

# On Yangian Symmetry in Planar $\mathcal{N} = 4$ SYM

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## Abstract

Planar  $\mathcal{N} = 4$  supersymmetric Yang–Mills theory appears to be perturbatively integrable. This work reviews integrability in terms of a Yangian algebra and compares the application to the problems of anomalous dimensions and scattering amplitudes.

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## 1 Introduction

Integrability is a very useful feature of selected physical models. It allows one to rely on certain algebraic properties to solve them exactly and to determine physical observables efficiently. Unfortunately, in general integrability is restricted to at most two-dimensional models. These can be discrete, e.g. spin chains, statistical physics models, or continuous, e.g. sigma models such as two-dimensional (super)gravity and worldsheet models string theory.

Despite this severe restriction, signs of integrability have been discovered in four-dimensional gauge theories: Lipatov noticed that the BFKL Hamiltonian [1] describing the evolution of reggeized gluons in QCD high-energy scattering is integrable [2] and closely related to the Heisenberg spin chain [3] (see also [4] for a recent account and additional references). The crucial additional assumption which enables integrability in this four-dimensional model is the 't Hooft large- $N_c$  or *planar limit* [5]. In this limit the gauge group dynamics reduces to two-dimensional surfaces on which the integrable structure lives.

Another instance of integrability in large- $N_c$  gauge theory is deep inelastic scattering where anomalous dimensions of local operators are responsible for scaling violations.

The anomalous dimensions of local operators can be described by the DGLAP evolution equation which was initiated by Gribov and Lipatov [6]. It was noticed that also these evolution equations are integrable to some extent [7, 8].

In 2002 a new line of developments started for a particular four-dimensional gauge theory, namely  $\mathcal{N} = 4$  maximally supersymmetric Yang-Mills ( $\mathcal{N} = 4$  SYM). This model, consisting of a  $U(N_c)$  gauge field, 4 flavours of massless adjoint fermions and 6 flavour of massless adjoint scalars, is relevant to the AdS/CFT string/gauge duality. Integrability was shown to apply to all leading-order planar anomalous dimensions [9, 10]. Unlike in the analogous problem in QCD, integrability was moreover demonstrated to survive in higher-order quantum corrections [11, 12] hinting at complete integrability of the planar sector of the theory.

In this paper we review integrability of planar  $\mathcal{N} = 4$  SYM in the guise of Yangian symmetry. We shall focus on the problems of anomalous dimensions of local operators (Sec. 2) and the spacetime scattering matrix (Sec. 3) in order to reveal the close similarities between them (Sec. 4).

## 2 Anomalous Dimensions of Local Operators

Scaling dimensions of local operators represent a key set of observables in a conformal field theory. They determine to a large extent the spacetime dependence of correlation functions. The spectrum of scaling dimensions can be viewed as the conformal analog of the mass spectrum of composite particles in a non-conformal field theory. In  $\mathcal{N} = 4$  SYM the planar spectrum turned out to be governed by an integrable system with an underlying Yangian algebra. In the following we shall review local operators and the role the Yangian algebra.

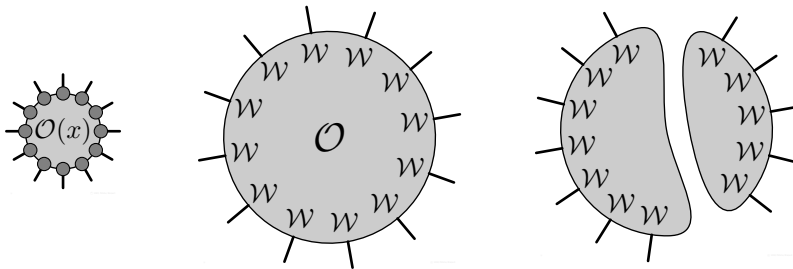
### 2.1 Framework

Local operators are local, gauge-invariant combinations of the scalars  $\mathcal{B}$ , fermions  $\mathcal{C}$  and gauge field strengths  $\mathcal{F}$  as well as their covariant derivatives  $\mathcal{D}$ . Gauge invariant combinations are constructed as traces of products of covariant fields, e.g.

$$\begin{aligned}\bar{\mathcal{O}}_1(x) &= \text{Tr}(\mathcal{B}_m(x)\mathcal{B}_m(x)), \\ \bar{\mathcal{O}}_2(x) &= \text{Tr}(\mathcal{D}^\mu\mathcal{B}_m(x)\mathcal{D}_\mu\mathcal{B}_n(x)), \\ \bar{\mathcal{O}}_3(x) &= \dots\end{aligned}\tag{2.1}$$

One can also construct multi-trace operators, such as  $\bar{\mathcal{O}}_1(x)\bar{\mathcal{O}}_2(x)$ , but in the planar limit these decouple and can be safely ignored, see Fig. 1.

In a perturbative QFT on flat Minkowski spacetime the correlator of two such operators takes the generic form  $\langle\bar{\mathcal{O}}_A(x)\bar{\mathcal{O}}_B(y)\rangle = F_{AB}(x-y, g, \mu, \epsilon)$  due to Poincaré symmetry. Here  $\mu$  is the regularisation scale and  $\epsilon$  is the parameter of dimensional regularisation. Importantly, the result is generically divergent as one removes the regulator, i.e. at  $\epsilon \rightarrow 0$ . This also applies to  $\mathcal{N} = 4$  SYM. Superficially, it contradicts the fact that  $\mathcal{N} = 4$  SYM is a finite CFT where two-point functions take a particular form which depends only on



**Figure 1:** Local operators as gauge invariant combination of fields at a common point in spacetime (left). Focusing on the  $U(N_c)$  gauge group structure alone, operators are classified by the number of traces: single-trace (middle) and multi-trace (right) operators.

the *scaling dimensions*  $D_A$  of the local operators  $\tilde{\mathcal{O}}_A$ <sup>1</sup>

$$\langle \tilde{\mathcal{O}}_A(x) \tilde{\mathcal{O}}_B(y) \rangle = \frac{\delta_{AB}}{|x - y|^{2D_A}}. \quad (2.2)$$

To recover this form one has to find the right linear combinations  $\tilde{\mathcal{O}}_A$  of the bare operators  $\bar{\mathcal{O}}_A$ , see e.g. [13]. This is usually done in two steps: First, renormalisation absorbs the divergencies into the definition of the operators  $\bar{\mathcal{O}}_A = Z_A^B \mathcal{O}_B$ . Then the operators are diagonalised to achieve the above form by means of another linear transformation  $\mathcal{O}_A \rightarrow \tilde{\mathcal{O}}_A$ . Of course the composition of the two maps is yet another linear map, but it still makes sense to distinguish the two steps: Renormalisation can be performed abstractly on a basis of states as in (2.1) to the end that one can enumerate renormalised operators  $\mathcal{O}_B$  in an equivalent basis. Conversely, diagonalisation requires the precise knowledge of the set of operators one is interested in. Moreover it requires to solve algebraic equations, potentially of very high degree. It should be noted though that the splitting remains somewhat ambiguous to the extent that  $Z$  is uniquely determined by the model only modulo some transformations (which are eventually compensated by the diagonalisation).

Next we establish a useful basis of single-trace local operators in  $\mathcal{N} = 4$  SYM

$$\mathcal{O} = \text{Tr}(\mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \dots \mathcal{W}_n), \quad \mathcal{W}_k \in \{\mathcal{D}^j \mathcal{B}, \mathcal{D}^j \mathcal{C}, \mathcal{D}^j \mathcal{F}\}. \quad (2.3)$$

The matrices  $\mathcal{W}_k$  represent the scalars  $\mathcal{B}$ , the fermions  $\mathcal{C}$ , the gauge field strengths  $\mathcal{F}$  or their (multiple) covariant derivatives  $\mathcal{D}$  (we hide the spacetime and internal indices). All fields are evaluated at a common point in spacetime which we need not specify further for the enumeration. Due to the trace, the definition of the local operators is invariant w.r.t. (graded) cyclic shifts  $\mathcal{W}_k \rightarrow \mathcal{W}_{k+1}$ ,  $\mathcal{W}_n \rightarrow \mathcal{W}_1$ .

In enumerating the local operators one should take the (quantum) equations of motions of the fields into account. For example,  $\mathcal{D}^2 \mathcal{B}$  can be expressed through products of the fields such as  $\mathcal{B}^3$  or  $\mathcal{C}^2$ . Such combinations are already accounted for in (2.3), so we can discard the term  $\mathcal{D}^2 \mathcal{B}$  (irrespectively of the precise form of the quantum equation

<sup>1</sup>This expression applies to scalar operators; spinning operators have a different, yet uniquely determined and  $x$ -dependent structure in the numerator.

of motion). Similarly, the terms  $\mathcal{D} \cdot \mathcal{C}$  and  $\mathcal{D} \cdot \mathcal{F}$ , as well as  $\mathcal{D} \wedge \mathcal{F}$  and  $\mathcal{D} \wedge \mathcal{D}$  can be dropped. A minimal basis for the fields  $\mathcal{W}$  can be expressed most conveniently using spinor indices  $\alpha, \beta, \dots = 1, 2$  and  $\dot{\alpha}, \dot{\beta} = 1, 2$  for the Lorentz algebra  $\mathfrak{so}(3, 1) = \mathfrak{sl}(2, \mathbb{C})$  as well as spinor indices  $a, b, \dots = 1, 2, 3, 4$  for the the internal algebra  $\mathfrak{so}(6) = \mathfrak{su}(4)$ . It turns out that in our basis Lorentz spinor indices are totally symmetric while internal spinor indices are totally antisymmetric. Such a basis can be represented through states of a supersymmetric oscillator [14] with two plus two bosonic operators  $a^{\dagger\alpha}, b^{\dagger\dot{\alpha}}$  and four fermionic operators  $d^{\dagger a}$ . Then the various fields of  $\mathcal{N} = 4$  SYM decompose as follows

$$\mathcal{F} \sim b^{\dagger} b^{\dagger}, \quad \mathcal{C} \sim b^{\dagger} d^{\dagger}, \quad \mathcal{B} \sim d^{\dagger} d^{\dagger}, \quad \bar{\mathcal{C}} \sim a^{\dagger} d^{\dagger} d^{\dagger} d^{\dagger}, \quad \bar{\mathcal{F}} \sim a^{\dagger} a^{\dagger} d^{\dagger} d^{\dagger} d^{\dagger} d^{\dagger}, \quad \mathcal{D} \sim a^{\dagger} b^{\dagger}, \quad (2.4)$$

where we have suppressed the indices. Note that all physical fields in (2.3) are uncharged w.r.t. the operator

$$C = 2 + N_a - N_b - N_d, \quad (2.5)$$

where the  $N_{a,b,d}$  measure the occupation numbers of the oscillators  $a, b, d$ . For local operators one introduces further indices for the sites, e.g.

$$\text{Tr } \mathcal{B}^{ab} \mathcal{B}^{cd} \sim d_1^{\dagger a} d_1^{\dagger b} d_2^{\dagger c} d_2^{\dagger d} |0\rangle. \quad (2.6)$$

Note that on the r.h.s. cyclicity is automatic while on the l.h.s. it must be imposed by hand. Altogether we have seen that local operators can be expressed through states of a supersymmetric harmonic oscillator subject to a charge and a cyclicity constraint.

## 2.2 One-Loop Hamiltonian

The anomalous dimensions of local operators originate from the divergent contributions to their two-point functions. They are therefore captured by the renormalisation matrix  $Z$ . More precisely, the matrix of anomalous dimensions for the  $\mathcal{O}_A$  is given by the logarithmic derivative of  $Z$  w.r.t. the logarithm of the renormalisation scale  $\mu$

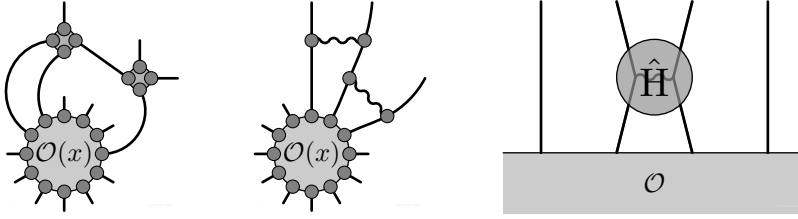
$$\delta D \sim Z^{-1} \frac{\mu dZ}{d\mu}. \quad (2.7)$$

The eigenvalues of the matrix  $\delta D$  represent the quantum corrections  $\delta D_A$  in the scaling dimensions  $D_A$  of the eigen-operators  $\hat{\mathcal{O}}_A$ . The matrix can be interpreted as a Hamiltonian of a quantum mechanical system: It acts on the states in a systematic fashion determined by connected Feynman diagrams attached to the fields constituting the local operators, see Fig. 2. The planar limit suppresses crossing lines in Feynman diagrams, therefore  $\delta D$  acts on a set of adjacent fields along the single-trace state (2.3).

The number of fields involved in the action of  $\delta D$  increases with the loop order. At the leading one-loop order the action is between nearest neighbours, cf. Fig. 2

$$\delta D^{(1)} = \hat{H} = \sum_{k=1}^n \hat{H}_{k,k+1}, \quad (2.8)$$

and it can be interpreted as the Hamiltonian of a quantum spin chain [9]. The one-loop Hamiltonian is invariant under the free superconformal symmetries. The representation



**Figure 2:** Non-planar (left) and planar (middle) gluing of interactions to a local operator  $\mathcal{O}$ . The planar Hamiltonian  $\hat{H}$  acts on a pair of nearest neighbours (right) when zooming into the trace structure of  $\mathcal{O}$ .

of the latter on the fields can be expressed conveniently in the oscillator framework using bilinears in the operators  $(a^\dagger, b, d)$  and  $(a, b^\dagger, d^\dagger)$  [14]

$$\begin{aligned}
L^\alpha{}_\beta &= a^{\dagger\alpha} a_\beta - \frac{1}{2} \delta_\beta^\alpha a^{\dagger\gamma} a_\gamma, & \bar{L}^{\dot{\alpha}}{}_{\dot{\beta}} &= b^{\dagger\dot{\alpha}} b_{\dot{\beta}} - \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} b^{\dagger\dot{\gamma}} b_{\dot{\gamma}}, \\
D &= \frac{1}{2} a^{\dagger\gamma} a_\gamma + \frac{1}{2} b_\gamma b^{\dagger\dot{\gamma}}, & R^a{}_b &= d^{\dagger a} d_b - \frac{1}{4} \delta_b^a d^{\dagger c} d_c, \\
\bar{Q}_b^{\dot{\alpha}} &= b^{\dagger\dot{\alpha}} d_b, & \bar{S}_\alpha^b &= d^{\dagger b} b_\alpha, \\
Q^{\beta a} &= a^{\dagger\beta} d^{\dagger a}, & S_{\beta a} &= a_\beta d_a, \\
P^{\beta\dot{\alpha}} &= a^{\dagger\beta} b^{\dagger\dot{\alpha}}, & K_{\beta\dot{\alpha}} &= a_\beta b_{\dot{\alpha}}.
\end{aligned} \tag{2.9}$$

Single-trace states (2.3) transform in tensor product representations of the above.

Invariance under free superconformal symmetry imposes strong constraints on  $\hat{H}$ . The crucial observation is that the tensor product of two field representations decomposes into a sequence of irreducible representations distinguished by their overall superconformal spin  $j$ . The latter can be measured using the quadratic Casimir of  $\mathfrak{psu}(2, 2|4)$  in analogy to the total spin of  $\mathfrak{su}(2)$ . Symmetry demands that that the Hamiltonian has a common eigenvalue for all components of an irreducible multiplet.<sup>2</sup> Hence it suffices to specify the eigenvalues, and we can write the nearest-neighbour Hamiltonian as [15]

$$\hat{H}_{k,k+1} = \sum_{j=0}^{\infty} c_j \hat{P}_{k,k+1;j}. \tag{2.10}$$

Here the operator  $\hat{P}_{k,k+1;j}$  projects a two-particle state to its components with superconformal spin  $j$ . Now there are several ways to determine the unspecified eigenvalues  $c_j$ : Direct calculation in the one-loop quantum field theory shows that the coefficients are given by the elements of the harmonic series [15, 16]

$$c_j \sim h(j) = \sum_{k=1}^j \frac{1}{k} = \Psi(j+1) - \Psi(1), \quad \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{2.11}$$

Two other methods of determining the coefficients purely algebraically are described in the following two subsections.

<sup>2</sup>This holds for multiplets of multiplicity 1; for higher multiplicity  $n$ , invariance allows an action equivalent to a  $n \times n$  matrix.

The analog of the above Hamiltonian for quasi-partonic operators in QCD is very similar, and it has a particular feature which was noticed in [7] and also in [8], see [17] for a review. Namely, the appearance of the digamma function  $\Psi$  hints at *integrability*, cf. [18]. In particular, Lipatov realised in [19] that for  $\mathcal{N} = 4$  SYM the Hamiltonian is particularly simple, and also made the prophetic connection to the newly proposed AdS/CFT correspondence to strings on  $AdS_5 \times S^5$ . Several years later and in a different context, integrability of the one-loop Hamiltonian was rediscovered in [9, 11]. Most importantly, it was also put to use by establishing a set of Bethe equations to determine the spectrum of planar one-loop anomalous dimensions very efficiently. In particular, the thermodynamic limit of long chains,  $n \rightarrow \infty$ , became accessible [9, 20] and could be compared to results from string theory [21],<sup>3</sup> see [23, 24] for reviews.

## 2.3 Leading-Order Yangian

Integrable spin chains with manifest Lie algebra symmetry  $\mathfrak{g}$  typically have a Yangian algebra  $Y$  underlying their structure [25]. The Yangian is a quantum algebra based on (half of) the affine extension of the Lie algebra. That is to say, next to the Lie generators  $J^A$ , there are level-one Yangian generators  $\widehat{J}^A$ . These obey similar commutation relations as the Lie generators, namely<sup>4</sup>

$$[J^A, J^B] = F_C^{AB} J^C, \quad [J^A, \widehat{J}^B] = F_C^{AB} \widehat{J}^C, \quad (2.12)$$

from which two sets of Jacobi-identities follow. However, a third Jacobi-identity involving two Yangian generators is quantum-deformed to the following Serre relation

$$[[J^A, \widehat{J}^B], \widehat{J}^C] + [[J^B, \widehat{J}^C], \widehat{J}^A] + [[J^C, \widehat{J}^A], \widehat{J}^B] = F_D^{AG} F_E^{BH} F_F^{CI} F_{GHI} J^{\{D} J^E J^F\}}. \quad (2.13)$$

A representation of a Lie algebra can sometimes be lifted to an evaluation representation of the corresponding Yangian. For these,  $J$  acts as in the Lie algebra and  $\widehat{J} \simeq uJ$  with  $u$  the spectral parameter of the evaluation representation. Clearly, the two commutation relations (2.12) are satisfied automatically, but in addition the r.h.s. of the the Serre relation (2.13) must vanish. This is true for the above superconformal representation [26], consequently the spin chain transforms in a representation of the Yangian. Due to homogeneity of the spin chain, the spectral parameters of all sites should be equal.

In addition to multiplication, a quantum algebra has a comultiplication operation  $\Delta : Y \rightarrow Y \otimes Y$  with

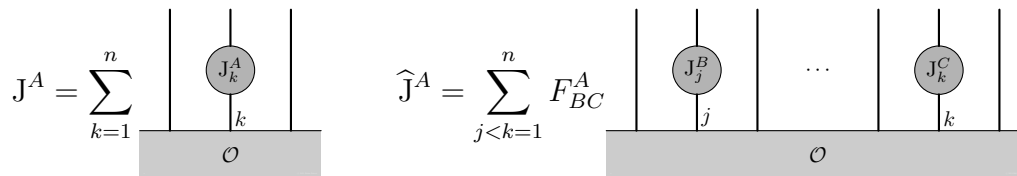
$$\Delta(J^A) = J^A \otimes 1 + 1 \otimes J^A, \quad \Delta(\widehat{J}^A) = \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + F_{BC}^A J^B \otimes J^C. \quad (2.14)$$

It is compatible with the multiplication, in particular with the Serre relation (2.13). Its main purpose is to define tensor products of representations, i.e. it determines how the

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<sup>3</sup>It turned out only later that the matching was more of a coincidence than a confirmation for AdS/CFT due to an order of limits issue, see [22].

<sup>4</sup>For reasons of clarity we treat all generators to be bosonic, the generalisation to superalgebras by insertion of appropriate sign factors is straight-forward.



**Figure 3:** Action of the free superconformal  $J^A$  and Yangian  $\hat{J}^A$  generators on spin chain state alias a local operators  $\mathcal{O}$ .

algebra acts on the spin chain. The action on the tensor product of  $n$  fields is determined by  $\Delta^{n-1}(J)$

$$\Delta^{n-1}(J^A) = \sum_{k=1}^n J_k^A, \quad \Delta^{n-1}(\hat{J}^A) = \sum_{k=1}^n \hat{J}_k^A + F_{BC}^A \sum_{j < k=1}^n J_j^B J_k^C. \quad (2.15)$$

When using an evaluation representation with homogeneous evaluation parameter  $u$ , we see that the first term in the action of  $\hat{J}^A$  equals the superconformal action  $uJ^A$ ; therefore nothing is lost by fixing  $u$  to a particular value, e.g.  $u = 0$ . Note that while the representation of the Lie generators  $J$  follows the usual pattern for tensor products, the representation of Yangian generators  $\hat{J}$  non-trivially combines the various sites of the chain. The action of  $J^A$  and  $\hat{J}^A$  is depicted in Fig. 3.

An integrable Hamiltonian  $\hat{H}$  is invariant under the Lie symmetries  $J$ , but it is typically not exactly invariant under the Yangian generators  $\hat{J}$ . It commutes up to a difference of two terms [27]

$$[\Delta(J^A), \hat{H}_{12}] = 0, \quad [\Delta(\hat{J}^A), \hat{H}_{12}] \sim J_1^A - J_2^A, \quad (2.16)$$

On a chain with  $n$  sites the commutator yields only boundary terms  $J_1 - J_n$  essentially because periodic boundary conditions are not compatible with the definition of the Yangian. This means that the spectrum of  $\hat{H}$  does not organise into multiplets of the Yangian, but merely of the Lie algebra. Nevertheless one can consider the Yangian to be a symmetry of the (bulk) Hamiltonian, because commutation (up to boundary terms) does yield non-trivial constraints on  $\hat{H}$  which guarantee its integrability. In particular, commutation requires the following recursion relation for the coefficients  $c_j$  of the Hamiltonian [27]

$$c_j = c_{j-1} + \frac{1}{j}. \quad (2.17)$$

This relation is precisely satisfied by the coefficients from field theory (2.11), and hence planar one-loop  $\mathcal{N} = 4$  SYM is integrable.

## 2.4 Higher Loops

Going to higher loops the above picture changes although integrability apparently remains valid. The symmetry generators as well as the Hamiltonian receive corrections in the coupling constant

$$J(g) = \sum_{k=0}^{\infty} g^k J^{(k/2)}, \quad \hat{J}(g) = \sum_{k=0}^{\infty} g^k \hat{J}^{(k/2)}, \quad \hat{H}(g) = \sum_{k=0}^{\infty} g^k \hat{H}^{(k/2)}. \quad (2.18)$$

$$J(g) = \begin{array}{cccccc}
\text{Diagram 1} & + g & \text{Diagram 2} & + g & \text{Diagram 3} & + g^2 & \text{Diagram 4} & + g^2 & \text{Diagram 5} & + \dots \\
\text{Circles: } J_{1 \rightarrow 1} & & J_{1 \rightarrow 2} & & J_{2 \rightarrow 1} & & J_{2 \rightarrow 2} & & J_{1 \rightarrow 1}^{(1)} & \\
\text{Bases: } \mathcal{O} & & \mathcal{O} & & \mathcal{O} & & \mathcal{O} & & \mathcal{O} &
\end{array}$$

**Figure 4:** Perturbative action of a superconformal generator on a spin chain state. The deformations involve long-range and dynamic interactions.

The structure of the operators must remain compatible with planar Feynman diagrams, therefore an operator at  $\mathcal{O}(g^k)$  involves at most  $k + 2$  ingoing plus outgoing fields, see Fig. 4. In particular, the number of sites of the chain is allowed to fluctuate.

Despite the above deformations of the representations, the algebra relations (2.12) and (2.13) should remain unchanged. Generally this involves cancellations between products of terms at various orders. These cancellations leave some space for ambiguities, and unfortunately the deformations cannot be defined uniquely. It turns out that the ambiguities correspond to perturbative similarity transformations  $J \rightarrow \mathcal{X}J\mathcal{X}^{-1}$  of the generators which leave all algebra relations invariant.<sup>5</sup> Only at low orders the set of permissible similarity transformations is empty and the algebra becomes unique.

We have seen that symmetry determines the one-loop Hamiltonian (2.10) up to a sequence of coefficients  $c_j$ . It turns out that the higher-loop corrections impose even stronger constraints: The point is that the Hilbert space of the spin chain decomposes into irreducible multiplets which are distinguished by their scaling dimension (among other quantum numbers). For the free superconformal algebra, the multiplets can be of short/atypical or of long/typical type [28]. Short multiplets must have (half) integral superconformal scaling dimension while long multiplets can have irrational scaling dimensions. However, the Hamiltonian attributes anomalous dimensions to almost all irreducible multiplets, long or short. Considering the spin chain Hamiltonian as the radiative correction  $\delta D$  to the dilatation generator  $D$  seemingly leads to a paradox. It is resolved if the right combination of short multiplets join to form a long multiplet.<sup>6</sup> This can only work if the short multiplets have coincident one-loop anomalous dimensions, which thus puts constraints on  $\hat{H}$ . On the level of the algebra, the joining of short multiplets into a long one is achieved through deformations of the superconformal generators  $Q, \bar{Q}, P$  and  $S, \bar{S}, K$  at order  $\mathcal{O}(g)$ . These map one site to two or vice versa, cf. Fig. 5. The algebra turns out to completely determine them. Invariance of the Hamiltonian then fixes the coefficients to the values of field theory (2.11) [23]

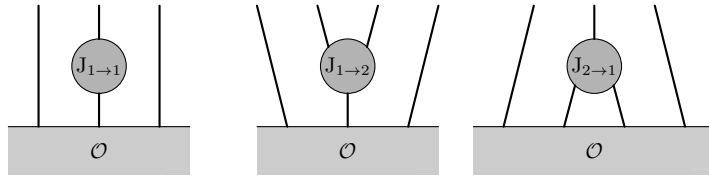
$$c_j \sim \sum_{k=1}^j \frac{1}{k}. \quad (2.19)$$

It is curious to see that integrability as well as higher-loop consistency lead to precisely the same constraints of the one-loop Hamiltonian. On the one hand, one can view it as

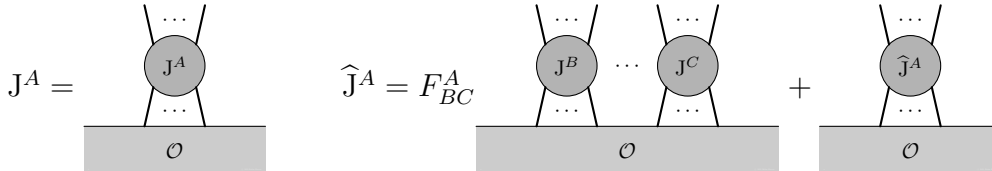
<sup>5</sup>Ambiguities (e.g. of ordering) are a generic problem of quantum algebras, which is also the reason why the Serre relation (2.13) is not formulated in the form of  $[\hat{J}^A, \hat{J}^B] = \dots$  analogously to (2.12). Hence quantum algebras are typically defined modulo certain types of deformations.

<sup>6</sup>A similar mechanism is required for the Higgs effect where a massless vector and a massless scalar combine into a massive vector.





**Figure 5:** Free action of a generic superconformal generator  $J$  (left) and leading-order corrections to  $Q, \bar{Q}, P$  (middle) and  $S, \bar{S}, K$  (right).



**Figure 6:** Structure of the deformed representation of the superconformal and Yangian algebra. The bilocal contributions to the Yangian are determined by the superconformal generators while the local contributions can be viewed as a short-distance regularisation thereof.

a consistency condition of higher-loop integrability, but on the other hand, the semantic relation between the two approaches remains somewhat obscure.

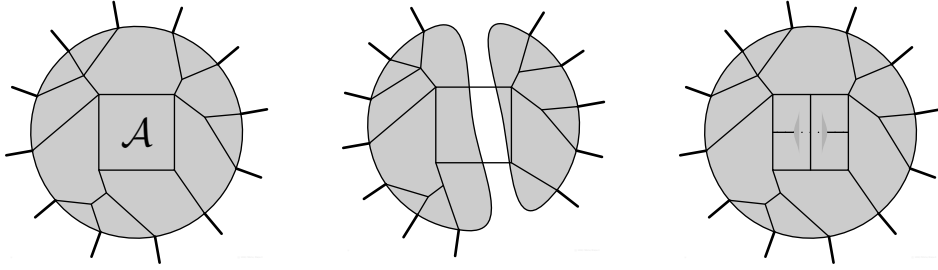
The leading-order deformation of the generators was established explicitly in [12, 29] for closed sectors of the full theory. In these sectors the construction was also continued by a few more perturbative orders. Knowledge of the higher-loop Hamiltonian revealed first strong hints that integrability survives [10] in perturbation theory. It was later shown that also the action of the Yangian can be deformed appropriately [30], which establishes integrability rigorously at a given perturbative order. We note that the structure of the deformed Yangian action always follows a pattern analogous to the coproduct rule (2.14): It consists of a bi-local combination of superconformal representations (properly expanded at each order of perturbation theory) and a local contribution which can be viewed as a short-distance regularisation of the bi-local term, see Fig. 6.

### 3 Scattering Amplitudes

A different type of observable which plays an important role in quantum field theories is the scattering matrix. Integrability has also been observed for  $\mathcal{N} = 4$  SYM in this context, and apparently it leads to substantial simplifications in their structure. We now review scattering amplitudes and their Yangian symmetry.

#### 3.1 Framework

A scattering amplitude of  $n$  particles is a function of the particle momenta  $p_k$ , spins or helicities, flavours and gauge degrees of freedom  $A_k$ . Statistics requires that this function is (graded) symmetric under the simultaneous interchange of all quantum numbers associated to any pair of particles. This symmetry can be enforced by summing over all



**Figure 7:** Colour-ordered scattering amplitudes expanded into  $U(N_c)$  traces: single-trace (left), multi-trace (middle) and non-planar (right) contributions.

(graded) permutations of particles with the associated quantum numbers

$$\mathcal{A}_{1\dots n}^{\text{full}} = \sum_{\pi \in S_n} \mathcal{A}_{\pi(1)\dots\pi(n)}^{\text{ordered}}. \quad (3.1)$$

In  $\mathcal{N} = 4$  SYM all particles transform in the adjoint representation of the gauge group. The gauge indices for a  $U(N_c)$  gauge group can be expanded in a basis of traces of  $U(N_c)$  generators  $T_k := T^{A_k}$  in the fundamental representation, see Fig. 7

$$\begin{aligned} \mathcal{A}_{1\dots n}^{\text{ordered}} &= \frac{1}{n} \text{Tr}(T_1 \dots T_n) \mathcal{A}_{1\dots n}^{\text{single-trace}} \\ &+ \sum_{k=1}^n \frac{1}{2k(n-k)} \text{Tr}(T_1 \dots T_k) \text{Tr}(T_{k+1} \dots T_n) \mathcal{A}_{1\dots k|k+1\dots n}^{\text{double-trace}} \\ &+ \dots \end{aligned} \quad (3.2)$$

The prefactors of  $1/k$  and  $1/2$  are the appropriate symmetry factors for  $\mathbb{Z}_k$  cyclic and  $S_2$  permutation symmetry. Note that the various multi-trace or *colour-ordered* contributions to the amplitude now just depend on the particle momenta, spins/helicities and flavours, but not on the gauge structure anymore.

Next, all particles in  $\mathcal{N} = 4$  SYM are massless. The on-shell momenta  $p_k$  are light-like and can be represented as bilinear combinations of bosonic spinors  $\lambda^\beta$  and  $\tilde{\lambda}^{\dot{\alpha}}$  [31]

$$p^{\beta\dot{\alpha}} = \sigma_\mu^{\beta\dot{\alpha}} p^\mu = \lambda^\beta \tilde{\lambda}^{\dot{\alpha}}. \quad (3.3)$$

The two spinors are complex conjugates,  $\tilde{\lambda}^{\dot{\alpha}} = \pm(\lambda^\alpha)^*$ , where the sign determines the sign of the particle energy. Furthermore, all flavours of on-shell particles – scalars  $\Phi$ , fermions  $\Psi$  and gluons  $\Gamma$  – can be conveniently combined into a field on superspace [32]

$$\Omega(\lambda, \tilde{\lambda}, \bar{\eta}) = \Gamma + \bar{\eta}^a \Psi_a + \frac{1}{2} \bar{\eta}^a \bar{\eta}^b \Phi_{ab} + \frac{1}{6} \varepsilon_{abcd} \bar{\eta}^a \bar{\eta}^b \bar{\eta}^c \bar{\eta}^d \bar{\Psi} + \frac{1}{24} \varepsilon_{abcd} \bar{\eta}^a \bar{\eta}^b \bar{\eta}^c \bar{\eta}^d \bar{\Gamma}. \quad (3.4)$$

This is useful because we can now scatter the superfield  $\Omega$  instead of the individual fields: The colour-ordered amplitudes then turn into plain function on the configuration superspace parametrised by  $\lambda_k$ ,  $\tilde{\lambda}_k$  and  $\bar{\eta}_k$  or, collectively,  $\Lambda_k$

$$\mathcal{A}^{\text{colour-ordered}}(\Lambda_1, \dots, \Lambda_n). \quad (3.5)$$

In particular, the flavour indices have been traded in completely for expansion coefficients in the  $\bar{\eta}_k$ . Also the helicity is determined by the flavour in  $\mathcal{N} = 4$  SYM, so that no indices remain. Such colour-ordered amplitudes on superspace will be the standard objects we shall consider.

Note that the multiplication of  $\lambda^\alpha$  by a complex phase does not change the momentum  $p^\mu$ . Such a multiplication is equivalent to a rotation about the particle momentum, and thus the amplitude must transform according to the particle helicity. In effect, the amplitude is constrained by

$$\mathcal{A}(\dots, \Lambda_k, \dots) = e^{2i\varphi} \mathcal{A}(\dots, e^{i\varphi} \Lambda_k, \dots), \quad e^{i\varphi}(\lambda, \tilde{\lambda}, \eta) := (e^{i\varphi} \lambda, e^{-i\varphi} \tilde{\lambda}, e^{i\varphi} \eta). \quad (3.6)$$

Put differently, the differential operator

$$C_k = 2 + \lambda_k^\alpha \frac{\partial}{\partial \lambda_k^\alpha} - \tilde{\lambda}_k^\alpha \frac{\partial}{\partial \tilde{\lambda}_k^\alpha} - \bar{\eta}_k^a \frac{\partial}{\partial \bar{\eta}_k^a} \quad (3.7)$$

acting on any leg  $k$  annihilates the amplitude.

Furthermore, one can classify amplitudes by an operator  $B$  which effectively measures the overall helicity of the particles

$$\mathcal{A}_n = \sum_{k=2}^{n-2} \mathcal{A}_{n,k}, \quad B \mathcal{A}_{n,k} = 4k \mathcal{A}_{n,k}, \quad B = \sum_{k=1}^n \bar{\eta}_k^a \frac{\partial}{\partial \bar{\eta}_k^a} = N_{\bar{\eta}}. \quad (3.8)$$

Note that due to  $\mathfrak{su}(4)$ -invariance of the amplitude, the  $\bar{\eta}$ 's can only appear in groups of four and due to supersymmetry there must be between 8 and  $4(n-2)$  of them. The amplitude with the minimum number of eight  $\bar{\eta}$ 's is called MHV and it is typically the simplest among those with the same number of legs.

## 3.2 Tree-Level Amplitudes

The MHV amplitudes  $\mathcal{A}_n^{\text{MHV}} := \mathcal{A}_{n,2}$  have particularly simple expressions [33, 31]

$$\mathcal{A}_n^{\text{MHV}} = \frac{\delta^4(P_n) \delta^8(Q_n)}{\langle 1, 2 \rangle \cdots \langle n, 1 \rangle} \quad (3.9)$$

with the overall momentum  $P_n$  and its fermionic partner  $Q_n$

$$P_n^{\beta\dot{\alpha}} = \sum_{k=1}^n \lambda_k^\beta \tilde{\lambda}_k^{\dot{\alpha}}, \quad Q_n^{\beta a} = \sum_{k=1}^n \lambda_k^\beta \bar{\eta}_k^a. \quad (3.10)$$

Furthermore invariants of the spinors are obtained by contraction with the antisymmetric invariant tensor  $\varepsilon_{\alpha\beta}$  or  $\varepsilon_{\dot{\alpha}\dot{\beta}}$

$$\langle \lambda, \mu \rangle := \varepsilon_{\alpha\beta} \lambda^\alpha \mu^\beta, \quad [\tilde{\lambda}, \tilde{\mu}] := \varepsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}^{\dot{\alpha}} \tilde{\mu}^{\dot{\beta}}. \quad (3.11)$$

We abbreviate  $\langle \lambda_j, \lambda_k \rangle$  as  $\langle jk \rangle$ . We can use the above expression  $\mathcal{A}^{\text{MHV}}$  to confirm  $\mathcal{N} = 4$  superconformal invariance of scattering amplitudes. The generators of  $\mathfrak{psu}(2, 2|4)$

acting on a single free field take particularly simple expressions using the spinor helicity superspace variables  $\lambda, \tilde{\lambda}, \bar{\eta}$  [34]

$$\begin{aligned}
L^\alpha{}_\beta &= \lambda^\alpha \partial_\beta - \frac{1}{2} \delta_\beta^\alpha \lambda^\gamma \partial_\gamma, & \bar{L}^{\dot{\alpha}}{}_{\dot{\beta}} &= \tilde{\lambda}^{\dot{\alpha}} \tilde{\partial}_{\dot{\beta}} - \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} \tilde{\lambda}^{\dot{\gamma}} \tilde{\partial}_{\dot{\gamma}}, \\
D &= \frac{1}{2} \partial_\gamma \lambda^\gamma + \frac{1}{2} \tilde{\lambda}^{\dot{\gamma}} \tilde{\partial}_{\dot{\gamma}}, & R^a{}_b &= \bar{\eta}^a \bar{\partial}_b - \frac{1}{4} \delta_b^a \bar{\eta}^c \bar{\partial}_c, \\
\bar{Q}_b^{\dot{\alpha}} &= \tilde{\lambda}^{\dot{\alpha}} \bar{\partial}_b, & \bar{S}_\alpha^b &= \bar{\eta}^b \bar{\partial}_\alpha, \\
Q^{\beta a} &= \lambda^\beta \bar{\eta}^a, & S_{\beta a} &= \partial_\beta \bar{\partial}_a, \\
P^{\beta \dot{\alpha}} &= \lambda^\beta \tilde{\lambda}^{\dot{\alpha}}, & K_{\beta \dot{\alpha}} &= \partial_\beta \tilde{\partial}_{\dot{\alpha}}.
\end{aligned} \tag{3.12}$$

The action on the amplitude is given by the standard tensor product rule as the sum over all fields

$$J^A = \sum_{k=1}^n J_k^A. \tag{3.13}$$

Invariance under the Lorentz  $L, \bar{L}$  and internal  $R$  rotations is manifest because the amplitude (3.9) is constructed only from scalar combinations.

The dilatation generator  $D$  counts the number of  $\lambda$ 's and  $\tilde{\lambda}$ 's. For scaling invariance the overall number must equal  $-2n$ . Using the degrees of homogeneity of the three components (3.9)

$$\delta^4(P) \sim \lambda^{-4} \bar{\lambda}^{-4}, \quad \delta^8(Q) \sim \bar{\eta}^8 \lambda^8, \quad \frac{1}{\langle 12 \rangle \dots \langle n1 \rangle} \sim \lambda^{-2n}. \tag{3.14}$$

invariance follows straight-forwardly. Furthermore, it is clear that each  $C_k$  annihilates the amplitude as desired because both delta-functions are invariant, and the denominator contributes  $\lambda_k^{-2}$  for each  $k$ .

Next, the translations  $P$  and the supertranslations  $Q$  annihilate the amplitude due to the two delta-functions  $\delta^4(P_n)$  and  $\delta^8(Q_n)$  because  $P$  acts through multiplication by the overall momentum  $P_n$  (analogously for  $Q$  and  $Q_n$ ).

Invariance under the conjugate supertranslation is less obvious. As it contains a derivative w.r.t.  $\bar{\eta}$ , it acts non-trivially only on the fermionic delta-function  $\delta^8(Q_n)$

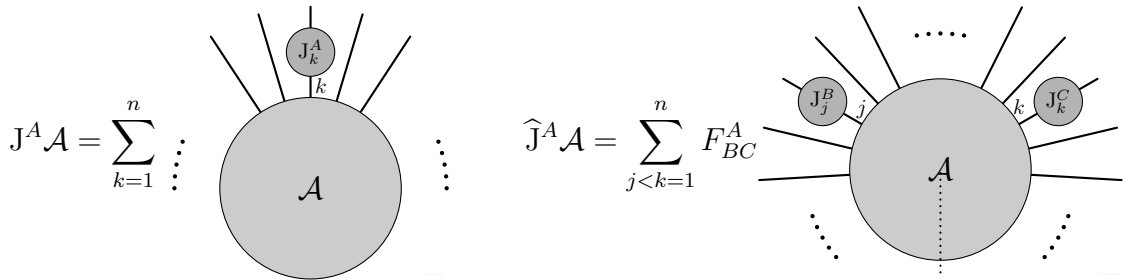
$$\bar{Q}_b^{\dot{\alpha}} \delta^8(Q_n) = \sum_{k=1}^n \tilde{\lambda}_k^{\dot{\alpha}} \bar{\partial}_{b,k} \delta^8(Q_n) = \sum_{k=1}^n \lambda_k^\gamma \tilde{\lambda}_k^{\dot{\alpha}} \frac{\partial \delta^8(Q_n)}{\partial Q_n^{\gamma b}} = P_n^{\gamma \dot{\alpha}} \frac{\partial \delta^8(Q_n)}{\partial Q_n^{\gamma b}}. \tag{3.15}$$

Due to the presence of the bosonic delta-function  $\delta^4(P_n)$  the conjugate supermomentum annihilates the amplitude. The derivation for invariance under the conjugate superboost  $\bar{S}$  is analogous, but there are important subtleties to be discussed in Sec. 3.4.

To show invariance under the conformal boost  $K$  and superboost  $S$  takes the largest number of steps. The two derivations are analogous and we consider only the superboost  $S$ . It contains a derivative w.r.t.  $\bar{\eta}$  which again only acts on the fermionic delta-function. By a sequence of transformation we can recombine the terms into useful combinations

$$S_{ab} \delta^8(Q_n) = \sum_{k=1}^n \partial_{\alpha,k} \lambda_k^\gamma \frac{\partial \delta^8(Q_n)}{\partial Q_n^{\gamma b}} = \left( L^\gamma{}_\alpha + \frac{1}{2} \delta_\alpha^\gamma \sum_{k=1}^n (\lambda_k^\delta \partial_{\delta,k} + 2) \right) \frac{\partial \delta^8(Q_n)}{\partial Q_n^{\gamma b}} \tag{3.16}$$

$$= \frac{\partial \delta^8(Q_n)}{\partial Q_n^{\gamma b}} L^\gamma{}_\alpha + \frac{1}{2} \frac{\partial \delta^8(Q_n)}{\partial Q_n^{\alpha b}} \left( \sum_{k=1}^n \lambda_k^\delta \partial_{\delta,k} - 3 + 7 + 2n \right). \tag{3.17}$$



**Figure 8:** Action of the free superconformal generator  $J$  and the Yangian generator  $\hat{J}$  on the colour-ordered amplitude  $\mathcal{A}$ .

The first step consists in rewriting  $\partial_\alpha \lambda^\gamma$  as a Lorentz generator  $L^{\gamma_\alpha}$ . In the next step these generators are commuted past the fermionic delta-function. This picks up  $-\frac{3}{2}$  from the Lorentz generator and  $\frac{7}{2}$  from the weight in  $\lambda$ . The point is then that the remaining denominator and bosonic delta-function in  $\mathcal{A}_n^{\text{MHV}}$  (3.9) are Lorentz invariant have overall weight  $\lambda^{-4-2n}$  according to (3.14). Hence the amplitude is annihilated.

### 3.3 Leading-Order Yangian

In addition to the standard superconformal symmetries a new type of superconformal symmetry has recently been discovered for planar scattering amplitudes in  $\mathcal{N} = 4$  [35,36]. Tree amplitudes were shown to be covariant with respect to these *dual superconformal* transformations [37,38], and also loop amplitudes appear to be substantially constrained. The dual superconformal algebras overlaps partially with the conventional one, and therefore the two algebras must close onto a bigger one. This algebra turns out to be a Yangian [39].

We now wish to extend the superconformal symmetry for amplitudes to a Yangian algebra. The fields transform in the superconformal representation specified in (3.12). It can be extended to an evaluation representation of the Yangian because the Serre relations are satisfied. Clearly, all external fields are on equal footing and should have coincident evaluation parameter. Again, its value does not play an important role because it merely multiplies the standard conformal generators; we can safely set it to zero. The representation of the Yangian generators from the coproduct (2.14) then becomes, see also Fig. 8,

$$\hat{J}^A = \frac{1}{2} F_{BC}^A J^B \wedge J^C, \quad \text{where} \quad J^B \wedge J^C := \sum_{j < k=1}^n (J_j^B J_k^C - J_k^B J_j^C). \quad (3.18)$$

Invariance of tree amplitudes under Yangian symmetry  $\hat{J}^A \mathcal{A} = 0$  [39] follows from their conventional and dual superconformal transformation properties [37,38].

Let us demonstrate Yangian invariance of the MHV tree amplitude. The simplest of the Yangian generators is the level-one momentum generator  $\hat{P}$ . Due to the adjoint transformation property of the Yangian generators (2.12) it suffices to show invariance w.r.t. this generator in addition to superconformal invariance in order to prove complete

Yangian invariance. The generator takes the explicit form

$$\widehat{\mathbf{P}}^{\beta\dot{\alpha}} = \mathbf{P}^{\beta\dot{\alpha}} \wedge \mathbf{D} + \mathbf{P}^{\delta\dot{\alpha}} \wedge \mathbf{L}^{\beta}_{\delta} + \mathbf{P}^{\beta\dot{\gamma}} \wedge \bar{\mathbf{L}}^{\dot{\alpha}}_{\dot{\gamma}} + \mathbf{Q}^{\beta c} \wedge \bar{\mathbf{Q}}^{\dot{\alpha}}_c. \quad (3.19)$$

This generator has one derivative which acts on the amplitude function (3.9). The action on delta-functions cancels straight-forwardly between the various contributions in (3.19). What remains is the action on the denominator terms

$$\begin{aligned} \widehat{\mathbf{P}}^{\beta\dot{\alpha}} \mathcal{A}_n^{\text{MHV}} &= \sum_{j < k} \left( -\lambda_k^{\beta} \tilde{\lambda}_j^{\dot{\alpha}} \frac{\langle j, k+1 \rangle}{\langle k, k+1 \rangle} - \lambda_k^{\beta} \tilde{\lambda}_j^{\dot{\alpha}} \frac{\langle j, k-1 \rangle}{\langle k, k-1 \rangle} + \lambda_j^{\beta} \tilde{\lambda}_j^{\dot{\alpha}} \right) \mathcal{A}_n^{\text{MHV}} \\ &+ \sum_{j < k} \left( +\lambda_j^{\beta} \tilde{\lambda}_k^{\dot{\alpha}} \frac{\langle k, j+1 \rangle}{\langle j, j+1 \rangle} + \lambda_j^{\beta} \tilde{\lambda}_k^{\dot{\alpha}} \frac{\langle k, j-1 \rangle}{\langle j, j-1 \rangle} - \lambda_k^{\beta} \tilde{\lambda}_k^{\dot{\alpha}} \right) \mathcal{A}_n^{\text{MHV}}. \end{aligned} \quad (3.20)$$

Shifting the summation variables appropriately we can use the spinor identity

$$\lambda_k^{\beta} \langle j, k+1 \rangle - \lambda_{k+1}^{\beta} \langle j, k \rangle = \lambda_j^{\beta} \langle k, k+1 \rangle \quad (3.21)$$

to combine several terms. What remains turns out to be proportional to the overall momentum and thus vanishes proving Yangian invariance for tree MHV amplitudes

$$\widehat{\mathbf{P}}^{\beta\dot{\alpha}} \mathcal{A}_n^{\text{MHV}} = \frac{\lambda_1^{\delta} \lambda_n^{\beta} + \lambda_1^{\beta} \lambda_n^{\delta}}{\langle n, 1 \rangle} \varepsilon_{\delta\gamma} P_n^{\gamma\dot{\alpha}} \mathcal{A}_n^{\text{MHV}} = 0. \quad (3.22)$$

Noting that colour-ordered amplitudes are cyclic, an important additional consideration is the cyclic behaviour of the Yangian [39]. The point is that the Yangian generators are typically not invariant under cyclic shifts: Let us compare the action on sites 1 through  $n$  with the action on sites 2 through  $n+1$

$$\widehat{\mathbf{J}}_{1,n}^A = \frac{1}{2} F_{BC}^A \sum_{k=1}^n \sum_{j=1}^{k-1} \mathbf{J}_j^B \mathbf{J}_k^C, \quad \widehat{\mathbf{J}}_{2,n+1}^A = \frac{1}{2} F_{BC}^A \sum_{k=2}^{n+1} \sum_{j=2}^{k-1} \mathbf{J}_j^B \mathbf{J}_k^C. \quad (3.23)$$

These two expressions are not equal, they differ by

$$\widehat{\mathbf{J}}_{2,n+1}^A - \widehat{\mathbf{J}}_{1,n}^A = -\frac{1}{2} F_{BC}^A \{ \mathbf{J}_1^B, \mathbf{J}_1^C \} = \frac{1}{2} F_{BC}^A F_D^{BC} \mathbf{J}_1^D - F_{BC}^A \mathbf{J}_1^B \mathbf{J}_1^C. \quad (3.24)$$

Hence the action typically maps cyclic states to non-cyclic ones. More importantly, the action of the Yangian on periodic states is not uniquely defined; it depends on the point where the periodic chain is cut open.

For amplitudes however the situation is better because both operators on the r.h.s. are symmetries. The second term vanishes because the amplitude is invariant under conventional superconformal symmetry. The first term contains  $F_{BC}^A F_D^{BC}$  which is proportional to the dual Coxeter number which equals zero for  $\mathfrak{psu}(2, 2|4)$ .

### 3.4 Collinearities and Higher Loops

The discussion of the free superconformal symmetries in Sec. 3.2 was not entirely honest to the end that the amplitude is *not exactly* invariant under them: The special

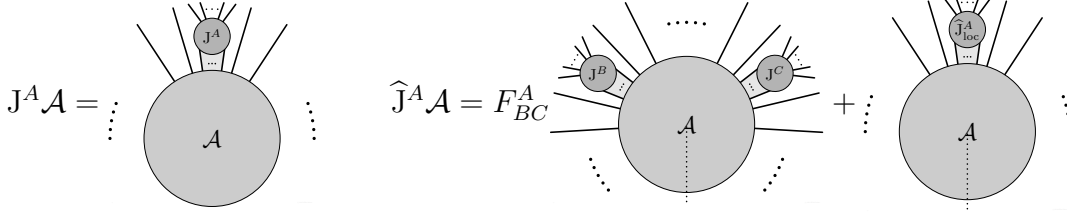
**Figure 9:** Exact invariance of amplitudes under the deformed superconformal representation at tree level.

superconformal symmetries  $S, \bar{S}, K$  acting on a colour-ordered amplitude leave behind a distributional remainder supported on configurations with a pair of adjacent particles being exactly collinear,  $p_k \sim p_{k+1}$ . In other words, generic amplitudes are indeed annihilated by the free symmetries as discussed above, but there exist special configuration where this is not case. The extra contributions originate in the analog of (3.15) for  $\bar{S}$  from the holomorphic anomaly in the complex spinor helicity space [40]

$$\frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} \frac{1}{\langle \lambda, \mu \rangle} = 2\pi \varepsilon_{\dot{\alpha}\dot{\gamma}} \tilde{\mu}^{\dot{\gamma}} \text{sign}(E(\lambda)E(\mu)) \delta^2(\langle \lambda, \mu \rangle). \quad (3.25)$$

The delta-function is supported on collinear spinors  $\lambda, \mu$  or, equivalently, when the associated momenta are collinear. Luckily these contributions can be compensated by deforming the representation of  $S, \bar{S}, K$  [41]. The additional contributions map one leg of the amplitude to two or three collinear particles, cf. Fig. 9. When acting with such an operator on an amplitude with fewer legs, one can cancel the contributions from the collinear anomaly. Altogether invariance at tree level is recovered only when acting on the superposition of all amplitudes with arbitrary numbers of legs (henceforth called *the amplitude*).

Non-invariance under the free superconformal generators turns out to be beneficial in several respects. While the free superconformal and Yangian generators only relate amplitudes with a common number of legs, the deformation introduce relations between amplitudes with different numbers of legs. Here the free and deformed generators serve two different purposes: The free superconformal generators are sensitive to the pole-like collinear singularities in the denominator of (3.9) through the collinear anomaly (3.25). The deformed generator provides the residues of the collinear singularities. Apart from those inherited from fewer-leg amplitudes, further collinear singularities are consequently prohibited by superconformal symmetry. In conclusion, conformal symmetry constrains and determines both the analytical structure and the structure of singularities of the amplitude. Together with Yangian symmetry it appears that the tree amplitude may be completely determined through symmetry arguments alone! Although there exist many ways to construct tree amplitudes conveniently, unique determination by symmetry is an interesting prospect because uniqueness is automatically inherited to all perturbative orders [41]. Nevertheless one has to bear in mind that this also requires understanding how the representation is deformed at loop level.



**Figure 10:** Perturbative action of the superconformal generator  $J$  and the Yangian generator  $\hat{J}$  on the colour-ordered amplitude  $\mathcal{A}$ .

Concerning the latter issues, it appears that the deformations at loop level depend to a large extent on the deformation at tree level or a suitable iteration thereof [42, 43]. In addition there are deformations due to infra-red divergences at loop level affecting all of the non-manifest symmetries. The divergent contributions to the one-loop planar amplitude  $\mathcal{A}_n^{(1)}$  are determined by the tree amplitude  $\mathcal{A}_n^{(0)}$

$$\mathcal{A}_n^{(1)} = \hat{Z}^{(1)} \mathcal{A}_n^{(0)} + \tilde{\mathcal{A}}_n^{(1)} \quad \text{with} \quad \hat{Z}^{(1)} = - \sum_{j=1}^n \frac{c_\epsilon}{\epsilon^2} \left( \frac{s_{j,j+1}}{-\mu^2} \right)^{-\epsilon}, \quad (3.26)$$

where  $s_{j,k} = (p_j + p_k)^2$  and where  $\tilde{\mathcal{A}}_n^{(1)}$  is finite. The anomaly of the dilatation operator due to the presence of the scale  $\mu$  is the clearest. It can be absorbed by a simple one-loop deformation  $D^{(1)}$  to the free dilatation generator  $D^{(0)}$  [43]

$$D^{(1)} = -[D^{(0)}, \hat{Z}^{(1)}] = -2 \sum_{j=1}^n \frac{c_\epsilon}{\epsilon} \left( \frac{s_{j,j+1}}{-\mu^2} \right)^{-\epsilon}, \quad (3.27)$$

so that  $D^{(0)} \mathcal{A}_n^{(1)} + D^{(1)} \mathcal{A}_n^{(0)} = 0$  because the finite contribution  $\tilde{\mathcal{A}}_n^{(1)}$  is scale invariant.

The situation for the Yangian momentum generator  $\hat{P}$  is similar. The anomaly due to the IR divergencies can be absorbed by a deformation analogous to the dilatation generator, but here the finite remainder is anomalous as well [36, 44, 37, 45]

$$(\hat{P}^{(0)})^{\beta\alpha} \tilde{\mathcal{A}}_n^{(1)} = 2 \sum_{j=1}^n p_j^{\beta\alpha} \log \left( \frac{s_{j,j+1}}{s_{j-1,j}} \right) \mathcal{A}_n^{(0)}. \quad (3.28)$$

Note that this anomaly depends only on neighbouring legs. Hence the total deformation of the Yangian momentum generator reads

$$\hat{P}^{(1)} = -[\hat{P}^{(0)}, \hat{Z}^{(1)}] + \hat{P}_{\text{loc}}^{(1)} \quad \text{with} \quad (\hat{P}_{\text{loc}}^{(1)})^{\beta\alpha} = 2 \sum_{j=1}^n (p_j^{\beta\alpha} - p_{j+1}^{\beta\alpha}) \frac{c_\epsilon}{\epsilon} \left( \frac{s_{j,j+1}}{-\mu^2} \right)^{-\epsilon}. \quad (3.29)$$

It follows the general structure of perturbative Yangian generators: The commutator generates the bi-local combinations of the deformed generators. In this case the dilatation generator  $D$  is the only deformed generator among the ones contributing to  $\hat{P}$  in (3.19) because the super-Poincaré generators  $P, Q, \bar{Q}$  are manifest symmetries. In particular the deformation reads simply, see Fig. 10

$$\hat{P}^{(1)} = P^{(0)} \wedge D^{(1)} + \hat{P}_{\text{loc}}^{(1)}. \quad (3.30)$$



The local contribution can be attributed to the anomaly of the finite remainder.

At higher loops we expect this general picture in Fig. 10 to remain valid. There are reasons to believe that the deformation of the dilatation  $D$  and Yangian momentum  $\widehat{P}$  generator at higher loops remains reasonably simple. For the other superconformal and Yangian generators, however, the deformation is already substantially more involved even at one loop [43]. And even though the deformations are known, it remains to be understood how the superconformal and Yangian algebra closes precisely.

## 4 Comparison and Summary

The attentive reader will have noticed that the discussions in Sec. 2 and in Sec. 3 were analogous to a large extent.

### 4.1 Analogies

Let us first concentrate on the representation of superconformal symmetry. The free representations on fields (2.9) and on external particles (3.12) are equivalent if one identifies oscillators with spinor-helicity variables

$$a \sim \lambda, \quad b \sim \tilde{\lambda}, \quad d \sim \bar{\eta}, \quad a^\dagger \sim \frac{\partial}{\partial \lambda}, \quad b^\dagger \sim \frac{\partial}{\partial \tilde{\lambda}}, \quad d^\dagger \sim \frac{\partial}{\partial \bar{\eta}}. \quad (4.1)$$

It is clear that their algebras coincide and hence the derived representations are equivalent.

This is not surprising because both describe free on-shell fields of  $\mathcal{N} = 4$  SYM: The spinor helicity superspace is designed to describe a field excitation with definite on-shell momentum. Conversely, a finite excitation of the supersymmetric oscillator describes a component of the fields expanded around a specific point in spacetime. Here, the free equations of motion are imposed through vanishing of certain components. A Fourier transformation translates between the two pictures

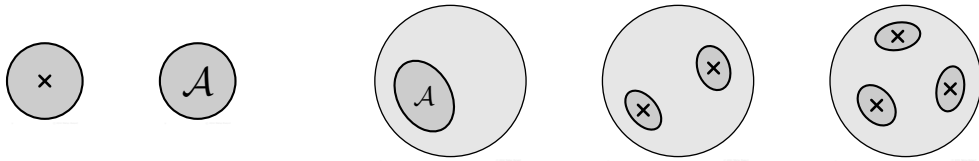
$$\mathcal{W}(x) \sim \int d^4\lambda e^{ix \cdot p(\lambda)} \Omega(\lambda), \quad \Omega(\lambda) \sim \int d^3x e^{-ix \cdot p(\lambda)} (\mathcal{W}(x, 0) - iE(\lambda)^{-1} \dot{\mathcal{W}}(x, 0)). \quad (4.2)$$

For the reverse transformation it suffices to use a time slice of  $\mathcal{W}$  at  $t = 0$ . It is however clear that for full equivalence between the two pictures one has to rely on distributions. Consequently the two representations are only equivalent in a physicist's sense or under additional assumptions.<sup>7</sup>

Once the equivalence of the free superconformal representations is established, it becomes straight-forward to lift the equivalence to the Yangian algebra. Indeed, the infinite-dimensional algebra appears to determine uniquely relevant structures such as the spin chain Hamiltonian as well as the scattering amplitude. There is however one crucial difference between the application of Yangians to local operators vs. scattering

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<sup>7</sup>It might be interesting to Fourier transform some of the structures from one picture between position and momentum space.



**Figure 11:** Large- $N_c$  topology representation of a single-trace local operator and a single-trace particle configuration (left two); a cross corresponds to a puncture of the surface serving as a source for charges. Quantum correlation functions insert a surface ending on the traces: planar scattering amplitude (middle) as well as two- and three-point correlators (right two); not shown are non-planar contributions where the surfaces have additional handles.

amplitudes. For the former it merely acts as a useful algebraic structure, cf. (2.16), while for the latter it is a true symmetry. This point will be discussed in more detail in the next subsection. A related issue is that Yangian symmetry is typically incompatible with cyclic symmetry. Only for the scattering amplitude it respects it due to superconformal invariance and due to vanishing dual Coxeter number, cf. (3.24).

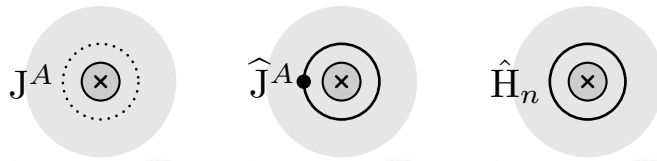
We have furthermore seen that the structure of the perturbative superconformal and Yangian representation is analogous in both pictures, cf. the pairs (2.15,3.18), Fig. 3,8, Fig. 5,9 and Fig. 6,10. The superconformal deformations act on several adjacent fields or legs in the trace of the local operator or colour-ordered amplitude. The deformed Yangian representation is constructed as a bi-local combination of deformed superconformal representations plus local terms which can be understood as a short-distance (along the trace) regularisation of the bi-local terms. One notable difference concerns the manifest symmetries which are not deformed by radiative corrections. For local operators only the Lorentz and internal symmetries  $L, \bar{L}, R$  are manifest, while for the scattering amplitude the full super-Poincaré algebra including the (super) momentum generators  $Q, \bar{Q}, P$  are undeformed.<sup>8</sup>

## 4.2 Large- $N_c$ Topology

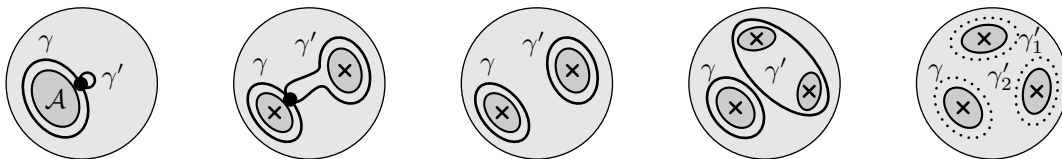
Integrability and the Yangian algebra is tightly related to the 't Hooft planar limit [5]. Let us therefore consider the large- $N_c$  expansion. Local operators and particle configurations can be viewed as closed one-dimensional contours such that each trace corresponds to one connected component; let us refer to them as *states*, see Fig. 11. Quantum correlation functions then span two-dimensional surfaces between the contours. The topology of these surfaces determines the suppression in powers of  $1/N_c$ . For example, a single-trace scattering amplitude as well as two- and three-point correlators of local operators are displayed in Fig. 11. This is in line with the picture from the AdS/CFT dual string theory on  $AdS_5 \times S^5$ , where the surface is the string world sheet and its boundaries (or punctures) correspond to states.

Algebra generators are represented by Wilson loops of the Lax family of flat connec-

<sup>8</sup>In fact, trying to construct a representation for local operators with manifest super-Poincaré symmetry implies vanishing anomalous dimensions [12].



**Figure 12:** Action of generators on a single-trace state: Superconformal  $J^A$ , Yangian  $\widehat{J}^A$  and local integrable Hamiltonians  $\widehat{H}_n$ . Fat lines represent Wilson lines; open loops have a marked base point, closed loops do not. Dotted lines are plain integrals which can be broken up.



**Figure 13:** Invariance conditions can be understood through deforming the contours  $\gamma \rightarrow \gamma'$  associated to generators on the surface. Yangian invariance follows from shifting the contour around the surface and shrinking it to a point (leftmost). For a two-point function Yangian invariance is broken because the marked base point cannot be moved (middle left); the integrable charges are nevertheless conserved (middle). For a three-point function even the integrable charges are not conserved (middle right), but the conformal ones are (rightmost).

tions on the world sheet [46]. Thus we should represent a generator by a closed loop on the surface. Due to flatness, the contour can be deformed smoothly. The action on a state corresponds to winding the loop around the trace, cf. Fig. 12.

Now we can consider invariance conditions for particular objects, see Fig. 13: In Sec. 3 we have seen that (tree-level) planar single-trace amplitudes are invariant under Yangian symmetry. In our picture we should wind an open Wilson loop around the trace of the particle configuration. We can unwind the Wilson loop on the disc ending on the trace without having to move the base point. The Wilson loop then shrinks to a point implying invariance. The unwinding would not be possible in the non-planar case of a disc with handles.

For a correlation function of two traces, e.g. a two-point function of single-trace local operators, the Yangian action on the two states is not equivalent because the base point is different. A closed Wilson loop can, however, be deformed from one trace to the other in the planar case of an annulus connecting the traces. This implies the integrability of the problem of planar anomalous dimensions.<sup>9</sup>

Finally, a correlation function of three traces does not have conserved charges. Merely superconformal symmetry survives, because for these generators the loop is an abelian contour integral which can be broken up into two pieces.

<sup>9</sup>For planar double-trace scattering amplitudes integrability implies the existence of a tower of conserved charges. It would be interesting to confirm them.

### 4.3 Summary and Outlook

In this paper we have reviewed Yangian symmetry which serves as an algebraic foundation of integrability in planar  $\mathcal{N} = 4$  maximally supersymmetric gauge theory.

We have seen that the Yangian is capable of uniquely determining certain physical observables by purely algebraic means. Even more importantly, there exist methods to exploit the uniqueness and obtain these observables very efficiently. Among them are the Bethe ansatz, spectral curves, asymptotic Bethe equations with Lüscher corrections, thermodynamic Bethe ansatz or Y-system and Graßmannians. Applying them one can avoid highly complicated calculations in quantum field theory and arrive at the correct final result much faster.

Although these methods are already being applied reliably, it still remains to be understood *why* they work. Why is planar  $\mathcal{N} = 4$  SYM governed by a Yangian algebra (technically as well as semantically)? How does it lead to the above methods? How is the algebra defined in the first place? As we have seen, at leading perturbative order the Yangian follows precisely from the established framework of quantum algebra. At higher loops the Yangian representation gets deformed, and some of the well-known rules have to be dropped in favour of new ones yet to be established.

We have discussed two subjects where the Yangian algebra makes a prominent appearance: anomalous dimensions of local operators and the spacetime scattering matrix. A third subject which was not discussed here is the worldsheet scattering matrix. The Yangian relevant to that problem is not based on  $\mathfrak{psu}(2, 2|4)$  but only on the subalgebra  $\mathfrak{psu}(2|2)$ . Although this is an exception case, several works have demonstrated that it can apparently be described by conventional quantum algebra methods, see the review [47]. Complete understanding the smaller Yangian may eventually lead to clues for the full perturbative Yangian for  $\mathcal{N} = 4$  SYM.

## References

- [1] L. N. Lipatov, “*Reggeization of the Vector Meson and the Vacuum Singularity in Nonabelian Gauge Theories*”, Sov. J. Nucl. Phys. 23, 338 (1976). • E. A. Kuraev, L. N. Lipatov and V. S. Fadin, “*Multi-Reggeon Processes in the Yang-Mills Theory*”, Sov. Phys. JETP 44, 443 (1976). • E. A. Kuraev, L. N. Lipatov and V. S. Fadin, “*The Pomeron Singularity in Nonabelian Gauge Theories*”, Sov. Phys. JETP 45, 199 (1977). • V. S. Fadin, E. A. Kuraev and L. N. Lipatov, “*On the Pomeron Singularity in Asymptotically Free Theories*”, Phys. Lett. B60, 50 (1975). • I. I. Balitsky and L. N. Lipatov, “*The Pomeron Singularity in Quantum Chromodynamics*”, Sov. J. Nucl. Phys. 28, 822 (1978).
- [2] L. N. Lipatov, “*High-energy asymptotics of multicolor QCD and exactly solvable lattice models*”, hep-th/9311037.
- [3] L. N. Lipatov, “*Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models*”, JETP Lett. 59, 596 (1994). • L. D. Faddeev and G. P. Korchemsky, “*High-energy QCD as a completely integrable model*”, Phys. Lett. B342, 311 (1995), hep-th/9404173.

- [4] L. N. Lipatov, “Integrability of scattering amplitudes in  $\mathcal{N} = 4$  SUSY”, J. Phys. A42, 304020 (2009), arxiv:0902.1444.
- [5] G. 't Hooft, “A Planar Diagram Theory for Strong Interactions”, Nucl. Phys. B72, 461 (1974).
- [6] V. N. Gribov and L. N. Lipatov, “Deep inelastic  $e p$  scattering in perturbation theory”, Sov. J. Nucl. Phys. 15, 438 (1972). • G. Altarelli and G. Parisi, “Asymptotic Freedom in Parton Language”, Nucl. Phys. B126, 298 (1977). • Yu. L. Dokshitzer, “Calculation of the Structure Functions for Deep Inelastic Scattering and  $e^+ e^-$  Annihilation by Perturbation Theory in Quantum Chromodynamics”, Sov. Phys. JETP 46, 641 (1977).
- [7] L. N. Lipatov, “Evolution equations in QCD”, in: “Perspectives in Hadronic Physics”, ed.: S. Boffi, C. Ciofi Degli Atti and M. Giannini, World Scientific (1998), Singapore.
- [8] V. M. Braun, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, “Baryon distribution amplitudes in QCD”, Nucl. Phys. B553, 355 (1999), hep-ph/9902375. • A. V. Belitsky, “Integrability and WKB solution of twist-three evolution equations”, Nucl. Phys. B558, 259 (1999), hep-ph/9903512.
- [9] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for  $\mathcal{N} = 4$  super Yang-Mills”, JHEP 0303, 013 (2003), hep-th/0212208.
- [10] N. Beisert, C. Kristjansen and M. Staudacher, “The Dilatation Operator of  $\mathcal{N} = 4$  Conformal Super Yang-Mills Theory”, Nucl. Phys. B664, 131 (2003), hep-th/0303060.
- [11] N. Beisert and M. Staudacher, “The  $\mathcal{N} = 4$  SYM Integrable Super Spin Chain”, Nucl. Phys. B670, 439 (2003), hep-th/0307042.
- [12] N. Beisert, “The  $SU(2/3)$  Dynamic Spin Chain”, Nucl. Phys. B682, 487 (2004), hep-th/0310252.
- [13] M. Bianchi, S. Kovacs, G. Rossi and Y. S. Stanev, “On the logarithmic behavior in  $\mathcal{N} = 4$  SYM theory”, JHEP 9908, 020 (1999), hep-th/9906188.
- [14] M. Günaydin and N. Marcus, “The Spectrum of the  $S^5$  Compactification of the Chiral  $\mathcal{N} = 2$ ,  $D = 10$  Supergravity and the Unitary Supermultiplets of  $U(2, 2/4)$ ”, Class. Quant. Grav. 2, L11 (1985).
- [15] N. Beisert, “The Complete One-Loop Dilatation Operator of  $\mathcal{N} = 4$  Super Yang-Mills Theory”, Nucl. Phys. B676, 3 (2004), hep-th/0307015.
- [16] A. V. Kotikov and L. N. Lipatov, “NLO corrections to the BFKL equation in QCD and in supersymmetric gauge theories”, Nucl. Phys. B582, 19 (2000), hep-ph/0004008. • A. V. Kotikov and L. N. Lipatov, “DGLAP and BFKL evolution equations in the  $\mathcal{N} = 4$  supersymmetric gauge theory”, hep-ph/0112346. • F. A. Dolan and H. Osborn, “Superconformal symmetry, correlation functions and the operator product expansion”, Nucl. Phys. B629, 3 (2002), hep-th/0112251.
- [17] A. V. Belitsky, V. M. Braun, A. S. Gorsky and G. P. Korchemsky, “Integrability in QCD and beyond”, Int. J. Mod. Phys. A19, 4715 (2004), hep-th/0407232.
- [18] L. D. Faddeev, “How Algebraic Bethe Ansatz works for integrable model”, hep-th/9605187, in: “Relativistic gravitation and gravitational radiation”, ed.: J.-A. Marck and J.-P. Lasota, Cambridge University Press (1997), Cambridge.
- [19] L. N. Lipatov, “Duality symmetry of Reggeon interactions in multicolour QCD”, Nucl. Phys. B548, 328 (1999), hep-ph/9812336.

- [20] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, “*Stringing Spins and Spinning Strings*”, JHEP 0309, 010 (2003), hep-th/0306139. • N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, “*Precision Spectroscopy of AdS/CFT*”, JHEP 0310, 037 (2003), hep-th/0308117.
- [21] D. Berenstein, J. M. Maldacena and H. Nastase, “*Strings in flat space and pp waves from  $\mathcal{N} = 4$  Super Yang Mills*”, JHEP 0204, 013 (2002), hep-th/0202021. • S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “*A semi-classical limit of the gauge/string correspondence*”, Nucl. Phys. B636, 99 (2002), hep-th/0204051. • S. Frolov and A. A. Tseytlin, “*Semiclassical quantization of rotating superstring in  $AdS_5 \times S^5$* ”, JHEP 0206, 007 (2002), hep-th/0204226.
- [22] N. Beisert and A. A. Tseytlin, “*On Quantum Corrections to Spinning Strings and Bethe Equations*”, Phys. Lett. B629, 102 (2005), hep-th/0509084.
- [23] N. Beisert, “*The Dilatation Operator of  $\mathcal{N} = 4$  Super Yang-Mills Theory and Integrability*”, Phys. Rept. 405, 1 (2004), hep-th/0407277.
- [24] J. Plefka, “*Spinning strings and integrable spin chains in the AdS/CFT correspondence*”, Living. Rev. Relativity 8, 9 (2005), hep-th/0507136. • G. Arutyunov and S. Frolov, “*Foundations of the  $AdS_5 \times S^5$  Superstring. Part I*”, J. Phys. A42, 254003 (2009), arxiv:0901.4937.
- [25] V. G. Drinfel’d, “*Hopf algebras and the quantum Yang-Baxter equation*”, Sov. Math. Dokl. 32, 254 (1985).
- [26] L. Dolan, C. R. Nappi and E. Witten, “*Yangian symmetry in  $D = 4$  superconformal Yang-Mills theory*”, hep-th/0401243, in: “*Quantum Theory and Symmetries*”, ed.: P. C. Argyres et al., World Scientific (2004), Singapore.
- [27] L. Dolan, C. R. Nappi and E. Witten, “*A Relation Between Approaches to Integrability in Superconformal Yang-Mills Theory*”, JHEP 0310, 017 (2003), hep-th/0308089.
- [28] V. K. Dobrev and V. B. Petkova, “*All positive energy unitary irreducible representations of extended conformal supersymmetry*”, Phys. Lett. B162, 127 (1985).
- [29] B. I. Zwiebel, “ *$\mathcal{N} = 4$  SYM to two loops: Compact expressions for the non-compact symmetry algebra of the  $su(1,1/2)$  sector*”, JHEP 0602, 055 (2006), hep-th/0511109.
- [30] D. Serban and M. Staudacher, “*Planar  $\mathcal{N} = 4$  gauge theory and the Inozemtsev long range spin chain*”, JHEP 0406, 001 (2004), hep-th/0401057. • B. I. Zwiebel, “*Yangian symmetry at two-loops for the  $su(2/1)$  sector of  $\mathcal{N} = 4$  SYM*”, J. Phys. A40, 1141 (2007), hep-th/0610283. • N. Beisert and D. Erkal, “*Yangian Symmetry of Long-Range  $gl(N)$  Integrable Spin Chains*”, J. Stat. Mech. 0803, P03001 (2008), arxiv:0711.4813. • T. Bargheer, N. Beisert and F. Loebbert, “*Boosting Nearest-Neighbour to Long-Range Integrable Spin Chains*”, J. Stat. Mech. 0811, L11001 (2008), arxiv:0807.5081. • T. Bargheer, N. Beisert and F. Loebbert, “*Long-Range Deformations for Integrable Spin Chains*”, J. Phys. A42, 285205 (2009), arxiv:0902.0956.
- [31] F. A. Berends and W. T. Giele, “*Recursive Calculations for Processes with  $n$  Gluons*”, Nucl. Phys. B306, 759 (1988).
- [32] V. P. Nair, “*A current algebra for some gauge theory amplitudes*”, Phys. Lett. B214, 215 (1988).

- [33] S. J. Parke and T. R. Taylor, “*An Amplitude for  $n$  Gluon Scattering*”, Phys. Rev. Lett. 56, 2459 (1986).
- [34] E. Witten, “*Perturbative gauge theory as a string theory in twistor space*”, Commun. Math. Phys. 252, 189 (2004), hep-th/0312171.
- [35] J. M. Drummond, J. Henn, V. A. Smirnov and E. Sokatchev, “*Magic identities for conformal four-point integrals*”, JHEP 0701, 064 (2007), hep-th/0607160. • Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower and V. A. Smirnov, “*The four-loop planar amplitude and cusp anomalous dimension in maximally supersymmetric Yang-Mills theory*”, Phys. Rev. D75, 085010 (2007), hep-th/0610248. • Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower, “*Maximally supersymmetric planar Yang-Mills amplitudes at five loops*”, Phys. Rev. D76, 125020 (2007), arxiv:0705.1864.
- [36] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, “*Dual superconformal symmetry of scattering amplitudes in  $\mathcal{N} = 4$  super-Yang-Mills theory*”, Nucl. Phys. B828, 317 (2010), arxiv:0807.1095.
- [37] A. Brandhuber, P. Heslop and G. Travaglini, “*A note on dual superconformal symmetry of the  $\mathcal{N} = 4$  super Yang-Mills  $S$ -matrix*”, Phys. Rev. D78, 125005 (2008), arxiv:0807.4097.
- [38] J. M. Drummond and J. M. Henn, “*All tree-level amplitudes in  $\mathcal{N} = 4$  SYM*”, JHEP 0904, 018 (2009), arxiv:0808.2475.
- [39] J. M. Drummond, J. M. Henn and J. Plefka, “*Yangian symmetry of scattering amplitudes in  $\mathcal{N} = 4$  super Yang-Mills theory*”, JHEP 0905, 046 (2009), arxiv:0902.2987.
- [40] F. Cachazo, P. Svrcek and E. Witten, “*Gauge theory amplitudes in twistor space and holomorphic anomaly*”, JHEP 0410, 077 (2004), hep-th/0409245. • F. Cachazo, “*Holomorphic anomaly of unitarity cuts and one-loop gauge theory amplitudes*”, hep-th/0410077.
- [41] T. Bargheer, N. Beisert, W. Galleas, F. Loebbert and T. McLoughlin, “*Extracting  $\mathcal{N} = 4$  Superconformal Symmetry*”, JHEP 0911, 056 (2009), arxiv:0905.3738.
- [42] A. Sever and P. Vieira, “*Symmetries of the  $\mathcal{N} = 4$  SYM  $S$ -matrix*”, arxiv:0908.2437.
- [43] N. Beisert, J. Henn, T. McLoughlin and J. Plefka, “*One-Loop Superconformal and Yangian Symmetries of Scattering Amplitudes in  $\mathcal{N} = 4$  Super Yang-Mills*”, JHEP 1004, 085 (2010), arxiv:1002.1733.
- [44] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, “*Generalized unitarity for  $\mathcal{N} = 4$  super-amplitudes*”, arxiv:0808.0491.
- [45] A. Brandhuber, P. Heslop and G. Travaglini, “*One-Loop Amplitudes in  $\mathcal{N} = 4$  Super Yang-Mills and Anomalous Dual Conformal Symmetry*”, JHEP 0908, 095 (2009), arxiv:0905.4377. • H. Elvang, D. Z. Freedman and M. Kiermaier, “*Dual conformal symmetry of 1-loop NMHV amplitudes in  $\mathcal{N} = 4$  SYM theory*”, JHEP 1003, 075 (2010), arxiv:0905.4379. • A. Brandhuber, P. Heslop and G. Travaglini, “*Proof of the Dual Conformal Anomaly of One-Loop Amplitudes in  $\mathcal{N} = 4$  SYM*”, JHEP 0910, 063 (2009), arxiv:0906.3552.
- [46] I. Bena, J. Polchinski and R. Roiban, “*Hidden symmetries of the  $AdS_5 \times S^5$  superstring*”, Phys. Rev. D69, 046002 (2004), hep-th/0305116.
- [47] A. Torrielli, review of Yangians in AdS/CFT, to appear.