[10]H. J. Kim, R. L. Robinson, W. T. Milner, and Z. W. Grabowski, Bull. Amer. Phys. Soc. 17, 530 (1972).
[11]P. H. Stelson, S. Raman, J. A. McNabb, R. W. Lide, and C. R. Bingham, Phys. Rev. C 8, 368 (1973).
[12]R. A. Sorensen, Rev. Mod. Phys. 45, 353 (1973).
[13]T. L. Khoo, F. M. Bernthal, J. S. Boyno, and R. A. Warner, Phys. Rev. Lett. 31, 1146 (1973).
[14]G. Scharff-Goldhaber, G. M. McKeown, A. H. Lumpkin, and W. G. Piel, Jr., Phys. Lett. 44B, 416 (1973).
[15]P. Thieberger, Phys. Lett. 45B, 417 (1973).
[16]K. Kumar, CERN Report No. 70-30, 1970 (unpublished), p. 779.

Axisymmetric Stability of Kerr Black Holes*

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A variational expression is used to prove that all axisymmetric modes of the Kerr sequence of rotating black holes are stable.

Recent theorems of Hawking[1] and Carter[2] indicate that the Kerr family of rotating black holes is likely to be the unique end point of any gravitational collapse in which an event horizon forms. The stability of these configurations is therefore a question of some astrophysical interest, at least to the extent that such collapse is itself a common occurrence. Numerical calculations by Press and Teukolsky[3] indicate that the entire physical part of the Kerr sequence (angular momentum $< M^2G/c$) is in fact stable, but as yet, no analytic proof has been found. In this Letter we report a proof that all axisymmetric modes of the Kerr sequence are stable. Full details will be published elsewhere.

There are three steps to the proof. We show first that the linearized field equations form a self-adjoint system for unstable axisymmetric modes, whose eigenfrequencies are therefore purely imaginary. Then, by examining some analytic properties of Teukolsky's[4] equations, we find that the eigenfrequency of each such mode varies continuously along the Kerr sequence. We infer that instability in an axisymmetric mode can only set in when its frequency vanishes, an eventuality which we exclude by invoking Carter's theorem.[2]

The Kerr metric has the form

$$ds^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2,$$

where, in Boyer-Lindquist[5] coordinates,

$$\omega^0 = (\Delta\rho^2/D)^{1/2}\,dt,$$

$$\omega^1 = \sin\theta\left(\frac{D}{\rho^2}\right)^{1/2}\left(d\varphi - \frac{2Mar}{D}\,dt\right),$$

$$\omega^2 = (\rho^2/\Delta)^{1/2}\,dr,$$

$$\omega^3 = \rho\,d\theta,$$

and where

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

$$\Delta = r^2 - 2Mr + a^2,$$

$$D = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta.$$

The Kerr parameters M and a are the mass and specific angular momentum of the solution they index, which has an event horizon at $r = r_+ \equiv M + (M^2 - a^2)^{1/2}$. The one-forms ω^i are a pseudo-orthonormal basis of locally nonrotating observers. All tensor indices below will refer to that basis; Latin indices run from 0 to 3 and Greek indices from 2 to 3.

Chandrasekhar and Friedman[6] derived a variational principle for the equations governing axisymmetric perturbations of any axisymmetric stationary solution of Einstein's equations. When no matter is present, their action can be written in the gauge-independent form

$$I = \int d\tau [h_{ij}{}^* \delta G^{ij} + \delta G_{1\alpha}{}^* \delta G^{1\alpha}], \qquad (1)$$

where h_{ij} is the perturbation in the metric, and δG^{ij} is the corresponding first-order perturbation in the Einstein tensor. The quadratic functional I has two important properties: First, it is Hermitian apart from integrations by parts over the spatial variables; and second, in a gauge where $h_{0\alpha} = 0$, only second time derivatives appear.

Suppose now that h_{ij} is a solution to the perturbed field equations, so that $I = 0$; further, choose a gauge in which $h_{0\alpha} = 0$, and assume a time dependence of the form $e^{-i\sigma t}$ for some complex eigenfrequency σ. Then, solving the equation $I=0$ for the squared frequency we find

$$\sigma^2 = \lim_{r\to r_+} \{[\int_r^\infty dr \int d\Omega B(h_{ij}^*, h_{ij}) + \{\oint d\Omega S(h_{ij}^*, h_{ij})\}|_r^\infty][\int_r^\infty dr \int d\Omega A(h_{ij}^*, h_{ij})]^{-1}\}, \qquad (2)$$

where the quantities A and B are Hermitian and S anti-Hermitian in their arguments. By a further gauge condition, for example, $h_\alpha{}^\alpha=0$, A can be made manifestly positive definite.[7] The integrals of A, B, and S are evaluated on a hypersurface $t=$ const; they generally diverge as $r\to r_+$, but the ratio of numerator and denominator in Eq. (2) is constant for all r.

Because A and B are both real, the imaginary part of σ^2 comes from the surface term S. This reflects the fact that S is a measure of the radiation crossing the surfaces at r and at infinity. We will now show that for *unstable* modes, the contribution to σ^2 from these surface terms vanishes, and that σ^2 is therefore real.

In the asymptotically flat region far from the horizon, the modes have the behavior $h_{ij} \sim r^{-1} \times \exp[-i\sigma(t-r-2M\ln r)]$. Then unstable modes (those for which $\mathrm{Im}\sigma>0$) fall off exponentially in r on the hypersurface $t=$ const, from which it follows that the surface term at spatial infinity vanishes. A detailed analysis of the linearized equations near the Kerr horizon shows similarly that the integral $\oint d\Omega S$ on a surface at $r=r_+ + \xi$ vanishes like $\xi^{2M\,\mathrm{Im}\sigma}$ as $r\to r_+$. In the limit $\mathrm{Im}\sigma \to 0$ the surface terms are finite and nonzero but the integral of the quantity A diverges, so that S again makes no contribution to σ^2. (Presumably these results do not depend on the precise nature of the Kerr horizon, and should be true, for example, of a horizon distorted by a distribution of mass outside it.) Thus the system of linearized equations is self-adjoint for unstable modes; σ^2 is then real, and $\sigma = ib$ for b real and positive.

We can now state the following lemma: If the unstable part of the trajectory $\sigma(a/M)$ of some axisymmetric mode is a continuous function of a/M along the Kerr sequence and has no endpoints for $a<M$, the instability in the mode must set in when its eigenfrequency vanishes (when $\sigma=0$). The lemma follows immediately from the above considerations and the fact that all modes of the Schwarzschild solution ($a/M=0$) are stable.[8] We may also note that because a formula of the form (2) can again be derived from an analogous variational expression when matter is present, a similar lemma applies to the axisymmetric modes of axisymmetric stellar models. That is, if the eigenfrequencies are continuous along some sequence of stellar models, then instability can set in only via a zero-frequency mode.

We must now prove that no eigenfrequency appears on the imaginary axis in a discontinuous way—without its first having passed through the origin, $\sigma=0$, for some smaller value of the sequence parameter a/M. One way to do this is to examine Teukolsky's[4] decoupled and separated equations for the Weyl tensor component ψ_0. The separation of variables gives two characteristic-value equations for the eigenfrequency σ and for an angular eigenvalue A. In each case, the condition that a particular A and σ be eigenvalues is that the Wronskian of the solution regular at one end of the domain with the solution regular at the other end vanish. Thus if $\Theta_0(a/M, \sigma, \theta)$ is the solution to the angular equation regular at $\theta=0$, and $\Theta_\pi(a/M, A, \sigma, \theta)$ is the solution regular at $\theta=\pi$, A and σ will be eigenvalues when

$$f(a/M, A, \sigma) \equiv W(\Theta_0, \Theta_\pi) = 0. \qquad (3)$$

Analogously, if $R_{r_+}(a/M, A, \sigma, r)$ is the solution to the radial equation corresponding to waves ingoing at the horizon, and $R_\infty(a/M, A, \sigma, r)$ the solution corresponding to waves outgoing at infinity, the eigenvalue condition is

$$g(a/M, A, \sigma) \equiv W(R_{r_+}, R_\infty) = 0. \qquad (4)$$

Following a method given, for example, in de Alfaro and Regge,[9] one can construct Born series for R_{r_+}, R_∞, Θ_0, and Θ_π that are uniformly convergent and term by term analytic in A, in a/M for $|a|<M$, and in σ for $\mathrm{Im}\sigma>0$. Therefore R_{r_+}, R_∞, Θ_0, and Θ_π, together with their Wronskians f and g, are similarly analytic.

Then if $(a_0/M, A_0, \sigma_0)$ is a solution to Eqs. (3) and (4) for $\sigma_0=ib$ ($b>0$) and for $a_0<M$, there must be a continuous set of solutions for nearby values of a/M. For if $f_{,A}(a/M, A_0, \sigma_0)\neq 0$, Eq. (3) can be locally inverted to give $A=F(a/M, \sigma)$ with F analytic, whence the analyticity of $g(a/M, F(a/M, \sigma), \sigma)$ guarantees a continuous solution $\sigma(a/M)$ near a_0/M. If, on the other hand, $f_{,A}=0$, then A as a function of σ has either a branch point or a pole. We exclude poles by noting that when $\sigma=ib$, A is real and positive; it is then easily shown that the radial equation has no solutions for sufficiently large A. In the neighborhood of a branch point, on the other hand, there will be a solution of the

form $A = H(a/M)(\sigma - \sigma_0)^\alpha$, and again the equation $g(a/M, H(a/M)(\sigma - \sigma_0)^\alpha, \sigma) = 0$ has a continuous solution for nearby a/M. Thus no trajectory can have an end point on the positive imaginary axis (except possibly at $a/M = 1$) and the eigenfrequency of any unstable mode must have passed through $\sigma = 0$.

Finally, we observe that Carter's theorem excludes such modes. That is, Carter proves that there are no stationary axisymmetric perturbations of Kerr. But if $h_{ij}(\sigma)e^{-i\sigma t}$ are a family of solutions to the time-dependent field equations, we can show that $h_{ij}(\sigma = 0)$ is a solution to the time-independent field equations as well. We conclude that there are no unstable axisymmetric modes of the Kerr geometry exterior to a black hole.

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[1]S. W. Hawking, Commun. Math. Phys. 25, 152 (1972).

[2]B. Carter, Phys. Rev. Lett. 26, 331 (1971).

[3]W. H. Press and S. A. Teukolsky, Astrophys. J. 185, 649 (1973).

[4]S. A. Teukolsky, Astrophys. J. 185, 635 (1973).

[5]R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).

[6]S. Chandrasekhar and J. L. Friedman, Astrophys. J. 177, 745 (1972).

[7]This gauge choice is analogous to a coordinate condition used to the same end in L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1965), p. 93, and brought to our attention by Rafael Sorkin.

[8]C. V. Vishveshwara, Phys. Rev. D 1, 2870 (1970).

[9]V. de Alfaro and T. Regge, *Potential Scattering* (Wiley, New York, 1965).

Nonrenormalizability of the Quantized Einstein-Maxwell System*

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The one-loop divergences of coupled general relativity and electrodynamics cannot be absorbed by renormalization.

Although quantization of general relativity has been discussed by many authors, it is only quite recently that the first complete calculation, at the one-loop level, of divergences due to gravitons has been supplied by 't Hooft and Veltman.[1] These authors used dimensional regularization and a general algorithm,[2] based on the background-field method,[3] to obtain the one-loop counterterms. In the case of pure gravitation, the divergent terms could be absorbed by a field renormalization; however, this was no longer possible upon coupling to a quantized scalar field.

While the scalar example is discouraging, one might hope that more realistic matter sources would keep one-loop renormalizability. We report here that this is *not* the case for the Einstein-Maxwell system. We also give some results for (1) Brans-Dicke theory (nonrenormalizable except for the singular case $\omega = -\frac{3}{2}$ which is equivalent to Einstein theory), (2) fermion loops, and (3) pure gravitation with a cosmological term (formally renormalizable).

In the coupled Einstein-Maxwell Lagrangian

$$L \equiv L_E + L_M = -(-\bar{g})^{1/2}[\kappa^{-2}R(\bar{g}) + \tfrac{1}{4}\bar{F}_{\mu\nu}\bar{F}_{\rho\sigma}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}], \qquad (1)$$

the fields $(\bar{g}_{\mu\nu},\ \bar{F}_{\mu\nu} \equiv \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu)$ are written as sums of background fields $(g_{\mu\nu},\ F_{\mu\nu})$ and quantum fields $(\kappa h_{\mu\nu},\ f_{\mu\nu})$. The Lagrangian being invariant under gravitational and electromagnetic gauge transformations of the quantum fields $h_{\mu\nu}$ and $f_{\mu\nu}$, we add to L the gauge-breaking terms

$$L_B \equiv L_{EB} + L_{MB} = -\tfrac{1}{2}\sqrt{-g}[(D^\nu h_{\mu\nu} - \tfrac{1}{2}D_\mu h_\alpha{}^\alpha)^2 + (D^\mu A_\mu)^2], \qquad (2)$$

where all tensor operations, including covariant differentiation D, are with respect to the background metric $g_{\mu\nu}$. The above choice corresponds to the usual de Donder (harmonic) and Lorentz gauges, which in turn give rise to a vector and a scalar ghost with Lagrangian

$$L_G = \sqrt{-g}\,[\eta^{*\alpha}(g_{\alpha\beta}D_\gamma D^\gamma - R_{\alpha\beta})\eta^\beta + \varphi^* D_\gamma D^\gamma \varphi]. \qquad (3)$$