

## A puzzle concerning the quadrupole formula for gravitational radiation

E. Balbinski and B. F. Schutz *Department of Applied Mathematics and Astronomy, University College, PO Box 78, Cardiff CF1 1XL*

Received 1982 May 6

**Summary.** A comparison is made between the damping of the fundamental mode of neutron star due to gravitational radiation reaction calculated by an approximate method based on the standard quadrupole formula for the energy loss, and the damping given by previous calculations using the full theory of general relativity. Numerical results for a mildly relativistic neutron star with  $2GM/Rc^2 < 10$  per cent show the surprising result that the damping times differ by a factor of about 3 by these two methods.

### Introduction

Little is known about the precise range of applicability of the standard quadrupole formula for the energy loss due to gravitational radiation from a weakly relativistic source. This is unfortunate since it is clearly easier and simpler to use the formalism described by Thorne (1969a), which involves Newtonian gravity plus the quadrupole formula where it is applicable, than the full theory of general relativity, (GR). The comparison reported here is an attempt to investigate the applicability of the formula to a weakly relativistic regime. Only brief details are given here, a fuller description is contained in Balbinski (1982).

Two neutron star models are considered, model I is a relatively low-density star with central density,  $\rho_c$ ,  $3 \times 10^{14} \text{ g cm}^{-3}$  and model II, a higher density star with  $\rho_c = 6 \times 10^{15} \text{ g cm}^{-3}$ . Radiation reaction damping times are compared with the fully relativistic calculations of Thorne (1969b) and Detweiler (1973). The neutron stars are the HWW models described in Harrison *et al.* (1965), but with the improved equation of state given in Hartle & Thorne (1968). Within the quadrupole radiation–reaction approximation, Schutz (1980) and Balbinski (1982) show that the damping time  $\tau$  is given (to first order in  $\omega_L$ , where  $\omega_L$  is the Newtonian eigenfrequency of the mode) by

$$\tau_L = (\omega_L/2\pi G) (c/\omega_L)^{2L+1} \frac{(L-1) [(2L+1)!!]^2}{L(L+1)(L+2)} \times \left\{ \int_0^R dr \rho r^{2L} \left[ u^2 + v^2 \left( \frac{L+1}{L} \right) \right] \right\} / \left\{ \int_0^R dr \rho r^{2L} \left[ u + v \left( \frac{L+1}{L} \right) \right] \right\}^2. \quad (1)$$

**Table 1.** A comparison of Newtonian and relativistic models with  $L = 2$  pulsations.

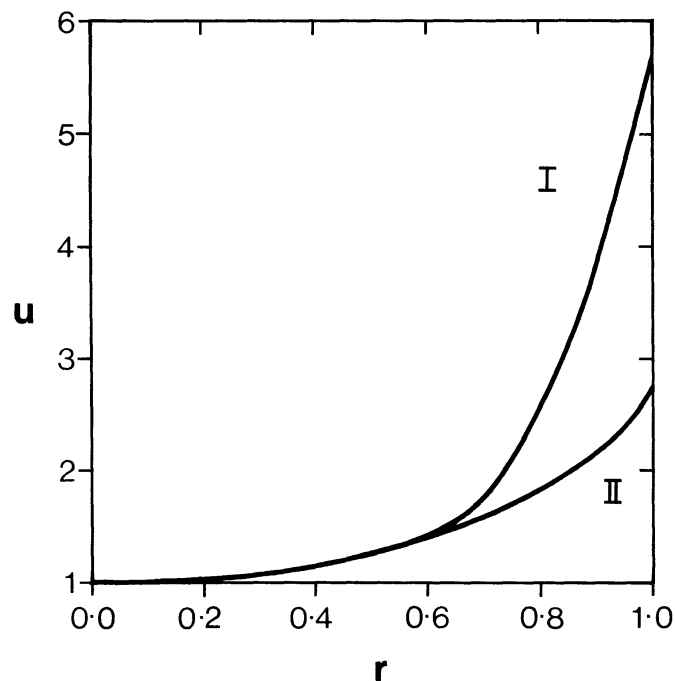
	MODEL I			MODEL II		
	Newtonian	Relativistic	%Difference	Newtonian	Relativistic	%Difference
$M/M_{\odot}$	0.473	0.405	17	1.36	0.682	100
$R$ (km)	21.6	20.9	3	10.1	8.4	20
$2GM/Rc^2$	0.070	0.057	22	0.398	0.240	67
$\omega^2$ (secs $^{-2}$ )	2.815(7)	2.755(7)	2	4.819(8)	4.085(8)	18
Period(secs)	1.184(-3)	1.197(-3)	1	2.862(-4)	3.109(-4)	9
$\tau$ (secs)	4.3	13 $\pm$ 3 (12.6)	67	1.7(-2)	0.19 $\pm$ 0.002 (0.18)	91

Here  $\rho$  is the density of the star,  $R$  its radius and  $u$  and  $v$  are defined in terms of the Newtonian Lagrangian displacement vector by

$$\xi(r, \theta, \phi, t) = \{r^{L-1}u(r)Y_L^M(\theta, \phi)\mathbf{e}_r + (r^L/L)v(r)\nabla Y_L^M(\theta, \phi)\} \exp(i\omega t). \quad (2)$$

### Results

A computer program was used to calculate the structure and normal modes of the two neutron star models in Newtonian gravity. The accuracy of the structure part of the program was checked by amending it to calculate the structure in GR; it then reproduced the values of mass and radius given in Thorne (1969b). The accuracy of the normal mode part of the program was checked by reproducing eigenfrequencies for polytropes given by Robe (1968), to the published accuracy.



**Figure 1.** Relative radial displacement,  $u$ , versus radial distance in units of the star's radius,  $r$ , for models I and II.

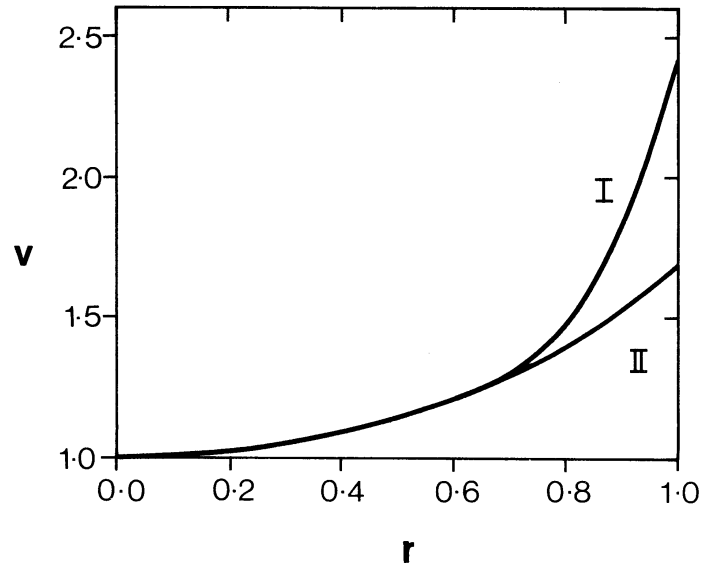


Figure 2. Relative tangential displacement,  $v$ , versus radial distance in units of the star's radius  $r$ , for models I and II.

Table 1 compares values calculated in the Newtonian approximation using equation (1) for the damping time with the fully relativistic calculation of Thorne (1969b), for spherical harmonic index  $L = 2$ . The figure in brackets in the relativistic column for the damping times is the result of Detweiler (1973). The figure in the percentage difference column is the modulus of the percentage difference of the Newtonian and relativistic results, to the nearest 1 per cent, using the relativistic result as base.

The Newtonian relative radial and tangential displacements  $u$  and  $v$  are shown in Figs 1 and 2.

For model I there is good agreement for the radius and eigenvalue,  $\omega^2$ , with Thorne's calculation, but the agreement for the mass is less good and, surprisingly, the damping time is a factor of 3 smaller than that given by Thorne and Detweiler. For model II, the agreement is poor, even for the radius and  $\omega^2$ , though this is to be expected since  $2GM/Rc^2$  is rather large for this model.

### Discussion

What could be the cause of the discrepancy for model I? The numerical errors have been carefully checked and are negligible here. Using the relativistic eigenfrequency and eigenfunctions calculated by Thorne does not significantly alter the damping time given by equation (1) for model I and neither does the inconsistent method of using a structure calculated in GR but eigenfrequencies and eigenfunctions calculated in Newtonian gravity. There is, of course, ambiguity in selecting a Newtonian model to compare with a relativistic one, since one cannot duplicate the density, mass and radius simultaneously. We have chosen to compare models with the same central density because for the less relativistic stars these have essentially the same eigenfrequencies and radii. Since one imagines that the bulk motions produce the emission of radiation, this comparison gives the closest similarity to these motions. It is true, however, that the Newtonian density falls off less steeply than the relativistic density, resulting in a larger mass, so we also computed a Newtonian star of the same mass as the relativistic model I. This has a value of  $\omega^2$  about 30 per cent smaller, but a damping time about double, in much better agreement with the relativistic calculation for

the latter. We have also looked at typical higher order corrections to the quadrupole formula and feel that they will be insufficient to remove the discrepancy for model I.

We are therefore led to conclude that either the relativistic calculations of Thorne and the independent one of Detweiler both had substantially greater numerical errors than they realized, or that the quadrupole formula must be used extremely cautiously, even in mildly relativistic situations where redshifts of order 5 per cent and ambiguities of order 20 per cent in comparisons of models, may lead to radiative fluxes that differ by factors of 3! In view of the importance of the quadrupole formula to predicting astrophysical radiation fluxes, such as in supernova explosions, we feel this question needs urgent examination.

### Acknowledgment

EB acknowledges the support of an SRC studentship.

### References

- Balbinski, E., 1982. *PhD thesis*, University College, Cardiff.  
Detweiler, S. L., 1973. *Astrophys. J.*, **197**, 203.  
Harrison, B. K., Thorne, K. S., Wakano, M. & Wheeler, J. A., 1965. *Gravitational Theory and Gravitational Collapse*, University of Chicago Press.  
Hartle, J. B. & Thorne, K. S., 1968. *Astrophys. J.*, **153**, 807.  
Robe, H., 1968. *Ann. Astrophys.*, **31**, 475.  
Schutz, B. F., 1980. *Mon. Not. R. astr. Soc.*, **190**, 21.  
Thorne, K. S., 1969a. *Astrophys. J.*, **158**, 1.  
Thorne, K. S., 1969b. *Astrophys. J.*, **158**, 997.

### Note added in proof

Since submitting this paper for publication we have received results of a recalculation of the damping time for model I, by Steven Detweiler and Lee Lindblom, using a method based on the numerical integration of the equations in GR. They find a  $\tau$  of about 7 seconds, which, though closer to the value given by the quadrupole formula, is still a factor of 2 different.