

Thermodynamic instability of doubly spinning black objects

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ABSTRACT: We investigate the thermodynamic stability of neutral black objects with (at least) two angular momenta. We use the quasilocal formalism to compute the grand canonical potential and show that the doubly spinning black ring is thermodynamically unstable. We consider the thermodynamic instabilities of ultra-spinning black objects and point out a subtle relation between the microcanonical and grand canonical ensembles. We also find the location of the black string/membrane phases of doubly spinning black objects.

KEYWORDS: Black Holes, Black Holes in String Theory

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1 Introduction

The physics of event horizons in higher-dimensional General Relativity (GR) is an interesting area of research not just for its intrinsic relevance to string theory. An investigation of black hole solutions in higher dimensions is also important because it has revealed new features: a richer rotation dynamics and the existence of regular extended black hole solutions.

It is clear by now that some of the remarkable properties of four-dimensional black holes do not hold in general. A notorious example of particular importance concerns their horizon topology. In four dimensions, the spherical S^2 topology is the only allowed horizon topology for asymptotically flat (AF) stationary black holes. A related result is the ‘uniqueness theorem’, which states that a stationary AF vacuum black hole in four dimensions is characterized by its mass and angular momentum and has no other independent characteristic (hair).

The spectrum of stationary black objects is far richer in dimensions bigger than four (see [1] for a concise review). The restrictions on the topology of AF black holes require that spatial sections of the event horizon must be positive Yamabe type [2] and if spinning (stationary) they have to be axisymmetric [3]. The most obvious indication is the existence of a neutral AF solution describing a spinning black ring in five dimensions [4, 5]. As it was

shown in [4], there are *three* solutions with the same asymptotic conserved charges (the same mass and angular momentum). On top of the well known Myers-Perry (MP) black hole [6] with an S^3 horizon there are two different black rings with an $S^1 \times S^2$ horizon ($S^1 \times S^{D-3}$ in D dimensions). Therefore, unlike in four dimensions, the black objects in higher dimensions are not completely determined by a few conserved asymptotic charges.¹ But the space of solutions of Einstein's equations in D dimensions also includes extended black holes. The black p-branes [8], dubbed black strings when $p = 1$ or otherwise membranes, are transverse asymptotically flat (only AF in $D-p$ directions), evade the no-hair theorems, and exhibit horizon topologies $S^{D-2-p} \times \mathbb{R}^p$. Interestingly enough, due to the richer rotational dynamics, in certain regimes in higher dimensions the thermodynamical properties of the compact black holes resemble those of the extended black objects. In this paper we aim to make progress towards a better understanding on these properties of black objects.

We first study in detail the thermodynamic instabilities of doubly spinning black objects. We identify the ultra-spinning regimes (in parameter space) from their thermodynamic quantities. Our motivation stems from the observation made in [9] that, in dimensions greater than four, the thermodynamics of certain fast spinning MP black holes show a qualitative change in behaviour. That is, there is a transition towards a black membrane-like behaviour. This is due to the fact that, as one increases the angular momentum, the temperature of these ultra-spinning black holes reaches a minimum and then starts to grow as expected for the black membrane.²

Black rings also exhibit a change in the thermodynamic behaviour that resembles the ultra-spinning regime of black holes when ultra-spinning along the S^1 direction. In the 'thin ring approximation', when the radius of S^1 is much larger than the radius of S^{D-3} , the singly spinning black ring can be approximated by a boosted black string. An interesting question we would like to address in this paper is how the ultra-spinning regime of the $D = 5$ black ring is affected by adding the second angular momentum, along the S^2 . In other words, whether neutral doubly spinning black ring is thermodynamically stable in the grand canonical ensemble and whether there is any connection with its ultra-spinning regime.

Since there is no known background subtraction method for these black objects, we use a slightly modified version of the quasilocal formalism of Brown and York [12] to address these questions. Supplemented with counterterms [13–15], the quasilocal formalism becomes a very powerful tool to study the thermodynamics of black objects that are AF. Recently, several concrete five-dimensional examples were discussed in detail in [7, 16] and in [17] for the unbalanced black ring.

The relevant thermodynamic potential in the grand canonical ensemble is the Gibbs potential that is the Euclidean action divided by the periodicity of the Euclidean time [18]. The Euclidean method was applied to the black ring thermodynamics in [19]. Since the black ring does not have a real non-singular Euclidean section, the 'quasi-Euclidean' method [20] was adopted to analyze the black ring thermodynamics. In this approach,

¹In [7] it was proposed that the necessary information to distinguish between black objects with different horizon topologies is encoded in the subleading terms in the boundary stress tensor.

²The numerical evidence of [10] suggests that the onset of the ultra-spinning regime for singly spinning MP black holes corresponds to a zero mode associated to the Gregory-Laflamme instability [11].

the horizon is described by the ‘bolt’ in a *complexified* Euclidean geometry rather than a real one.³ It was also pointed out in [19] that the neutral black ring with one angular momentum is unstable to angular fluctuations — a more detailed analysis can be found in [7, 21, 22].

By employing this method to compute the Gibbs potential, we find the response functions directly in the grand canonical ensemble. We observe that the second angular momentum changes the situation in the sense that, unlike the black ring with one angular momentum, the doubly spinning ring is stable against perturbations in the angular velocity in some specific region of the parameter space.⁴ However, a careful analysis of all response functions that characterize the system reveals that the doubly spinning black ring is thermodynamically *unstable* in the grand canonical ensemble. That is, there is no region in the parameter space in which all response functions are positive definite.

On general physical grounds it is expected that the microcanonical ensemble of asymptotically flat black holes is dominated by diffuse radiation states rather than black hole states [23]. In other words, it is favorable for the black hole to decay away (the heat capacity is negative — see appendix B for a detailed discussion on local thermodynamic conditions) and so pure thermal radiation is a local equilibrium state. Indeed, it was shown in [24] that, for any vacuum black hole characterized by its mass and angular momenta, the Hessian of the entropy always has a negative eigenvalue. Since the Hessian of the entropy is related to the inverse of the Hessian of the Gibbs potential [25], this implies at generic points in the moduli space, i.e. away from the hypersurfaces defined by a vanishing eigenvalue, an instability in the grand canonical ensemble [24].

Identifying which doubly spinning black hole solutions exhibit this ultra-spinning regime will be our next objective. We will first find the threshold of the black membrane regime for MP black holes with at least two large angular momenta. To compare the ultra-spinning regimes of both, the black hole and black ring, we present a careful analysis of their ultra-spinning regime. For the black ring we observe a similar but slightly different ultra-spinning phase. That is, the temperature does not have a minimum but rather there is a ‘turning point’, which is responsible for the boosted black string behaviour. We also find that even after adding the second angular momentum the ring can have a membrane-like behavior. However, for large enough values of the second angular momentum the ‘membrane phase’ disappears.

The dynamical instabilities were related to the thermodynamic ones by (a conjecture of) Gubser and Mitra [26]: gravitational backgrounds with translationally invariant horizon develop a tachyonic mode (negative non-conformal mode) whenever the specific heat of the black brane geometry becomes negative. Our goal is not to study dynamical instabilities, which would require a perturbative analysis, but rather to generalize the arguments of [9] to black rings. However, to connect the work of [9] to [26], one has to understand the black hole thermodynamics in the grand canonical ensemble (see, e.g., [27]). At first sight, it is surprising that an analysis in the microcanonical ensemble [9] provides information

³The complex geometry is obtained by the usual analytic continuation of time coordinate, $\tau = it$.

⁴For a black ring with one angular momentum the ‘isothermal compressibility’ (moment of inertia) is always negative [7].

about the membrane phase. We point out a subtle relation between the microcanonical and grand canonical ensembles, which is valid for the analytic solutions we are interested in, but it may be more general, and argue that this is why the microcanonical ensemble also encodes information about the membrane phase.

The remainder of this paper is organized as follows. We start in section 2 with a brief review of the counterterm method for asymptotically flat spacetimes. We then compute the stress tensor and the corresponding asymptotic charges for black objects with two angular momenta: Myers-Perry black hole, black ring, and black branes. In section 3, we present an analysis of the thermodynamic stability of the doubly spinning ring. We compute the thermodynamic action and check the quantum statistical relation. We also analyze in great detail the response functions. In section 4, we investigate the black string/membrane phases of doubly spinning black objects. Finally, we conclude with a discussion of our results. In appendix A we present the expressions of angular velocities and temperature for a general metric with two angular momenta. Appendix B contains some general aspects of black holes thermodynamics, the local stability conditions, and concrete expressions for some of the response functions used in section 2.

2 Stress tensor and conserved charges

In this section we apply the counterterm method to doubly spinning five-dimensional vacuum solutions of Einstein gravity. We explicitly show how to compute the boundary stress tensor and the conserved charges for Myers-Perry black hole, doubly spinning black ring, and doubly spinning black branes.

2.1 Quasilocal formalism and conserved charges

To begin our considerations on thermodynamics of doubly spinning black objects in five dimensions, we recall the description of quasilocal formalism [12] supplemented with counterterms.

To define the conserved charges we use the divergence-free boundary stress tensor proposed in [19]:

$$\tau_{ij} \equiv \frac{2}{\sqrt{|\hbar|}} \frac{\delta I}{\delta h^{ij}} = \frac{1}{8\pi G_5} \left(K_{ij} - h_{ij} K - \Psi (\mathcal{R}_{ij} - \mathcal{R} h_{ij}) - h_{ij} \square \Psi + \Psi_{;ij} \right) \quad (2.1)$$

where $\Psi = \sqrt{3}/\sqrt{2\mathcal{R}}$, h_{ij} is the induced boundary metric, and \mathcal{R}_{ij} is its Ricci scalar. A rigorous justification and more details about this proposal can be found in [28–30].

Here, I is the *renormalized* action that includes counterterms,

$$I = \frac{1}{16\pi G_5} \int_M R \sqrt{-g} d^5x + \frac{\epsilon}{8\pi G_5} \int_{\partial M} \left(K - \sqrt{\frac{3\mathcal{R}}{2}} \right) \sqrt{|\hbar|} d^4x \quad (2.2)$$

K is the extrinsic curvature of ∂M and $\epsilon = +1(-1)$ if ∂M is timelike (spacelike).

The boundary metric can be written locally in ADM-like form

$$h_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{ab} (dy^a + N^a dt)(dy^b + N^b dt) \quad (2.3)$$

where N and N^a are the lapse function and the shift vector respectively and $\{y^a\}$ are the intrinsic coordinates on a (closed) hypersurface Σ . If the boundary geometry has an isometry generated by a Killing vector ξ^i , a conserved charge

$$\mathfrak{Q}_\xi = \oint_\Sigma d^3y \sqrt{\sigma} n^i \tau_{ij} \xi^j \tag{2.4}$$

can be associated with the hypersurface Σ (with normal n^i).

2.2 Doubly spinning solutions

2.2.1 Black hole

The Einstein equations in higher dimensions have spinning black hole solutions [6]. In five dimensions, the Myers-Perry black hole in Boyer-Lindquist type coordinates is

$$ds_{BH}^2 = -dt^2 + \Sigma \left(\frac{r^2}{\Delta} dr^2 + d\theta^2 \right) + (r^2 + a^2) \sin^2 \theta d\phi^2 + (r^2 + b^2) \cos^2 \theta d\psi^2 + \frac{m}{\Sigma} (dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi)^2 \tag{2.5}$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Delta = (r^2 + a^2)(r^2 + b^2) - m r^2 \tag{2.6}$$

and m is a parameter related to the physical mass of the black hole, while the parameters a and b are associated with its two independent angular momenta. This metric depends only on two coordinates, $0 < r < \infty$ and $0 \leq \theta \leq \pi/2$, and it is independent of time, $-\infty < t < \infty$, and the azimuthal angles, $0 < \phi, \psi < 2\pi$.

Since r is playing the role of a radial coordinate in this coordinate system, the event horizon is also the null surface determined by the equation $g^{rr} = 0$. So, the event horizon of the black hole can be computed by using (A.5), which implies $\Delta = 0$. The largest root of this equation gives the radius of the black hole's outer event horizon

$$r_h^2 = \frac{1}{2} \left(m - a^2 - b^2 + \sqrt{(m - a^2 - b^2)^2 - 4a^2 b^2} \right) \tag{2.7}$$

Notice that the horizon exists if and only if

$$a^2 + b^2 + 2|ab| \leq m \tag{2.8}$$

so that the condition $m = a^2 + b^2 + 2|ab|$ or, equivalently, $r_h^2 = |ab|$ defines the extremal horizon of a five dimensional black hole (when one angular momentum vanishes, the horizon area goes to zero in the extremal limit). Otherwise, the metric describes a naked singularity.

In the asymptotic limit, $r \rightarrow \infty$, the metric (2.5) approaches Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2) \tag{2.9}$$

We use the expression of black hole metric in Boyer-Lindquist coordinates to compute the boundary stress tensor and we obtain the following non-vanishing components:

$$\begin{aligned}
 \tau_{tt} &= \frac{1}{8\pi G_5} \left(-\frac{3}{2} m \frac{1}{r^3} - \frac{5}{3} (a^2 - b^2) \frac{\cos 2\theta}{r^3} + \mathcal{O}(1/r^5) \right), \\
 \tau_{t\phi} &= \frac{1}{8\pi G_5} \left(-2 a m \frac{\sin^2 \theta}{r^3} + \mathcal{O}(1/r^5) \right), \\
 \tau_{t\psi} &= \frac{1}{8\pi G_5} \left(-2 b m \frac{\cos^2 \theta}{r^3} + \mathcal{O}(1/r^5) \right), \\
 \tau_{\theta\theta} &= \frac{1}{8\pi G_5} \left(\frac{2}{3} (a^2 - b^2) \frac{\cos 2\theta}{r} + \mathcal{O}(1/r^3) \right), \\
 \tau_{\phi\phi} &= \frac{1}{8\pi G_5} \left(\frac{2}{3} (a^2 - b^2) \frac{(-1 + 2 \cos 2\theta) \sin^2 \theta}{r} + \mathcal{O}(1/r^3) \right), \\
 \tau_{\psi\psi} &= \frac{1}{8\pi G_5} \left(\frac{2}{3} (a^2 - b^2) \frac{(1 + 2 \cos 2\theta) \cos^2 \theta}{r} + \mathcal{O}(1/r^3) \right), \\
 \tau_{\phi\psi} &= \frac{1}{8\pi G_5} \left(-4 a b m \frac{\cos^2 \theta \sin^2 \theta}{r^3} + \mathcal{O}(1/r^5) \right)
 \end{aligned} \tag{2.10}$$

This stress tensor is covariantly conserved with respect to the boundary metric (2.9). We also notice that, for equal angular momenta, the diagonal ‘angular’ components of the stress tensor vanish — this is intuitively expected due to the enhanced symmetry.

Using the definition (2.4), it is straightforwardly to obtain the conserved charges associated with the surface Σ as

$$M = \oint_{\Sigma} d^3 y \sqrt{\sigma} n^i \tau_{ij} \xi_t^j, \quad J_{\phi} = \oint_{\Sigma} d^3 y \sqrt{\sigma} n^i \tau_{ij} \xi_{\phi}^j, \quad J_{\psi} = \oint_{\Sigma} d^3 y \sqrt{\sigma} n^i \tau_{ij} \xi_{\psi}^j$$

where the normalized Killing vectors associated with the mass and angular momenta are $\xi_t = \partial_t$, $\xi_{\phi} = \partial_{\phi}$, and $\xi_{\psi} = \partial_{\psi}$ respectively. We find

$$M = \frac{3 \pi m}{8 G_5}, \quad J_{\phi} = \frac{\pi m a}{4 G_5}, \quad J_{\psi} = \frac{\pi m b}{4 G_5} \tag{2.11}$$

which is in perfect agreement with the *ADM* calculation.

2.2.2 Black ring

A black ring is a five-dimensional black hole with an event horizon of topology $S^1 \times S^2$ and the metric was presented in [4] — the solution of Emparan and Reall has one angular momentum. In five dimensions, a more general solution for a black ring with two angular momenta was presented by Pomeransky and Sen’kov [5]. Some properties of the solution including the structure of the phases in the microcanonical ensemble are discussed in [31]. A study of its geodesics has been performed in [32] and a careful investigation of global properties appeared recently in [33]. We provide here a brief account of the doubly spinning black ring solution and compute the boundary stress tensor and the conserved charges.

We will use the solution in the form presented in [5]. The metric depends just on the coordinates x and y defined within the following intervals $-1 \leq x \leq 1$ and $-\infty < y < -1$.

Notice that x is like an angular coordinate — this observation will be useful when we will define new coordinates that make asymptotic flatness clear.

The metric has a coordinate singularity where g_{yy} diverges. The event horizon of the doubly spinning black ring is located at the smallest absolute value of $1 + \lambda y + \nu y^2 = 0$, namely

$$y_h = \frac{-\lambda + \sqrt{\lambda^2 - 4\nu}}{2\nu} \quad (2.12)$$

For a regular black ring solution, the parameters ν and λ are constrained to satisfy [5]:

$$0 \leq \nu < 1, \quad 2\sqrt{\nu} \leq \lambda < 1 + \nu \quad (2.13)$$

In the limit $\nu \rightarrow 0$ the black ring with one angular momentum (J_ϕ) is recovered (J_ψ is the angular momentum on S^2). The limit $\lambda \rightarrow 2\sqrt{\nu}$ was carefully studied in [31] and shown to correspond to regular extremal black rings.

We use a coordinate transformation similar to the one in [31]:

$$x = -1 + 4k^2 \alpha^2 \frac{\cos^2 \theta}{r^2}, \quad y = -1 - 4k^2 \alpha^2 \frac{\sin^2 \theta}{r^2}, \quad \alpha = \sqrt{\frac{1 + \nu - \lambda}{1 - \nu}} \quad (2.14)$$

In these coordinates $\partial_t, \partial_\phi,$ and ∂_ψ are Killing vectors and the asymptotic metric is the same as (2.9).

The boundary stress tensor in these new coordinates is

$$\begin{aligned} \tau_{tt} &= \frac{1}{8\pi G_5} \left(-\frac{12k^2 \lambda}{(1 + \nu - \lambda)} \frac{1}{r^3} - \frac{8k^2 F_1[\nu, \lambda]}{3(1 + \nu - \lambda)(1 - \nu)^2} \frac{\cos 2\theta}{r^3} + \mathcal{O}(1/r^5) \right), \\ \tau_{t\phi} &= \frac{1}{8\pi G_5} \left(\frac{16k^3 \lambda (1 + \lambda - 6\nu + \lambda\nu + \nu^2) \sqrt{(1 + \nu)^2 - \lambda^2} \sin^2 \theta}{(1 + \nu - \lambda)^2 (1 - \nu)^2} \frac{1}{r^3} + \mathcal{O}(1/r^5) \right), \\ \tau_{t\psi} &= \frac{1}{8\pi G_5} \left(\frac{32k^3 \lambda \sqrt{\nu [(1 + \nu)^2 - \lambda^2]} \cos^2 \theta}{(1 + \nu - \lambda)(1 - \nu)^2} \frac{1}{r^3} + \mathcal{O}(1/r^5) \right), \\ \tau_{\theta\theta} &= \frac{1}{8\pi G_5} \left(\frac{2k^2 F_2[\nu, \lambda]}{3(1 + \nu - \lambda)(1 - \nu)^2} \frac{\cos 2\theta}{r} + \mathcal{O}(1/r^3) \right), \\ \tau_{\phi\phi} &= \frac{1}{8\pi G_5} \left(\frac{k^2 (-F_3[\nu, \lambda] + F_4[\nu, \lambda] \cos 2\theta) \sin^2 \theta}{3(1 + \nu - \lambda)(1 - \nu)^2} \frac{1}{r} + \mathcal{O}(1/r^3) \right), \\ \tau_{\psi\psi} &= \frac{1}{8\pi G_5} \left(\frac{k^2 (F_3[\nu, \lambda] + F_4[\nu, \lambda] \cos 2\theta) \cos^2 \theta}{3(1 + \nu - \lambda)(1 - \nu)^2} \frac{1}{r} - \mathcal{O}(1/r^3) \right), \\ \tau_{\phi\psi} &= \frac{1}{8\pi G_5} \left(\frac{128k^4 \lambda \sqrt{\nu} (4\lambda\nu - (\lambda^2 + (1 - \nu)^2)(1 + \nu)) \cos^2 \theta \sin^2 \theta}{(1 + \nu - \lambda)(1 - \nu)^4} \frac{1}{r^3} + \mathcal{O}(1/r^5) \right) \end{aligned} \quad (2.15)$$

where

$$\begin{aligned} F_1[\nu, \lambda] &= 1 - 5\nu - \nu^2 + 5\nu^3 + \lambda^2(3 + 7\nu) + \lambda(1 - 14\nu - 7\nu^2), \\ F_2[\nu, \lambda] &= 1 - 11\nu - \nu^2 + 11\nu^3 + \lambda^2(3 + 13\nu) + 4\lambda(1 - 5\nu - 4\nu^2), \\ F_3[\nu, \lambda] &= 5 - 7\nu - 5\nu^2 + 7\nu^3 + \lambda^2(15 + 17\nu) - 4\lambda(1 + 13\nu + 2\nu^2), \\ F_4[\nu, \lambda] &= 7 - 29\nu - 7\nu^2 + 29\nu^3 + \lambda^2(21 + 43\nu) + \lambda(4 - 92\nu - 40\nu^2) \end{aligned}$$

As in the case of doubly spinning black hole, this stress tensor is covariantly conserved with respect to the boundary metric (2.9). However, since for the doubly spinning black ring the angular momenta can not be equal, namely

$$3 J_\psi \leq J_\phi, \tag{2.16}$$

there is no similar symmetry enhancement as in the black hole case in the angular part.

By plugging the expressions of the boundary stress-energy components (2.15) in (2.4) we find the following expressions for the conserved charges:

$$M = \frac{3\pi k^2}{G_5} \frac{\lambda}{1 + \nu - \lambda}, \quad J_\psi = \frac{4\pi k^3 \lambda \sqrt{\nu[(1 + \nu)^2 - \lambda^2]}}{G_5 (1 + \nu - \lambda)(1 - \nu)^2}, \tag{2.17}$$

$$J_\phi = \frac{2\pi k^3 \lambda (1 + \lambda - 6\nu + \nu\lambda + \nu^2) \sqrt{(1 + \nu)^2 - \lambda^2}}{G_5 (1 + \nu - \lambda)^2 (1 - \nu)^2} \tag{2.18}$$

As expected, the charges computed by using the quasilocal formalism recover correctly the ADM results [5].

In principle, one can obtain a black hole and a black ring with the same conserved charges. However, an asymptotic observer can not distinguish between a black hole and a black ring just by computing the conserved asymptotic charges. We would like to emphasize that it is expected that the subleading terms of the quasilocal stress tensor encode the information necessary to distinguish between black objects with different horizon topologies.

2.2.3 Black membrane

Here we would like to apply the quasilocal formalism to doubly spinning black p-branes, dubbed also black membranes (BM) or black strings if $p = 1$. The black membrane metric we are interested in is obtained by adding flat directions to a 5-dimensional black hole with two angular momenta. Therefore, the metric is

$$ds_{BM}^2 = ds_{BH}^2 + \sum_{i=1}^{D-5} dx_i^2 \tag{2.19}$$

where ds_{BH}^2 is the black hole metric defined in (2.5).

Since the number of dimensions and the topology are changed, one expects changes with respect to the former discussion. For example, the form of the counterterm leading to a finite actions may be different when the number of dimensions is increased. However, in this particular case, what is important is the ‘seed’ 5-dimensional solution to which we add the flat directions. Thus, the form of the counterterm does not change but the stress tensor will have new components.

A similar computation as for the doubly spinning black hole reveals that the stress tensor of the BM is the one in (2.10) supplemented with the components in the new directions:

$$\tau_{x_i x_i} = \frac{1}{8\pi G_5} \left(-\frac{3}{2} m \frac{1}{r^3} - \frac{5}{3} (a^2 - b^2) \frac{\cos 2\theta}{r^3} + \mathcal{O}(1/r^5) \right) \tag{2.20}$$

This result resembles the tension (per unit length) of the black string.

3 Thermodynamic instability of the black ring

In this section, we discuss the thermodynamics of a doubly spinning ring in the grand canonical ensemble.

So far, we have computed the conserved charges of neutral spinning black objects with two angular momenta by using the quasilocal formalism. However, the quasilocal formalism is a very powerful tool for understanding the thermodynamics in more detail. In particular, one can compute the action and, therefore, the thermodynamic potential.

In what follows, we present a detailed analysis of thermodynamic stability of the doubly spinning black ring — an analysis of the thermodynamic stability of Myers-Perry black hole with two angular momenta can be found in [22].

Let us start by computing the angular velocities and the temperature for this solution. From (A.6) we obtain the following expressions for the angular velocities:

$$\Omega_\psi = \frac{\lambda(1+\nu) - (1-\nu)\sqrt{\lambda^2 - 4\nu}}{4\lambda\sqrt{\nu}k} \sqrt{\frac{1+\nu-\lambda}{1+\nu+\lambda}}, \quad \Omega_\phi = \frac{1}{2k} \sqrt{\frac{1+\nu-\lambda}{1+\nu+\lambda}} \quad (3.1)$$

The area of the event horizon and the temperature (A.7) are

$$\mathcal{A}_H = \frac{32\pi^2 k^3 \lambda(1+\lambda+\nu)}{(1-\nu)^2(y_h^{-1} - y_h)}, \quad T = \frac{\sqrt{\lambda^2 - 4\nu}(1-\nu)(y_h^{-1} - y_h)}{8\pi k \lambda(1+\lambda+\nu)} \quad (3.2)$$

Note that $y_h = \frac{-\lambda + \sqrt{\lambda^2 - 4\nu}}{2\nu}$ is the biggest root of (A.5) which corresponds to the outer event horizon — at this point, it might be useful to emphasize again that $-\infty < y < -1$.

The starting point of the Euclidean approach to black hole thermodynamics is the partition function [18]⁵

$$Z(\beta) = \int d[g, \phi] e^{-I[g, \phi]} \quad (3.3)$$

where ϕ is a collective notation for the matter fields, $d[g, \phi]$ is the measure, and $I[g, \phi]$ is the Euclidean classical action. The gravitational partition function is defined by a sum over all *smooth* geometries (including black holes) that are periodic with period $\beta = T^{-1}$ in the same class of boundary conditions e.g., AF spacetimes.

For our purpose it is enough to consider the saddle point approximation. The grand canonical partition function is then $Z = \text{Tr} e^{-\beta(H - \Omega_a J_a)} \simeq e^{-I_{cl}}$ (here we are interested in black objects with two angular momenta), where I_{cl} is the classical action. The saddle point is usually referred to as a *gravitational instanton*.⁶

The thermodynamic (effective) potential associated to grand canonical ensemble is

$$G[T, \Omega_a] \equiv \frac{I_{cl}}{\beta} = M - TS - \Omega_a J_a \quad (3.4)$$

⁵It should be understood as a low energy effective theory rather than a proper theory of quantum gravity.

⁶A quantum field can be treated as a small perturbation about the gravitational instanton. The next order contribution, which gives the one loop correction, includes also the thermal radiation outside the black hole.

On the Euclidean section, the topology near the horizon is modified⁷ and one has to deal with manifolds with conical singularities. It was shown in [34, 35] that the conical defect has a contribution to the curvature and, consequently, the path integral is rescaled by e^S . However, this can be intuitively interpreted as a consequence of a trace over the macroscopically indistinguishable microstates.

Let us now compute the action for the doubly spinning black ring. Since the Ricci scalar vanishes on-shell, the only contribution to the action is coming from the surface terms. To evaluate these terms, it is convenient to use the (r, θ) coordinate system instead of the (x, y) coordinates — the reason is that the normal to the boundary has just one non-vanishing component. We find

$$\lim_{r \rightarrow \infty} \sqrt{|h|} \left(\sqrt{\frac{3}{2}} \mathcal{R} - K \right) = \frac{2k^2 (\lambda(1 - \nu) - F_5[\nu, \lambda] \cos 2\theta) \sin 2\theta}{(1 + \nu - \lambda)(1 - \nu)} + \mathcal{O}(1/r) \quad (3.5)$$

where $F_5[\nu, \lambda] = 1 + 3\lambda^2 + 6\nu + 5\nu^2 - 4\lambda - 8\lambda\nu$. The expression for the total action is

$$I_{cl} = \beta \frac{\pi k^2}{G_5} \frac{\lambda}{(1 + \nu - \lambda)} \quad (3.6)$$

and satisfies (3.4), which is the quantum statistical relation for the doubly spinning black ring. This can also be regarded as a non-trivial check that the entropy S of this solution is, indeed, one quarter of the event horizon area \mathcal{A}_H .

We have checked that the usual thermodynamic relations

$$S = - \left(\frac{\partial G}{\partial T} \right)_{\Omega_a}, \quad J_a = - \left(\frac{\partial G}{\partial \Omega_a} \right)_{T, \Omega_b} \quad (3.7)$$

are satisfied and so the Gibbs potential $G[T, \Omega_\phi, \Omega_\psi]$ is indeed the Legendre transform of the energy $M[S, J_\phi, J_\psi]$ with respect to S , J_ϕ , and J_ψ .

We want to also point out that, in the light of the new developments in understanding the balance condition for gravity solutions [17], the form of quantum statistical relation hints to the fact that this solution is balanced. Indeed, our results are in perfect agreement with the recent detailed analysis of the global properties of the doubly spinning black ring [33].

Now, we are ready to discuss the thermodynamic stability in the grand canonical ensemble — in appendix B we summarize the thermal stability conditions and present explicit expressions for some response functions we are interested in. We analyze in detail the response functions that signal the (in)stability of the black ring against fluctuations.

We consider first the specific heat at constant angular velocities

$$C_\Omega \equiv T \left(\frac{\partial S}{\partial T} \right)_{\Omega_\phi, \Omega_\psi} \quad (3.8)$$

⁷The origin in the Euclidean spacetime translates to the horizon surface in the Lorentzian spacetime. The Euclidean section can be understood as an effective description where the microstates can not be distinguished.

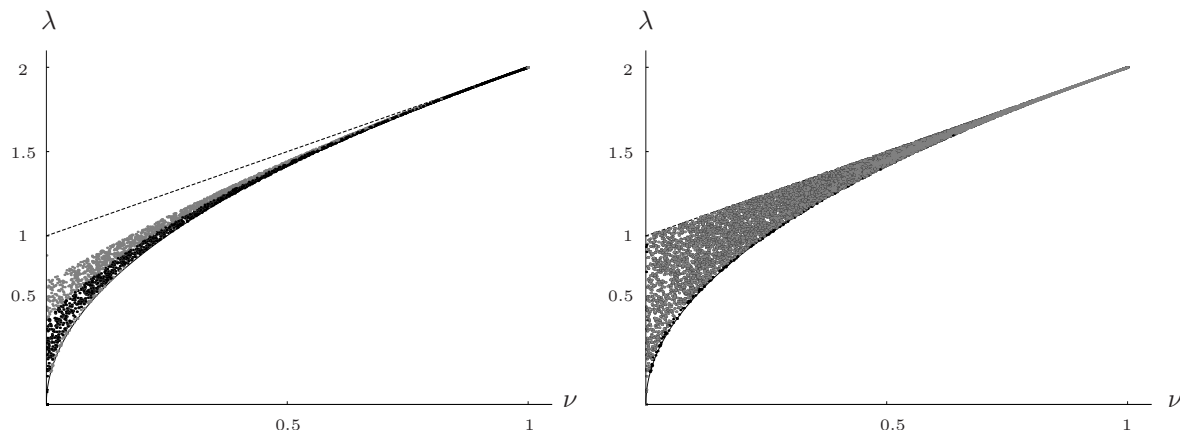


Figure 1. Scatter plots in parameter phase space (ν, λ) for the doubly spinning black ring. The plot on the left shows the regions (10,000 points) where the heat capacities are negative, $C_\Omega < 0$ (*gray*) and $C_J < 0$ (*black*). The regions where the compressibility $\epsilon_{\phi\phi}$ (*gray*) and the $\det[\epsilon]$ (*black*) are negative cover the entire parameter space (plot on the right) implying the local thermal instability of the doubly spinning black ring. The region in the parameter space is bounded: $0 \leq \nu < 1$ and λ by the functions $1 + \nu$ and $2\sqrt{\nu}$, shown as the *dashed* and *solid* lines respectively.

The analytic form of this quantity is too complicated to be written down here. Instead, we show on the left hand side in figure 1 a scatter region in the parameter space of the doubly spinning black ring where this heat capacity C_Ω is negative (*gray* — 10,000 points). Note also that the parameters in the solution (2.13) are constrained and represented as a dashed line for $\lambda = 1 + \nu$ and solid line for the extremal black ring with $\lambda = 2\sqrt{\nu}$.

In a similar way, we explore the region where the specific heat at constant angular momentum

$$C_J \equiv T \left(\frac{\partial S}{\partial T} \right)_{J_\phi, J_\psi} \tag{3.9}$$

is negative. The region (in black) where $C_J < 0$ is shown in the scatter plot on the left of figure 1.

We observe a region in the parameter space of the doubly spinning black hole where both specific heats, C_Ω and C_J , can be positive simultaneously. However, this condition is not sufficient to draw the conclusion of thermodynamic stability: one should also investigate the matrix of ‘isothermal moment of inertia’.

These response functions are defined as

$$\epsilon_{ab} \equiv \left(\frac{\partial J_a}{\partial \Omega_b} \right)_{T, \Omega_{a \neq b}} \tag{3.10}$$

We observe in figure 1 that the spectrum of the matrix of isothermal moment of inertia, $\text{spec}[\epsilon_{ab}]$, is nowhere positive definite in the parameter space.

Since there is no overlap region in parameter space in which all the response functions of interest are positive definite, we conclude that the doubly spinning black ring is unstable in the grand canonical ensemble.

4 Instabilities from thermodynamics

Many known stationary black holes in higher dimensions present a black string or, more general, black brane phase — we will refer to it as the ‘membrane phase’. That is, as the angular momenta are sufficiently increased (the ultra-spinning regime), the behaviour of some black holes and black rings changes to that of extended black branes and strings.

In the next subsection, we deal with the ultra-spinning black holes. From the study of the Gibbs potential’s Hessian, we show the existence and find the locus of the transition points to the membrane phase. We also argue that there is a subtle relation between the microcanonical and grand canonical ensembles that may be at the basis of some of the results for ultra-spinning black holes discussed recently in [10].

The analysis can be extended to (doubly) spinning black rings.⁸ These results are presented in section 4.2.

To compare different examples we will make use of the *dimensionless* expressions for the temperature t , the spin j , and the angular velocity ω defined by

$$t^{D-3} = c_t GMT^{D-3} \quad \omega^{D-3} = c_\omega GM\Omega^{D-3}, \quad j^{D-3} = c_j \frac{J^{D-3}}{GM^{D-2}} \quad (4.1)$$

where the numerical constants are

$$c_t = \frac{2}{(D-2)} \frac{(4\pi)^{D-3}}{\Omega_{D-3}} \left(\frac{D-3}{D-4} \right)^{\frac{D-3}{2}}, \quad c_\omega = \frac{16}{(D-2)} \frac{(D-3)^{\frac{D-3}{2}}}{\Omega_{D-3}}, \quad c_j = \frac{\Omega_{D-3} (D-2)^{D-2}}{2^{D+1} (D-3)^{\frac{D-3}{2}}}. \quad (4.2)$$

4.1 Ultra-spinning black holes

Due to the qualitative changing behaviour of black holes as the dimensions are increased, the authors of [9] have argued that the ultra-spinning black holes — those in $D \geq 6$ dimensions that can have arbitrary large angular momentum per unit mass [6] — become unstable. The transition of these black holes, from behaving like a spherical black hole to behaving like a black membrane as the spin grows, was established to be at the minimum of the temperature. From that point onwards, the temperature increases in a similar way as for the black brane temperature.

The minimum of the temperature where the behavior of the singly spinning black hole changes is determined as [9]

$$\frac{a^2}{r_h^2} = \frac{D-3}{D-5} \quad (4.3)$$

This result was also obtained by using a different method, namely finding the divergences of the ‘Ruppeiner curvature’ [25]. It was shown in [37] that, for a singly spinning Myers-Perry black hole, this curvature⁹ blows-up exactly at the value (4.3) signaling a thermal instability of the system.

⁸The doubly spinning solutions we consider are with the spins in orthogonal planes. Other black hole solutions, where the two spins are parallelly oriented, were also studied [36].

⁹The Ruppeiner curvature is the scalar curvature of the Hessian matrix of the entropy.

A qualitative understanding of this fact is related to the observation that, as the spin becomes large, the event horizon spreads out in the plane of rotation: it becomes a higher dimensional ‘pancake’ approaching the geometry of a black brane.

The existence of the ultra-spinning limit resembling black branes has a remarkable consequence. Black branes were shown to be classically unstable [11] so that the ultra-spinning black holes would inherit the Gregory-Laflamme instability. The threshold of the classical instabilities and the connection to the thermal instability as conjectured by [26] (see, also, [27]) requires a linearized analysis of the perturbations of the black hole solutions.

However, the transition to a membrane-like phase of the rapidly spinning black holes can be established from the study of the thermodynamics of the system. The existence and location of the threshold of this regime is signaled by the minimum of the temperature and the maximum angular velocity as functions of the angular momentum.

It was observed in [10] that, for ultra-spinning black holes, this is in tight correspondence with a vanishing eigenvalue of the Hessian of the Gibbs potential. A complete thermodynamic analysis, though, should be based on the full Hessian of the thermodynamic potential rather than only a study of the determinant.¹⁰ We will see in the next subsection that the membrane phase of a doubly spinning ring is not signaled by a zero-eigenvalue of the Gibbs potential’s Hessian.

For ultra-spinning black holes, there is a *direct* relation between (some response functions in) the microcanonical and grand canonical ensembles.¹¹ To see that, let us compare the expressions of two particular response functions in these two ensembles:

$$\left(\frac{\partial^2 S}{\partial J^2}\right)_M = -\frac{1}{T} \left(\frac{\partial \Omega}{\partial J}\right)_M + \frac{\Omega}{T^2} \left(\frac{\partial T}{\partial J}\right)_M \quad \text{and} \quad \left(\frac{\partial^2 G}{\partial \Omega^2}\right)_T = -\left(\frac{\partial J}{\partial \Omega}\right)_T \quad (4.4)$$

We have checked that in the particular case of the singly spinning black hole, indeed, these two response functions are inverse proportional at the particular point where the temperature has a minimum. Therefore, an inflexion point in the microcanonical ensemble corresponds to a divergence of the corresponding response function in the grand canonical ensemble. This may well be an explanation for the results obtained in [10]. Moreover, this point should not be considered as a sign for an instability or a new branch but a transition to an infinitesimally nearby solution along the same family of solutions. The numerical evidence of [10] supports this connection with the zero-mode perturbation of the solution.

We now examine the situation for a more general family of ultra-spinning Myers-Perry black holes with multiple spin parameters, a_i , where $i = 1, 2, \dots, N$ and $N = [(D - 1)/2]$. The black hole is characterized by the mass parameter μ and the horizon radius r_h (the largest root of)

$$\mu = \frac{1}{r_h^{1+\epsilon}} \prod_{i=1}^N (r_h^2 + a_i^2), \quad (4.5)$$

¹⁰A spinodal is defined as a line separating the regions of stability and instability of a homogeneous system. It is important to emphasize that all spinodals are zero-determinant lines, but in general not all zero-determinant lines are spinodal.

¹¹Different ensembles correspond to different physical conditions and so, in more general cases, one does not expect such a relation.

by which we can express the thermodynamics

$$M = \frac{\Omega_{D-2}}{16\pi G_D} (D-2) \mu, \quad J_i = \frac{\Omega_{D-2}}{16\pi G_D} a_i \mu, \quad \Omega_i = \frac{a_i}{r_h^2 + a_i^2},$$

$$\mathcal{A}_H = \Omega_{D-2} \mu r_h, \quad T = \frac{1}{2\pi r_h} \left(r_h^2 \sum_{i=1}^N \frac{1}{r_h^2 + a_i^2} - \frac{1+\epsilon}{2} \right), \quad (4.6)$$

where $\epsilon = \text{mod}_2 D$. A sufficient, but not necessary, condition for the existence of ultra-spinning black holes was given in [9]. In even(odd) dimensions at least one(two) of the spins should be much smaller than the rest. The ultra-spinning regime is obtained in the limit

$$0 \leq a_1, a_2, \dots, a_k \ll a_{k+1}, \dots, a_N \rightarrow \infty \quad (4.7)$$

where $N-1 \geq k \geq 1 + \epsilon$. The generic limiting black brane metric whether static, with all finite angular momenta a_1, \dots, a_k vanishing, or spinning, with some a_1, \dots, a_k non-vanishing, is the product $S^{D-2(N-k+1)} \times \mathbb{R}^{2(N-k)}$.

Our focus will be on the case in which the black hole has at least two large spins and we set the remaining angular momenta to zero. When the angular momenta are equal, $J_{k+1} = \dots = J_N = J$, the Ruppeiner curvature scalar blows up at¹²

$$\frac{a^2}{r_h^2} = \frac{D-3}{2k-1-\epsilon}. \quad (4.8)$$

According with the arguments in [25], this signals a thermodynamic instability. However, the expected new phase should correspond to the black membrane phase of ultra-spinning black holes and not to a new branch of solutions.

This is further supported by examining the eigenvalues of the Hessian of the Gibbs potential. Indeed, we find that the divergences of the Ruppeiner curvature pinpoint the zero of the determinant of the Gibbs potential's Hessian.

Also, by studying the temperature

$$T = \frac{(D-3) \left(1 + \frac{n}{(D-3)} \frac{4J^2}{S^2} \right)}{4S^{\frac{1}{D-2}} \left(1 + \frac{4J^2}{S^2} \right)^{\frac{D-1+n}{2(D-2)}}}, \quad n = 2k-1-\epsilon \quad (4.9)$$

we find that the temperature has a minimum at exactly (4.8), while the angular velocity Ω reaches its maximum value. Therefore for these more general ultra-spinning black holes, similarly to the singly spinning situation discussed in [9], once the minimum is reached the temperature increases and the angular velocity decreases signaling a transition to a membrane phase. This conduct is shown in figure 2 (I points on the solid thin line) for the particular case of $D=7$, $k=2$ so $j_1 = j_2 \equiv j_\phi$ and $j_3 = 0$.

Another case of interest is the ultra-spinning black holes that resemble spinning black branes, when some of the slower spins are non-zero $a_1, \dots, a_k \neq 0$. It is not our goal to make a detailed analysis of this case here. Nevertheless, in all the cases where the non-vanishing spins are set equal, we find divergences of the Ruppeiner scalar curvature which could help to detect the threshold of their membrane phase.

¹²Note that $\frac{a^2}{r_h^2} = \frac{4J^2}{S^2}$ and in the particular case when $k = k_{\max}$ our results agree with those of [37].

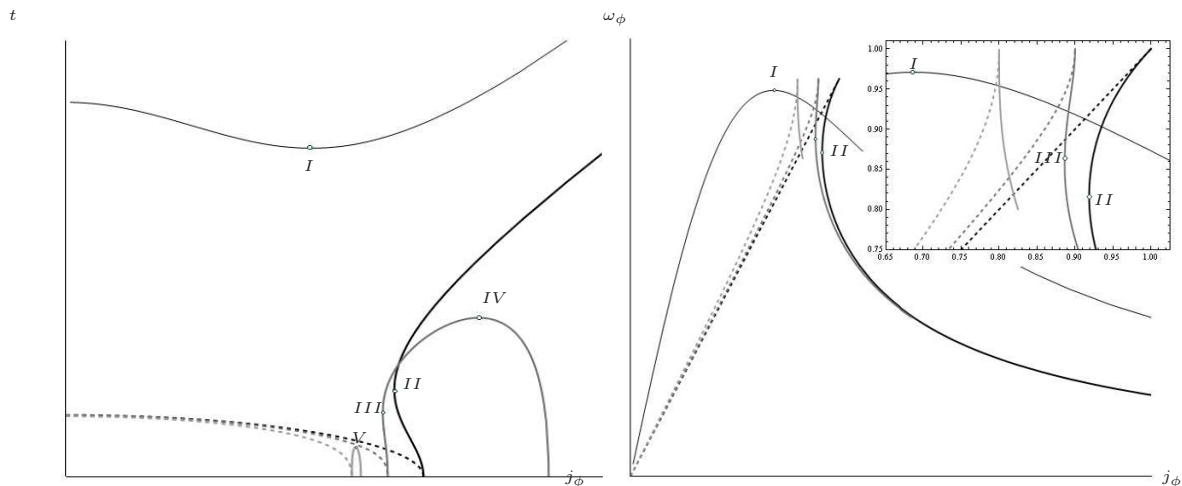


Figure 2. Plots of the temperature (left hand side) and angular velocity (right hand side) as functions of the angular momentum j_ϕ , for a fixed mass, for different black objects. These include the singly spinning Myers-Perry black holes in five dimensions of space-time (*black dashed line*) and its seven dimensional cousin (*solid thin line*). The singly (*solid thick*) and doubly spinning black ring (*light gray*) for different values of angular momenta (right towards left) $j_\psi = 0.1, 0.2$ on the S^2 , are also shown here. For this same values of the second angular momenta the five-dimensional doubly spinning black holes are represented by the *dashed light gray lines*.

4.2 Membrane phase of black rings

Other solutions, e.g. the black ring with one angular momentum, also exhibit an ultra-spinning behavior. The black ring, which is characterized by the radii r_0 and R of the spheres S^{D-3} and S^1 , respectively, becomes *thin* in this limit (when $r_0 \ll R$).¹³

Since the final expressions for the response functions are very complicated for the doubly spinning ring (see appendix B), we prefer to present the ‘conjugacy diagram’ of the angular velocity versus the angular momentum and the plot of the temperature as a function of the angular momentum, both for a fixed mass (see figure 2).

For the singly spinning black ring, an analysis of the temperature as a function of the angular momentum was presented in [40]. In this case (solid thick line in the plot on the left hand side of figure 2), the temperature does not have a minimum, but there exists a turning point.¹⁴ In our analysis, the turning point *II* for the black ring plays a similar role as the minimum of the temperature for the black hole. That is, it signals a change in the thermodynamic behaviour of the black ring. In fact, it is the starting point of the ultra-spinning regime where the black ring can be approximated by a boosted black string.

Using the Poincaré ‘turning point’ method, this special point was carefully studied in [40]. In particular, they found a divergence of the Ruppeiner curvature. In the conjugacy diagram (on the right) there is also a turning point *II* at the same minimum value of the angular momentum j_ϕ .

¹³This *thin* regime was essential to find perturbatively the higher dimensional black rings cousins. Moreover, a generalization of this construction to black branes led to the construction of blackfolds [38, 39].

¹⁴We would like to point out that in (Sherk-Schwarz-)Anti-de Sitter, there is also a turning point [41].

The question is then if there still is a relation between the microcanonical and grand canonical ensembles in this case. We have explicitly checked, using the results of [7], that one of the eigenvalues of the Hessian of the Gibbs potential is zero at this specific point II while the second eigenvalue never changes its sign. Therefore, we conclude that the turning point is the onset of the ultra-spinning black string phase.

A far more richer structure is found for the doubly spinning black ring. The angular momentum on S^2 is bounded as $j_\psi \in [0, 1/4]$ and for a specific j_ψ the black ring can always be extremal (in the limit $\lambda \rightarrow 2\sqrt{\nu}$ as shown in [31]). But besides extremality, according to how large j_ψ is, the behaviour of the doubly spinning black ring changes. There are two distinctive regions in the microcanonical ensemble. On one hand, for $0 \leq j_\psi < 1/5$, there are phases with the characteristic cusp for black rings with the two (fat and thin) branches. But on the other hand, for $1/5 \leq j_\psi \leq 1/4$, the fat black ring branch disappears and so there are no cusps. As we will discuss in what follows, this will become relevant to understand the physics and regimes of the doubly spinning black ring.

To explore how the physics of the black rings at fixed mass is modified as we turn on the angular momenta along the S^2 , j_ψ , we will study the temperature and angular velocity as functions of the S^1 angular momentum, j_ϕ , for different fixed values of j_ψ .

Doubly spinning black rings with $0 \leq j_\psi < 1/5$. In this case the situation is similar, to some extent, to what we found before for the singly spinning black ring. In this range, as we turn on j_ψ , there also are turning points with tangents of infinite slope signaling the onset of the black membrane phase that coincide with the cusps (in parameter space), namely at $\lambda = -(1/4)(1 + \nu - (9 + \nu)^{1/2}(1 + 9\nu)^{1/2})$. Figure 2 shows this change in behavior explicitly. On the left (light gray curve) in the t vs. j_ϕ diagram, we observe the turning point III that corresponds to the minimum value of j_ϕ angular momentum. In the conjugacy diagram ω vs. j_ϕ (also the light gray curve), the corresponding point (III) is a turning point.

But an interesting difference with the single spin black ring occurs for a large enough S^1 angular momentum. The temperature of the black ring in its membrane phase increases while the area of the event horizon decreases up to a point where the spin-spin interaction is large enough making a turn to abruptly become extremal with zero temperature. This is the maximum critical value labeled IV in figure 2. Therefore, the black membrane phase exists between points III and IV .

Finally, as for the singly spinning black ring, we computed the eigenvalues of the Hess[G] and found that none of them are zero the relevant turning points.

Doubly spinning black rings with $1/5 \leq j_\psi \leq 1/4$. This black rings (lightest gray line in figure 2) with larger S^2 angular momentum show no turning points and therefore have no membrane phase. This mimics its behavior in the phase diagram of microcanonical ensemble where the cusp and fat branch of these doubly spinning black ring disappear [31]. The lack of a fat black ring branch seems to coincide with the lack of a black membrane phase. Therefore such solutions would never be captured with long distance effective approaches [38, 39].

Note that for certain fixed $j_\psi \in [1/5, 1/4]$ the temperature grows, reaching a maximum at V and rapidly decreasing to zero to become extremal. It would be interesting to explore the physical meaning of these points which we observe to correspond to an inflection point $\partial^2 S / \partial^2 J = 0$ for fixed mass.

In summary, we have found examples for which the zero eigenvalues of the Hessian of the Gibbs potential can also be turning points (with tangents of infinite slope) and not just critical points as for the Myers-Perry black holes. Other less symmetric solutions, such as the doubly spinning black ring, do not show this connection between the zeros of the Hess[G] and the onset of the black membrane phase. Moreover, we showed that certain doubly spinning black ring, those with $1/5 \leq j_\psi \leq 1/4$, have no membrane phase. Therefore, these particular solutions fall into the same category as other black holes with no membrane phase as the four dimensional Kerr black hole and the five dimensional Myers-Perry black hole.

5 Discussion

In this paper we have analyzed in detail the thermodynamic stability of neutral doubly spinning black objects. We have analytically computed the response functions and presented strong evidence showing that the doubly spinning black ring is thermodynamically unstable. That is, there is no region in the parameter space in which all the response functions are positive definite.

We have provided an explanation of why the microcanonical and grand canonical ensembles for ultra-spinning black holes are related in a very specific way. An inflexion point in the microcanonical ensemble corresponds to a divergence of the corresponding response function in the grand canonical ensemble. We will comment on the validity of this argument and on the significance of these results in the last part of this section.

The onset of a membrane phase of different doubly spinning black objects was identified. We have found that the onset of the black membrane phase for all black holes that we have studied (except for the less symmetric doubly spinning black ring) is characterized by at least one zero eigenvalue of the Hessian of the Gibbs potential. A tight relation with the classical perturbations, where a transition to an infinitesimally nearby solution of the same family branch happens, is expected in the cases where the connection between membrane and zero-eigenvalue of Hess[G] exists. The numerical evidence of [10, 24] supports this connection precisely with the zero-mode perturbation of the solution.

We now discuss the thermodynamics of doubly spinning black rings in the grand canonical ensemble. An analysis of this solution in the microcanonical ensemble was presented in [31]. In general, for black holes, the entropy is used to obtain the phase diagrams in the microcanonical ensemble while the mass is kept fixed. However, in general relativity it makes more sense to use the total energy instead of the entropy. The reason is that this would require appropriate boundary conditions.

It is important to emphasize that it is not known how the background subtraction method can be applied to black rings, because it is not clear how to choose the background

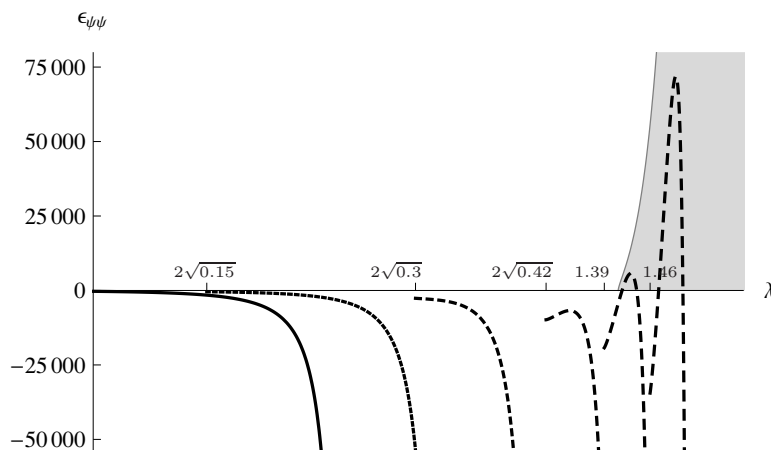


Figure 3. Plot of the response function $\epsilon_{\psi\psi}$ as a function of λ . The *gray* curve corresponds to the compressibility of the singly spinning black ring, namely $\nu = 0$. As the angular momentum along the S^2 is increased (*dashed line* left towards right) the isothermal moment of inertia for different values of $\nu = 0.15, 0.3, 0.42, 0.48, 0.53$ changes and becomes positive for values of $\nu > 0.46690042$ and $\lambda > 1.40685$ (shaded *gray region*).

solution. We therefore used the counterterm method to compute the action and the grand thermodynamic potential (which is a Legendre transform of the energy). Since the expressions of the response functions are too complicated for analytical treatment (see appendix B), we have plotted the regions in the parameter space where they are positive.

Let us now compare some of our results with other well understood examples — at this point we are interested just in the thermodynamic instabilities, not in the relation with dynamical ones. The fact that the Schwarzschild black hole has a negative heat capacity means that the thermodynamic ensemble is dominated by diffuse radiation states rather than black holes states but is classically dynamically stable at the linearized level. When adding angular momentum the situation changes and the heat capacity can be positive for a large enough angular momentum. However, this condition is not enough to conclude that the system is thermodynamically stable. The stability also implies that when angular momentum is added to the system the angular velocity goes up.

For a black ring with one angular momentum the heat capacity can be positive, but the momentum of inertia is always negative [7]. Therefore, the singly spinning black ring is thermodynamically unstable. As in the case of one angular momentum, the heat capacity of a doubly spinning black ring can be positive in some region of the parameter space. However, there is a key difference when the second angular momentum is turned on. That is, the component of the momentum of inertia associated to S^1 of the black ring can become positive — this is explicitly shown in figure 3.

Since there are two angular momenta one should also investigate the effect of coupled ‘angular’ inhomogeneities. A careful study of the determinant of the momentum of inertia matrix shows that there is no region in the parameter space with the desired properties and so the doubly spinning black ring is also thermodynamically unstable.

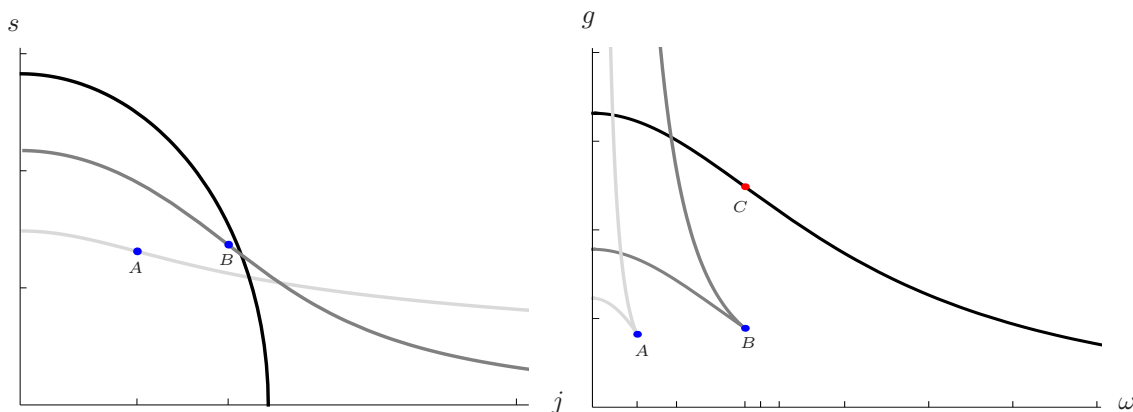


Figure 4. The microcanonical phase diagram - entropy, s , as a function of the angular momentum, j - for fixed mass (on the *left*) of the singly spinning Myers-Perry black hole in $D = 5, 6, 10$ dimensions of space time (*black, gray and lightgray* lines respectively). The grand canonical phase diagram (on the *right*) - Gibbs potential, g , as a function of the angular velocity, ω - for fixed temperature of the black hole. The points A, B correspond to $a/r_h = \sqrt{3}, \sqrt{\frac{7}{5}}$ for six and ten dimensions, and in general for $D > 5$ the points are $a/r_h = \sqrt{(D-3)/(D-5)}$; the change in convexity of the entropy corresponds to a blowing up of the convexity of the Gibbs potential. The point C instead, at $a/r_h = 1/\sqrt{3}$ and where the Gibbs potential has a change in convexity, has no analog in the microcanonical scheme.

We would like now to discuss in more detail some of the results for ultra-spinning black holes presented in section 4. In figure 4, we show the points (A, B) in the grand canonical ensemble that correspond to inflexion points (A, B) in the microcanonical ensemble. This can be quantitatively understood by comparing the particular response functions in eq. (4.4) at a very special point in the parameter space, namely where the temperature has a minimum.

We have explained that this argument applies to this particular case but not in general. A counterexample is the 5-dimensional black hole with one angular momentum. In this case there is no relation between ensembles in the sense that there is no special point in the microcanonical ensemble which corresponds to the inflexion point (C) in the grand canonical ensemble. Moreover we have checked and there are no points where an eigenvalue of the Hessian of the Gibbs potential vanishes. Therefore, it should not be considered as a sign for a membrane phase — most probably, it is similar with the Schwarzschild black hole example for which the thermodynamical instability is not related to a dynamical one.

One can also consider the ultra-spinning black holes with some of the finite angular momenta non-zero. In odd(even) spacetime dimensions, the metric of an ultra-spinning black hole with all but two(one) of the spins finite and non-zero will reduce to that of a spinning black brane. As we have shown in section 2, the counterterm method can also be applied to spinning black branes and the results are similar to the ones for the ‘seed’ spinning black hole solution. We have computed the *renormalized* action to find the Gibbs potential and we expect similar thermal instabilities as for the corresponding black holes. As we already emphasized, though, in all these examples we expect that the thermodynamic instabilities do not signal a dynamical instability or a new branch, but rather a transition to an infinitesimally nearby solution along the same family of solutions [10].

It is remarkable that our study of thermodynamic instabilities provides to some extent information about the zero-mode of the ‘Gregory-Laflamme instability’ and it may well be the starting point for a more detailed study of dynamical instabilities. The extension of the dynamical stability studies to spinning black branes has not been yet developed. The analytic theory of perturbations is much more involved. However, we expect that the spinning black branes suffer from similar instabilities as the static ones. We hope that the observations made in this paper will be useful in future investigations of the perturbations of higher-dimensional spinning black rings, black holes and spinning black branes.

Acknowledgments

We would like to thank Ido Adam for interesting conversations. DA and MJR would also like to thank Robb Mann and Cristian Stelea for collaboration on related projects and valuable discussions.

A Temperature and angular velocities

Consider a general stationary 5-dimensional metric that corresponds to a black object with two angular momenta:¹⁵

$$ds^2 = g_{tt}(\vec{x})dt^2 + 2g_{t\phi}(\vec{x})dtd\phi + 2g_{t\psi}(\vec{x})dtd\psi + g_{\phi\phi}(\vec{x})d\phi^2 + 2g_{\phi\psi}(\vec{x})d\phi d\psi + g_{\psi\psi}(\vec{x})d\psi^2 + g_{\alpha\beta}(\vec{x})dx^\alpha dx^\beta \quad (\text{A.1})$$

∂_t , ∂_ϕ , and ∂_ψ are Killing vectors. Rewrite the metric in the ADM form

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad (\text{A.2})$$

with lapse function

$$N^2 = -g_{tt} + g_{\phi\phi}(N^\phi)^2 + g_{\psi\psi}(N^\psi)^2 + 2g_{\phi\psi}N^\phi N^\psi \quad (\text{A.3})$$

and shift vector

$$N^\phi = \frac{g_{t\psi}g_{\phi\psi} - g_{\psi\psi}g_{t\phi}}{g_{\phi\psi}^2 - g_{\phi\phi}g_{\psi\psi}}, \quad N^\psi = \frac{g_{t\phi}g_{\phi\psi} - g_{\phi\phi}g_{t\psi}}{g_{\phi\psi}^2 - g_{\phi\phi}g_{\psi\psi}} \quad (\text{A.4})$$

The event horizon is obtained for

$$N^2 = 0 \quad (\text{A.5})$$

In other words, it is a Killing horizon of $\partial_t + \Omega_\phi \partial_\phi + \Omega_\psi \partial_\psi$, where Ω_ϕ and Ω_ψ are the angular velocities defined as the shift vectors at the horizon:

$$\Omega_\phi = -N^\phi \Big|_H, \quad \Omega_\psi = -N^\psi \Big|_H \quad (\text{A.6})$$

¹⁵A similar analysis for one angular momentum can be found in [7, 42].

Black holes are thermodynamic objects: the causal structure of spacetime can influence the physics of a quantum field. The vacuum fluctuations near the event horizon cause the black hole to emit particles with a thermal spectrum. The Euclidean regularity at the horizon is equivalent to the condition that the black hole is in thermodynamical equilibrium.

By a straightforward computation one can eliminate the conical singularity in the Euclidean section, (it, r) , to obtain the periodicity of the Euclidean time. In this way, one obtains the following expression for the temperature of the black hole:

$$T = \frac{(N^2)'}{4\pi\sqrt{g_{rr}N^2}} \Big|_H \tag{A.7}$$

We have used these definitions to compute the corresponding physical quantities of the doubly spinning ring.

B Conditions for thermodynamic stability

In this appendix, we present the conditions for the thermodynamic stability and we also give some useful explicit expressions for the response functions used in section 4 — we follow closely [43].

For simplicity, let us start with a black hole with one angular momentum. We are interested in the thermodynamic potentials: the energy and its Legendre transforms.

The basic extremum principle of thermodynamics (for the entropy S) implies both that $dS = 0$ and that $d^2S < 0$. The second condition determines the stability of predicted equilibrium states. The stability criterion in energy representation requires that an equilibrium state at fixed S and J is a state of minimum energy, namely a minimum of $E[S, J]$. The local stability conditions ensure that inhomogeneities of either S and J separately

$$\left(\frac{\partial^2 E}{\partial S^2}\right)_J = \left(\frac{\partial T}{\partial S}\right)_J \geq 0, \quad \left(\frac{\partial^2 E}{\partial J^2}\right)_S = \left(\frac{\partial \Omega}{\partial J}\right)_S \geq 0 \tag{B.1}$$

and *also* that a coupled inhomogeneity of S and J together

$$\det(\text{Hess}(E)) = \frac{\partial^2 E}{\partial S^2} \frac{\partial^2 E}{\partial J^2} - \left(\frac{\partial^2 E}{\partial S \partial J}\right)^2 \geq 0 \tag{B.2}$$

do not decrease the energy.

In more generality the stability criterion states that the thermodynamic potentials are *convex* functions of their *extensive* variables and *concave* functions of their *intensive* variables (see, e.g., [43]).

For a grand canonical ensemble defined at fixed temperature T and angular velocities Ω_a (intensive variables) the associated potential, the Gibbs free energy, satisfies the following relations:

$$G[T, \Omega] = E - TS - \Omega J, \quad dG = -SdT - Jd\Omega \tag{B.3}$$

In this case, the local stability conditions following from the convexity of the Gibbs function yield

$$\left(\frac{\partial^2 G}{\partial T^2}\right)_\Omega = -\left(\frac{\partial S}{\partial T}\right)_\Omega \leq 0, \quad \left(\frac{\partial^2 G}{\partial \Omega^2}\right)_T = -\left(\frac{\partial J}{\partial \Omega}\right)_T \leq 0 \quad (\text{B.4})$$

and

$$\det(\text{Hess}(G)) = \frac{\partial^2 G}{\partial T^2} \frac{\partial^2 G}{\partial \Omega^2} - \left(\frac{\partial^2 G}{\partial T \partial \Omega}\right)^2 \geq 0 \quad (\text{B.5})$$

Equivalently, the heat capacities (C_Ω , C_J) and the ‘isothermal moment of inertia’ or ‘compressibility’ $\epsilon \equiv (\partial J / \partial \Omega)_T$ should be positive definite.

A generalization for two angular momenta is straightforward (see, also, [22]). The Hessian is a 3×3 matrix

$$\text{Hess}(G) = (-1) \begin{pmatrix} C_\Omega T^{-1} & \alpha_b \\ \alpha_a & \epsilon_{ab} \end{pmatrix}$$

where the matrix components are $\alpha_a = \left(\frac{\partial J_a}{\partial T}\right)_\Omega$, $\epsilon_{ab} = \left(\frac{\partial J_a}{\partial \Omega_b}\right)_T$, and the indices cover the angular directions $a, b = \phi, \psi$.

Considering the relationship between the specific heats $C_\Omega = C_J + T(\epsilon^{-1})_{ab} \alpha^a \alpha^b$ it can be shown that a thermodynamically stable system is characterized by positive heat capacities $C_\Omega > 0$ and $C_J > 0$ and, also, a positive definite matrix of isothermal momenta of inertia, i.e. $\text{spec}[\epsilon_{ab}] > 0$.

The $(\psi\psi)$ -component of the isothermal moment of inertia tensor is

$$\epsilon^{\psi\psi} = -\frac{4k^4 \pi \lambda (1 + \lambda + \nu)}{G_5 (\nu - 1)^4 (1 - \lambda + \nu)^2} \frac{F(\lambda, \nu)}{\sqrt{\lambda^2 - 4\nu} (\lambda^2 + \lambda^2 \nu - 8\nu) + \lambda^3 (1 + \nu) - 2\lambda (4\nu + \nu^2 - 1)}$$

where

$$\begin{aligned} F(\lambda, \nu) = & \lambda^6 (1 + \nu)^2 + \lambda^5 (1 + \nu)^2 (1 + \sqrt{\lambda^2 - 4\nu} + \nu) \\ & + \lambda^4 (1 + \nu) [4 + \sqrt{\lambda^2 - 4\nu} + \nu (2(1 + \sqrt{\lambda^2 - 4\nu}) + \nu(-24 + \sqrt{\lambda^2 - 4\nu} + 2\nu))] \\ & - 16\sqrt{\lambda^2 - 4\nu} + \nu [1 + \nu(-14 + \nu(18 + (-14 + \nu)\nu))] \\ & + 2\lambda^3 (1 + \nu) [2 + \sqrt{\lambda^2 - 4\nu} + \nu(-25 + \sqrt{\lambda^2 - 4\nu} + \nu(13 - 11\sqrt{\lambda^2 - 4\nu} \\ & + \nu(-7 + \sqrt{\lambda^2 - 4\nu} + \nu))] + 2\lambda^2 (1 + \nu) [2 + \sqrt{\lambda^2 - 4\nu} + \nu(-32 - 26\sqrt{\lambda^2 - 4\nu} \\ & + \nu(32 + 14\sqrt{\lambda^2 - 4\nu} + \nu(16 - 6\sqrt{\lambda^2 - 4\nu} + (-2 + \sqrt{\lambda^2 - 4\nu})\nu))] \\ & - 4\lambda [-1 + \nu(10 + 4\sqrt{\lambda^2 - 4\nu} + \nu(-89 - 12\sqrt{\lambda^2 - 4\nu} + \nu(-12(-6 + \sqrt{\lambda^2 - 4\nu}) \\ & + \nu(-23 + 4\sqrt{\lambda^2 - 4\nu} + (-2 + \nu)\nu))] \end{aligned}$$

The determinant is

$$\epsilon = \frac{4k^8 \pi^2 \lambda^2 (\lambda - \sqrt{\lambda^2 - 4\nu}) (\lambda + \sqrt{\lambda^2 - 4\nu})^2 \sqrt{\nu} (1 + \lambda + \nu)^5}{(G_5)^2 (-1 + \nu)^4 [-\lambda^2 + (1 + \nu)^2]^{3/2} [\nu(-\lambda^2 + (1 + \nu)^2)]^{3/2}} \frac{G(\lambda, \nu)}{Z(\lambda, \nu)} \quad (\text{B.6})$$

where

$$\begin{aligned}
 Z(\lambda, \nu) &= 8\sqrt{\lambda^2 - 4\nu}\nu - \lambda^3(1 + \nu) - \lambda^2\sqrt{\lambda^2 - 4\nu}(1 + \nu) + 2\lambda(-1 + 4\nu + \nu^2) \\
 G(\lambda, \nu) &= \lambda^5 + 7\lambda^4(1 + \nu) - \lambda^2(1 + \nu)[1 + 3\sqrt{\lambda^2 - 4\nu} + (2 - 3\sqrt{\lambda^2 - 4\nu})\nu + \nu^2] \\
 &\quad + \lambda^3[5 - 6\sqrt{\lambda^2 - 4\nu} + (26 + 6\sqrt{\lambda^2 - 4\nu})\nu + 5\nu^2] - 8\nu(1 + 11\nu + 11\nu^2 + \nu^3) \\
 &\quad - \lambda[(8 - 27\sqrt{\lambda^2 - 4\nu})\nu + 9(16 + 3\sqrt{\lambda^2 - 4\nu})\nu^2 + (8 + 3\sqrt{\lambda^2 - 4\nu})\nu^3 - 3\sqrt{\lambda^2 - 4\nu}]
 \end{aligned}$$

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