ON THE CLASSICAL GEOMETRY OF EMBEDDED MANIFOLDS IN TERMS OF NAMBU BRACKETS

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Abstract. We prove that many aspects of the differential geometry of embedded Riemannian manifolds can be formulated in terms of a multi-linear algebraic structure on the space of smooth functions. In particular, we find algebraic expressions for Weingarten's formula, the Ricci curvature and the Codazzi-Mainardi equations.

1. Introduction

Given a manifold Σ, it is interesting to study in what ways information about the geometry of Σ can be extracted as algebraic properties of the algebra of smooth functions $C^\infty(\Sigma)$. In case Σ is a Poisson manifold, this algebra has a second (apart from the commutative multiplication of functions) bilinear (non-associative) algebra structure realized as the Poisson bracket. The bracket is compatible with the commutative multiplication via Leibniz rule, thus carrying the basic properties of a derivation.

On a surface Σ, with local coordinates u^1 and u^2, one may define

$$\{f,h\} = \frac{1}{\sqrt{g}}\left(\frac{\partial f}{\partial u^1}\frac{\partial h}{\partial u^2} - \frac{\partial h}{\partial u^1}\frac{\partial f}{\partial u^2}\right),$$

where g is the determinant of the induced metric tensor, and one can readily check that $\big(C^\infty(\Sigma), \{\cdot,\cdot\}\big)$ is a Poisson algebra. Having only this very particular combination of derivatives at hand, it seems at first unlikely that one can encode geometric information of Σ in Poisson algebraic expressions. Surprisingly, it turns out that many differential geometric quantities can be computed in a completely algebraic way, cp. Theorem 3.6 and Theorem 5.6. For instance, the Gaussian curvature of a surface embedded in $\mathbb{R}^3$ can be written as

$$K = -\frac{1}{2}\sum_{i,j=1}^{3}\{x^i,n^j\}\{x^j,n^i\},$$

where $x^i(u^1,u^2)$ are the embedding coordinates and $n^i(u^1,u^2)$ are the components of a unit normal vector at each point of Σ.

For a general n-dimensional manifold Σ, we are led to consider Nambu brackets [Nam73], i.e. multi-linear n-ary maps from $C^\infty(\Sigma)\times\cdots\times C^\infty(\Sigma)$ to $C^\infty(\Sigma)$, defined by

$$\{f_1,\ldots,f_n\} = \frac{1}{\sqrt{g}}\varepsilon^{a_1\cdots a_n}\big(\partial_{a_1}f_1\big)\cdots\big(\partial_{a_n}f_n\big).$$

Our initial motivation for studying this problem came from matrix regularizations of Membrane Theory. Classical solutions in Membrane Theory are 3-manifolds with vanishing mean curvature in $\mathbb{R}^{1,d}$. Considering one of the coordinates to be time, the problem can also be formulated in a dynamical way as surfaces sweeping out volumes of vanishing mean curvature. In this context, a regularization was introduced replacing the infinite dimensional function algebra on the surface by an algebra of $N \times N$ matrices [Hop82]. If we let $T^{(N)}$ be a linear map from smooth functions to hermitian $N \times N$ matrices, the regularization is required to fulfill

$$\lim_{N\to\infty} \left|\left|T^{(N)}(f)T^{(N)}(g) - T^{(N)}(fg)\right|\right| = 0,$$
$$\lim_{N\to\infty} \left|\left|N[T^{(N)}(f), T^{(N)}(h)] - iT^{(N)}(\{f,h\})\right|\right| = 0,$$

where $||\cdot||$ denotes the operator norm, and therefore it is natural to regularize the system by replacing (commutative) multiplication of functions by (non-commutative) multiplication of matrices and Poisson brackets of functions by commutators of matrices.

Although we may very well consider $T^{(N)}(\frac{\partial f}{\partial u^1})$, its relation to $T^{(N)}(f)$ is in general not simple. However, the particular combination of derivatives in $T^{(N)}(\{f,h\})$ is expressed in terms of a commutator of $T^{(N)}(f)$ and $T^{(N)}(h)$. In the context of Membrane Theory, it is desirable to have geometrical quantities in a form that can easily be regularized, which is the case for any expression constructed out of multiplications and Poisson brackets.

The paper is organized as follows: In Section 2 we introduce the relevant notation by recalling some basic facts about submanifolds. In Section 3 we formulate several basic differential geometric objects in terms of Nambu brackets, and in Section 4 we provide a construction of a set of orthonormal basis vectors of the normal space. Section 5 is devoted to the study of the Codazzi-Mainardi equations and how one can rewrite them in terms of Nambu brackets. Finally, in Section 6 we study the particular case of surfaces, for which many of the introduced formulas and concepts are particularly nice and in which case one can construct the complex structure in terms of Poisson brackets.

2. Preliminaries

To introduce the relevant notations, we shall recall some basic facts about submanifolds, in particular Gauss' and Weingarten's equations (see e.g. [KN96a, KN96b] for details). For $n \geq 2$, let Σ be a n-dimensional manifold embedded in a Riemannian manifold M with $\dim M = n + p \equiv m$. Local coordinates on M will be denoted by $x^1, \ldots, x^m$, local coordinates on Σ by $u^1, \ldots, u^n$, and we regard $x^1, \ldots, x^m$ as being functions of $u^1, \ldots, u^n$ providing the embedding of Σ in M. The metric tensor on M is denoted by $\bar{g}_{ij}$ and the induced metric on Σ by g_{ab}; indices i, j, k, l, n run from 1 to m, indices a, b, c, d, p, q run from 1 to n and indices A, B, C, D run from 1 to p. Furthermore, the covariant derivative and the Christoffel symbols in M will be denoted by $\bar{\nabla}$ and $\bar{\Gamma}^i_{jk}$ respectively.

The tangent space $T\Sigma$ is regarded as a subspace of the tangent space TM and at each point of Σ one can choose $e_a = (\partial_a x^i)\partial_i$ as basis vectors in $T\Sigma$, and in this basis we define $g_{ab} = \bar{g}(e_a, e_b)$. Moreover, we choose a set of normal vectors N_A, for $A = 1, \ldots, p$, such that $\bar{g}(N_A, N_B) = \delta_{AB}$ and $\bar{g}(N_A, e_a) = 0$.

The formulas of Gauss and Weingarten split the covariant derivative in M into tangential and normal components as

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X,Y) \tag{2.1}$$

$$\bar{\nabla}_X N_A = -W_A(X) + D_X N_A \tag{2.2}$$

where $X,Y \in T\Sigma$ and $\nabla_X Y$, $W_A(X) \in T\Sigma$ and $\alpha(X,Y)$, $D_X N_A \in T\Sigma^\perp$. By expanding $\alpha(X,Y)$ in the basis $\{N_1,\dots,N_p\}$ one can write (2.1) as

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{A=1}^{p} h_A(X,Y) N_A, \tag{2.3}$$

and we set $h_{A,ab} = h_A(e_a,e_b)$. From the above equations one derives the relation

$$h_{A,ab} = -\bar{g}\big(e_a, \bar{\nabla}_b N_A\big), \tag{2.4}$$

as well as Weingarten's equation

$$h_A(X,Y) = \bar{g}\big(W_A(X),Y\big), \tag{2.5}$$

which implies that $(W_A)^a_b = g^{ac} h_{A,cb}$, where g^{ab} denotes the inverse of g_{ab}.

From formulas (2.1) and (2.2) one obtains Gauss' equation, i.e. an expression for the curvature R of Σ in terms of the curvature $\bar{R}$ of M, as

$$\begin{aligned} g\big(R(X,Y)Z,V\big) = \bar{g}\big(\bar{R}(X,Y)Z,V\big) - \bar{g}\big(\alpha(X,Z),\alpha(Y,V)\big) \\ + \bar{g}\big(\alpha(Y,Z),\alpha(X,V)\big), \end{aligned} \tag{2.6}$$

where $X,Y,Z,V \in T\Sigma$. As we shall later on consider the Ricci curvature, let us note that (2.6) implies

$$\mathcal{R}^p_b = g^{pd} g^{ac} \bar{g}\big(\bar{R}(e_c,e_d)e_b,e_a\big) + \sum_{A=1}^{p} \Big[(W_A)^a_a (W_A)^p_b - (W_A^2)^p_b\Big] \tag{2.7}$$

where $\mathcal{R}$ is the Ricci curvature of Σ considered as a map $T\Sigma \to T\Sigma$. We also recall the mean curvature vector, defined as

$$H = \frac{1}{n} \sum_{A=1}^{p} \big(\operatorname{tr} W_A\big) N_A. \tag{2.8}$$

3. Algebraic formulation

In this section we will prove that one can express many aspects of the differential geometry of an embedded manifold Σ in terms of a Nambu bracket introduced on $C^\infty(\Sigma)$. Let $\rho : \Sigma \to \mathbb{R}$ be an arbitrary non-vanishing density and define

$$\{f_1,\dots,f_n\} = \frac{1}{\rho} \varepsilon^{a_1\cdots a_n} \big(\partial_{a_1} f_1\big) \cdots \big(\partial_{a_n} f_n\big) \tag{3.1}$$

for all $f_1,\dots,f_n \in C^\infty(\Sigma)$, where $\varepsilon^{a_1\cdots a_n}$ is the totally antisymmetric Levi-Civita symbol with $\varepsilon^{12\cdots n} = 1$. Together with this multi-linear map, Σ is a Nambu-Poisson manifold.

The above Nambu bracket arises from the choice of a volume form on Σ. Namely, let ω be a volume form and define $\{f_1,\dots,f_n\}$ via the formula

$$\{f_1,\dots,f_n\}\omega = df_1 \wedge \cdots \wedge df_n. \tag{3.2}$$

Writing $\omega = \rho\, du^1 \wedge \cdots \wedge du^n$ in local coordinates, and evaluating both sides of (3.2) on the tangent vectors $\partial_{u^1},\dots,\partial_{u^n}$ gives

$$\{f_1,\dots,f_n\} = \frac{1}{\rho} \det\left(\frac{\partial(f_1,\dots,f_n)}{\partial(u^1,\dots,u^n)}\right) = \frac{1}{\rho} \varepsilon^{a_1\cdots a_n} \big(\partial_{a_1} f_1\big) \cdots \big(\partial_{a_n} f_n\big).$$

To define the objects which we will consider, it is convenient to introduce some notation. Let $x^1(u^1,\dots,u^n),\dots,x^m(u^1,\dots,u^n)$ be the embedding coordinates of

Σ into M, and let $n^i_A(u^1,\dots,u^n)$ denote the components of the orthonormal vectors N_A, normal to $T\Sigma$. Using multi-indices $I = i_1\cdots i_{n-1}$ and $\vec{a} = a_1\cdots a_{n-1}$ we define

$$\begin{aligned}\{f,\vec{x}^I\} &\equiv \{f,x^{i_1},x^{i_2},\dots,x^{i_{n-1}}\}\\ \{f,\vec{n}^I_A\} &\equiv \{f,n^{i_1}_A,n^{i_2}_A,\dots,n^{i_{n-1}}_A\},\end{aligned}$$

together with

$$\begin{aligned}&\partial_{\vec{a}}\vec{x}^I \equiv \big(\partial_{a_1}x^{i_1}\big)\big(\partial_{a_2}x^{i_2}\big)\cdots\big(\partial_{a_{n-1}}x^{i_{n-1}}\big)\\ &\big(\bar\nabla_{\vec{a}}\vec{n}_A\big)^I \equiv \big(\bar\nabla_{a_1}N_A\big)^{i_1}\big(\bar\nabla_{a_2}N_A\big)^{i_2}\cdots\big(\bar\nabla_{a_{n-1}}N_A\big)^{i_{n-1}}\\ &\bar g_{IJ} \equiv \bar g_{i_1j_1}\bar g_{i_2j_2}\cdots\bar g_{i_{n-1}j_{n-1}}.\end{aligned}$$

We now introduce the main objects of our study

$$\mathcal{P}^{iJ} = \frac{1}{\sqrt{(n-1)!}}\{x^i,\vec{x}^J\} = \frac{1}{\sqrt{(n-1)!}}\frac{\varepsilon^{a\vec{a}}}{\rho}\big(\partial_a x^i\big)\big(\partial_{\vec{a}}\vec{x}^J\big) \tag{3.3}$$

$$\mathcal{S}^{iJ}_A = \frac{(-1)^n}{\sqrt{(n-1)!}}\frac{\varepsilon^{a\vec{a}}}{\rho}\big(\partial_a x^i\big)\big(\bar\nabla_{\vec{a}}\vec{n}_A\big)^J \tag{3.4}$$

$$\mathcal{T}^{Ij}_A = \frac{(-1)^n}{\sqrt{(n-1)!}}\frac{\varepsilon^{\vec{a}a}}{\rho}\big(\partial_{\vec{a}}\vec{x}^I\big)\big(\bar\nabla_a N_A\big)^j \tag{3.5}$$

from which we construct

$$\big(\mathcal{P}^2\big)^{ik} = \mathcal{P}^{iI}\mathcal{P}^{kJ}\bar g_{IJ} \tag{3.6}$$

$$\big(\mathcal{B}_A\big)^{ik} = \mathcal{P}^{iI}(\mathcal{T}_A)^{Jk}\bar g_{IJ} \tag{3.7}$$

$$\big(\mathcal{S}_A\mathcal{T}_A\big)^{ik} = (\mathcal{S}_A)^{iI}(\mathcal{T}_A)^{Jk}\bar g_{IJ}. \tag{3.8}$$

By lowering the second index with the metric $\bar g$, we will also consider $\mathcal{P}^2$, $\mathcal{B}_A$ and $\mathcal{T}_A\mathcal{S}_A$ as maps $TM\to TM$. Note that both $\mathcal{S}_A$ and $\mathcal{T}_A$ can be written in terms of Nambu brackets, e.g.

$$\mathcal{T}^{Ij}_A = \frac{(-1)^n}{\sqrt{(n-1)!}}\Big[\{\vec{x}^I,n^j_A\} + \{\vec{x}^I,x^k\}\bar\Gamma^j_{kl}n^l_A\Big].$$

Let us now investigate some properties of the maps defined above. As it will appear frequently, we define

$$\gamma = \frac{\sqrt{g}}{\rho}. \tag{3.9}$$

It is useful to note that (cp. Proposition 3.3)

$$\gamma^2 = \sum_{i,I=1}^{m}\frac{1}{n!}\{x^i,\vec{x}^I\}\{\vec{x}^I,x^i\},$$

and to recall the cofactor expansion of the inverse of matrix:

Lemma 3.1. *Let g^{ab} denote the inverse of g_{ab} and $g = \det(g_{ab})$. Then*

$$gg^{ba} = \frac{1}{(n-1)!}\varepsilon^{aa_1\cdots a_{n-1}}\varepsilon^{bb_1\cdots b_{n-1}}g_{a_1b_1}g_{a_2b_2}\cdots g_{a_{n-1}b_{n-1}}. \tag{3.10}$$

Proposition 3.2. *For $X\in TM$ it holds that*

$$\mathcal{P}^2(X) = \gamma^2\bar g(X,e_a)g^{ab}e_b \tag{3.11}$$

$$\mathcal{B}_A(X) = -\gamma^2\bar g(X,\bar\nabla_a N_A)g^{ab}e_b \tag{3.12}$$

$$\mathcal{S}_A\mathcal{T}_A(X) = \gamma^2(\det W_A)\bar g(X,\bar\nabla_a N_A)h^{ab}_A e_b, \tag{3.13}$$

and for $Y \in T\Sigma$ one obtains

$$\mathcal{P}^2(Y) = \gamma^2 Y \tag{3.14}$$

$$\mathcal{B}_A(Y) = \gamma^2 W_A(Y) \tag{3.15}$$

$$\mathcal{S}_A\mathcal{T}_A(Y) = -\gamma^2(\det W_A)Y. \tag{3.16}$$

Proof. Let us provide a proof for equations (3.11) and (3.14); the other formulas can be proven analogously.

$$\begin{aligned}\mathcal{P}^2(X) = \mathcal{P}^{iI}\mathcal{P}^{jJ}\bar{g}_{IJ}\bar{g}_{jk}X^k\partial_i &= \frac{\varepsilon^{a\vec{a}}\varepsilon^{c\vec{c}}}{\rho^2(n-1)!}\big(\partial_a x^i\big)\big(\partial_{\vec{a}}x^I\big)\big(\partial_c x^j\big)\big(\partial_{\vec{c}}x^J\big)\bar{g}_{IJ}\bar{g}_{jk}X^k\partial_i\\ &= \frac{\varepsilon^{a\vec{a}}\varepsilon^{c\vec{c}}}{\rho^2(n-1)!}g_{a_1c_1}\cdots g_{a_{n-1}c_{n-1}}\big(\partial_a x^i\big)\big(\partial_c x^j\big)\bar{g}_{jk}X^k\partial_i\\ &= \gamma^2 g^{ac}\big(\partial_a x^i\big)\big(\partial_c x^j\big)\bar{g}_{jk}X^k\partial_i = \gamma^2\bar{g}(X,e_c)g^{ca}e_a.\end{aligned}$$

Choosing a tangent vector $Y = Y^c e_c$ gives immediately that $\mathcal{P}^2(Y) = \gamma^2 Y$. □

For a map $\mathcal{B} : TM \to TM$ we denote the trace by $\operatorname{Tr}\mathcal{B} \equiv \mathcal{B}^i_i$ and for a map $W : T\Sigma \to T\Sigma$ we denote the trace by $\operatorname{tr} W \equiv W^a_a$.

Proposition 3.3. *It holds that*

$$\frac{1}{n}\operatorname{Tr}\mathcal{P}^2 = \gamma^2 \tag{3.17}$$

$$\operatorname{Tr}\mathcal{B}_A = \gamma^2\operatorname{tr}W_A \tag{3.18}$$

$$\frac{1}{n}\operatorname{Tr}\mathcal{S}_A\mathcal{T}_A = -\gamma^2(\det W_A). \tag{3.19}$$

A direct consequence of Propositions 3.2 and 3.3 is that one can write the projection onto $T\Sigma$, as well as the mean curvature vector, in terms of Nambu brackets.

Proposition 3.4. *The map*

$$\gamma^{-2}\mathcal{P}^2 = \frac{n}{\operatorname{Tr}\mathcal{P}^2}\mathcal{P}^2 : TM \to T\Sigma \tag{3.20}$$

is the orthogonal projection of TM onto $T\Sigma$. Furthermore, the mean curvature vector can be written as

$$H = \frac{1}{\operatorname{Tr}\mathcal{P}^2}\sum_{A=1}^{p}\big(\operatorname{Tr}\mathcal{B}_A\big)N_A.$$

Proposition 3.2 tells us that $\gamma^{-2}\mathcal{B}_A$ equals the Weingarten map W_A, when restricted to $T\Sigma$. What is the geometrical meaning of $\mathcal{B}_A$ acting on a normal vector? It turns out that the maps $\mathcal{B}_A$ also provide information about the covariant derivative in the normal space. If one defines $(D_X)_{AB}$ through

$$D_X N_A = \sum_{B=1}^{p}(D_X)_{AB}N_B$$

for $X \in T\Sigma$, then one can prove the following relation to the maps $\mathcal{B}_A$.

Proposition 3.5. *For $X \in T\Sigma$ it holds that*

$$\bar{g}\big(\mathcal{B}_B(N_A), X\big) = \gamma^2\big(D_X\big)_{AB}. \tag{3.21}$$

Proof. For a vector $X = X^a e_a$, it follows from Weingarten's formula (2.2) that

$$(D_X)_{AB} = \bar{g}\big(\bar{\nabla}_X N_A, N_B\big).$$

On the other hand, with the formula from Proposition 3.2, one computes

$$\begin{aligned}\bar{g}\big(\mathcal{B}_B(N_A), X\big) &= -\gamma^2 \bar{g}\big(N_A, \bar{\nabla}_a N_B\big) g^{ab} g_{bc} X^c = -\gamma^2 \bar{g}\big(N_A, \bar{\nabla}_X N_B\big)\\ &= -\gamma^2 (D_X)_{BA} = \gamma^2 (D_X)_{AB}.\end{aligned}$$

The last equality is due to the fact that D is a covariant derivative, which implies that $0 = D_X \bar{g}(N_A, N_B) = \bar{g}(D_X N_A, N_B) + \bar{g}(N_A, D_X N_B)$. □

Thus, one can write Weingarten's formula as

$$\gamma^2 \bar{\nabla}_X N_A = -\mathcal{B}_A(X) + \sum_{B=1}^{p} \bar{g}\big(\mathcal{B}_B(N_A), X\big) N_B, \tag{3.22}$$

and since $h_A(X, Y) = \gamma^{-2} \bar{g}(\mathcal{B}_A(X), Y)$ Gauss' formula becomes

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{\gamma^2} \sum_{A=1}^{p} \bar{g}\big(\mathcal{B}_A(X), Y\big) N_A. \tag{3.23}$$

Let us now turn our attention to the curvature of Σ. Since Nambu brackets involve sums over all vectors in the basis of $T\Sigma$, one can not expect to find expressions for quantities that involve a choice of tangent plane, e.g. the sectional curvature (unless Σ is a surface). However, it turns out that one can write the Ricci curvature as an expression involving Nambu brackets.

Theorem 3.6. *Let $\mathcal{R}$ be the Ricci curvature of Σ, considered as a map $T\Sigma \to T\Sigma$. For any $X \in T\Sigma$ it holds that*

$$\mathcal{R}(X) = g^{pd} g^{ac} \bar{g}\big(\bar{R}(e_c, e_d)e_b, e_a\big) X^b e_p + \frac{1}{\gamma^4} \sum_{A=1}^{p} \Big[(\operatorname{Tr} \mathcal{B}_A)\mathcal{B}_A(X) - \mathcal{B}_A^2(X)\Big],$$

where $\bar{R}$ is the curvature tensor of M.

Proof. The Ricci curvature of Σ is defined as

$$\mathcal{R}^p_b = g^{ac} g^{pd} g\big(R(e_c, e_d)e_b, e_a\big)$$

and from Gauss' equation (2.6) it follows that

$$\mathcal{R}^p_b = g^{pd} g^{ac} \bar{g}\big(\bar{R}(e_c, e_d)e_b, e_a\big) + g^{ac} g^{pd} \sum_{A=1}^{p} \Big(h_{A,bd} h_{A,ac} - h_{A,bc} h_{A,ad}\Big).$$

Since $(W_A)^a_b = g^{ac} h_{A,cb}$ one obtains

$$\mathcal{R}^p_b = g^{ac} g^{pd} \bar{g}\big(\bar{R}(e_c, e_d)e_b, e_a\big) + \sum_{A=1}^{p} \Big[\big(\operatorname{tr} W_A\big)(W_A)^p_b - (W_A^2)^p_b\Big],$$

and as $\mathcal{B}_A(X) = \gamma^2 W_A(X)$ for any $X \in T\Sigma$, and $\operatorname{Tr} \mathcal{B}_A = \gamma^2 \operatorname{tr} W_A$, one has

$$\mathcal{R}(X) = g^{ac} g^{pd} \bar{g}\big(\bar{R}(e_c, e_d)e_b, e_a\big) X^b e_p + \frac{1}{\gamma^4} \sum_{A=1}^{p} \Big[\big(\operatorname{Tr} \mathcal{B}_A\big)\mathcal{B}_A(X) - \mathcal{B}_A^2(X)\Big]. \quad \square$$

4. Construction of normal vectors

The results in Section 3 involve Nambu brackets of the embedding coordinates and the components of the normal vectors. In this section we will prove that one can replace sums over normal vectors by sums of Nambu brackets of the embedding coordinates, thus providing expressions that do not involve normal vectors.

It will be convenient to introduce yet another multi-index; namely, we let $\alpha = i_1 \dots i_{p-1}$ consist of $p-1$ indices all taking values between 1 and m.

Proposition 4.1. *For any value of the multi-index α, the vector*

$$Z_\alpha = \frac{1}{\gamma\big(n!\sqrt{(p-1)!}\big)}\bar{g}^{ij}\varepsilon_{jk_1\cdots k_n\alpha}\{x^{k_1},\dots,x^{k_n}\}\partial_i, \tag{4.1}$$

where $\varepsilon_{i_1\cdots i_m}$ is the Levi-Civita tensor of M, is normal to $T\Sigma$, i.e. $\bar{g}(Z_\alpha, e_a) = 0$ for $a = 1, 2, \dots, n$. For hypersurfaces ($p = 1$), equation (4.1) defines a unique normal vector of unit length.

Proof. To prove that Z_α are normal vectors, one simply notes that

$$\gamma\big(n!\sqrt{(p-1)!}\big)\bar{g}(Z_\alpha, e_a) = \frac{1}{\rho}\varepsilon^{a_1\cdots a_n}\varepsilon_{jk_1\cdots k_n\alpha}\big(\partial_a x^j\big)\big(\partial_{a_1}x^{k_1}\big)\cdots\big(\partial_{a_n}x^{k_n}\big) = 0,$$

since the $n+1$ indices $a, a_1, \dots, a_n$ can only take on n different values and since $(\partial_a x^j)(\partial_{a_1}x^{k_1})\cdots(\partial_{a_n}x^{k_n})$ is contracted with $\varepsilon_{jk_1\cdots k_n\alpha}$ which is completely anti-symmetric in $j, k_1, \dots, k_n$. Let us now calculate $|Z|^2 \equiv \bar{g}(Z, Z)$ when $p = 1$. Using that[1]

$$\varepsilon_{ik_1\cdots k_n}\varepsilon^{il_1\cdots l_n} = \delta^{[l_1}_{[k_1}\cdots\delta^{l_n]}_{k_n]}$$

one obtains

$$\begin{aligned}
|Z|^2 &= \frac{1}{\gamma^2 n!^2}\bar{g}_{l_1l_1'}\cdots\bar{g}_{l_nl_n'}\varepsilon_{ik_1\cdots k_n}\varepsilon^{il_1\cdots l_n}\{x^{k_1},\dots,x^{k_n}\}\{x^{l_1'},\dots,x^{l_n'}\}\\
&= \frac{1}{\gamma^2 n!^2}\bar{g}_{l_1l_1'}\cdots\bar{g}_{l_nl_n'}\delta^{[l_1}_{[k_1}\cdots\delta^{l_n]}_{k_n]}\{x^{k_1},\dots,x^{k_n}\}\{x^{l_1'},\dots,x^{l_n'}\}\\
&= \frac{1}{\gamma^2 n!}\{x^{l_1},\dots,x^{l_n}\}\bar{g}_{l_1l_1'}\cdots\bar{g}_{l_nl_n'}\{x^{l_1'},\dots,x^{l_n'}\}\\
&= \frac{1}{\gamma^2 n!}(n-1)!\operatorname{Tr}\mathcal{P}^2 = \frac{1}{\gamma^2 n!}(n-1)!n\gamma^2 = 1,
\end{aligned}$$

which proves that Z has unit length. □

If the codimension is greater than one, Z_α defines more than p non-zero normal vectors that do not in general fulfill any orthonormality conditions. In principle, one can now apply the Gram-Schmidt orthonormalization procedure to obtain a set of p orthonormal vectors. However, it turns out that one can use Z_α to construct another set of normal vectors, avoiding explicit use of the Gram-Schmidt procedure; namely, introduce

$$\mathcal{Z}_\alpha^\beta = \bar{g}(Z_\alpha, Z^\beta),$$

and consider it as a matrix over multi-indices α and β. As such, the matrix is symmetric (with respect to $\bar{g}_{\alpha\beta} \equiv \bar{g}_{i_1j_1}\cdots\bar{g}_{i_{p-1}j_{p-1}}$) and we let E_α, μ_α denote orthonormal eigenvectors (i.e. $\bar{g}_{\delta\sigma}E^\delta_\alpha E^\sigma_\beta = \delta_{\alpha\beta}$) and their corresponding eigenvalues.

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[1] In our convention, no combinatorial factor is included in the antisymmetrization; for instance, $\delta^{[i}_{[k}\delta^{j]}_{l]} = \delta^i_k\delta^j_l - \delta^i_l\delta^j_k$.

Using these eigenvectors to define

$$\hat{N}_\alpha = E_\alpha^\beta Z_\beta$$

one finds that $\bar{g}(\hat{N}_\alpha, \hat{N}_\beta) = \mu_\alpha \delta_{\alpha\beta}$, i.e. the vectors are orthogonal.

Proposition 4.2. *For $\mathcal{Z}_\alpha^\beta = \bar{g}_{ij} Z_\alpha^i Z^{j\beta}$ it holds that*

$$\mathcal{Z}_\alpha^\delta \mathcal{Z}_\delta^\beta = \mathcal{Z}_\alpha^\beta \tag{4.2}$$

$$\mathcal{Z}_\alpha^\alpha = p. \tag{4.3}$$

Proof. Both statements can be easily proven once one has the following result

$$Z_\alpha^i Z^{j\alpha} = \bar{g}^{ij} - \frac{1}{\gamma^2}\big(\mathcal{P}^2\big)^{ij},$$

which is obtained by using that

$$\varepsilon_{kk_1\cdots k_n\alpha}\varepsilon^{ll_1\cdots l_n\alpha} = (p-1)!\Big(\delta^{[l}_{[k}\delta^{l_1}_{k_1}\cdots\delta^{l_n]}_{k_n]}\Big).$$

Formula (4.3) is now immediate, and to obtain (4.2) one notes that since $\mathcal{Z}_\alpha \in T\Sigma^\perp$ it holds that $\mathcal{P}^2(\mathcal{Z}_\alpha) = 0$, due to the fact that $\mathcal{P}^2$ is proportional to the projection onto $T\Sigma$. □

From Proposition 4.2 it follows that an eigenvalue of $\mathcal{Z}$ is either 0 or 1, which implies that $\hat{N}_\alpha = 0$ or $\bar{g}(\hat{N}_\alpha, \hat{N}_\alpha) = 1$, and that the number of non-zero vectors is $\operatorname{Tr}\mathcal{Z} = \mathcal{Z}_\alpha^\alpha = p$. Hence, the p non-zero vectors among $\hat{N}_\alpha$ constitute an orthonormal basis of $T\Sigma^\perp$, and it follows that one can replace any sum over normal vectors N_A by a sum over the multi-index of $\hat{N}_\alpha$. As an example, let us work out some explicit expressions in the case when $M = \mathbb{R}^m$.

Proposition 4.3. *Assume that $M = \mathbb{R}^m$ and that all repeated indices are summed over. For any $X \in T\Sigma$ one has*

$$\sum_{A=1}^p \big(\operatorname{Tr}\mathcal{B}_A\big)\mathcal{B}_A(X)^i = \frac{\varepsilon_{jj'K\alpha}\varepsilon_{klL\alpha}}{\gamma^2 c(n,p)^2}\{x^j, \vec{x}^I\}\{\vec{x}^I, \{x^{j'}, \vec{x}^K\}\}\{x^i, \vec{x}^J\}\{\vec{x}^J, \{x^l, \vec{x}^L\}\}X^k$$

$$\sum_{A=1}^p \mathcal{B}_A^2(X)^i = \frac{\varepsilon_{jj'K\alpha}\varepsilon_{klL\alpha}}{\gamma^2 c(n,p)^2}\{x^i, \vec{x}^I\}\{\vec{x}^I, \{x^{j'}, \vec{x}^K\}\}\{x^j, \vec{x}^J\}\{\vec{x}^J, \{x^l, \vec{x}^L\}\}X^k$$

$$\sum_{A=1}^p \big(\operatorname{Tr}\mathcal{B}_A\big)N_A = \frac{(-1)^n}{n}\frac{\varepsilon_{ikK\alpha}\varepsilon_{jlL\alpha}}{\gamma^2 c(n,p)^2}\{x^i, \vec{x}^I\}\{\vec{x}^I, \{x^k, \vec{x}^K\}\}\{x^l, \vec{x}^L\}\partial_j$$

where

$$c(n,p) = n!(n-1)!\sqrt{(p-1)!}.$$

Proof. Let us prove the formula involving $(\operatorname{Tr}\mathcal{B}_A)\mathcal{B}_A(X)$; the other formulas can be proven analogously. For $\mathbb{R}^m$ one has

$$\sum_A \big(\operatorname{Tr}\mathcal{B}_A\big)\mathcal{B}_A(X)^i = \frac{1}{(n-1)!^2}\sum_A \{x^j, \vec{x}^I\}\{\vec{x}^I, n_A^j\}\{x^i, \vec{x}^J\}\{\vec{x}^J, n_A^k\}X^k,$$

and since the non-zero vectors in the set $\{\hat{N}_\alpha\}$ consist of exactly p orthonormal vectors one can write

$$\begin{aligned}\sum_A \big(\operatorname{Tr}\mathcal{B}_A\big)\mathcal{B}_A(X)^i &= \frac{1}{(n-1)!^2}\sum_\alpha \{x^j, \vec{x}^I\}\{\vec{x}^I, \hat{N}_\alpha^j\}\{x^i, \vec{x}^J\}\{\vec{x}^J, \hat{N}_\alpha^k\}X^k \\ &= \frac{1}{(n-1)!^2}\sum_\alpha \{x^j, \vec{x}^I\}\{\vec{x}^I, E_\alpha^\beta Z_\beta^j\}\{x^i, \vec{x}^J\}\{\vec{x}^J, E_\alpha^\epsilon Z_\epsilon^k\}X^k.\end{aligned}$$

Now, one notes that

$$\sum_j \{x^j, \vec{x}^I\}\{\vec{x}^I, E^\beta_\alpha Z^j_\beta\} = \sum_j E^\beta_\alpha \{x^j, \vec{x}^I\}\{\vec{x}^I, Z^j_\beta\}$$
$$\sum_i \{x^i, \vec{x}^J\}\{\vec{x}^J, E^\epsilon_\alpha Z^k_\epsilon\}X^k = \sum_i E^\epsilon_\alpha \{x^j, \vec{x}^I\}\{\vec{x}^I, Z^k_\epsilon\}X^k$$

since the terms with Z^j_β and Z^k_ϵ outside the Poisson bracket vanish due to the appearance of a scalar product with a tangent vector. Thus, one obtains

$$\sum_A \big(\operatorname{Tr}\mathcal{B}_A\big)\mathcal{B}_A(X)^i = \frac{1}{(n-1)!^2}\sum_\alpha E^\beta_\alpha E^\epsilon_\alpha \{x^j, \vec{x}^I\}\{\vec{x}^I, Z^j_\beta\}\{x^i, \vec{x}^J\}\{\vec{x}^J, Z^k_\epsilon\}X^k,$$

and since $\sum_\alpha E^\beta_\alpha E^\epsilon_\alpha = \delta^{\beta\epsilon}$ the result is

$$\sum_A \big(\operatorname{Tr}\mathcal{B}_A\big)\mathcal{B}_A(X)^i = \frac{1}{(n-1)!^2}\sum_\alpha \{x^j, \vec{x}^I\}\{\vec{x}^I, Z^j_\alpha\}\{x^i, \vec{x}^J\}\{\vec{x}^J, Z^k_\alpha\}X^k,$$

from which the statement follows by inserting the definition of Z_α. □

For hypersurfaces in $\mathbb{R}^{n+1}$, the "Theorema Egregium" states that the determinant of the Weingarten map, i.e the "Gaussian curvature", is an invariant (up to a sign when Σ is odd-dimensional) under isometries (this is in fact also true for hypersurfaces in a manifold of constant sectional curvature). From Proposition 3.3 we know that one can express $\det W_A$ in terms of $\operatorname{Tr}\mathcal{S}_A\mathcal{T}_A$.

Proposition 4.4. *Let Σ be a hypersurface in $\mathbb{R}^{n+1}$ and let W denote the Weingarten map with respect to the unit normal*

$$Z = \frac{1}{\gamma n!}\bar{g}^{ij}\varepsilon_{jkK}\{x^k, \vec{x}^K\}.$$

Then one can write $\det W$ *as*

$$\det W = -\frac{1}{\gamma(\gamma n!)^{n+1}}\sum \varepsilon_{ilL}\varepsilon_{j_1k_1K_1}\cdots\varepsilon_{j_{n-1}k_{n-1}K_{n-1}}$$
$$\times \{x^i, \{x^{k_1}, \vec{x}^{K_1}\}, \ldots, \{x^{k_{n-1}}, \vec{x}^{K_{n-1}}\}\}\{\vec{x}^J, \{x^l, \vec{x}^L\}\}.$$

In fact, one can express all the elementary symmetric functions of the principle curvatures in terms of Nambu brackets as follows: The k'th elementary symmetric function of the eigenvalues of W is given as the coefficient of t^k in $\det(W - t\mathbb{1})$. Since $\mathcal{B}(X) = 0$ for all $X \in T\Sigma^\perp$ and $\mathcal{B}(X) = \gamma^2 W(X)$ for all $X \in T\Sigma$, it holds that

$$-t\det(W - t\mathbb{1}_n) = \det(\gamma^{-2}\mathcal{B} - t\mathbb{1}_{n+1}) = \frac{1}{\gamma^{2(n+1)}}\det(\mathcal{B} - t\gamma^2\mathbb{1}_{n+1})$$

which implies that the k'th symmetric function is given by the coefficient of t^{k+1} in $-\det(\mathcal{B} - t\gamma^2\mathbb{1})\gamma^{2(n-k)}$.

5. The Codazzi-Mainardi equations

When studying the geometry of embedded manifolds, the Codazzi-Mainardi equations are very useful. In this section we reformulate these equations in terms of Nambu brackets.

The Codazzi-Mainardi equations express the normal component of $\bar{R}(X,Y)Z$ in terms of the second fundamental forms; namely

$$\begin{aligned}\bar{g}\big(\bar{R}(X,Y)Z,N_A\big) &= \big(\nabla_X h_A\big)(Y,Z) - \big(\nabla_Y h_A\big)(X,Z)\\ &\quad + \sum_{A=1}^{p}\Big[\bar{g}(D_X N_B,N_A)h_B(Y,Z) - \bar{g}(D_Y N_B,N_A)h_B(X,Z)\Big],\end{aligned} \tag{5.1}$$

for $X,Y,Z\in T\Sigma$ and $A=1,\dots,p$. Defining

$$\begin{aligned}\mathcal{W}_A(X,Y) &= \big(\nabla_X W_A\big)(Y) - \big(\nabla_Y W_A\big)(X)\\ &\quad + \sum_{B=1}^{p}\Big[\bar{g}(D_X N_B,N_A)W_B(Y) - \bar{g}(D_Y N_B,N_A)W_B(X)\Big]\end{aligned} \tag{5.2}$$

one can rewrite the Codazzi-Mainardi equations as follows.

Proposition 5.1. *Let Π denote the projection onto $T\Sigma^\perp$. Then the Codazzi-Mainardi equations are equivalent to*

$$\mathcal{W}_A(X,Y) = -(\mathbb{1}-\Pi)\big(\bar{R}(X,Y)N_A\big) \tag{5.3}$$

for $X,Y\in T\Sigma$ and $A=1,\dots,p$.

Proof. Since $h_A(X,Y)=\bar{g}(W_A(X),Y)$ (by Weingarten's equation) one can rewrite (5.1) as

$$\bar{g}\big(\mathcal{W}_A(X,Y),Z\big) = \bar{g}\big(\bar{R}(X,Y)Z,N_A\big), \tag{5.4}$$

and since $\bar{g}(\bar{R}(X,Y)Z,N_A) = -\bar{g}(\bar{R}(X,Y)N_A,Z)$ this becomes

$$\bar{g}\big(\mathcal{W}_A(X,Y)+\bar{R}(X,Y)N_A,Z\big)=0. \tag{5.5}$$

That this holds for all $Z\in T\Sigma$ is equivalent to saying that

$$(\mathbb{1}-\Pi)\big(\mathcal{W}_A(X,Y)+\bar{R}(X,Y)N_A\big)=0, \tag{5.6}$$

from which (5.3) follows since $\mathcal{W}_A(X,Y)\in T\Sigma$. □

Note that since $\gamma^{-2}\mathcal{P}^2$ is the projection onto $T\Sigma$ one can write (5.3) as

$$\gamma^2\mathcal{W}_A(X,Y) = -\mathcal{P}^2\big(\bar{R}(X,Y)N_A\big). \tag{5.7}$$

Since both W_A and D_X can be expressed in terms of $\mathcal{B}_A$, one obtains the following expression for $\mathcal{W}_A$:

Proposition 5.2. *For $X,Y\in T\Sigma$ one has*

$$\begin{aligned}\gamma^2\mathcal{W}_A(X,Y) =& \big(\bar{\nabla}_X\mathcal{B}_A\big)(Y) - \big(\bar{\nabla}_Y\mathcal{B}_A\big)(X)\\ &- \frac{1}{\gamma^2}\Big[\big(\nabla_X\gamma^2\big)\mathcal{B}_A(Y) - \big(\nabla_Y\gamma^2\big)\mathcal{B}_A(X)\Big]\\ &+ \frac{1}{\gamma^2}\sum_{B=1}^{p}\Big[\bar{g}\big(\mathcal{B}_A(N_B),X\big)\mathcal{B}_B(Y) - \bar{g}\big(\mathcal{B}_A(N_B),Y\big)\mathcal{B}_B(X)\Big].\end{aligned}$$

As the aim is to express the Codazzi-Mainardi equations in terms of Nambu brackets, we will introduce maps $\mathcal{C}_A$ that is defined in terms of $\mathcal{W}_A$ and can be written as expressions involving Nambu brackets.

Definition 5.3. The maps $\mathcal{C}_A : C^\infty(\Sigma)\times\cdots\times C^\infty(\Sigma)\to T\Sigma$ are defined as

$$\mathcal{C}_A(f_1,\dots,f_{n-2}) = \frac{1}{2\rho}\varepsilon^{aba_1\cdots a_{n-2}}\mathcal{W}_A(e_a,e_b)\big(\partial_{a_1}f_1\big)\cdots\big(\partial_{a_{n-2}}f_{n-2}\big) \tag{5.8}$$

for $A = 1, \ldots, p$ and $n \geq 3$. When $n = 2$, $\mathcal{C}_A$ is defined as

$$\mathcal{C}_A = \frac{1}{2\rho}\varepsilon^{ab}\mathcal{W}_A(e_a, e_b).$$

Proposition 5.4. *Let* $\{g_1, g_2\}_f \equiv \{g_1, g_2, f_1, \ldots, f_{n-2}\}$. *Then*

$$\mathcal{C}_A(f_1,\ldots,f_{n-2})^i = \left\{\gamma^{-2}(\mathcal{B}_A)^i_k, x^k\right\}_f + \frac{1}{\gamma^2}\left\{x^j, x^l\right\}_f\left[\bar{\Gamma}^i_{jk}(\mathcal{B}_A)^k_l - (\mathcal{B}_A)^i_k\bar{\Gamma}^k_{jl}\right]$$
$$-\frac{1}{\gamma^2}\sum_{B=1}^{p}\Big[\left\{n^k_A, x^l\right\}_f(\mathcal{B}_B)^i_l + \bar{\Gamma}^k_{lj}\left\{x^l, x^m\right\}_f n^j_A(\mathcal{B}_B)^i_m\Big](n_B)_k.$$

Remark 5.5. In case Σ is a hypersurface, the expression for $\mathcal{C} \equiv \mathcal{C}_1$ simplifies to

$$\mathcal{C}(f_1,\ldots,f_{n-2})^i = \left\{\gamma^{-2}\mathcal{B}^i_k, x^k\right\}_f + \frac{1}{\gamma^2}\left\{x^j, x^l\right\}_f\left[\bar{\Gamma}^i_{jk}\mathcal{B}^k_l - \mathcal{B}^i_k\bar{\Gamma}^k_{jl}\right],$$

since $D_X N = 0$.

It follows from Proposition 5.1 that we can reformulate the Codazzi-Mainardi equations in terms of $\mathcal{C}_A$:

Theorem 5.6. *For all* $f_1, \ldots, f_{n-2} \in C^\infty(\Sigma)$ *it holds that*

$$\gamma^2\mathcal{C}_A(f_1,\ldots,f_{n-2}) = (\mathcal{P}^2)^i_j\Big[\{x^k, \bar{\Gamma}^j_{kj'}\}_f - \left\{x^k, x^l\right\}_f\bar{\Gamma}^m_{lj'}\bar{\Gamma}^j_{km}\Big]n^{j'}_A\partial_i, \tag{5.9}$$

for $A = 1, \ldots, p$, *where* $\{g_1, g_2\}_f = \{g_1, g_2, f_1, \ldots, f_{n-2}\}$.

Proof. As noted previously, one can write the Codazzi-Mainardi equations as

$$\gamma^2\mathcal{W}_A(X, Y) = -\mathcal{P}^2\big(\bar{R}(X, Y)N_A\big).$$

That the above equation holds for all $X, Y \in T\Sigma$ is equivalent to saying that

$$\gamma^2\frac{1}{2\rho}\varepsilon^{aba_1\cdots a_{n-2}}\mathcal{W}_A(e_a, e_b) = -\frac{1}{2\rho}\varepsilon^{aba_1\cdots a_{n-2}}\mathcal{P}^2\big(\bar{R}(e_a, e_b)N_A\big)$$

for all values of $a_1, \ldots, a_{n-2} \in \{1, \ldots, n\}$; furthermore, this is equivalent to

$$\gamma^2\mathcal{C}_A(f_1,\ldots,f_{n-2}) = -\frac{1}{2\rho}\varepsilon^{aba_1\cdots a_{n-2}}\mathcal{P}^2\big(\bar{R}(e_a, e_b)N_A\big)(\partial_{a_1}f_1)\cdots(\partial_{a_{n-2}}f_{n-2})$$

for all $f_1, \ldots, f_{n-2} \in C^\infty(\Sigma)$. It is now straightforward to show that

$$-\frac{1}{2\rho}\varepsilon^{aba_1\cdots a_{n-1}}\big(\bar{R}(e_a, e_b)N_A\big)^i(\partial_{a_1}f_1)\cdots(\partial_{a_{n-2}}f_{n-2})$$
$$= \Big(\{x^k, \bar{\Gamma}^i_{kj}\}_f - \left\{x^k, x^l\right\}_f\bar{\Gamma}^m_{lj}\bar{\Gamma}^i_{km}\Big)n^j_A,$$

which proves the statement. □

If M is a space of constant curvature (in which case $\bar{g}(\bar{R}(X, Y)Z, N_A) = 0$), then Theorem 5.6 states that

$$\mathcal{C}_A(f_1,\ldots,f_{n-2}) = 0 \tag{5.10}$$

for all $f_1, \ldots, f_{n-2} \in C^\infty(\Sigma)$. Furthermore, if $M = \mathbb{R}^m$, then (5.9) becomes

$$\gamma^2\left\{\gamma^{-2}(\mathcal{B}_A)^i_k, x^k\right\}_f - \sum_{B=1}^{p}\Big[\left\{n^k_A, x^l\right\}_f(\mathcal{B}_B)^i_l\Big](n_B)_k = 0. \tag{5.11}$$

6. Embedded surfaces

Let us now turn to the special case when Σ is a surface. For surfaces, the tensors $\mathcal{P}$, $\mathcal{S}_A$ and $\mathcal{T}_A$ are themselves maps from TM to TM, and $\mathcal{S}_A$ coincides with $\mathcal{T}_A$. Moreover, since the second fundamental forms can be considered as 2×2 matrices, one has the identity

$$2\det W_A = \big(\operatorname{tr} W_A\big)^2 - \operatorname{tr} W_A^2,$$

which implies that the scalar curvature can be written as

$$\begin{aligned} R &= g^{ac}g^{bd}\bar{g}\big(\bar{R}(e_c,e_d)e_b,e_a\big) + 2\sum_{A=1}^{p}\det W_A\\ &= 2\frac{\bar{g}\big(\bar{R}(e_1,e_2)e_2,e_1\big)}{g} + 2\sum_{A=1}^{p}\det W_A. \end{aligned}$$

Thus, defining the Gaussian curvature K to be one half of the above expression (which also coincides with the sectional curvature), one obtains

$$K = \frac{\bar{g}\big(\bar{R}(e_1,e_2)e_2,e_1\big)}{g} - \frac{1}{2\gamma^2}\sum_{A=1}^{p}\operatorname{Tr}\mathcal{S}_A^2, \tag{6.1}$$

which in the case when $M=\mathbb{R}^m$ becomes

$$K = -\frac{1}{2\gamma^2}\sum_{A=1}^{p}\sum_{i,j=1}^{m}\{x^i,n_A^j\}\{x^j,n_A^i\}, \tag{6.2}$$

and by using the normal vectors Z_α the expression for K can be written as

$$K = -\frac{1}{8\gamma^4(p-1)!}\sum \varepsilon_{jklI}\varepsilon_{imnI}\{x^i,\{x^k,x^l\}\}\{x^j,\{x^m,x^n\}\}. \tag{6.3}$$

To every Riemannian metric on Σ one can associate an almost complex structure $\mathcal{J}$ through the formula

$$\mathcal{J}(X) = \frac{1}{\sqrt{g}}\varepsilon^{ac}g_{cb}X^b e_a,$$

and since on a two dimensional manifold any almost complex structure is integrable, $\mathcal{J}$ is a complex structure on Σ. For $X\in TM$ one has

$$\mathcal{P}(X) = -\frac{1}{\gamma\sqrt{g}}\bar{g}\big(X,e_a\big)\varepsilon^{ab}e_b, \tag{6.4}$$

and it follows that one can express the complex structure in terms of $\mathcal{P}$.

Theorem 6.1. *Defining $\mathcal{J}_M(X)=\gamma\mathcal{P}(X)$ for all $X\in TM$ it holds that $\mathcal{J}_M(Y)=\mathcal{J}(Y)$ for all $Y\in T\Sigma$. That is, $\gamma\mathcal{P}$ defines a complex structure on $T\Sigma$.*

Let us now turn to the Codazzi-Mainardi equations for surfaces. In this case, the map $\mathcal{C}_A$ becomes a tangent vector and one can easily see in Proposition 5.4 that the sum in the expression for $\mathcal{C}_A$ can be written in a slightly more compact form, namely

$$\begin{aligned} \mathcal{C}_A = \big\{\gamma^{-2}(\mathcal{B}_A)^i_k,x^k\big\}\,\partial_i + \frac{1}{\gamma^2}\left\{x^j,x^l\right\}\left[\bar{\Gamma}^i_{jk}(\mathcal{B}_A)^k_l - (\mathcal{B}_A)^i_k\bar{\Gamma}^k_{jl}\right]\\ + \frac{1}{\gamma^2}\sum_{B=1}^{p}\mathcal{B}_B\mathcal{S}_A(N_B). \end{aligned}$$

Thus, for surfaces embedded in $\mathbb{R}^m$ the Codazzi-Mainardi equations become

$$\sum_{j,k=1}^{m} \big\{\gamma^{-2}\{x^i,x^j\}\{x^j,n_A^k\},x^k\big\}\,\partial_i + \frac{1}{\gamma^2}\sum_{B=1}^{p}\mathcal{B}_B\mathcal{S}_A(N_B) = 0,$$

and in $\mathbb{R}^3$ one has

$$\sum_{j,k=1}^{3} \big\{\gamma^{-2}\{x^i,x^j\}\{x^j,n^k\},x^k\big\} = 0. \tag{6.5}$$

Let us note that one can rewrite these equations using the following result:

Proposition 6.2. *For $M=\mathbb{R}^m$ and $i=1,\ldots,m$ it holds that*

$$\sum_{j,k=1}^{m} \big\{f\{x^i,x^j\}\{x^j,n^k\},x^k\big\} = \sum_{j,k=1}^{m} \big\{f\{x^i,x^j\}\{x^j,x^k\},n^k\big\} \tag{6.6}$$

for any normal vector $N=n^i\partial_i$ and any $f\in C^\infty(\Sigma)$.

Proof. We start by recalling that for any $g\in C^\infty(\Sigma)$ it holds that $\sum_{i=1}^m\{g,x^i\}n^i = 0$, since it involves the scalar product $\bar{g}(e_a,N)$. Moreover, one also has

$$\begin{aligned}\sum_{k=1}^{m}\{x^k,n^k\} &= \sum_{k=1}^{m}\frac{1}{\rho}\varepsilon^{ab}(\partial_a x^k)(\partial_b n^k) = \sum_{k=1}^{m}\frac{1}{\rho}\varepsilon^{ab}\Big(\partial_b\big(n^k\partial_a x^k\big) - n^k\partial^2_{ab}x^k\Big)\\ &= -\sum_{k=1}^{m}\frac{1}{\rho}\varepsilon^{ab}n^k\partial^2_{ab}x^k = 0,\end{aligned}$$

which implies that $\sum_{k=1}^m\{x^k,gn^k\}=0$ for all $g\in C^\infty(\Sigma)$. By using the above identities together with the Jacobi identity, one obtains

$$\begin{aligned}\big\{f\{x^i,x^j\}\{x^j,n^k\},x^k\big\} &= f\{x^i,x^j\}\big\{\{x^j,n^k\},x^k\big\} + \{x^j,n^k\}\big\{f\{x^i,x^j\},x^k\big\}\\ &= -f\{x^i,x^j\}\big\{\{x^k,x^j\},n^k\big\} - n^k\big\{x^j,\{f\{x^i,x^j\},x^k\}\big\}\\ &= -f\{x^i,x^j\}\big\{\{x^k,x^j\},n^k\big\} + n^k\big\{f\{x^i,x^j\},\{x^k,x^j\}\big\}\\ &= -f\{x^i,x^j\}\big\{\{x^k,x^j\},n^k\big\} - \{x^k,x^j\}\big\{f\{x^i,x^j\},n^k\big\}\\ &= \big\{f\{x^i,x^j\}\{x^j,x^k\},n^k\big\}.\end{aligned}$$

□

Hence, one can rewrite the Codazzi-Mainardi equations for a surface in $\mathbb{R}^3$ as

$$\sum_{j,k=1}^{3}\big\{\gamma^{-2}(\mathcal{P}^2)^{ik},n^k\big\} = 0, \tag{6.7}$$

and it is straight-forward to show that

$$\sum_{i,j,k=1}^{3}\big(\partial_c x^i\big)\big\{\gamma^{-2}(\mathcal{P}^2)^{ik},n^k\big\} = \frac{1}{\rho}\varepsilon^{ab}\nabla_a h_{bc},$$

thus reproducing the classical form of the Codazzi-Mainardi equations.

Is it possible to verify (6.7) directly using only Poisson algebraic manipulations? It turns out that that the Codazzi-Mainardi equations in $\mathbb{R}^3$ is an identity for arbitrary Poisson algebras, if one assumes that a normal vector is given by $\frac{1}{2\gamma}\varepsilon_{ijk}\{x^j,x^k\}\partial_i$.

Proposition 6.3. *Let $\{\cdot,\cdot\}$ be an arbitrary Poisson structure on $C^\infty(\Sigma)$. Given $x^1,x^2,x^3\in C^\infty(\Sigma)$ it holds that*

$$\sum_{j,k,l,n=1}^{3}\frac{1}{2}\varepsilon_{kln}\big\{\gamma^{-2}\{x^i,x^j\}\{x^j,x^k\},\gamma^{-1}\{x^l,x^n\}\big\}=0$$

for $i=1,2,3$, where

$$\gamma^2=\{x^1,x^2\}^2+\{x^2,x^3\}^2+\{x^3,x^1\}^2.$$

Proof. Let u,v,w be a cyclic permutation of $1,2,3$. In the following we do not sum over repeated indices u,v,w. Denoting by CM^i the i'th component of the Codazzi-Mainardi equation, one has

$$\begin{aligned}\mathrm{CM}^u &= -\big\{\gamma^{-2}\big(\{x^u,x^v\}^2+\{x^w,x^u\}^2\big),\gamma^{-1}\{x^v,x^w\}\big\}\\
&\quad+\big\{\gamma^{-2}\{x^u,x^v\}\{x^v,x^w\},\gamma^{-1}\{x^u,x^v\}\big\}+\big\{\gamma^{-2}\{x^u,x^w\}\{x^w,x^v\},\gamma^{-1}\{x^w,x^u\}\big\}\\
&= -\big\{1-\gamma^{-2}\{x^v,x^w\}^2,\gamma^{-1}\{x^v,x^w\}\big\}+\gamma^{-1}\{x^u,x^v\}\big\{\gamma^{-1}\{x^v,x^w\},\gamma^{-1}\{x^u,x^v\}\big\}\\
&\quad+\gamma^{-1}\{x^u,x^w\}\big\{\gamma^{-1}\{x^w,x^v\},\gamma^{-1}\{x^w,x^u\}\big\}\\
&= \frac{1}{2}\big\{\gamma^{-1}\{x^v,x^w\},\gamma^{-2}\big(\gamma^2-\{x^v,x^w\}^2\big)\big\}=0.\end{aligned}$$

□

Let us end by noting that these results generalize to arbitrary hypersurfaces in $\mathbb{R}^{n+1}$. Namely,

$$\begin{aligned}&\big\{\gamma^{-2}\{x^i,\vec{x}^J\}\{\vec{x}^J,n^k\},x^k\big\}_f=\big\{\gamma^{-2}\{x^i,\vec{x}^J\}\{\vec{x}^J,x^k\},n^k\big\}_f,\\
&(\partial_c x^i)\big\{\gamma^{-2}\big(\mathcal{P}^2\big)^{ik},n^k\big\}_f=-\frac{1}{\rho}\varepsilon^{aba_1\cdots a_{n-2}}\big(\nabla_a h_{bc}\big)\big(\partial_{a_1}f_1\big)\cdots\big(\partial_{a_{n-2}}f_{n-2}\big),\end{aligned}$$

and

$$\varepsilon_{klL}\big\{\gamma^{-2}\{x^i,\vec{x}^J\}\{\vec{x}^J,x^k\},\gamma^{-1}\{x^l,\vec{x}^L\}\big\}_f=0$$

for arbitrary $x^1,\dots,x^{n+1}\in C^\infty(\Sigma)$.

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