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Calculus of Variations

The gradient flow of the Möbius energy near local minimizers

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Abstract In this article we show that for initial data close to local minimizers of the Möbius energy the gradient flow exists for all time and converges smoothly to a local minimizer after suitable reparametrizations. To prove this, we show that the heat flow of the Möbius energy possesses a quasilinear structure which allows us to derive new short-time existence results for this evolution equation and a Łojasiewicz-Simon gradient inequality for the Möbius energy.

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1 Introduction

The search for nice representatives of a given knot class led to the invention of a variety of new energies which are subsumed under the term knot energies. These new energies were needed basically due to the fact that other well known candidates like the elastic energy cannot be minimized within a given knot class [14] or at least their gradient flow can leave the given knot class (which can be seen for the elastic energy along the lines in [3]).

This article deals with the gradient flow of the Möbius energy proposed by Jun O'Hara in [12]. For curves $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ this energy is defined as

$$E(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^2} \left(\frac{1}{|\gamma(u) - \gamma(v)|^2} - \frac{1}{d_\gamma(u,v)^2} \right) |\gamma'(u)|\,|\gamma'(v)|\,du\,dv,$$

where $d_\gamma(u, v)$ denotes the distance of the points $\gamma(u)$ and $\gamma(v)$ along the curve γ. E is called Möbius energy due to the fact that it is invariant under Möbius transformations.

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Freedman et al. showed in [6] that the differential of the Möbius energy can be represented as

$$E'(\gamma)h = \int_{\mathbb{R}/\mathbb{Z}} \langle H\gamma(u), h(u)\rangle \cdot |\gamma'(u)| du$$

for all imbedded regular curves $\gamma \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and $h \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, where

$$H\gamma(u) = 2 \lim_{\varepsilon \searrow 0} \int_{\substack{\mathbb{R}/\mathbb{Z} \\ |v-u| \geq \varepsilon}} \left(2 \frac{P^{\perp}_{\gamma'(u)}(\gamma(v) - \gamma(u))}{|\gamma(v) - \gamma(u)|^2} - \frac{1}{|\gamma'(u)|} \frac{d}{du} \left(\frac{\gamma'(u)}{|\gamma'(u)|} \right) \right) \frac{|\gamma'(v)| dv}{|\gamma(v) - \gamma(u)|^2}$$

and $P^{\perp}_{\gamma'(u)}(w) := w - \left\langle w, \frac{\gamma'(u)}{|\gamma'(u)|} \right\rangle \frac{\gamma'(u)}{|\gamma'(u)|}$ for all $w \in \mathbb{R}^n$ (for a rigorous argument for the fact that this is the L^2- gradient of E see [13, Chapter 1])

In this paper we will investigate time dependent families of curves $\gamma(t)$ which move in the direction of $-H\gamma(t)$, i.e. for which

$$\partial_t \gamma = -H\gamma, \tag{1.1}$$

or whose normal velocity is given $-H\gamma$, i.e. for which

$$\partial_t^{\perp} \gamma = -H\gamma. \tag{1.2}$$

where $\partial_t^{\perp}\gamma = P^{\perp}_{\gamma'} \partial_t \gamma$.

The key to the results in this article is the observation that the evolution Eq. 1.1 can be written in a quasilinear form. To formulate this result let $C^{\alpha}_{i,r}$ for $\alpha \geq 1$ denote the set of all embedded and regular curves in C^{α} and

$$Q\gamma(u) := \lim_{\varepsilon \searrow 0} \int_{I_\varepsilon} \left(2 \frac{\gamma(u+w) - \gamma(u) - w\gamma'(u)}{|w|^2} - \gamma''(u) \right) \frac{dw}{|w|^2},$$

for $I_\varepsilon := [-1/2, 1/2] \backslash [-\varepsilon, \varepsilon]$. The functional Q is well-defined for $f \in C^{3+\alpha}(\mathbb{R}/\mathbb{Z})$ for $\alpha > 0$. Note that $C^{\alpha}_{i,r}$ is an open subset of C^{α}, $\forall \propto \geq 1$.

Theorem 1.1 (Quasilinear structure) *There is a mapping*

$$F \in \bigcap_{\alpha > \beta > 0} C^{\omega}(C^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), C^{\beta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$$

such that

$$H\gamma = \frac{2}{|\gamma'|^3} P^{\perp}_{\gamma'} Q\gamma + F\gamma$$

For all $\gamma \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.[1]

Here, C^{ω} stands for real analytic mappings.

In [9], He calculated the linearization of H around curves parametrized by arc-length and found that

$$\nabla_h H(\gamma) = \frac{2\pi}{3}(-\Delta)^{3/2} h + \tilde{L}_{\gamma}(h)$$

[1] In fact, one can even exchange Q by $\frac{1}{3\pi}(-\Delta)^{3/2}$ using [9, Lemma 2.3] and regularity results for pseudo-differential operators. We do not prove this here, since we do not need it, but this relation is the motivation for many manipulations we make.

where $\tilde{L}$ is a differential operator of order ≤ 2 and $\Delta = \frac{\partial^2}{\partial x^2}$. Unfortunately, in He's calculations the symbol of the operator $\tilde{L}$ depends on the third derivatives of γ and thus he could show short time existence only for smooth initial data using the Nash-Moser implicit function theorem.

The structure in Theorem 1.1 enables us to show the following short time existence result which contains smoothing effects for $C^{2+\alpha}$ initial data. To state it, let $h^{k+\alpha}(\mathbb{R}/\mathbb{Z})$ for $k \in \mathbb{N}, \alpha \in (0,1)$ denote the little Hölder spaces, i.e. the closure of $C^\infty(\mathbb{R}/\mathbb{Z})$ under $\|\cdot\|_{C^{k+\alpha}}$, and let $h^{k+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ denote the set of all injective and regular curves in $h^{k+\alpha}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. Given a curve $\gamma \in C^1_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, let $h^{k+\alpha}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)^\perp_\gamma$ denote the space of all vector fields in $h^{k+\alpha}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ that are normal to γ.

Theorem 1.2 (Short time existence) *For every $\alpha > 0, \alpha \notin \mathbb{N}$ there is a strictly positive, upper continuous function $r : h^{2+\alpha}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n) \to (0,\infty)$ with the following property:*

Let $\gamma_0 \in C^\infty_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. Then for every

$$N_0 \in \mathcal{V}_r(\gamma_0) := \left\{ f \in \left(h^{2+\alpha}\left(\mathbb{R}/\mathbb{Z},\mathbb{R}^n\right)\right)^\perp_{\gamma_0} : \|f\|_{C^1} \leq r(\gamma_0)\right\}$$

there is a constant $T = T(N_0) > 0$ and a neighborhood of $U \subset \mathcal{V}_r(\gamma_0)$ of N_0 such that for every $\tilde{N}_0 \in U$ there is a unique solution $N_{\tilde{N}_0} \in C([0,T), h^{2+\alpha}(\mathbb{R}/\mathbb{Z})^\perp_{\gamma_0}) \cap C^1((0,T), C^\infty(\mathbb{R}/\mathbb{Z})^\perp_{\gamma_0})$ of

$$\begin{cases} \partial_t^\perp(\gamma_0 + N) = -H(\gamma_0 + N) & t \in [0,T], \\ N(0) = \tilde{N}_0. \end{cases} \tag{1.3}$$

Furthermore, the flow $(\tilde{N}_0, t) \mapsto N_{\tilde{N}_0}(t)$ is in $C^1((U \times (0,T)), C^\infty(\mathbb{R}/\mathbb{Z}))$.

As for every $\gamma \in h^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ there is a $\gamma_0 \in C^\infty_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $N \in \mathcal{V}_r(\gamma_0)$ and a diffeomorphism $\psi \in C^{2+\alpha}(\mathbb{R}/\mathbb{Z},\mathbb{R}/\mathbb{Z})$ such that $\gamma \circ \psi = \gamma_0 + N$, (cf. Lemma 3.1), Theorem 1.2 gives us a solution to the gradient flow for every initial data in $h^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ (cf. Corollary 3.2 for a precise statement). Due to the uniqueness mentioned in this result, it makes even sense to speak of maximal solution of (1.1) in the sense that the time of existence is maximal. Such a maximal solution is unique up to reparametrizations.

Using the analyticity of the term F in Theorem 1.1, we will then derive the following Łojasiewicz-Simon gradient estimate for the Möbius energy

Lemma 1.3 (Łojasiewicz-Simon gradient estimate) *Let $\gamma_M \in C^\infty_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ be a stationary point of the Möbius energy. Then there are constants $\theta \in [0,1/2], \sigma, c > 0$, such that every $\gamma \in H^3_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ with $\|\gamma - \gamma_M\|_{H^3} \leq \sigma$ satisfies*

$$|E(\gamma) - E(\gamma_M)|^{1-\theta} \leq c \cdot \left(\int_{\mathbb{R}/\mathbb{Z}} |(H\gamma)(x)|^2 |\gamma'(x)| dx \right)^{1/2}.$$

Combining Theorem 1.2 with Lemma 1.3, we get the main result of this article - long time existence for (1.1) near local minimizers

Theorem 1.4 (Long time existence) *Let $\gamma_M \in C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ be a local minimizer of the Möbius energy in $C^k(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ for some $k \in \mathbb{N}_0$, i.e. let there be a neighborhood U of γ_M in $C^k(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ such that*

$$E(\gamma) \geq E(\gamma_M), \quad \forall \gamma \in U.$$

Then for every $\beta > 0$ there is a neighborhood V of γ_M in $C^{2+\beta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ such that for all $\gamma_0 \in V$ the heat flow (1.1) with initial data γ_0 exists for all times and converges after suitable reparametrizations to a stationary point γ_∞ satisfying

$$E(\gamma_\infty) = E(\gamma_M).$$

In [6], Freedman, He, and Wang showed that there are in fact minimizers of the Möbius energy in all prime knot classes. As He showed that all these minimizers are smooth (cf. [9]), the above theorem tells us that the gradient flow converges to such a minimizer if we start near to one.

Please note, that the precise value of constants in the proofs may change from line to line and that the summation convention is used in this article.

2 Quasilinear structure

The proof of Theorem 1.1 basically relies on Taylor approximation and estimates for the multilinear Hilbert transform which can be found in the appendix (Lemma A.5). Furthermore, it uses the relation of H to the operator $\tilde{H}$ we will introduce now.

Using $I_\varepsilon = [-1/2, 1/2]/[-\varepsilon, \varepsilon]$ we can write

$$\begin{aligned} &H\gamma(u) \\ &=2 \lim_{\varepsilon \searrow 0} \int_{I_\varepsilon} \left(2\frac{P^{\perp}_{\gamma'(u)}(\gamma(u+w)-\gamma(u))}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{|\gamma'(u)|}\frac{d}{du}\left(\frac{\gamma'(u)}{|\gamma'(u)|}\right)\right)\frac{|\gamma'(u+w)|dw}{|\gamma(u+w)-\gamma(u)|^2}. \end{aligned}$$

Setting

$$\begin{aligned} &(\tilde{H}\gamma)(u) \\ &\quad := 2 \lim_{\varepsilon \searrow 0} \int_{I_\varepsilon} \left(2\frac{\gamma(u+w) - \gamma(u) - w\gamma'(u)}{|\gamma(u+w) - \gamma(u)|^2} - \frac{\gamma''(u)}{|\gamma'(u)|^2}\right)\frac{|\gamma'(u+w)|dw}{|\gamma(u+w) - \gamma(u)|^2}, \end{aligned}$$

one easily sees that

$$H(\gamma) = P^{\perp}_{\gamma'}\tilde{H}(\gamma). \tag{2.1}$$

The first step in the proof of Theorem 1.1 is to show that the difference between $\tilde{H}\gamma$ and $\frac{2}{|\gamma'|^3}Q\gamma$—which is well-defined for $\gamma \in C^{3+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), \alpha > 0$—can be extended to an analytic operator from $C^{2+\alpha}$ to C^β for all $\alpha > \beta$.

Lemma 2.1 *There is a map $\tilde{F} \in \bigcap_{\alpha>\beta>0} C^\omega(C^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), C^\beta(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ such that*

$$\tilde{H}\gamma = \frac{2}{|\gamma'(u)|^3}Q\gamma + \tilde{F}(\gamma) \quad \forall \gamma \in H^3_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n).$$

Proof We have

$$
\begin{aligned}
&\frac{1}{2}\tilde{H}\gamma(u) - \frac{1}{|\gamma'(u)|^3} Q\gamma(u) \\
&\xleftarrow{\varepsilon\searrow 0} 2\int_{I_\varepsilon}\left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{|\gamma'(u)|^2 w^2}\right)\big(\gamma(u+w)-\gamma(u)-w\gamma'(u)\big) \\
&\quad\times \frac{|\gamma'(u+w)|}{|\gamma(u+w)-\gamma(u)|^2} dw + \int_{I_\varepsilon}\left(2\frac{\gamma(u+w)-\gamma(u)-w\gamma'(u)}{|\gamma'(u)|^2 w^2} - \frac{\gamma''(u)}{|\gamma'(u)|^2}\right) \\
&\quad\times\left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{|\gamma'(u)|^2 w^2}\right)|\gamma'(u+w)| dw \\
&\quad+\int_{I_\varepsilon}\left(2\frac{\gamma(u+w)-\gamma(u)-w\gamma'(u)}{|\gamma'(u)|^2 w^2} - \frac{\gamma''(u)}{|\gamma'(u)|^2}\right)\frac{|\gamma'(u+w)|-|\gamma'(u)|}{|\gamma'(u)|^2 w^2} dw.
\end{aligned}
$$

Let us denote the first integral on the right hand side of the last equation by $I_1(\gamma;\varepsilon)$, the second integral by $I_2(\gamma;\varepsilon)$, and last one by $I_3(\gamma;\varepsilon)$.

We define

$$
\begin{aligned}
(\alpha_w\gamma)(u) &:= \int_0^1 (1-t)\gamma''(u+tw)dt, \\
(\beta_w\gamma)(u) &:= \int_0^1 (1-t)(\gamma''(u+tw)-\gamma''(u))dt = (\alpha_w\gamma)(u) - \gamma''(u) \\
(\delta_w\gamma)(u) &:= \int_0^1 \left\langle \gamma''(u+tw, \frac{\gamma'(u+tw)}{|\gamma'(u+tw)|}\right\rangle dt \\
w(X_w\gamma)(u) &:= 1 - \frac{|\gamma(u+w)-\gamma(u)|^2}{w^2|\gamma'(u)|^2} \\
(\tilde{X}_w\gamma(u)) &:= \frac{|\gamma(u+w)-\gamma(u)|^2}{w^2|\gamma'(u)|^2}
\end{aligned}
$$

for regular injective curves $\gamma \in C^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ and $w \in [-1/2, 1/2]$. Taylor approximation of γ yields

$$
\begin{aligned}
|\gamma(u+w)-\gamma(u)|^2 &= \left| w\gamma'(u) + w^2\int_0^1 (1-t)\gamma''(u+tw)dt\right|^2 \\
&= w^2\gamma'(u)^2 + 2w^3\gamma'(u)\int_0^1 (1-t)\gamma''(u+tw)dt \\
&\quad + w^4\left(\int_0^1 (1-t)\gamma''(u+tw)dt\right)^2
\end{aligned}
$$

$$= w^2\gamma'(u)^2\left(1+\frac{2w}{|\gamma'(u)|^2}\gamma'(u)\cdot(\alpha_w\gamma)(u)+\frac{w^2}{|\gamma'(u)|^2}\cdot(\alpha_w\gamma)(u)^2\right).$$
$$= w^2\gamma'(u)^2(1-w(X_w\gamma)(u)). \tag{2.2}$$

We will also need this in the version

$$(\tilde{X}_w\gamma)(u) = 1 - w(X_w\gamma)(u)) = 1 + \frac{2w}{|\gamma'(u)|^2}\gamma'(u)\cdot(\alpha_w\gamma)(u) + \frac{w^2}{|\gamma'(u)|^2}\cdot(\alpha_w\gamma)(u)^2. \tag{2.3}$$

Using

$$(1-x)^{-1} = 1 + x + \frac{x^2}{1-x},$$

we get

$$\frac{1}{|\gamma(u+w)-\gamma(u)|^2} = \frac{1}{|\gamma'(u)|^2\,w^2} + \frac{1}{|\gamma'(u)|^2}\left(\frac{(X_w\gamma)(u)}{w} + \frac{(X_w\gamma)(u)^2}{1-w(X_w\gamma)(u)}\right) \tag{2.4}$$

for all $w > 0$ as the injectivity of the curve γ implies

$$(\tilde{X}_w\gamma)(u) := 1 - wX(\gamma; u, w) = \frac{|\gamma(u+w)-\gamma(u)|^2}{w^2|\gamma'(u)|^2} > 0.$$

Together with

$$\gamma(u+w) - \gamma(u) - w\gamma'(u) = w^2\int_0^1 (1-t)\gamma''(u+tw)dt = w^2(\alpha_w\gamma)(u),$$

$$2\frac{\gamma(u+w)-\gamma(u) - w\gamma'(u)}{w^2} - \gamma''(u) = \int_0^1 (1-t)\left(\gamma''(u+tw)-\gamma''(u)\right)dt = (\beta_w\gamma)(u),$$

and

$$\left|\gamma'(u+w)\right| - \left|\gamma'(u)\right| = w\int_0^1 \left\langle \frac{\gamma'(u+tw)}{|\gamma'(u+tw)|}, \gamma''(u+tw)\right\rangle dt =: w(\delta_w\gamma)(u),$$

we get

$$I_1(\gamma;\varepsilon) = \frac{2}{|\gamma'(u)|^4}\int_{I_\varepsilon}\left(\frac{(X_w\gamma)(u))}{w} + \frac{(X_w\gamma)(u)^2}{1-w(X_w\gamma)(u)}\right)(\alpha_w\gamma)(u)$$
$$\times\left(1 + w(X_w\gamma)(u) + \frac{w^2(X_w\gamma)(u)}{1-w(X_w\gamma)(u)}\right)|\gamma'(u+w)|dw,$$
$$I_2(\gamma;\varepsilon) = \frac{2}{|\gamma'(u)|^4}\int_{I_\varepsilon}(\beta_w\gamma)(u)\left(\frac{(X_w\gamma)(u)}{w} + \frac{(X_w\gamma)(u)^2}{1-w(X_w\gamma)(u)}\right)|\gamma'(u+w)|dw$$

and

$$I_3(\gamma;\varepsilon) = \frac{2}{|\gamma'(u)|^4} \int\limits_{I_\varepsilon} \frac{(\beta_w\gamma)(u)}{w} \left(1 + w(X_w\gamma)(u) + \frac{w^2(X_w\gamma)(u)^2}{1 - w(X_w\gamma)(u)}\right)(\delta_w\gamma)(u)dw.$$

After factoring out these expressions using Eq. 2.3 and $|\gamma'(u+w)| = w\delta_w\gamma(u) + |\gamma'(u)|$, Lemma 2.2 together with Lemma A.2 tell us that these integrals define analytic operators from $C^{2+\alpha}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ to $C^\beta(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ for all $\alpha > \beta$. □

Lemma 2.2 *Let $l_1, l_2, l_3, l_4 \in \mathbb{N}_0$ and $M : (\mathbb{R}^n)^{l_2} \to \mathbb{R}^k$ be a l_2-linear mapping. Then*

$$(T\gamma)(u) := \lim_{\varepsilon\searrow 0} \int\limits_{I_\varepsilon} \frac{w^{l_1-1} M((\alpha_w\gamma)(u),\ldots,(\alpha_w\gamma)(u))((\delta_w\gamma)(u))^{l_3}}{(1 - w(X_w\gamma)(u))^{l_4}} dw$$

defines an analytic map in $C^\omega(C^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n), C^\beta(\mathbb{R}/\mathbb{Z}), \mathbb{R}^k)$ for all $\alpha > \beta > 0$.

Proof We will use the shorthand $M(v)$ for $M(v,\ldots,v)$.

Step 1: The non-singular case $l_1 > 0$

Let γ_0 be fixed. For $w \in [-1/2, 1/2]$ we define the function $T_w : C^{2,\alpha}_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n) \to C^\alpha(\mathbb{R}/\mathbb{Z},\mathbb{R})$ by

$$T_w\gamma(u) := w^{l_1-1} \frac{M(\alpha_w\gamma(u))(\delta_w\gamma(u))^{l_3}}{(\tilde{X}_w\gamma(u))^{l_4}}$$

for all $u \in \mathbb{R}/\mathbb{Z}$.

Since the function α_w is linear, we get a $C < \infty$ such that

$$\|D\alpha_w(\gamma)\|_{L(C^{2+\alpha},C^\alpha)} = \|\alpha_w\|_{L(C^{2+\alpha},C^\alpha)} \le C \quad \forall w \in [-1/2, 1/2],$$
$$D^m\alpha = 0 \quad \forall m \ge 2. \tag{2.5}$$

We write δ_w as

$$\delta_w\gamma(u) = \int\limits_0^1 \tilde{\delta}_{tw}(u)dt$$

where $\tilde{\delta}_{\tilde{w}}(u) := \left\langle \gamma''(u+\tilde{w}), \frac{\gamma'(u+\tilde{w})}{|\gamma'(u+\tilde{w})|}\right\rangle$. Using Lemma A.3, Lemma A.2, and writing $\tilde{\delta}_0\gamma = g(\gamma',\gamma'')$ where

$$g_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
$$g_1(x,y) := \left\langle x, \frac{y}{|y|}\right\rangle$$

is analytic away from $\{0\}\times\mathbb{R}^n$, we get that δ_0 is analytic on the set $C^{2,\alpha}_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. Hence, there is an open neighborhood U of γ_0 and a constant $C < \infty$ such that

$$\|D^m\tilde{\delta}_{tw}(\gamma)\|_{L(C^{\alpha+2},C^\alpha)} = \|D^m\tilde{\delta}_0(\gamma)\|_{L(C^{\alpha+2},C^\alpha)} \le C^m m! \quad \forall \gamma \in U, t \in [0,1].$$

Applying Lemma A.3, we get

$$\|D^m\delta_w(\gamma)\|_{L(C^{\alpha+2},C^\alpha)} = \|D^m\delta_0(\gamma)\|_{L(C^{\alpha+2},C^\alpha)} \le C^m m! \quad \forall \gamma \in U. \tag{2.6}$$

To get an estimate for $\tilde{X}_w$ we first observe that

$$(\tilde{X}_w\gamma)(u) = \frac{\left(\int_0^1 \gamma'(u+tw)dt\right)^2}{|\gamma'(u)|^2} = g_2((\hat{X}_w\gamma)(u), \gamma'(u))$$

where $g_2(x, y) := \frac{|x|^2}{|y|^2}$ and $(\hat{X}_w\gamma)(u) := \int_0^1 \gamma'(u+tw)dt$. Since $\hat{X}_w$ is linear from $C^{\alpha+1}$ to C^α and with operator norm bounded by 1, we get using Lemma A.2 again that there is a neighborhood $U' \subset U$ of γ_0 and a constant $C < \infty$ such that

$$\|D^m X_w(\gamma)\|_{L(C^{\alpha+2},C^\alpha)} \leq C^m m! \tag{2.7}$$

for all $\gamma \in U'$

Using that $x \to x^{-l_4}$ is analytic on $(0, \infty)$ and that there is an open neighborhood $U'' \subset U'$ such that $0 < \inf_{\gamma\in U'', u\in\mathbb{R}/\mathbb{Z}}(\tilde{X}_w\gamma)(u))$ and $\sup_{\gamma\in U'',u\in\mathbb{R}/\mathbb{Z}}(X_w\gamma)(u)) < \infty$, we see that there is a constant $C < \infty$ such that

$$\|D^m((\tilde{X}_w)^{-l_4})(\gamma)\|_{L(C^{\alpha+2},C^\alpha)} \leq C^m m! \quad \forall \gamma \in U''. \tag{2.8}$$

Combining Eqs. 2.5, 2.6, and 2.8 with Lemma A.2, we get

$$\|D^m T_w(\gamma)\|_{L(C^{\alpha+2},C^\alpha)} \leq m^3 C^m m! \quad \forall \gamma \in U''$$

and hence by Lemma A.3 T is analytic on U'' from $C^{2+\alpha}$ to C^α.

Step 2: The singular case $l_1 = 0$

Using the Taylor expansion

$$(1-x)^{-l_4} = 1 + l_4 x + l_4(l_4+1)x^2 \int_0^1 (1-t)(1-tx)^{-l_4-2}dt$$

for $x = wX_w(\gamma)(u)$, we get

$$\begin{aligned}
T_w(\gamma)u &= w^{-1}M(\alpha_w\gamma(u))(\delta_w\gamma(u))^{l_3}\Bigg(1 + l_4 w X_w(\gamma)(u) + l_4(l_4+1)w^2 X_w(\gamma)(u)^2 \\
&\quad \times \int_0^1 (1-t)(1-twX_w(\gamma)(u))^{-l_4-2}dt\Bigg) \\
&= (T_w^1\gamma)(u) + (T_w^2\gamma)(u) + (T_w^3\gamma)(u),
\end{aligned}$$

where

$$\begin{aligned}
T_w^1\gamma(u) &:= w^{-1}M(\alpha_w\gamma(u))(\delta_w\gamma(u))^{l_3} \\
T_w^2\gamma(u) &:= l_4 M(\alpha_w\gamma(u))(\delta_w\gamma(u))^{l_3} X_w(\gamma)(u) \\
T_w^3\gamma(u) &:= l_4(l_4+1)wM(\alpha_w\gamma(u))(\delta_w\gamma(u))^{l_3} X_w(\gamma)(u)^2 \\
&\quad \times \int_0^1 (1-t)(1-twX_w(\gamma)(u))^{-l_4-2}dt.
\end{aligned}$$

Plugging in the definition of α_w and δ_w and interchanging the order of integration we get

$$\int_{I_\varepsilon} T_w^1 = \int_{[0,1]^{l_2}} \int_{[0,1]^{l_3}} \frac{1}{w}(\Pi_{i=1}^{2l_2+1}(1-t_i))$$
$$\times \int_{I_\varepsilon} M(\gamma''(u+t_1 w),\dots,\gamma''(u+t_{l_2}w))$$
$$\times \left(\Pi_{i=1}^{l_3}\left\langle \gamma''(u+s_i w), \frac{\gamma'(u+s_i w)}{|\gamma'(u+s_i w)|}\right\rangle\right) dw ds^{l_3} dt^{l_2}$$
$$= \int_{[0,1]^{l_2}} \int_{[0,1]^{l_3}} \int_{I_\varepsilon} (\tilde{T}_{\gamma,u})(w,t_1,\dots t_{l_2},s_1,s_{l_3}) dw ds^{l_3} dt^{2l_2+1}$$

where

$$(\tilde{T}_{\gamma,u})(w,t_1,\dots t_{l_2},s_1,s_{l_3})$$
$$:= \frac{1}{w}(\Pi_{i=1}^{l_2}(1-t_i))\left(M(\gamma''(u+t_1 w),\dots,\gamma''(u+t_{l_2}w))\Pi_{i=1}^{l_3}\right.$$
$$\left.\times \left\langle \gamma''(u+s_i w), \frac{\gamma'(u+s_i w)}{|\gamma'(u+s_i w)|}\right\rangle - M\left(\gamma''(u)\right)\langle\gamma''(u), \frac{\gamma'(u)}{|\gamma'(u)|}\rangle^{l_3}\right).$$

Form the Hölder regularity of γ we deduce that $\tilde{T}_{\gamma,u}$ is integrable and hence

$$\lim_{\varepsilon\searrow 0}\int_{I_\varepsilon} T_w^1\gamma(u)dw$$
$$= \int_{[0,1]^{l_2}} \int_{[0,1]^{l_3}} \left(\lim_{\varepsilon\searrow 0}\int_{I_\varepsilon} (\tilde{T}_{\gamma,u})(w,t_1,\dots t_{l_2},s_1,s_{l_3})dw\right) ds^{l_3} dt^{2l_2+1}.$$

Applying Lemma A.6 for the inner integral and then Lemma A.3, we get $(T^1\gamma)(u) = \lim_{\varepsilon\downarrow 0}\int_{I_\varepsilon} T_w^1(\gamma)(u)dw$ is analytic from $C^{2+\alpha}$ to C^β for all $\alpha > \beta > 0$.

Step 1 implies that the term $(T^2\gamma)(u) := \lim_{\varepsilon\searrow 0}\int_{I_\varepsilon}(T_w\gamma)(u)dw$ is an analytic operator from $C^{2+\alpha}$ to C^α for all $\alpha > 0$.

The term T_w^3 can be written using Eq. 2.3 as a sum of expressions of the form

$$cw\int_0^1 (1-t)\frac{M(\alpha_w\gamma(u))\delta_w^{\tilde{l}_3}}{(1-twX_w)^{\tilde{l}_4}}dt =: c\int_0^1(1-t)I_{t,w}dt$$

where $\tilde{l}_2 \in \mathbb{N}$. Since X_w is analytic and

$$\inf\left\{1-tw(X_w\gamma)(u) : \gamma\in U'', w\in[-1/2,1/2], u\in\mathbb{R}/\mathbb{Z}, t\in[0,1]\right\} > 0,$$

there is a constant C such that

$$\|D^m I_{t,w}\|_{L(C^{2+\alpha},C^\alpha} \le C^m m! \quad \forall t\in[0,1], w\in\mathbb{R}/\mathbb{Z}.$$

Using Lemma A.3 twice, we get that T^3 is analytic from $C_{i,r}^{2+\alpha}$ to C^α. This finishes the proof of the statement for T_w. $\square$

Proof of Theorem 1.1 From Lemma 2.1 we get an $\tilde{F} \in \bigcap_{\alpha>\beta>0} C^w(C^{2+\alpha}, C^\beta)$ such that for all $\gamma \in H^3_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$

$$\tilde{H}(\gamma) = \frac{1}{|\gamma'|^3} Q(\gamma) + \tilde{F}(\gamma).$$

Hence,

$$H(\gamma) = P^\perp_{\gamma'}(\tilde{H}(\gamma)) = \frac{1}{|\gamma'|^3} P^\perp_{\gamma'} Q(\gamma) + F(\gamma),$$

where

$$F(\gamma) = P^\perp_{\gamma'} \tilde{F}(\gamma) = \tilde{F}(\gamma) - \left\langle \tilde{F}(\gamma), \frac{\gamma'}{|\gamma'|} \right\rangle \frac{\gamma'}{|\gamma'|}.$$

From $\tilde{F} \in \bigcap_{\alpha>\beta>0} C^w(C^{2+\alpha}, C^\beta)$ we hence deduce that $F \in \bigcap_{\alpha>\beta>0} C^w(C^{2+\alpha}, C^\beta)$. □

3 Short time existence

In this section we derive short time existence results for the gradient flow of the Möbius energy. For this we will work with families of curves that are normal graphs over a fixed smooth curve. We show that for initial data γ_0 that can be written as this fixed smooth reference curve plus a vector field that belongs to a certain open neighborhood around 0 in the space of $h^{2+\alpha}$ vector fields, there is a family γ_1 of curves with normal velocity $-H(\gamma_t)$ and converging to γ_0 in $h^{2,\alpha}$ as $t \searrow 0$.

To describe these neighborhoods, note that there is a strictly positive, lower semi-continuous function $r : C^2_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to (0, \infty)$ such that

$$\gamma + \{N \in C^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^\perp_{\gamma_0} : \|N\|_{C^1} < r(\gamma)\} \subset C^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$$

for all $\gamma \in C^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and

$$r(\gamma) \leq 1/2 \inf_{x \in \mathbb{R}/\mathbb{Z}} |\gamma'(x)|. \tag{3.1}$$

Here, $C^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^\perp_\gamma$ denotes the space of all vector fields $N \in C^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ which are normal to γ, i.e. for which $\langle \gamma'(u), N(u)\rangle = 0$ for all $u \in \mathbb{R}/\mathbb{Z}$. Letting

$$\mathcal{V}_r(\gamma) := \{N \in h^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^\perp_\gamma : \|N\|_{C^1} < r(\gamma)\}$$

we have for all $\gamma \in h^{2,\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$

$$\gamma + \mathcal{V}_r(\gamma) \subset h^{2,\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n). \tag{3.2}$$

Let $N \in \mathcal{V}_r(\gamma)$. Equation 3.1 guarantues that $P^\perp_{(\gamma+N)'(u)}$ is an isomorphism from the normal space along γ at u to the normal space along $\gamma + N$. Otherwise there would be a $v \neq 0$ in the normal space of γ at u such that

$$0 = P^\perp_{(\gamma+N)'(u)}(v) = v - \left\langle v, \frac{(\gamma+N)'(u)}{|(\gamma+N)'(u)|} \right\rangle \frac{(\gamma+N)'(u)}{|(\gamma+N)'(u)|}$$

which would contradict

$$\left| v - \left\langle v, \frac{(\gamma+N)'(u)}{|(\gamma+N)'(u)|} \right\rangle \frac{(\gamma+N)'(u)}{|(\gamma+N)'(u)|} \right| \geq |v| - \left| \langle v, \frac{N'(u)}{|(\gamma+N)'(u)|} \rangle \right| \geq |v|/2 > 0.$$

For $\gamma \in C^1((0,T), C^1_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ we denote by

$$\partial_t^\perp \gamma = P^\perp_{\gamma'}(\partial_t \gamma)$$

the normal velocity of the family of curves.

We prove the following strengthened version of the short time existence result mentioned in the introduction

Theorem 1.2 (Short time existence for normal graphs) *Let $\gamma_0 \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be an embedded regular curve and $\alpha > 0, \alpha \notin \mathbb{N}$. Then for every $N_0 \in \mathcal{V}_r(\gamma_0)$ there is a constant $T = T(N_0) > 0$ and a neighborhood of $U \subset \mathcal{V}_r$ of N_0 such that for every $\tilde{N}_0 \in U$ there is a unique solution $N_{\tilde{N}_0} \in C([0,T), h^{2+\alpha}(\mathbb{R}/\mathbb{Z})^\perp_{\gamma_0}) \cap C^1((0,T), C^\infty(\mathbb{R}/\mathbb{Z})^\perp_{\gamma_0})$ of*

$$\begin{cases} \big(\partial_t^\perp(\gamma_0 + N\big) = -H(\gamma_0 + N) & t \in [0,T], \\ N(0) = \tilde{N}_0. \end{cases} \tag{3.3}$$

Furthermore, the flow $(\tilde{N}_0, t) \mapsto N_{\tilde{N}_0}(t)$ is in $C^1((U \times (0,T)), C^\infty(\mathbb{R}/\mathbb{Z}))$.

The proof of Theorem 1.2 consists of two steps. First we show that (3.3) can be transformed into an abstract quasilinear system of parabolic type. The second step is to establish short time existence results for the resulting equation.

The second step can be done using general results about analytic semigroups, regularity of pseudo-differential operators with rough symbols [4], and the short time existence results for quasilinear equations in [2] or [1]. Furthermore, we need continuous dependence of the solution on the data and smoothing effects in order to derive the long time existence results in Sect. 5.

For the convenience of the reader, we go a different way here and present a self-contained proof of the short time existence that only relies on a characterization of the little Hölder spaces as trace spaces. In Subsect. 3.1, we deduce a maximal regularity result for solutions of linear equations of type $\partial_t u + a(t)Qu + b(t)u = f$ in little Hölder spaces using heat kernel estimates. Following ideas from [2], we then prove short time existence and differentiable dependence on the data for the quasilinear equation.

The following lemma (the proof of which we postpone till the end of the section), will allow us to solve the gradient flow for all initial data in $h^{2,\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

Lemma 3.1 *Let $r : h^{2,\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to (0,\infty)$ be a lower semi-continuous function. Then for every $\tilde{\gamma} \in h^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ there is a $\gamma \in C^\infty_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $N \in \mathcal{V}_r(\gamma)$ and a diffeomorphism $\psi \in C^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ such that $\tilde{\gamma} \circ \psi = \gamma + N$*

Combining Theorem 1.2 with Lemma 3.1 we immediately get

Corollary 3.2 (Short time existence) *Let $\gamma_0 \in h^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $\alpha > 0, \alpha \notin \mathbb{N}$. Then there is a constant $T > 0$ and a reparametrization $\phi \in C^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ such that there is a solution $\gamma \in C([0,T), h^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \cap C^1((0,T), C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ of the initial value problem*

$$\begin{cases} \partial_t^\perp \gamma = -H(\gamma) & \forall t \in [0,T], \\ \gamma(0) = \gamma_0 \circ \phi. \end{cases}$$

This solution is unique in the sense that for each other solution

$$\tilde{\gamma} \in C([0,\tilde{T}), h^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \cap C^1((0,\tilde{T}), C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$$

and each time $t \in (0, \min(T, \tilde{T})]$ *there is a smooth diffeomorphism* $\phi_t \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ *such that*

$$\gamma(t, \cdot) = \tilde{\gamma}(t, \phi_t(\cdot)).$$

3.1 The linear equation

In this subsection we derive a priori estimates and existence results for linear equations of the type

$$\begin{cases} \partial_t u + aQu + bu = f \text{ in } \mathbb{R}/\mathbb{Z} \times (0, T) \\ u(0) = u_0 \end{cases}$$

where $a(t) \in h^\alpha(\mathbb{R}/\mathbb{Z}, (0, \infty))$, $b(t) \in L(h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), C^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$.

We will use the trace method of the theory of real interpolation theory (cf. [11, Sect. 1.2.2]). For the precise statements, we need for $\theta \in (0, 1)$, $\alpha > 0$, and $T > 0$ the space

$$X_T^{\theta,\alpha} := \left\{ g \in C((0, T), h^{\alpha+3}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \cap C^1((0, T), h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) : \sup_{t \in (0,T)} t^{1-\theta} \left(\|\partial_t g(t)\|_{C^\alpha} + \|g(t)\|_{C^{3+\alpha}} \right) < \infty \right\}$$

equipped with the norm

$$\|g\|_{X_T^{\theta,\alpha}} := \sup_{t \in (0,T)} t^{1-\theta} \left(\|\partial_t g(t)\|_{C^\alpha} + \|g(t)\|_{C^{3+\alpha}} \right)$$

and the space

$$Y_T^{\theta,\alpha} := \left\{ g \in C((0, T), h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) : \sup_{t \in (0,T)} t^{1-\theta} \|g(t)\|_{C^\alpha} < \infty \right\}$$

equipped with the norm

$$\|g\|_{Y_T^{\theta,\alpha}} := \sup_{t \in (0,T)} t^{1-\theta} \|g(t)\|_{C^\alpha}.$$

Note that $X_T^{\theta,\alpha} \subset C^\theta((0, T), C^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$. From the trace method in the theory of interpolation spaces (cf. [11, Sect. 1.2.2]), it is well known that for all $u \in X_T^{\theta,\alpha}$ the pointwise limit $u(0)$ of $u(t)$ for $t \to 0$ satisfies $u(0) \in h^{\alpha+3\theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and

$$\|u(0)\|_{C^{\alpha+3\theta}} \leq C\|u\|_{X_{\theta,\alpha}^T} \tag{3.4}$$

if $\alpha + 3\theta \notin \mathbb{N}$.

The aim of this subsection is to prove that for

$$a \in C^1([0, T], h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)), \qquad b \in C^0((0, T), L(h^\alpha(\mathbb{R}/\mathbb{Z}, \quad \mathbb{R}^n), \quad h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$$

with $\sup_{t \in [0,T]} t^{1-\theta} \|b(t)\|_{L(h^\alpha, h^\alpha)} < \infty$ the map

$$J : u \mapsto (u(0), \partial_t u + aQu + bu)$$

is an isomorphism between $X_T^{\theta,\alpha}$ and $h^{\alpha+3\theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \times Y_T^{\theta,\alpha}$. That J is a bounded linear operator follows from Eq. 3.4. That it is onto will be shown using a priori estimates together with the method of continuity.

To derive these estimates, we will freeze the coefficients and use a priori estimates for $\partial_t u + \lambda(-\Delta)^{3/2} u = f$ on $\mathbb{R}$ where $\lambda > 0$ is a constant. We will use the formula

$$(-\Delta)^{3/2} u = \frac{3}{\pi} \lim_{\varepsilon \searrow 0} \int_{\substack{w \in \mathbb{R} \\ |w| > \varepsilon|}} \left(2 \frac{u(x+w) - u(x) - w u'(x)}{|w|^2} - u''(x) \right) \frac{dw}{|w|^2} \tag{3.5}$$

observed by He in [8] for $u \in H^3(\mathbb{R}, \mathbb{R}^n)$. Using the Taylor expansion

$$u(x+w) = \gamma(x) + w u'(x) + \frac{1}{2} w^2 u'''(x) + \frac{1}{2} \int_0^1 (1-t)^2 u'''(x+tw) dt$$

and the Hilbert transform $\mathcal{H}$, one can derive this formula for $u \in C^{3,\alpha}(\mathbb{R})$ with compact support spt u calculating

$$\begin{aligned}
(-\Delta)^{3/2} u &= \mathcal{H}(u''')(x) \\
&= \frac{3}{\pi} \int_0^1 (1-t)^2 \int_{w \in \mathbb{R}} \frac{1}{w} \big(u'''(x+tw) - u'''(x) \big) dw dt \\
&= \frac{3}{\pi} \int_{w \in \mathbb{R}} \frac{1}{w} \int_0^1 (1-t)^2 (u'''(x+tw) - u'''(x)) dt dw \\
&= \frac{3}{\pi} \lim_{\varepsilon \searrow 0} \int_{\substack{w \in \mathbb{R} \\ |w| > \varepsilon}} \frac{1}{w} \int_0^1 (1-t)^2 u'''(x+tw) dt dw \\
&= \frac{3}{\pi} \lim_{\varepsilon \searrow 0} \int_{\substack{w \in \mathbb{R} \\ |w| > \varepsilon}} \left(2 \frac{u(x+w) - u(x) - w u'(x)}{|w|^2} - u''(x) \right) \frac{dw}{|w|^2}.
\end{aligned}$$

Using the boundedness of the Hilbert transform, we furthermore deduce that the operator $(-\Delta)^{3/2}$ is bounded from $C_0^{3+\alpha}(\mathbb{R}, \mathbb{R}^n)$ to $C^\alpha(\mathbb{R}, \mathbb{R}^n)$.

Let us consider the heat kernel of the equation $\partial_t u + (-\Delta)^{3/2} u = 0$ which is given by

$$G_t(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{2\pi i k x} e^{-t|2\pi k|^3} dk. \tag{3.6}$$

for all $t > 0$ and $x \in \mathbb{R}$.

Note that since $k \mapsto e^{-t|2\pi k|^3}$ is a Schwartz function, its inverse Fourier transform G_t is a Schwartz function as well. Furthermore, one easily sees using the Fourier transformation that

$$\partial_t G_t + (-\Delta)^{3/2} G_t = 0 \quad \text{on } \mathbb{R} \quad \forall t > 0. \tag{3.7}$$

The most important property for us is the scaling

$$G_t(x) = t^{-1/3} G_1(t^{-1/3} x), \tag{3.8}$$

from which we deduce

$$\partial_x^k G_t(x) = t^{-(1+k)/3} (\partial_x^k G_1)(t^{-1/3} x) \tag{3.9}$$

and hence

$$\|\partial_x^k G_t\|_{L^1(\mathbb{R})} \le C_k t^{-k/3} \|\partial_x^k G_1\|_{L^1(\mathbb{R})} \le C_k t^{-k/3}. \tag{3.10}$$

Combining these relations with standard interpolation techniques, we get the following estimates for the heat kernel

Lemma 3.3 (heat kernel estimates) *For all* $0 \le \alpha_1 \le \alpha_2$, *and* $T > 0$ *there is a constant* $C = C(\alpha_1, \alpha_2, T) < \infty$ *such that*

$$\|G_t * f\|_{C^{\alpha_2}} \le C t^{-(\alpha_2 - \alpha_1)/3} \|f\|_{C^{\alpha_1}} \quad \forall f \in C^{\alpha_1}(\mathbb{R}, \mathbb{R}^n), t \in (0, T].$$

Proof Let $k_i \in \mathbb{N}_0$ and $\tilde{\alpha}_i \in [0, 1)$ be such that $\alpha_i = k_i + \tilde{\alpha}_i$ for $i = 1, 2$.

For $l > m$ and $\alpha \in (0, 1)$, we deduce from $\int_{\mathbb{R}} \partial_x^{l-m} G_t(y) dy = 0$ and that G_1 is a Schwartz function

$$\begin{aligned}
|\partial_x^l (G_t * f)(x)| &= \left| \int_{\mathbb{R}} \partial_x^{l-m} G_t(y) (\partial_x^m f(x-y) - \partial_x^m f(x)) dy \right| \\
&\overset{(3.9)}{\le} \int_{\mathbb{R}} t^{-(1+l-m)/3} |(\partial_x^{l-m} G_1)(y/t^{1/3})| \cdot |(\partial_x^m f(x-y) - \partial_x^m f(x))| dy \\
&\overset{z=y/t^{1/3}}{\le} \int_{\mathbb{R}} t^{-(l-m)/3} |(\partial_x^{l-m} G_1)(z)| \cdot |\partial_x^m f(x - t^{1/3} z) - \partial_x^m f(x))| dz \\
&\le t^{-(l-(m+\alpha))/3} \operatorname{höl}_\alpha(\partial_x^m f) \int_{\mathbb{R}} |(\partial_x^{l-m} G_1)(z)| |z|^\alpha dz \\
&\le C(l, m, \alpha) t^{-(l-(m+\alpha))/3} \operatorname{höl}_\alpha(\partial_x^m f).
\end{aligned}$$

For all $l \ge m, \alpha \in (0, 1)$ we have

$$\operatorname{höl}_\alpha(\partial_x^l (G_t * f)) \le C(l-m) t^{l-m} \operatorname{höl}_\alpha(\partial_x^m f)$$

as for all $x_1, x_2 \in \mathbb{R}$

$$\begin{aligned}
|\partial_x^l (G_t * f)(x_1) - \partial_x^l (G_t * f)(x_2)| &= \left| \int_{\mathbb{R}} \partial_x^{l-m} G_t(y) (\partial_x^m f(x_1 - y) - \partial_x^m f(x_2 - y)) dy \right| \\
&\le \|\partial_x^{l-m} G_t\|_{L^1} \operatorname{höl}_\alpha(\partial_x^m f) |x_1 - x_2|^\alpha \\
&\overset{(3.10)}{\le} C(l-m) t^{-((l-m))/3} \operatorname{höl}_\alpha(\partial_x^m f) |x_1 - x_2|^\alpha.
\end{aligned}$$

In a similar way we obtain for all $l \ge m$

$$\|\partial_x^l (G_t * f)\|_{L^\infty} \le C(l-m) t^{-(l-m)/3} \|\partial_x^m f\|_{L^\infty}.$$

Combining these three estimates, we get

$$\|G_t * f\|_{C^{k_2+\tilde{\alpha}_1}} \le C t^{-(k_2-k_1)/3} \|f\|_{C^{k_1+\tilde{\alpha}_1}}, \tag{3.11}$$

$$\|G_t * f\|_{C^{k_2+1}} \le C t^{-((k_2+1)-(k_1+\tilde{\alpha}_1)/3} \|f\|_{C^{k_1+\tilde{\alpha}_1}}, \tag{3.12}$$

and if $k_2 > k_1$

$$\|G_t * f\|_{C^{k_2}} \le C t^{-(k_2-(k_1+\tilde{\alpha}_1))/3} \|f\|_{C^{k_1+\tilde{\alpha}_1}} \tag{3.13}$$

Furthermore, we will use that for $0 \leq \alpha \leq \beta \leq \gamma \leq 1, \alpha \neq \gamma$, and $f \in C^{\alpha}$ we have the interpolation inequality

$$\|f\|_{C^{\beta}} \leq 2\|f\|_{C^{\gamma}}^{\frac{\beta-\alpha}{\gamma-\alpha}} \|f\|_{C^{\alpha}}^{\frac{\gamma-\beta}{\gamma-\alpha}}. \tag{3.14}$$

which can in the case of $\alpha > 0$ be obtained from

$$\begin{aligned}|f(x_1) - f(x_2)| &\leq |f(x_1) - f(x_2)|^{\frac{\beta-\alpha}{\gamma-\alpha}} |f(x_1) - f(x_2)|^{\frac{\gamma-\beta}{\gamma-\alpha}} \\ &\leq (\text{höl}_{\gamma}(f)|x_1 - x_2|^{\gamma})^{\frac{\beta-\alpha}{\gamma-\alpha}} (\text{höl}_{\alpha}(f)|x_1 - x_2|^{\alpha})^{\frac{\gamma-\beta}{\gamma-\alpha}}\end{aligned}$$

and in the case that $\alpha = 0$ from

$$\begin{aligned}|f(x_1) - f(x_2)| &\leq |f(x_1) - f(x_2)|^{\frac{\beta}{\gamma}} |f(x_1) - f(x_2)|^{\frac{\gamma-\beta}{\gamma}} \\ &\leq (\text{höl}_{\gamma}(f)|x_1 - x_2|^{\gamma})^{\frac{\beta}{\gamma}} (2\|f\|_{L^{\infty}})^{\frac{\gamma-\beta}{\gamma}}.\end{aligned}$$

For $\tilde{\alpha}_2 \geq \tilde{\alpha}_1$ we get

$$\begin{aligned}\|G_t * f\|_{C^{k_2+\tilde{\alpha}_2}} &\leq C\left(\|\partial_x^{k_2}(G_t * f)\|_{C^{\tilde{\alpha}_2}} + \|G_t * f\|_{L^{\infty}}\right) \\ &\overset{(3.14)}{\leq} C(\|\partial_x^{k_2}(G_t * f)\|_{C^{\tilde{\alpha}_1}}^{\frac{1-\tilde{\alpha}_2}{1-\tilde{\alpha}_1}} \|\partial_x^{k_2+1}(G_t * f)\|_{C^0}^{\frac{\tilde{\alpha}_2-\tilde{\alpha}_1}{1-\tilde{\alpha}_1}} + \|f\|_{L^{\infty}}) \\ &\overset{(3.11)\&(3.12)}{\leq} C(t^{-(\alpha_2-\alpha_1)/3} + 1)\|f\|_{C^{\alpha_1}}.\end{aligned}$$

For $\tilde{\alpha}_1 > \tilde{\alpha}_2$ and hence $k_1 < k_2$, we obtain

$$\begin{aligned}\|G_t * f\|_{C^{k_2+\tilde{\alpha}_2}} &\leq C\left(\|\partial_x^{k_2}(G_t * f)\|_{C^{\tilde{\alpha}_2}} + \|G_t * f)|_{L^{\infty}}\right) \\ &\overset{(3.14)}{\leq} C(\|\partial_x^{k_2}(G_t * f)\|_{C^{\tilde{\alpha}_1}}^{\frac{\tilde{\alpha}_2}{\tilde{\alpha}_1}} \|\partial_x^{k_2}(G_t * f)\|_{C^0}^{\frac{\tilde{\alpha}_1-\tilde{\alpha}_2}{\tilde{\alpha}_1}} + \|G_t * f\|_{L^{\infty}}) \\ &\overset{(3.11)\&(3.13)}{\leq} C(t^{-(\alpha_2-\alpha_1)/3} + 1)\|f\|_{C^{\alpha_1}}.\end{aligned}$$

□

To derive a representation formula for the solution of $\partial_t u + (-\Delta)^{3/2} u = f$, we need

Lemma 3.4 *For all t we have*

$$\int_{\mathbb{R}} G_t(x)dx = 1.$$

Furthermore, for all $f \in h^{\alpha}(\mathbb{R}, \mathbb{R}^n)$, $\alpha \notin \mathbb{N}$ we have

$$G_t * f \xrightarrow{t \downarrow 0} f \quad \text{in } h^{\alpha}(\mathbb{R}, \mathbb{R}^n).$$

Proof For $g \in L^2(\mathbb{R})$ let $\hat{g}$ denote the Fourier transform of g.

For $t > 0$ and $f \in L^2(\mathbb{R})$ we obtain from Lebesgue's theorem of dominated convergence

$$(G_t * f)^{\wedge} = e^{-t|2\pi\cdot|^3} \hat{f} \xrightarrow{t \downarrow 0} \hat{f} \quad \text{in } L^2.$$

Hence, Plancherel's formula shows

$$G_t * f \to f \quad \text{in } L^2.$$

Setting $f = \chi_{[-1,1]}$ and observing

$$\lim_{t\searrow 0}(G_t * f)(x) = \lim_{t\searrow 0} \int\limits_{[t^{-1/3}(x-1),t^{1/3}(x+1)]} G_1 dy = \int\limits_{\mathbb{R}} G_1 dy, \quad \forall x \in (-1,1),$$

we deduce that

$$\int\limits_{\mathbb{R}} G_1 dy = 1.$$

To prove the second part, let $f \in h^\alpha(\mathbb{R}, \mathbb{R}^n)$. From convergence results for smoothing kernels we get for all $\tilde{f} \in C^\infty(\mathbb{R})$

$$\begin{aligned} \limsup_{t\downarrow 0} \|f - G_t * f\|_{C^\alpha} &\leq \limsup_{t\downarrow 0} \|(f-\tilde{f}) - G_t * (f-\tilde{f})\|_{C^\alpha} + \|\tilde{f} - G_t * \tilde{f}\|_{C^\alpha} \\ &= \limsup_{t\downarrow 0} \|(f-\tilde{f}) - G_t * (f-\tilde{f})\|_{C^\alpha} \\ &\overset{\text{Lemma 3.3}}{\leq} C\|(f-\tilde{f})\|_{C^\alpha}. \end{aligned}$$

Since $h^\alpha(\mathbb{R}, \mathbb{R}^n)$ is the closure of $C^\infty(\mathbb{R}, \mathbb{R}^n)$ under $\|\cdot\|_{C^\alpha}$, this proves the statement. □

Linking the heat kernel G_t to the evolution equation $\partial_t + \lambda(-\Delta)^{3/2} = f$ for constant $\lambda > 0$ we derive the following a priori estimates

Lemma 3.5 (Maximal regularity for constant coefficients) *For all $\alpha > 0, \theta \in (0,1)$ with $\alpha + 3\theta \notin \mathbb{N}$, and $0 < T < \infty, \lambda > 0$ there is a constant $C = C(\alpha, \theta, T, \lambda)$ such that the following holds:*

Let $u \in C^1((0,T), h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \cap C^0((0,T), h^{3+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \cap C^0([0,T), h^{\alpha+3\theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ such that $u(t)$ has compact support for all $t \in (0,T)$. Then

$$\begin{aligned} &\sup_{t\in(0,T]} t^{1-\theta}\left(\|\partial_t u\|_{C^\alpha} + \|u\|_{C^{3+\alpha}}\right) \\ &\leq C\left(\sup_{t\in(0,T]} t^{1-\theta}\|\partial_t u + \lambda(-\Delta)^{3/2})u\|_{C^\alpha} + \|u(0)\|_{h^{\alpha+3\theta}}\right) \end{aligned} \tag{3.15}$$

Proof Setting $\tilde{u}(t,x) := u(t, \lambda^{1/3}x)$ and observing that $\partial_t\tilde{u}(x,t) + (-\Delta)^{3/2}\tilde{u}(x,t) = \partial_t u(t, \lambda^{1/3}x) + \lambda(-\Delta)^{3/2}u(t,\lambda^{1/3}x)$, one sees that it is enough to prove the lemma for $\lambda = 1$

To this end, we first show that u can be written as

$$u(t,\cdot) = \int\limits_0^t G_{t-s} * f(s,\cdot)ds + G_t * u(0) \tag{3.16}$$

where $f = \partial_t u + (-\Delta)^{3/2}u$. For fixed $t > 0$ we decompose the integral in Eq. 3.16 into

$$I_\varepsilon := \int\limits_{t-\varepsilon}^t G_{t-s} * f(s,\cdot)dxds$$

and

$$J_\varepsilon := \int\limits_0^{t-\varepsilon} G_{t-s} * f(s,\cdot)dxds$$

and see

$$\|I_\varepsilon\|_{L^\infty} \overset{\text{Lemma 3.3}}{\leq} C\varepsilon \sup_{s\in(t-\varepsilon,t)} \|f(s,\cdot)\|_{L^\infty} \xrightarrow{\varepsilon\downarrow 0} 0.$$

As our assumptions imply that $u(t) \in H^3(\mathbb{R}, \mathbb{R}^n)$, we get comparing the Fourier transform of both sides

$$\big(G_{t-s} * ((-\Delta)^{3/2} u(s,\cdot))\big)(x) = \big(((-\Delta)^{3/2} G_{t-s}) * u(s,\cdot)\big)(x). \tag{3.17}$$

We get using partial integration in time and Eq. 3.17

$$\begin{aligned} J_\varepsilon &= \int_0^{t-\varepsilon} G_{t-s} * \partial_t u(s,\cdot) ds + \int_0^{t-\varepsilon} G_{t-s} * (-\Delta)^{3/2} u(s,\cdot) ds \\ &= G_\varepsilon * u(t-\varepsilon,\cdot) - G_t * u(0,\cdot) + \int_0^{t-\varepsilon} (\partial_s(G_{t-s}) + (-\Delta)^{3/2} G_{t-s}) * u ds \\ &\overset{(3.7)}{=} G_\varepsilon * u(t-\varepsilon,\cdot) - G_t * u(0,\cdot) \xrightarrow{\text{Lemma 3.4}} u(t,\cdot) - G_t * u(0,\cdot). \end{aligned}$$

in C^α as $\varepsilon \searrow 0$. This proves Eq. 3.16.

From Lemma 3.3 we get

$$\|G_t * u_0\|_{C^{3+\alpha}} \leq C t^{\theta-1} \|u_0\|_{C^{\alpha+3\theta}} \tag{3.18}$$

We decompose $v(t) := \int_0^t G_{t-s} * f(s,\cdot) ds = v_1(t) + v_2(t)$ where

$$v_1(t) = G_{t/2} * v(t/2) \qquad v_2(t) = \int_{s=t/2}^{t} G_{t-s} * f(s,\cdot) ds.$$

Then the definition of $\|\cdot\|_{Y_T^{\alpha,\theta}}$ and the estimates for the heat kernel in Lemma 3.3 lead to

$$\|v_1(t)\|_{C^{3+\alpha}} \leq C(t/2)^{-1} \|f\|_{Y_T^{\alpha,\theta}} \int_0^{t/2} s^{\theta-1} ds \leq C(t/2)^{\theta-1} \|f\|_{Y_T^{\alpha,\theta}}. \tag{3.19}$$

For $\xi > 0$ and $\eta \in (0,1)$ we get

$$\begin{aligned} \|\xi^{1-\eta}(G_\xi * v_2(t))\|_{C^{6+\alpha-3\eta}} &= \left\| \xi^{1-\eta} \int_{t/2}^{t} (G_{t-s+\xi} * f) ds \right\|_{C^{6+\alpha-3\eta}} \\ &\overset{\text{Lemma 3.3}}{\leq} C\xi^{1-\eta} \int_{t/2}^{t} (t-s+\xi)^{-2+\eta} s^{\theta-1} ds \|f\|_{Y_T^{\alpha,\theta}} \\ &\leq C(t/2)^{\theta-1} \|f\|_{Y_T^{\alpha,\theta}} \end{aligned}$$

and

$$\|\xi^{1-\eta}\frac{d}{d\xi}(G_\xi * v_2)\|_{C^{3+\alpha-3\eta}} = \left\|\xi^{1-\eta}\int\limits_{t/2}^{t}(\partial_t G_{t-s+\xi} * f\right\|_{C^{3+\alpha-3\eta}}$$

$$\leq C\xi^{1-\eta}\int\limits_{t/2}^{t}(t-s+\xi)^{-2+\eta}s^{\theta-1}ds\|f\|_{Y_T^{\alpha,\theta}}$$

$$\leq C(t/2)^{\theta-1}\|f\|_{Y_T^{\alpha,\theta}}$$

as

$$\xi^{1-\eta}\int\limits_{\frac{t}{2}}^{t}(t-s+\xi)^{-2+\eta}ds = \frac{\xi^{1-\eta}}{1-\eta}(\xi^{\eta-1}-(\frac{t}{2}+\xi)^{\eta-1}) \leq \frac{1}{1-\eta}.$$

Hence, by the estimate (3.4)

$$\begin{aligned}&\|v_2(t)\|_{C^{3+\alpha}}\\ &\leq C\sup_{\xi\in(0,T/2)}\|\xi^{1-\eta}\left(\|(G_\xi * v_2(t))\|_{C^{6+\alpha-3\eta}}\| + \|\partial_\xi(G_\xi * v_2(t))\|_{C^{3+\alpha-3\eta}}\right)\\ &\leq Ct^{\theta-1}\|f\|_{Y_T^{\alpha,\theta}}.\end{aligned} \tag{3.20}$$

From (3.16), (3.18), (3.19), and (3.20) we obtain the desired estimate for $\|u\|_{C^{3+\alpha}}$.

The estimate for $\partial_t u$ then follows from $\partial_t u = f - (-\Delta)^{3/2}u$ and the triangle inequality. □

Lemma 3.6 (Maximal regularity) *For all* $\Lambda, T > 0, n \in \mathbb{N}$, *and* $\alpha > 0, \theta \in (0,1)$ *with* $\alpha + 3\theta \notin \mathbb{N}$ *there is a constant* $C = C(\Lambda, \alpha, \theta, n, T), < \infty$ *such that the following holds:*

For all $a \in C^1([0,T], h^\alpha(\mathbb{R}/\mathbb{Z}, [1/\Lambda, \infty)))$, $b \in C^0((0,T), L(h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)))$ *with* $\|a\|_{C^1([0,T],C^\alpha)} + t^{1-\theta}\|b(t)\|_{L(h^\alpha,h^\alpha)} \leq \Lambda$ *and all* $u \in C^1((0,T), h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \cap C^0((0,T), h^{3+\alpha}) \cap C^0([0,T], h^{\alpha+3\theta})$ *we have*

$$\begin{aligned}&\sup_{t\in[0,T]} t^{1-\theta}\left\{\|\partial_t u(t)\|_{C^\alpha} + \|u(t)\|_{C^{3+\alpha}}\right\}\\ &\leq C\left(\sup_{t\in[0,T]} t^{1-\theta}\|\partial_t u(t) + a(t)Qu(t) + b(t)u(t)\|_{C^\alpha} + \|u(0)\|_{h^{\alpha+3\theta}}\right).\end{aligned}$$

Proof Note that it is enough to prove the statement for small T. Let us fix $T_0 > 0$ and assume that $T \leq T_0$. Furthermore, we use the embedding $h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to h^\alpha(\mathbb{R}, \mathbb{R}^n)$ and extend the definition of Q to functions f defined on $\mathbb{R}$ by setting

$$Qf(x) := \lim_{\varepsilon\searrow 0}\int\limits_{[-1/2,1/2]-[\varepsilon,\varepsilon]}\left(2\frac{f(u+w)-f(u)-wf'(u)}{w^2} - f''(x)\right)\frac{dw}{w^2}.$$

Step 1: $\alpha \in (0,1)$ **and** $b = 0$

Let $\phi, \psi \in C^\infty(\mathbb{R})$ be two cutoff functions satisfying

$$\chi_{B_{1/2}(0)} \leq \phi \leq \chi_{B_1(0)}$$
$$\chi_{B_2(0)} \leq \psi \leq \chi_{B_4(0)}.$$

and $\phi_r(x) := \phi(x/r)$, $\psi_r(x) = \psi(x/r)$. We set

$$f = \partial_t u + aQu.$$

Then for $r < 1/8$ we set $a_0 = a(0,0)$ and calculate

$$\begin{aligned}\partial_t(u\phi_r) + \frac{\pi}{3}a(0)(-\Delta_{\mathbb{R}})^{3/2}(u\phi_r) &= (\partial_t u + aQu)\phi_r - a(Q(u)\phi_r - Q(u\phi_r)) \\ &\quad -(a-a(0))Q(u\phi_r) - a_0(Q(u\phi_r) - \frac{\pi}{3}(-\Delta_{\mathbb{R}})^{3/2}(u\phi_r) \\ &= f\phi_r - f_1 - f_2 - f_3\end{aligned}$$

where

$$\begin{aligned}f_1 &:= a(Q(u)\phi_r - Q(u\phi_r)) \\ f_2 &:= (a - a(0))Q(u\phi_r) \\ f_3 &:= a_0(Q(u\phi_r) - \frac{\pi}{3}(-\Delta_{\mathbb{R}})^{3/2}(u\phi_r)\end{aligned}$$

From Lemma 3.5 we get

$$\begin{aligned}&t^{1-\theta}(\|\partial_t u(t)\phi_r\|_{C^\alpha} + \|u(t)\phi_r\|_{C^{3+\alpha}}) \\ &\leq C\Bigg(\sup_{s\in[0,T]} s^{1-\theta}(\|f(s)\phi_r\|_{C^\alpha} + \|f_1(s)\|_{C^\alpha} + \|f_2(s)\|_{C^\alpha} + \|f_3(s)\|_{C^\alpha}) \\ &\quad + \|u(0)\|_{C^{\alpha+3\theta}}.\Bigg)\end{aligned}$$

Using Lemma A.8, we get

$$\|f_1(s)\|_{C^\alpha} \leq C\Lambda\|u(s)\|_{C^{2+\alpha}}\|\phi_r\|_{C^{3+\alpha}}.$$

Using $|a(x,t) - a_0| \leq \Lambda(|x|^\alpha + T)$, we derive

$$\begin{aligned}\|f_2\|_{C^\alpha} &\leq \|\psi_r(a-a(0))Q(u\phi_r)\|_{C^\alpha} + \|(\psi_r - 1)(a-a(0))Q(u\phi_r)\|_{C^\alpha} \\ &\leq C_1\Lambda((2r)^{\tilde{\alpha}} + T)\|u\phi_r\|_{C^{3+\alpha}} + \|(\psi_r - 1)(a-a(0))Q(u\phi_r)\|_{C^\alpha}\end{aligned}$$

where C_1 does not depend on r or T. Since $\operatorname{spt} 1 - \psi_r \subset \mathbb{R} - B_{4r}(0)$ and $\operatorname{spt}\phi_r \subset B_r(0)$, we see that

$$\begin{aligned}&(\psi_r - 1)(a - a(0))Q(u\phi_r)(x) \\ &= (\psi_r(x) - 1)(a(x) - a(0)) \int_{[-1/2,1/2]-[-r,r]} \frac{u(x+w)\phi_r(x+r)}{w^2}dw\end{aligned}$$

and hence

$$\|(\psi_r - 1)(a - a(0))Q(u\phi_r)\|_{C^\alpha} \leq C(\Lambda, \psi, \phi, r)\|u\|_{C^\alpha}.$$

This leads to

$$\|f_2(s)\|_{C^\alpha} \leq C_1\Lambda((2r)^{\tilde{\alpha}} + T)\|u(s)\phi_r\|_{C^{3+\alpha}} + C\|u(s)\|_{C^\alpha}.$$

Furthermore,

$$\|f_3\|_{C^\alpha} \leq C(\Lambda, \phi, r)\|u\|_{C^{2+\alpha}}$$

as for $v \in C^{3+\alpha}(\mathbb{R})$ with compact support we have

$$Q(v) - \frac{\pi}{3}(-\Delta_{\mathbb{R}})^{3/2}(v) = - \int_{\mathbb{R}-[-1/2,1/2]} \left(2\frac{v(u+w)-v(u)}{w^2} - v''(u) \right) \frac{dw}{w^2}$$

and hence

$$\|Q(v) - \frac{1}{6}(-\Delta_{\mathbb{R}})^{3/2}(v)\|_{C^\alpha} \leq (4\|v\|_{C^\alpha} + \|v\|_{C^{2+\alpha}}) \cdot 2\int_{\frac{1}{2}}^{\infty} \frac{1}{w^2} dw \leq 24\|v\|_{C^{2+\alpha}}.$$

Summing up, we thus get

$$\sup_{t\in(0,T]} t^{1-\theta}(\|\partial_t(u\phi_r)(t)\|_{C^\alpha} + \|u(t)\phi_r\|_{C^{3+\alpha}}) \leq C_1\Lambda((2r)^{\tilde{\alpha}} + T) \sup_{s\in(0,T]} s^{1-\theta}\|u\phi_r\|_{C^{3+\alpha}}$$
$$+C(\phi,\psi,r,\Lambda))\left(\sup_{s\in(0,T]}(s^{1-\theta}\|f(s)\|_{C^\alpha} + s^{1-\theta}\|u(s)\|_{C^3}) + \|u_0\|_{C^{\alpha+3\theta}}\right)$$

where C_1 does not depend on r. Choosing r and T small enough and absorbing the first term on the right hand side leads to

$$\sup_{t\in(0,T]} t^{1-\theta}\left(\|\partial_t u\|_{C^\alpha(B_{r/2}(0))} + \|u\|_{C^{3+\alpha}(B_{r/2}(0))}\right)$$
$$\leq C(\phi,\psi,r,\Lambda))\left(\sup_{s\in(0,T]}(s^{1-\theta}\|f(s)\|_{C^\alpha} + s^{1-\theta}\|u(s)\|_{C^3}) + \|u(0)\|_{C^{\alpha+3\theta}}\right).$$

Of course, the same inequality holds for all balls of radius $r/4$. Thus, covering $[0, 1]$ with balls of radius $r/4$ we obtain

$$\sup_{t\in(0,T]} t^{1-\theta}\left(\|\partial_t u(t)\|_{C^\alpha} + \|u(t)\|_{C^{3+\alpha}}\right)$$
$$\leq C\left(\sup_{s\in(0,T]}(s^{1-\theta}\|f(s)\|_{C^\alpha} + s^{1-\theta}\|u(s)\|_{C^3}) + \|u(0)\|_{C^{\alpha+3\theta}}\right).$$

Using the interpolation inequality for Hölder spaces

$$\|u\|_{C^3} \leq \varepsilon\|u\|_{C^{3+\alpha}} + C(\varepsilon)\|u\|_{C^\alpha}$$

and absorbing, this leads to

$$\sup_{t\in(0,T]} t^{1-\theta}\left(\|\partial_t u(t)\|_{C^\alpha} + \|u(t)\|_{C^{3+\alpha}}\right)$$
$$\leq C\left(\sup_{s\in(0,T]}\left(s^{1-\theta}\|f(s)\|_{C^\alpha} + \|u(s)\|_{C^\alpha}\right) + \|u(0)\|_{C^{\alpha+3\theta}}\right).$$

Since

$$\begin{aligned}\|u(s)\|_{C^\alpha} &\le \int_0^s \|\partial_t u(\tau)\|_{C^\alpha} d\tau + \|u(0)\|_{C^{\alpha+3\theta}} \\ &\le \int_0^T \tau^{\theta-1} d\tau \sup_{\tau\in[0,T]} \tau^{1-\theta} \|\partial_t u(\tau)\|_{C^\alpha} + \|u(0)\|_{C^{\alpha+3\theta}} \\ &\le \frac{1}{\theta} T^\theta \sup_{\tau\in(0,T]} \tau^{1-\theta} \|\partial_t u(\tau)\|_{C^\alpha} + \|u(0)\|_{C^{\alpha+3\theta}}.\end{aligned}$$

we can absorb the first term for $T > 0$ small enough to obtain

$$\begin{aligned}&\sup_{s\in(0,T)} s^{1-\theta} \left(\|\partial_t u(s)\|_{C^\alpha} + \|u(s)\|_{C^{3+\alpha}}\right) \\ &\quad \le C(\Lambda)) \left(\left(\sup_{s\in[0,T]} s^{1-\theta} \|f\|_{C^\alpha}\right) + \|u(0)\|_{C^{\alpha+3\theta}}\right).\end{aligned}$$

Step 2: General α but $b = 0$

Let $k \in \mathbb{N}_0, \tilde{\alpha} \in (0, 1)$ and let the lemma be true for $\alpha = k + \tilde{\alpha}$. We deduce the statement for $\alpha = k + 1 + \tilde{\alpha}$.

From $\partial_t u + aQu = f$ we deduce that

$$\partial_t(\partial_x u) + aQ\partial_x u = \partial_x f - (\partial_x a)Qu$$

and we obtain applying the induction hypothesis

$$\begin{aligned}\|\partial_x u\|_{X_T^{k+\alpha,\theta}} &\le C\left(\|\partial_x f\|_{Y_T^{k+\tilde{\alpha},\theta}} + \|(\partial_x a)Qu\|_{Y_T^{k+\tilde{\alpha},\theta}} + \|\partial_x u(0)\|_{C^{\alpha+3\theta}}\right) \\ &\le C\left(\|f\|_{Y_T^{k+1+\tilde{\alpha},\theta}} + \Lambda\|u\|_{X_T^{k+\tilde{\alpha},\theta}} + \|\partial_x u(0)\|_{C^{\alpha+3\theta}}\right) \\ &\le C\left(\|f\|_{Y_T^{k+1+\tilde{\alpha},\theta}} + \|\partial_x u(0)\|_{C^{\alpha+3\theta}}\right).\end{aligned}$$

Step 3: General α and b

From Step 2 we get

$$\|u\|_{X_T^{\alpha,\theta}} \le C\left(\|f\|_{Y_T^{\alpha,\theta}} + \|((t,x) \mapsto b(t)(u(t))(x))\|_{Y_T^{\alpha,\theta}} + \|u_0\|_{C^{\alpha+3\theta}}\right).$$

As

$$\|((t,x) \mapsto b(t)(u(t))(x))\|_{Y_T^{\alpha,\theta}} = \sup_{s\in(0,T]} t^{1-\theta} \|b(t)(u(t))\|_{C^\alpha} \le \Lambda \sup_{s\in(0,T]} \|u(s)\|_{C^\alpha}$$

and

$$\begin{aligned}\|u(s)\|_{C^\alpha} &\leq \int_0^s \|\partial_t u(\tau)\|_{C^\alpha} d\tau + \|u_0\|_{C^{\alpha+3\theta}} \\ &\leq \int_0^T \tau^{\theta-1} d\tau \sup_{\tau\in[0,T]} \tau^{1-\theta} \|\partial_t u(\tau)\|_{C^\alpha} + \|u(0)\|_{C^{\alpha+3\theta}} \\ &\leq \frac{1}{\theta} T^\theta \sup_{\tau\in(0,T]} \tau^{1-\theta} \|\partial_t u(\tau)\|_{C^\alpha} + \|u(0)\|_{C^{\alpha+3\theta}},\end{aligned}$$

we get absorbing the first term for $T > 0$ small enough

$$\|u\|_{X_T^{\alpha,\theta}} \leq C \left(\|f\|_{Y_T^{\alpha,\theta}} + \|u_0\|_{C^{\alpha+3\theta}} \right).$$

□

Lemma 3.7 *For all $T > 0, \alpha > 0, \theta \in (0,1)$ with $\alpha + 3\theta \notin \mathbb{N}$, a and b as in Lemma 3.6, the map $J : u \mapsto (u(0), \partial_t u + aQu + bu)$ defines an isomorphism between $X_T^{\theta,\alpha}$ and $h^\beta(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \times Y_T^{\theta,\alpha}$.*

Proof The only thing left to show is that this map is onto. To prove this, we use the method of continuity on the family of operators $J_\tau : u \mapsto (u(0), \partial_t u + ((1-\tau)\lambda Qu + \tau(aQu + bu)$. In view of Lemma 5.2 in [7], the only thing to show is that J_0 is onto. For u_0, f in C^∞ a smooth solution of the equation

$$\begin{cases} \partial_t u + \lambda Q u = f & \forall t \in (0, T] \\ u(0) = u_0 \end{cases}$$

is given by

$$u(t,x) = \sum_{k\in\mathbb{Z}} \hat{u}_0(k) e^{-t\lambda\lambda_k|2\pi k|^3} e^{2\pi i k x} + \int_0^t \sum_{k\in\mathbb{Z}} (f(s))^\wedge(k) e^{-(t-s)\lambda\lambda_k|2\pi k|^3} ds$$

where

$$\lambda_k := \frac{2}{3} \int_0^{\pi k} \frac{1}{t} \left(1 - \frac{1}{k\pi}\right)^3 \sin(t) dt = \frac{\pi}{3} + O(\frac{1}{k}).$$

This follows from the fact that for $f \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ we have by [9, Lemma 2.3]

$$Q(f)^\wedge(k) = \lambda_k |2\pi k|^3 \hat{f}(k).$$

Let now $u_0 \in h^{\alpha+3\theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and $f \in Y_T^{\alpha,\theta}$. We set $f_k(t) := f(t + 1/k)$ and observe that

$$f_k \to f \quad \text{in } C^0((0, T], h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)).$$

Since $f_k \in C^0([0, T - 1/k], h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ we can find functions $f_{n,k} \in C^\infty([0, T - 1/k] \times \mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ such that $f_{n,k} \to f_n$ in $C^0([0, T - 1/k], C^\alpha)$ for $n \to \infty$ and smooth $u_0^{(k)}$ converging to u_0 in $h^{\alpha+3\theta}$. Let $u_{n,k} \in C^\infty$ be the solution of

$$\begin{cases} \partial_t u_{n,k} + Q u_{n,k} = f_{n,k} \\ u_{n,k}(0) = u_0^{(k)}. \end{cases}$$

Using the a priori estimate of Lemma 3.6, one deduces that the sequence $\{u_{n,k}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $X^{\alpha,\theta}_{T-\varepsilon}$ for every $\varepsilon > 0$. The limit u_n solves the equation

$$\begin{cases} \partial_t u_n + Q u_n = f_n \\ u_n(0) = u_0. \end{cases}$$

Using the a priori estimates again, one sees that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $X^{\alpha,\theta}_{T-\varepsilon}$. Since $X^{\alpha,\theta}_{T-\varepsilon}$ embeds continuously into $C^{\theta/2}([0,T-\varepsilon], h^{\alpha+3/2\theta})$ and $C^{1-\eta}([\delta, T-\varepsilon], h^{\alpha+3\eta})$ for all $\eta \in (0,1)$, $\varepsilon, \delta > 0$, we can assume after going to a subsequence that there is a $u_\infty \in X^{\alpha,\theta}_T$ such that

$$u_n \to u_\infty \text{ in } C^0((0,T-\varepsilon), C^{3+\beta})$$

for $0 \le \beta < \alpha$, $\varepsilon > 0$ and

$$u_\infty(0) = u_0.$$

Hence we get

$$\partial_t u_n = f_n - \lambda Q u_n \to f + \lambda Q u_\infty \text{ in } C^0((0,T-\varepsilon), C^{3+\beta})$$

for all $\varepsilon > 0$ which implies that u_∞ solves

$$\begin{cases} \partial_t u_\infty + \lambda Q u_\infty = f \\ u_\infty(0) = u_0. \end{cases}$$

□

3.2 The quasilinear equation

Proposition 3.8 (Short time existence) *Let $0 < \alpha, 0 < \theta < \sigma < 1, \alpha, \alpha + 3\theta, \alpha + 3\theta \notin \mathbb{N}_0, U \subset C^{\alpha+3\theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be open and let $a \in C^1(U, C^\alpha(\mathbb{R}/\mathbb{Z}, (0,\infty)))$, $f \in C^1(U, C^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$.*

Then for every $u_0 \in h^{\alpha+3\sigma}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ there is a constant $T > 0$ and a unique $u \in C^0([0,T), h^{\alpha+3\sigma}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \cap C^1((0,T), h^{3+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ such that

$$\begin{cases} \partial_t u + a(u) Q(u) = f(u) \\ u(0) = u_0. \end{cases}$$

Proof Let us first prove the existence. We set $\tilde{X}^{\alpha,\theta}_T := \{w \in X^{\alpha,\theta}_T : w(0) = u_0\}$. For $w \in \tilde{X}^{\alpha,\theta}_T(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ let Φw denote the solution of the problem

$$\begin{cases} \partial_t u + A_0 u = B(w) w + f(w) \\ u(0) = u_0 \end{cases}$$

where $A_0 = a(u_0) Q$ and $B(w) = (a(u_0) - a(w)) Q$.

Let v be the solution of

$$\begin{cases} \partial_t \tilde{u} + a(u_0) Q(\tilde{u}) = f(u_0) \\ \tilde{u}(0) = u_0 \end{cases}$$

and $\mathcal{B}_r(v) := \{w \in \tilde{X}^{\alpha,\theta}_T : \|w - v\|_{X^{\alpha,\theta}_T} \le r\}$. We will show that Φ defines a contraction on $\mathcal{B}_r(v)$ if $r, T > 0$ are small enough.

Since $a \in C^1(U, h^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$, we get $\|B(z)\|_{L(C^{\alpha+3}, C^\alpha)} \le C\|z - u\|_{C^{\alpha+3\theta}}$ for all $z \in C^{\alpha+3\theta}$.

Let $w_1, w_2 \in \mathcal{B}_r(v)$, $r \le 1$. Using that the space $X_T^{\alpha,\sigma}$ embeds continuously into $C^{\sigma-\theta}([0,T], C^{\alpha+3\theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ and $w_1(0) = w_2(0) = v(0) = u_0$ we get

$$\|w_2(t) - u_0\|_{C^{\alpha+3\theta}} \le Ct^{\sigma-\theta}\|w_2\|_{X_T^{\alpha,\sigma}} \le Ct^{\sigma-\theta}(\|v\|_{X_T^{\alpha,\sigma}} + r), \tag{3.21}$$

$$\|w_1(t) - w_2(t)\|_{C^{\alpha+3\theta}} \le Ct^{\sigma-\theta}\|w_1 - w_2\|_{X_T^{\alpha,\sigma}}. \tag{3.22}$$

We estimate using Lemma 3.6

$$\|\Phi w_1 - \Phi w_2\|_{X_T^{\alpha,\sigma}} \le C\|B(w_1)w_1 - B(w_2)w_2\|_{Y_T^{\alpha,\theta}} + C\|fw_1 - fw_2\|_{Y_T^{\alpha,\sigma}}.$$

and

$$\begin{aligned} t^{1-\sigma}\|f(w_1(t)) - f(w_2(t))\|_{C^\alpha} &\le Ct^{1-\sigma}\|w_1(t) - w_2(t)\|_{C^{\alpha+3\theta}} \\ &\overset{(3.22)}{\le} CT^{1-\theta}\|w_1 - w_2\|_{X_T^{\alpha,\sigma}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\|B(w_1)w_1 - B(w_2)w_2\|_{Y_T^{\alpha,\sigma}} \\ &\quad \le \|(B(w_1) - B(w_2))w_1\|_{Y_T^{\alpha,\sigma}} + \|B(w_2)(w_1 - w_2)\|_{Y_T^{\alpha,\sigma}} \\ &\quad \le C \sup_{t\in(0,T]} t^{1-\sigma}\left(\|w_1(t) - w_2(t)\|_{C^{\alpha+3\theta}}\|w_1(t)\|_{C^{3+\alpha}}\right. \\ &\qquad \left. + \|w_2(t) - u_0\|_{C^{\alpha+3\theta}}\|w_1(t) - w_2(t)\|_{C^{3+\alpha}}\right) \\ &\quad \overset{(3.22)\&(3.21)}{\le} C \sup_{t\in(0,T]}\left(t^{\sigma-\theta}\|w_1 - w_2\|_{X_T^{\alpha,\theta}}\|w_1\|_{X_T^{\alpha,\sigma}}\right. \\ &\qquad \left. + t^{\sigma-\theta}(\|v\|_{X_T^{\alpha,\sigma}} + r)t^{1-\sigma}\|w_1(t) - w_2(t)\|_{C^{3+\alpha}}\right) \\ &\qquad C(T^{\sigma-\theta}(\|v\|_{X_T^{\alpha,\sigma}} + r)\|w_1 - w_2\|_{X_T^{\alpha,\sigma}}. \end{aligned}$$

and thus

$$\|\Phi(w_1) - \Phi(w_2)\|_{Y_T^{\alpha,\theta}} \le C(T^{1-\sigma} + T^{\sigma-\theta}(\|v\|_{X_T^{\alpha,\sigma}} + r))\|w_1 - w_2\|_{Y_T^{\alpha,\theta}}$$

and hence Φ is a contraction on $\mathcal{B}_r(v)$, if T and r are small enough.

Similarly, we deduce from the definition of v that

$$\begin{aligned} \|\Phi(w) - v\|_{X_T^{\alpha,\sigma}} &\le C\|B(w)w\|_{Y_T^{\alpha,\sigma}} + \|f(w) - f(u_0)\|_{Y_T^{\alpha,\sigma}} \\ &\le CT^{\sigma-\theta}\|w\|_{X_T^{\alpha,\sigma}}\|w - v\|_{X_T^{\alpha,\theta}} + T^{1-\theta}\|w - v\|_{X_T^{\alpha,\sigma}} + \|v - u_0\|_{C^{\alpha+3}} \\ &< \|w - v\|_{X_T^{\alpha,\sigma}} \end{aligned}$$

if T and r are small enough. Then $\phi(\mathcal{B}_r(v)) \subset \phi(\mathcal{B}_r(v))$. Hence, Banach's fixed point theorem tell us that there is a unique $u \in B_r(v)$ with $\partial_t u + a(u)Q(u)u = f(u)$.

For the uniqueness statement, we only have to show that every solution is in $Y_T^{\alpha,\theta}$. But this follows from Lemma 3.6. ◻

Proposition 3.9 (Dependence on the data) *Let a, b be as in Proposition 3.8 and $u \in Y_T^{\theta,\alpha}$ be a solution of the quasilinear equation*

$$\begin{cases} \partial_t u + a(u)Q(u) = 0 \\ u(0) = u_0. \end{cases}$$

Then there is a neighborhood U of u_0 in $h^{\alpha+3\theta}$ such that for all $x \in U$ there is a solution u_x of

$$\begin{cases} \partial_t u + a(u)Qu = 0 \\ u(0) = x \end{cases}$$

Furthermore, the map

$$\begin{aligned} U &\to Y_T^{\theta,\alpha} \\ x &\mapsto u_x \end{aligned}$$

is C^1.

Proof We define $\Phi : h^{\alpha+3\theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \times X_T^{\alpha,\theta} \to Y_T^{\alpha,\theta}$ by

$$\Phi(x, u) := (u(0) - x, \partial_t u + a(u)Qu)$$

Then the Fréchet derivative of ϕ with respect to u reads

$$\frac{\partial \phi(x,u)}{\partial u}(h) = (h, \partial_t h + a(u)Qh + a'(u)hQu).$$

Setting $a(t) = a(u(t))$ and $b(t)(h) = a'(u)hQu$, Lemma 3.6 tells us that this is an isomorphism between $X_T^{\theta,\alpha}$ and $h^\beta \times Y_T^{\theta,\alpha}$. Hence, the statement of the lemma follows from the implicit function theorem on Banach spaces. □

3.3 Proof of Theorem 1.2 and Lemma 3.1

Finally, we are in a position to prove Theorem 1.2

Proof of Theorem 1.2 Since the normal bundle of a curve is trivial, we can find smooth normal vector fields $\nu_1, \ldots, \nu_{n-1} \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ such that for each of $u \in \mathbb{R}/\mathbb{Z}$ the vectors $\nu_1(u), \ldots, \nu_{n-1}(u)$ form an orthonormal basis of the space of all normal vectors to γ_0 at u. Let $\tilde{\mathcal{V}}_r(\gamma) := \{(\phi_1, \ldots, \phi_{n-1} \in h^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{n-1}) : \sum_{i=1}^{n-1} \phi_i \nu_i \in \mathcal{V}_r(\gamma)\}$.

Now let $\alpha - 1 < \beta < \alpha$, $\beta \notin \mathbb{N}$. If we have $N_t = \sum_{i=0}^{n-1} \phi_t^i \nu_i$, $(\phi_{1,t}, \ldots \phi_{n-1,t}) \in \tilde{\mathcal{V}}_r(\gamma)$, then (3.3) reads

$$\begin{aligned} \left(\partial_t \phi_{i,t}\right)\left(P^\perp_{\gamma'(u)}\nu_r\right) &= -\frac{2}{|\gamma'|^3} P^\perp_{\gamma'}\left(Q\left(\gamma_0 + \phi_{i,t}\nu_i\right)\right) + F(\gamma_0 + \phi_{i,t}\nu_i) \\ &= -\frac{2}{|\gamma'|^3}\left(Q\phi_{i,t}\right)P^\perp_{\gamma'}\nu_i - F(\gamma_0 + \phi_{i,t}\nu_i) \\ &\quad -\frac{2}{|\gamma'|^3} P^\perp_{\gamma'}\left(Q\left(\phi_{i,t}\nu_i\right) - \left(Q\phi_{i,t}\right)\nu_i + Q\gamma_0\right) \\ &\quad -F(\gamma_0 + \phi_{i,t}\nu_i) \\ &= -\frac{2}{|\gamma'|^3}\left(Q\phi_{i,t}\right)P^\perp_{\gamma'}\nu_i + \tilde{F}_{\gamma_0}(\phi_t) \end{aligned}$$

where

$$\begin{aligned} \tilde{F}_{\gamma_0}(\phi_t) = -F(\gamma_0 + \phi_{r,t}\nu_r) &- \frac{2}{|\gamma'|^3} P^\perp_{\gamma'}\left(Q\left(\phi_{r,t}\nu_r\right) - \left(Q\phi_{r,t}\right)\nu_r + Q\gamma_0\right) \\ &+ \frac{2}{|\gamma'|^3}\left(\left(Q\phi_{i,t}\right)P^\perp_{\gamma'}\nu_i - P^\perp_{\gamma'}\left(Q\phi_{i,t}\nu_i\right)\right). \end{aligned}$$

Using Lemma A.8 one sees that $\tilde{F} \in C^w(C^{2+\alpha}, C^\beta)$.

Since (3.1) implies that $\{P^\perp_{\gamma'} \nu_r : r = 1, \dots, n-1\}$ is a basis of the normal space of the curve γ at the point u, the map $A : \mathbb{R}^{n-1} \to (\gamma'(u))^\perp$, $(x_1, \dots, x_{n-1}) \mapsto x_i P^\perp_{\gamma'(u)} \nu_i$ is invertible as long as $\|\gamma' - \gamma_0'\|_{L^\infty} < 1$.

Hence, we derive

$$\partial_t \phi_t = \frac{2}{|\gamma'|^3} Q\phi_t + A^{-1}\left(\tilde{F}(\phi_t)\right)$$

where $A^{-1}(\tilde{F}(\phi_t)) \in C^\omega(h^{2+\alpha}, h^\beta)$. Now the statement follows easily from Proposition 3.8 and Proposition 3.9 and a standard bootstrapping argument..

To be more precise, one gets immediately form the short time existence result that for every $\phi_0 \in h^{2+\alpha}$ there is a solution in $\phi \in C^0([0,T), h^{2+\alpha}) \cap C^1((0,T), h^{3+\beta}))$ for all $\max\{0, \alpha - 1\} < \beta < \alpha$, $\alpha \notin \mathbb{N}$ and the C^1 dependence on the data. Bootstrapping, we get $\phi_t \in C^0((0,T), C^\infty)$ and the corresponding C^1 dependence on the data. □

We conclude this section proving Lemma 3.1

Proof of Lemma 3.1 Let $\tilde{\gamma} \in h^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and let us set $\gamma_\varepsilon := \phi_\varepsilon * \tilde{\gamma}$ where $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$ is a smooth smoothing kernel. Since $h^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is an open subset of $h^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and $(\varepsilon \to \gamma_\varepsilon) \in C^0([0,\infty), h^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, we get $\gamma_\varepsilon \in h^{2+\alpha}_{i,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for ε small enough.

Furthermore, it can be deduced from $\gamma_\varepsilon \in C^0([0,\infty), h^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$, that there is an open neighborhood U of the set $\gamma(\mathbb{R}/\mathbb{Z})$ and an $\varepsilon_0 > 0$ such that the nearest neighborhood retract $r_\varepsilon : U \to \mathbb{Z}/\mathbb{R}$ onto γ_ε is defined on U simultaneously for all $\varepsilon < \varepsilon_0$. Note, that these retracts r_ε are smooth as the curves γ_ε are smooth.

We set $\psi_\varepsilon(x) := r_\varepsilon(\gamma(x))$ and $N_\varepsilon(x) = \gamma_\varepsilon(\psi_\varepsilon^{-1}(x)) - \gamma_0(x)$ and want to show that $\gamma := \gamma_\varepsilon$, $N := N_\varepsilon$, and $\psi := \psi_\varepsilon$ satisfy the statement of the lemma if ε is small enough.

Using the fact that $(\varepsilon \mapsto r_\varepsilon) \in C((0,\varepsilon_0), C^{1,\alpha}(U, \mathbb{R}/\mathbb{Z}))$ and making U smaller if necessary, we get that $\varepsilon \to \psi_\varepsilon$ belongs to $C^0((0,\varepsilon_0), C^{1+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ and $\psi_0 = id_{\mathbb{R}/\mathbb{Z}}$. Hence, ψ_ε is a $C^{1+\alpha}$ diffeomorphism for $\varepsilon > 0$ small enough as the subset of diffeomorphism is open in $C^{1+\alpha}$. From $\psi_\varepsilon(x) := r_\varepsilon(\gamma(x))$ we deduce that ψ_ε is in fact a $C^{2+\alpha}$ diffeomorphism, as r_ε is smooth.

Furthermore, as

$$N_\varepsilon = \gamma_\varepsilon \circ \psi_\varepsilon^{-1} - \gamma_0 \in C^0([0,\infty), C^{1+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$$

and $N_0 = 0$ we get

$$\|N_\varepsilon\|_{C^1} \xrightarrow{\varepsilon \searrow 0} 0.$$

Since r is lower semicontinuous and $r(\gamma_0) > 0$, we hence get $\|N_\varepsilon(x)\|_{C^1} < r(\gamma_\varepsilon)$ for small ε. As $N_\varepsilon \in h^{2,\alpha}$ we deduce that $N_\varepsilon \in \mathcal{V}_r(\gamma_\varepsilon)$ if ε is small enough. □

4 The Łojasiewicz-Simon gradient estimate

Proof of Lemma 1.3 We can assume without loss of generality that γ_M is parametrized by arc-length and that the length of the curve is 1.

Let $H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M}$ denote the space of all vector fields $N \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ which are orthogonal to γ_M. We first show that the functional

$$\begin{aligned} \tilde{E} : H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M} &\to \mathbb{R} \\ N &\mapsto E(\phi + N) \end{aligned}$$

satisfies a Łojasiewicz-Simon gradient estimate using [5, Corollary 3.11].

To prove that the derivative $\tilde{E}''(0)$ defines a Fredholm operator from $H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M}$ to $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M} \subset (H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))'$, we calculate using $|\gamma_M'| = 1$

$$\begin{aligned} &\tilde{E}''(0)(h_1, h_2) \\ &= \lim_{t\to 0} \frac{\int_{\mathbb{R}/\mathbb{Z}} \langle H(\gamma_M + th_1), h_2\rangle \, |\gamma_M d' + th_1'| dw - \int_{\mathbb{R}/\mathbb{Z}} \langle H(\gamma_M), h_2\rangle \, |\gamma_M'| dw}{t} \\ &= \left\langle \nabla_{h_1} H\gamma_M, h_2\right\rangle_{L^2} + \langle L_1 h_1, h_2\rangle_{L^2} \end{aligned}$$

where $L_1 h_1 = H\gamma_M \cdot \left\langle \gamma_M', h_1'\right\rangle$ is a differential operator of order 1 in h_1.

So we have to show that $P_{\gamma_M'}(\nabla H(\gamma_M))$ is a Fredholm operator from $H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M}$ to $l^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M}$. We know from Theorem 1.1 that

$$H\gamma = \frac{2}{|\gamma'|^3} P^{\perp}_{\gamma'}(Q\gamma) + F(\gamma)$$

where $F \in C^{\omega}(C^{2+\alpha}, C^{\beta})$ for all $\alpha > \beta > 0$. Thus

$$P^{\perp}_{\gamma_M'}(\nabla_h H(\gamma_M')) = \frac{2}{|\gamma'|^3}(P^{\perp}_{\gamma_M'}(Qh) + \tilde{F}_{\gamma_M}(h)) \tag{4.1}$$

where

$$\begin{aligned} \tilde{F}_{\gamma_M}(h) = &-\frac{6}{|\gamma_M'|^5}\left\langle \gamma_M', h'\right\rangle P^{\perp}_{\gamma_M'}(Q\gamma_M) + P^{\perp}_{\gamma_M'}\Big(\nabla_h F(\gamma_M) + (\nabla_h P^{\perp}_{\gamma_M'})(Q\gamma_M)\Big) \\ &\in C^{\omega}(C^{2+\alpha}, C^{\beta}). \end{aligned}$$

Now let ν_i be a smooth functions such that $\nu_1(u), \ldots, \nu_{n-1}(u)$ is an orthonormal basis of the normal space on γ at u. Then each $\phi \in H^s(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}$ can be written in the form

$$\phi = \phi_i \nu_i$$

where $\phi_i := \langle \phi, \nu_i\rangle \in H^s(\mathbb{R}/\mathbb{Z})$. We calculate

$$\begin{aligned} P^{\perp}_{\gamma_M'}(Q\phi) = P^{\perp}_{\gamma_M'}(Q\phi_i \nu_i) &= Q\phi_r\left(P^{\perp}_{\gamma_M'}\nu_i\right) + P^{\perp}_{\gamma_M'}(Q(\phi_i\nu_i) - (Q\phi_r)\nu_r) \\ &= Q\phi_i\nu_i + F_3(\phi) \end{aligned} \tag{4.2}$$

where $F_3 \in C^{\omega}(C^{2+\alpha}, C^{\beta})$ by Lemma A.8. From [9, Lemma 2.3] we know that $Q - \frac{3}{\pi}(-\Delta)^{3/2}$ is a bounded linear operator from H^2 to L^2. Combining this with the fact that $(-\Delta)^{3/2}$ is a Fredholm operator of index zero from $H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ to $L^2(\mathbb{R}/\mathbb{Z}R^n)$, we get that the operator

$$\begin{aligned} A : H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M} &\to L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M} \\ \phi &\mapsto Q\phi_i\nu_i \end{aligned}$$

is Fredholm of order 0. As the Eqs. 4.2 and 4.1 tell us that $P^{\perp}_{\gamma'_M}(\nabla H(\gamma_M))$ is a compact perturbation of A, this is a Fredholm operator as well. Hence, E'' is a Fredholm operator from $H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M}$ to $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M}$ of index 0.

That H is an analytic operator from $(H^3)^{\perp}_{\gamma_M}$ to $(L^2)^{\perp}_{\gamma_M}$ can be seen from Lemma 2.1, using the fact that $H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ embeds into $C^{2+\alpha}$ for every $\alpha \in (0, 1/2)$ and C^{β} embeds into L^2.

Now, [5, Corollary 3.11] tells us that

$$|E(\gamma_M + N) - E(\gamma_M)|^{1-\theta} \leq c \cdot \left(\int_{\mathbb{R}/\mathbb{Z}} |H(\gamma_M + N)(x)|^2 dxq \right)^{1/2}$$

for all $\phi \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}$ with $\|\phi\|_{H^3} \leq \tilde{\sigma}$.[2]

To prove the full estimate, note that since γ_M is C^{∞} Lemma A.9 tells us how to write nearby curves as normal graphs. More precisely, there is a $\sigma > 0$ such that for all $\gamma \in H^3(\mathbb{R}/\mathbb{Z})$ with $\|\gamma - \gamma_M\|_{H^3} \leq \sigma$ we have $|\gamma'(x)| \geq c_0 > 0$ and there is a reparametrization $\psi \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ and a $\phi \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}$ such that

$$\gamma \circ \psi = \gamma_M + \phi$$

and

$$\|\phi\|_{H^3} \leq C \cdot \|\gamma - \gamma_M\|_{H^3}.$$

Assuming $\sigma < 1/2$ we furthermore have

$$1/2 < |\gamma'|$$

and hence

$$\begin{aligned}(E(\gamma) - E(\gamma_M))^{1-\theta} &= \left(E(\gamma_M + N_\gamma) - E(\gamma_M)\right)^{1-\theta} \\ &\leq c \cdot \left(\int_{\mathbb{R}/\mathbb{Z}} |H(\gamma_M + N_\gamma)(x)|^2 dx \right)^{1/2} \\ &\leq 2c \cdot \int_{\mathbb{R}/\mathbb{Z}} |H(\gamma_M + N_\gamma)(x)|^2 |\gamma'(x)| dx.\end{aligned}$$

□

5 Long time existence results

In this section we will prove the following more general version of Theorem 1.4

Theorem 5.1 (Long time existence) *Let $\gamma_M \in C^{\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be a stationary point of the Möbius energy and let $k \in \mathbb{N}, \delta > 0$ and $\alpha > 0$. Then there is a constant $\varepsilon > 0$*

[2] We apply [5, Corollary 3.11], for $W = (L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M} \subset (H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp})'$ and let P denote the orthogonal projection onto $ker(E''(\phi))$

such that the following is true: Suppose that $(\gamma_t)_{t\in[0,T)}$ *is a maximal solution of the gradient flow for the Möbius energy with smooth initial data satisfying*

$$\|\gamma_0 - \gamma_M\|_{C^{2+\alpha}} \leq \varepsilon$$

for an $\alpha > 0$ *and*

$$E(\gamma_t) \geq E(\gamma_M)$$

whenever there is a diffeomorphism $\phi_t : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *such that* $\|\gamma_t \circ \phi_t - \gamma_M\|_{C^k} \leq \delta$. *Then the flow* $(\gamma_t)_t$ *exists for all times and converges, after suitable reparametrizations, smoothly to a stationary point* γ_∞ *satisfying*

$$E(\gamma_\infty) = E(\gamma_M).$$

This theorem will follow easily from the following long time existence result for normal graphs over a stationary point of the Möbius energy

Theorem 5.2 (Long time existence for normal graphs) *Let* $\gamma_M \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ *be a stationary point of the Möbius energy and let* $k \in \mathbb{N}$, $\delta > 0$ *and* $\alpha > 0$. *Then there is a constant* $\varepsilon > 0$ *such that the following is true:*

Suppose that $N \in C([0, T), h^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^\perp_{\gamma_M}) \cap C([0, T), C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^\perp_{\gamma_M})$ *is a maximal solution of the equation*

$$\partial_t^\perp(\gamma_M + N_t) = H(\gamma_M + N_t)$$

with

$$\|N(0)\|_{C^{2+\alpha}} \leq \varepsilon$$

and

$$E(\gamma_t) \geq E(\gamma_M)$$

whenever $\|N(t)\|_{C^k} \leq \delta$.

Then $T = \infty$ *and* $N(t)$ *converges smoothly to a* $N_\infty \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^\perp_{\gamma_M}$ *satisfying*

$$E(\gamma_\infty) = E(\gamma_M).$$

Furthermore $\gamma_M + N_\infty$ *is a stationary point of the Möbius energy.*

Proof From Theorem 1.2 we see that we can exchange the condition $\|N(0)\|_{C^{2+\alpha}} \leq \varepsilon$ by the stronger condition $\|N(0)\|_{C^{3+\alpha}} \leq \varepsilon$. Using the smoothing properties again, we can furthermore exchange the property

$$E(\gamma_t) \geq E(\gamma_M)$$

whenever $\|N(t)\|_{C^k} \leq \delta$ by

$$E(\gamma_t) \geq E(\gamma_M)$$

whenever $\|N(t)\|_{C^{2+\alpha}} \leq \delta$.

Theorem 1.2 tells us that there is a δ such that every maximal solution of the gradient flow of the Möbius energy for normal graphs $N \in C([0, T_{max}), h^{2+\alpha}) \cap C^\infty((0.T_{max}), C^\infty)$ that satisfies

$$\|N(t)\|_{C^3} \leq \delta \quad \forall t \in [0, T)$$

exists for all time, i.e. $T_{max} = \infty$.

Making $\delta > 0$ smaller if necessary, we can use Lemma 2.1 and a short calculation to get constants $\theta \in [0, 1/2]$, $c > 0$ such that for every $N \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M}$ with $\|N\|_{C^3} \leq \sigma$ we have

$$|E(\gamma_M + N) - E(\gamma_M)|^{1-\theta} \leq c \cdot \left(\int\limits_{\mathbb{R}/\mathbb{Z}} |H(\gamma + N)|^2 |\gamma' + N'| \right)^{1/2}, \tag{5.1}$$

$$\|P^{\perp}_{\gamma_M' + N'} - P^{\perp}_{\gamma_M'}\| \leq 1/2 \tag{5.2}$$

and

$$|\gamma_M' + N'| \geq \frac{1}{2} \inf |\gamma_M'| > 0. \tag{5.3}$$

Now let $N \in C([0, T_{max}), h^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M}) \cap C((0, T_{max}), C^{\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)^{\perp}_{\gamma_M})$ be a maximal solution of the equation

$$\partial_t^{\perp}(\gamma_M + N_t) = H(\gamma_M + N_t)$$

with

$$\|N(0)\|_{C^{3+\alpha}} \leq \varepsilon$$

and

$$E(\gamma_t) \geq E(\gamma_M)$$

whenever $\|N(t)\|_{C^k} \leq \delta$.

Let us assume that $\varepsilon < \delta/2$ and that there is a smallest $t_0 \in (0, T)$ such that

$$\|N(t)\|_{C^3} \geq \delta.$$

We will derive a contradiction, if ε is small enough, which implies that $\|N_t\|_{C^3} \leq \delta, \forall t \in [0, T_{max})$. By our choice of δ this implies that the solution exists for all time.

By making $\varepsilon > 0$ smaller if necessary and using the smoothing from Theorem 1.2 we can furthermore achieve that there are constants $C = C(\gamma_M)$ such that

$$\|N(t)\|_{C^{3+\alpha}} \leq C \quad \forall t \in [0, t_0). \tag{5.4}$$

For $\tilde{\gamma}_t := \gamma_M + N$ and $t \in (0, t_0)$ we calculate

$$\begin{aligned} \frac{d}{dt} E(\tilde{\gamma}_t) &= - \int\limits_{\mathbb{R}/\mathbb{Z}} \langle \partial_t^{\perp} \tilde{\gamma}_t, H(\tilde{\gamma}_t) \rangle |\tilde{\gamma}_t'| \\ &= - \int\limits_{\mathbb{R}/\mathbb{Z}} |\partial_t^{\perp} \tilde{\gamma}_t|^2 |\tilde{\gamma}_t'| \\ &= - \int\limits_{\mathbb{R}/\mathbb{Z}} |H \tilde{\gamma}_t|^2 |\tilde{\gamma}_t'| \end{aligned}$$

and hence

$$-\frac{d}{dt}\left(E(\tilde{\gamma}_t)-E(\gamma_M)\right)^{\theta} = -\theta\left(E(\tilde{\gamma}_t)-E(\gamma_M)\right)^{\theta-1}\frac{d}{dt}E(\tilde{\gamma}_t)$$
$$\geq \frac{\theta}{\sigma}\left(\int_{\mathbb{R}/\mathbb{Z}} |\partial_t^{\perp}\tilde{\gamma}_t|^2|\tilde{\gamma}_t'|\right)^{1/2}$$
$$\geq c\left(\int_{\mathbb{R}/\mathbb{Z}} |\partial_t\tilde{\gamma}_t|^2\right)^{1/2}.$$

Integrating the above inequality over $(0, t)$ yields

$$\|\tilde{\gamma}_t-\gamma_M\|_{L^2} \leq \|\tilde{\gamma}_0-\tilde{\gamma}_t\|_{L^2} + C\left(E(\tilde{\gamma}_0)-E(\gamma_M)\right)^{\theta}$$
$$\leq C\|\tilde{\gamma}_0-\gamma_M\|_{C^2}^{\theta}.$$

Using the interpolation inequality

$$\|f\|_{C^3} \leq \|f\|_{C^{3+\alpha}}^{(1-\beta)}\|f\|_{L^2}^{\beta}$$

where $\beta = \frac{\alpha}{\alpha+7/2}$ we get for $t \in [0, t_0]$

$$\|\tilde{\gamma}_t-\gamma_M\|_{C^3(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} \leq C\|\tilde{\gamma}_t-\gamma_M\|_{C^{3+\alpha}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)}^{1-\beta}\|\tilde{\gamma}_t-\gamma_M\|_{L^2}^{\beta} \leq C\|\tilde{\gamma}_0-\gamma_M\|_{C^2}^{\theta\beta}$$
$$\leq C\varepsilon^{\theta\beta} < \delta/2$$

if $\varepsilon > 0$ is small enough, which leads to a contradiction to the definition of t_0. This proves the long time existence.

Since

$$\partial_t\tilde{\gamma} \in L^1([0,\infty), L^2)$$

we get that there is a $\gamma_\infty \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ such that

$$\tilde{\gamma}_t \to \gamma_\infty.$$

From Theorem 1.2 we get $\sup_t \|\tilde{\gamma}_t\|_{C^l} < \infty$ for all $l \in \mathbb{N}$ and hence this convergence is even smooth and $\delta E(\gamma_\infty) = 0$. Using the Łojasievicz-Simon gradient inequality again we get

$$\left(E(\gamma_\infty)-E(\gamma_M)\right)^{1-\theta} \leq c\left(\int_{\mathbb{R}/\mathbb{Z}} |H\gamma_\infty|^2|\gamma_\infty'|\right)^{1/2} = 0$$

and hence $E(\gamma_\infty) = E(\gamma_M)$. □

Proof of Theorem 5.1 Due to Lemma A.9 for all $\gamma \in C^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $\|\gamma-\gamma_M\|_{C^{2+\alpha}} \leq \varepsilon$ there is a diffeomorphism ϕ_γ and a vector field $N_\gamma \in C^{2+\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ normal to γ_M such that

$$\gamma \circ \phi_\gamma = \gamma_M + N_\gamma \tag{5.5}$$

and

$$\|N_\gamma\|_{C^{2+\alpha}} \leq C\|\gamma-\gamma_M\|_{C^{2+\alpha}} \tag{5.6}$$

if $\varepsilon > 0$ is small enough.

For $\gamma \in C^{2+\alpha}$ with

$$\|\gamma - \gamma_M\|_{C^{2+\alpha}} \leq \varepsilon$$

let $(N_t)_{t\in[0,\tilde{T})}$ be the maximal solution of

$$\begin{cases} \partial_t^\perp(\gamma_M + N_t) = H(\gamma_M + N_t) \\ N_0 = N_\gamma. \end{cases}$$

Then $\tilde{T} \leq T$ and for all $t \in [0, \tilde{T})$ there are diffeomorphisms ϕ_t such that $\gamma_t = (\gamma_M + N_t)(\phi_t)$. Hence N_t satisfies all the assumptions of Theorem 5.1 if ε is small enough and thus $\infty = \tilde{T}$. Form $\tilde{T} \leq T$ we deduce $T = \infty$. □

A Appendix

A.1 Analytic functions on banach spaces

We briefly prove some lemmata about analytic functions on Banach spaces. A thorough discussion of this subject can be found in [10, Chapter 3, Sect. 3].

Definition A.1 (Analytic operator) Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be real Banach spaces. A function $f \in C^\infty(A, Y)$, $A \subset X$ open, is called real analytic, if for every $a \in A$ there is a open neighborhood U of a in X and a constant $C < \infty$ such that

$$\left\|D^m f(x)\right\| \leq C^m m! \quad \forall m \in \mathbb{N}, x \in U.$$

In this context $\|\cdot\|$ stands for the operator norm. The next lemmata show how to construct analytic functions:

Lemma A.2 *Let $g : U \to \mathbb{R}^k$ be a real analytic function, $U \subset \mathbb{R}^n$ be an open subset, and let $V \subset C^{k,\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be an open subset such that* $\mathrm{im}(f) \subset U$ *for all $f \in V$. Then*

$$\begin{aligned} T : V &\to C^{k,\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^l) \\ x &\to g \circ x \end{aligned}$$

defines a real analytic function.

Proof Let $f_0 \in V$. Since im f_0 is a compact subset of the open set U there is an $\varepsilon > 0$ such that $K_\varepsilon := \bigcup_{y\in \mathrm{im}\, f_0} \overline{B_\varepsilon(y)} \subset U$.

Since g is real analytic and $K_\varepsilon \subset U$ is compact, there is a constant $C \leq \infty$ such that

$$\|D^m g(y)\| \leq C^m m!, \quad \forall m \in \mathbb{N},\ y \in K_\varepsilon.$$

As

$$D^m T(y)(h_1, \ldots, h_m) = D^m g(y)(h_1, \ldots, h_m)$$

(can easily be deduced from the Taylor expansion of g) and since $C^\alpha(\mathbb{R}/\mathbb{Z})$ is a Banach algebra, we get

$$\|D^m T(f)\| \leq (k+1)C^{m+k+1}(m+k+1)! \leq \tilde{C}^m m!$$

for all $f \in B_\varepsilon(f_0) = \{y \in C^\alpha : \|y\|_{C^\alpha} \leq \varepsilon\}$ where we put $\tilde{C} := \sup_{m\in\mathbb{N}_0} \left(\frac{(k+1)C^{m+k+1}(m+k+1!)}{m!}\right)^{1/m}$. □

Lemma A.3 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and assume that $T_t \in C^\omega(X, Y)$ for $t \in I$ is such that the functions $t \to T_t$ are measurable and for all $a \in X$ there is a neighborhood U of a in X such that*

$$\int_I \left(\sup_{y \in U} \|D^m T_t(y)\| \right) dt \leq C^m m!. \tag{A.1}$$

Then the mapping $T : X \to Y$ defined by

$$Tx = \int_I T_t x dt$$

is real analytic.

Proof We want to show that

$$D^m Tx(h_1, \ldots, h_m) = \int_I D^m T_t x(h_1, \ldots, h_m) dt.$$

from which we get that $T \in C^\omega(X, Y)$ using the estimates (A.1). In fact this follows from well known facts about differentiation of parameter dependent integrals. □

Remark A.4 In the case that $Y = C^{k,\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ it is well known, that

$$(Tx)(u) = \int_I (T_t) x(u) dt$$

i.e. the value of function Tx given by the Bochner integral at the point u is equal to the Lebesque integral of the functions $T_t(u)$ evaluated at the point u.

A.2 Estimates for the multilinear Hilbert transform

Lemma A.5 *For $1 \geq \alpha > \beta > 0, n, m \in \mathbb{N}$, and $t_i \in (0, 1)$ the singular integral*

$$T(\gamma_1, \ldots \gamma_m)(u) := \lim_{\varepsilon \searrow 0} \int_{I_\varepsilon} \frac{1}{w} \prod_{i=1}^m \gamma_i(u + t_i w) dw$$

defines a bounded multilinear operator from $C^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ to $C^\beta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$.

Proof For $u, v \in \mathbb{R}/\mathbb{Z}$ and $a \in (0, \frac{1}{2})$ we get

$$\begin{aligned}
&|T(\gamma_1, \ldots, \gamma_m)(u) - T(\gamma_1, \ldots, \gamma_m)(v)| \\
&\leq \int_{1/2 \geq |w| \geq a} \frac{1}{w} \left| \prod_{i=1}^m \gamma_i(u + t_i w) - \prod_{i=1}^m \gamma_i(v + t_i w) \right| dw \\
&\quad + \int_{a \geq |w|} \frac{1}{w} \left| \prod_{i=1}^m \gamma_i(u + t_i w) - \prod_{i=1}^m \gamma_i(u) \right| dw \\
&\quad + \int_{a \geq |w|} \frac{1}{w} \left| \prod_{i=1}^m \gamma_i(v + t_i w) - \prod_{i=1}^m \gamma_i(v) \right| dw
\end{aligned}$$

Since $|t_i| \leq 1$ we get

$$\left|\prod_{i=1}^{m} \gamma_i(u + t_i w) - \prod_{i=1}^{m} \gamma_i(u)\right| \leq m \prod_{i=0}^{m} \|\gamma_i\|_{C^\alpha(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} |w|^\alpha$$

and hence, using the Hölder-continuity of the γ_i,

$$\begin{aligned}
&|T(\gamma_1, \ldots, \gamma_m)(u) - T(\gamma_1, \ldots, \gamma_m)(v)| \\
&\leq m \prod_{i=0}^{m} \|\gamma_i\|_{C^\alpha(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} \left(\int\limits_{1\geq|w|\geq a} \frac{1}{w} |u - v|^\alpha \, dw + 2 \int\limits_{|w|\leq a} \frac{1}{w^{1-\alpha}} dw \right) \\
&\leq m \prod_{i=0}^{m} \|\gamma_i\|_{C^\alpha(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} \left(-2\log(a)\, |u - v|^\alpha \, dw + 4\alpha a^\alpha\right).
\end{aligned}$$

Choosing $a = |u - v|$ we get

$$\begin{aligned}
&|T(\gamma_1, \ldots, \gamma_m)(u) - T(\gamma_1, \ldots, \gamma_m)(v)| \\
&\leq m \prod_{i=0}^{m} \|\gamma_i\|_{C^\alpha(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} (-2\log|u - v| + 4\alpha)|u - v|^\alpha \\
&\leq C \prod_{i=0}^{m} \|\gamma_i\|_{C^\alpha(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} |u - v|^\beta
\end{aligned}$$

where $C = m \sup_{x\in[0,1]}(-2\log \mathrm{x}+4\alpha)x^{\alpha-\beta} < \infty$. □

Lemma A.6 *For arbitrary $\alpha > \beta > 0, n, m \in \mathbb{N}$, and $t_i \in (0, 1)$ the singular integral*

$$T(\gamma_1, \ldots \gamma_m)(u) := \lim_{\varepsilon \searrow 0} \int\limits_{I_\varepsilon} \frac{1}{w} \prod_{i=1}^{m} \gamma_i(u + t_i w) dw$$

defines a bounded multilinear operator from $C^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ to $C^\beta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$.

Proof First let us note that it is enough to prove the statement for $\alpha = \tilde{\alpha} + n$, $\beta = \tilde{\beta} + n$, $n \in \mathbb{N}_0$, $1 \geq \tilde{\alpha} > \tilde{\beta} > 0$ and we will use induction on n to prove this statement. For $n = 0$ the claim is the content of Lemma A.5. Let $(\tau_h f)(x) := f(x + h)$. Using the relation $\tau_h(T(\gamma_1, \ldots, \gamma_m)) = T(\tau_h(\gamma_1) \ldots \tau_h(\gamma_m))$ and the multilinearity of T, the difference quotient can be written as

$$\begin{aligned}
\frac{\tau_h(T(\gamma_1, \ldots, \gamma_m) - T(\gamma_1, \ldots, \gamma_m))}{h} &= \frac{T(\tau_n(\gamma_1), \ldots, \tau_h(\gamma_m)) - T(\gamma_1, \ldots, \gamma_m))}{h} \\
&\sum_{i=1}^{m} T\left(\gamma_1, \ldots, (\tau_h(\gamma_i) - \gamma_i)/h, \ldots, \tau_h(\gamma_m)\right).
\end{aligned}$$

Now let $\tilde{\alpha}\beta' > \beta$. Since

$$\begin{aligned}
(\tau_h(\gamma_i) - \gamma_i)/h &\xrightarrow{h\to 0} \gamma' \quad \text{in } C^\beta \\
(\tau_h(\gamma_i)) &\xrightarrow{h\to 0} \gamma \quad \text{in } C^\beta
\end{aligned}$$

and T a bounded linear operator form $C^{\beta'}$ to C^{β}, we get that

$$\frac{\tau_h(T(\gamma_1,\ldots,\gamma_m)-T(\gamma_1,\ldots,\gamma_m))}{h}\xrightarrow{h\to 0}\sum_{i=1}^{m}T\left(\gamma_1,\ldots,\gamma_i',\ldots,\gamma_m\right)$$

in $C^{\tilde{\beta}}$. Hence, T is a bounded multilinear map from $C^{t+\tilde{\alpha}}$ to $C^{1+\tilde{\beta}}$. Using induction, one gets the full statement. □

Remark A.7 Let us state a simple extension of Lemma A.6. Given a multilinear form $M\colon\mathbb{R}^n\times\cdots\times\mathbb{R}^n\to\mathbb{R}^k$, $\alpha>\beta>0$, $m\in\mathbb{N}$, and $t_i\in(0,1)$ the singular integral

$$T(\gamma_1,\ldots\gamma_m)(u):=\lim_{\varepsilon\searrow 0}\int_{I_\varepsilon}\frac{1}{w}M(\gamma_1(u+t_1w),\ldots,\gamma_m(u+t_mw)))dw$$

defines a bounded multilinear operator from $C^{\alpha}(\mathbb{R}/\mathbb{Z},\mathbb{R})$ to $C^{\beta}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. This can be deduced plugging

$$M(\gamma_1(u+t_1w),\ldots,\gamma_m(u+t_mw)))=\sum_{j_1,\ldots,j_m=1}^{n}M(e_{i_1},\ldots,e_{j_m})\Pi_{i=1}^{m}\left\langle\gamma_j(u+t_iw),e_{j_i}\right\rangle$$

into the definition of T, where $e_1,\ldots e_n$ is the standard basis of $\mathbb{R}^n$, and applying Lemma A.6 to all the coordinates of the resulting summands.

A.3 Facts about the functional Q

In this section we prove a commutator inequality that serves us as a substitute for the Leibniz rule for Q.

Lemma A.8 (Leibniz rule for Q) *For $f,g\in C^{\alpha+3}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ and $\alpha>\beta$ we have*

$$\|Q(fg)-Q(f)g\|_{C^{\beta}}\le C(\alpha,\beta)\|f\|_{C^{\alpha+2}}\|g\|_{C^{\alpha+3}}.$$

Proof We have

$$\begin{aligned}
&(Q(fg)-Q(f)g)(u)\\
&=\int_{[-1/2,1/2]}\left\{2\frac{f(u+w)g(u+w)-f(u)g(u)-w(f'(u)g(u)-f(u)g'(u))}{w^2}\right.\\
&\quad\left.-(f''(u)+2g'(u)f'(u)+f(u)g''(u))\right\}\frac{dw}{w^2}\\
&\quad-\int_{[-1/2,1/2]}\left\{2\frac{f(u+w)g(u)-f(u)g(u)-wf'(u)g(u)}{w^2}-f''(u)g(u)\right\}\frac{dw}{w^2}\\
&=\int_{[-1/2,1/2]}\left\{2\frac{(f(u+w)-f(u))(g(u+w)-g(u))}{w^2}-2f'(u)g'(u)\right.\\
&\quad\left.+f(u)\left(2\frac{g(u+w)-g(u)-wg'(u)}{w^2}-g''(u)\right)\right\}\frac{dw}{w^2}\\
&=2\int_{[-1/2,1/2]}\left\{\frac{(f(u+w)-f(u))(g(u+w)-g(u))}{w^2}-f'(u)g'(u)\right\}\\
&\quad+(fQ(g))(u).
\end{aligned}$$

Taylor expansion yields

$$\frac{f(u+w)-f(u)}{w} = f'(u) + w\int_0^1 (1-t)f''(u+tw)dt$$

$$\frac{g(u+w)-g(u)}{w} = g'(u) + w\int_0^1 (1-t)g''(u+tw)dt$$

and hence the first term in Eq. A.2 can be written as

$$\begin{aligned}
&2\int_{[-1/2,1/2]}\left\{\frac{(f(u+w)-f(u))(g(u+w)-g(u))}{w^2} - f'(u)g'(u)\right\}\\
&= 2f'(u)\int_{[-1/2,1/2]}\frac{\int_0^1(1-t)g''(u+tw)dt}{w}dw\\
&+2g'(u)\int_{[-1/2,1/2]}\frac{\int_0^1(1-t)f''(u+tw)dt}{w}dw\\
&+2\int_{[-1/2,1/2]}\int_{[0,1]}\int_{[0,1]}(1-t)(1-s)g''(u+tw)f''(u+sw)dtdsdw.
\end{aligned}$$

Using the boundedness of the Hilbert transform to estimate the first two terms, we get

$$\|Q(fg)-Q(f)g-gQ(f)\|_{C^\alpha} \le C\left(\|f'\|_{C^\alpha}\|g''\|_{C^\alpha} + \|g'\|_{C^\alpha}\|f''\|_{C^\alpha}\right)$$

and hence

$$\|Q(fg)-Q(f)g-gQ(f)\|_{C^\alpha} \le \|f\|_{C^{\alpha+2}}\|g\|_{C^{\alpha+3}}.$$

□

A.4 Normal graphs

The following lemma is used in the proofs of Lemma 1.3 and Theorem 5.1.

Lemma A.9 *Let $\gamma_0 \in C^\infty_{i,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. Then for every $\alpha \ge 1$ there is an $\varepsilon > 0$ such that for all $\gamma \in C^\alpha(\mathbb{R}/\mathbb{Z})$ with*

$$\|\gamma-\gamma_0\|_{C^\alpha} \le \varepsilon,$$

there is a reparametrization ϕ and a function $N \in C^\alpha(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ normal to γ_0 such that

$$\gamma\circ\phi = \gamma_0 + N$$

and

$$\|N\| \le C\|\gamma-\gamma_0\|_{C^\alpha}.$$

Proof Let U be an open neighborhood of γ_0 such that there is a nearest point retract r_U, i.e. a C^α function $r_U : U \to \mathbb{R}/\mathbb{Z}$ such that

$$\gamma\circ r_U\circ\gamma = \gamma$$

and

$$|g(r_U(p)) - p| = \inf_{x \in \mathbb{R}/\mathbb{Z}} |\gamma(x) - p|.$$

We then set

$$\psi_\gamma(x) = r_U(\gamma(x))$$

and deduce, since $\psi_{\gamma_0} = id$ and the space of diffeomorphism is open in C^α, that ψ_γ is a diffeomorphism if ε small enough. Furthermore, let us note that $\gamma \to \psi_\gamma$ is smooth from $C^\alpha(\mathbb{R}/\mathbb{R})$ to $C^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$. We set

$$\phi_\gamma = \psi_\gamma^{-1}$$

and

$$N_\gamma(x) = \gamma \circ \psi^{-1} - \gamma_0.$$

The claim now follows from the fact that $\gamma \to N_\gamma$ is a smooth function with $N_{\gamma_0} = 0$. □

References

1. Amann, H.: Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In: Function Spaces, Differential Operators and Nonlinear Analysis (Friedrichroda, 1992), vol. 133 of Teubner-Texte Math, pp. 9–126. Teubner, Stuttgart (1993)
2. Angenent, S.B.: Nonlinear analytic semiflows. Proc. R. Soc. Edinburgh Sect. A **115**(1–2), 91–107 (1990)
3. Blatt, S.: Loss of convexity and embeddedness for geometric evolution equations of higher order. J. Evol. Equ. **10**, 21–27 (2010)
4. Bourdaud, G.: Une algèbre maximale d'opérateurs pseudo-différentiels. Comm. Partial Differ. Equ. **13**(9), 1059–1083 (1988)
5. Chill, R.: On the łojasiewicz-simon gradient inequality. J. Funct. Anal. **201**, 572–601 (2003)
6. Freedman, M.H., He, Z.-X., Wang, Z.: Möbius energy of knots and unknots. Ann. Math. (2) **139**(1), 1–50 (1994)
7. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. In: Classics in Mathematics. Springer-Verlag, Berlin (2001). Reprint of the 1998 edition
8. He, Z.: A formula for the non-integer powers of the Laplacian. Acta Math. Sin. (Engl. Ser.) **15**(1), 21–24 (1999)
9. He, Z.-X.: The Euler-Lagrange equation and heat flow for the Möbius energy. Comm. Pure Appl. Math. **53**(4), 399–431 (2000)
10. Hille, E., Phillips, R.S.: Functional Analysis and Semi-Groups. American Mathematical Society (1957)
11. Lunardi, A.: Analytic semigroups and optimal regularity in parabolic problems. In: Progress in Nonlinear Differential Equations and their Applications, vol. 16. Birkhäuser Verlag, Basel (1995)
12. O'Hara, J.: Energy of a knot. Topology **30**(2), 241–247 (1991)
13. Reiter, P.: Repulsive knot energies and pseudodifferential calculus: rigorous analysis and regularity theory for O'Hara's knot energy family $E^{(\alpha)}$, $\alpha \in [2, 3)$. PhD thesis, RWTH Aachen (2009)
14. von der Mosel, H.: Minimizing the elastic energy of knots. Asymptot. Anal. **18**(1–2), 49–65 (1998)