 9$), 90 ($n = 10$), 110 ($n = 11$) and exceed 128 otherwise.

If $n \geq 8$ and the weight has ≥ 4 sectors, its superdimension exceeds 128.

D.2.5 Comparison with the exceptional cases

We compare the superdimensions above with the dimensions of the exceptional cases in Lemma 8.1. Except for the cases where the superdimension is 20 or 56 the dimensions are different. If the dimension is 20, then the irreducible representation is $\Lambda^3(st)$ for $SL(6)$. If the dimension is 56, then the irreducible representation is either $\Lambda^3(st)$ for $SL(8)$ or the irreducible representation of minimal dimension of E_7.

If V_λ or W is of the form $\Lambda^3(st)$, then so is its restriction $Res(W)$ to $G_{\lambda'}$ since Λ^3 commutes with Res, in contradiction to the induction assumption.

In the dim $= 56$-case with V_λ irreducible upon restriction to G_λ, the corresponding $L(\lambda)$ is the hook weight $[n-1, 2, 1, \dots, 1, 0, \dots, 0]$ for $n \geq 8$. For $n = 8$

$$DS(L(\lambda)) \cong Ber \otimes S^6 \oplus (Ber \otimes S^6)^\vee \oplus [7, 1, 1, 1, 1, 1, 0].$$

The connected derived Tannaka group of $[7, 1, 1, 1, 1, 1, 0]$ is either $SO(42)$, $Sp(42)$ or $SL(24)$ and doesn't embed into E_7. If $n \geq 9$, the hook weight $[n-1, 2, 1, \ldots, 1, 0, \ldots, 0]$ has one sector and therefore one derivative, hence the corresponding Tannaka group contains either $SO(42)$, $Sp(42)$ or $SL(24)$.

If V_λ decomposes as $W \oplus W^\vee$ and $\dim(W) = 56$, then $\dim(V_\lambda) = 112$. This happens for $L(\lambda) \cong [n-1, 2, 1, \ldots, 1, 0]$ for $n = 8$. In this case

$$DS(L(\lambda)) \cong [7, 2, 1, \ldots, 1] \oplus [7, 2, 1, \ldots, 1]^\vee.$$

Since this weight is NSD, its connected derived Tannaka group is $SL(56)$ which doesn't embed into E_7.

D.2.6 The regular cases

We can now assume that we are in one of the regular cases of Lemma 8.1. If V is either $S^2(st)$, $S^2(st^\vee)$, $\Lambda^2(st)$, $\Lambda^2(st^\vee)$ or the nontrivial irreducible representation of $\Lambda^2(st)$ in the C_r-case, we get a contradiction to the induction assumption since restriction commutes with Schur functors. Therefore the representation is a standard representation or its dual for type A, B, C, D.

Corollary D.7 *If the selfdual irreducible representation V_λ is irreducible upon restriction to G_λ or splits in the form $W \oplus W^\vee$, the group G_λ is a simple group of type ABCD and V_λ and W are its standard representation (or its dual.)*

Appendix E. Clean decomposition

The ambiguity in the determination of G_λ is only due to the fact that we cannot exclude special elements with 2-torsion in $\pi_0(H_n)$. We show that $I \cong \mathbf{1}$ if $I \otimes I^\vee \simeq \mathbf{1} \oplus Proj$ holds. We then discuss the occurence of projective summands in tensor products of irreducible modules and show that $I \cong \mathbf{1}$ in some cases for $n = 4$.

E.1 The module I

The object I from Sect. 11.7 that realizes the surjective projection $H_\lambda = \mu_2 = \langle w \rangle$ corresponds to an element with the following properties.

Lemma E.1 *The module I is indecomposable with* $\mathrm{sdim}(I) = 1$. *It satisfies $I^\vee \cong I$ and $I^* \cong I$. Moreover:*

(1) *There exists an irreducible object L of $\mathcal{T}_n^+$. such that I occurs (with multiplicity one) as a direct summand in $L \otimes L^\vee$.*
(2) *$L \otimes I \cong L \oplus N$ for some negligible object N.*
(3) *$DS(I)$ is $\tilde{I} \oplus$ negligible for an indecomposable object $\tilde{I}$ concentrated in degree 1 of superdimension 0 satisfying $\tilde{I}^\vee = \tilde{I}$ and $\tilde{I}^* \cong \tilde{I}$. If we assume the stronger structure theorem for $n-1$ by induction, $DS(I)$ is $\mathbf{1}$ plus some negligible object.*

Proof That I is indecomposable of superdimension 1 is obvious. For (2) notice that by our analysis in Clifford theory in Sect. 10 we got

$$V_\lambda \cong Ind_{H_1}^H(W)$$

for a subgroup H_1 of index 2 between H^0 and H. Here H_1 is simply $ker(\varepsilon)$. This implies that I appears as an indecomposable constituent of superdimension 1 in $L(\lambda) \otimes L(\lambda)^\vee$ that is not isomorphic to the trivial representation. Since $Ind_{H_1}^H(W)$ is irreducible, Clifford theory implies that $V_\lambda \otimes \mu \cong V_\lambda$ which implies (2). Since $(L \otimes L^\vee)^\vee \cong L \otimes L^\vee$ and $I^\vee$ cannot be isomorphic to the trivial representation, this implies $I^\vee \cong I$, and similarly $(L \otimes L^\vee)^* \cong L \otimes L^\vee$ implies $I^* \cong I$. Property (3) follows since I is selfdual of superdimension 1, and so its cohomology is concentrated in degree 0. By definition I is a retract of $L \otimes L^\vee$, so $DS(I)$ is a retract of $DS(L) \otimes DS(L)^\vee$. If we assume that the stronger structure theorem holds for $n-1$, the only summands of superdimension 1 in a tensor product $L(\lambda_i) \otimes L(\lambda_i)^\vee$ are Berezin powers by Corollary 12.4, hence $DS(I) \cong \mathbf{1} \oplus N$. □

Conjecture E.2 $I \simeq \mathbf{1}$.

We are unable to prove this result at the moment. This conjecture immediately implies that V_λ stays irreducible under restriction to G_λ and therefore would prove the stronger version of the structure theorem.

E.2 Endotrivial modules

Our condition $I^{\otimes 2} \simeq \mathbf{1} \oplus N$ resembles the definition of an endotrivial representation.

Lemma E.3 *The following conditions are equivalent:*

(1) $I^{\otimes 2} \simeq \mathbf{1} \oplus Proj$.
(2) $DS(I) = \mathbf{1}$.

Proof If $DS(I) = \mathbf{1}$, then $\text{sdim}(I) = 1$, hence $I^{\otimes 2} \simeq \mathbf{1} \oplus negl$. But $ker(DS)$ (restricted to $\mathcal{T}_n^+$) is $Proj$. □

Modules M with the property $M \otimes M^\vee \simeq \mathbf{1} \oplus Proj$ are called endotrivial. If I satisfies $I^{\otimes 2} \simeq \mathbf{1} \oplus Proj$ or equivalently $DS(I) \simeq \mathbf{1}$, I is endotrivial (since $I^\vee \simeq I$).

Theorem E.4 [49] *All endotrivial modules for $\mathcal{T}_n$ are of the form $Ber^j \otimes \Omega^i(\mathbf{1}) \oplus Proj$ or $\Pi(Ber^j \otimes \Omega^i(\mathbf{1})) \oplus Proj$ for some $i, j \in \mathbb{Z}$ where $\Omega^i(\mathbf{1})$ denotes the i-th syzygy of* $\mathbf{1}$.

We remark that we can split the projective resolution defining the $\Omega^i(M)$ into exact sequences

$$1 \to \Omega^i(M) \to P \to \Omega^{i-1}(M) \to 1$$

with some projective object P. It follows $\text{sdim}(\Omega^i(M) = -\text{sdim}(\Omega^{i-1}(M)$ since $\text{sdim}(P) = 0$.

Lemma E.5 *If $I^{\otimes 2} \simeq \mathbf{1} \oplus Proj$ with I as above, then $I \simeq \mathbf{1}$.*

Proof By restricting to $SL(n|n)$ we can ignore Berezin twists. By the classification of endotrivial modules $I \simeq \Pi^j \Omega^i(\mathbf{1}) \oplus Proj$ for some $i, j \geq 0$. Hence according to our list of properties of I

$$L \otimes \Pi^j \Omega^i(\mathbf{1}) \oplus Proj \cong L \oplus Proj.$$

On the other hand $\Omega^i(M) \otimes N \simeq \Omega^i(M \otimes N) \oplus Proj$ holds for all N and i. Hence for $M \simeq \mathbf{1}$ we would have (using $\Pi\Omega^i(M)) = \Omega^i(\Pi M))$

$$L \otimes \Pi^i \Omega^i(\mathbf{1}) \simeq \Omega^i(\Pi^j L) \oplus Proj \simeq L \oplus Proj$$

which is absurd since $\Omega^i(\Pi^j L) \not\cong L \oplus Proj$ for $i > 0$ (the latter is impossible by [38, Theorem 11.3]). In fact using the short exact sequences

$$1 \to \Omega^i(M) \to P \to \Omega^{i-1}(M) \to 1$$

and $DS(Proj) = 0$ we obtain

$$H^l(\Omega^i(L)) \simeq H^{l+i}(L).$$

Hence $i = 0$ and so $I \simeq \Omega^0(\mathbf{1}) \simeq \mathbf{1}$. □

E.3 Clean decomposition

We say a direct sum is *clean* if none of the summands is negligible. We say a negligible module N in $\mathcal{T}_n$ is potentially projective of degree r if $DS^{n-r}(N) \in \mathcal{T}_r$ is projective and $DS^i(N)$ is not for $i \leq n - r$.

Now consider the special representations S^i. Then we proved in [37] the surprising fact that the projection of $S^i \otimes S^j$ or $S^i \otimes (S^j)^\vee$ on the maximal atypical block is clean. To prove the result we establish the $n = 2$-case by a brute force calculation. The theory of mixed tensors [35] then shows that the Loewy length of any summand in $S^i \otimes S^j$ is less or equal to 5. This implies the result since the Loewy length of a projective cover is $2n + 1$.

Lemma E.6 *Every maximal atypical negligible summand in a tensor product $L(\lambda) \otimes L(\mu)$ is potentially projective of degree at least 3.*

Proof The decomposition of $S^i \otimes S^j$ in $\mathcal{R}_2$ is clean. Further DS sends negligible modules in $\mathcal{T}_n^+$ to negligible modules in $\mathcal{T}_{n-1}^+$ and the kernel of DS on $\mathcal{T}_n^+$ consists of the projective elements. Since $DS^{n-2}(L(\lambda) \otimes L(\mu)) \in \mathcal{T}_2$ splits into a direct sum of irreducible representations of the form $Ber^j S^i$ for some $i, j \in Z$ by the main theorem of [36], $DS^{n-2}(N) = 0$. □

We show below that the decomposition of the tensor product $L(\lambda) \otimes L(\mu)$ is also clean in the case $n = 3$ unless $\lambda_{basic} = \mu_{basic} = (2, 1, 0)$ and that projective summands can occur only under strong restrictions in the case $n = 4$.

Question. Let $L(\lambda)$, $L(\mu)$ be maximal atypical. Is the projection of the decomposition of $L(\lambda) \otimes L(\mu)$ on the maximal atypical block always clean?

Lemma E.7 *If all tensor product decompositions $L(\lambda) \otimes L(\mu)$ for maximal atypical $L(\lambda)$, $L(\mu)$ are clean, $I \simeq \mathbf{1}$.*

Proof We assume by induction that the stronger structure theorem is proven for $\mathcal{T}^+_{n-1}$. Then by Lemma E.1 we have $DS(I) \cong \mathbf{1} \oplus N$ for some negligible object N. Note that $DS(I)$ is a direct summand in $DS(L(\lambda)) \otimes DS(L(\lambda))^\vee$. If this tensor product decomposition is clean, we obtain $N = 0$, hence $DS(I) \cong \mathbf{1}$. But then Lemma E.3 implies $I \otimes I^\vee \cong \mathbf{1} \oplus Proj$, hence I is endotrivial. However the higher syzigies of Theorem E.4 are not $*$-invariant and non-trivial Berezin twists are not selfdual, hence $I \cong \mathbf{1}$. □

To prove that decompositons are always clean, it would be enough to prove that the tensor product of two irreducible maximally atypical representations never contains a maximally atypical projective summand since repeated applications of DS to a negligible representation results in a direct sum of projective representations. A positive answer to this question would also imply that the tensor product decomposition of two maximal atypical irreducible representations behaves classically after projection to the maximal atypical block (and not just modulo vanishing superdimension).

Example E.8 If I is a direct summand in $[2, 2, 0, 0]^{\otimes 2}$ as above, then $I \cong \mathbf{1}$. This follows from the S^i-computations. Consider $L = [2, 2, 0, 0] \in \mathcal{R}_n$. Then $DS(I) \simeq \mathbf{1}$. In fact $DS(L) = [2, 2, 0] + B^{-1}S^3$. Hence $DS(L) \otimes DS(L)$ is a tensor product involving only S^i's and their duals (or their Berezin twists). Their decomposition is clean according to [37]. Hence any negligible module in $[2, 2, 0, 0] \otimes [2, 2, 0, 0]$ maps to zero under DS. In particular $DS(I) = \mathbf{1}$ and hence $I \simeq \mathbf{1}$.

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