Born–Oppenheimer Decomposition for Quantum Fields on Quantum Spacetimes

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Abstract

Quantum Field Theory on Curved Spacetime (QFT on CS) is a well established theoretical framework which intuitively should be a an extremely effective description of the quantum nature of matter when propagating on a *given background spacetime*. If one wants to take care of backreaction effects, then a theory of quantum gravity is needed. It is now widely believed that such a theory should be formulated in a non – perturbative and therefore *background independent* fashion. Hence, it is a priori a puzzle how a background dependent QFT on CS should emerge as a semiclassical limit out of a background independent quantum gravity theory.

In this article we point out that the Born – Oppenheimer decomposition (BOD) of the Hilbert space is ideally suited in order to establishing such a link, provided that the Hilbert space representation of the gravitational field algebra satisfies an important condition. If the condition is satisfied, then the framework of QFT on CS can be, in a certain sense, embedded into a theory of quantum gravity.

The unique representation of the holonomy – flux algebra underlying Loop Quantum Gravity (LQG) violates that condition. While it is conceivable that the condition on the representation can be relaxed, for convenience in this article we consider a new classical gravitational field algebra and a Hilbert space representation of its restriction to an algebraic graph for which the condition is satisfied. An important question that remains and for which we have only partial answers is how to construct eigenstates of the full gravity – matter Hamiltonian whose BOD is confined to a small neighbourhood of a physically interesting vacuum spacetime.

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1 Introduction

Quantum Field Theory (QFT) on Curved Spacetime (CS) is nowadays a well established discipline of mathematical physics (see [1] for a recent review and [2] for a functorial formulation). It is designed to describe the regime of physics in which backreaction effects of matter on geometry and the quantum fluctuations of geometry itself can be safely neglected. In order to go beyond that, a theory of quantum gravity is needed. It is nowadays widely believed that perturbative quantum gravity is non renormalisable and therefore a non perturbative formulation has to be found. Of course, such a theory of quantum gravity must contain QFT on CS in its semiclassical limit. The puzzle that poses itself is, how a background independent theory can describe a background dependent theory such as QFT on CS *all*. This question is the central topic of the present paper.

We still do not have a working theory of quantum gravity, however, there are several proposals which are still under construction. One of them, Loop Quantum Gravity (LQG) [3, 4] indeed is a background independent approach to the quantisation of general relativity and in the last 20 years a lot of effort has been put in the rigorous definition of the mathematical framework of this theory. Background independence is a feature that distinguishes LQG from most other quantum field theories constructed so far, as for example the standard model of elementary particle physics. When examining ordinary quantum field theory with care one realises that much of the structure we are used to from the particle physics point of view, including for instance the construction of a Fock space as the fundamental Hilbert space of the theory, the concept of a particle as a certain mode–excitation of the field, the notion of causality and the notion of a vacuum state, are strongly linked to the presence of a given background spacetime. For a quantum theory of general relativity there is no such background spacetime available because we are quantising the dynamics of spacetime itself, and thus it should not be too surprising that the fundamental structures appearing in a quantisation of general relativity differ a lot from the ones used in ordinary quantum field theory.
The natural Hilbert space to describe the quantum dynamics of LQG is not a Fock space as one might expect – in fact the Hilbert $\mathcal{H}_{\rm AIL}$ space used in LQG, named the Ashtekar–Isham–Lewandowski Hilbert space [5, 6], is the space of square–integrable functions over a set of appropriately generalised $SU(2)$–connections on a 3 dimensional manifold and does not bear a lot of similarity with the Hilbert spaces known from ordinary QFT. However, when trying to find irreducible representations of the kinematical algebra of GR, also called the holonomy – flux – algebra in the LQG context, one is naturally led to this space. Analogous to the well known Stone–von Neumann theorem in quantum mechanics, there exists even an uniqueness result [7, 8] , that states that under certain assumptions (spatial diffeomorphism invariance being the main one) the representation used in LQG associated with $\mathcal{H}_{\rm AL}$ is unique. The structure of this Hilbert space is quite different from the ones known from ordinary quantum field theory on a given background: It is manifestly background independent and its elements can be written in terms of so called spin network functions, functions living on certain embedded graphs which are labeled by irreducible representations of $SU(2)$. Furthermore, the flux operators, corresponding to the momentum operators in the canonical picture, encode geometrical information about the quantum state turn out to fulfil a non–commutative algebra.
The relation between LQG and classical general relativity has been studied in detail [9, 10, 11, 12, 13] and it turns out that LQG, or rather its recently proposed modification coined algebraic quantum gravity, see [14, 13, 15]), has the right semiclassical limit at least as far as the infinitesimal dynamics is concerned.
However, if LQG is a fundamental quantum theory for gravity and matter, then it should be possible to rediscover ordinary quantum field theory on a fixed background within a sector of the fully background independent framework of LQG. There has been some work into this direction [16, 17], but some issues remained unresolved: First, one did quantise at the level of the kinematical Hilbert space. Therefore all operators in question were not gauge invariant, the coherent states that one was using were not gauge invariant and thus one might rightfully ask whether the constructions displayed there would survive upon passage to the physical Hilbert space. Next, the matter Hamiltonians used there were just the matter contributions to the Hamiltonian constraint and one did not have a convincing argument why those should be used as Hamiltonians driving the physical time evolution of the system. Finally, backreaction effects were not taken into account.

The aim of this article is to try to improve the understanding of how a quantum field theory of matter on a given background can emerge out of a theory of quantum gravity. In order to deal with the afore mentioned constraints we carry out a reduced phase space quantisation of general relativity plus matter degrees of freedom using a dynamical reference frame given by some dust fields using methods from [18, 19, 20]. Thus we arrive at a true (i.e. non–vanishing) physical Hamiltonian that generates time evolution in the reference frame described by these dust fields. Since in contrast to the total Hamiltonian constraint the physical Hamiltonian is non – vanishing, in general, the physical Hamiltonian operator will involve more complicated terms than just the total Hamiltonian constraint. In particular it involves a square root of polynomials of the contributions to the constraints associated with the non – dust degrees of freedom. To deal with this square root is a difficult technical subject of its own which is not the main concern of this paper. Thus, we consider a regime of the theory for which the Taylor expansion of the square root can be

restricted to the dominant term which is just the integrated non – dust contributions to the Hamiltonian constraint. Therefore we get an a posteriori justification for why one can build a matter QFT just using the matter parts of the Hamiltonian constraints as dynamical operators which otherwise would seem rather ad hoc. Of course, there is also the gravitational contribution which we will take care of in this paper. That this is possible relies on the judicious choice of the Brown – Kuchař dust fields as rods and clocks.
When examining LQG and ordinary QFT from a mathematical point of view it turns out that there are two main issues which need to be clarified when the latter should emerge out of the former in some appropriate limit: First, although LQG is a true continuum theory, its basic building blocks are defined on graph like structures. On the other hand, the Fock space of ordinary QFT uses the continuum properties of the underlying spacetime and one has to understand how one can arrive at such a space starting with a discrete theory at least in some limit.
Second, the flux operators in LQG, which are the basic building blocks if we were to construct a 'metric operator' as a quantum analog of the classical metric form a non–abelian operator algebra in LQG. This non–commutativity originates in the fact that the fluxes are obtained by smearing the sensitised triads along hypersurfaces of codimension one (and thus, singular ones from the three dimensional point of view) [21] which causes the associated Hamiltonian vector fields to be non commutative. Hence in any representation of this algebra on a Hilbert space, the non commutativity prohibits that the three geometry operators which couple to the matter fields can be simultaneously diagonalised. This implies an obstacle to directly applying the framework of QFT on CS when choosing a representation for the matter fields because there one relies on a commutative background spacetime. While much work on QFT on non commutative spacetimes has been conducted (see e.g. [22, 23, 24, 25, 26] and references therein), there one assumes a non commutative structure in the coordinates rather than the metric and therefore we are not in the position to apply that framework. Thus, if one wishes to import QFT on CS techniques into LQG, one first has to understand how to "Abelianise" the metric operators in appropriate regime.

In this article we will focus on the first point and leave the latter open for future research: Namely we will construct a quantum theory for gravity plus matter that is close to LQG in the sense that it is defined on a fundamentally discrete entities (algebraic graphs), but the geometrical operators are defined as ordinary (commuting) multiplication operators. This is possible by choosing an algebra different from the holonomy – flux algebra which shares some of its features (namely the discreteness of its fundamental building blocks) but deviates from it in other essential points (mainly the non–commutativity of geometrical operators is avoided). We choose a representation of this algebra which is very close to a (background independent) Fock representation[1] and therefore the discreteness of the spectrum of geometrical operators of LQG is not true in this representation[2] Using this algebra we examine how a Fock space for the matter degrees of freedom on a fixed background can emerge out of a full quantum theory for gravity plus matter.

The idea then for how to import QFT on CS techniques into quantum gravity is to employ a Born–Oppenheimer Decomposition (BOD) of the Hilbert space: In the chosen representation for the gravitational field, the wave functions depend on the co-triad configuration e. For each co – triad configuration we consider a matter Hilbert space $\mathcal{H}_e$ labelled by the co – triad. That matter Hilbert space is naturally chosen as a Fock space defined by the natural creation and annihilation operators on the formally ultrastatic[3] spacetime defined by the co – triad, whenever it is non – degenerate. When the co – triad is degenerate, the QFT on CS construction does not work and in fact the Hamiltonian operator is ill – defined in this case. Its domain of definition therefore must be chosen as to exclude this case. The fibre Hilbert spaces $\mathcal{H}_e$ are then glued together in a direct integral with respect to the spectral measure defined by the projection valued measures of the commutative algebra of co – triad operators. The cotriad operator and the momentum operators respectively act by multiplication and (suitable) derivation with respect to e in the fibres respectively. Also the matter field operators, being expressible as linear combinations (with e dependent "coefficients") of creation and annihilation operators, preserve these. This works when the Hilbert space for the total system is separable. Conversely, choosing a spectral measure and an action of the momenta conjugate to the triads with respect to which they are self – adjoint defines a separable Hilbert space representation. Notice that in this construction the question of unitary equivalence of the fibres $\mathcal{H}_e$ never arises (they will typically be unitarily inequivalent). We will generalise this construction and start with a non – separable Hilbert space based on von Neumann's Infinite Tensor Product. Now the separable Hilbert spaces $\mathcal{H}_e$ (and uncountably infinitely many more) arise as strong equivalence class separable sectors in the matter

[1] Essentially, we smear the continuum co-triad along paths and the conjugate momentum along surfaces, similar as connections and densitised triads in LQG. Notice that the smeared co-triads in contrast to the holonomy of a connection do not transform locally under gauge transformations which makes it difficult to construct gauge invariant objects from them, in contrast to the Wilson loops. However, in this paper we consider a top to bottom approach and define an abstract algebra on an algebraic graph which approximates the continuum algebra and which transforms locally under gauge transformations.

[2] There is no contradiction to the uniqueness theorem because we use a different algebra.

[3] A spacetime metric is called ultrastatic if it is static (i.e. stationary and the shift vanishes) and the lapse equals unity. Thus the non – trivial components of the four – metric are encoded in the three – metric which is not explicitly time dependent. There is no explicitly time dependence here because our Hamiltonian is not explicitly time dependent by virtue of the choice of the material reference frame.

ITP labelled by the corresponding vacua Ω_e. They are automatically orthogonal in the ITP inner product when the Ω_e lie in different strong equivalence classes. The advantage of the ITP over the separable Hilbert space construction is that we do not need to choose a spectral measure (and the $\mathcal{H}_e$ for degenerate $\mathcal{H}_e$ which could be defined e.g. by an e independent Fock representation) and that it is easier to find a self – adjoint action of the operator conjugate to the triad. Moreover, the ITP contains vectors of the form $\Omega_g \otimes \Omega_m$ where Ω_g, Ω_m lie in the gravitational and matter ITP respectively. This not possible in the separable construction (at least not for non compact spacetimes for which an infinite algebraic graph is essential) because this would mean that the same Ω_m lies in all the $\mathcal{H}_e$ which is a contradiction when the $\mathcal{H}_e$ are not all unitarily equivalent.

The use of Born–Oppenheimer Decomposition in the context of quantum gravity has a long tradition, see for example [27] and references therein for a historical review of the Born–Oppenheimer method in quantum geometrodynamics or [28] for a recent application in the context of spinfoam models. However, to the best of our knowledge, the connection between the BOD and QFT on CS has not been much exploited yet. As already mentioned, one would like to use the BOD also for the representation underlying LQG, however, at the moment there seems to be no promising idea for how to do this when those slow degrees of freedom that couple to the fast degrees of freedom cannot be simultaneously diagonalised.

The structure of this article is as follows:
In section 2 we very briefly review how one can construct a true physical Hamiltonian for general relativity plus arbitrary matter at the classical level using a preferred reference frame given by four dust fields. For details concerning this construction see [29, 18, 19]. Then we define a regime for which it is justified to approximate the physical Hamiltonian by just the integrated contributions of the non – dust degrees of freedom to the Hamiltonian constraint. In the remainder of the article we will use this approximation to define dynamics in the dust frame. Next, we rewrite classical GR in terms of an $SU(2)$–cotriad and its canonically conjugate momentum which is essentially a densitised extrinsic curvature. The smearing of these objects along one dimensional paths and two dimensional surfaces respectively defines a new Poisson algebra for classical GR which separates the points of the phase space and which we will call the *(extrinsic) curvature – length algebra* because the square root of the trace of the square of the cotriad along short paths is approximately the length along that path[4]. In section 3 we construct a representation of the curvature – length algebra and of a scalar field of Klein–Gordon type[5] on an abstract algebraic graph in the spirit of [14, 13, 15, 20] and the essential feature which distinguishes this theory from ordinary LQG is that cotriads are quantised as multiplication operators which will be essential for the semiclassical analysis in the second half of this article.
In section 4 we construct semiclassical states for the gravitational sector of this quantum theory.
In section 5 we introduce the Born – Oppenheimer Decomposition for our system and the associated approximation scheme for the full spectral problem, which still takes backreaction effects into account. Here the gravitational variables are treated as the ‘heavy’ ones and the matter variables as the ‘light’ ones. Such an approximation scheme is justified in regimes where a description in terms of interacting quantum fields on an almost classical background spacetime makes sense because for time scales on which particle interactions take place the change in geometry are truly small. We illustrate how this works using a simple minisuperspace model. In the appendix we list some facts about the ITP for the benefit of the reader.

2 Classical theory

2.1 Brown–Kuchǎr dust reduction and observables for General Relativity

The problem of observables, that is, calculating quantities which are invariant under the gauge group of the theory is much more severe in general relativity compared to other gauge theories such as electrodynamics or $SU(N)$ Yang–Mills theories. The reason is twofold: First, in general relativity the gauge group is given by Diff(M), the diffeomorphism group of the four dimensional underlying manifold. This leads to a rather complicated constraint algebra (often called the *hypersurface deformation algebra* or *Dirac algebra* in the literature) which is neither finite dimensional nor an honest Lie algebra (it has phase space dependent structure functions and not structure constants). Second, the issue of observables is strongly related to the background independent nature of the theory: Taking the manifold M just

[4]It is straightforward to construct a representation of this algebra in the continuum very close to the one of LQG by taking as configuration variables the exponentials of the cotriads (one for each direction in the Lie algebra) along paths which would substitute the holonomies of LQG and then one would use the Ashtekar – Lewandowski measure for $U(1)^3$. However, the construction of gauge invariant functions in the continuum from these Abelian holonomies works only in the limit of short paths which is a limit too singular in this representation because it is not strongly continuous under path deformations. This is why we proceed to the algebraic formulation here.

[5]A generalisation to gauge fields or fermions is straightforward but we will only consider a scalar field for conceptual clarity.

as an auxiliary structure, it is not meaningful to talk about fields evaluated at some point on this manifold. The true gauge dependent degrees of freedom should be defined as *relations between dynamical fields.*

Such relational observables were introduced first in [30, 31, 32, 33] and later refined in [34, 35, 36, 37, 38]. In retrospect it turns out that there is a one to one correspondence between a choice of gauge fixing conditions with an associated reduced phase space and a choice of relational observables. The correspondence is established by choosing as rods and clocks of the relational observables precisely the gauge fixing conditions of the reduced framework, see e.g. [18, 39]. The explicit expression of these observables in terms of the non gauge invariant degrees of freedom is rather involved, see e.g. [40, 41] for a perturbative treatment. Fortunately, it is not needed for what matters is the Poisson algebra between the observables (i.e. true degrees of freedom) and the associated physical Hamiltonian (i.e. reduced Hamiltonian) [42].

The reduced phase space for vacuum general relativity has a very complicated symplectic structure because the rods and clocks that one can construct necessarily lead to spatially non − local objects. However, in the presence of suitable matter these complications can be avoided. Here by suitable we mean matter that comes as close as possible to the mathematical idealisation of a test observer fluid and such that the physical Hamiltonian that drives the physical time evolution of the observables comes as close as possible to the standard model Hamiltonian [43] when the spacetime is close to flat. In [18, 19] such a suitable matter system was identified as the pressureless dust matter fields introduced by Brown and Kuchǎr in [29]. One considers the enlarged system $S_{\rm EH} + S_{\rm matter} + S_{\rm dust}$ where $S_{\rm EH}$ is the usual Einstein–Hilbert action, $S_{\rm matter}$ includes all possible other standard matter that one likes to couple to gravity and the dust action $S_{\rm dust}$ is given by

$$S_{\rm dust} = -\frac{1}{2}\int\limits_M d^4X\sqrt{|\det(g)|}\rho[g^{\mu\nu}U_\mu U_\nu + 1] \tag{2.1}$$

Here M is the four dimensional (spacetime) manifold which can topologically be identified with $\mathbb{R}\times\mathcal{X}$ for some three dimensional manifold $\mathcal{X}$ of arbitrary topology and $X \in M$ are local coordinates on M. $g_{\mu\nu}(X)$ with $\mu,\nu = 0,1,2,3$ denotes a (Pseudo–) Riemannian metric on M. $U \in T^*M$ is a one form defined as $U = -dT + W_j dS^j$, with $j = 1,2,3$, for some scalar fields $T, W_j, S^j \in C^\infty(M)$. So finally the action (2.1) is a functional of $g_{\mu\nu}$ and eight scalar fields ρ, T, W_i, S^i.

The coupled system $S_{\rm EH} + S_{\rm matter} + S_{\rm dust}$ seems to be very complicated at first sight, and indeed when performing a Hamiltonian analysis one realises that it is a second class constrained system. However, the second class constraints can be solved and on the reduced phase space the Dirac bracket is identical to the usual Poisson bracket. Moreover, if one chooses T, S^j as the four clock fields then it turns out that one can explicitly construct observables associated to the spatial three metric $q_{ab}(x)$ and their conjugate momenta $p^{ab}(x)$ in the dynamical reference frame (τ,σ^a) defined through the readings of the dust fields T, S^j. We will not give the details of this construction here and the interested reader should consult [18, 19] for all the details. What will be important for the purpose of this article is that we can construct a *physical Hamiltonian* $H_{\rm phys}$ that generates *physical equations of motion* for the *gauge invariant*[6] 3–metrics $Q_{ab}(\sigma)$ and their respective momenta $P^{ab}(\sigma)$. These observables are functions on the "dust manifold" Σ, i.e. $\sigma\in\Sigma$ does not label a point in $\mathcal{X}$ but a point in the range of the clock fields S^j. The explicit form of the physical Hamiltonian reads

$$H_{\rm phys} := \int\limits_\Sigma d\sigma\sqrt{C^2 - Q^{ab}C_aC_b} = \int\limits_\Sigma d\sigma H(\sigma) \tag{2.2}$$

where $C := C^g(Q,P) + C^m(Q,\Pi,\Phi)$ and $C_a := C^g_a(Q,P) + C^m_a(Q,\Pi,\Phi)$ are the contributions to the Hamiltonian and diffeomorphism constraints of general relativity and the matter sector written in terms of gauge invariant quantities $Q_{ab}(\sigma), P^{ab}(\sigma)$ and the corresponding matter observables denoted by $\Pi(\sigma), \Phi(\sigma)$. Also we introduced the Hamiltonian density denoted by $H(\sigma)$.

Note that C and C_a are non vanishing because only the total Hamiltonian and diffeomorphism constraint including the contribution of the dust vanishes and the energy momentum density of the dust, which are constants of the physical motion, must be non vanishing anywhere in order that it provides a bona fide system of coordinates. Note that the matter degrees of freedom should be just understood symbolically since we have not yet decided which matter components we actually want to consider. In general the algebra of observables involves a Dirac bracket rather than the Poisson bracket. However, for this particular choice of clock variables, Q_{ab} and P^{ab} as well as the corresponding matter observables form mutually commuting canonical pairs, that is their Poisson bracket is given by

[6] We will use capital letters to denote the gauge invariant observables in the reference frame given by the dust clocks T, S^j. Thus, by Q^{ab} we denote the gauge invariant three metric associated to the gauge variant 3–metric q_{ab} and by P^{ab} the gauge invariant canonical momenta associated to the gauge variant momenta p_{ab}. For the exact construction of these gauge invariant quantities see the articles cited above.

$\{Q_{ab}(\sigma), P^{cd}(\sigma')\} = \delta^c_{(a}\delta^d_{b)}\delta(\sigma,\sigma')$ and one can show that the physical equations of motion are given by

$$\dot{Q}_{ab} = \{H_{\rm phys}, Q_{ab}\}, \quad \dot{P}^{ab} = \{H_{\rm phys}, P^{ab}\} \tag{2.3}$$

and likewise for the matter sector.

The consistency of the dust picture with the usual gauge variant description of black hole and cosmological spacetimes has been verified in [18, 19, 44].

2.2 Approximation for the physical Hamiltonian

The physical Hamiltonian shown in equation (2.2) includes a square root under the integral. This square root originates from solving the Hamiltonian constraint for GR coupled to dust and possibly other standard matter for the dust momentum P, the momentum conjugate to the dust time T. We want to discuss how the reduced framework when quantised is related to quantum field theory on curved spacetimes. In contrast to the standard framework we will also quantise the gravitational sector of the theory by quantising $H_{\rm phys}$ which contains all (gauge -invariant) degrees of freedom except the dust ones. Although the physical Hamiltonian $H_{\rm phys}$ has been quantised in its full form using Loop Quantum Gravity techniques and spectral theory [20], in this paper we will consider $H_{\rm phys}$ in a certain approximation which we will now discuss.

First of all, the derivation of $H_{\rm phys}$ in [18] shows that one has the two anholonomic constraints $C^2 - Q^{ab}C_aC_b \geq 0$ and $C > 0$ on the constraint surface as otherwise the momentum P conjugate to T becomes imaginary and because one was using phantom dust rather than dust. This is why the argument of the square root is classically forced to be non negative. Thus, classically we can write $H = C\sqrt{1 - \frac{Q^{ab}C_aC_b}{C^2}}$ and know that the Taylor expansion of the square root

$$\begin{aligned} H(\sigma) &= C\sqrt{1 - \frac{Q^{ab}C_aC_b}{C^2}}(\sigma) \\ &\simeq C\left(1 - \frac{1}{2}\frac{Q^{ab}C_aC_b}{C^2} + \frac{1}{8}\left[\frac{Q^{ab}C_aC_b}{C^2}\right]^2\right)(\sigma) + \mathcal{O}([Q^{ab}C_aC_b/C^2]^3) \end{aligned} \tag{2.4}$$

converges. One could now argue that square root and its Taylor expansion (2.4) provide classically equivalent starting points and therefore instead of defining the square root via the spectral theorem, one could quantise the Taylor expansion directly. To do the latter, one would need to order the terms in each order of the expansion symmetrically and one would need to define the operator corresponding to $1/C$ on a dense domain where the expansion is valid. To that end, notice the identity

$$\frac{1}{C} = \lim_{\epsilon\to 0+}\frac{C}{C^2+\epsilon^2} = \lim_{\epsilon\to 0+}\frac{1}{2}[\frac{1}{C+i\epsilon} + \frac{1}{C-i\epsilon}] = \lim_{\epsilon\to 0+}\frac{1}{2}[R(i\epsilon) + R(-i\epsilon)] \tag{2.5}$$

where $R(z) = (C+z)^{-1}$ denotes the resolvent of C at $z \notin \text{spec}(C)$. Thus if we can quantise (the smeared) density $C(\sigma)$ as a self − adjoint operator, then the resolvent $R(\pm i\epsilon)$ is a bounded operator at finite ϵ. The removal of the regulator will of course only be possible on a suitable dense domain of the Hilbert space, for instance on analytic vectors for C.

In order not to deviate from the main thrust of the paper, we will neglect the higher order contributions in (2.4) by considering a subspace of the Hilbert space for which in the sense of expectation values $< Q^{ab}C_aC_b > \ll < C >^2$. We will come back to the higher order terms in a future publication. Thus for the remainder of the paper we will assume that

$$H_{\rm phys} \simeq \int_\Sigma d\sigma C(\sigma) \tag{2.6}$$

is a good approximation. For instance, for spacetimes close to a homogeneous one the approximation should be very good as one can see by expanding in terms of the number of spatial derivatives. Then the zeroth order contribution to the total diffeomorphism constraint vanishes identically so that $H_{\rm phys} = \sqrt{C^2} = C$ is true without any approximation in zeroth order. To define the sectors of validity of the approximation in the quantum theory is more delicate but one could use semiclassical perturbation theory [45]. See also [46] for related ideas.

2.3 Canonical transformation

Using the gauge fixed formalism described above, where one has a preferred reference frame given by the Brown–Kuchăr dust clocks, general relativity in terms of observables can be defined as follows: Let $\mathcal{M}$ be an (infinite dimensional) symplectic manifold and $(Q_{ab}(\sigma), P^{ab}(\sigma))$ a coordinate basis of $\mathcal{M}$. $Q_{ab}(\sigma)$ can be interpreted as a (physical, i.e. gauge independent) Riemannian metric[7] on a 3–dimensional manifold Σ with local coordinate system $\{\sigma^a\}_{a=1}^3$ defined through the readings of the dust fields and $P^{ab}(\sigma)$ is related to the extrinsic curvature of Σ in a $4-$dimensional globally hyperbolic manifold M which in turn is isomorphic to $\mathbb{R}\times\Sigma$. Σ as a Riemannian manifold carries geometric structures such as a unique torsionfree covariant derivative D or the Ricci scalar $R(Q)$. $Q_{ab}(\sigma)$ and $P^{ab}(\sigma)$ form a canonical pair, i.e. the symplectic 2–form $\Omega \in T^*\mathcal{M}\times T^*\mathcal{M}$ is given by $\Omega := \int d\sigma dP^{ab}(\sigma)\wedge dQ_{ab}(\sigma)$. Furthermore let

$$C^g \;:=\; \frac{\kappa}{\sqrt{\det Q}}\big[Q_{ab}Q_{cd}-\frac{1}{2}Q_{ac}Q_{bd}\big]P^{ac}P^{bd}-\frac{1}{\kappa}\sqrt{\det Q}R(Q) \tag{2.7}$$

$$C^g_a \;:=\; -2Q_{ab}D_cP^{bc} \tag{2.8}$$

be C^∞ functions on $\mathcal{M}$, X_{C^g}, $X_{C^g_a}$ their respective Hamiltonian vector fields and $\kappa = 8\pi G/c^3$ the gravitational coupling constant. These functions are the gravitational parts of the *Hamiltonian* and *diffeomorphism constraint* respectively. If we would work in the standard formalism of GR where q_{ab}, p^{ab} are the gauge dependent 3–metric and its canonical momentum then C^g and C^g_a (after the replacement $Q_{ab}\to q_{ab}, P^{ab}\to p^{ab}$) would generate gauge transformations and only quantities which are constants along the flow lines of their respective Hamiltonian vector fields would have a physical interpretation. However, in the enlarged phase space described in the last section, C^g and C^g_a are just the *gravitational parts* of the constraints and, without taking into account the parts coming from the dust fields, do not generate gauge transformations. Due to the gauge fixing and construction of manifestly gauge invariant quantities $Q_{ab}(\sigma), P^{ab}(\sigma)$ we have a physical Hamiltonian that generates time evolution in the dust–frame, see equation (2.2).
For the purpose of what follows we introduce an orthonormal frame at each point $\sigma\in\Sigma$ and define the cotriad[8] E^i_a through $Q_{ab}(\sigma) = \delta_{ij}E^i_aE^j_b$. It can easily be seen that this definition is invariant under $SO(3)$ or more general $SU(2)$ rotations, thus the cotriads can be interpreted as $su(2)$–valued one forms with $i,j,k,\dots = 1,2,3$ $su(2)$–indices. Furthermore define the triad E^a_i to be the inverse of the cotriad through $E^a_iE^i_b = \delta^b_a, E^a_iE^j_a = \delta^j_i$.
Now we perform the following coordinate transformation:

$$(Q_{ab}, P^{ab}) \quad\to\quad (E^j_a, P^a_j := 2P^{ab}E^j_b) \quad . \tag{2.9}$$

One can easily check that this transformation is indeed a canonical one, i.e. it leaves the symplectic 2–form invariant and thus (E^i_a, P^a_i) form a canonical pair

$$\{P^a_k(\sigma), E^i_b(\sigma')\} \;=\; \kappa\delta^a_b\delta^i_k\delta(\sigma,\sigma') \quad . \tag{2.10}$$

However, the phase space spanned by (P^a_k, E^k_a) is larger than the one spanned by (P^{ab}, Q_{ab}): Only the symmetrised product $P^{ab} = \frac{1}{2}P^{(a}_jE^{b)}_j := \frac{1}{4}(P^a_jE^b_j + P^b_jE^a_j)$ survives when we invert the canonical transformation and want to transform back to ADM variables, which means that the canonical transformation $(Q_{ab}, P^{ab})\to(E^j_a, P^a_j)$ is not one–to–one but one–to–many. To get rid of the additional degrees of freedom we must demand $P^{[a}_jE^{b]}_j := \frac{1}{2}(P^a_jE^b_j - P^b_jE^a_j) = 0$ which after some algebraic manipulations leads to a constraint

$$G_i := \epsilon_{ijk}P^a_jE^k_a \quad , \tag{2.11}$$

the so called SU(2)–Gauss constraint[9].
Modulo Gauss constraint the gravitational parts of the Hamiltonian and diffeomorphism constraints can be written as $C^g = \frac{\kappa}{4\det(E)}[E^i_aE^i_b\delta^k_m - \frac{1}{2}E^k_aE^m_b]P^a_kP^b_m - \frac{\det(E)}{\kappa}R(E)$ and $C^g_a = -E^i_aD_b(e)P^b_i$ respectively where $R(E)$ denotes the Ricci scalar written in terms of the cotriad and $D(E)$ is the covariant derivative compatible with the cotriad.
So finally we arrive at a classically equivalent formulation of general relativity in the dust–frame, which we want to summarise in the following paragraph:

[7] This identification can be made precise by demanding appropriate boundary conditions for Q_{ab}, P^{ab} and specifying the function space in which they live. See [18] for more details and appropriate boundary conditions.

[8] Again, we use a capital letter E^i_a to make clear that this is a gauge invariant version of the cotriad. It should not be mistaken for the densitised triad used in LQG.

[9] In analogy with the formulation of GR in terms of Ashtekar's variables we still call this the Gauss–constraint although it does not have the form known from Yang–Mills theories.

The phase space $\mathcal{M}'$ is spanned by an $su(2)$–valued one–form E_a^i and its canonically conjugate momentum P_j^a, a vector density which takes values in $su(2)$ as well. The phase space is subject to the Gauss constraint

$$G_i = \epsilon_{ijk} P_j^a E_a^k \quad , \tag{2.12}$$

which means that only quantities which are invariant along the flow lines of its associated Hamiltonian vector field have a physical interpretation. In the gauge fixed formalism we have a physical Hamiltonian

$$H_{\rm phys} = \int_\Sigma d\sigma \sqrt{(C^g + C^m)^2 - Q^{ab}(C_a^g + C_a^m)(C_b^g + C_b^m)} \tag{2.13}$$

that generates motion in dust–time and in our variables the gravitational parts of the Hamiltonian and diffeomorphism constraints are given by

$$C^g = \kappa G_{ab}^{ij} P_i^a P_j^b - \frac{1}{\kappa} \det(E) R(E) \tag{2.14}$$

$$C_a^g = -E_a^i D_b(E) P_i^b \quad , \tag{2.15}$$

As before C^m and C_a^m denote all possible standard matter contributions that one might couple to gravity and we defined the deWitt–type supermetric $G_{ab}^{ij} := \frac{1}{4\det(E)}[E_a^k E_b^k \delta^{ij} - \frac{1}{2} E_a^i E_b^j]$. Thus, in contrast to the formulation of general relativity in terms of Ashtekar's variables the Hamiltonian constraint looks similar to a standard Hamiltonian composed out of of a kinetic term $\propto P^2$ and a potential $\det(E)R(E)$.

The choice of these "hybrid variables" lies in between those of connection dynamics and geometrodynamics. Geometrodynamics is not sufficient if we are interested in coupling fermionic matter. Furthermore, it is much harder to smear symmetric tensors over submanifolds in a spatially diffeomorphism covariant manner than it is possible for the one and two forms e, $*P$ respectively. The drawback of using cotriads rather than connections, as already mentioned in the introduction, is that the holonomy transforms locally under SU(2) gauge transformations while the integral of the cotriad along one dimensional curves does not. However, in the algebraic setting that we choose in the next section, this will not pose any problem.

For what follows we need the explicit expression of the curvature $R(E)$–term in the Hamiltonian constraint. The cotriad E_a^i together with the compatibility condition $D_a E_b^i \overset{!}{=} 0$ uniquely defines a spin connection Γ_a^i through $D_a E_b^i =: \partial_a E_b^i + \Gamma_{ab}^c E_c^i + \epsilon_{ijk} \Gamma_a^j E_b^k$ where Γ_{ab}^c are the components of the Levi–Civita connection. In terms of the cotriad and its inverse this spin connection can be written as

$$\Gamma_a^i = -\frac{1}{2} \epsilon_{ijk} E_j^b (E_{a,b}^k - E_{b,a}^k + E_k^c E_a^m E_{c,b}^m) \quad , \tag{2.16}$$

where we used $E_{a,b}^i$ as a short hand notation for $\partial_b E_a^i$.
Using the above covariant derivative one can define a curvature tensor R_{ab}^j in the obvious way as[10]

$$[D_a, D_b] v_j =: \epsilon_{jkl} R_{ab}^k v_l \tag{2.17}$$

for every $su(2)$–valued scalar function v_j on Σ. In terms of the spin connection this curvature tensor can be given explicitly as

$$R_{ab}^l = 2\partial_{[a} \Gamma_{b]}^l + \epsilon_{lkj} \Gamma_a^k \Gamma_b^j \quad . \tag{2.18}$$

Finally the Ricci scalar in terms of cotriads can be written as

$$R(E) = \epsilon_{jkl} R_{ab}^l E_j^a E_k^b = 2\epsilon_{jkl} E_j^a E_k^b \partial_a \Gamma_b^l + (E_k^a \Gamma_a^j)^2 - E_j^a E_k^b \Gamma_a^k \Gamma_b^j \quad . \tag{2.19}$$

In this form the Ricci scalar is a function of the cotriad E_a^i and its inverse E_i^a. Having a quantised version of this theory in mind, it will turn out to be more convenient to rewrite R in terms of the cotriad and its determinant only which can be achieved by employing the identity $E_i^a = \frac{1}{2\det(E)} \epsilon^{abc} \epsilon_{ijk} E_b^j E_c^k$. Thus the 3 terms in the preceding expression

[10] This definition is only meaningful if $\dim(\Sigma) = 3$ where the fundamental and the adjoint representation of the gauge group have the same dimension.

take the form

$$
\begin{aligned}
(E^a_i\Gamma^i_a)^2 &= \frac{1}{4(\det(E))^2}\epsilon^{abc}\epsilon^{def}E^k_aE^l_dE^k_{b,c}E^l_{e,f} && (2.20)\\
-E^a_iE^b_j\Gamma^j_a\Gamma^i_b &= -\frac{1}{4(\det(E))^2}\epsilon^{abc}\epsilon^{def}\left[E^i_c(E^i_{d,b}-E^i_{b,d})+E^j_dE^j_{c,b}\right]\left[E^k_f(E^k_{a,e}-E^k_{e,a})+E^l_aE^l_{f,e}\right] && (2.21)\\
2\epsilon_{ijk}E^a_iE^b_j\partial_a\Gamma^k_b &= -\frac{1}{(\det(E))^2}\epsilon^{abc}\epsilon^{def}E^i_cE^i_f\left[E^j_eE^j_{b,da}+E^j_{b,a}E^j_{e,d}+E^j_bE^j_{e,da}\right]\\
&\quad+\frac{1}{2(\det(E))^3}\epsilon^{abc}\epsilon^{def}\epsilon^{a'b'c'}\epsilon_{jkl}E^i_cE^i_fE^j_{b'}E^k_{c'}\\
&\qquad\left[E^m_b(E^l_{d,a}E^m_{e,a'}+E^l_{e,a}E^m_{a',d})+E^m_eE^l_{d,a}(E^m_{b,a'}-E^m_{a',b})\right] \quad . && (2.22)
\end{aligned}
$$

This means that the "potential"–term in the Hamiltonian constraint is symbolically of the form

$$
\det(E)R(E) \quad\propto\quad \frac{E^2(\partial E)^2}{\det(E)}+\frac{E^3(\partial\partial E)}{\det(E)}+\frac{E^5(\partial E)^2}{(\det(E))^2} \quad . \tag{2.23}
$$

In terms of the momenta P^a_i, the cotriad E^i_a and its determinant the diffeomorphism constraint can be written as

$$
\begin{aligned}
C^g_a &= -E^i_aD_bP^b_i=-E^i_a\partial_bP^b_i-\epsilon_{ijk}E^i_a\Gamma^j_bP^b_k\\
&= -E^i_a\partial_bP^b_i+\frac{1}{2}P^b_i(E^i_{a,b}-E^i_{b,a})+\frac{1}{4\det(E)}\epsilon^{cdg}\epsilon_{ijk}E^j_dE^k_gP^b_i\left[E^l_a(E^l_{b,c}-E^l_{c,b})+E^l_b(E^l_{a,c}-E^l_{c,a})\right] \quad . && (2.24)
\end{aligned}
$$

Throughout this article we will work in units where $c=1$ and the coordinates of Σ have the dimension of length i.e. $[\sigma]=cm$. Thus, the metric Q_{ab} is dimensionless as well as the cotriads, $[E^i_a]=1$. The extrinsic curvature of Σ is measured in units of cm^{-1} which directly carries over to our canonical momenta, $[P^a_i]=cm^{-1}$. The gravitational coupling constant carries units $[\kappa]=cm\cdot kg^{-1}$ and we will leave Planck's constant dimensionful, $[\hbar]=kg\cdot cm$.

2.4 Coupling matter

Now we want to couple matter to the gravitational sector. In principle one can allow all (standard model) matter but for illustrative purposes we want to restrict ourselves to the case of a minimally coupled scalar field of Klein–Gordon type in this article: The basic variables are given by a scalar field Φ and its canonically conjugate momentum Π, both assumed to be in $C^\infty(\Sigma)$:

$$
\{\Pi(\sigma),\Phi(\sigma)\}=\lambda\delta(\sigma,\sigma') \tag{2.25}
$$

with the matter coupling constant λ which we will set equal to one for the remainder of this article.
In ADM–variables the Hamiltonian and diffeomorphism constraints in the matter sector are given by

$$
\begin{aligned}
C^\phi &= \frac{\Pi^2}{2\lambda\sqrt{\det Q}}+\frac{\sqrt{\det Q}}{2\lambda}(Q^{ab}\partial_a\Phi\partial_b\Phi+\frac{m^2}{\hbar^2}\Phi^2) && (2.26)\\
C^\phi_a &= \Pi\partial_a\Phi && (2.27)
\end{aligned}
$$

respectively which can directly be rewritten in terms of our gravitational variables (E^i_a,P^a_i) as

$$
\begin{aligned}
C^\phi &= \frac{1}{2\lambda\det(E)}\Pi^2+\frac{1}{2\lambda\det(E)}\epsilon^{acd}\epsilon^{bef}E^k_cE^l_dE^k_eE^l_f\Phi_{,a}\Phi_{,b}+\frac{1}{2\lambda}\det(E)\frac{m^2}{\hbar^2}\Phi^2 && (2.28)\\
C^\phi_a &= \Pi\partial_a\Phi \quad . && (2.29)
\end{aligned}
$$

In the above definitions we keep the factor of $\frac{1}{\hbar^2}$ in the definition of the massive term to make sure that Π,Φ have the right dimensions: It can easily be seen that $[\Phi]=kg^{-1/2}\cdot cm^{-1/2}$ and $[\Pi]=kg^{1/2}\cdot cm^{-3/2}$.
The physical Hamiltonian, which generates evolution in dust–time for the whole system gravity plus matter is now given by

$$
H_{\rm phys}=\int_\Sigma d\sigma\sqrt{(C^g+C^\phi)^2-Q^{ab}(C^g_a+C^\phi_a)(C^g_b+C^\phi_b)} \quad . \tag{2.30}
$$

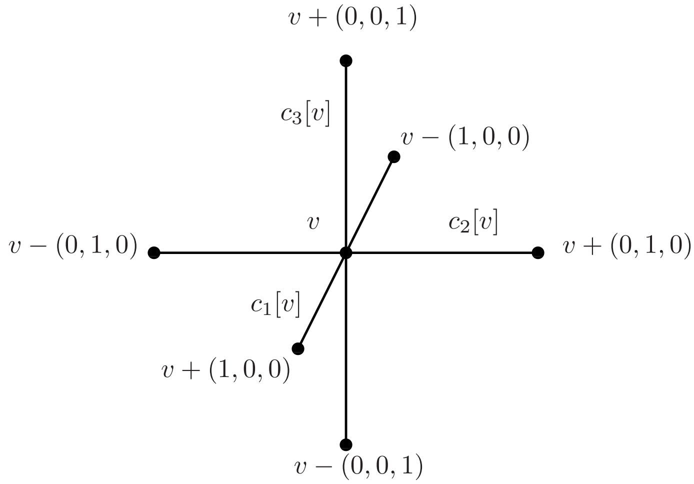

Figure 1: A part of the graph γ with vertices $v, v \pm (1,0,0), v \pm (0,1,0), v \pm (0,0,1)$ and (outgoing) edges $c_1(v), c_2(v), c_3(v)$.

After the canonical transformation in the gravitational sector the matter degrees of freedom only couple to the cotriads E_a^i and not to their momenta P_i^a. When turning to the quantum theory in section 3 we will choose a representation where the cotriads are quantised as multiplication operators as opposed to the LQG-representation where (smeared) densitised triads act as (non–commuting) right–invariant vector fields on SU(2). Albeit loosing the non–commutative structure (which might turn out to be important in the long run) this is one of the main advantages of working in this representation because having to deal only with gravitational multiplication operators in the analysis of the matter sector makes it much easier to analyse the QFT on CS limit in this framework.

2.5 Discretisation

Having rewritten general relativity in terms of the cotriad E_a^j and its canonical momentum P_j^a we now want to explain how one can discretise this theory on a regular cubic lattice in 3 spatial dimensions. This will result in a classical dynamical system of an at most countable number of degrees of freedom and it is exactly this dynamical system which we will compare the quantum theory with that we will define in subsequent sections.
Let γ be a piecewise analytic graph, embedded in Σ and of cubic topology (that is, each vertex is 6–valent). For details concerning piecewise analytic graphs of the type usually employed in LQG see for instance [7]. For our purposes it is sufficient[11] to think of γ as a collection $V(\gamma)$ of points $v \in \Sigma$, called vertices, together with analytic edges $c_I[v] : (0,1) \to \Sigma; \quad s \mapsto c_I^a[v](s)$ such that $c_I[v](0) = v$ and $c_I[v](1) = v + I$. The requirement of cubic topology means that each vertex v has exactly 3 outgoing edges, that is, $I, J, K = 1, 2, 3$. Hence, each vertex v has exactly 6 neighbouring vertices, which we will denote by $v \pm (1,0,0), v \pm (0,1,0)$ and $v \pm (0,0,1)$, or simply as $v \pm I$ for the neighbouring vertex in direction I.
Furthermore we will need the dual graph γ^* whose faces are defined through a smooth embedding

$$S_{c_I[v]} : (0,1) \times (0,1) \to \Sigma; \quad s_1, s_2 \mapsto S_{c_I[v]}(s_1, s_2) \quad . \tag{2.31}$$

The embedding is chosen in such a way that each surface $S_{c_I[v]} \in \gamma^*$ intersects γ only in exactly one point p_c, which is an interior point of $S_{c_I[v]}$ as well as $c_I[v]$.
A discretised version of general relativity using the variables defined above should not depend on any background metric, since there simply is none. In order to define lattice variables that do not show any dependence on a metric one proceeds as follows: E_a^j is an $su(2)$–valued one–form, so we can naturally (that is without reference to any metric) integrate it along one–dimensional submanifolds of Σ, namely the edges. P_j^a is an $su(2)$–valued vector density, so we can naturally integrate it over the faces of the dual graph. Similiar reasoning holds for the matter variables: Π is a scalar density, thus it can naturally be integrated over the volumina of the dual graph, and for Φ as a scalar field of

[11]We will be rather informal here: Of course one should not consider paths as we do here but certain equivalence classes thereof. But in this article these details play a minor role so we will drop them for the sake of clarity.

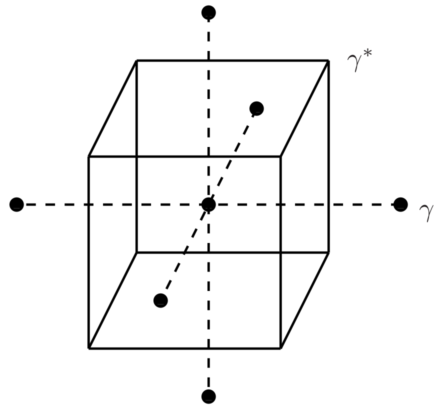

Figure 2: A part of the graph γ and the corresponding part of its dual graph γ^*.

density weight zero the natural thing to do is to just take the value of the field at each vertex. Thus, we define

$$E_I^i(v) := \int_0^1 ds \partial_s c_I^a[v](s) E_a^i(c_I[v](s)) \quad , \tag{2.32}$$

$$P_i^I(v) := \int_0^1 ds_1 \int_0^1 ds_2 \frac{\partial S^a_{c_I[v]}}{\partial s_1} \frac{\partial S^b_{c_I[v]}}{\partial s_2} \epsilon_{abc} P_i^c(S_{c_I[v]}(s_1, s_2)) \quad , \tag{2.33}$$

$$\Phi(v) := \Phi(\sigma)|_{\sigma=v} \quad , \tag{2.34}$$

$$\Pi(v) := \int_{V(v)} d\sigma \Pi(\sigma) \quad . \tag{2.35}$$

These discrete variables fulfil the Poisson algebra[12]

$$\begin{aligned} \{P_i^I(v), E_J^j(v')\} &= \kappa \delta_J^I \delta_i^j \delta_{vv'} \quad , \\ \{P_i^I(v), P_j^J(v')\} = \{E_I^i(v), E_J^j(v')\} &= 0 \quad , \\ \{\Pi(v), \Phi(v')\} &= \delta_{vv'} \quad , \\ \{\Pi(v), \Pi(v')\} = \{\Phi(v), \Phi(v')\} &= 0 \quad . \end{aligned} \tag{2.36}$$

Poisson brackets between matter and geometry variables vanish obviously and the $\delta_{vv'}$ appearing in the above formulas is now the Kronecker delta. The Poisson brackets lose their distributional form exactly because we have chosen to smear the dynamical variables in dimensions which add up to $\dim(\Sigma)$ for canonical pairs. Due to the smearing procedure the discrete variables carry different units compared to the continuum variables, namely $[E_I^i(v)] = cm$, $[P_i^I(v)] = cm$, $[\Phi(v)] = kg^{1/2} \cdot cm^{-1/2}$ and $[\Pi(v)] = kg^{1/2} \cdot cm^{3/2}$.
Now we can define discrete analogues of the constraints from the last section by simply replacing

$$\begin{aligned} E_a^i(\sigma) &\to E_I^i(v) \quad , \\ P_i^a(\sigma) &\to P_i^I(v) \quad , \\ \Phi(\sigma) &\to \Phi(v) \quad , \\ \Pi(\sigma) &\to \Pi(v) \quad , \end{aligned} \tag{2.37}$$

and replacing the integrals $\int_\Sigma d\sigma$ by Riemann sums $\sum_{V(\alpha)}$. Of course there is an ambiguity in defining these discrete analogues just because there is no unique way of replacing the derivatives of $E_a^i(\sigma)$: One could replace $\partial_a E_b^i(\sigma) \to \nabla_I^+ E_J^i(v)$ or $\partial_a E_b^i(\sigma) \to \nabla_I^- E_J^i(v)$ where ∇_I^+, ∇_I^- are the lattice forward and backward derivatives into direction I respectively defined through $\nabla_I^+ f(v) := f(v+I) - f(v)$ and $\nabla_I^- f(v) := f(v) - f(v-I)$ for functions $f : V(\gamma) \to \mathcal{T}$ (in our case either $\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = su(2)$ but one could have different topological spaces $\mathcal{T}$). Both definitions (and others involving

[12]Here we fix the matter coupling constant $\lambda = 1$.

not only next – neighbouring terms) are equivalent when taking the continuum limit.

Remark:
The algebra of the naively discretised Hamiltonian densities or any other quantities involving spatial derivatives that are replaced by difference operators will differ from their continuum limits even in the classical theory. This poses no consistency problem because these are no constraints anymore but true Hamiltonian densities, there are no anomalies. However, the densities will no longer be constants of the motion and one may want to improve the situation. If we would have constraints then one could fix the corresponding Lagrange multipliers by the method proposed in [47, 48, 49, 50]. However, we are not in this situation and thus one would need to rather change the naive discretisation into a more subtle one as suggested in [20], possibly along the lines of [51]. Notice that we still do have a constraint, namely the Gauss constraint, but in our variables, in contrast to the connection variables, this does not involve spatial derivatives and thus there are no anomalies.

3 Quantum theory

In this section we want to define a quantum theory of General Relativity in the formulation developed in section 2. To be more specific, we will start with the algebra (2.36) and explain how one can represent it on an appropriately chosen Hilbert space. First we will define a quantum theory on a *purely algebraic* level, that is, we will start with an elementary algebra of observables $\mathfrak{A}$ labeled by an algebraic graph α and then construct a representation of this algebra on some Hilbert space $\mathcal{H}$ defined on that graph. Thus, dynamics will be defined on an abstract graph, which does not know anything about embedding, differential or even topological structures of a manifold, since there is no manifold at this stage. In the framework of LQG many things do actually not depend on the embedding and this was one of the motivations in [14] to formulate an LQG inspired quantum theory purely on the algebraic level. However, if we want to interpret this theory as a quantum theory of General Relativity we need to make sure that there exists a large enough class of semiclassical states (and these states will carry information about the embedding of the graph α into the manifold Σ to be approximated) and it can indeed be shown that such semiclassical states exist (see section 4). If the quantum system is in such a state the expectation values of geometrical observables will be sharply peaked on classical values and thus, we are getting classical discretised gravity as a classical limit of our algebraic theory.

Now we want to define the quantum theory living on an algebraic graph and we will do this in a similar way as in [14]. First we need to define what we mean by an algebraic graph as opposed to the embedded, piecewise analytic graph γ from section 2.5: Let α be an oriented algebraic graph with N vertices, i.e. a countable set $V(\alpha)$ of N points ("vertices") together with an $N \times N$ matrix α whose entries $\alpha_{vv'}, v, v' = 1, \ldots, N$ take non–negative integer values. In general $\alpha_{vv'}$ is not a symmetric matrix for an oriented graph and the total number of "edges" connecting vertex v and vertex v' is given by $c_{vv'} := \alpha_{vv'} + \alpha_{v'v}$. To be more intuitive, one can think of α as all the information that remains if one starts with an embedded, piecewise analytic graph γ, consisting of N vertices and $\alpha_{vv'}$ embedded edges between vertex v and v' and then drops everything that γ knows about metric, differentiable or even topological structures of Σ. α neither contains any information about the braiding of its edges nor about smoothness or n–times differentiability usually associated to edges which are embedded in a manifold.
For the remainder of this article we will be interested in the case $N = \aleph$ and graphs of cubic topology, i.e. each vertex $v \in V(\alpha)$ is 6 valent and $c_{vv'} \in \{0, 1\}\, \forall v, v'$. Thus, each vertex v has three outgoing edges, which we will label by $I, J, K = 1, 2, 3$ for bookmarking purposes and three ingoing edges which will be counted as outgoing edges for the respective neighbouring vertex in direction $-I, -J, -K$.

3.1 The algebra

Now we define an abstract *−algebra $\mathfrak{A}$ whose generators are labeled by vertices of the algebraic graph α and which implements the *−relations and canonical commutation relations that arise from the classical reality conditions (all discretised fields are real valued) and the Poisson brackets (2.36):

3.1.1 Gravitational sector

Let α be the cubic algebraic graph with $N = \aleph$ as described above. With each vertex v of α associate a triple $E^i_I(v)$ of $su(2)$–valued quantities where $I, J, K, \ldots$ label the 3 outgoing edges at each vertex and $i, j, k, \ldots$ are $su(2)$–indices. Furthermore associate a triple of $su(2)$–valued quantities $P^I_i(v)$ with each vertex where again capital letters denote outgoing edges and small letters are $su(2)$–indices. Let these basic variables be subject to the following algebraic relations:

$$\begin{aligned}
\left[E^i_I(v), E^j_J(v')\right] &= 0 \\
\left[P^I_i(v), P^J_j(v')\right] &= 0 \\
\left[P^I_i(v), E^j_J(v')\right] &= \kappa\delta_{IJ}\delta_{ij}\delta_{vv'} \quad .
\end{aligned} \tag{3.1}$$

Furthermore we can define an involution on the algebra and will demand the (trivial) *–relations

$$(E^i_I(v))^* = E^i_I(v), \quad (P^I_i(v))^* = P^I_i(v) \quad . \tag{3.2}$$

Thus $E^i_I(v)$ and $P^I_i(v)$ form an abstract *–algebra with the above relations which we want to denote as $\mathfrak{A}^g$.

3.1.2 Matter sector

The same thing can be done for the matter degrees of freedom: With each vertex v of α associate a pair of real valued quantities $\Phi(v)$ and $\Pi(v)$. Let these variables be subject to the algebraic relations

$$\begin{aligned}
\left[\Phi(v), \Phi(v')\right] &= 0 \\
\left[\Pi(v), \Pi(v')\right] &= 0 \\
\left[\Pi(v), \Phi(v')\right] &= \delta_{vv'} \quad ,
\end{aligned} \tag{3.3}$$

and again define a (trivial) involution * through

$$(\Phi(v))^* = \Phi(v), \quad (\Pi(v))^* = \Pi(v) \quad . \tag{3.4}$$

Again $\Phi(v)$ and $\Pi(v)$ together with the above relations form an abstract *–algebra which we want to denote by $\mathfrak{A}^\phi$. So the whole theory is described by an abstract *–algebra $\mathfrak{A} := \mathfrak{A}^g \otimes \mathfrak{A}^\phi$.

3.2 The Hilbert space

$\mathfrak{A}$ can be represented as an algebra of linear operators on an ITP Hilbert space $\mathcal{H}_\otimes := \otimes_v \mathcal{H}_v$ where $\mathcal{H}_v = \mathcal{H}^g_v \otimes \mathcal{H}^m_v$. where the individual Hilbert spaces on each vertex, $\mathcal{H}^g_v$ and $\mathcal{H}^\phi_v$, are chosen as explained below. Note that this ITP is different from

$$\mathcal{H}^g_\otimes \otimes \mathcal{H}^\phi_\otimes := (\otimes_v \mathcal{H}^g_v) \otimes (\otimes_v \mathcal{H}^\phi_v) \quad , \tag{3.5}$$

because of the non associativity of the infinite tensor product. The bracketing chosen by us is preferred because it allows to construct gauge invariant quantities. More information about the ITP construction can be found in appendix A.

3.2.1 Gravitational sector

Let $\mathcal{H}^g_v := L_2(\mathbb{R}^9, d\mu)$ be a Hilbert space associated to each vertex v with Lebesgue measure μ on $\mathbb{R}^9$ and consider the ITP Hilbert space $\mathcal{H}^g_\otimes$. On $\mathcal{H}^g_\otimes$ the algebra $\mathfrak{A}^g$ can be represented by choosing a representation $\rho : \mathfrak{A}^g \to \mathcal{L}(\mathcal{H}^g_\otimes)$ in which $E^i_I(v)$ acts as a multiplication operator and $P^I_i(v)$ as a derivative operator,

$$\hat{E}^i_I(v) \cdot \psi(E) := \left[\rho(E^i_I(v))\psi\right](E) \quad := \quad E^i_I(v) \cdot \psi(E) \tag{3.6}$$

$$\hat{P}^I_i(v) \cdot \psi(E) := \left[\rho(P^I_i(v))\psi\right](E) \quad := \quad i\ell_P^2 \frac{\partial}{\partial E^i_I(v)} \psi(E) \quad , \tag{3.7}$$

for all $\psi(E) \in \mathcal{H}^g_\otimes$ and $\ell_P = \sqrt{\kappa\hbar}$ is the Planck length[13]. Note that the derivative in the second line is indeed just the partial derivative and not a functional derivative. As operators on $\mathcal{H}^g_\otimes$ the cotriad $E^i_I(v)$ and its conjugate momentum $P^I_i(v)$ fulfil the commutator relations

$$\left[\hat{P}^I_i(v), \hat{E}^j_J(v')\right] \quad = \quad i\ell_P^2 \delta_{IJ}\delta_{ij}\delta_{vv'} \quad . \tag{3.8}$$

In what follows we will mostly omit the hat above the operators and use the same symbols $E^i_I(v), P^I_i(v)$ for elements of the abstract *–algebra $\mathfrak{A}^g$ and their representatives as linear operators on the Hilbert space $\mathcal{H}^g_\otimes$ but it should be clear from the context which one we mean.

[13]We need to include a factor of ℓ_P^2 into the definition of the operator $P^I_i(v)$ in order to get the same dimensions as its lattice–analogue.

3.2.2 Matter sector

Again, a natural representation of $\mathfrak{A}^\phi$ is given by the ITP representation on $\mathcal{H}^\phi := \otimes_v \mathcal{H}^\phi_v$ where each vertex labels a Hilbert space $\mathcal{H}^\phi_v := L_2(\mathbb{R}, d\nu)$ with ν being the Lebesgue measure on $\mathbb{R}$. We choose a representation such that

$$\begin{aligned} \hat{\Phi}(v) \cdot \psi(\Phi) &:= \Phi(v)\psi(v) \quad , & (3.9) \\ \hat{\Pi}(v) \cdot \psi(\Phi) &:= i\hbar\partial_{\Phi(v)}\psi(\Phi) \quad , & (3.10) \end{aligned}$$

for $\psi \in \mathcal{H}^\phi_\otimes$. As linear operators on $\mathcal{H}^\phi_\otimes$ these fulfil the commutator relations

$$[\hat{\Pi}(v), \hat{\Phi}(v)] = i\hbar\delta_{vv'} \quad . \tag{3.11}$$

3.3 The Hamiltonian Density

In section 2.2 we saw that, given that $\frac{Q^{ab}C_aC_b}{C^2}$ is small, one can approximate the physical Hamiltonian by $H_{\text{phys}} \approx \int_\Sigma d\sigma \left(C^g(\sigma) + C^\phi(\sigma)\right)$ plus higher order terms. So in order to define physical dynamics in the quantum sector, the first step is to have a well defined quantum operator $\hat{C}$ associated to the non dust contributions of the classical Hamiltonian constraint $\int_\Sigma d\sigma \left(C^g(\sigma) + C^\phi(\sigma)\right)$. For higher order corrections we will need the diffeomorphism constraint as well, so for completeness we will also give a quantum operator $\hat{C}_a(v)$ associated to the non dust contributions to the classical diffeomorphism constraint density.
The crucial step in defining these operators is to be able to define operator analogues for functions involving inverse powers of $\det(E)$. In this section we will just formally define these operators, in the next section we will show in detail that they are in fact symmetric operators on a dense subspace of $\mathcal{H}_\otimes$.
All the operators will be of the form $O = O^g \otimes O^\phi$, where the first part, O^g, acts only on $\mathcal{H}^g_\otimes$ and the second part, O^ϕ, acts only on $\mathcal{H}^\phi_\otimes$.

3.3.1 Gravitational sector

As the operator analogue for the (integrated) gravitational part of the Hamiltonian constraint we define[14]

$$\begin{aligned}
\hat{C}^g &:= (\hat{C}^g_{\text{kin}} - \hat{C}^g_{\text{pot}}) \otimes id_\phi & (3.12)\\
\hat{C}^g_{\text{pot}} &:= \hat{C}^g_{\text{pot}_1} + \hat{C}^g_{\text{pot}_2} + \hat{C}^g_{\text{pot}_3} & (3.13)\\
\hat{C}^g_{\text{kin}} &:= \kappa \sum_v \sum_{I,J} \left[i\ell_P^2 \partial_{E^i_I(v)}\right] G^{ij}_{IJ} \left[i\ell_P^2 \partial_{E^j_J(v)}\right] \quad , & (3.14)\\
\hat{C}^g_{\text{pot}_1} &:= \frac{1}{4\kappa \det E(v)} \sum_{I,J,K,L,M,N} \epsilon^{IJK}\epsilon^{LMN} E^k_I(v) E^l_L(v) \big(\nabla^+_K E^k_J(v)\big)\big(\nabla^+_N E^l_M(v)\big) \quad , & (3.15)\\
\hat{C}^g_{\text{pot}_2} &:= -\frac{1}{4\kappa \det E(v)} \sum_{I,J,K,L,M,N} \epsilon^{IJK}\epsilon^{LMN} \\
&\quad \left[E^k_K(v)\big(\nabla^+_J E^k_L(v) - \nabla^+_L E^k_J(v)\big) + E^k_L(v)\big(\nabla^+_J E^k_K(v)\big)\right] \times \\
&\quad \times \left[E^k_N(v)\big(\nabla^+_M E^k_I(v) - \nabla^+_I E^k_M(v)\big) + E^l_I(v)\big(\nabla^+_M E^l_N(v)\big)\right] \quad , & (3.16)\\
\hat{C}^g_{\text{pot}_3} &:= -\frac{1}{\kappa \det E(v)} \sum_{I,J,K,L,M,N} \epsilon^{IJK}\epsilon^{LMN} E^k_K(v) E^k_N(v) \times \\
&\quad \times \left[E^l_M(v)\nabla^-_I\nabla^+_L E^l_J(v) + \big(\nabla^+_I E^l_J(v)\big)\big(\nabla^+_L E^l_M(v)\big) + E^l_J(v)\big(\nabla^-_I\nabla^+_L E^l_M(v)\big)\right] \\
&\quad + \frac{1}{2\kappa\big(\det E(v)\big)^2} \sum_{I,J,K,L,M,N,O,P,Q} \epsilon^{IJK}\epsilon^{LMN}\epsilon^{OPQ}\epsilon_{jkl} E^i_K(v) E^i_N(v) E^j_P(v) E^k_Q(v) \times \\
&\quad \times \left[E^m_J(v)\left[\big(\nabla^+_I E^l_L(v)\big)\big(\nabla^+_O E^m_M(v)\big) + \big(\nabla^+_I E^l_M(v)\big)\big(\nabla^+_L E^m_O(v)\big)\right]\right. \\
&\quad \left. + E^m_M(v)\big(\nabla^+_I E^l_L(v)\big)\left[\big(\nabla^+_O E^m_J(v)\big) - \big(\nabla^+_J E^m_O(v)\big)\right]\right] \quad . & (3.17)
\end{aligned}$$

[14] For the most part of what follows we will again drop the hats over the constraint operators when there is no ambiguity in mistaking them for the classical functions.

id_ϕ is the identity operator on $\mathcal{H}^\phi_\otimes$. The multiplication operator in the kinetic term is given by

$$G^{ij}_{IJ}[E(v)] := \frac{1}{4\det E(v)}(E^k_I(v)E^k_J(v)\delta^{ij} - \frac{1}{2}E^i_I(v)E^j_J(v)) \tag{3.18}$$

and by ∇^+_I and ∇^-_I we denote the lattice forward and backward derivatives into direction I respectively defined through $\nabla^+_I f(v) := f(v+I) - f(v)$ and $\nabla^-_I f(v) := f(v) - f(v-I)$ for functions $f : V(\alpha) \to su(2)$.
The operator $\hat{C}^g_I(v) = \hat{\hat{C}}^g_I(v) \otimes id_\phi$, associated to the classical gravitational diffeomorphism constraint density, can be defined as

$$\begin{aligned}
\hat{\hat{C}}^g_I(v) \;\; := \;\; & -\frac{1}{2}\sum_{J,j} E^j_I(\nabla^+_J P^J_j(v)) + (\nabla^+_J P^J_j(v))E^j_I(v) \\
& +\frac{1}{4}\sum_{J,j} P^J_j(v)\left[(\nabla^+_J E^j_I(v)) - (\nabla^+_I E^j_J(v))\right] + \left[(\nabla^+_J E^j_I(v)) - (\nabla^+_I E^j_J(v))\right] P^J_j(v) \\
& +\frac{1}{8}\sum_{\substack{JKLM \\ ijkl}} \epsilon^{KLM}\epsilon_{ijk}\Big[\\
& P^J_i(v)\left[\frac{1}{\det(E)}E^j_L(v)E^k_M(v)\Big[E^l_I(v)\Big((\nabla^+_K E^l_J(v)) - (\nabla^+_J E^l_K(v))\Big) + E^l_J(v)\Big((\nabla^+_K E^l_I(v)) - (\nabla^+_I E^l_K(v))\Big)\Big]\right] \\
& + \left[\frac{1}{\det(E)}E^j_L(v)E^k_M(v)\Big[E^l_I(v)\Big((\nabla^+_K E^l_J(v)) - (\nabla^+_J E^l_K(v))\Big) + E^l_J(v)\Big((\nabla^+_K E^l_I(v)) - (\nabla^+_I E^l_K(v))\Big)\Big]\right] P^J_i(v)\Big] \quad .
\end{aligned} \tag{3.19}$$

Here we used the notation $\nabla^+_I P^J_j(v) := P^J_j(v+I) - P^J_j(v) = i\ell_P^2\partial_{E^j_J(v+I)} - i\ell_P^2\partial_{E^j_J(v)}$. This operator, besides the terms proportional to $\nabla^+_I E^j_J(v)$, causes $\hat{C}^g_I(v)$ to act not only on vertex v but also on its neighbouring ones.
Finally the gravitational $SU(2)$–Gauss constraint operator can be defined as

$$\hat{G}[\Lambda] \;\; = \;\; \mathrm{id}_\phi \otimes \frac{1}{2}\sum_{v\in V(\alpha)} \epsilon_{ijk}\left[P^I_j(v)E^k_I(v)\Lambda_i(v) + E^k_I(v)\Lambda_i(v)P^I_j(v)\right] \tag{3.20}$$

for $\Lambda : V(\alpha) \to SU(2); \quad v \mapsto \Lambda_i(v)$.

3.3.2 Matter sector

As operator analogues for the (integrated) matter Hamiltonian constraint and the matter diffeomorphism constraint density we define:

$$\begin{aligned}
\hat{C}^\phi \;\; := \;\; & \frac{1}{2}\sum_v \Big[\frac{1}{\det E(v)} \otimes \Pi^2(v) \\
& + \sum_{I,J,K,L,M,N} \frac{1}{2\det E(v)}\epsilon^{IKL}\epsilon^{JMN}E^k_K(v)E^l_L(v)E^k_M(v)E^l_N(v) \otimes (\nabla^+_I\Phi(v))(\nabla^+_J\Phi(v)) \\
& + \det E(v) \otimes \frac{m^2}{\hbar^2}\Phi^2(v) \quad \Big]
\end{aligned} \tag{3.21}$$

$$\hat{C}^\phi_I(v) \;\; := \;\; id_g \otimes \frac{i\hbar}{2}\left[(\nabla^+_I\Phi(v))\partial_{\Phi(v)} + \partial_{\Phi(v)}(\nabla^+_I\Phi(v))\right] \quad . \tag{3.22}$$

One can rewrite (3.21) in a slightly different way which will become more convenient for the analysis of the matter sector which we want to perform later on

$$\hat{C}^\phi \;\; = \;\; \frac{1}{2}\sum_{v\in V(\alpha)} \Pi(v)\frac{1}{\det E(v)}\Pi(v) + \Phi(v)\det E(v)(-\Delta + \frac{m^2}{\hbar^2})\Phi(v) \quad , \tag{3.23}$$

with the lattice Laplace–Beltrami operator Δ defined as

$$\Delta \;\; = \;\; \frac{1}{\det E(v)}\sum_{I,J,K,L,M,N} \nabla^-_I\left(\frac{1}{2\det E(v)}\epsilon^{IKL}\epsilon^{JMN}E^k_K(v)E^l_L(v)E^k_M(v)E^l_N(v)\nabla^+_J\right) \quad . \tag{3.24}$$

Here we used that $\nabla_I^+ = (\nabla_I^-)^\dagger$ if interpreted as an operator on $l_2(V(\alpha))$, the Hilbert space of square–summable functions on the set of vertices $V(\alpha)$. This Hilbert space, $l_2(V(\alpha))$, should not be mistaken for the ITP Hilbert space which we based our quantum theory on. But it will play a role as the "one particle" Hilbert space in the construction of an appropriate Fock space.

3.4 Domains of Definition

In order to show that the operators written down in the last section are actually well defined symmetric operators on a dense subspace of $\mathcal{H}_\otimes$ we proceed as follows: As can easily be seen the parts acting on $H_\otimes^\phi$ will cause no problems. For each vertex v we can define the matter part of (3.21) and (3.22) on the Schwarz space $\mathcal{S}(\mathbb{R})$ of smooth functions of rapid decrease which is a dense subspace of $\mathcal{H}_v^\phi = L_2(\mathbb{R}, d\nu)$.
In order to show that the operators (3.12), (3.19) as well as the gravitational part of (3.21) and (3.22) is well defined it is sufficient to show that $A^n(v) := \frac{1}{(\det(E)(v))^n}$ for $n = 1, 2$ can be defined on a dense subspace of $\mathcal{H}_v^g = L_2(\mathbb{R}^9, d\mu)$ and that the entire operators are symmetric on that subspace.
Let us analyse the determinant of the cotriad in more detail: The operator $E_I^i(v)$ (associated to the cotriad) is densely defined on $\mathcal{S}(\mathbb{R}^9)$, so we need to show that this holds also for powers of the inverse of its determinant . If we view the determinant $\det : \mathbb{R}^9 \to \mathbb{R}; \quad E_I^i(v) \mapsto \det(E)(v)$ as a real–valued function on $\mathbb{R}^9$ then the singularity structure of $\frac{1}{\det(E)(v)}$ can most easily be analysed if we perform a singular value decomposition as follows:

$$E \quad =: \quad LDR \quad , \tag{3.25}$$

where $L, D \in SO(3)$ and D is a diagonal 3×3–matrix with real eigenvalues[15]. Explicitly these matrices are given by

$$E \;:=\; \begin{pmatrix} E_1^1 & E_1^2 & E_1^3 \\ E_2^1 & E_2^2 & E_2^3 \\ E_3^1 & E_3^2 & E_3^3 \end{pmatrix} \tag{3.26}$$

$$L \;:=\; \begin{pmatrix} \cos(\beta)\cos(\gamma) & -\cos(\beta)\sin(\gamma) & \sin(\beta) \\ \sin(\alpha)\sin(\beta)\cos(\gamma) + \cos(\alpha)\sin(\gamma) & -\sin(\alpha)\sin(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & -\sin(\alpha)\cos(\beta) \\ -\cos(\alpha)\sin(\beta)\cos(\gamma) + \sin(\alpha)\sin(\gamma) & \cos(\alpha)\sin(\beta)\sin(\gamma) + \sin(\alpha)\cos(\gamma) & \cos(\alpha)\cos(\beta) \end{pmatrix}$$

$$D \;:=\; \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$R \;:=\; \begin{pmatrix} \cos(\theta)\cos(\varphi) & -\cos(\theta)\sin(\varphi) & \sin(\theta) \\ \sin(\delta)\sin(\theta)\cos(\varphi) + \cos(\delta)\sin(\varphi) & -\sin(\delta)\sin(\theta)\sin(\varphi) + \cos(\delta)\cos(\varphi) & -\sin(\delta)\cos(\theta) \\ -\cos(\delta)\sin(\theta)\cos(\varphi) + \sin(\delta)\sin(\varphi) & \cos(\delta)\sin(\theta)\sin(\varphi) + \sin(\delta)\cos(\varphi) & \cos(\delta)\cos(\theta) \end{pmatrix}$$

That means we perform a coordinate transformation
$\varphi : \mathbb{R}^9 \to \mathbb{R}^9; \quad (E_1^1, E_1^2, E_1^3, E_2^1, E_2^2, E_2^3, E_3^1, E_3^2, E_3^3) \mapsto (\lambda_1, \lambda_2, \lambda_3, \alpha, \beta, \gamma, \delta, \theta, \phi)$, where the range of the new coordinates is $\alpha, \gamma, \delta, \phi \in (0, 2\pi], \quad \beta, \theta \in (0, \pi]$ and $\lambda_1, \lambda_2, \lambda_3 \in (-\infty, +\infty)$ and the Jacobian of this coordinate transformation can be computed to be

$$\det(\Phi) := \det(\partial\varphi/\partial(\lambda, \alpha, .., \phi)) = \cos(\beta)\cos(\theta)(-\lambda_3^4\lambda_2^2 + \lambda_1^4\lambda_2^2 - \lambda_1^4\lambda_3^2 + \lambda_3^2\lambda_2^4 + \lambda_3^4\lambda_1^2 - \lambda_1^2\lambda_2^4) \quad . \tag{3.27}$$

$L, R \in SO(3)$, thus the determinant in this coordinate system is simply given by

$$\det(E) = \lambda_1\lambda_2\lambda_3 \quad . \tag{3.28}$$

So if we compute $||A^n(v)\psi||_{L_2}$ for $\psi \in \mathcal{S}(\mathbb{R}^9)$ we realise

$$||A^n(v)\psi||_{L_2}^2 := \int d\Omega\; |\cos(\beta)\cos(\theta)|\; d\lambda_1 d\lambda_2 d\lambda_3 \frac{\lambda_1^4(\lambda_2^2 - \lambda_3^2) + \lambda_2^4(\lambda_3^2 - \lambda_1^2) + \lambda_3^4(\lambda_1^2 - \lambda_2^2)}{\lambda_1^{2n}\lambda_2^{2n}\lambda_3^{2n}} |\Psi(\Omega, \lambda_i)|^2 \tag{3.29}$$

where we used Ω as an abbreviation for all angles and $d\Omega := d\alpha d\beta d\gamma d\delta d\theta d\phi$. The denominator in the last integral is zero if either λ_1, λ_2 or λ_3 is zero whereas the numerator has its roots at $\lambda_1 = \pm\lambda_2, \lambda_2 = \pm\lambda_3$ or $\lambda_1 = \pm\lambda_3$. This

[15] We could have absorbed the signs of the eigenvalues into L, R by having them take values in $O(3)$. But it is more convenient to have them take values on the whole real axis. Notice that we do not insist on definite sign of $\det(E)$ on the whole phase space.

means that the argument of the integral (including the factor coming from the measure) diverges for $\lambda_i \to 0$ and thus $A^n(v) := \frac{1}{(\det(E)(v))^n}$ does not have finite norm on all $\psi \in \mathcal{S}(\mathbb{R}^9)$.

To cure this problem we need to change the domain of the operator $A^n(v)$, and the heuristic idea is to choose as its domain functions $\psi \in \mathcal{S}(\mathbb{R}^9)$ with the additional condition that ψ vanishes faster than any power of λ_i in a neighbourhood of the coordinate hypersurfaces defined through $\lambda_i = 0$.
In order to do this we will modify the function ψ in a way that they have the required properties and still approximate the original functions well. For this purpose we adopt a standard regularisation procedure. Let us introduce a characteristic function χ_V with V being the interval $[-2\epsilon, 2\epsilon]$. χ_V is equal to one if x lies in V and zero otherwise and is not smooth. Now consider the family of functions

$$\rho_\epsilon(x) \quad := \quad \begin{cases} C_\epsilon \exp\left(-\frac{\epsilon^2}{\epsilon^2 - x^2}\right) & \text{for} \quad |x| \leq \epsilon \\ 0 & \text{for} \quad |x| \geq \epsilon \end{cases} \quad . \tag{3.30}$$

The constant C_ϵ is determined by the condition $\int_{\mathbb{R}} dx \rho_\epsilon = 1$ and is explicitly given by $C_\epsilon^{-1} = \epsilon \int\limits_{|y|\leq 1} \exp(-1/(1-y^2)) dy$.
This function is smooth, has compact support since supp $\rho_\epsilon \subset [-\epsilon, \epsilon]$ and on all of $\mathbb{R}$ we have $\rho_\epsilon \geq 0$. We use ρ_ϵ to regularise the characteristic function χ_V and obtain a family of smooth versions of the characteristic function which is equal to one if x lies within the interval $[-\epsilon, \epsilon]$ and vanishes for all $|x| > 3\epsilon$. Let us denote the regularised function as χ_ϵ then

$$\chi_\epsilon(x) \quad := \quad \int\limits_{\mathbb{R}} dy \chi_V(y) \rho_\epsilon(x-y) = \int\limits_{-2\epsilon}^{2\epsilon} dy \rho_\epsilon(x-y) \tag{3.31}$$

With this definition of χ_ϵ we can easily show that it vanishes if $|x| > 3\epsilon$. Since the aim is to set the functions at the singularities to zero in a smooth manner we define the function

$$s_\epsilon(x) \quad := \quad 1 - \chi_\epsilon(x) = \begin{cases} 0 & \text{for} \quad |x| \leq \epsilon \\ 1 - \int\limits_{-2\epsilon}^{+2\epsilon} dy \rho_\epsilon(x-y) & \text{for} \quad \epsilon \leq |x| \leq 3\epsilon \\ 1 & \text{for} \quad |x| \geq 3\epsilon \end{cases} \quad . \tag{3.32}$$

Notice that $0 \leq \chi_\epsilon,\ s_\epsilon \leq 1$. This function is smooth by construction and $s_\epsilon(0) = 0$. Thus, we can use this function to modify our original functions $\psi \in \mathcal{S}(\mathbb{R}^9)$ such that they vanish at $\lambda_i = 0$ in a smooth way. Define

$$\psi_\epsilon : \mathbb{R}^9 \to \mathbb{R}; \quad \Omega, \lambda_i \mapsto \psi_\epsilon(\Omega, \lambda_i) = s_\epsilon(\lambda_1) s_\epsilon(\lambda_2) s_\epsilon(\lambda_2) \psi(\Omega, \lambda_i) \tag{3.33}$$

for any $\psi \in \mathcal{S}(\mathbb{R}^9)$. The product of two smooth functions is still smooth, hence the set $\mathcal{F} := \{\psi_\epsilon\}$ is a subset of $\mathcal{S}(\mathbb{R}^9)$ and one can easily see that $\mathcal{F}$ is dense in $\mathcal{S}(\mathbb{R}^9)$ and therefore also in $L_2(\mathbb{R}^9)$ in the L_2–norm:

$$\begin{aligned} ||\psi - \psi_\epsilon||_{L_2} &= \left[\int d\Omega \int\limits_{-\infty}^{+\infty} d\lambda_1 \int\limits_{-\infty}^{+\infty} d\lambda_2 \int\limits_{-\infty}^{+\infty} d\lambda_3 \det\Phi |\psi(\Omega, \lambda_i) - \psi_\epsilon(\Omega, \lambda_i)|^2 \right]^{\frac{1}{2}} \\ &= \left[\int d\Omega \int\limits_{-3\epsilon}^{+3\epsilon} d\lambda_1 \int\limits_{-3\epsilon}^{+3\epsilon} d\lambda_2 \int\limits_{-3\epsilon}^{+3\epsilon} d\lambda_3 \det\Phi |\psi(\Omega, \lambda_i)|^2 |1 - s_\epsilon(\lambda_1) s_\epsilon(\lambda_2) s_\epsilon(\lambda_3)|^2 \right]^{\frac{1}{2}} \\ &\leq \left[\int d\Omega \int\limits_{-3\epsilon}^{+3\epsilon} d\lambda_1 \int\limits_{-3\epsilon}^{+3\epsilon} d\lambda_2 \int\limits_{-3\epsilon}^{+3\epsilon} d\lambda_3 \det\Phi |\psi(\Omega, \lambda_i)|^2 \right]^{\frac{1}{2}} \\ &\leq \mathcal{I}_{\max} \sqrt{6^3 \epsilon^3} \left(\int d\Omega |\cos(\beta)\cos(\theta)| \right)^{\frac{1}{2}} \propto \epsilon^{\frac{3}{2}} \quad , \end{aligned} \tag{3.34}$$

where $\mathcal{I}_{\max}$ is the supremum of $\left[|\frac{\det\Phi}{\cos(\beta)\cos(\theta)}| \right]^{\frac{1}{2}} |\psi|$ in the compact set $|\lambda_i| \leq 3\epsilon$ which is assured to be finite for every smooth function of rapid decrease. Thus, we can approximate every $\psi \in \mathcal{S}$ through some $\psi_\epsilon \in \mathcal{F}$ to arbitrary precision. Hence, A^n with domain $\mathcal{F}$ is a densely defined operator in $L_2(\mathbb{R}^9)$ and therefore all the operators defined in the last section are actually well defined and even more they leave the domain invariant.

4 Coherent states for the gravitational sector

In this section we want to construct coherent states for the gravitational sector: The gravitational Hilbert space introduced in section 3.2 is at each vertex of the algebraic graph given by $\mathcal{H}_v^g = L_2(\mathbb{R}^9, d\mu)$ and thus consists of nine copies of usual Schrödinger representation Hilbert spaces. We will construct coherent states for each vertex Hilbert space $\mathcal{H}_v^g$ and the coherent state for the complete gravitational sector are then given by the infinite tensor product of the vertex coherent states. Since the quantum theory is formulated on a purely algebraic level, the coherent states will also be the bridge between the algebraic quantum theory and the embedded classical theory of General Relativity.

In order to construct coherent states we first need to choose an embedding X of our algebraic graph α in to a given manifold Σ and we call its image $\gamma := X(\alpha)$. We will choose embeddings X such that γ is dual to a certain triangulation γ^* . Thus, for each embedded edge $X(c)$ there is a face S_c in γ^* which intersects $X(c)$ only in an interior point p_c of both S_c and $X(c)$. Next we need to choose a classical cotriad E_0 (suppressing indices) and an $su(2)$ – valued vector density P_0. With the manifold, the embedding and the classical data (E_0, P_0) we can define the smeared quantities

$$\begin{aligned} E^i_{I0}(v) &= \int\limits_{X(c_I)} E_0 \\ P_i^I{}_0(v) &= \int\limits_{S_{c_I}} \epsilon_{abc} d\sigma^a \wedge d\sigma^b P_i^c(\sigma) \end{aligned} \tag{4.1}$$

We follow the complexifier method for coherent states introduced in [12, 9, 10, 11]. We complexify the configuration space $\mathbb{R}^9$ to $\mathbb{C}^9$ by introducing

$$Z^j_J(v) := \frac{1}{\ell_p\sqrt{2}}\left(E^j_{J0}(v) - iP_j^J{}_0(v)\right) \tag{4.2}$$

From now on we will suppress indices and denote the classical data simply by $Z(v) = (E_0(v), P_0(v))$. This complexification can be obtained from a classical complexifier of the form

$$C := \frac{1}{2}\frac{1}{\kappa\ell_p\sqrt{2}}\sum_{v\in V(\alpha)} P_j^J(v)P_J^j(v) \tag{4.3}$$

with

$$Z^j_J(v) = \sum_{n=0}^{\infty}\frac{i}{n!}\{E^j_J(v), C\}_{(n)} \tag{4.4}$$

where $\{.,.\}_{(n)}$ denotes the iterative Poisson bracket defined as $\{E^j_J(v), f\}_{(0)} = E^j_J(v)$ and then iteratively by $\{E^j_J(v), f\}_{(n+1)} = \{E^j_J(v), \{E^j_J(v), f\}_{(n)}\}$. Similarly we can define a corresponding complexifer for the quantum configuration space

$$\hat{C} := \frac{1}{2}\frac{1}{\ell_p^2\sqrt{2}}\sum_{v\in V(\alpha)}\left(i\ell_p^2\frac{\partial}{\partial E^j_J(v)}\right)\left(i\ell_p^2\frac{\partial}{\partial E^k_K(v)}\right)\delta_{jk}\delta^{JK} \tag{4.5}$$

from which we obtain the form of the annihilation and creation operators

$$A^j_J(v) := \frac{1}{\ell_p\sqrt{2}}\left(E^i_I(v) - i\left(i\ell_p^2\frac{\partial}{\partial E^{j_J}(v)}\right)\right) \tag{4.6}$$

$$(A^j_J)^\dagger(v) := \frac{1}{\ell_p\sqrt{2}}\left(E^i_I(v) + i\left(i\ell_p^2\frac{\partial}{\partial E^{j_J}(v)}\right)\right) \tag{4.7}$$

that satisfy the algebra $[A^j_J(v), (A^k_K)^\dagger(v')] = \delta_{vv'}\delta^{jk}\delta_{JK}$. Note that the characteristic length that enters into the definition of the coherent states is ℓ_p in our case and not as for the harmonic oscillator $\ell = \sqrt{\hbar/m\omega}$. The reason for this is simply that the squared Planck length ℓ_p^2 also occurs in the definition of the momentum operator. One of the defining properties of coherent states is that they are eigenstates of the annihilation operator A^j_J with eigenvalue

Z, the classical phase space point that they are labeled by and around which they are sharply peaked. They can be expressed as

$$|\psi_Z\rangle \;\; := \;\; \sum_{n=0}^{\infty} \exp(-\frac{1}{2}|Z|^2)\frac{Z^n}{n!}\Big[\Big(A^j_J(v)\Big)^\dagger\Big]^n|0\rangle = \exp\Big(-\frac{1}{2}|Z|^2\Big)\exp\Big(Z(A^j_J)^\dagger(v)\Big)\,|0\rangle \tag{4.8}$$

where $|0\rangle$ is the vacuum state defined by $A^j_J(v)|0\rangle = 0$. Further properties of coherent states are that they satisfy a resolution of identity, they form an overcomplete basis, and that they are not mutually orthogonal but $\langle\psi_{Z'}\,,\,\psi_Z\rangle = \exp(-|Z - Z'|^2)$.
In position space representation we get using $Z(v) = (E_0(v), P_0(v))$

$$\psi_Z(E^j_J(v)) \;\; = \;\; \frac{1}{\sqrt{\ell_p\sqrt{\pi}}}\exp\Big(-\frac{i}{2\ell_p^2}P_0(v)E_0(v) + \frac{i}{\ell_p^2}P_0(v)E^j_J(v)\Big)\exp\Big(-\frac{1}{2\ell_p^2}\Big(E^j_J(v) - E_0(v)\Big)^2\Big) \tag{4.9}$$

An easy calculation shows that $\psi_Z(E^j_J)$ is indeed an eigenstate of $A^j_J(v)$ with eigenvalue $Z(v)$. The states $\psi_Z(E^j_J)$ are the ordinary harmonic oscillator coherent states and thus we can construct the coherent states associated to each vertex as a product of nine harmonic oscillator coherent states and the total states for the gravitational sector then as infinite product of states over all vertices of the graph

$$\psi_Z(v) \;\; := \;\; \prod_{j,J=1}^{3}\psi_Z(E^j_J(v)) \in \mathcal{H}^g_v \quad \text{and} \quad \psi_{\rm Z} := (\otimes_{\rm v}\psi_{\rm Z}({\rm v})) \in \mathcal{H}^{\rm g}_\otimes \tag{4.10}$$

The fact that ψ_Z is an eigenstate of the annihilation operator A^j_J with eigenvalue Z has the consequence that it approximates expectation values of E^j_J and P^J_j semiclassically well. Precisely, we have

$$\langle\psi_Z\,,\,E^j_J\,\psi_Z\rangle \;\; = \;\; E_0\,, \quad \langle\psi_Z\,,\,P^J_j\,\psi_Z\rangle = P_0 \tag{4.11}$$

However, when we consider the inverse operator $1/\det(E^j_J)$ it turns out its expectation value with respect to Ψ_Z diverges. A simple way to see why this happens is to take a cotriad configuration at one vertex where all but two triads are vanishing. Then we have at least one integral of the symbolic form $\int_{\mathbb{R}} e^{-x^2}/(x-x_0)dx$ which does not converge. For this reason we have to introduce modified coherent states along the lines of the modified Schwartz functions of rapid decrease in section 3.4. Thus we define the modified coherent states by means of the function s_ϵ in equation (3.32) as

$$\psi^\epsilon_Z(v) \;\; := \;\; \prod_{j,J=1}^{3} s_\epsilon(\det(E^j_J(v))\psi_Z(E^j_J(v)) \quad \text{and} \quad \psi^\epsilon_{\rm Z} := (\otimes_{\rm v}\psi^\epsilon_{\rm Z}({\rm v})) \tag{4.12}$$

More precisely

$$\psi^\epsilon_Z(v): \mathbb{R}^9 \to \mathbb{R}; \quad \Omega, \lambda_i \mapsto \psi^\epsilon_Z(\Omega,\lambda_i) = s_\epsilon(\lambda_1)s_\epsilon(\lambda_2)s_\epsilon(\lambda_2)\psi_Z(\Omega,\lambda_i) \tag{4.13}$$

Using the same steps as in equation 3.34 we obtain

$$||\psi^\epsilon_Z - \psi_Z|| \;\; \leq \;\; \mathcal{I}_{\max}\sqrt{6^3\epsilon^3}\Big(\int d\Omega|\cos(\beta)\cos(\theta)|\Big)^{\frac{1}{2}} \propto \epsilon^{\frac{3}{2}} \tag{4.14}$$

where here $\mathcal{I}_{\max}$ is the supremum of $\Big[\frac{\det\Phi}{\cos(\beta)\cos(\theta)}\Big]^{\frac{1}{2}}|\psi_Z|$ in $|\lambda_i| \leq 3\epsilon$ which is finite for smooth functions. Hence, for sufficiently small ϵ we can approximate the states ψ_Z by the states ψ^ϵ_Z for all values of the classical phase space point label Z. Furthermore, because the coherent states have the property of being sharply peaked around Z, the absolute values of ψ_Z that appears in $\mathcal{I}_{\max}$ in the estimate above becomes smaller and smaller the farther the classical phase space point Z is away from a singular configuration. Consequently as long as we are mainly interested in states which are not peaked around singularities the approximation works very well. Indeed we are actually interested in those states because we want to use these states to mimic a classical geometry background when investigating the semiclassical limit of this theory understood as QFT on curved spacetimes in section 5.

We will now verify that the expectation values for E^j_J and P^J_j are also well approximated for sufficiently small ϵ. The modified states are no longer eigenstates of A^j_J with eigenvalue Z but we get a correction involving the derivative of s_ϵ and the original state ψ_Z.

$$A^j_J \psi^\epsilon_Z \quad = \quad Z\psi^\epsilon_Z + \frac{\ell_p}{\sqrt{2}} \frac{\partial s_\epsilon(\det(E^j_J))}{\partial E^j_J} \psi_Z \tag{4.15}$$

This correction term only appears when $|\det(E^j_J)| \leq \epsilon$ since otherwise $s_\epsilon(\det(E^j_J)) = 1$. For the error of the expectation value of $\hat{E}^j_J$ we therefore get

$$\begin{aligned}
&\Big|\langle \psi^\epsilon_Z,\, \hat{E}^j_J\, \psi^\epsilon_Z\rangle - \langle \psi_Z \hat{E}^j_J\, \psi_Z\rangle\Big| = \Big|\langle (s_\epsilon + 1)\, \psi_Z,\, \hat{E}^j_J\ (s_\epsilon - 1)\, \psi_Z\rangle\Big| \\
&= \Big|\int\limits_{\mathbb{R}^9} d^9E^j_J \left(s_\epsilon(\det(E^j_J)) + 1\right) \bar{\psi}_Z(E^j_J) E^j_J \left(s_\epsilon(\det(E^j_J)) - 1\right) \psi_Z(E^j_J)\Big| \\
&= \Big|\int\limits_{\mathbb{R}^9} d\Omega d^3\lambda \det(\Phi) \left((s_\epsilon(\lambda_1)s_\epsilon(\lambda_2)s_\epsilon(\lambda_2))^2 - 1\right) \bar{\psi}_Z(\lambda_i,\Omega) E^j_J(\lambda_i,\Omega)\psi_Z(\lambda_i,\Omega)\Big| \\
&= \int\limits_{[S_3/Z_2]\times[S_3/Z_2]} d\Omega \int\limits_{-3\epsilon}^{3\epsilon} d\lambda_1 \int\limits_{-3\epsilon}^{3\epsilon} d\lambda_2 \int\limits_{-3\epsilon}^{3\epsilon} d\lambda_3 \big|\det(\Phi)\big| \left|\left((s_\epsilon(\lambda_1)s_\epsilon(\lambda_2)s_\epsilon(\lambda_2))^2 - 1\right)\right| \big|\bar{\psi}_Z(\lambda_i,\Omega) E^j_J(\lambda_i,\Omega)\psi_Z(\lambda_i,\Omega)\big| \\
&\leq \int\limits_{[S_3/Z_2]\times[S_3/Z_2]} d\Omega \int\limits_{-3\epsilon}^{3\epsilon} d\lambda_1 \int\limits_{-3\epsilon}^{3\epsilon} d\lambda_2 \int\limits_{-3\epsilon}^{3\epsilon} d\lambda_3 \big|\det(\Phi)\big| \big|\bar{\psi}_Z(\lambda_i,\Omega) E^j_J(\lambda_i,\Omega)\psi_Z(\lambda_i,\Omega)\big| \\
&\leq \mathcal{I}_{\max} 6^3 \epsilon^3 \left(\int d\Omega\ |\cos(\beta)\cos(\theta)|\right) \propto \epsilon^3
\end{aligned} \tag{4.16}$$

with $\mathcal{I}_{\max}$ is the supremum of $\left[\frac{\det\Phi}{\cos(\beta)\cos(\theta)}\right]^{\frac{1}{2}} |\bar{\psi}_Z E^j_j(\lambda_i)\psi_Z|$ in $|\lambda_i| \leq 3\epsilon$. We used that E, s_ϵ commute and in step three we performed again a transformation to the (λ,Ω) coordinates and considered E^j_J as a function of these variables. Similarly we obtain for the momentum operator

$$\begin{aligned}
\Big|\langle \psi^\epsilon_Z,\, \hat{P}^J_j\, \psi^\epsilon_Z\rangle - \langle \psi_Z, \hat{P}^J_j\, \psi_Z\rangle\Big| &= \Big|\langle [\psi^\epsilon_Z - \psi_Z],\, \hat{P}^J_j\, \psi^\epsilon_Z\rangle + \langle \psi_Z, \hat{P}^J_j\, [\psi^\epsilon_Z - \psi_Z]\rangle\Big| \qquad (4.17)\\
&= \Big|\langle [\psi^\epsilon_Z - \psi_Z],\, [[\hat{P}^J_j s_\epsilon]\, \psi_Z + s_\epsilon\, [\hat{P}^J_j \psi_Z]]\rangle + \langle \hat{P}^J_j \psi_Z,\, [\psi^\epsilon_Z - \psi_Z]\rangle\Big| \\
&= \Big|\langle \psi_Z,\, \frac{1}{2}[[\hat{P}^J_j (s_\epsilon - 1)^2]\, \psi_Z + [s_\epsilon - 1]\, s_\epsilon\, [\hat{P}^J_j \psi_Z]]\rangle + \langle \hat{P}^J_j \psi_Z,\, [s_\epsilon - 1]\, \psi_Z\rangle\Big| \\
&\leq \int\limits_{\mathbb{R}^9} d\Omega d^3\lambda \big|\det(\Phi)\big|\ \left(s_\epsilon(\lambda_1)s_\epsilon(\lambda_2)s_\epsilon(\lambda_2) - 1\right)^2\ \Big|i\ell_p^2 \frac{\partial}{\partial E^j_J(\lambda_i,\Omega)} |\psi_Z(\lambda_i,\Omega)|^2\Big| \\
&\quad + \Big|\int\limits_{\mathbb{R}^9} d\Omega d^3\lambda\ |\det(\Phi)|\ |\bar{\psi}_Z(\lambda_i,\Omega)|\ |\frac{\partial}{\partial E^j_J(\lambda_i,\Omega)}\psi_Z(\lambda_i,\Omega)|\ \Big|s_\epsilon(\lambda_1)s_\epsilon(\lambda_2)s_\epsilon(\lambda_2)\Big| \qquad (4.18)
\end{aligned}$$

where we introduced the abbreviation $\partial E^j_J(\lambda_i,\Omega) = (\partial\lambda_i/\partial E^j_J)\partial\lambda_i + (\partial\Omega_i/\partial E^j_J)\partial\Omega_i$ for the transformed partial derivative. Here in the last step we have performed an integration by parts. As before, both terms above are only non – vanishing in the compact interval $\lambda_i \in [-3\epsilon, 3\epsilon]$. Restricting the integration domain and using the upper bounds $|s_\epsilon|,\ |1 - s_\epsilon| \leq 1$ we see that both terms are bounded by $[6\epsilon]^3\ [\int d\Omega\ |\cos(\beta)\cos(\theta)|]$ times the supremum of the respective integrand on the domain $|\lambda_i| \leq 3\epsilon$. Since that supremum does not increase if we lower ϵ we see that the bound can be arbitrarily small. Thus also for the expectation value of P^J_j we are arbitrarily close to the original value when using the modified coherent states. Finally, let us compute the expectation value for the operator $A^n := [1/\det(E^j_J)]^n$ with respect to the modified states. To display the exact formulas requires a lot of notation, so we satisfy ourselves with giving the flavour of it by considering a simpler example which nevertheless contains all the essential features. The exact calculation proceeds completely analogously. We model the expectation value of A^n by

$$< \frac{1}{|.|^n} > := \int_{\mathbb{R}} \frac{dx}{\sqrt{\pi}\delta}\ e^{-(x-x_0)^2/\delta^2}\ \frac{s_\epsilon(x)^2}{|x|^n} \tag{4.19}$$

and want to compare its value with $1/|x_0|^n$. We write

$$< \frac{1}{|.|^n} >:= I_1 + I_2 := \int_{\epsilon \le |x| \le 3\epsilon} \frac{dx}{\sqrt{\pi}\delta}\; e^{-(x-x_0)^2/\delta^2}\; \frac{s_\epsilon(x)^2}{|x|^n} + \int_{|x|\ge 3\epsilon} \frac{dx}{\sqrt{\pi}\delta}\; e^{-(x-x_0)^2/\delta^2}\; \frac{1}{|x|^n} \tag{4.20}$$

and have due to $|s_\epsilon| \le 1$

$$\begin{aligned} |I_1| &\le \int_{\epsilon \le |x| \le 3\epsilon} \frac{dx}{\sqrt{\pi}\delta}\; e^{-(x-x_0)^2/\delta^2}\; \frac{1}{|x|^n} \\ &\le \frac{1}{\epsilon^n} \int_\epsilon^{3\epsilon} \frac{dx}{\sqrt{\pi}\delta}\; [e^{-(x-|x_0|)^2/\delta^2} + e^{-(x+|x_0|)^2\delta^2}] \\ &\le \frac{2}{\epsilon^{n-1}\sqrt{\pi}\delta}\; [e^{-(|x_0|-3\epsilon)^2/\delta^2} + e^{-(|x_0|+\epsilon)^2/\delta^2}] \end{aligned} \tag{4.21}$$

where we have assumed that $|x_0| \gg \epsilon$. Next

$$I_2 - \frac{1}{|x_0|^n} := -I_3 + I_4 := -\int_{|x|\le 3\epsilon} \frac{dx}{\sqrt{\pi}\delta}\; e^{-(x-x_0)^2/\delta^2}\; \frac{1}{|x_0|^n} + \int_{|x|\ge 3\epsilon} \frac{dx}{\sqrt{\pi}\delta}\; e^{-(x-|x_0|)^2/\delta^2}\; \left[\frac{1}{|x|^n} - \frac{1}{|x_0|^n}\right] \tag{4.22}$$

where in the last integral we have exploited the invariance under $x \to -x$. We can estimate

$$|I_3| \le \frac{3\epsilon}{\sqrt{\pi}\delta |x_0|^n}\; [e^{-(|x_0|-3\epsilon)^2/\delta^2} + e^{-(|x_0|)^2/\delta^2}] \tag{4.23}$$

To estimate I_4 consider

$$f(x) := -\frac{(x-|x_0|)^2}{\delta^2} - n\ln(|x|) \tag{4.24}$$

The extremum y of f satisfies $2x(x-|x_0|) = -n\delta^2$ i.e. $y_\pm = [|x_0| \pm \sqrt{x_0^2 - n\delta^2}]/2$ assuming $|x_0| \gg \delta$. Hence $y_+ \approx |x_0|,\ y_- \approx n\delta^2/4|x_0|$. Since $f''(x) = -2/\delta^2 + n/x^2$ we find

$$f''(y_+) \approx -2/\delta^2 + n/x_0^2 < 0,\quad f''(y_-) \approx 1/\delta^4[-2\delta^2 + 16x_0^2/n] > 0 \tag{4.25}$$

Thus only y_+ is a maximum and it also satisfies $|y_+| \ge 3\epsilon$. Accordingly, a saddle point estimate of I_4 yields

$$\begin{aligned} |I_4| &\approx \left| 2\int_\epsilon^\infty \frac{dx}{\sqrt{\pi}\delta}\; \left[e^{f(y_+)+f''(y_+)[x-y_+]^2} - e^{-(x-|x_0|)^2/\delta^2}/|x_0|^n\right] \right| \\ &\approx 2\left|\int_\epsilon^\infty \frac{dx}{\sqrt{\pi}\delta}\; e^{-(x-|x_0|)^2/\delta^2}/|x_0|^n\; \left(\left[\frac{|x_0|}{y_+}\right]^n - 1\right)\right| \\ &\le \left|\frac{1}{|x_0|^n}\left(\left[\frac{|x_0|}{y_+}\right]^n - 1\right)\right| \approx \frac{1}{|x_0|^n}\; \frac{n^2\delta^2}{4|x_0|^2} \end{aligned} \tag{4.26}$$

Summarising

$$\left|< \frac{1}{|.|^n} > |x_0|^n - 1\right| \le (|I_1| + |I_3| + |I_4|)\; |x_0|^n\; \le \frac{n^2\delta^2}{4|x_0|^2} + \frac{6\epsilon}{\sqrt{\pi}\delta}\; e^{-|x_0|^2/\delta^2} + \frac{4|x_0|^n}{\epsilon^{n-1}\sqrt{\pi}\delta}\; e^{-|x_0|^2/\delta^2} \tag{4.27}$$

In our application $\delta = \ell_P$ is fixed while $\epsilon \to 0$. We have already seen that $|x_0| \gg \epsilon, \delta$ in order that (4.27) is a good estimate. Let $y = |x_0|/\delta$ then the third term in (4.27) is of the form

$$\left[\frac{\epsilon}{\delta}\right]^{n-1}\; y^n\; e^{-y^2/2} \tag{4.28}$$

which should be small as compared to the fluctuation term $n^2\delta^2/4|x_0|^2$. This shows that we cannot let $\epsilon \to 0$ for our approximate coherent states. However, choosing e.g. $\epsilon = \delta$ we see that for n of the order unity already for y of the order $\sqrt{n}$ the third term in (4.27) is subdominant.

5 Analysis of the system

5.1 Born–Oppenheimer approximation

To analyse the interplay between gravitational and matter dynamics and to understand how a (matter–) quantum field theory on a fixed classical spacetime can emerge from the fully quantum gravitational setup we want to employ an approximation scheme of Born–Oppenheimer type. As we will see in the following for such an approximation scheme to work it is essential that the geometrical variables are quantised as multiplication operators, not as derivative operators. This is the main motivation for not using the LQG–representation but the one that we constructed in previous sections. This representation captures some of the features of LQG (or the recently proposed generalisation coined Algebraic Quantum Gravity [14, 13, 15]), especially the fundamentally discrete graph–like structures which appear in the very definition of the kinematical Hilbert space. In order to be able to use a Born–Oppenheimer approximation scheme we have chosen a representation of the gravitational Poisson algebra which is commutative in the geometrical variables, unlike the non–commutative flux operators in ordinary LQG. Thus, the theory constructed in this paper is a close relative to LQG when defined on an algebraic graph which shares some of its features (based on graph–like structures) but deviates from it in other essential characteristics (abelian geometrical operators). Of course, the long term goal will be to generalise the methods derived in this section to full LQG also taking into account the non–commutative structure of the holonomy–flux algebra. But, as we will see below, due to the specific assumptions made in the derivation of the Born–Oppenheimer approximation such a generalisation is not straightforward and will require substantially more work. One step into that direction is the construction of a non–commutative flux representation for the holonomy–flux algebra, where the flux–operators act as multiplication operators on an appropriately defined Hilbert space and the non–abelian nature of the flux–operators is taken into account through a specific $*$–product on that space [52].
The use of Born–Oppenheimer approximation schemes in the context of quantum gravity has a long tradition, especially in the older Wheeler-deWitt approach (for a historical account see for example Kiefer's book [27] and references therein, for a recent application in the framework of spinfoam models see [28]). Most of the results so far were obtained on a rather formal level, because the Hilbert space on which Wheeler-deWitt theory is assumed to be defined is not known at all, but the application to minisuperspace models for FRW–universes gives reasonable results and can shed some light on conceptual questions such as the often discussed problem of time.
Concerning this issue we are in a better position: The Hilbert space of the quantum gravity theory we consider is explicitly known and all the operators we use are densely defined symmetric operators on that space. Therefore the derivations which follow are all well defined and work beyond the formal level – at least in principle when neglecting practical difficulties in actually performing the calculations. Moreover, we are advocating a reduced phase space quantisation of General Relativity where the preferred reference frame is given by Brown–Kuchăr dust fields. Thus we have a true physical Hamiltonian that generates evolution in dust–time, not only a vanishing Hamiltonian constraint. The problem of time is therefore solved already on the classical level and does not import any extra obstructions into the quantum theory.

5.1.1 General Framework

For simplicity we will first explain the Born–Oppenheimer approximation using a very simple quantum mechanical system and later generalise to the quantum gravitational setup. Assume that our classical phase space is four dimensional and coordinatised by two pairs of canonically conjugate variables (Q, P) and (q, p) (a generalisation to an arbitrary number of configuration variables is straightforward but will not be considered here for pedagogical reasons). Further, assume that dynamics is generated by a Hamiltonian of the form

$$H := \frac{P^2}{2M} + \frac{p^2}{2m} + V(Q, q) \quad , \tag{5.1}$$

where M and m are some parameters describing the system under investigation (in the case of two particles these will be their respective masses) and $V(Q, q)$ is a potential term which depends only on the configuration variables, not on their momenta.
This system can be quantised on a Hilbert space $\mathcal{H} := \mathcal{H}^Q \otimes \mathcal{H}^q$ where $\mathcal{H}^Q$ and $\mathcal{H}^q$ are taken to be the usual spaces of square integrable functions over the real line. On $\mathcal{H}$ one defines operators in the usual way as $(\hat{q}\Psi)(Q, q) := q\Psi(Q, q), (\hat{Q}\Psi)(Q, q) := Q\Psi(Q, q), (\hat{p}\Psi)(Q, q) := i\hbar\partial_q\Psi(Q, q), (\hat{P}\Psi)(Q, q) := i\hbar\partial_Q\Psi(Q, q)$ for functions $\Psi \in \mathcal{H}$. Thus, the Hamilton operator of the total system is given by

$$\hat{H} := -\frac{\hbar^2}{2M}\Delta_Q - \frac{\hbar^2}{2m}\Delta_q + V(Q, q) \quad , \tag{5.2}$$

where Δ_Q, Δ_q denotes the Laplacian with respect to Q and q respectively.
In general, depending on the exact expression for the potential term $V(Q, q)$, it can be difficult to obtain solutions to the full eigenvalue problem

$$\hat{H}\Psi^\alpha = \Lambda^\alpha \Psi^\alpha \quad , \tag{5.3}$$

where α is a set of quantum numbers labelling the eigenvalues Λ^α and the corresponding eigenfunctions Ψ^α. However, under certain assumptions it is possible to obtain approximate solutions: Assume that (i) $m \ll M$, which means that there are two clearly separated energy scales in the problem at hand, and (ii) we have some control over one half of the problem, namely we can solve the eigenvalue problem for the light variables

$$\hat{\hat{H}}(Q)\chi_i(q; Q) = \lambda_i(Q)\chi_i(q; Q) \quad , \tag{5.4}$$

where $\hat{\hat{H}}(Q) := -\frac{\hbar^2}{2m}\Delta_q + V(Q, q)$ is an operator acting on the space $\mathcal{H}^q$ that depends on an external parameter Q and $\chi_i(q; Q)$ (labeled by a set of quantum numbers i) are assumed to be square integrable in $\mathcal{H}^q$ and also depending on Q in a parametrical way. So in a sense this means we can solve 5.4 for each Q separately.
Let us first stick to the first half of the problem and analyse a little further equation (5.4): Assume that we can solve the eigenvalue problem for each value of Q and we have obtained eigenfunctions $\chi(q; Q) \in \mathcal{H}^q$ and eigenvalues $\lambda(Q)$ that depend on the external parameter Q. Then a priori, it is not clear how χ and λ behave if we change Q[16]. Berry [53] showed that this question can be answered in a nice geometric way: Under the assumption of adiabaticity[17] (i.e. assuming that the change in Q is so slow that the quantum system at time t is in a state $\chi(q; Q(t))$) it turns out that when moving the quantum system around a closed curve C in parameter space (in our simple example the parameter space would just be $\mathbb{R}$ but the same phenomenon holds also for higher dimensional parameter spaces) the wave function χ picks up a non–trivial phase factor. That means when the system is prepared to be in an energy eigenstate $\chi_i(0)$ for $t = 0$, after having followed the loop C in a time interval T it turns out to be in the state $\psi(T) = \exp\left[\frac{i}{\hbar}\gamma_i(C)\right] \exp\left[-\frac{i}{\hbar}\int\limits_0^T dt\lambda_i(Q(t))\right]\chi_i(0)$. The second exponential is just the usual dynamical phase factor, but the first one, explicitly given by

$$\gamma_i(C) = i\hbar \oint\limits_C \langle \chi_i(Q) \mid \partial_Q \mid \chi_i(Q)\rangle_q \quad , \tag{5.5}$$

is indeed non–trivial. This phase factor, known as *Berry's phase* in the literature was experimentally confirmed in many experiments and can be used to explain a variety of physical phenomena, including for example the Aharonov–Bohm effect (see [54] for more information on this broad topic). From a geometrical point of view the situation is quite interesting (see for example [55] and references in there for a modern explanation using the language of fiber bundles), because it turns out that $A := i\hbar \langle \chi_i \mid \partial_Q \mid \chi_i\rangle_q$ transforms as a $U(1)$–connection under coordinate transformations in parameter space. Generalised to a higher–dimensional parameter space this connection is given by $A_\mu := i\hbar \left\langle \chi_i \mid \frac{\partial}{\partial Q^\mu} \mid \chi_i \right\rangle_q$ in a local coordinate system Q^μ. Thus, $\exp\left[\frac{i}{\hbar}\gamma_i(C)\right]$ is nothing else than the holonomy of a $U(1)$–connection along the path C in parameter space.
One essential point in the derivation of the above formulas is that Q is treated as an external classical parameter: To be able to talk about a path in parameter–space one must assume that Q is a classical variable so that it makes sense to demand differentiability of the paths. Strictly speaking, this assumption is violated when we take into account the quantum nature of Q itself, because in quantum theory particles do not follow differentiable paths anymore[18], and taking this quantum nature into account is a highly non–trivial task (see again [54] and especially [56] for elaborations on this topic). However, as long as we are interested in a system where the respective energy scales are well separated ($m \ll M$) and we are working in a representation where $\hat{Q}$ acts as a multiplication operator and not as a derivative operator the approximation seems to be valid and is confirmed by many experiments in molecular physics.

Now that we understood the first half of the problem, let us go back to the full problem (5.3) and see how we can obtain solutions to the eigenvalue problem for the full Hamilton operator $\hat{H}$ acting on the fullg space $\mathcal{H}$: Assuming that we

[16]To get some intuition for this problem one could think of a quantum particle under the influence of an external magnetic field Q. Assume that we can solve the Schroedinger equation for each fixed value of the magnetic field. Then the question would be: How does the quantum system describing the particle change when we manually change the external magnetic field?
[17]The argument turns out to work out in more general situations, see for example [54].
[18]In fact, the path integral measure is concentrated on paths that are arbitrarily discontinuous.

have solved the eigenvalue problem for $\hat{\hat{H}}(Q)$ with Q as an external parameter and we already know its eigenfunctions $\chi_i(q;Q)$ we start with the Ansatz

$$\Psi^\alpha(Q,q) := \sum_i \xi_i^\alpha(Q)\chi_i(q;Q) \quad . \tag{5.6}$$

Plugging this into (5.3) leads to

$$\hat{H}\Psi^\alpha = \sum_i (-\frac{\hbar^2}{2M}\Delta_Q + \lambda_i(Q))\xi_i^\alpha(Q)\chi_i(q;Q) = \Lambda^\alpha \sum_i \xi_i^\alpha(Q)\chi_i(Q;q) \quad . \tag{5.7}$$

Now we multiply by χ_k from the left and take the scalar product in $\mathcal{H}^q$ to receive an equation for the Q–dependent coefficients ξ_i^α

$$\sum_i \left(\left\langle \chi_k \mid \left(-\frac{\hbar^2}{2M}\Delta_Q\right) \mid \chi_i \right\rangle_q + \lambda_i(Q)\delta_{ik} \right) \xi_i^\alpha = \Lambda^\alpha \xi_k^\alpha \tag{5.8}$$

where both χ and ξ are functions of Q and the Laplacian Δ_Q acts on everything on its right. This can be rewritten as

$$\sum_i \left(\sum_l \frac{1}{2M}(-i\hbar\delta_{kl}\partial_Q - A_{kl})(-i\hbar\delta_{li}\partial_Q - A_{li}) + \lambda_i(Q)\delta_{ik} \right) \xi_i^\alpha = \Lambda^\alpha \xi_k^\alpha \tag{5.9}$$

when using the connection

$$A_{kl} := i\hbar \left\langle \chi_k \mid \partial_Q \mid \chi_l \right\rangle_q \quad . \tag{5.10}$$

Thus we see that the effect of the light variables on the quantum–dynamics of the heavy system is twofold: First, there is an effective potential $\lambda_i(Q)\delta_{ik}$ which can be interpreted as the heavy variables experiencing the presence of the light variables only through an average. Second, the light variables effectively curve the space as seen by the heavy variables, because the ordinary momentum operator $\hat{P}_{kl} = -i\hbar\delta_{kl}\partial_Q$ gets replaced by the covariant momentum operator $\hat{\hat{P}}_{kl} := -i\hbar(\delta_{kl}\partial_Q - \frac{i}{\hbar}A_{kl})$. Using the covariant momentum operators and defining $\hat{\hat{P}}^2_{ik} := \sum_l \hat{\hat{P}}_{il}\hat{\hat{P}}_{lk}$ the effective Schroedinger equation for the Q–dependent coefficients reads

$$\sum_i \left[\frac{\hat{\hat{P}}^2_{ik}}{2M} + \lambda_i(Q)\delta_{ik} \right] \xi_i^\alpha = \Lambda^\alpha \xi_k^\alpha \quad . \tag{5.11}$$

So far, apart from the assumption that Q can be treated as a classical external parameter (i.e. showing no quantum behaviour as long as the dynamics of the q–variables is concerned), everything was exact. The next step would be to solve (5.11) and thus to compute the coefficients ξ_i^α in the ansatz (5.6). Thus we would arrive at a complete solution $\Psi^\alpha(Q,q)$ of the quantum system.
However, for general potentials $V(Q,q)$ (5.11) turns out to be too complicated to be solved exactly, thus one has to rely on perturbation theory. The easiest route one can follow, which was originally proposed by Born and Oppenheimer in the context of molecule physics [57] and carries their name in the literature is to simply approximate

$$\hat{\hat{P}}_{ik} \approx \hat{P}\delta_{ik} \tag{5.12}$$

which diagonalises the operator in (5.11) and leads to simply

$$\left[\frac{\hat{P}^2}{2M} + \lambda_k(Q) \right] \xi_k^\alpha = \Lambda^\alpha \xi_k^\alpha \quad . \tag{5.13}$$

This means that the influence of the q–variables onto the quantum dynamics in the Q–sector is just taken into account via the effective potential term $\lambda_k(Q)$. One can show that this approximation is justified whenever $|\lambda_i - \lambda_j| \gg |\Lambda^\alpha - \Lambda^\beta|$ (see for example [56]) and better approximations can be obtained by taking into account off-diagonal matrix elements or not setting the connection A_{kl} to zero.
As we said before for the Born–Oppenheimer approximation to work one must assume that the Q–variables can be treated classically when we are only interested in the quantum dynamics of the q–system. This assumption is surely

justified when working in a representation where Q acts as a multiplication operator and furthermore the characteristic energy scales are well separated ($|\lambda_i - \lambda_j| \gg |\Lambda^\alpha - \Lambda^\beta|$): To illustrate such a situation with a well known example consider a molecule with a number of nuclei with masses M as Q–variables and a number of electrons with masses m as q–variables. Semiclassically speaking, the electrons move much faster than the nuclei, therefore when treating the dynamics in the electron–sector one can safely treat the slow nuclei as classical variables.

There have been some efforts to generalise this picture and take into account the quantum nature of the Q–variables when treating the dynamics in the q–sector (see [54]) but the methods described above do not easily generalise to this setting. At least to our knowledge there are no results concerning the applicability of the Born–Oppenheimer method for non–commuting Q–variables and a generalisation into that direction seems to require some substantially new input.

The difficulties with the Born–Oppenheimer method for non–commuting variables was one of the main reasons to not choose the Ashtekar–Lewandowski–representation for gravity but to work with the one we constructed in this article. Here our geometrical variables (given by the cotriads) are represented as ordinary multiplication operators and thus the Born–Oppenheimer approximation is directly applicable.

5.1.2 Application to gravity

We want to employ an approximation scheme of the type described above to analyse the quantum theory for gravity plus matter which we have constructed in section 3. Matter couples to gravity only via the cotriads, not via their canonically conjugate momenta, and due to the fact that we have chosen to quantise the cotriads as commuting multiplication operators a Born–Oppenheimer approximation scheme can directly be applied.

We note that the system gravity plus matter has two widely separated energy scales: The *matter part* of the Hamiltonian carries information about the mass of the scalar field, which defines a length scale $\ell_\Phi := \frac{\hbar}{m}$. On the other hand, the Planck length ℓ_P enters the definition of the *gravitational part* of the Hamiltonian. For typical values of m the quotient $\frac{\ell_P}{\ell_\Phi} \ll 1$. In fact this quotient will be much smaller than the quotient $\frac{m_{e^-}}{M_{\text{nuc}}}$ in molecule physics, so one should expect that the Born–Oppenheimer approximation gives results of very good precision.

One can interpret this large separation of energy scales in a very intuitive way: During typical interaction processes between matter particles the geometry of spacetime changes very slowly. Of course, this will not be the case in the deep Planck regime, when considering processes with energies comparable to the Planck energy such that the quantum nature of spacetime itself has to be taken into account. But for all particle interactions in the semiclassical regime with centre of mass energies much lower than the Planck energy this should be a good approximation. Especially when considering a wave function that approximates a classical geometry (like the coherent states for the gravitational sector considered in section 4) the influence of quantum (gravitational) fluctuations will be small. So in a sense this method allows us to compute the QFT on CS limit of quantum gravity. However, the method we will describe below allows us to go beyond this approximation and at least in principle compute quantum gravitational corrections to QFT on CS.

Let us now explain the Born–Oppenheimer approximation adapted to our gravitational theory: The role of the heavy variables is played by the cotriads $E^i_I(v)$ and the role of the light variables is played by the matter fields $\Phi(v)$. To be more precise we are considering the Hilbert space $\mathcal{H}_\otimes = \otimes_v \mathcal{H}_v = \otimes_v(\mathcal{H}^g_v \otimes \mathcal{H}^\phi_v)$, where $\mathcal{H}^g_v = L_2(\mathbb{R}^9)$, $\mathcal{H}^\phi_v = L_2(\mathbb{R})$. $\hat{E}^i_I(v)$ acts as a multiplication operator on $\mathcal{H}_\otimes$, $\hat{P}^I_i(v) := i\ell_P^2 \partial_{E^i_I(v)}$ as a derivative operator thereon, $\hat{\Phi}(v)$ acts as a multiplication operator on $\mathcal{H}_\otimes$ and $\hat{\Pi}(v) = i\hbar\partial_{\Phi(v)}$ as a derivation operator thereon. The Hamiltonian is given approximately[19] given by

$$\hat{H}(\hat{P}, \hat{E}, \hat{\Pi}, \hat{\Phi}) \quad \approx \quad \left[\hat{C}^g_{\text{kin}}(\hat{P}, \hat{E}) + \hat{C}^g_{\text{pot}}(\hat{E})\right] \otimes id_\Phi + \hat{C}^\phi(\hat{E}, \hat{\Pi}, \hat{\Phi}) \quad , \tag{5.14}$$

where $\hat{C}^\phi(\hat{E}, \hat{\Pi}, \hat{\Phi}) = \hat{C}^\phi_{\text{kin}}(\hat{E}, \hat{\Pi}) + \hat{C}^\phi_{\text{pot}}(\hat{E}, \hat{\Phi})$ factorizes into products of the form $\hat{O}^g \otimes \hat{O}^\phi$ with $\hat{O}^g \in \mathcal{L}(\mathcal{H}^g_\otimes)$ and $\hat{O}^\phi \in \mathcal{L}(\mathcal{H}^\phi_\otimes)$. One difference to the Hamiltonian in the toy model described above is that the kinetic terms $\hat{C}^g_{kin}$ and $\hat{C}^\phi_{\text{kin}}$ are not functions of the momenta alone but are functions of the cotriad–operators $\hat{E}^i_I(v)$ as well. This does not spoil the general argument made above, however, the nice expressions in terms of a gauge potential A_{kl} and its associated covariant derivative is not sufficient anymore and there will be extra terms.

We want to solve the eigenvalue problem

$$\hat{H}(\hat{P}, \hat{E}, \hat{\Pi}, \hat{\Phi})\Psi^\alpha(E, \Phi) = \Lambda^\alpha \Psi^\alpha(E, \Phi) \quad , \tag{5.15}$$

[19] In this approximation we keep only the first term in the power series $\bar{H} := \hat{C}\left(1 + \mathcal{O}\left(\frac{\widehat{Q^{ab}C_aC_b}}{C^2}\right)\right)$ as explained earlier.

and for illustrative purposes we will assume that $\hat{H}$ has discrete spectrum such that $\Psi^\alpha(E,\Phi)$ are proper eigenfunctions and Λ^α their respective eigenvalues. At least as far as the matter part of the Hamiltonian is concerned, this will be the case whenever the spatial manifold of the continuum theory which arises as an approximation to the discrete theory is compact without boundary. The non–compact case has to be treated with more care and one needs to perform a direct integral decomposition of $\mathcal{H}_\otimes$. However, these technical details will not be essential for our method and we will restrict ourselves to the case of compact spatial manifolds without boundary.
The strategy is exactly the same as in the toy model discussed before: We start with the Ansatz

$$\Psi^\alpha(E,\Phi) := \sum_i \xi_i^\alpha(E)\chi_i(\Phi;E) \quad , \tag{5.16}$$

for states $\Psi \in \mathcal{H}_\otimes$ and assume that we already have some knowledge about the partial eigenvalue problem

$$\hat{C}^\phi(\hat{\Phi},\hat{\Pi};E)\chi_i(\Phi;E) = \lambda_i(E)\chi_i(\Phi;E) \quad . \tag{5.17}$$

Here the dependence on E has again to be understood in a parametric sense, i.e. we keep E fixed and regard $\chi_i(\Phi;E)$ as an element of $\mathcal{H}_\otimes^\phi$. Thus, because $\hat{C}^\phi(\hat{\Phi},\hat{\Pi};E)$ is just the standard discrete matter Hamiltonian, this amounts to solving a certain type of lattice–QFT for an arbitrary (but fixed) background. Of course, in general this cannot be accomplished analytically and one will have to employ further approximation techniques, but the interesting point is that QFT on a curved spacetime emerges in the analysis of a theory of quantum gravity in the same way as the theory of quantised electrons in a given classical potential emerges out of the full quantum treatment of a molecule. For technical reasons one has to assume that $\chi_i(\Phi;E)$ depends on E in an at least twice differentiable manner.
In order to get the full wave functions $\Psi^\alpha(E,\Phi)$ we follow the Born–Oppenheimer approach as discussed for the toy model above, so we want to compute the coefficients $\xi_i^\alpha(E)$. This can be done by starting with (5.15), multiplying by $\chi_k(\Phi;E)$ from the left and taking the scalar product in $\mathcal{H}_\otimes^\phi$. Assuming that $\chi_i(\Phi;E)$ is an orthonormal basis of $\mathcal{H}_\otimes^\phi$ we get

$$\begin{aligned}\Big\langle \chi_k(\Phi;E) \mid \hat{H} \mid \Psi^\alpha(E,\Phi)\Big\rangle_\phi &= \sum_i \Big\langle \chi_k(\Phi;E) \mid \Big[\hat{C}^g_{\rm kin}(\hat{P},\hat{E}) + \hat{C}^g_{\rm pot}(\hat{E})\Big] \mid \chi_i(\Phi;E)\Big\rangle_\phi \xi_i^\alpha(E) + \lambda_k(E)\xi_k^\alpha(E) \\ &= \Lambda^\alpha \xi_k^\alpha(E)\end{aligned} \tag{5.18}$$

If we define $\hat{\mathbb{C}}^g_{ki} := \Big\langle \chi_k(\Phi;E) \mid \Big[\hat{C}^g_{\rm kin}(\hat{P},\hat{E}) + \hat{C}^g_{\rm pot}(\hat{E})\Big] \mid \chi_i(\Phi;E)\Big\rangle_\phi$ as an operator in $\mathcal{L}(\mathcal{H}^g_\otimes)$ then we can write (5.18) as

$$\sum_i \Big(\hat{\mathbb{C}}^g_{ki} + \delta_{ik}\lambda^i(E)\Big)\xi_i^\alpha(E) = \Lambda^\alpha \xi_k^\alpha(E) \quad , \tag{5.19}$$

which is a coupled system of eigenvalue equations in $\mathcal{H}^g_\otimes$ and its solutions ξ_k^α are the appropriate coefficients in the expansion (5.16).
So far everything is exact and one could obtain the full set of solutions by first solving (5.17) and then use χ_i and λ_i to compute the coefficients in the expansion (5.16). However, even in our toy model we saw that there is in general no chance of obtaining analytic solutions. Quantum gravity is surely more difficult than a single molecule, so it would be surprising if one could make progress in the full theory without further simplifications. Thus, the crudest approximation is to use again the Born–Oppenheimer approximation which in this case amounts to neglecting all the off–diagonal terms in the matrix $\hat{\mathbb{C}}^g_{ki}$, that is we neglect the action of $\hat{P}$ on χ_i. Then we can write $\Big\langle \chi_k(\Phi;E) \mid \Big[\hat{C}^g_{\rm kin}(\hat{P},\hat{E}) + \hat{C}^g_{\rm pot}(\hat{E})\Big] \mid \chi_i(\Phi;E)\Big\rangle_\phi \approx \langle \chi_k(\Phi;E) \mid \chi_i(\Phi;E)\rangle_\phi \Big[\hat{C}^g_{\rm kin}(\hat{P},\hat{E}) + \hat{C}^g_{\rm pot}(\hat{E})\Big] = \delta_{ki}\Big[\hat{C}^g_{\rm kin}(\hat{P},\hat{E}) + \hat{C}^g_{\rm pot}(\hat{E})\Big]$. Thus we are left with

$$\Big[\hat{C}^g_{\rm kin}(\hat{P},\hat{E}) + \hat{C}^g_{\rm pot}(\hat{E}) + \lambda^i(\hat{E})\Big]\xi_i^\alpha(E) = \Lambda^\alpha \xi_k^\alpha(E) \quad . \tag{5.20}$$

In the toy model we saw that using the pure Born–Oppenheimer approximation means that the influence of the light variables onto the quantum dynamics of the heavy variables is only effectively taken into account via their eigenvalues. This is also the case here: If we would set $\lambda^i(E) = 0$ then we would describe vacuum gravity and our solutions in the gravitational sector would not know anything about the matter content of the theory. One can regard (5.20) as a first step towards quantum gravitational solutions which do not neglect the presence of matter fields, however matter is only 'effectively' taken into account and not with respect to full dynamics. In a sense this is the quantum analog of the semiclassical Einstein equations $G_{\mu\nu} = \langle T_{\mu\nu}\rangle_\phi$ where the expectation value of the stress energy tensor is used as a source in Einstein's equations.

At least in principle the strategy to solve dynamics in the *fully quantum gravitational setup* of the theory defined above would be the following:

1. As a first step consider the Schrödinger equation (5.17) for the matter part. This can be interpreted as a fundamentally discrete quantum theory for matter degrees of freedom on a fixed background whose continuum limit would be quantum field theory (QFT) on a curved spacetime (CS). However, in the discrete setup we are not restricted to smooth metrics since the differential operators have been replaced by difference operators. Hence, $\hat{C}^\phi$ is a well defined operator for arbitrary values of $E^i_I(v)$ as long as it is non degenerate[20]. Thus, in a sense this can be seen as a generalisation of QFT from spacetimes with a curved smooth metric to "spacetimes" with a curved and even discontinuous metric. Nevertheless, if we consider $E^i_I(v)$ which approximate a smooth metric then (5.17) is a 'lattice version' of QFT on CS. Note however, that the interpretation is quite opposite to the one usually employed for lattice theories: This theory of quantum gravity is fundamentally discrete and a continuum theory should be regarded as an approximation which is good for a certain set of states (the semiclassical ones) and bad for other states.

2. After analysing the matter dynamics on a fixed background one would obtain eigenfunctions $\chi_i(\Phi; E)$ and eigenvalues $\lambda_i(E)$ which are simply functions of E, i.e. it is not enough to consider QFT on CS for a single fixed background, but one would need knowledge about the whole class of quantum field theories on an arbitrary background.
Using these functions we could go ahead and analyse the gravitational sector of the theory, i.e. we would use equation (5.19) to compute the coefficients ξ^α_i in ansatz (5.16) and thus find solutions to the full quantum theory of gravity plus matter. However, from a practical point of view, there is no chance that one can exactly solve such a complicated system of coupled differential equations as (5.19), so we must resort to approximation methods such as the pure Born–Oppenheimer approximation as described above. Thus, one would use equation (5.20) to calculate the coefficients ξ^α_i. In this approximation the only imprint that matter leaves on the gravitational sector is the term proportional to $\lambda_i(E)$ in equation (5.20).

However, from a practical point of view the situation is less clear: Even if we would succeed to somehow compute the eigenfunctions of the matter Hamiltonian $\chi_i(\Phi; E)$ (either numerically or using perturbative approaches such as an expansion in terms of Feynman diagrams) there still remains one big problem: In step 2 of the scheme described above we would need to solve the eigenvalue problem for the operator

$$\hat{C}^g_{\text{kin}}(\hat{P}, \hat{E}) + \hat{C}^g_{\text{pot}}(\hat{E}) + \lambda^i(\hat{E}) \tag{5.21}$$

in order to get the correct coefficients ξ^α_i. This operator is tremendously complicated, as we have seen in (3.12) and even setting $\lambda_i(E)$ to zero does not substantially simplify the problem. For $\lambda_i(E) = 0$ (5.21) is equivalent to the Hamiltonian constraint of a pure gravity theory on an algebraic lattice, and there is no hope that one can gain enough information about the space of its eigenfunctions in an analytic way. This is the reason why it is so complicated to incorporate the matter influence on the gravitational sector in the full theory on a more than formal level. Even considering $\lambda_i(E)$ just as a small perturbation does not help much: Without any knowledge about the spectrum of the unperturbed operator it is not meaningful to start a perturbative analysis.
At this point we want to stress that from a conceptual point of view there are no obstacles in the scheme we proposed above, it's just that the operator (5.21) is too complicated without further simplifications. One could for example employ numerical methods to approximately solve (5.20) in certain situations. A second option is to further exploit the fact that we want to describe situations where geometry can be treated almost classically, thus it makes sense to use a semiclassical approximation, which we will describe in the next paragraph.

5.1.3 Born–Oppenheimer in the semiclassical regime

What we are interested in is not the complete solution to the spectral problem. It would be sufficient to know the $\lambda_i(E)$ in the "neighbourhood" of some three geometry E_0 and seek for coefficients $\xi_i(E)$ that at the same time solve the eigenvalue equation, say in the pure Born – Oppenheimer approximation, and in addition die off sufficiently quickly away from E_0. The problem is of course 1. to actually compute the $\lambda_i(E)$ in a suitable neighbourhood of some exactly solusolvableable E_0 with sufficient analytical control and 2. to find the eigenvectors ξ_i.

[20]In the degenerate case, QFT on CS methods cannot be used. By construction of our domain of definition, this case does not arise when restricting the action of the Hamiltonian operator to that domain.

As we are unable at present to say much about the solution of either of these problems we must resort here to a very crude approximation:
Pick any solvable E_0 (say Minkowski, FRW, ...) and choose as $\chi_i(E)$ a coherent state Ψ_{Z_i} from the gravitational sector alone peaked on a classical phase space point $Z^j_J(v) := \frac{1}{\ell_P\sqrt{2}}\left(E^j_{J0} - iP^J_{j\ i}\right)$ (see section 4). Notice that we pick $\Re(Z_i) = E_0$ independent of i but $\Im(Z_i) = P_i$ is not specified at the moment. These states are kinematical coherent states, not dynamical ones, which means that they are not granted to be stable under the dynamics for a very long time. However, when considering particle interactions the typical timescales on which such interactions happen are rather short, and at this stage we want to assume that the gravitational coherent states keep sharply peaked on a classical trajectory during these time intervals. This assumption seems natural from a particle physics point of view. Clearly, the coherent states Ψ_Z are not exact eigenstates of the Hamiltonian operator but at least approximately so. Moreover, they are orthogonal to a very good approximation whenever the Z_i lie in different quantum cells of the phase space. As we will see, this condition will hold automatically whenever the $\lambda_i(E_0)$ differ sufficiently from each other. We now make the following Ansatz for the "eigenstate" of the physical Hamiltonian

$$\Psi(E,\phi) = \sum_i c_i\ \Psi_{Z_i}(E)\ \chi_i(\phi; E_0) \tag{5.22}$$

for certain complex numbers c_i (the Ψ_{Z_i} themselves are normalised). Notice that we have frozen the E dependence of $\chi_i(\phi; E)$ at E_0 so that C^g actually does not act on it. Therefore the Ansatz (5.22) is not really of Born – Oppenheimer type. Inserting into the eigenvalue equation we obtain the exact equation

$$H\Psi = \sum_i c_i\ [(C^g + \lambda_i(E_0))\Psi_{Z_i}](E)\ \chi_i(\phi; E_0) = \Lambda\Psi \tag{5.23}$$

and thus due to the orthonormality of the χ_i

$$c_i\ [(C^g + \lambda_i(E_0) - \Lambda)\Psi_{Z_i}](E) = 0 \tag{5.24}$$

for all i. Taking the inner product with Ψ_{Z_j} we see that, due to our assumption on the $\lambda_i(E_0)$ the equation for $j \neq i$ is trivially satisfied (approximately) while for $i = j,\ c_i \neq 0$ we obtain

$$< \Psi_{Z_i}, C^g\Psi_{Z_i} > \approx C^g(E_0, P_i) = -\lambda_i(E_0) + \Lambda \tag{5.25}$$

which we use as a condition on the P_i which therefore will mutually different from each other as long as the $\lambda_i(E_0)$ are. Notice that our assumption $< \psi_{Z_i}, \psi_{Z_j} > \approx \delta_{ij}$ is therefore self – consistent.

However, as already stated, this procedure is unsatisfactory for several reasons of which we mention two: First it does not really follow the Born – Oppenheimer spirit and does not respect the true interaction between geometry and matter. This is also obvious from the fact that we nowhere used that the geometry operators were commuting, what we did could have been done in LQG as well and has already been done [16]. The only new ingredient is that we took into account the backreaction in the sense of expectation values. Second, the above method can never reveal the spectral values of the physical Hamiltonian, because we can always choose the P_i to obey (5.25) for any choice of Λ.

Suffice it to say that a lot of work has to be invested in order to make the Born Oppenheimer Ansatz unfold its true power. To get more insight into the problem, we consider a truncation of the gravitational phase space (FRW spacetimes) in section 5.3.

5.2 Construction of a Fock space

Now we want to describe how one can in general construct a Fock space for a linear (i.e. the Hamiltonian is quadratic in the dynamical variables) classical theory. As explained above this situation emerges as step 1 in the Born–Oppenheimer analysis of our theory: When solving (5.17) and only considering matter dynamics the cotriads can be regarded as fixed non–dynamical functions because we have chosen a representation where they act as multiplication operators in the full theory. We follow largely [16] and then apply this scheme to the present situation. We assume the reader is familiar with the basic definitions concerning Fock spaces, for a more detailed review on the construction of Fock spaces in the context of QFT on CS we refer the reader to [58].

Let $\mathcal{H}_1$ be a Hilbert space (usually called the *one particle Hilbert space*) with inner product $\langle \cdot \mid \cdot \rangle_1$, $\psi = [\pi, \phi]$ a basis of $\mathcal{H}_1$ and $\mathcal{Q}, \mathcal{P} \in \mathcal{L}(\mathcal{H}_1)$ linear symmetric operators thereon (i.e. $\mathcal{Q}^\dagger = \mathcal{Q}, \mathcal{P}^\dagger = \mathcal{P}$). Assume further that their inverses $\mathcal{Q}^{-1}, \mathcal{P}^{-1}$ exist on $\mathcal{H}_1$. The Hamiltonian H of a linear classical dynamical system can then be interpreted as a functional

$$H : \mathcal{H}_1 \to \mathbb{R};\ \psi = [\pi, \phi] \mapsto H(\psi) = \frac{1}{2}\langle \pi \mid \mathcal{P} \mid \pi \rangle_1 + \frac{1}{2}\langle \phi \mid \mathcal{Q} \mid \phi \rangle_1 \tag{5.26}$$

for given operators $\mathcal{Q}, \mathcal{P}$.
Further assume that π, ϕ form a canonical pair, that is, the Hilbert space $\mathcal{H}_1$ has also the structure of a symplectic manifold[21] (if $\dim(\mathcal{H}_1)$ is finite and l, l' label its basis elements we have $\{\pi_l, \phi_{l'}\} = \delta_{ll'}$ and $\{\pi_l, \pi_{l'}\} = \{\phi_l, \phi_{l'}\} = 0$). We are now aiming for a 'quantisation' of that classical linear dynamical system, namely we want to construct a Hilbert space $\mathcal{H}$ which carries a representation of the classical observable algebra given by the Poisson algebra of π and ϕ.
One convenient way of doing this, which naturally allows for an interpretation in terms of 'particles' is the construction of a Fock space: That is, we are implementing a unitary map U on $\mathcal{H}_1$ and use this map to construct complex functions $z(\pi, \phi)$ such that $\{\bar{z}_l, z_{l'}\} = i\delta_{ll'}$ where $\bar{z}$ is the complex conjugate to z. But this is just the commutator algebra of creation and annihilation operators and it is well known that such an algebra can naturally be represented on $\mathcal{F}_s(\mathcal{H}_1)$, the symmetric Fock space over $\mathcal{H}_1$.
Let us start with the Hilbert space $\mathcal{H}_1$ and complexify it. Now choose a subspace $\mathcal{H}_1^{\mathbb{C}+} \subset \mathcal{H}_1^{\mathbb{C}}$ such that there exists a unitary, real–linear map $U : \mathcal{H} \mapsto \mathcal{H}_1^{\mathbb{C}+}$. Using this map we can rewrite the Hamiltonian in the form

$$H = \langle \bar{z} \mid \omega \mid z \rangle_{\mathcal{H}_1^{\mathbb{C}+}} \tag{5.27}$$

where $z \in \mathcal{H}_1^{\mathbb{C}+}$ and $\bar{z}$ is the complex conjugate to z in $\mathcal{H}_1^{\mathbb{C}+}$ and $\omega \in \mathcal{L}(\mathcal{H}_1^{\mathbb{C}+})$ as defined below. z is obtained by using this unitary map U through[22]

$$\begin{aligned} z &:= \frac{1}{\sqrt{2}} U(\mathcal{D}\phi - i\mathcal{D}^{-1}\pi) && (5.28)\\ \mathcal{D} &:= [\mathcal{P}^{-1/2}(\mathcal{P}^{1/2}\mathcal{Q}\mathcal{P}^{1/2})^{1/2}\mathcal{P}^{-1/2}]^{1/2} \quad , && (5.29)\end{aligned}$$

and ω is given through

$$\omega := U\mathcal{D}^{-1}\mathcal{Q}\mathcal{D}^{-1}U^{-1} \tag{5.30}$$

Note that $\mathcal{D}$ is a symmetric operator with well defined inverse if $\mathcal{Q}$ and $\mathcal{P}$ are.
Then one gets

$$\begin{aligned} \langle \bar{z} \mid \omega \mid z \rangle_{\mathcal{H}_1^{\mathbb{C}+}} &= \frac{1}{2}\left\langle U(\mathcal{D}\phi + i\mathcal{D}^{-1}\pi) \mid U\mathcal{D}^{-1}\mathcal{Q}\mathcal{D}^{-1}U^{-1}U \mid (\mathcal{D}\phi - i\mathcal{D}^{-1}\pi)\right\rangle_{\mathcal{H}_1^{\mathbb{C}+}} \\ &= \frac{1}{2}\left\langle \mathcal{D}\phi + i\mathcal{D}^{-1}\pi \mid \mathcal{D}^{-1}\mathcal{Q}\mathcal{D}^{-1}(\mathcal{D}\phi - i\mathcal{D}^{-1}\pi)\right\rangle_1 \\ &= \frac{1}{2}\left\langle (\mathcal{D}^{-1})^\dagger\mathcal{D}\phi \mid \mathcal{Q}\phi\right\rangle_1 + \frac{1}{2}\left\langle \pi \mid (\mathcal{D}^{-1})^\dagger\mathcal{D}^{-1}\mathcal{Q}\mathcal{D}^{-1}\mathcal{D}^{-1}\pi\right\rangle_1 \\ &\quad + \frac{i}{2}\left\langle \pi \mid (\mathcal{D}^{-1})^\dagger\mathcal{D}^{-1}\mathcal{Q}\phi\right\rangle_1 - \frac{i}{2}\left\langle (\mathcal{D}^{-1})^\dagger(\mathcal{D}^{-1})^\dagger\mathcal{Q}^\dagger(\mathcal{D}^{-1})^\dagger\mathcal{D}\phi \mid \pi\right\rangle_1 \\ &= \frac{1}{2}\left\langle \phi \mid \mathcal{Q}\phi\right\rangle_1 + \frac{1}{2}\left\langle \pi \mid \mathcal{D}^{-2}\mathcal{Q}\mathcal{D}^{-2}\pi\right\rangle_1 \\ &\quad + \frac{i}{2}\left\langle \pi \mid \mathcal{D}^{-2}\mathcal{Q}\phi\right\rangle_1 - \frac{i}{2}\overline{\left\langle \pi \mid \mathcal{D}^{-2}\mathcal{Q}\phi\right\rangle_1} \\ &= \frac{1}{2}\left\langle \phi \mid \mathcal{Q}\phi\right\rangle_1 + \frac{1}{2}\left\langle \pi \mid \mathcal{P}\pi\right\rangle_1 \\ &= H \end{aligned} \tag{5.31}$$

where in the second equality we used that U is a unitary map, in the fourth equality that $\mathcal{Q}, \mathcal{P}$ and $\mathcal{D}$ are symmetric and in the fifth equality that $\mathcal{D}$ is real and $\mathcal{D}^{-2}\mathcal{Q}\mathcal{D}^{-2} = \mathcal{P}$. This last equality can be seen by explicitly writing $\mathcal{D}$ in terms of $\mathcal{Q}, \mathcal{P}$ and then multiplying both sides first by $\mathcal{P}^{-1/2}$ and then by $(\mathcal{P}^{1/2}\mathcal{Q}\mathcal{P}^{1/2})^{1/2}$.
From equation (5.28) one can verify that $\bar{z}, z$ indeed form the Poisson algebra of latter operators and therefore the Hamiltonian (5.26) can be defined as a linear operator on $\mathcal{F}_s(\mathcal{H}_1^{\mathbb{C}+})$ in the form (5.27).

In our case the one particle Hilbert space is given by $\mathcal{H}_1 = l_2(V(\alpha))$, the space of square–summable functions over the set of vertices $V(\alpha)$. The inner product on this space is simply given by $\langle \cdot, \cdot \rangle_1 : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}; \quad f, f' \mapsto \sum_{v \in V(\alpha)} \bar{f}(v) f'(v)$.

[21]We will omit the details here, but in the field–theoretic case one has to consider smeared versions of π, ϕ with appropriately chosen test functions to turn $\mathcal{H}_1$ into an symplectic manifold.
[22]It will become obvious why the operator $\mathcal{D}$ must have this rather complicated form in the next paragraph.

Field coordinates are given by $\pi(v), \phi(v)$ and these form a canonical pair with the Poisson bracket $\{\pi(v), \phi(v')\} = \delta_{vv'}$. If we consider the cotriads on each vertex as prescribed functions then the "classical" Hamiltonian is given by

$$H = \frac{1}{2}\sum_{v\in V(\alpha)} \frac{1}{\det(E)(v)}\pi^2(v) + \det(E)(v)\phi(v)(-\Delta + m^2)\phi(v) \quad , \tag{5.32}$$

with the discrete Laplace–Beltrami operator Δ given by

$$\Delta = \frac{1}{2\det(E)(v)}\nabla_I^+[\det(E)(v)\epsilon^{IKL}\epsilon^{JMN}E_K^k(v)E_L^l(v)E_M^k(v)E_N^l(v)\nabla_J^-] \tag{5.33}$$

with the forward and backward derivative, ∇_I^+ and ∇_I^-, acting on functions $f \in l_2(V(\alpha))$ as $\nabla_I^+ f(v) := f(v+I) - f(v)$ and $\nabla_I^- f(v) := f(v) - f(v-I)$ respectively. Thus $\mathcal{Q}$ and $\mathcal{P}$ are given as linear symmetric operators on $\mathcal{H}_1$ as

$$\mathcal{Q} = \frac{1}{\det(E)(v)} \quad , \tag{5.34}$$

$$\mathcal{P} = \det(E)(v)(-\Delta + m^2) \quad . \tag{5.35}$$

So we can construct $\bar{z}, z$ such that they have the desired Poisson brackets and we can represent them as creation and annihilation operators on the Fock space $\mathcal{F}_s(l_2(V(\alpha)))$.

5.3 A minisuperspace example: QFT on FRW–spacetimes

We want to illustrate the Born Oppenheimer Decomposition as well as its semiclassical approximation explained in section 5.1.3 for a case for which we have sufficient mathematical control over the equations. We will choose a "hybrid model", that is, homogeneous, isotropic FRW – minisuperspace model coupled to inhomogeneous matter which recently was advocated in the context of Gowdy models [59]. That is, in the language of the previous section, we pick as a spatial geometry a homogeneous cotriad E_0 and pick as a neighbourhood around it all homogeneous cotriads E. This neighbourhood confines us to the homogeneous sector of GR while matter is treated as a QFT. In the classical theory this leads to consistent equations of motion because the equations of motion for geometry only depend on the integrated matter fields.

We will assume that the spatial topology of the universe which arises in the classical limit of this quantum theory is given by $\Sigma = T^3$. In the classical continuum theory for these model the cotriad is given by

$$E_a^i(\sigma) := E\delta_a^i \tag{5.36}$$

where E is a homogeneous function of time only, related to the scale factor by $a = |E|$. Its canonical momentum

$$P_i^a(\sigma) := P\delta_i^a \tag{5.37}$$

is also homogeneous and, in terms of τ–derivatives takes the form $P = -4E\dot{E}$. Assuming that these variables emerge from some averaging procedure

$$P := \frac{1}{3V_0}\int_{T^3} d^3\sigma P_i^a(\sigma)\delta_a^i$$

$$E := \frac{1}{3V_0}\int_{T^3} d^3\sigma E_a^i(\sigma)\delta_i^a \tag{5.38}$$

with V_0 the coordinate volume of T^3 they have canonical Poisson brackets

$$\{P, E\} = \frac{\kappa}{3V_0} \tag{5.39}$$

In the FRW – minisuperspace model the physical Hamiltonian H_{phys} reduces to $H_{\text{phys}} = \frac{1}{3V_0}\int_{T^3} d\sigma\sqrt{(C^g + C^\phi)^2(\sigma)} = C^g + C^\phi$ because the spatial diffeomorphism constraint identically vanishes. Inserting the ansatz for the elementary phase space variables in the general expression (2.14) the gravitational part of the integrated Hamiltonian is simply

$$C^g = -\frac{3V_0}{8\kappa}\frac{P^2}{|E|} \tag{5.40}$$

Later we will consider spacetimes with non–vanishing cosmological constant $\Lambda \neq 0$, in this case the gravitational part of the Hamiltonian constraint is given by

$$C^g_\Lambda = -\frac{3V_0}{8\kappa}\frac{P^2}{|E|} + \frac{1}{\kappa}\Lambda V_0 |E|^3 \quad . \tag{5.41}$$

Passing to the quantum theory this means we have operators $\hat{E}^i_I(v) = \hat{E}\delta^i_I$ and $\hat{P}^I_i(v) = \hat{P}\delta^i_i$ which do not depend on the algebraic lattice point v, $\hat{E} = E\cdot$ acts as a multiplication operator and $\hat{P} = i\ell_P^2\partial_E$ as a derivative operator. As in the Wheeler–de Witt theory the Hilbert space of this minisuperspace model is given by $\mathcal{H} = L_2(\mathbb{R})$, the space of square–integrable functions over the real line. In the absence of matter fields dynamics in the gravitational sector is then generated by an operator $\hat{C}^g_\Lambda$ that is obtained from the classical expression by replacing $E \to \hat{E}, P \to \hat{P}$ in some suitable operator ordering. The solutions $e_\lambda(E)$ to the eigenvalue problem

$$\hat{C}^g_\Lambda e_\lambda(E) = \lambda e_\lambda(E) \tag{5.42}$$

can be given in terms of some hypergeometric functions. If we choose a symmetric operator ordering the eigenvalue problem reads

$$-\frac{3V_0\ell_P^4}{8\kappa}E^2\frac{\partial^2}{\partial E^2}e_\lambda(E) + \frac{3\kappa V_0\ell_P^4}{8}|E|\frac{\partial}{\partial E}e_\lambda(E) - \frac{V_0\Lambda}{\kappa}E^6 e_\lambda(E) + \lambda|E|^3 e_\lambda(E) = 0 \tag{5.43}$$

and the solutions $e_\lambda(E)$ can explicitly be written in terms of Kummer's functions of the first and second kind (see [60] for their properties). For the special case $\Lambda = 0$, $\lambda = 0$ the solutions reduce to ordinary Bessel functions and in [61] it was shown that in this basis one can construct 'wave packet solutions' whose quantum evolution stays sharply peaked on a classical trajectory in phase space. Thus, these quantum solutions are close to classical ones not only at one instant of time but also under dynamical evolution. We will follow here a somewhat different route.

On such a FRW–spacetime one can explicitly carry out the construction of a Fock space using the method explained in the last section: If we want this theory to describe quantum field theory on an FRW universe with spatial topology $\Sigma = T^3$, the number of vertices of α should be finite. Assuming further that the number of vertices in each spatial direction is N we have $|V(\alpha)| = N^3$. Let $\Phi(v)$ and $\Pi(v)$ be the scalar field and its canonical momentum at each vertex v of the algebraic graph. Then the classical Hamiltonian for a scalar field on a space with cotriad E is given by

$$C^\phi := \frac{1}{2}\sum_{v\in V(\alpha)} \Pi(v)\frac{1}{|E|^3}\Pi(v) + \Phi(v)|E|^3\left[\frac{-\Delta_{\text{flat}}}{E^2} + m^2\right]\Phi(v) \tag{5.44}$$

with the ordinary flat lattice Laplacian $\Delta_{\text{flat}} := \delta^{IJ}\nabla^-_I\nabla^+_J$. Using spatial homogeneity and flatness one can introduce a discrete Fourier transform from $V(\alpha)$ to $K(\alpha)$, the associated momentum space. $K(\alpha)$ is discrete (because $V(\alpha)$ is bounded) and bounded (because $V(\alpha)$ is discrete). If we choose each of the dimensions in T^3 to be of unit coordinate length and choose to embed $V(\alpha)$ uniformly in T^3, then the vertices are labeled by triples $v = (v_1, v_2, v_3)$ with $v_1, v_2, v_3 \in \{\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}\}$. Thus, for each function $f(v)$ we can define its Fourier transform

$$f_k := \sum_v f(v)e^{-ik\cdot v} \tag{5.45}$$

with $k = (k_1, k_2, k_3)$ and $k_1, k_2, k_3 \in \{2\pi, 4\pi, \ldots, 2N\pi\} =: K(\alpha)$. The inverse transformation is then given by

$$f(v) = \frac{1}{N^3}\sum_{k\in K(\alpha)} f_k e^{+ik\cdot v} \quad . \tag{5.46}$$

Furthermore we have the identities $\frac{1}{N^3}\sum_k e^{k\cdot(v-v')} = \delta_{v,v'}$ and $\frac{1}{N^3}\sum_v e^{(k-k')\cdot v} = \delta_{k,k'}$ where the deltas on the right hand side are both Kronecker deltas.
One can show that the Fourier transform of the flat Laplacian acting on a function $f(v)$ is given by

$$\Delta_{\text{flat}} f(v) := \frac{1}{N^3}\sum_{k\in K(\alpha)}\left[-4N^2\sum_{I=1,2,3}\sin^2\left(\frac{k_I}{2N}\right)\right] f_k e^{+ik\cdot v} \quad , \tag{5.47}$$

which in the continuum limit just gives the usual expression $\lim_{N\to\infty}\Delta_{\text{flat}} f_k = -k^2 f_k$. Here we chose a naive discretisation for implementing the derivative on the lattice. Improved discretisations can be found in the context of perfect actions

[51] that provide a framework in which discretisation artefacts can be avoided. For this article we will restrict our discussion to the standard (naive) discretisation procedure and will present the discussion on improved actions elsewhere.
Thus, we can use the Fourier transform and write the Hamiltonian as

$$C^{\phi} = \frac{1}{2N^3} \sum_{k \in K(\alpha)} \Pi_k \frac{1}{|E|^3} \Pi_{-k} + \Phi_k |E|^3 \left[\frac{4N^2}{E^2} \sum_{I=1,2,3} \sin^2(\frac{k_I}{2N}) + m^2 \right] \Phi_{-k}. \tag{5.48}$$

In Fourier space the symplectic structure is given by $\{\Pi_k, \Phi_{k'}\} = N^3 \delta_{k,-k'}$ where $-k \stackrel{!}{=} (2\pi N - k)$. Introducing the Hilbert space $\mathcal{H}_1 := l_2(K(\alpha))$ with inner product $\langle f \mid g \rangle_k := \frac{1}{N^3} \sum_{k \in K(\alpha)} f_k g_{-k}$ we can write

$$C^{\phi} = \frac{1}{2} \langle \Pi \mid \mathcal{P} \mid \Pi \rangle_k + \frac{1}{2} \langle \Phi \mid \mathcal{Q} \mid \Phi \rangle_k \tag{5.49}$$

with $\mathcal{P} := \frac{1}{E^3}$ and $\mathcal{Q} := |E|^3 \left[\frac{\gamma^2(k)}{E_0^2} + m^2 \right]$ with $\gamma^2(k) := 4N^2 \sum_{I=1,2,3} \sin^2(\frac{k_I}{2N})$ both simply acting by multiplication.
Here we are in the special situation that $[\mathcal{P}, \mathcal{Q}] = 0$ as linear operators on $\mathcal{H}_1$, thus we can explicitly construct the algebra of creation and annihilation variables: For commuting $\mathcal{P}$ and $\mathcal{Q}$ we get

$$\mathcal{D} = \left[\frac{\mathcal{Q}}{\mathcal{P}} \right]^{1/4} = |E|^{3/2} \left[\frac{\gamma^2(k)}{E^2} + m^2 \right]^{1/4} \tag{5.50}$$

and thus

$$z_k := \frac{1}{\sqrt{2}} \left[|E_0|^{3/2} \left[\frac{\gamma^2(k)}{E^2} + m^2 \right]^{1/4} \Phi_k - i |E|^{-3/2} \left[\frac{\gamma^2(k)}{E^2} + m^2 \right]^{-1/4} \Pi_k \right] \tag{5.51}$$

One can easily check that $\{\bar{z}_k, z_{k'}\} = i \delta_{k,-k'}$. The standard representation of this algebra is as creation and annihilation operators on the Fock space $\mathcal{F}(\mathcal{H}_1)$.
Using these variables the Hamiltonian is given by

$$C^{\phi} = \langle \bar{z} \mid \omega_k \mid z \rangle_k \tag{5.52}$$

with $\omega_k = \sqrt{\frac{\gamma^2(k)}{E^2} + m^2}$. Compared to standard QFT on Minkowski space, where the spectrum takes values $\omega_k = \sqrt{k^2 + m^2}$ there are two differences: First, we are considering a discrete theory, thus the momentum squared gets replaced by $\gamma^2(k)$, a bounded expression that converges against k^2 in the limit $N \to \infty$. Second, the z_k, ω_k and therefore the vacuum $|0>$ depends explicitly on E.
If we choose this representation and denote by $\hat{a}_k^\dagger, \hat{a}_k$ the corresponding ladder operators in the Fock space can be characterised through a cyclic vector ('the vacuum') $|0\rangle$ defined through the condition $\hat{a}_k |0\rangle = 0 \quad \forall k$. All other elements of $\mathcal{F}$ can then be constructed by letting the creation operators $\hat{a}^\dagger$ act on $|0\rangle$ and the Hamiltonian operator is given by

$$\hat{C}^{\phi} := \sum_k \omega_k \, \hat{a}_k^\dagger \, \hat{a}_k \tag{5.53}$$

For the vacuum state $|0\rangle$ the expectation value of the Hamiltonian is simply $\left\langle 0 \mid \hat{H} \mid 0 \right\rangle = 0$[23] and if we denote n–particle states for particles with momenta $k_1, \ldots, k_n$ by

$$|n\rangle := \hat{a}_{k_1}^\dagger \ldots \hat{a}_{k_n}^\dagger |0\rangle \tag{5.54}$$

we get

$$\hat{H} |n\rangle = [\sum_{i=1}^{n} \omega_{k_i}] \; |n\rangle =: \lambda_n(E) \, |n> \tag{5.55}$$

[23]Here the inner product $\langle . \mid . \rangle$ is taken on the full Fock space $\mathcal{F}$ as opposed to the one–particle inner product $\langle . \mid . \rangle_k$

where the role of the index n is played by the labels $k_1, .., k_n$. Let us now analyse the coupled quantum system gravity (in the minisuperspace approximation) plus matter according to the scheme we proposed in section 5.1.3: To ensure that we have nontrivial dynamics for the gravity part already in the absence of matter we choose to work with the Hamiltonian C^g_Λ with non–vanishing cosmological constant.

In order to carry out the Born Oppenheimer programme, we have to construct states $\Psi(E,\Phi) = \sum_n \xi(E)\,|n\rangle\,(E,\Phi)$ and to compute the coefficients $\xi(a)$ in the Born–Oppenheimer ansatz. It is convenient to first pass to a new set of canonical variables which are adapted to the specific form C^g_Λ and then to quantise the system in these new variables for which coherent states can be more easily constructed. We introduce the following variables

$$\tilde{P}_{:} = \sqrt{3V_0}\frac{P}{\sqrt{|E|}}, \quad \tilde{E} := \sqrt{3V_0}\frac{2}{3}\mathrm{sgn}(E)|E|^{\frac{3}{2}}, \quad \text{then} \quad \{\tilde{P},\tilde{E}\} = \kappa \tag{5.56}$$

and all other Poisson brackets are vanishing. Here we neglected terms involving a derivative of the signum function since these terms only contributes when $E = 0$. The quantity $\tilde{P}$ can be defined as long as $E \neq 0$ which is given in the classical theory. In the quantum theory we will modify the corresponding Schwartz functions as before by means of the function $s_\epsilon(\tilde{E})$. In these variables C^g_Λ takes the form

$$C^g_\Lambda = -\frac{1}{8\kappa}\tilde{P}^2 + \frac{3}{4\kappa}\Lambda\tilde{E}^2 \tag{5.57}$$

which is almost of form of a harmonic oscillator except that it has the wrong sign in the kinetic term. We now choose a representation on $L_2(\mathbb{R}, d\tilde{E})$ in which P is implemented as $-i\ell_P^2\partial/\partial\tilde{E}$ and $\tilde{E}$ is represented as a multiplication operator. The quantum Hamiltonian is thus

$$\hat{C}^g_\Lambda = \frac{\ell_P^4}{8\kappa}\frac{\partial^2}{\partial\tilde{E}^2} + \frac{3}{4\kappa}\Lambda\hat{\tilde{E}}^2 \tag{5.58}$$

This motivates to construct coherent states which simply those of that harmonic oscillator which results when switching to a positive sign in front of the kinetic term in 5.57. It is then obvious that these coherent states approximate the operators $\tilde{P}$ and $\tilde{E}$ semiclassically well i.e.

$$\langle\tilde{\Psi}^\epsilon_{Z_0}\,,\,\tilde{E}\tilde{\Psi}^\epsilon_{Z_0}\rangle = \tilde{E}_0 = \sqrt{3V_0}\frac{2}{3}\mathrm{sgn}(E_0)E_0^{\frac{3}{2}}, \quad \langle\tilde{\Psi}^\epsilon_{Z_0}\,,\,\tilde{P}\tilde{\Psi}^\epsilon_{Z_0}\rangle = \tilde{P}_0 = \sqrt{3V_0}\frac{P_0}{\sqrt{|E_0|}} \tag{5.59}$$

as long as we are not close to a classical singularity $E_0 = 0$ where the function s_ϵ is constant, see also discussion in section 4. Therefore these states will also give the correct classical value for the expectation value of C^g_Λ.

The exact Born Oppenheimer programme now consists in the following: We are looking for the (generalised) eigenvalues μ and the system of corresponding eigenfunctions $n \mapsto \xi_n(E)$ of the equation

$$[C^g_\Lambda + \lambda_n]\xi_n = \mu\chi_n \tag{5.60}$$

where λ_n is a multiplication operator. This can now be done by using quantum mechanical perturbation theory: Multiply this equation by κ and treat κC^g_Λ as the "free" Hamiltonian and $\kappa\lambda_n$ as a perturbation potential (notice that κ is an extremely small parameter. Assuming that the spectral problem for C^g_Λ has been solved we can then compute the μ and the corresponding systems ξ_n. This is beyond the scope of the present paper and we plan to come back to this problem in a future publication. From the resulting eigenfunctions $\Psi_\mu(E,\phi)$ one then has to select those which are peaked on a given spatial geometry E_0 and a given extrinsic curvature P_0.
The approximate (semiclassical) programme consists in choosing coherent states $\chi_n := \psi_{Z_n}$ with $\Re(Z_n) = E_0,\ \Im(Z_n) = P_n$ for some given E_0 and to satisfy the eigenvalue equation in the sense of expectation values only which results in conditions which are approximately given by

$$C^g_\Lambda(E_0, P_n) + \lambda_n(E_0) = \mu \tag{5.61}$$

which in this case can be trivially solved for P_n.

6 Conclusions and Outlook

In this article we investigated the question of how one can understand (matter) QFT on a fixed background starting from first principles, i.e. from a theory where gravitational and matter degrees of freedom are treated as quantum

variables. In contrast to ordinary QFT on (curved) spacetimes such a theory describes quantum matter on a quantum spacetime. We performed an analysis using Born–Oppenheimer methods and it turns out that (matter) QFT on a fixed background emerges as an intermediate step when trying to find solutions for the full quantum theory. Within the Born–Oppenheimer approximation one assumes that the change in spacetime geometry is very small on typical timescales of particle interactions. This is obviously the case in situations where one expects ordinary QFT to be valid but will be violated in the Planck scale regime where the notion of a smooth background spacetime is not appropriate any longer because quantum fluctuations of the quantum geometry can not be neglected even for very short timescales. However, within its regime of applicability, this assumption means that the geometrical operators, which encode the gravitational degrees of freedom, can be well approximated by their classical counterparts as long as we are only interested in matter dynamics.
Classically, gravity couples via the spatial metric to standard matter (at least for scalar fields and gauge bosons) and not via its canonical momentum. In order to be able to apply the Born–Oppenheimer approximation scheme we need to require that the operator analog of the spatial metric can be well approximated by the classical metric in some regime. This is surely the case when geometrical quantities are represented as multiplication operators, but is not obvious when these quantities are represented as derivative operators because the derivative operators might not commute among each other.
In ordinary LQG we are exactly in the latter situation: The flux operators, which are the quantum versions of densitised triads smeared along 2–dimensional surfaces, are represented as derivative operators on the Ashtekar–Lewandowski Hilbert space $\mathcal{H}_{\rm AL}$ and furthermore form a non–commuting operator algebra. Thus it is not immediately obvious in which sense they can be used to approximate a classical geometry in the sense of the Born–Oppenheimer approximation.

This led us to consider a new algebra of operators and a representation thereof which is similar to ordinary LQG in the sense that its Hilbert space carries a basis of states which are defined on certain graphs but deviates from LQG in the choice of representation: We worked in a representation where cotriads are represented as ordinary multiplication operators and their canonical momenta as derivation operators. By this we obtain a theory of quantum gravity defined on an algebraic graph to which we can directly apply the Born – Oppenheimer methods because the matter degrees of freedom now couple to ordinary multiplication operators. Since the quantum theory defined in this article is an AQG – inspired quantisation on a fixed algebraic graph where the notion of a manifold and therefore a classical spacetime has a priori no interpretation, we need to analyse its semiclassical sector in order to be able to make contact to ordinary QFT.
Using these methods we saw how QFT on a fixed background emerges out of a full quantum theory and moreover were able to give a conceptually framework that in principle allows to calculate backreactions from quantum matter onto the quantum dynamics within the gravitational sector.
However, from a practical point of view there is little chance that one will be able to perform these calculations analytically as they require a substantial amount of input concerning the quantum dynamics from the gravitational sector. This means that in order to be able to fully understand the problem and to be able to calculate corrections to particle physics processes from first principles one will need to get more control over the solution space of the gravitational Hamiltonian.
As in previous contributions to this subject, there exists a 'semiclassical route', i.e. one can consider states that can be written as a tensor product of certain appropriately defined Fock states for the matter sector on a fixed background and coherent states (which are sharply peaked on a classical geometry around this background) for the gravitational sectors. However, while backreaction effects can be taken into account in the sense of expectation values, these states are not eigenstates of the quantum Hamiltonian, therefore do not shed light on the true spectral problem and in particular do not really follow the Born – Oppenheimer scheme.
Our analysis aims at a complete understanding of particle scattering processes within a full theory of quantum gravity for which some knowledge on the spectrum of the physical Hamiltonian and in particular its vacuum sector might prove useful. In order to illustrate our method we tested the framework using a FRW minisuperspace model for this quantisation in section 5.3. There we could see more explicitly how the presence of quantum matter fields can be taken into account into the dynamics of the gravitational sector using the Born–Oppenheimer method.
Of course, in principle there is no obstruction to use this method to analyse how a quantum field theory for matter emerges out of a quantum gravitational setting as loop quantum cosmology. Contrary to full LQG the geometrical operators are represented as multiplication operators in loop quantum cosmology as well and it will be interesting to study the relation[24] between our method and the results obtained in [46].
It is challenging to generalise the Born Oppenheimer method to representations for which the three geometry is represented by non commuting operators such as the non–commuting flux operators of LQG. But this requires a completely

[24]LQC is a based on a representation that mimics that of LQC, it is not of Schrödinger type in contrast to the representation used in this paper.

new input.

Acknowledgements

JT wants to thank Aristide Baratin, Carla Cederbaum, Bianca Dittrich and Hanno Sahlmann for valuable discussions. KG wants to thank Stefan Hofmann for discussions. The part of the research performed at the Perimeter Institute for Theoretical Physics was supported in part by funds from the Government of Canada through NSERC and from the Province of Ontario through MEDT.

A Infinite tensor product Hilbert spaces

Infinite tensor product (ITP) Hilbert spaces emerge as a generalisation of the tensor product of a finite number of Hilbert spaces. They were already studied by von Neumann in the thirties and we want to give a brief summary of the material that is relevant for our purposes. For a more detailed discussion see for instance [12] or von Neumann's original article [62].

Let $\mathcal{H}_i$ be Hilbert spaces and $i \in \mathcal{I}$ for some index set $\mathcal{I}$. In general one can allow for arbitrary cardinality of $\mathcal{I}$ but for our purposes it will be enough to consider $|\mathcal{I}| = \aleph$.

The ITP Hilbert space $\mathcal{H}_\otimes$ is defined as the closure of the finite linear span of vectors of the form $\otimes_f := \otimes_i f_i$ for $f_i \in \mathcal{H}_i$ with respect to the inner product

$$\langle \otimes_f, \otimes'_f \rangle_{\mathcal{H}_\otimes} := \prod_i \langle f_i, f'_i \rangle_{\mathcal{H}_i} \quad . \tag{A.1}$$

Here the infinite product $\prod_i z_i$ of complex numbers $z_i = |z_i| \exp(i\phi_i)$ is defined via

$$\prod_i z_i := \left[\prod_i |z_i|\right] \exp\left(i \sum_i \phi_i\right) \tag{A.2}$$

if the absolute value $\prod_i |z_i|$ and the phase $\sum_i \phi_i$ both converge. In this case we call $z := \prod_i z_i$ convergent. Otherwise we set $\prod_i z_i = 0$. If the absolute value $\prod_{i\in\mathcal{I}} |z_i|$ converges but not necessarily the phase, we say that $\prod_{i\in\mathcal{I}} z_i$ is *quasi convergent*. One can show that for $z = \prod_i z_i \neq 0$ given any $\delta > 0$ there exists a finite subset $\mathcal{I}_\delta \subset \mathcal{I}$ such that $|z - \prod_{i\in\mathcal{I}_\delta}| < \delta$. We will only be interested in elements $\otimes_f$ that have non–vanishing norm.

On $\mathcal{H}_\otimes$ one can define different notions of equivalence: Two vectors $\otimes_f$ and $\otimes_{f'}$ are said to be *strongly equivalent* if and only if $|\sum_{i\in\mathcal{I}} \langle f_i, f'_i \rangle_{\mathcal{H}_i} - 1|$ converges. Strong equivalence classes of vectors $\otimes_f$ will be denoted by $[f]$. It follows that $\langle \otimes_f, \otimes_{f'} \rangle_{\mathcal{H}_\otimes} = 0$ if either $[f] \neq [f']$ or $[f] = [f']$ and $\langle f_i, f'_i \rangle_{\mathcal{H}_i} = 0$ for at least one i.

If we set $(z \cdot f)_i := z_i f_i$ then $\otimes_{z\cdot f} = (\prod_{i\in\mathcal{I}}) \otimes_f$ fails to hold if $\prod_{i\in\mathcal{I}} z_i$ is not convergent. Provided that there exists such z we say that $\otimes_f$ and $\otimes_{f'}$ are *weakly equivalent* if and only if $[z \cdot f] = [f]$. These weak equivalence classes will be denoted by (f). One can show that to functions $\otimes_f, \otimes_{f'}$ are in the same weak equivalence class if and only if $\sum_{i\in\mathcal{I}} |\ |\langle f_i, f'_i \rangle_{\mathcal{H}_i}| - 1|$ converges. Hence, strong equivalence implies weak equivalence.

One can show that the closure of the span of all vectors in the same strong equivalence class $[f]$, denoted by $\mathcal{H}^\otimes_{[f]}$ is separable, consisting of the completion of the finite linear span of vectors of the form $\otimes_{f'}$ where $f'_i = f_i$ for all but finitely many i. The whole ITP Hilbert space is the direct sum (in general an infinite sum) of these separable subspaces, $\mathcal{H}_\otimes = \sum_{[f]} \mathcal{H}^\otimes_{[f]}$. Let also $\mathcal{H}^\otimes_{(f)}$ be the closure of the finite linear span of vectors $\otimes_{f'}$ with $(f') = (f)$. Then the strong equivalence subspaces of $\mathcal{H}^\otimes_{(f)}$ are unitarily equivalent with corresponding unitary operators of the form $U_z \otimes_f := \otimes_{z\cdot f}$ with $\prod_{i\in\mathcal{I}} z_i$ quasi convergent.

We see that although the ITP Hilbert space $\mathcal{H}_\otimes$ is not separable and might look "too large" at first sight, it decomposes into a direct sum of separable Hilbert spaces $\mathcal{H}^\otimes_{[f]}$ when taking into account strong equivalence classes of vectors $[f]$. In QFT on a fixed curved background the theory is naturally formulated on a separable Hilbert space, one for each background spacetime. So in a sense these ITP Hilbert spaces emerge as a natural generalisation when going from QFT on CS to quantum gravity, which should incorporate *QFT on all possible backgrounds simultaneously*.

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