Group Field Theory with Noncommutative Metric Variables

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We introduce a dual formulation of group field theories as a type of noncommutative field theories, making their simplicial geometry manifest. For Ooguri-type models, the Feynman amplitudes are simplicial path integrals for BF theories. We give a new definition of the Barrett-Crane model for gravity by imposing the simplicity constraints directly at the level of the group field theory action.

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Group field theories [1] (GFTs) are developing into a promising formalism for quantum gravity, combining elements from several approaches [2]. They are a higher-dimensional generalization of matrix models for 2D gravity and build up on the achievements of loop quantum gravity [3] and spin foam models [4]. Loop quantum gravity describes quantum space in terms of spin networks; spin foam models define its dynamics in a covariant language. GFTs subsume this dynamics, as every spin foam model can be interpreted as a GFT Feynman amplitude.

Several results in spin foam models, and the historic roots in matrix models, suggest a close relation between GFTs and simplicial gravity path integrals, as used in other discrete approaches [5]. The spin foam quantization is based on the geometric quantization of simplicial geometry [4,6], and there are close relations between simplicial and loop gravity canonical data [7]. The resulting amplitudes contain the Regge action in a semiclassical limit [8] and can be analyzed from a path integral perspective [9]. However, much remains to be understood, and GFTs, which already realize a duality between spin foam models and lattice path integrals in connection variables, seem a convenient setting to do so.

In parallel, interesting connections between GFT and spin foam models and noncommutative geometry have been discovered. Noncommutative matter field theories, interesting for quantum gravity phenomenology [10], can be derived either from coupling particles to spin foam amplitudes [11] or from GFT actions, as perturbations around classical GFT solutions [12]. These results suggest that noncommutative structures lie hidden at the very foundations of the GFT formalism.

In this Letter, we recast GFTs as nonlocal, noncommutative field theories on Lie algebras, which we relate to the B variables of simplicial BF theory. We prove that the Feynman amplitudes for arbitrary diagram are simplicial BF path integrals. This new representation of GFTs gives an explicit duality between spin foam models and simplicial gravity path integrals and clarifies the encoding of simplicial geometry in the action. We illustrate this by giving a new GFT definition of the Barrett-Crane model.

Noncommutative representation of 3D GFT.—3D GFTs are defined [13] in terms of fields $\varphi_{123} := \varphi(g_1, g_2, g_3)$ on $\mathrm{SO}(3)^3$ satisfying the invariance

$$\varphi(g_1, g_2, g_3) = \varphi(hg_1, hg_2, hg_3) \tag{1}$$

$\forall\ h \in \mathrm{SO}(3)$. The dynamics is governed by the action

$$S = \frac{1}{2}\int [dg]^3 \varphi_{123}^2 - \frac{\lambda}{4!}\int [dg]^6 \varphi_{123}\varphi_{345}\varphi_{526}\varphi_{641}. \tag{2}$$

The Feynman graphs generated by this theory are two-complexes dual to 3D triangulations: The combinatorics of the field arguments in the interaction vertex is that of a tetrahedron, while the kinetic term dictates the gluing rule for tetrahedra along triangles. The Peter-Weyl theorem gives an expansion of φ in terms of functions on irreducible representations $j_i \in \mathbb{N}$ of SO(3). In such a representation, the field is pictured as a 3-valent spin network vertex and interpreted as a quantized triangle; a generic Feynman amplitude gives the well-known Ponzano-Regge spin foam model [13].

We now introduce an alternative formulation of the model, obtained by a "group Fourier transform" [11,14] mapping functions on a group G to (noncommutative) functions on its Lie algebra $\mathfrak{g}$. The transform is based on plane waves $e_g(x) = e^{i\vec{p}_g\cdot\vec{x}}$, labeled by $g \in G$, as functions on $\mathfrak{g} \sim \mathbb{R}^n$, depending on a choice of coordinates $\vec{p}_g$ on the group manifold. In what follows, we will identify functions of SO(3) with functions of SU(2) invariant under $g \to -g$.

We choose the coordinates $\vec{p}_g = \mathrm{Tr}(|g|\vec{\tau})$, where $|g| := \mathrm{sgn}(\mathrm{Tr}g)g$, $\vec{\tau}$ are i times the Pauli matrices, and "Tr" is the trace in the fundamental representation. For $x = \vec{x}\cdot\vec{\tau}$ and $g = e^{\theta\vec{n}\cdot\vec{\tau}}$, we thus have

$$e_g(x) = e^{i\,\mathrm{Tr}(x|g|)} = e^{-2i\sin\theta\vec{n}\cdot\vec{x}}. \tag{3}$$

The Fourier transform of functions f on SU(2) is defined by

$$\hat{f}(x) = \int dg f(g) e_g(x), \tag{4}$$

where dg is the normalized Haar measure.

The image of the Fourier transform inherits an algebra structure from the convolution product on the group, given by the $\star$ product defined on plane waves as

$$e_{g_1} \star e_{g_2} = e_{g_1 g_2}. \tag{5}$$

On functions of SO(3), the Fourier is invertible:

$$f(g) = \frac{1}{\pi} \int d^3x (\hat{f} \star e_{g^{-1}})(x). \tag{6}$$

With a bit more work, the above construction is extended to an invertible SU(2) Fourier transform [14].

The Fourier transform and $\star$ product extend to functions of several variables like the Boulatov field as

$$\begin{aligned} \hat{\varphi}_{123} &:= \hat{\varphi}(x_1, x_2, x_3) \\ &= \int [dg]^3 \varphi_{123} e_{g_1}(x_1) e_{g_2}(x_2) e_{g_3}(x_3). \end{aligned} \tag{7}$$

The first feature of the dual formulation is that the constraint (1) acts on dual fields as a "closure constraint" for the variables x_j. Indeed, given the projector $P\varphi_{123} := \int dh \varphi(hg_1, hg_2, hg_3)$ onto gauge invariant fields, a simple calculation gives

$$\widehat{P\varphi} = \hat{C} \star \hat{\varphi}, \qquad \hat{C}(x_1, x_2, x_3) = \delta_0(x_1 + x_2 + x_3), \tag{8}$$

where δ_0 is the element $x = 0$ of the family of functions:

$$\delta_x(y) := \frac{1}{\pi} \int_{\mathrm{SO}(3)} dh e_{h^{-1}}(x) e_h(y). \tag{9}$$

These play the role of Dirac distributions in the noncommutative setting, in the sense that

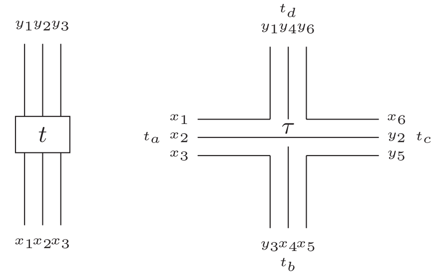
$$\int d^3y (\delta_x \star f)(y) = \int d^3y (f \star \delta_x)(y) = f(x). \tag{10}$$

We may thus interpret the variables of the Boulatov dual field as the edge vectors of a triangle in $\mathbb{R}^3$ and the dual fields themselves as (noncommutative) triangles.

Since the $\star$ product is dual to group convolution, the combinatorial structure of the action in terms of the dual field matches the one in (2). We may thus show that

$$\begin{aligned} S = \frac{1}{2} \int [dx]^3 \hat{\varphi}_{123} \star \hat{\varphi}_{123} - \frac{\lambda}{4!} \int [dx]^6 \hat{\varphi}_{123} \star \hat{\varphi}_{345} \\ \star \hat{\varphi}_{526} \star \hat{\varphi}_{641}, \end{aligned} \tag{11}$$

where it is understood that $\star$ products relate repeated indices as $\phi_i \star \phi_i := (\phi \star \phi_-)(x_i)$, with $\phi_-(x) := \phi(-x)$. The structure of this action is best visualized in terms of diagrams. Thus, kinetic and interaction terms identify a propagator and a vertex given by



(12)

$$\begin{aligned} &\int dh_t \prod_{i=1}^{3} (\delta_{-x_i} \star e_{h_t})(y_i), \\ &\int \prod_t dh_{t\tau} \prod_{i=1}^{6} (\delta_{-x_i} \star e_{h_{tt'}})(y_i), \end{aligned} \tag{13}$$

with $h_{tt'} := h_{t\tau} h_{\tau t'}$. We have used "$t$" for triangle and "$\tau$" for tetrahedron. The group variables h_t and $h_{t\tau}$ arise from gauge invariance (1).

The integrands in (13) factorize into a product of functions associated to strands (one for each field argument), with a clear geometrical meaning. Just like in the standard group representation [1], the group elements h_t and $h_{t\tau}$ are interpreted as parallel transports through the triangle t and from the center of the tetrahedron τ to triangle t, respectively. The pair of variables (x_i and y_i) associated to the same edge i corresponds to the edges vectors seen from the frames associated to the two triangles t and t' sharing it. The vertex functions state that the two variables are identified, up to parallel transport $h_{tt'}$ and up to a sign labeling the two opposite edge orientations inherited by the triangles t and t'. The propagator encodes a similar gluing condition, allowing for the possibility of a further mismatch between the reference frames associated to the same triangle in different tetrahedra.

Feynman amplitudes as simplicial path integrals.—In building up a closed Feynman graph, propagator and vertex strands are joined to one another by using the $\star$ product, keeping track of the ordering of functions associated to the various building blocks of the graph. Each loop of strands bound a face of the two-complex, which is dual to an edge of the triangulation.

Under the integration over the group variables h_t and $h_{t\tau}$, the amplitude factorizes into a product of *face* amplitudes. Let f_e be a face of the two-complex, dual to an edge e in the triangulation, and consider the loop of strands that bound it. The choice of an orientation and a reference vertex defines an ordered sequence $\{\tau_j\}_{0 \leq j \leq N}$ of vertices on the loop (equivalently, an ordered set of tetrahedra around e). By using (10), each vertex τ_j, after contraction with the propagator t_j joining τ_j and τ_{j+1}, contributes with $(\delta_{x_j} \star e_{h_{jj+1}})\ (x_{j+1})$ to the face amplitude, where $h_{jj+1} = h_{\tau_j t_j} h_{t_j} h_{t_j \tau_{j+1}}$ parallel transports j to $j + 1$.

The face amplitude $A_{f_e}[h]$ is then the cyclic $\star$ product of all these contributions:

$$A_{f_e}[h] = \int \prod_{j=0}^{N} d^3x_j \overset{\rightarrow}{\underset{j=0}{\overset{N+1}{\bigstar}}} (\delta_{x_j} \star e_{h_{jj+1}})(x_{j+1}), \qquad (14)$$

where $x_{N+1} := x_0$. This amplitude encodes the identification, up to parallel transport, of the metric variables associated to e in different tetrahedron frames. Integrating over all metric variables x_j in $A_{f_e}[h]$, except for that of the reference frame, we obtain the Feynman amplitude:

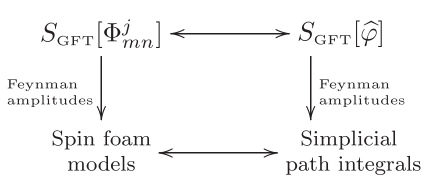
$$Z(\Gamma) = \frac{1}{\pi^{|e|}} \int \prod_t dh_t \prod_e d^3x_e e^{i\sum_e \mathrm{Tr} x_e H_e}, \qquad (15)$$

where h_t is the parallel transport between the two tetrahedra sharing t, H_e is the holonomy around the boundary of f_e, computed from a given tetrahedron, and $|e|$ is the number of edges of the triangulation.

Equation (15) is the usual expression for the simplicial path integral of first order 3D gravity. The Lie algebra variables x_e, one per edge of the simplicial complex, play the role of discrete triad; the group elements h_t, one per triangle or link of the dual two-complex, play the role of discrete connection, defining the discrete curvature H_e through holonomy around the faces dual to the edges of the simplicial complex.

Open GFT Feynman amplitudes have fixed boundary simplicial data. The one-vertex contribution to the 4-point functions, for example, is the function of 12 metric variables x_i and x_i^l obtained by acting with a closure operator $\hat{C}$ (propagator) on each external 3-stranded leg of the vertex diagram in (12), building up four triangles. The amplitude is the $\star$ product of functions $\delta_{-x_i^l}(x_i)$ of the boundary metric with the BF action $e^{i\sum_i \mathrm{Tr} x_i h_i}$ for a single simplex, where $h_i = h_{t\tau} h_{\tau t'}$ is the parallel transport between the two triangles sharing i. The amplitude of generic open graphs is then given by a path integral for the BF action augmented by the appropriate boundary terms. Note that the BF action for a single simplex is already explicitly present in the GFT action. This can be useful to study the link with semiclassical or continuum gravity at the GFT level.

These results show an exact duality between spin foam models and simplicial gravity path integrals, stemming from two equivalent representations of the GFT field: as a function Φ^j_{mn} of representation labels, obtained by harmonic analysis, and as a noncommutative function $\hat{\varphi}$ of Lie algebra variables, interpreted as metric variables:



Towards 4D gravity models.—4D GFTs for SO(4) BF theory are defined in terms of fields $\varphi_{1234} = \int dh \varphi(hg_1, hg_2, hg_3, hg_4)$ by the action

$$S = \frac{1}{2} \int \varphi_{1234}^2 - \frac{\lambda}{5!} \int \varphi_{1234}\varphi_{4567}\varphi_{7389}\varphi_{96210}\varphi_{10851}. \quad (16)$$

The Feynman graphs are two-complexes dual to 4D simplicial complexes: The combinatorics of the interaction term is that of a four-simplex; the kinetic terms dictates the gluing rules for four-simplices along tetrahedra. By using harmonic analysis on SO(4), the Feynman amplitudes take the form of the Ooguri state sum model.

The SO(3) group Fourier transform naturally extends to a Fourier transform on $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2$, which is invertible on even functions $f(g) = f(-g)$. In what follows, we assume the further invariance of the Ooguri field under $g_i \to -g_i$ in each of the variables.

The dual Ooguri field is a function of four $\mathfrak{so}(4)$ Lie algebra elements, or bivectors, associated to the four triangles of each tetrahedron. Gauge invariance translates into a closure constraint for the bivectors, meaning that the four triangles close to form a tetrahedron. Kinetic and vertex terms encode the identification, up to parallel transport, of the bivectors associated to the same triangle in different tetrahedral frames. As in 3D, Feynman amplitudes are simplicial path integrals for BF theory.

The new representation of the Ooguri model provides a convenient starting point for imposing in a geometrically transparent manner the discrete simplicity constraint that turn BF theory into 4D simplicial gravity [6]. Using the decomposition of $x \in \mathfrak{so}(4)$ into self-dual x^+ and anti-self-dual x^- $\mathfrak{su}(2)$ components, we impose that the four bivectors in each tetrahedron are orthogonal to the same vector $k \in \mathcal{S}^3 \sim \mathrm{SU}(2)$ normal to the tetrahedron, by means of the constraint projector

$$\hat{S}_k(x_j^-, x_j^+) = \prod_{j=1}^{4} \delta_{-kx_j^-k^{-1}}(x_j^+), \qquad (17)$$

where the δ functions are given by (9). One can show that $\hat{S}_k$ acts dually as the projector onto fields on the homogeneous space $\mathcal{S}^3 \sim \mathrm{SO}(4)/\mathrm{SO}(3)_k$, where $\mathrm{SO}(3)_k$ is the stabilizer of k. The case $k = 1$ reproduces the standard Barrett-Crane projector.

By combining the simplicity projector $\hat{S} := \hat{S}_1$ with closure, one may build up the field $\hat{\Psi} := \hat{S} \star \hat{C} \star \hat{\varphi}$ of the standard GFT formulation of the Barrett-Crane model. More precisely, combining the interaction term

$$\frac{\lambda}{5!} \int \hat{\Psi}_{1234} \star \hat{\Psi}_{4567} \star \hat{\Psi}_{7389} \star \hat{\Psi}_{96210} \star \hat{\Psi}_{10851} \qquad (18)$$

with the possible kinetic terms

$$\frac{1}{2} \int \hat{\Psi}_{1234}^{\star 2}, \qquad \frac{1}{2} \int (\hat{C} \star \hat{\varphi})_{1234}^{\star 2}, \qquad \text{or} \qquad \frac{1}{2} \int \hat{\varphi}_{1234}^{\star 2} \quad (19)$$

gives the versions of the Barrett-Crane model derived in Refs. [9,15,16], respectively. The origin of these different versions can be understood geometrically, thanks to the new GFT representation. Given $h \in \mathrm{SO}(4)$, one has

$$(e_h \star \hat{S}_k)(x) = (\hat{S}_{h \rhd k} \star e_h)(x), \tag{20}$$

with $h \rhd k := h^+ k (h^-)^{-1}$. This expresses the fact that, after rotation by h, simple bivectors with respect to the normal k become simple with respect to the rotated normal $h \rhd k$. Therefore closure and simplicity constraints do not commute. Moreover, whereas the model couples correctly the bivector variables x across simplices, the integration over holonomies effectively decorrelates the normal vectors k associated to the same tetrahedron in different four-simplices. This implies a missing geometric condition on connection variables $h_{\tau\sigma}$. Work on a GFT model where simplicity constraints are imposed covariantly is currently in progress.

A simplicial path integral formulation of the Barrett-Crane model, in, say, its version [9], is obtained by using the Feynman rules for the propagator and vertex:

$$\prod_{i=1}^{4} (\delta_{-x_i})(y_i), \quad \int \prod_t dh_{\tau\sigma} \prod_{i=1}^{10} (\delta_{-x_i} \star \hat{S} \star e_{h_{\tau\tau'}})(y_i). \tag{21}$$

The amplitude of a graph dual to a triangulation Δ takes the form of the $\star$ evaluation of a noncommutative observable in BF theory:

$$Z_{\mathrm{BC}}(\Delta) = \int \prod_{\tau\sigma} dh_{\tau\sigma} \int \prod_t d^6 x_t (\mathcal{O}_t \star e_{H_t})(x_t), \tag{22}$$

where the functions $\mathcal{O}_t(x_t)$ implement simplicity $\delta_{-h_{0j}^{-\,-1} x_t^- h_{0j}^-}\ (h_{0j}^{+\,-1} x_t^+ h_{0j}^+)$ of the bivectors x_t in each of the four-simplex frames $j = 0 \ldots N$ around t:

$$\mathcal{O}_t = \mathop{\bigstar}_{j=0}^{N} \delta_{-h_{0j}^{-\,-1} x_j^- h_{0j}^-}(h_{0j}^{+\,-1} x_j^+ h_{0j}^+). \tag{23}$$

Conclusions and perspectives.—The new noncommutative representation of GFTs introduced in this Letter, based on the group Fourier transform, realizes an explicit GFT duality between spin foam models and simplicial gravity path integrals. It also makes explicit how simplicial geometry is encoded in the GFT formalism.

The interpretation of GFTs as a 2nd quantization of spin networks suggests to apply the group Fourier transform to generic loop quantum gravity states. This should give a *flux* representation of the theory, usually assumed to be intractable precisely because of the noncommutativity of flux operators.

The new representation should also help the identification of spacetime symmetries (e.g., diffeomorphisms) which act on the B variables, at the level of the GFT action. Understanding the role of diffeomorphisms can then guide the study of the relation between GFTs and continuum general relativity.

Obviously, the goal is the construction of a satisfactory GFT model for quantum gravity in 4 dimensions. In the new GFT representation, guided by the manifest geometric meaning of variables and amplitudes, simplicity constraints on the B variables, with and without an Immirzi parameter, can be imposed in a natural way. This is work in progress and can lead either to a new spin foam model for 4D quantum gravity or to a geometrically clear GFT formulation of the recently proposed ones. It can also, in one stroke, give a reformulation of these models as simplicial path integrals.

The new representation may help also the study of GFT renormalization [17,18] and that of their phase structure and continuum approximation [19]. It can also be used for the introduction of scales by reexpressing the star product in terms of differential operators [14].

Finally, it should reinforce the links between the GFT formalism and noncommutative geometry, as well as the approach to quantum gravity phenomenology [10] based on effective noncommutative matter field theories.

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