

The inner Cauchy horizon of axisymmetric and stationary black holes with surrounding matter‡

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Abstract. We investigate the interior of regular axisymmetric and stationary black holes surrounded by matter and find that for non-vanishing angular momentum of the black hole the space time can always be extended regularly up to and including an inner Cauchy horizon. We provide an explicit relation for the regular metric at the inner Cauchy horizon in terms of that at the event horizon. As a consequence, we obtain the universal equality $(8\pi J)^2 = A^+ A^-$ where J is the black hole's angular momentum and A^- and A^+ denote the horizon areas of inner Cauchy and event horizon, respectively. We also find that in the limit $J \rightarrow 0$ the inner Cauchy horizon becomes singular.

PACS numbers: 04.70.Bw, 04.40.-b, 04.20.Cv

1. Introduction

An interesting feature of the well-known Kerr solution is the existence of a Cauchy horizon \mathcal{H}^- inside the black hole. While outside the black hole the two Killing vectors ξ and η , describing stationarity and axisymmetry, can always be linearly combined to form a time-like vector, any such non-trivial linear combination inevitably leads to a space-like vector when performed in some interior neighbourhood of the event horizon. As a consequence, the axisymmetric and stationary Einstein equations, being elliptic in the black hole's exterior, become hyperbolic in its interior. Hence, for the Kerr solution a boundary of the future domain of dependence of the event horizon \mathcal{H}^+ can be identified, and this is the inner Cauchy horizon in question. As far as the mathematical form of the field equations is concerned, \mathcal{H}^- is completely equivalent to \mathcal{H}^+ . However, from a physical point of view, the inner Cauchy horizon is a *future* horizon whereas the event horizon is a *past* one. While the space-time is always regular at \mathcal{H}^+ , it is regular at \mathcal{H}^- only if the black hole's angular momentum J does not vanish, i.e. for $J \rightarrow 0$ the horizon \mathcal{H}^- becomes singular.

In this paper we find that this picture also holds true for the black hole's interior of general axisymmetric and stationary space-times which contain a regular black hole and

‡ This paper is dedicated to Reinhard Meinel on the occasion of his 50th birthday.

surrounding matter[§]. We are able to provide an explicit relation between the metric at the inner Cauchy horizon and that at the event horizon. Moreover a universal equality

$$(8\pi J)^2 = A^+ A^- \quad (1)$$

results where A^- and A^+ denote the horizon areas of inner Cauchy and event horizon respectively.

The paper is organized as follows. In Sec. 2 we recall Weyl's coordinates which cover an exterior vacuum vicinity of the black hole. In order to describe the black hole interior we introduce Boyer-Lindquist type coordinates. We revisit the formulation in terms of the complex Ernst potential for which the Einstein equations can be combined in the complex Ernst equation. In Sec. 3 we use this formulation to write the Ernst potential f , describing the exterior vicinity of a black hole, as a *Bäcklund transform* of another Ernst potential f_0 which corresponds to a space-time without a black hole, but with a regular central vacuum region^{||}. In Sec. 4 we take the Bäcklund representation in order to expand f into the interior of the black hole. It turns out that by utilizing appropriate symmetry properties of f_0 , an explicit formula of the Ernst potential f at the inner Cauchy horizon \mathcal{H}^- in terms of that at the event horizon \mathcal{H}^+ can be derived. Finally, from this formula we are able to conclude the universal equality (1), see Sec. 5.

2. Weyl coordinates and Ernst equation

In a vacuum vicinity of the black hole's event horizon[¶] the Ernst potential is most easily introduced by utilizing the line element in Weyl coordinates $(\varrho, \zeta, \varphi, t)$:

$$ds^2 = e^{-2U} [e^{2k}(d\varrho^2 + d\zeta^2) + \varrho^2 d\varphi^2] - e^{2U} (dt + a d\varphi)^2, \quad (2)$$

where the metric potentials U , k and a are functions of ϱ and ζ alone. Along the rotation axis, $\varrho = 0$, $|\zeta| \geq 2r_h$, the axial Killing vector η vanishes identically. The event horizon \mathcal{H}^+ is a degenerate surface when considered in Weyl coordinates. It is located at $\varrho = 0$, $-2r_h \leq \zeta \leq 2r_h$, see Fig. 1, left panel.

The constant $r_h > 0$ describes a coordinate radius of the event horizon in Boyer-Lindquist type coordinates (R, θ, φ, t) which are introduced via

$$\varrho^2 = 4(R^2 - r_h^2) \sin^2 \theta, \quad \zeta = 2R \cos \theta. \quad (3)$$

These coordinates allow us to expand the metric coefficients into the interior of the black hole⁺. The horizons \mathcal{H}^\pm are located at $R = \pm r_h$, see Fig. 1, right panel.

The complex Ernst potential f combines metric functions,

$$f = e^{2U} + ib, \quad (4)$$

[§] Note that we concentrate here on pure gravity (i.e. no electromagnetic fields) with vanishing cosmological constant.

^{||} A well-known example for this procedure is the construction of the Kerr solution from Minkowski space $f_0 = 1$, see e.g. [13].

[¶] For a stationary black hole space-time, the immediate vicinity of the event horizon must be vacuum, see. e.g. [4].

⁺ The interior ($-r_h < R < r_h$) corresponds to negative values of ϱ^2 .

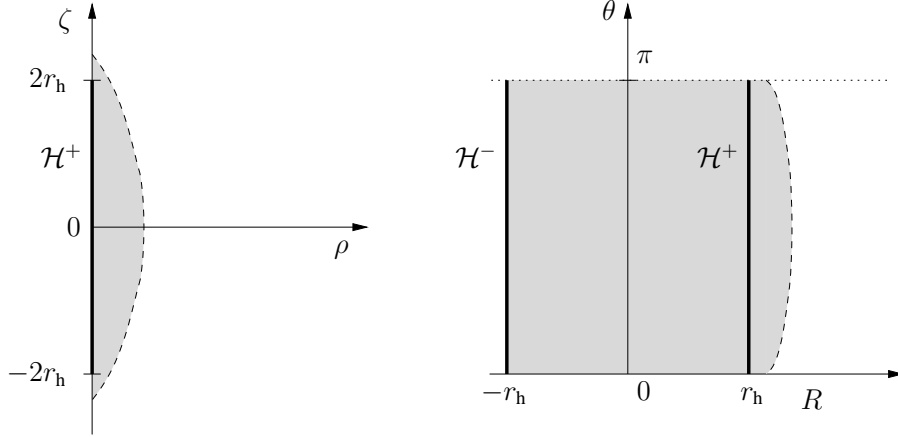


Figure 1. Sketch of a part of a black hole space-time in Weyl coordinates (*left panel*) and Boyer-Lindquist type coordinates (*right panel*). We study a vacuum region in an exterior vicinity of the event horizon and the interior of the black hole (grey areas).

where the *twist potential* b is related to the coefficient a via

$$a_{,\varrho} = \varrho e^{-4U} b_{,\zeta}, \quad a_{,\zeta} = -\varrho e^{-4U} b_{,\varrho}, \quad (5)$$

or, in terms of R and θ ,

$$a_{,R} = -2 \sin \theta e^{-4U} b_{,\theta}, \quad a_{,\theta} = 2(R^2 - r_h^2) \sin \theta e^{-4U} b_{,R}. \quad (6)$$

The vacuum Einstein equations are equivalent to the *Ernst equation* [7], which reads in Weyl coordinates as

$$(\Re f) \left(f_{,\varrho\varrho} + f_{,\zeta\zeta} + \frac{1}{\varrho} f_{,\varrho} \right) = f_{,\varrho}^2 + f_{,\zeta}^2 \quad (7)$$

and in Boyer-Lindquist type coordinates:

$$(\Re f) \left[(R^2 - r_h^2) f_{,RR} + 2R f_{,R} + f_{,\theta\theta} + \cot \theta f_{,\theta} \right] = (R^2 - r_h^2) f_{,R}^2 + f_{,\theta}^2. \quad (8)$$

Note that k can be calculated by a line integral once f is known, see e.g. [2].

For convenience we introduce the following metric functions that are, for a regular black hole, positive and analytic in terms of R and $\cos \theta$ in the black hole vicinity, see [4, 10]:

$$\hat{\mu} := 4e^{2k-2U} (R^2 - r_h^2 \cos^2 \theta), \quad \hat{u} := 4(R^2 - r_h^2) e^{-2U} - \frac{a^2}{\sin^2 \theta} e^{2U}. \quad (9)$$

At \mathcal{H}^+ , the gravito-magnetic potential ω , likewise analytic in R and $\cos \theta$ and defined through

$$\omega := \frac{ae^{4U}}{4(R^2 - r_h^2) \sin^2 \theta - a^2 e^{4U}}, \quad (10)$$

assumes the constant value ω^+ describing the angular velocity of the event horizon.

Because of the degeneracy of \mathcal{H}^+ in Weyl coordinates, the potential f is, for $\varrho = 0$, only a C^0 -function in terms of ζ . However, as the functions $\hat{\mu}$, \hat{u} and ω , also f is analytic with respect to the Boyer-Lindquist type coordinates R and $\cos \theta$.

In the following sections we shall see that our conclusions only work if (i) $\omega^+ \neq 0$ and (ii) $b(\varrho = 0, \zeta = 2r_h) \neq b(\varrho = 0, \zeta = -2r_h)$. However, both situations may occur. If a rotating black hole with $J \neq 0$ is dragged along by the motion of a surrounding, sufficiently relativistic counter-rotating torus then (i) the horizon angular velocity or (ii) the black hole's Komar mass may vanish (see [3]) which would correspond to the two situations in question. Nevertheless, in such a case our considerations can still be applied if one uses the Ernst formulation in a *rotating* frame of reference $\varphi' = \varphi + \Omega t$, $\Omega = \text{constant}$. It can be shown that any such rotating system with $\Omega \neq \pm \kappa^+ A^+ / (8\pi J)$ (' \pm ' for the cases (i) and (ii) respectively) and $\Omega \neq 0$ could then be taken, where κ^+ is the black hole's surface gravity of the event horizon. Therefore, without loss of generality we shall henceforth assume that $\omega^+ \neq 0$ and $b(\varrho = 0, \zeta = 2r_h) \neq b(\varrho = 0, \zeta = -2r_h)$.

3. Bäcklund transformation

The Bäcklund transformation is a particular soliton method, which creates a new solution from a previously known one. For the Ernst equation this technique can be applied to construct a large number of axisymmetric and stationary space-time metrics [13, 8, 12, 1, 2]. In this paper we consider the Bäcklund transformation in order to write an arbitrary regular axisymmetric, stationary black hole solution f in terms of a potential f_0 , which describes a space-time without a black hole, but with a completely regular central vacuum region.

Theorem 3.1. *Consider a regular axisymmetric and stationary black hole solution f describing a sufficiently small exterior vacuum vicinity V of the event horizon \mathcal{H}^+ . Then an Ernst potential $f_0 = e^{2U_0} + ib_0$ of a space-time without a black hole can be identified with the following properties:*

- (i) f_0 is defined in a vicinity of the axis section $\varrho = 0, |\zeta| \leq 2r_h$.
- (ii) In this vicinity, f_0 is an analytic function of ϱ and ζ and an even function of ϱ .
- (iii) The axis values of f_0 in terms of those of f for $\varrho = 0, |\zeta| \leq 2r_h$ are given by

$$f_0 = \frac{i [2r_h(b_N^+ + b_S^+) - (b_N^+ - b_S^+)\zeta] f + 4r_h b_N^+ b_S^+}{4r_h f - i [2r_h(b_N^+ + b_S^+) + (b_N^+ - b_S^+)\zeta]}, \quad (11)$$

where $b_N^+ = b(\varrho = 0, \zeta = 2r_h)$ and $b_S^+ = b(\varrho = 0, \zeta = -2r_h)$ (twist potential values at north and south pole of \mathcal{H}^+).

From this Ernst potential f_0 the original potential f can be recovered in all of V by means of an appropriate Bäcklund transformation of the following form:

$$f = \frac{\begin{vmatrix} f_0 & 1 & 1 \\ \bar{f}_0 & \alpha_1 \lambda_1 & \alpha_2 \lambda_2 \\ f_0 & \lambda_1^2 & \lambda_2^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ -1 & \alpha_1 \lambda_1 & \alpha_2 \lambda_2 \\ 1 & \lambda_1^2 & \lambda_2^2 \end{vmatrix}}, \quad (12)$$

where

$$\lambda_i = \sqrt{\frac{K_i - i\bar{z}}{K_i + iz}}, \quad i = 1, 2, \quad K_1 = -2r_h, \quad K_2 = 2r_h \quad (13)$$

with the complex coordinates $z = \varrho + i\zeta$, $\bar{z} = \varrho - i\zeta$, and α_1, α_2 are solutions to the Riccati equations

$$\alpha_{i,z} = -(\lambda_i \alpha_i^2 + \alpha_i) \frac{f_{0,z}}{2e^{2U_0}} + (\alpha_i + \lambda_i) \frac{\bar{f}_{0,z}}{2e^{2U_0}}, \quad (14)$$

$$\alpha_{i,\bar{z}} = -\left(\frac{1}{\lambda_i} \alpha_i^2 + \alpha_i\right) \frac{f_{0,\bar{z}}}{2e^{2U_0}} + \left(\alpha_i + \frac{1}{\lambda_i}\right) \frac{\bar{f}_{0,\bar{z}}}{2e^{2U_0}} \quad (15)$$

with

$$\alpha_i \bar{\alpha}_i = 1. \quad (16)$$

Proof. At first we show that the axis values of f_0 as given in (11) form an analytic function with respect to ζ . Using the analyticity of \hat{u} (being strictly positive) and ω with respect to $\cos\theta$ as well as Eqs. (9) and (10), we may express e^{2U} and b on \mathcal{H}^+ as

$$e^{2U} = -(\omega^+)^2 \hat{u} \sin^2\theta, \quad b = \frac{1}{2} [b_N^+ + b_S^+ + (b_N^+ - b_S^+) \cos\theta] + A \sin^2\theta, \quad (17)$$

with A also being an analytic function in $\cos\theta$. Then (11) leads to

$$f_0 = \frac{(b_N^+ - b_S^+)^2}{4[(\omega^+)^2 \hat{u} - iA]} + \frac{i}{2} [b_N^+ + b_S^+ - (b_N^+ - b_S^+) \cos\theta], \quad (18)$$

which is analytic in $\cos\theta$ (and hence in ζ) and has a strictly positive real part (recall that, without loss of generality, $\omega^+ \neq 0$ and $b_N^+ \neq b_S^+$, see end of Sec. 2). Starting from the boundary values (11), we can expand f_0 analytically and uniquely into some neighborhood of the axis part $\varrho = 0$, $|\zeta| \leq 2r_h$ by virtue of the Ernst equation, see [9, 14]. This means that in contrast to f the Ernst potential f_0 is analytic with respect to the Weyl coordinates (ϱ, ζ) within this neighborhood. Most important for later use, this axisymmetric expansion of f_0 is *even* in ϱ .

Now we show that a Bäcklund transformation, applied to f_0 , returns our original Ernst potential f . To this end we need to choose appropriate integration constants for the above Riccati equations.

The equations (14) and (15) can be solved explicitly on the horizon \mathcal{H}^+ , where $\lambda_i = \pm 1$ holds. We are free to choose the sign convention $\lambda_1 = -1$, $\lambda_2 = 1$. Then on \mathcal{H}^+ the equations (14) and (15) reduce to

$$\alpha_{i,\zeta} = \frac{\alpha_i + \lambda_i}{2e^{2U_0}} (-\lambda_i \alpha_i f_{0,\zeta} + \bar{f}_{0,\zeta}) \quad (19)$$

with the solution

$$\alpha_1(\zeta) = -\frac{\bar{f}_0(\zeta) + i\gamma_1}{f_0(\zeta) - i\gamma_1}, \quad \alpha_2(\zeta) = \frac{\bar{f}_0(\zeta) + i\gamma_2}{f_0(\zeta) - i\gamma_2}. \quad (20)$$

The integration constants γ_i are *real* numbers in order to guarantee (16).

In terms of the coordinates R and θ , the above Bäcklund transformation Eq. (12) reads generally as follows:

$$f = \frac{[\alpha_1(R + r_h \cos \theta) - \alpha_2(R - r_h \cos \theta)]f_0 + 2r_h \bar{f}_0}{\alpha_1(R + r_h \cos \theta) - \alpha_2(R - r_h \cos \theta) - 2r_h}. \quad (21)$$

On the horizon \mathcal{H}^+ ($R = r_h$), Eqns. (20) and (21) lead to an Ernst potential with the values $i\gamma_1$ and $i\gamma_2$ on the north and south pole of \mathcal{H}^+ respectively. Now, if we choose consistently $\gamma_1 = b_N^+$ and $\gamma_2 = b_S^+$, then (21) becomes equivalent to (11). Since in a vicinity of \mathcal{H}^+ the Ernst potential f is uniquely determined by its horizon values (due to a theorem by Hauser and Ernst [9]), we recover the original solution f in this vicinity. \square

4. The Ernst potential on the Cauchy horizon

In this section we expand the exterior Ernst potential f into the interior of the black hole, i.e. to the region $R \in [-r_h, r_h]$. As mentioned in Sec. 2, for regular black holes the Ernst potential f is analytic with respect to R and $\cos \theta$ in an exterior vicinity of \mathcal{H}^+ . Hence we can expand it analytically into an *interior* vicinity of \mathcal{H}^+ . Then, due to a theorem by Chruściel (theorem 6.3 in [6]*), the potential f exists as a regular solution of the interior Ernst equation for all values $(R, \cos \theta) \in (-r_h, r_h] \times [-1, 1]$, i.e. within a region that only excludes the Cauchy horizon \mathcal{H}^- ($R = -r_h$). In the following we obtain an explicit formula for f on \mathcal{H}^- in terms of the boundary data on \mathcal{H}^+ , which shows that f is also regular on \mathcal{H}^- provided that $J \neq 0$ holds.

A crucial role for our considerations is played by the fact that f_0 is even in ϱ , see discussion in Sec. 3. Hence, in terms of the Boyer-Lindquist type coordinates (3), f_0 is an analytic function of $(R^2 - r_h^2) \sin^2 \theta$ and $R \cos \theta$. The analytic expansion of f_0 into the region $R < r_h$ retains this property. As a consequence we find that the interior boundary values $f_0(R, \cos \theta = \pm 1)$, $-r_h \leq R \leq r_h$, as well as $f_0(-r_h, \cos \theta)$ are given in terms of the values at $R = r_h$. Also it follows that f_0 is regularly defined in a sufficiently small vicinity of the boundary of the interior region, see figure 2. Specifically we obtain

$$f_0(R = -r_h, \cos \theta) = f_0(R = +r_h, -\cos \theta). \quad (22)$$

These properties allow us to construct f on \mathcal{H}^- from f_0 via the Bäcklund transformation (12). As done for the proof of theorem 3.1, we solve the Riccati equations (14) and (15), now on \mathcal{H}^- and for $\cos \theta = \pm 1$ in accordance with the solution on \mathcal{H}^+ , thereby obtaining a unique continuous function f defined on the entire boundary of the interior region. In particular we find the following.

Theorem 4.1. *Any Ernst potential f of a regular axisymmetric and stationary black hole space-time with angular momentum $J \neq 0$ can be regularly extended into the interior of the black hole up to and including an interior Cauchy horizon, described by $R = -r_h$*

* We note that the interior space-time region of axisymmetric and stationary black holes is closely related to Gowdy space-times. In particular, we obtain Chruściel's form of the Gowdy space-time metric by substituting $R = r_h \cos T$ and $\theta = \psi$. More information will be presented in [11].

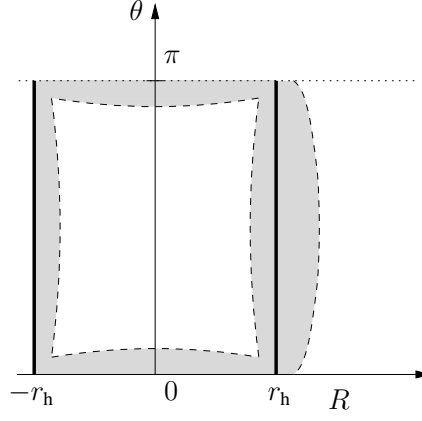


Figure 2. The seed function f_0 can be regularly defined in at least the grey areas.

in Boyer-Lindquist type coordinates (R, θ) . The values of f on the Cauchy horizon are given by

$$f(R = -r_h, \cos \theta) = \frac{i[\delta_1 + \delta_2 - (\delta_1 - \delta_2) \cos \theta] f_0(R = r_h, -\cos \theta) + 2\delta_1 \delta_2}{2f_0(R = r_h, -\cos \theta) - i[\delta_1 + \delta_2 + (\delta_1 - \delta_2) \cos \theta]} \quad (23)$$

with \ddagger

$$\delta_1 = \frac{b_S^+(b_N^+ - b_S^+) + 2b_N^+(b_{,\theta\theta})_N^+}{b_N^+ - b_S^+ + 2(b_{,\theta\theta})_N^+}, \quad \delta_2 = \frac{b_N^+(b_N^+ - b_S^+) + 2b_S^+(b_{,\theta\theta})_N^+}{b_N^+ - b_S^+ + 2(b_{,\theta\theta})_N^+} \quad (24)$$

where the scripts ‘+’ and ‘N/S’ indicate that the corresponding value of b or its second θ -derivative has to be taken at the event horizon’s north or south pole respectively. The values of the seed solution f_0 for $R = r_h$ follow via (11) from f on the event horizon. For $J \rightarrow 0$ the Cauchy horizon becomes singular.

5. A universal equality

With the relation between $f|_{\mathcal{H}^+}$ and $f|_{\mathcal{H}^-}$ we are able to prove the equality (1). Angular momentum J and horizon areas A^\pm are given as follows in terms of the Ernst potential:

$$J = \frac{1}{8\pi} \oint_{\mathcal{H}^\pm} \eta^{a;b} dS_{ab} = -\frac{1}{16} \int_0^\pi \hat{u}^2 \omega_{,R}|_{\mathcal{H}^\pm} \sin^3 \theta d\theta = 2 \frac{b_S^+ - b_N^+}{(b_{,R})_N^+} - \frac{r_h}{4} (b_{,R})_N^+, \quad (25)$$

$$A^\pm = 2\pi \int_0^\pi \sqrt{\hat{\mu}\hat{u}}|_{\mathcal{H}^\pm} \sin \theta d\theta = 4\pi \hat{u}_N^\pm = \pm \frac{32\pi r_h}{(e^{2U})_N^\pm}, \quad (26)$$

Here we used (9), (10) and equation (18) of [10] (together with a corresponding version valid on \mathcal{H}^-). The derivative $f_{,R}$ on \mathcal{H}^\pm can be calculated from f and its θ -derivatives by considering the Ernst equation (8) which becomes degenerate at $R = \pm r_h$. In particular, at the north and south poles where $e^{2U} = 0$ and $f_{,\theta} = 0$, we obtain via L’Hospital’s rule

$$f_{,R} = \pm i \frac{f_{,\theta\theta} b_{,\theta\theta}}{r_h e^{2U}_{,\theta\theta}} \quad \text{for } R = \pm r_h, \quad \sin \theta = 0. \quad (27)$$

\ddagger As we shall see in (29), δ_1 and δ_2 are well-defined because $b_N^+ - b_S^+ + 2(b_{,\theta\theta})_N^+ \neq 0$ for $J \neq 0$. For $J \rightarrow 0$ we have $|\delta_{1/2}| \rightarrow \infty$ and $f|_{\mathcal{H}^-}$ diverges.

With the formula (23) for the Ernst potential on the Cauchy horizon, we finally arrive after some calculation at

$$A^+ = -32\pi r_h^2 \left. \frac{e^{2U},_{\theta\theta}}{b_{,\theta\theta}^2} \right|_N^+, \quad A^- = -8\pi r_h^2 \frac{(b_N^+ - b_S^+ + 2(b_{,\theta\theta})_N^+)^2}{(e^{2U},_{\theta\theta} b_{,\theta\theta}^2)_N^+}, \quad (28)$$

$$J = -2r_h^2 \frac{b_N^+ - b_S^+ + 2(b_{,\theta\theta})_N^+}{(b_{,\theta\theta})_N^+}, \quad \omega^+ = \frac{(b_{,\theta\theta})_N^+}{4r_h} \neq 0. \quad (29)$$

Note that $e^{2U},_{\theta\theta} < 0$ as $A^+ > 0$ for regular black holes. Together with the results in [10] we thus find the following.

Theorem 5.1. *Every regular axisymmetric and stationary black hole with non-vanishing angular momentum J satisfies the relation $(8\pi J)^2 = A^+ A^-$ where A^\pm are the horizon areas of event (\mathcal{H}^+) and Cauchy horizon (\mathcal{H}^-). If in addition the black hole is sub-extremal (i.e. if there exist trapped surfaces in every sufficiently small interior vicinity of \mathcal{H}^+ , see [5]), then the following inequalities hold: $A^- < 8\pi|J| < A^+$. Moreover, sub-extremal black holes with $J \neq 0$ have no trapped surfaces in sufficiently small interior vicinities of \mathcal{H}^- .*

Acknowledgments

We would like to thank Vincent Moncrief and Alan Rendall for many valuable discussions. This work was supported by the Deutsche Forschungsgemeinschaft (DFG) through the Collaborative Research Centre SFB/TR7 ‘‘Gravitational wave astronomy’’.

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