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Thermodynamic instability of doubly spinning black objects

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ABSTRACT

We investigate the thermodynamic stability of neutral black objects with (at least) two angular momenta. We use the quasilocal formalism to compute the grand canonical potential and show that the doubly spinning black ring is thermodynamically unstable. We consider the thermodynamic instabilities of ultra-spinning black objects and point out a subtle relation between the microcanonical and grand canonical ensembles. We also find the location of the black string/membrane phases of doubly spinning black objects.

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1 Introduction

The physics of event horizons in higher-dimensional general relativity is an interesting area of research not just for its intrinsic relevance to string theory. An investigation of black hole solutions in higher dimensions is also important because it has revealed new intriguing aspects.

It is clear by now that some of the remarkable properties of four-dimensional black holes do not hold in general. A notorious example of particular importance concerns their horizon topology. In four dimensions, the spherical topology (S^2) is the only allowed horizon topology (for asymptotically flat black holes). A related result is the 'uniqueness theorem', which states that a vacuum black hole in four dimensions is characterized by its mass and angular momentum and has no other independent characteristic (hair).

The spectrum of black objects is far richer in dimensions bigger than four (see, e.g., [1] for a concise review) and, consequently, the notion of uniqueness is very much weaker. The most obvious indication supporting this point is provided by the existence of an asymptotically flat solution describing a spinning black ring in five dimensions [2] (whose horizon topology is $S^2 \times S^1$).

The Euclidean approach was applied to the black ring thermodynamics for the first time in [3]. Since the black ring does not have a real non-singular Euclidean section, the 'quasi-Euclidean' method [4] was adopted to analyze the black ring thermodynamics. In this approach, the horizon is described by the 'bolt' in a *complexified* Euclidean geometry (rather than a real one).[1]

It was also pointed out in [3] that the neutral black ring with one angular momentum is unstable to angular fluctuations — a more detailed analysis can be found in [5, 6, 7]. A natural question one would like to answer is if a second angular momentum will change the situation, in other words if a neutral doubly spinning black ring is thermodynamically stable.

Understanding the thermodynamics of the doubly spinning black ring is important from a different perspective. It was observed in [8] that, in dimensions greater than five, the thermodynamics of the (ultra-)spinning black holes show a qualitative change in behaviour.

Even at a relatively low value of the angular momentum, there is a transition toward a black membrane-like behaviour. This is due to the fact that the temperature has a minimum: after the temperature reaches a minimum, it starts growing as expected for the black membrane.[2]

In the 'thin ring approximation' (that resembles the ultra-spinning regime for black holes), the radius of S^1 is much larger than the radius of S^2 and so the black ring is approximated by a boosted black string (black 1-brane). Another interesting question we would like to address in this paper is how the ultra-spinning regime of the black ring is affected by adding the second angular momentum.

Remarkably, Pomeransky and Sen'kov managed to find the exact solution [11] for a *balanced* (neutral) doubly spinning black ring. Some properties of the solution including the structure of the phases (in the microcanonical ensemble) are discussed in [12]. A study of its geodesics has been performed in [13] and a careful investigation of global properties appeared recently in [14].

Within the quasilocal formalism [15] there exist all ingredients necessary to study in detail the thermodynamics of the doubly spinning ring. Supplemented with counterterms [16], the quasilocal formalism becomes a very powerful tool to study the thermodynamics of black objects that are asymptotically flat (several concrete five-dimensional examples were discussed in detail in [7, 17]).

For thin black ring solutions one can use the results for black strings to study the thermodynamics. However, it was emphasized in [18] that the quasilocal stress tensor can

[1] The complex geometry is obtained by the usual analytic continuation of time coordinate, $\tau = it$.

[2] Recently, it was understood [9] that it corresponds to a zero mode associated with the onset of the Gregory-Laflamme instability [10].

be used to study any general (thin or fat) black ring solution (even the unbalanced black ring).

In this paper, we use the quasilocal formalism supplemented with counterterms (counterterm method) to investigate the thermodynamic stability of a neutral doubly spinning black ring. The second angular momentum changes, indeed, the situation in the sense that all response functions can be now positive definite in some regions of the parameter space.[3]

Unlike the black ring with one angular momentum, the doubly spinning ring is stable against perturbations in the angular velocity in some specific region of the parameter space. However, a careful analysis of all response functions that characterize the system reveals that the doubly spinning black ring is thermodynamically *unstable* in the grand canonical ensemble. That is, there is no overlap region in the parameter space in which all response functions are positive definite.

We also present a careful analysis of the ultra-spinning regime for both, the black hole and black ring. We show that there is a similar ultra-spinning limit for the black ring. However, there is a key difference in this case. That is, the temperature does not have a minimum but rather there is a 'turning point', which is responsible for the (boosted) black membrane behaviour.

Interestingly enough, we will see that even after adding the second angular momentum the ring can have a membrane-like behaviour. However, for large enough values of the second angular momentum the 'membrane phase' will disappear.

The remainder of this paper is organized as follows: we start in Section 2 with a brief review of the counterterm method for asymptotically flat spacetimes. We then compute the stress tensor and the corresponding asymptotic charges for black objects with two angular momenta: Myers-Perry black hole, black ring, and black branes. In Section 3, we present an analysis of the thermodynamic stability of the doubly spinning ring. We compute the thermodynamic action and check the quantum statistical relation. We also analyze in great detail the response functions. In Section 4, we investigate the black string/membrane phases of (doubly) spinning black objects. Finally, we conclude with a discussion of our results. In Appendix A we present the expressions of angular velocities and temperature for a general metric with two angular momenta. Appendix B contains some general aspects of black holes thermodynamics, the local stability conditions, and concrete expressions for some of the response functions used in Section 2.

2 Stress tensor and conserved charges

In this section we apply the counterterm method to doubly spinning five-dimensional vacuum solutions of Einstein gravity. We explicitly show how to compute the boundary stress

[3]For a black ring with one angular momentum, the 'isothermal compressibility' (moment of inertia) is always negative.

tensor and the conserved charges for Myers-Perry black hole, doubly spinning black ring, and doubly spinning black branes.

2.1 Quasilocal formalism and conserved charges

To begin our considerations on thermodynamics of doubly spinning black objects in five dimensions, we recall the description of quasilocal formalism [15] supplemented with counterterms.

To define the conserved charges we use the divergence-free boundary stress tensor proposed in [3][4]:

$$\tau_{ij} \equiv \frac{2}{\sqrt{|h|}}\frac{\delta I}{\delta h^{ij}} = \frac{1}{8\pi G_5}\Big(K_{ij} - h_{ij}K - \Psi(\mathcal{R}_{ij} - \mathcal{R}\, h_{ij}) - h_{ij}\Box\Psi + \Psi_{;ij}\Big) \tag{2.1}$$

where $\Psi = \sqrt{3}/\sqrt{2\mathcal{R}}$, h_{ij} is the induced boundary metric, and $\mathcal{R}_{ij}$ is its Ricci scalar.

Here, I is the *renormalized* action that includes counterterms,

$$I \;=\; \frac{1}{16\pi G_5}\int_M R\sqrt{-g}\, d^5x + \frac{\epsilon}{8\pi G_5}\int_{\partial M}\left(K - \sqrt{\frac{3\,\mathcal{R}}{2}}\right)\sqrt{|h|}\, d^4x \tag{2.2}$$

K is the extrinsic curvature of ∂M and $\epsilon = +1(-1)$ if ∂M is timelike (spacelike).

The boundary metric can be written locally in ADM-like form

$$h_{ij}dx^i dx^j = -N^2\, dt^2 + \sigma_{ab}\,(dy^a + N^a\, dt)(dy^b + N^b\, dt) \tag{2.3}$$

where N and N^a are the lapse function and the shift vector respectively and $\{y^a\}$ are the intrinsic coordinates on a (closed) hypersurface Σ.

If the boundary geometry has an isometry generated by a Killing vector ξ^i, a conserved charge

$$\mathfrak{Q}_\xi = \oint_\Sigma d^3y\,\sqrt{\sigma} n^i \tau_{ij}\,\xi^j \tag{2.4}$$

can be associated with the hypersurface Σ (with normal n^i).

2.2 Doubly spinning solutions

2.2.1 Black hole

The Einstein equations in higher dimensions have spinning black hole solutions [22]. In five dimensions, the Myers-Perry black hole in Boyer-Lindquist type coordinates is

$$\begin{aligned} ds^2_{BH} \;&=\; -dt^2 + \Sigma\left(\frac{r^2}{\Delta}dr^2 + d\theta^2\right) + (r^2+a^2)\,\sin^2\theta\, d\phi^2 + (r^2+b^2)\,\cos^2\theta\, d\psi^2 \\ &\quad + \frac{m}{\Sigma}\left(dt - a\,\sin^2\theta\, d\phi - b\,\cos^2\theta\, d\psi\right)^2 \end{aligned} \tag{2.5}$$

[4] A rigorous justification and more details about this proposal can be found in [19, 20, 21].

where

$$\Sigma = r^2 + a^2 \cos^2\theta + b^2 \sin^2\theta, \qquad \Delta = (r^2+a^2)(r^2+b^2) - m\, r^2 \tag{2.6}$$

and m is a parameter related to the physical mass of the black hole, while the parameters a and b are associated with its two independent angular momenta. This metric depends only on two coordinates, $0 < r < \infty$ and $0 \le \theta \le \pi/2$, and it is independent of time, $-\infty < t < \infty$, and the azimuthal angles, $0 < \phi, \psi < 2\pi$.

The event horizon of the black hole can be computed by using (A.5), which implies $\Delta = 0$.[5] The largest root of this equation gives the radius of the black hole's outer event horizon

$$r_h^2 = \frac{1}{2}\left(m - a^2 - b^2 + \sqrt{(m-a^2-b^2)^2 - 4\, a^2\, b^2}\right) \tag{2.7}$$

Notice that the horizon exists if and only if

$$a^2 + b^2 + 2|a\, b| \le m \tag{2.8}$$

so that the condition $m = a^2 + b^2 + 2|a\, b|$ or, equivalently, $r_h^2 = |a\, b|$ defines the extremal horizon of a five dimensional black hole (when one angular momentum vanishes, the horizon area goes to zero in the extremal limit). Otherwise, the metric describes a naked singularity.

In the asymptotic limit, $r \to \infty$, the metric (2.5) approaches Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2) \tag{2.9}$$

We use the expression of black hole metric in Boyer-Lindquist coordinates to compute the boundary stress tensor and we obtain the following non-vanishing components:

$$\begin{aligned}
\tau_{tt} &= \frac{1}{8\pi G_5}\left(-\frac{3}{2}\, m\, \frac{1}{r^3} - \frac{5}{3}(a^2-b^2)\frac{\cos 2\theta}{r^3} + \mathcal{O}(1/r^5)\right), \\
\tau_{t\phi} &= \frac{1}{8\pi G_5}\left(-2\, a\, m \frac{\sin^2\theta}{r^3} + \mathcal{O}(1/r^5)\right), \\
\tau_{t\psi} &= \frac{1}{8\pi G_5}\left(-2\, b\, m \frac{\cos^2\theta}{r^3} + \mathcal{O}(1/r^5)\right), \\
\tau_{\theta\theta} &= \frac{1}{8\pi G_5}\left(\frac{2}{3}\left(a^2-b^2\right)\frac{\cos 2\theta}{r} + \mathcal{O}(1/r^3)\right), \\
\tau_{\phi\phi} &= \frac{1}{8\pi G_5}\left(\frac{2}{3}\left(a^2-b^2\right)\frac{(-1+2\cos 2\theta)\, \sin^2\theta}{r} + \mathcal{O}(1/r^3)\right), \\
\tau_{\psi\psi} &= \frac{1}{8\pi G_5}\left(\frac{2}{3}\left(a^2-b^2\right)\frac{(1+2\cos 2\theta)\, \cos^2\theta}{r} + \mathcal{O}(1/r^3)\right), \\
\tau_{\phi\psi} &= \frac{1}{8\pi G_5}\left(-4\, a\, b\, m \frac{\cos^2\theta\, \sin^2\theta}{r^3} + \mathcal{O}(1/r^5)\right)
\end{aligned} \tag{2.10}$$

[5]Since r is playing the role of a radial coordinate in this coordinate system, the event horizon is also the null surface determined by the equation $g^{rr} = 0$.

This stress tensor is covariantly conserved with respect to the boundary metric (2.9). We also notice that, for equal angular momenta, the diagonal 'angular' components of the stress tensor vanish — this is intuitively expected due to the enhanced symmetry.

Using the definition (2.4), it is straightforwardly to obtain the conserved charges associated with the surface Σ as

$$M = \oint_{\Sigma} d^3y\sqrt{\sigma}n^i\,\tau_{ij}\,\xi_t^j\,, \qquad J_\phi = \oint_{\Sigma} d^3y\sqrt{\sigma}n^i\,\tau_{ij}\,\xi_\phi^j\,, \qquad J_\psi = \oint_{\Sigma} d^3y\sqrt{\sigma}n^i\,\tau_{ij}\,\xi_\psi^j$$

where the normalized Killing vectors associated with the mass and angular momenta are $\xi_t = \partial_t$, $\xi_\phi = \partial_\phi$, and $\xi_\psi = \partial_\psi$ respectively. We find

$$M = \frac{3\,\pi\,m}{8\,G_5}\,, \qquad J_\phi = \frac{\pi\,m\,a}{4\,G_5}\,, \qquad J_\psi = \frac{\pi\,m\,b}{4\,G_5} \tag{2.11}$$

which is in perfect agreement with the ADM calculation.

2.2.2 Black ring

A black ring is a five-dimensional black hole with an event horizon of topology $S^1 \times S^2$ and the metric was presented in [2] — the solution of Emparan and Reall has one angular momentum. In five dimensions, a more general solution (with two angular momenta) for a black ring was presented by Pomeransky and Sen'kov [11]. We provide here a brief account of the doubly spinning black ring solution and compute the boundary stress tensor and the conserved charges.

We will use the solution in the form presented in [11]. The metric depends just on the coordinates x and y defined within the following intervals $-1 \leq x \leq 1$ and $-\infty < y < -1$. Notice that x is like an angular coordinate — this observation will be useful when we will define new coordinates that make asymptotic flatness clear.

The metric has a coordinate singularity where g_{yy} diverges. The event horizon of the doubly spinning black ring is located at the smallest absolute value of $1 + \lambda\,y + \nu\,y^2 = 0$, namely

$$y_h = \frac{-\lambda + \sqrt{\lambda^2 - 4\nu}}{2\nu} \tag{2.12}$$

For a regular black ring solution, the parameters ν and λ are constrained to satisfy [11]:

$$0 \leq \nu < 1\,, \qquad 2\sqrt{\nu} \leq \lambda < 1 + \nu \tag{2.13}$$

In the limit $\nu \to 0$ the black ring with one angular momentum (J_ϕ) is recovered (J_ψ is the angular momentum on S^2). The limit $\lambda \to 2\sqrt{\nu}$ was carefully studied in [12] and shown to correspond to regular extremal black rings.

We use a coordinate transformation similar to the one in [12]:

$$x = -1 + 4k^2\,\alpha^2\,\frac{\cos^2\theta}{r^2}\,, \qquad y = -1 - 4k^2\,\alpha^2\,\frac{\sin^2\theta}{r^2}\,, \qquad \alpha = \sqrt{\frac{1+\nu-\lambda}{1-\nu}} \tag{2.14}$$

In these coordinates $\partial_t, \partial_\phi$, and ∂_ψ are Killing vectors and the asymptotic metric is the same as (2.9).

The boundary stress tensor in these new coordinates is

$$\begin{aligned}
\tau_{tt} &= \frac{1}{8\pi G_5}\left(-\frac{12k^2\lambda}{(1+\nu-\lambda)}\frac{1}{r^3} - \frac{8\,k^2 F_1[\nu,\lambda]}{3(1+\nu-\lambda)(1-\nu)^2}\frac{\cos 2\theta}{r^3} + \mathcal{O}(1/r^5)\right) , \\
\tau_{t\phi} &= \frac{1}{8\pi G_5}\left(\frac{16k^3\lambda\,(1+\lambda-6\nu+\lambda\nu+\nu^2)\sqrt{(1+\nu)^2-\lambda^2}}{(1+\nu-\lambda)^2(1-\nu)^2}\frac{\sin^2\theta}{r^3} + \mathcal{O}(1/r^5)\right) , \\
\tau_{t\psi} &= \frac{1}{8\pi G_5}\left(\frac{32k^3\lambda\sqrt{\nu\left[(1+\nu)^2-\lambda^2\right]}}{(1+\nu-\lambda)(1-\nu)^2}\frac{\cos^2\theta}{r^3} + \mathcal{O}(1/r^5)\right) , \\
\tau_{\theta\theta} &= \frac{1}{8\pi G_5}\left(\frac{2k^2F_2[\nu,\lambda]}{3(1+\nu-\lambda)(1-\nu)^2}\frac{\cos 2\theta}{r} + \mathcal{O}(1/r^3)\right) , \qquad (2.15) \\
\tau_{\phi\phi} &= \frac{1}{8\pi G_5}\left(\frac{k^2\,(-F_3[\nu,\lambda]+F_4[\nu,\lambda]\,\cos 2\theta)}{3(1+\nu-\lambda)(1-\nu)^2}\frac{\sin^2\theta}{r} + \mathcal{O}(1/r^3)\right) , \\
\tau_{\psi\psi} &= \frac{1}{8\pi G_5}\left(\frac{k^2\,(F_3[\nu,\lambda]+F_4[\nu,\lambda]\,\cos 2\theta)}{3(1+\nu-\lambda)(1-\nu)^2}\frac{\cos^2\theta}{r} - \mathcal{O}(1/r^3)\right) , \\
\tau_{\phi\psi} &= \frac{1}{8\pi G_5}\left(\frac{128k^4\lambda\sqrt{\nu}\,(4\lambda\nu-(\lambda^2+(1-\nu)^2)(1+\nu))}{(1+\nu-\lambda)(1-\nu)^4}\frac{\cos^2\theta\sin^2\theta}{r^3} + \mathcal{O}(1/r^5)\right)
\end{aligned}$$

where

$$\begin{aligned}
F_1[\nu,\lambda] &= 1-5\nu-\nu^2+5\nu^3+\lambda^2(3+7\nu)+\lambda(1-14\nu-7\nu^2) , \\
F_2[\nu,\lambda] &= 1-11\nu-\nu^2+11\nu^3+\lambda^2(3+13\nu)+4\lambda(1-5\nu-4\nu^2) , \\
F_3[\nu,\lambda] &= 5-7\nu-5\nu^2+7\nu^3+\lambda^2(15+17\nu)-4\lambda(1+13\nu+2\nu^2) , \\
F_4[\nu,\lambda] &= 7-29\nu-7\nu^2+29\nu^3+\lambda^2(21+43\nu)+\lambda(4-92\nu-40\nu^2)
\end{aligned}$$

As in the case of doubly spinning black hole, this stress tensor is covariantly conserved with respect to the boundary metric (2.9). However, since for the doubly spinning black ring the angular momenta can not be equal, there can not be a similar enhanced symmetry in the angular part as in the black hole case.

By plugging the expressions of the boundary stress-energy components (2.15) in (2.4) we find the following expressions for the conserved charges:

$$M = \frac{3\pi\,k^2}{G_5}\frac{\lambda}{1+\nu-\lambda}, \qquad J_\psi = \frac{4\pi\,k^3}{G_5}\frac{\lambda\sqrt{\nu\left[(1+\nu)^2-\lambda^2\right]}}{(1+\nu-\lambda)(1-\nu)^2}, \qquad (2.16)$$

$$J_\phi = \frac{2\pi\,k^3}{G_5}\frac{\lambda\,(1+\lambda-6\nu+\nu\,\lambda+\nu^2)\sqrt{(1+\nu)^2-\lambda^2}}{(1+\nu-\lambda)^2(1-\nu)^2} \qquad (2.17)$$

As expected, the charges computed by using the quasilocal formalism recover correctly the ADM results [11].

In principle, one can obtain a black hole and a black ring with the same conserved charges. However, an asymptotic observer can not distinguish between a black hole and a black ring just by computing the (conserved) asymptotic charges. We would like to emphasize that it is expected that the quasilocal stress tensor (the subleading terms) should encode the information necessary to distinguish between black objects with different horizon topologies.

2.2.3 Black brane

Here we would like to apply the quasilocal formalism to doubly spinning black branes. The black brane (BB) metric we are interested in is obtained by adding flat directions to a 5-dimensional black hole with two angular momenta. Therefore, the metric is

$$ds^2_{BB} = ds^2_{BH} + \sum_{i=1}^{D-5} dx_i^2 \tag{2.18}$$

where ds^2_{BH} is the black hole metric defined in (2.5).

Since the number of dimensions and the topology are changed, one expects changes with respect to the former discussion. For example, the form of the counterterm leading to a finite actions may be different when the number of dimensions is increased. However, in this particular case, what is important is the 'seed' 5-dimensional solution to which we add the flat directions. Thus, the form of the counterterm does not change but the stress tensor will have new components.

A similar computation as for the doubly spinning black hole reveals that the stress tensor of the BB is the one in (2.10) supplemented with the components in the new directions:

$$\tau_{x_i x_i} = \frac{1}{8\pi G_5}\left(-\frac{3}{2}m\,\frac{1}{r^3} - \frac{5}{3}(a^2-b^2)\frac{\cos 2\theta}{r^3} + \mathcal{O}(1/r^5)\right) \tag{2.19}$$

This result resembles the tension (per unit length) of the black string.

3 Thermodynamic instability for black ring

In this section, we discuss the thermodynamics of a doubly spinning ring in the grand canonical ensemble.

So far, we have computed the conserved charges of neutral spinning black objects with two angular momenta by using the quasilocal formalism. However, the quasilocal formalism is a very powerful tool for understanding the thermodynamics in more detail. In particular, one can compute the action and, therefore, the thermodynamic potential.

In what follows, we present a detailed analysis of thermodynamic stability of the doubly spinning black ring — an analysis of the thermodynamic stability of Myers-Perry black hole with two angular momenta can be found in [6].

Let us start by computing the angular velocities and the temperature for this solution. From (A.6) we obtain the following expressions for the angular velocities:

$$\Omega_\psi = \frac{\lambda(1+\nu)-(1-\nu)\sqrt{\lambda^2-4\nu}}{4\lambda\sqrt{\nu}k}\sqrt{\frac{1+\nu-\lambda}{1+\nu+\lambda}}\,, \qquad \Omega_\phi = \frac{1}{2k}\sqrt{\frac{1+\nu-\lambda}{1+\nu+\lambda}} \tag{3.1}$$

The area of the event horizon and the temperature (A.7) are

$$\mathcal{A}_H = \frac{32\pi^2 k^3\,\lambda(1+\lambda+\nu)}{(1-\nu)^2(y_h^{-1}-y_h)}\,, \qquad T = \frac{\sqrt{\lambda^2-4\nu}(1-\nu)(y_h^{-1}-y_h)}{8\pi k\,\lambda(1+\lambda+\nu)} \tag{3.2}$$

Note that $y_h = \frac{-\lambda+\sqrt{\lambda^2-4\nu}}{2\nu}$ is the biggest root of (A.5) which corresponds to the outer event horizon — at this point, it might be useful to emphasize again that $-\infty < y < -1$.

The starting point of the Euclidean approach to black hole thermodynamics is the partition function [23][6]

$$Z(\beta) = \int d[g,\phi]e^{-I[g,\phi]} \tag{3.3}$$

where ϕ is a collective notation for the matter fields, $d[g,\phi]$ is the measure, and $I[g,\phi]$ is the Euclidean classical action. The gravitational partition function is defined by a sum over all *smooth* geometries (including black holes) that are periodic with period $\beta = T^{-1}$ in the same class of boundary conditions (e.g., asymptotically flat spacetimes).

For our purpose it is enough to consider the saddle point approximation. The grand canonical partition function is then $Z = Tre^{-\beta(H-\Omega_a J_a)} \simeq e^{-I_{cl}}$ (here we are interested in black objects with two angular momenta), where I_{cl} is the classical action. The saddle point is usually reffered to as a *gravitational instanton*.[7]

The thermodynamic (effective) potential associated to grand canonical ensemble is

$$G[T,\Omega_a] \equiv \frac{I_{cl}}{\beta} = M - TS - \Omega_a\, J_a \tag{3.4}$$

On the Euclidean section, the topology near the horizon is modified[8] and one has to deal with manifolds with conical singularities. It was shown in [24, 25] that the conical defect has a contribution to the curvature and, consequently, the path integral is rescaled by e^S. However, this can be intuitively interpreted as a consequence of a trace over the macroscopically indistinguishable microstates.

Let us now compute the action for the doubly spinning black ring. Since the Ricci scalar vanishes on-shell, the only contribution to the action is coming from the surface terms. To

[6]It should be understood as a low energy effective theory rather than a proper theory of quantum gravity.

[7]A quantum field can be treated as a small perturbation about the gravitational instanton. The next order contribution, which gives the one loop correction, includes also the thermal radiation outside the black hole.

[8]The origin in the Euclidean spacetime translates to the horizon surface in the Lorentzian spacetime. The Euclidean section can be understood as an effective description where the microstates can not be distinguished.

evaluate these terms, it is convenient to use the (r, θ) coordinate system instead of the (x, y) coordinates — the reason is that the normal to the boundary has just one non-vanishing component. We find

$$\lim_{r\to\infty} \sqrt{|h|}\left(\sqrt{\frac{3}{2}\mathcal{R}} - K\right) = \frac{2k^2\left(\lambda(1-\nu) - F_5[\nu,\lambda]\cos 2\theta\right)\sin 2\theta}{(1+\nu-\lambda)(1-\nu)} + \mathcal{O}(1/r) \quad (3.5)$$

where $F_5[\nu, \lambda] = 1 + 3\lambda^2 + 6\nu + 5\nu^2 - 4\lambda - 8\lambda\nu$. The expression for the total action is

$$I_{cl} = \beta \frac{\pi k^2}{G_5} \frac{\lambda}{(1+\nu-\lambda)} \quad (3.6)$$

and satisfies (3.4) (with $\beta = T^{-1}$), which is the quantum statistical relation for the doubly spinning black ring. This can also be regarded as a non-trivial check that the entropy of this solution is, indeed, one quarter of the event horizon area.

We have checked that the usual thermodynamic relations

$$S = -\left(\frac{\partial G}{\partial T}\right)_{\Omega_a}, \qquad J_a = -\left(\frac{\partial G}{\partial \Omega_a}\right)_{T,\Omega_b} \quad (3.7)$$

are satisfied and so the Gibbs potential $G[T, \Omega_\phi, \Omega_\psi]$ is indeed the Legendre transform of the energy $M[S, J_\phi, J_\psi]$ with respect to S, J_ϕ, and J_ψ.

We want to also point out that, in the light of the new developments in understanding the balance condition for gravity solutions [18], the form of quantum statistical relation hints to the fact that this solution is balanced. Indeed, our results are in perfect agreement with the recent detailed analysis of the global properties of the doubly spinning black ring [14].

Now, we are ready to discuss the thermodynamic stability in the grand canonical ensemble — in Appendix B we summarize the thermal stability conditions and present explicit expressions for some response functions we are interested in. We analyze in detail the response functions that signal the (in)stability of the black ring against fluctuations.

We consider first the specific heat at constant angular velocities

$$C_\Omega \equiv T\left(\frac{\partial S}{\partial T}\right)_{\Omega_\phi,\Omega_\psi} \quad (3.8)$$

The analytic form of this quantity is too complicated to be written down here. Instead, we show on the left hand side in Fig. 1 a scatter region in the parameter space of the doubly spinning black ring where this heat capacity C_Ω is negative (*gray* – 10,000 points). Note also that the parameters in the solution (2.13) are constrained and represented as a dashed line for $\lambda = 1 + \nu$ and solid line for the extremal black ring with $\lambda = 2\sqrt{\nu}$.

In a similar way, we explore the region where the specific heat at constant angular momentum

$$C_J \equiv T\left(\frac{\partial S}{\partial T}\right)_{J_\phi,J_\psi} \quad (3.9)$$

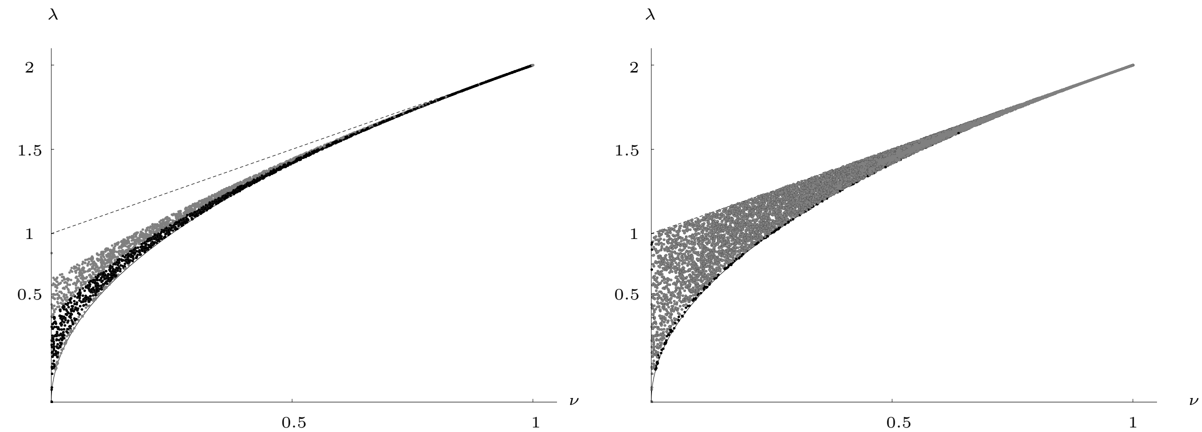

Figure 1: Scatter plots in parameter phase space (ν, λ) for the doubly spinning black ring. The plot on the left shows the regions (10,000 points) where the heat capacities are negative, $C_\Omega < 0$ (*gray*) and $C_J < 0$ (*black*). The regions where the compressibility $\epsilon_{\phi\phi}$ (*gray*) and the det$[\epsilon]$ (*black*) are negative cover the entire parameter space (plot on the right) implying the local thermal instability of the doubly spinning black ring. The region in the parameter space is bounded: $0 \leq \nu < 1$ and λ by the functions $1 + \nu$ and $2\sqrt{\nu}$, shown as the *dashed* and *solid* lines respectively.

is negative. The region (in black) where $C_J < 0$ is shown in the scatter plot on the left of Fig. 1.

We observe a region in the parameter space of the doubly spinning black hole where both specific heats, C_Ω and C_J, can be positive simultaneously. However, this condition is not sufficient to draw the conclusion of thermodynamic stability: one should also investigate the matrix of 'isothermal moment of inertia'.

These response functions are defined as

$$\epsilon_{ab} \equiv \left(\frac{\partial J_a}{\partial \Omega_b}\right)_{T, \Omega_{a \neq b}} \tag{3.10}$$

We observe in Fig. 1 that the spectrum of the matrix of isothermal moment of inertia, *spec*$[\epsilon_{ab}]$, is nowhere positive definite in the parameter space.

Since there is no overlap region in which all the response functions of interest are positive definite, we conclude that the doubly spinning black ring is unstable in the grand canonical ensemble.

4 Instabilities from thermodynamics

Many known stationary black holes in higher dimensions present a black string or, more general, black brane phase — we will refer to it as the 'membrane phase'. That is, as the angular momenta are sufficiently increased, the behaviour of some black holes and black rings changes to that of extended black branes and strings (the utra-spinning regime).

In Section 4.1, we deal with the ultra-spinning black holes. From the study of the Gibbs potential's hessian, we show the existence and find the locus of the transition points to the membrane phase. We also argue that there is a subtle relation between the microcanonical and grand canonical ensembles that may be at the basis of some of the results for ultra-spinning black holes discussed recently in [9].

The analysis can be extended to (doubly) spinning black rings. These results are presented in Section 4.2.

4.1 Ultra-spinning black holes

Due to the qualitative changing behaviour of black holes as the dimensions are increased, the authors of [8] have argued that the ultra-spinning black holes — those in $D \geq 6$ dimensions that can have arbitrary large angular momentum per unit mass [22] — become unstable. The transition of these black holes, from behaving like a spherical black hole to behaving like a black membrane as the spin grows, was established to be at the minimun of the temperature. From that point onwards, the temperature increases in a similar way as for the black brane temperature.

The minimum of the temperature where the behaviour of the singly spinning black hole changes is determined as [8]

$$\frac{a^2}{r_h^2} = \frac{D-3}{D-5} \tag{4.1}$$

This result was also obtained by using a different method, namely finding the divergences of the 'Ruppeiner curvature' [26]. It was shown in [27] that, for a singly spinning Myers-Perry black hole, this curvature[9] blows-up exactly at the value (4.1) signalling a thermal instability of the system.

A qualitative understanding of this fact is related to the observation that, as the spin becomes large, the event horizon spreads out in the plane of rotation: it becomes a higher dimensional 'pancake' approaching the geometry of a black brane.

The existence of the ultra-spinning limit resembling black branes has a remarkable consequence. Black branes were shown to be classically unstable [10] so that the ultra-spinning black holes would inherit the Gregory-Laflamme instability. The threshold of the classical instabilities and the connection to the thermal instability as conjectured by [28] (see, also, [29]) requires a linearized analysis of the perturbations of the black hole solutions.

However, the transition to a membrane-like phase of the rapidly spinning black holes can be established from the study of the thermodynamics of the system. The existence and location of the threshold of this regime is signalled by the minimum of the temperature and the maximum angular velocity as functions of the angular momentum.

It was observed in [9] that, for ultra-spinning black holes, this is in tight correspondence with a vanishing eigenvalue of the Hessian of the Gibbs potential. A complete thermodynamic analysis, though, should be based on the full hessian of the thermodynamic potential rather than only a study of the determinant.[10] We will see in the next subsection that the

[9]The Ruppeiner curvature is the scalar curvature of the Hessian matrix of the entropy.

[10]A spinodal is defined as a line separating the regions of stability and instability of a homogenous

mebrane phase of a doubly spinning ring is not signalled by a zero-eigenvalue of the Gibbs potential's hessian.

For ultra-spinning black holes, there is a *direct* relation between (some response functions in) the microcanonical and grand canonical ensembles.[11] To see that, let us compare the expressions of two (particular) response functions in these two ensembles:

$$\left(\frac{\partial^2 S}{\partial J^2}\right)_M = -\frac{1}{T}\left(\frac{\partial \Omega}{\partial J}\right)_M + \frac{\Omega}{T^2}\left(\frac{\partial T}{\partial J}\right)_M \qquad \text{and} \qquad \left(\frac{\partial^2 G}{\partial \Omega^2}\right)_T = -\left(\frac{\partial J}{\partial \Omega}\right)_T \tag{4.2}$$

We have checked that in the particular case of the singly spinning black hole, indeed, these two response functions are inverse proportional at the particular point where the temperature has a minimum. Therefore, an inflexion point in the microcanonical ensemble corresponds to a divergence of the corresponding response function in the grand canonical ensemble. This may well be an explanation for the results obtained in [9]. Moreover, this point should not be considered as a sign for an instability or a new branch but a transition to an infinitesimally nearby solution along the same family of solutions. The numerical evidence of [9] supports this connection with the zero-mode perturbation of the solution.

We now examine the situation for a more general family of ultra-spinning Myers-Perry black holes with multiple spin parameters, a_i, where $i = 1, 2, ..., N$ and $N = [(D-1)/2]$. The black hole is characterized by the mass parameter μ and the horizon radius r_h (the largest root of)

$$\mu = \frac{1}{r_h^{1+\epsilon}} \prod_{i=1}^{N} (r_h^2 + a_i^2)\,, \tag{4.3}$$

by which we can express the thermodynamics

$$\begin{aligned} M &= \frac{\Omega_{D-2}}{16\pi G_D}(D-2)\,\mu, \qquad J_i = \frac{\Omega_{D-2}}{16\pi G_D} a_i\,\mu, \qquad \Omega_i = \frac{a_i}{r_h^2 + a_i^2}\,, \\ \mathcal{A}_H &= \Omega_{D-2}\,\mu\, r_h, \qquad T = \frac{1}{2\pi r_h}\left(r_h^2 \sum_{i=1}^{N} \frac{1}{r_h^2 + a_i^2} - \frac{1+\epsilon}{2}\right)\,, \end{aligned} \tag{4.4}$$

where $\epsilon = mod_2 D$. A sufficient, but not necessary, condition for the existence of ultra-spinning black holes was given in [8]. In even(odd) dimensions at least one(two) of the spins should be much smaller than the rest. The ultra-spinning regime is obtained in the limit

$$0 \leq a_1, a_2, ..., a_k << a_{k+1}, ..., a_N \to \infty \tag{4.5}$$

where $N - 1 \geq k \geq 1 + \epsilon$. The generic limiting black brane metric whether static, with all finite angular momenta $a_1, .., a_k$ vanishing, or spinning, with some $a_1, .., a_k$ non-vanishing, is the product $S^{D-2(N-k+1)} \times R^{2(N-k)}$.

---

system. It is important to emphasize that all spinodals are zero-determinant lines, but in general not all zero-determinant lines are spinodal.

[11] Different ensembles correspond to different physical conditions and so, in more general cases, one does not expect such a relation.

Our focus will be on the case in which the black hole has at least two large spins and we set the remaining angular momenta to zero. When the angular momenta are equal, $J_{k+1} = ... = J_N = J$, the Ruppeiner curvature scalar blows up at[12]

$$\frac{a^2}{r_h^2} = \frac{D-3}{2k-1-\epsilon} \,. \tag{4.6}$$

According with the arguments in [26], this signals a thermodynamic instability. However, the expected new phase should correspond to the black membrane phase of ultra-spinning black holes and not to a new branch of solutions.

This is further supported by examining the eigenvalues of the Hessian of the Gibbs potential. Indeed, we find that the divergences of the Ruppeiner curvature pinpoint the zero of the determinant of the Gibbs potential's hessian.

Also, by studying the temperature

$$T = \frac{(D-3)\left(1 + \frac{n}{(D-3)}\frac{4J^2}{S^2}\right)}{4S^{\frac{1}{D-2}}\left(1 + \frac{4J^2}{S^2}\right)^{\frac{D-1+n}{2(D-2)}}} \,, \qquad n = 2k-1-\epsilon \tag{4.7}$$

we find that the temperature has a minimum at exactly (4.6), while the angular velocity Ω reaches its maximum value.

For ultra-spinning black holes, similar to the singly spinning situation discussed in [8], once this minimum is reached the temperature increases and the angular velocity decreases signalling a transition to a membrane phase.

Another case of interest is the ultra-spinning black holes that resemble spinning black branes, when some of the slower spins are non-zero $a_1, ..., a_k \neq 0$. It is not our goal to make a detailed analysis of this case here. Nevertheless, in all the cases where the non-vanishing spins are set equal, we find divergences of the Ruppeiner scalar curvature which could help to detect the threshold of their membrane phase.

4.2 Membrane phase of black rings

Other solutions, e.g. the black ring with one angular momentum, also exhibit an ultra-spinning behaviour. The black ring, which is characterized by the radii r_0 and R of the spheres S^{D-3} and S^1, respectively, becomes *thin* in this limit (when $r_0 << R$).[13]

Since the final expressions for the response functions are very complicated for the doubly spinning ring (see Appendix B), we prefer to present the 'conjugacy diagram' of the angular velocity versus the angular momentum and the plot of the temperature as a function of the angular momentum, both for a fixed mass.

[12]Note that $\frac{a^2}{r_h^2} = \frac{4J^2}{S^2}$ and in the particular case when $k = k_{max}$ our results agree with those of [27].

[13]This regime was the starting point to finding perturbatively the higher dimensional cousins of the black rings. Moreover, a generalization of this construction to black branes led to the construction of blackfolds [30, 31].

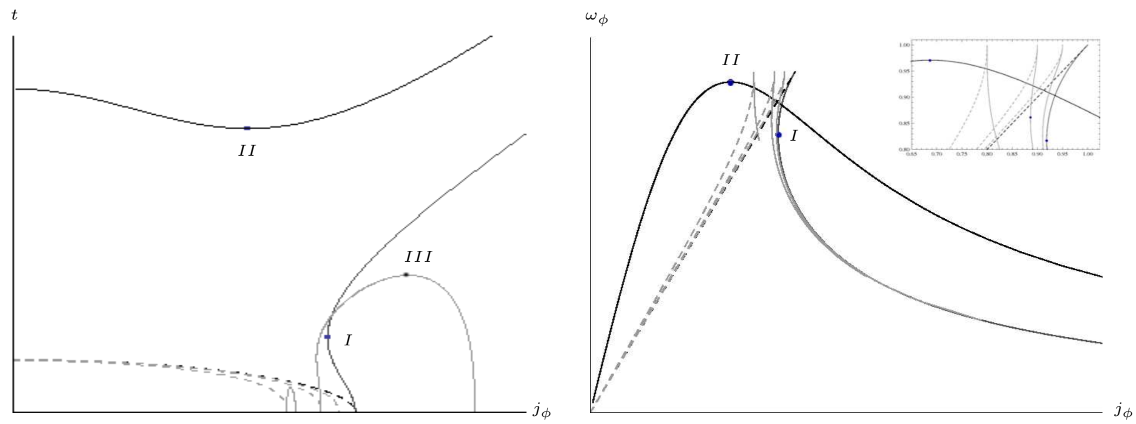

Figure 2: Plots of the temperature (left hand side) and angular velocity (right hand side) as functions of the angular momentum j_ϕ, for a fixed mass, for different black objects. These include the singly spinning Myers-Perry black holes in five dimensions of space-time (*black dashed line*) and its seven dimensional cousin (*solid line*). The singly (*dark gray*) and doubly spinning black ring (*light gray*) for different values of angular momenta (right towards left) $j_\psi = 0.05, 0.1, 0.2$ on the S^2, are also shown here. The five-dimensional doubly spinning black holes are respresented by *dashed light gray* lines.

The *dimensionless* expressions for the temperature t, the spin j, and the angular velocity ω are

$$t^{D-3} = c_t\, GMT^{D-3} \qquad \omega^{D-3} = c_\omega\, GM\Omega^{D-3}\,, \qquad j^{D-3} = c_j\, \frac{J^{D-3}}{GM^{D-2}} \tag{4.8}$$

where the numerical constants are

$$c_t = \frac{2}{(D-2)}\frac{(4\pi)^{D-3}}{\Omega_{D-3}}\left(\frac{D-3}{D-4}\right)^{\frac{D-3}{2}}\,,\ c_\omega = \frac{16}{(D-2)}\frac{(D-3)^{\frac{D-3}{2}}}{\Omega_{D-3}},\ c_j = \frac{\Omega_{D-3}}{2^{D+1}}\frac{(D-2)^{D-2}}{(D-3)^{\frac{D-3}{2}}} \tag{4.9}$$

For the singly spinning black ring, an analysis of the temperature as a function of the angular momentum was presented in [32] — a similar discussion in Anti-de Sitter spacetime can be found in [33].

In this case the temperature does not have a minimum, but there exists a turning point (see Fig. 2). In our analysis, the turning point (I) for the black ring plays a similar role as the minimum of the temperature for the black hole. That is, it signals a change in the behaviour of the black ring. In fact, it is the starting point of the ultra-spinning regime where the black ring can be approximated by a black string.

Using the Poincaré ('turning point') method, this special point was carefully studied in [32]. In particular, they found a divergence of the Ruppeiner curvature. In the conjugacy diagram there is also a turning point (I) at the same minimum value of the angular momentum j_ϕ.

The question is then if there still is a relation between the microcanonical and grand canonical ensembles in this case. We have explicitly checked, using the results of [7], that one of the eigenvalues of the hessian of the Gibbs potential is zero at this specific point

I (while the second eigenvalue never changes its sign). Therefore we conclude that the turning point is the onset of the ultra-spinning black string phase.

A far more richer structure is found for the doubly spinning black ring for which the angular momentum on S^2 is bounded as $j_\psi \in [0, 1/4]$. In this case, for a fixed value of the mass and j_ψ in the range of $0 \leq j_\phi \leq 1/5$, there also exist turning points signalling the onset of the black membrane phase.

In the plot of the temperature as a function of the angular momentum j_ϕ, there is a turning point that corresponds to the minimum value of the S^1 angular momentum. In the conjugacy diagram ω vs. j, the minimum angular momentum is also a turning point. We would like to emphasize that this point does not correspond to a zero-eigenvalue of the Gibbs potential's hessian.

The situation is similar, to some extent, to what we found before for the singly spinning black ring but there also are some important differences. Let us now discuss the main differences between the singly and doubly spinning black rings.

First, the angular momentum of S^2 is bounded and for a specific j_ψ the black ring can always be extremal (in the limit $\lambda \to 2\sqrt{\nu}$ as shown in [12]).

For a large enough S^1 angular momentum, the temperature of the black ring (in its membrane phase) increases while the area of the event horizon decreases up to a point where the spin-spin interaction is large enough making a turn to abruptly become extremal (zero temperature). This is the maximum critical value labeled III in Fig.2. Therefore, the black membrane phase exists between points I and III.

Second, the cusps in the phase diagram (area vs. the S^1 angular momentum) of the doubly spinning black ring disappear for $1/5 < j_\psi \leq 1/4$ [12]. In other words, there is no fat black ring branch and so the fat black rings have no black membrane phase. Such solutions would never be captured with long distance effective approaches [30, 31].

In summary, we have found that the zero eigenvalues of the hessian of the Gibbs potential can also be turning points (with tangents of infinite slope) and not just critical points as for the Myers-Perry black holes. For certain values of the angular momenta, the doubly spinning black ring has no membrane phase. Therefore, these particular solutions with $j_\psi > 1/5$ fall into the same category as other black holes with no membrane phase as the four dimensional Kerr black hole and the five dimensional Myers-Perry black hole.

5 Discussion

In this paper we have analyzed in detail the thermodynamic stability of *neutral* doubly spinning black objects.

We have proved that the doubly spinning black ring is thermodynamically unstable. That is, there is no region in the parameter space in which all response functions are positive definite.

We have provided an explanation of why the microcanonical and grand canonical ensembles for ultra-spinning black holes are related in a very special way. This argument is not generally valid and we will comment in more detail on the significance of these results in the last part of this section.

We have also identified the onset of membrane phase of different doubly spinning black objects. There at least one of the eigenvalues of the Hessian of the Gibbs potential is zero.

Let us start with a discussion on the doubly spinning black ring. An analysis of this solution in the microcanonical ensemble was presented in [12]. In general, for black holes, the entropy is used to obtain the phase diagrams in the microcanonical ensemble while the mass/energy is kept fixed. However, within general relativity it makes more sense to use the total energy instead of the entropy. The reason is that this would require appropriate boundary conditions.

We have presented a careful study of doubly spinning black ring thermodynamics in the grand canonical ensemble. We have used the counterterm method to compute the action and so the grand thermodynamic potential (which is a Legendre transform of the energy). Since the concrete expressions of the response functions are complicated, we have plotted the regions in the parameter space where they are positive definite.

It is well known that Schwarzschild black hole has a negative heat capacity, which means that the thermodynamic ensemble would be dominated by diffuse radiation states rather than black holes states. In other words, it is favorable for the black hole to decay away and so pure thermal radiation is a local equilibrium state.

When adding angular momentum the situation is changing and the heat capacity can become positive for large enough angular momentum. However, this condition is not enough to conclude that the system is thermodynamically stable. The stability also implies that when angular momentum is added to the system the angular velocity goes up.

For a black ring with one angular momentum the heat capacity can be positive definite but the momentum of inertia is always negative. Therefore, the singly spinning black ring is thermodynamically unstable.

As in the case of one angular momentum, the heat capacity of a doubly spinning black ring can be positive in some region of the parameter space. However, there is a key difference when the second angular momentum is turned on. That is, the component of the momentum of inertia associated to S^1 of the black ring can become positive — this is explicitly shown in the Figure 2.

Since there are two angular momenta one should also investigate the effect of coupled 'angular' inhomogeneities. A careful study of the determinant of the momentum of inertia matrix shows that there is no overlap region in the parameter space with the desired properties and so the doubly spinning black ring is also thermodynamically unstable.

We would like now to discuss in more detail some of the results for ultraspinning black

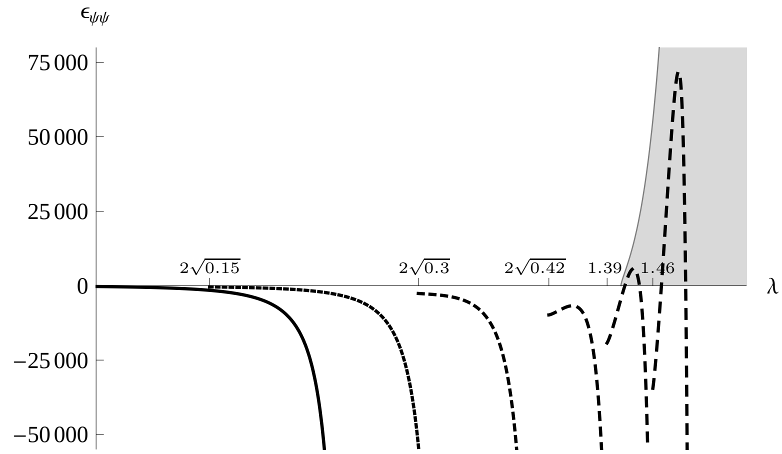

Figure 3: Plot of the response function $\epsilon_{\phi\phi}$ as a function of λ. The *gray* curve corresponds to the compressibility of the singly spinning black ring, namely $\nu = 0$. As the angular momentum along the S^2 is increased (*dashed line* left towards right) the isothermal moment of inertia for different values of $\nu = 0.15, 0.3, 0.42, 0.48, 0.53$ changes and becomes positive for values of $\nu > 0.46690042$ and $\lambda > 1.40685$ (shaded *gray region*).

holes presented in Section 4. In Fig. 4, we show the points (A,B) in the grand canonical ensemble that correspond to inflexion points (A,B) in the microcanonical ensemble. This can be quantitatively understood by comparing the (particular) response functions in (4.2) at a very special point in the parameter space (where the temperature has a minimum).

However, we have explained that this argument applies just in this particular case, not in general. A counterexample is the 5-dimensional black hole with one angular momentum. In this case there is no relation between ensembles in the sense that there is no special corresponding point (where the response function is divergent or zero) in the microcanonical ensemble which corresponds to the inflexion point (C) in the grand canonical ensemble. Moreover we have checked and there are no points where an eigenvalue of the Hessian of the Gibbs potential vanishes. Therefore, it should not be considered as a sign for a membrane phase.

It would be interesting to see if this inflexion point in the grand canonical ensemble is related in any way to the dynamical stability.

One can also consider the ultra-spinning black holes with some of the finite angular momenta non-zero. In odd(even) spacetime dimensions, the metric of an ultra-spinning black hole with all but two(one) of the spins finite and non-zero will reduce to that of a spinning black brane.

As we have shown in Section 2, the counterterm method can also be applied to spinning black branes and the results are similar with the ones for the 'seed' spinning black hole solution. We have computed the *renormalized* action to find the Gibbs potential and we expect similar thermal instabilities as for the corresponding black holes.

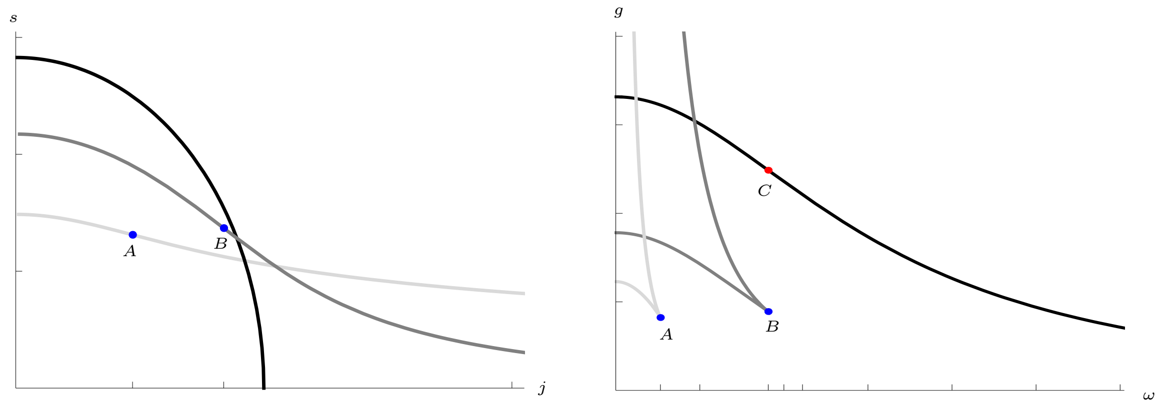

Figure 4: The microcanonical phase diagram - entropy, s, as a function of the angular momentum, j - for fixed mass (on the *left*) of the singly spinning Myers-Perry black hole in $D = 5, 6, 10$ dimensions of space time (*black*, *gray* and *lightgray* lines respectively). The grand canonical phase diagram (on the *right*) - Gibbs potential, g, as a function of the angular velocity, ω - for fixed temperature of the black hole. The points A, B correspond to $a/r_h = \sqrt{3}, \sqrt{\frac{7}{5}}$ for six and ten dimensions, and in general for $D > 5$ the points are $a/r_h = \sqrt{(D-3)/(D-5)}$; the change in convexity of the entropy corresponds to a blowing up of the convexity of the Gibbs potential. The point C instead, at $a/r_h = 1/\sqrt{3}$ and where the Gibbs potential has a change in convexity, has no analog in the microcanonical scheme.

This is the starting point for studying the instabilities. The extension of the dynamical stability studies to spinning black branes has not been yet developed. The analytic theory of perturbation is much more involved. However, as for the static black branes, we expect that the spinning black branes suffer of similar instabilities.

We anticipate that the observations made in this paper will be also useful in the future investigations of the perturbations of higher-dimensional spinning black/rings holes as well as for spinning black branes.

Acknowledgments

We would like to thank Ido Adam for interesting conversations. DA and MJR would also like to thank Robb Mann and Cristian Stelea for collaboration on related projects and valuable discussions.

A Temperature and angular velocities

Consider a general stationary 5-dimensional metric that corresponds to a black object with two angular momenta[14] :

$$\begin{aligned} ds^2 &= g_{tt}(\vec{x})dt^2 + 2g_{t\phi}(\vec{x})dtd\phi + 2g_{t\psi}(\vec{x})dtd\psi + g_{\phi\phi}(\vec{x})\,d\phi^2 \\ &+ 2g_{\phi\psi}(\vec{x})\,d\phi d\psi + g_{\psi\psi}(\vec{x})\,d\psi^2 + g_{\alpha\beta}(\vec{x})\,dx^\alpha dx^\beta \end{aligned} \tag{A.1}$$

∂_t, ∂_ϕ, and ∂_ψ are Killing vectors. Rewrite the metric in the ADM form

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i\,dt)(dx^j + N^j\,dt) \tag{A.2}$$

with lapse function

$$N^2 = -g_{tt} + g_{\phi\phi}\,(N^\phi)^2 + g_{\psi\psi}\,(N^\psi)^2 + 2\,g_{\phi\psi}\,N^\phi N^\psi \tag{A.3}$$

and shift vector

$$N^\phi = \frac{g_{t\psi}g_{\phi\psi} - g_{\psi\psi}g_{t\phi}}{g_{\phi\psi}^2 - g_{\phi\phi}g_{\psi\psi}}\,, \qquad N^\psi = \frac{g_{t\phi}g_{\phi\psi} - g_{\phi\phi}g_{t\psi}}{g_{\phi\psi}^2 - g_{\phi\phi}g_{\psi\psi}} \tag{A.4}$$

The event horizon is obtained for

$$N^2 = 0 \tag{A.5}$$

In other words, it is a Killing horizon of $\partial_t + \Omega_\phi\,\partial_\phi + \Omega_\psi\,\partial_\psi$, where Ω_ϕ and Ω_ψ are the angular velocities defined as the shift vectors at the horizon:

$$\Omega_\phi = -N^\phi\big|_H\,, \qquad \Omega_\psi = -N^\psi\big|_H \tag{A.6}$$

Black holes are thermodynamic objects: the causal structure of spacetime can influence the physics of a quantum field. The vacuum fluctuations near the event horizon cause the black hole to emit particles with a thermal spectrum. The Euclidean regularity at the horizon is equivalent to the condition that the black hole is in thermodynamical equilibrium.

By a straightforward computation one can eliminate the conical singularity in the Euclidean section, (it, r), to obtain the periodicity of the Euclidean time. In this way, one obtains the following expression for the temperature of the black hole:

$$T = \left.\frac{(N^2)'}{4\pi\sqrt{g_{rr}\,N^2}}\right|_H \tag{A.7}$$

We have used these definitions to compute the corresponding physical quantities of the doubly spinning ring.

[14] A similar analysis for one angular momentum can be found in [34, 7].

B Conditions for thermodynamic stability

In this appendix, we present the conditions for the thermodynamic stability and we also give some useful explicit expressions for the response functions used in Section 4 — we follow closely [35].

For simplicity, let us start with a black hole with one angular momentum. We are interested in the thermodynamic potentials: the energy and its Legendre transforms.

The basic extremum principle of thermodynamics (for the entropy S) implies both that $dS = 0$ and that $d^2S < 0$. The second condition determines the stability of predicted equilibrium states. The stability criterion in energy representation requires that an equilibrium state at fixed S and J is a state of minimum energy, namely a minimum of $E[S, J]$. The local stability conditions ensure that inhomogeneities of either S and J separately

$$\left(\frac{\partial^2 E}{\partial S^2}\right)_J = \left(\frac{\partial T}{\partial S}\right)_J \geq 0, \qquad \left(\frac{\partial^2 E}{\partial J^2}\right)_S = \left(\frac{\partial \Omega}{\partial J}\right)_S \geq 0 \tag{B.1}$$

and *also* that a coupled inhomogeneity of S and J together

$$\det(\text{Hess}(E)) = \frac{\partial^2 E}{\partial S^2}\frac{\partial^2 E}{\partial J^2} - \left(\frac{\partial^2 E}{\partial S \partial J}\right)^2 \geq 0 \tag{B.2}$$

do not decrease the energy.

In more generality the stability criterion states that the thermodynamic potentials are *convex* functions of their *extensive* variables and *concave* functions of their *intensive* variables (see, e.g., [35]).

For a grand canonical ensemble defined at fixed temperature T and angular velocities Ω_a (intensive variables) the associated potential, the Gibbs free energy, satisfies the following relations:

$$G[T, \Omega] = E - TS - \Omega J\ , \qquad dG = -SdT - Jd\Omega \tag{B.3}$$

In this case, the local stability conditions following from the convexity of the Gibbs function yield

$$\left(\frac{\partial^2 G}{\partial T^2}\right)_\Omega = -\left(\frac{\partial S}{\partial T}\right)_\Omega \leq 0, \qquad \left(\frac{\partial^2 G}{\partial \Omega^2}\right)_T = -\left(\frac{\partial J}{\partial \Omega}\right)_T \leq 0 \tag{B.4}$$

and

$$\det(\text{Hess}(G)) = \frac{\partial^2 G}{\partial T^2}\frac{\partial^2 G}{\partial \Omega^2} - \left(\frac{\partial^2 G}{\partial T \partial \Omega}\right)^2 \geq 0 \tag{B.5}$$

Equivalently, the heat capacities (C_Ω, C_J) and the 'isothermal moment of inertia' or 'compressibility' $\epsilon \equiv (\partial J/\partial \Omega)_T$ should be positive definite.

A generalization for two angular momenta is straightforward (see, also, [6]). The Hessian is a 3×3 matrix

$$\text{Hess}(G) = (-1)\begin{pmatrix} C_\Omega T^{-1} & \alpha_a \\ \alpha_a & \epsilon_{ab} \end{pmatrix}$$

where the matrix components are $\alpha_a = \left(\frac{\partial J_i}{\partial T}\right)_\Omega$, $\epsilon_{ab} = \left(\frac{\partial J_a}{\partial \Omega_b}\right)_T$, and the indices cover the angular directions $a, b = \phi, \psi$.

Considering the relationship between the specific heats $C_\Omega = C_J + T\,(\epsilon^{-1})_{ab}\,\alpha^a \alpha^b$ it can be shown that a thermodynamically stable system is characterized by positive heat capacities $C_\Omega > 0$ and $C_J > 0$ and, also, a positive definite matrix of isothermal momenta of inertia, i.e. $\mathrm{spec}[\epsilon_{ab}] > 0$.

The $(\psi\psi)$-component of the isothermal moment of inertia tensor is

$$\epsilon^{\psi\psi} = -\frac{4k^4\pi\lambda(1+\lambda+\nu)}{G_5(-1+\nu)^4(1-\lambda+\nu)^2}\,\frac{F(\lambda,\nu)}{\sqrt{\lambda^2-4\nu}(-8\nu+\lambda^2+\lambda^2\nu)+\lambda^3(1+\nu)-2\lambda(-1+4\nu+\nu^2)}$$

where

$$\begin{aligned}
F(\lambda,\nu) = {} & \lambda^6(1+\nu)^2+\lambda^5(1+\nu)^2(1+\sqrt{\lambda^2-4\nu}+\nu)+ \\
& +\lambda^4(1+\nu)[4+\sqrt{\lambda^2-4\nu}+\nu(2(1+\sqrt{\lambda^2-4\nu})+\nu(-24+\sqrt{\lambda^2-4\nu}+2\nu))]- \\
& -16\sqrt{\lambda^2-4\nu}+\nu[1+\nu(-14+\nu(18+(-14+\nu)\nu))]+ \\
& +2\lambda^3(1+\nu)[2+\sqrt{\lambda^2-4\nu}+\nu(-25+\sqrt{\lambda^2-4\nu}+\nu(13-11\sqrt{\lambda^2-4\nu}+ \\
& +\nu(-7+\sqrt{\lambda^2-4\nu}+\nu)))]+2\lambda^2(1+\nu)[2+\sqrt{\lambda^2-4\nu}+\nu(-32-26\sqrt{\lambda^2-4\nu}+ \\
& +\nu(32+14\sqrt{\lambda^2-4\nu}+\nu(16-6\sqrt{\lambda^2-4\nu}+(-2+\sqrt{\lambda^2-4\nu})\nu)))]- \\
& -4\lambda[-1+\nu(10+4\sqrt{\lambda^2-4\nu}+\nu(-89-12\sqrt{\lambda^2-4\nu}+\nu(-12(-6+\sqrt{\lambda^2-4\nu})+ \\
& +\nu(-23+4\sqrt{\lambda^2-4\nu}+(-2+\nu)\nu))))]
\end{aligned}$$

The determinant is

$$\epsilon = \frac{4k^8\pi^2\lambda^2(\lambda-\sqrt{\lambda^2-4\nu})(\lambda+\sqrt{\lambda^2-4\nu})^2\sqrt{\nu}(1+\lambda+\nu)^5}{(G_5)^2(-1+\nu)^4[-\lambda^2+(1+\nu)^2]^{3/2}[\nu(-\lambda^2+(1+\nu)^2)]^{3/2}}\,\frac{G(\lambda,\nu)}{Z(\lambda,\nu)} \tag{B.6}$$

where

$$\begin{aligned}
Z(\lambda,\nu) = {} & 8\sqrt{\lambda^2-4\nu}\,\nu-\lambda^3(1+\nu)-\lambda^2\sqrt{\lambda^2-4\nu}(1+\nu)+2\lambda(-1+4\nu+\nu^2) \\
G(\lambda,\nu) = {} & \lambda^5+7\lambda^4(1+\nu)-\lambda^2(1+\nu)[1+3\sqrt{\lambda^2-4\nu}+(2-3\sqrt{\lambda^2-4\nu})\nu+\nu^2]+ \\
& +\lambda^3[5-6\sqrt{\lambda^2-4\nu}+(26+6\sqrt{\lambda^2-4\nu})\nu+5\nu^2]-8\nu(1+11\nu+11\nu^2+\nu^3)- \\
& -\lambda[-3\sqrt{\lambda^2-4\nu}+(8-27\sqrt{\lambda^2-4\nu})\nu+9(16+3\sqrt{\lambda^2-4\nu})\nu^2+(8+3\sqrt{\lambda^2-4\nu})\nu^3]
\end{aligned}$$

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