

Null data characterization of asymptotically flat stationary spacetimes

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Abstract. Using two sequences of symmetric trace free tensors, called the ‘null data’, we give a complete characterization of the asymptotics of all asymptotically flat, stationary solutions with non-vanishing ADM mass to Einstein’s vacuum field equations. We present necessary and sufficient growth estimates on the null data for the existence of such a solution.

1. Introduction

Since Penrose’s [16] proposal to use conformal compactification as a way to describe isolated systems in General Relativity a lot of effort has been devoted to understand which are the spacetimes that belong to the group of asymptotically flat spacetimes, which are the properties of those spacetimes, and to know if this group includes the spacetimes that are physically relevant as isolated systems.

Considering in particular the time independent solutions of Einstein’s vacuum field equations, Geroch [11] has given a definition of multipoles for static asymptotically flat spacetimes, which was later generalized by Hansen [13] to the stationary case. Beig and Simon [4], [5] and Kundu [15] add to this construction by showing that static or stationary solutions to Einstein’s vacuum field equations that satisfy some decay rate at infinity are analytic even at infinity. These results show that the given definitions of the multipoles make sense. They also show that given two solutions, if they have the ‘same’ multipoles, then the solutions are isomorphic, thus representing the ‘same’ spacetime. Despite these important results the question about existence of a spacetime given the multipoles plus some convergence criteria on them remained open.

Taking a different approach, Reula [17] has shown existence and uniqueness of asymptotically flat stationary spacetimes in terms of a boundary value problem, when data are prescribed in the sphere that encompasses the asymptotic end. However, in order to be able to control the precise asymptotic behaviour of the spacetime, it would be convenient to have a complete description of the asymptotically flat stationary spacetimes in terms of asymptotic quantities, in the spirit of Hansen’s multipoles.

Some progress towards such a result have been done by Bäckdahl and Herberthson [3], who found necessary bounds on the multipoles. Bäckdahl [2] has also found necessary and sufficient bounds on the multipoles in the axisymmetric case.

Concerned with the question of existence of static solutions with a given asymptotic behaviour, Friedrich [10] has used as data a sequence of trace-free symmetric tensors defined at infinity, referred to as *null data*, which are different but related to the multipoles. He has shown that imposing certain types of estimates on the null data is necessary and sufficient for the existence of asymptotically flat static spacetimes. This result has been used by Herberthson [14] to find necessary and sufficient bounds on Geroch’s multipoles for the existence of a static solution.

In this article we present results obtained in [1], where generalizing Friedrich's null data [10] we use two sequences of trace-free symmetric tensors, referred again to as null data, to characterize all asymptotically flat stationary vacuum solutions of Einstein's field equations. We present necessary and sufficient conditions for the null data to determine apart from gauge conditions unique real analytic solutions and thus to provide a complete characterization of all possible asymptotically flat solution to the stationary vacuum field equations. For technical details and the proof of the result the reader is referred to [1].

2. Asymptotically flat stationary spacetimes

We consider a stationary vacuum spacetime $(\tilde{M}, \tilde{g}_{\mu\nu}, \xi^\mu)$. Here \tilde{M} is a four-dimensional manifold, $\tilde{g}_{\mu\nu}$ is a Lorentzian metric with signature $(+ - - -)$ that satisfy Einstein's vacuum equations $Ric[\tilde{g}] = 0$, and ξ^μ is a time-like Killing vector field with complete orbits. As shown by Geroch [12] the description of this spacetime can be done in terms of fields defined on the quotient space of \tilde{M} with respect to the trajectories of ξ^μ , \tilde{N} . As we are interested in the asymptotics of the spacetime at spatial infinity, \tilde{N} is assumed to be diffeomorphic to the complement of a closed ball $\tilde{B}_R(0)$ in \mathbb{R}^3 . This assumption is natural in the present context, it corresponds to the idea of an isolated system, where the material sources are confined to a bounded region outside of which is vacuum. Then, following Hansen [13], it is possible starting from $(\tilde{M}, \tilde{g}_{\mu\nu}, \xi^\mu)$ to define a negative definite metric \tilde{h}_{ab} and two potentials $\tilde{\phi}_M, \tilde{\phi}_S$ on \tilde{N} . Einstein's vacuum field equations on \tilde{M} imply then the following equations for the potentials and the metric on \tilde{N} ,

$$\Delta_{\tilde{h}} \tilde{\phi}_A = 2R[\tilde{h}] \tilde{\phi}_A, \quad A = M, S, \quad (1)$$

$$R_{ab}[\tilde{h}] = 2[(\tilde{D}_a \tilde{\phi}_M)(\tilde{D}_b \tilde{\phi}_M) + (\tilde{D}_a \tilde{\phi}_S)(\tilde{D}_b \tilde{\phi}_S) - (\tilde{D}_a \tilde{\phi}_K)(\tilde{D}_b \tilde{\phi}_K)], \quad (2)$$

where $\tilde{\phi}_K = \left(\frac{1}{4} + \tilde{\phi}_M^2 + \tilde{\phi}_S^2\right)^{\frac{1}{2}}$. Reciprocally, if one has a solution $(\tilde{N}, \tilde{h}_{ab}, \tilde{\phi}_M, \tilde{\phi}_S)$ of (1), (2) then a stationary spacetime $(\tilde{M}, \tilde{g}_{\mu\nu}, \xi^\mu)$ can be constructed (cf. [9] for a detailed discussion). As it is equivalent but simpler to work in the quotient manifold \tilde{N} than in the spacetime \tilde{M} , we are looking for solutions of (1) and (2).

We also consider $(\tilde{N}, \tilde{h}_{ab})$ to be asymptotically flat, i.e., we assume $(\tilde{N}, \tilde{h}_{ab})$ to admit a smooth conformal extension in the following way: there exist a smooth Riemannian manifold (N, h_{ab}) and a function $\Omega \in C^2(N) \cap C^\infty(\tilde{N})$ such that $N = \tilde{N} \cup \{i\}$, where i is a single point,

$$\begin{aligned} \Omega > 0 \text{ on } \tilde{N}, \quad h_{ab} = \Omega^2 \tilde{h}_{ab} \text{ on } \tilde{N}, \\ \Omega|_i = 0, \quad D_a \Omega|_i = 0, \quad D_a D_b \Omega|_i = -2h_{ab}|_i, \end{aligned}$$

where D is the covariant derivative operator defined by h . This makes N diffeomorphic to an open ball in \mathbb{R}^3 , with centre at the point i , which represents space-like infinity.

3. The null data characterization

As said, it would be convenient to have a complete description of the asymptotically flat solutions to (1), (2) in terms of asymptotic quantities. In our attempt for such a characterization it turns out convenient to use as asymptotic quantities a generalization of Friedrich's static null data [10] to the stationary case. Using as conformal factor

$$\Omega = \frac{1}{2} m^{-2} \left[\left(1 + 4\tilde{\phi}_M^2 + 4\tilde{\phi}_S^2 \right)^{\frac{1}{2}} - 1 \right], \quad (3)$$

where m is a non-zero constant and defining the conformal potentials as

$$\phi_M = \Omega^{-\frac{1}{2}} \tilde{\phi}_M, \quad \phi_S = \Omega^{-\frac{1}{2}} \tilde{\phi}_S,$$

we define the following two sequences of trace-free symmetric tensors at infinity

$$\begin{aligned} \mathcal{D}_n^\phi &= \{ \mathcal{C}(D_{a_1} \phi)(i), \mathcal{C}(D_{a_2} D_{a_1} \phi)(i), \mathcal{C}(D_{a_3} D_{a_2} D_{a_1} \phi)(i), \dots \}, \\ \mathcal{D}_n^S &= \{ S_{a_2 a_1}(i), \mathcal{C}(D_{a_3} S_{a_2 a_1})(i), \mathcal{C}(D_{a_4} D_{a_3} S_{a_2 a_1})(i), \dots \}, \end{aligned}$$

where $\phi = \phi_S$ and S_{ab} is the trace free part of the Ricci tensor of h . These two sequences are referred to as the *null data*.

The natural question now is to what extent do the null data determine the metric h and the potentials ϕ_M, ϕ_S , or even the definition of the null data makes sense. For this to be the case the metric and the potentials should be real analytic even at i in suitable coordinates and conformal rescaling. Beig and Simon [5] (using as conformal factor (3)) and Kundu [15] have shown that the metric and the potentials do extend in a suitable gauge as real analytic fields to i if it is assumed that

$$(\phi_M(i))^2 + (\phi_S(i))^2 \neq 0.$$

As explained in [18] (cf. also [6]), in order for a solution of (1), (2) to lead to an asymptotically flat spacetime \tilde{M} it is necessary that $\phi_S(i) = 0$. So, we assume from now on that

$$\phi_M(i) \neq 0, \quad \phi_S(i) = 0.$$

We express now the tensors in $\mathcal{D}_n^\phi, \mathcal{D}_n^S$ in terms of an h -orthonormal frame $c_a, a = 1, 2, 3$, at i . Denoting by D_a the covariant derivative in the direction of c_a ,

$$\begin{aligned} \mathcal{D}_n^{\phi*} &= \{C(D_{a_1}\phi)(i), C(D_{a_2}D_{a_1}\phi)(i), C(D_{a_3}D_{a_2}D_{a_1}\phi)(i), \dots\}, \\ \mathcal{D}_n^{S*} &= \{S_{a_2a_1}(i), C(D_{a_3}S_{a_2a_1})(i), C(D_{a_4}D_{a_3}S_{a_2a_1})(i), \dots\}. \end{aligned}$$

These tensors are defined uniquely up to rigid rotations in \mathbb{R}^3 .

From Cauchy estimates on analytic functions it is possible to show that if the metric h and the potential ϕ are real analytic near i , then there exist constants $M, r > 0$ such that the components of these tensors satisfy the estimates

$$\begin{aligned} |C(D_{a_p} \dots D_{a_1}\phi)(i)| &\leq \frac{Mp!}{r^p}, \quad a_p, \dots, a_1 = 1, 2, 3, \quad p \geq 0, \\ |C(D_{a_p} \dots D_{a_1}S_{bc})(i)| &\leq \frac{Mp!}{r^p}, \quad a_p, \dots, a_1, b, c = 1, 2, 3, \quad p \geq 0. \end{aligned}$$

The main result to be presented in the following theorem is the statement that these estimates are not only necessary but also sufficient to have an analytic solution of the stationary field equations.

Theorem. *Suppose $m \neq 0$ and*

$$\begin{aligned} \hat{\mathcal{D}}_n^\phi &= \{\psi_{a_1}, \psi_{a_2a_1}, \psi_{a_3a_2a_1}, \dots\}, \\ \hat{\mathcal{D}}_n^S &= \{\Psi_{a_2a_1}, \Psi_{a_3a_2a_1}, \Psi_{a_4a_3a_2a_1}, \dots\}, \end{aligned}$$

are two infinite sequences of symmetric, trace free tensors given in an orthonormal frame at the origin of a 3-dimensional Euclidean space. If there exist constants $M, r > 0$ such that the components of these tensors satisfy the estimates

$$\begin{aligned} |\psi_{a_p \dots a_1}| &\leq \frac{Mp!}{r^p}, \quad a_p, \dots, a_1 = 1, 2, 3, \quad p \geq 1, \\ |\Psi_{a_p \dots a_1 bc}| &\leq \frac{Mp!}{r^p}, \quad a_p, \dots, a_1, b, c = 1, 2, 3, \quad p \geq 0, \end{aligned}$$

then there exists an analytic, asymptotically flat, stationary vacuum solution $(\tilde{h}, \tilde{\phi}_M, \tilde{\phi}_S)$ with mass monopole m and zero angular momentum monopole, unique up to isometries, so that the null data implied by $h = \frac{1}{4}m^{-4}[(1+4\tilde{\phi}_M^2+4\tilde{\phi}_S^2)^{\frac{1}{2}}-1]^2\tilde{h}$ and $\phi_S = 2^{\frac{1}{2}}m[(1+4\tilde{\phi}_M^2+4\tilde{\phi}_S^2)^{\frac{1}{2}}-1]^{-\frac{1}{2}}\tilde{\phi}_S$ in a suitable frame c_a as described above satisfy

$$\begin{aligned} C(D_{a_q} \dots D_{a_1}\phi_S)(i) &= \psi_{a_q \dots a_1}, \quad a_q, \dots, a_1 = 1, 2, 3, \quad q \geq 1, \\ C(D_{a_q} \dots D_{a_3}S_{a_2a_1})(i) &= \Psi_{a_q \dots a_1}, \quad a_q, \dots, a_1 = 1, 2, 3, \quad q \geq 2. \end{aligned}$$

The type of estimates imposed here on the abstract null data does not depend on the orthonormal frame in which they are given. Since these estimates are necessary as well as sufficient, all possible asymptotically flat solutions of the stationary vacuum field equations are characterized by the null data.

4. Conclusions

The result presented closes in some sense the quest to find a characterization of asymptotically flat stationary vacuum spacetimes.

This work contains the static case as a particular case, and is a generalization of Friedrich's work [10] from the static to the stationary case.

Corvino and Schoen [8] and Chruściel and Delay [7] have proven that it is possible to deform given general asymptotically flat vacuum data in an annulus in order to glue that data to stationary vacuum data in the asymptotic region. In relation with those works, as they need a family of asymptotically flat stationary solutions to perform the gluing procedure, our result gives a complete survey of the possible stationary asymptotics that can be attained beyond the known exact solutions.

With respect to Hansen's multipoles, as both, the multipoles and the null data, determine the metric and the potentials, there is a bijective map between them. Therefore the multipoles and the null data are from a mathematical point of view equivalent, although the equivalent result as the one here presented has so far not been found in terms of the multipoles.

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