AN ADI SCHEME FOR A BLACK HOLE PROBLEM

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Abstract. We outline an implicit finite differencing scheme for solving hyperbolic partial differential equations, and describe a 3-D black hole problem to which the scheme is applied.

1 INTRODUCTION

One issue in numerical relativity is how to solve the hyperbolic partial differential equations (PDEs) which appear as evolution equations for the geometric and matter variables. Typically a finite grid is introduced, and the PDEs are replaced by finite difference approximations, so that the geometric and matter variables are numerically calculated at the grid points. However, once we choose to use finite differencing we have to decide between an *explicit* or *implicit* scheme.

Traditionally explicit differencing schemes have been used, these are easy to derive and to implement in a code. They suffer, however, from a time step constraint needed to maintain stability. For the d-dimensional wave equation this constraint is, $\Delta t \leq \Delta/c\sqrt{d}$, where Δ is the grid spacing and c is the wave speed. Alternatively an implicit scheme is in general *unconditionally stable*, that is the size of the time step may be decided by accuracy considerations alone. Implicit schemes are usually avoided because they involve inverting huge matrices, however in Cardiff we have been developing *Alternating Direction Implicit* or *ADI* schemes. These ADI schemes can be constructed to remain as accurate as explicit or implicit schemes, as stable as implicit schemes, and yet as computationally tractable as explicit schemes.

2 AN ADI SCHEME FOR THE WAVE EQUATION

Consider the usual 3-D wave equation with sources

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = 4\pi\rho(\mathbf{x},t). \tag{1}$$

Defining

$$\Phi^n \equiv \Phi^n_{i,j,k} := \Phi(n\Delta t, i\Delta, j\Delta, k\Delta),$$
$$\delta_x^2\Phi^n := \Phi^n_{i+1,j,k} - 2\Phi^n_{i,j,k} + \Phi^n_{i-1,j,k}, \quad \text{etc.},$$

we can write one possible ADI scheme for (1), due to Lees (1962), as a system of three tridiagonal matrix equations, each of which must be solved over the grid to advance the solution by one timestep

$$(1-\theta r^2\delta_x^2)\Phi^{*n+1} = 2\Phi^n - \Phi^{n-1} + r^2(\delta_x^2+\delta_y^2+\delta_z^2)[(1-2\theta)\Phi^n + \theta\Phi^{n-1}] + r^2\theta(\delta_y^2+\delta_z^2)\Phi^{n-1} + 4\pi c^2\Delta t^2\rho^n \tag{2a}$$

$$(1-\theta r^2\delta_y^2)\Phi^{**n+1} = \Phi^{*n+1} - \theta r^2\delta_y^2\Phi^{n-1} \tag{2b}$$

$$(1-\theta r^2\delta_z^2)\Phi^{n+1} = \Phi^{**n+1} - \theta r^2\delta_z^2\Phi^{n-1}. \tag{2c}$$

Here r is the *Courant parameter*, $r = c\Delta t/\Delta$ and θ is the *implicit weighting parameter.*

Eliminating the intermediate solutions Φ^{*n+1} and Φ^{**n+1} reduces (2) to the standard second order centred implicit approximation with the addition of an extra term of order $O(\Delta t^4)$, thus overall a second order scheme is retained. A stability analysis reveals that if $\theta \geq .25$ this scheme is unconditionally stable.

3 THE BLACK HOLE PROBLEM

To test our numerical schemes on a non-linear relativistic problem we are now applying them to a perturbed Schwarzschild black hole problem. Using a linearised formulation we find that the evolution equations resemble the 3-D wave equation making the application of the ADI scheme trivial. Also modelling the linearised problem we are free from the problems associated with evolving black holes. This problem has been investigated previously using Green-function techniques *e.g.* Oohara (1986).

3.1 The Evolution Equations

To derive the field equations for the perturbations we expand the Einstein field equations, $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi T_{\alpha\beta}$, $(c = G = 1)$, about a given background metric $g^{(0)}_{\alpha\beta}$, and background matter distribution $T^{(0)}_{\alpha\beta}$. We thus write

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h_{\alpha\beta}, \qquad T_{\alpha\beta} = T^{(0)}_{\alpha\beta} + \theta_{\alpha\beta}, \tag{3}$$

where $h_{\alpha\beta} \ll g^{(0)}_{\alpha\beta}$ and $\theta_{\alpha\beta}$ describes the matter causing the perturbations. We can then write the zeroth and first order field equations

$$R^{(0)}_{\alpha\beta} - \frac{1}{2}g^{(0)}_{\alpha\beta}R^{(0)} = 8\pi T^{(0)}_{\alpha\beta}, \tag{4}$$

$$\begin{aligned} -\frac{1}{2}\nabla^\epsilon\nabla_\epsilon h_{\alpha\beta} + \frac{1}{2}\nabla_\alpha\nabla_\gamma h^\gamma{}_\beta + \frac{1}{2}\nabla_\beta\nabla_\gamma h^\gamma{}_\alpha - \frac{1}{2}\nabla_\beta\nabla_\alpha h \\ +\frac{1}{2}g^{(0)}_{\alpha\beta}\nabla^\epsilon\nabla_\epsilon h - \frac{1}{2}g^{(0)}_{\alpha\beta}\nabla_\gamma\nabla_\delta h^{\gamma\delta} + \frac{1}{2}g^{(0)}_{\alpha\beta}R^{(0)}_{\gamma\delta}h^{\gamma\delta} - \frac{1}{2}R^{(0)}h_{\alpha\beta} \\ +\frac{1}{2}R^{(0)}_{\alpha\gamma}h^\gamma{}_\beta + \frac{1}{2}R^{(0)}_{\beta\gamma}h^\gamma{}_\alpha + R^{(0)}_{\gamma\alpha\beta\delta}h^{\gamma\delta} = 8\pi\theta_{\alpha\beta}. \end{aligned} \tag{5}$$

Here the Riemann tensor, $R^{(0)}_{\alpha\beta\gamma\delta}$, and the covariant derivatives, ∇_α, are calculated with respect to the background metric, $g^{(0)}_{\alpha\beta}$, only.

We have to choose coordinate gauges for both the background spacetime and the perturbation field. Our ADI schemes have been formulated in Cartesian coordinates since we believe that for generality the coordinate system used should have no special association with the black hole(s). This has lead us to use quasi-Cartesian isotropic coordinates (t, x, y, z) for the background Schwarzschild spacetime, giving the background line element

$$ds^2 = -\left(\frac{2r-M}{2r+M}\right)^2 + \left(1+\frac{M}{2r}\right)^4 (dx^2+dy^2+dz^2) \tag{6}$$

where $r = \sqrt{(x^2+y^2+z^2)}$ and M is the black hole mass. For the linearised equations we work in the *harmonic* or *de Donder* gauge which constrains $h_{\alpha\beta}$ via

$$\nabla_\alpha(h^{\alpha\beta} - \frac{1}{2}g^{(0)\alpha\beta}h) = 0. \tag{7}$$

This condition greatly simplifies the perturbation field equation (5), which reduces to a wave-like equation for each of the ten components $h_{\alpha\beta}$

$$\nabla^\gamma\nabla_\gamma\bar{h}_{\alpha\beta} = 2R^{(0)}_{\gamma\alpha\beta\delta}(h^{\gamma\delta} - \frac{1}{2}g^{(0)\gamma\delta}\bar{h}) - 16\pi\theta_{\alpha\beta} \tag{8}$$

where $\bar{h}_{\alpha\beta}$ is the trace-reversed perturbation tensor, and the fact that the background spacetime is Ricci-flat, $R^{(0)}_{\alpha\beta} = 0$, has been used.

The evolution equations, (8), in the quasi-Cartesian isotropic background coordinates were expanded using Macsyma. The TEX output generated by Macsyma for the evolution equation of the $\bar{h}_{xy}$ component is shown in Figure 1.

3.2 The Constraint Equations

The perturbation tensor, $\bar{h}_{\alpha\beta}$, must satisfy constraint equations on the initial timeslice and subsequently on following timeslices. The constraint equations are the harmonic gauge condition and its first time derivative

$$\bar{h}^{\alpha\beta}{}_{;\beta} = 0, \qquad (\bar{h}^{\alpha\beta}{}_{;\beta})_{,t} = 0. \tag{9}$$

It can be shown that the harmonic gauge condition (9) is preserved by the evolution equations (5). Thus we only need solve (9) to obtain initial values for the simulation. From the constraint equations we obtain a set of elliptic equations. These are solved by assuming that initially all the spatial perturbations, $\bar{h}_{ij}$, are zero, leading to four coupled second order elliptic equations to be solved for $\bar{h}_{tt}, \bar{h}_{tx}, \bar{h}_{ty}, \bar{h}_{tz}$. The unphysical waves on the grid caused by setting $\bar{h}_{ij} = 0$ will 'wash' off the grid at the start of the simulation. The constraint equations have been solved using an SOR method.

$$
\begin{aligned}
&\frac{\frac{d^2\bar{h}_{12}}{dt^2}\,(2\,r+M)^2}{2\,(2\,r-M)^2}-\frac{8\,\frac{d^2\bar{h}_{12}}{dz^2}\,r^4}{(2\,r+M)^4}-\frac{8\,\frac{d^2\bar{h}_{12}}{dy^2}\,r^4}{(2\,r+M)^4}-\frac{8\,\frac{d^2\bar{h}_{12}}{dx^2}\,r^4}{(2\,r+M)^4}=\\
&-\frac{32\,\bar{h}_{12}\,M\,\left(24\,r^3\,z^2-40\,M\,r^2\,z^2+18\,M^2\,r\,z^2-3\,M^3\,z^2-8\,r^5+12\,M\,r^4-6\,M^2\,r^3+2\,M^3\,r^2\right)}{(2\,r-M)^2\,(2\,r+M)^6}\\
&+\frac{32\,\bar{h}_{13}\,M\,\left(24\,r^3-40\,M\,r^2+18\,M^2\,r-3\,M^3\right)\,y\,z}{(2\,r-M)^2\,(2\,r+M)^6}+\frac{32\,\frac{d\bar{h}_{23}}{dz}\,M\,r^2\,x}{(2\,r+M)^5}\\
&+\frac{16\,\frac{d\bar{h}_{12}}{dz}\,M\,r^2\,(8\,r-3\,M)\,z}{(2\,r-M)\,(2\,r+M)^5}-\frac{32\,\frac{d\bar{h}_{23}}{dx}\,M\,r^2\,z}{(2\,r+M)^5}-\frac{32\,\frac{d\bar{h}_{13}}{dy}\,M\,r^2\,z}{(2\,r+M)^5}-\frac{4\,\bar{h}_{00}\,M\,\left(12\,r^2-8\,M\,r+3\,M^2\right)\,x\,y}{r^3\,(2\,r-M)^4}\\
&-\frac{32\,\bar{h}_{22}\,M^2\,\left(8\,r^2-4\,M\,r+M^2\right)\,x\,y}{(2\,r-M)^2\,(2\,r+M)^6}-\frac{32\,\bar{h}_{11}\,M^2\,\left(8\,r^2-4\,M\,r+M^2\right)\,x\,y}{(2\,r-M)^2\,(2\,r+M)^6}-\frac{64\,\bar{h}_{33}\,M\,(3\,r-M)\,x\,y}{(2\,r+M)^6}\\
&+\frac{16\,\frac{d\bar{h}_{12}}{dy}\,M\,r^2\,(8\,r-3\,M)\,y}{(2\,r-M)\,(2\,r+M)^5}+\frac{4\,\frac{d\bar{h}_{01}}{dt}\,M\,(2\,r+M)\,y}{r\,(2\,r-M)^3}-\frac{32\,\frac{d\bar{h}_{22}}{dx}\,M\,r^2\,y}{(2\,r+M)^5}+\frac{32\,\frac{d\bar{h}_{13}}{dz}\,M\,r^2\,y}{(2\,r+M)^5}\\
&+\frac{32\,\frac{d\bar{h}_{11}}{dx}\,M\,r^2\,y}{(2\,r+M)^5}+\frac{16\,\frac{d\bar{h}_{12}}{dx}\,M\,r^2\,(8\,r-3\,M)\,x}{(2\,r-M)\,(2\,r+M)^5}+\frac{4\,\frac{d\bar{h}_{02}}{dt}\,M\,(2\,r+M)\,x}{r\,(2\,r-M)^3}\\
&+\frac{32\,\bar{h}_{23}\,M\,\left(24\,r^3-40\,M\,r^2+18\,M^2\,r-3\,M^3\right)\,x\,z}{(2\,r-M)^2\,(2\,r+M)^6}+\frac{32\,\frac{d\bar{h}_{22}}{dy}\,M\,r^2\,x}{(2\,r+M)^5}-\frac{32\,\frac{d\bar{h}_{11}}{dy}\,M\,r^2\,x}{(-2\,r+M)^5}+8\,\pi\,\theta_{12}
\end{aligned}
$$

Figure 1. The evolution equation for the $\bar{h}_{xy}$ component of the perturbation, calculated using Macsyma.

3.3 The Matter Equations

The matter making up $\theta_{\alpha\beta}$ is assumed to follow geodesics of the background metric $g^{(0)}_{\alpha\beta}$. The matter is taken to be an extended point particle, with rest mass μ ,($\mu \ll M$), and proper volume V. The energy-momentum tensor is then defined as

$$\theta^{\alpha\beta} = \frac{\mu}{V}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\left(\frac{dt}{d\tau}\right)^2 . \tag{10}$$

The geodesic equations are solved for the position and velocity of the particle using a second order leapfrog scheme.

4 IMPLEMENTING THE ADI SCHEME

The gauge constraints (9) were used to eliminate the first order time derivatives from the evolution equations (8), giving 10 second order hyperbolic equations of the form

$$\frac{\partial^2 \bar{h}_{\alpha\beta}}{\partial t^2} - 16r^4\frac{(2r-M)^2}{(2r+M)^6}\nabla^2\bar{h}_{\alpha\beta} = 16\pi\theta'_{\alpha\beta} \tag{11}$$

where $\theta'_{\alpha\beta}$ is now a 'source' term which includes terms in $\bar{h}_{\alpha\beta}$ and $\bar{h}_{\alpha\beta,i}$, as well as the matter source term $\theta_{\alpha\beta}$. Comparing (11) with the usual wave equation (1) it is seen that the ADI scheme (2) is now easy to apply. The new 'source' terms $\theta'_{\alpha\beta}$ were finite differenced and written into Fortran code using Macsyma.

Outgoing wave boundary conditions have been applied to the grid exterior. They fit in with the ADI scheme in a natural way, the condition on the boundaries at constant x is applied with the first equation of (2) *etc.* The boundary conditions used are perfectly absorbing for spherically symmetric waves. A static boundary condition has been applied at the event horizon, $r = M/2$.

5 CURRENT WORK AND CONCLUSION

We hope to soon demonstrate that it is possible to include ADI schemes in a black hole problem, and that using such schemes we will be able to take much longer timesteps than those possible with explicit schemes while retaining a second order computationally feasible stable scheme.

We will also be investigating this problem in a rotating frame, using new ADI schemes developed by Alcubierre (1992). These schemes use causal reconnection to allow grid velocities faster than the speed of light, whilst maintaining stability and second order convergence.

Ultimately we want to model a coalescing black hole problem where the black holes orbit a number of times on roughly circular paths before coalescing. In this regime, in a frame co-rotating with the black holes, ADI schemes would appear to be a necessity.

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