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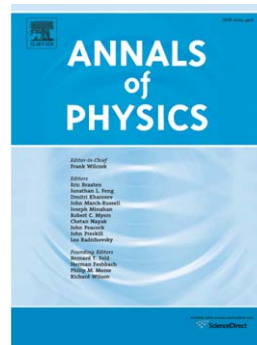
Causal structure and algebraic classification of non-dissipative linear optical media

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Causal structure and algebraic classification of non-dissipative linear optical media

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Abstract

In crystal optics and quantum electrodynamics in gravitational vacua, the propagation of light is not described by a metric, but an area metric geometry. In this article, this prompts us to study conditions for linear electrodynamics on area metric manifolds to be well-posed. This includes an identification of the timelike future cones and their duals associated to an area metric geometry, and thus paves the ground for a discussion of the related local and global causal structure in standard fashion. In order to provide simple algebraic criteria for an area metric manifold to present a consistent spacetime structure, we develop a complete algebraic classification of area metric tensors up to general transformations of frame. This classification, valuable in its own right, is then employed to prove a theorem excluding the majority of algebraic classes of area metrics as viable spacetimes. Physically, these results classify and drastically restrict the viable constitutive tensors of non-dissipative linear optical media.

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1. Introduction

Electrodynamics, as originally conceived by Maxwell, has been interpreted by Einstein and others to reveal a flat Minkowskian geometry of spacetime. Soon afterwards it was realized that in order to describe light propagation in some crystals or in the presence of gravity, the flat Minkowskian geometry needs to be replaced by a curved metric geometry. Not all metric geometries, however, are suitable backgrounds for electrodynamics. Of all algebraic classes of metrics, distinguished by their signature, only Lorentzian metrics may render the initial value problem of Maxwell theory well-posed. This result is obtained in two steps. First, it is recognized that a necessary condition for the initial value problem to be well-posed is that the electromagnetic equations are strongly hyperbolic. But then the metrics allowing

for strongly hyperbolic Maxwell equations are precisely the Lorentzian ones. In other words, for a metric spacetime geometry one is fortunate to have available, on the one hand, a full algebraic classification of the metrics, and, on the other hand, a theorem linking the strong hyperbolicity of Maxwell theory to precisely one of these algebraic classes. All this is well-known.

However, the reformulation of the original Maxwell equations as gauge field dynamics on a metric spacetime is unnecessarily restrictive. In fact, inspection of the Maxwell action on a Lorentzian manifold reveals that there is no coupling to the actual metric tensor, but only to the area measure induced from it. Moreover, the electromagnetic potential is a covector field so that its coupling to a charged vector current does not depend on the background geometry at all. Thus there is no way to experimentally distinguish whether the underlying spacetime structure is metric or area metric, unless one discovers phenomena that are captured only by an area metric but not a metric.

In crystals, such non-metric electromagnetic phenomena are well-known: for instance, birefringence, i.e., the splitting of one light ray into two polarized rays. Of course, from a fundamental perspective, this is generally viewed as an effect emerging from the quantum interaction of the electromagnetic field with the constituent matter of the crystal. But for the effective classical wave theory, and in particular its dynamical equations, this is irrelevant. Their properties are governed solely by the area metric tensor whose particular form is then determined by the material properties of the optical medium.

But this observation raises a crucial question. Is there an algebraic classification of area metric tensors in analogy to that of metrics? And can these classes similarly be linked to hyperbolicity properties of the electromagnetic equations on such more general backgrounds? It is the purpose of this paper to answer both questions affirmatively. In particular, we provide a full algebraic classification of four-dimensional area metrics and prove a theorem that excludes, out of a total of 23 algebraic classes, 16 classes that cannot provide viable backgrounds. In physical terms, these results amount to a complete classification of non-dissipative linear optical media and an identification of many formally conceivable electromagnetic constitutive tensors which are in fact unphysical, so that materials with corresponding properties cannot exist or be engineered.

Area metric geometry makes another appearance in standard physics when one considers photons propagating in a vacuum gravitational field. Due to the reaction of virtual electron-positron pairs in the dressed photon propagator to the Weyl curvature of the spacetime, which has been calculated in

an influential paper by Drummond and Hathrell [1], photons do not follow null geodesics with respect to the metric spacetime geometry, but rather with respect to an area metric geometry defined by the metric and the Weyl tensor. Again, the electromagnetic field equations do not care about how the effect emerges. The well-posedness of the effective field equations and indeed the causal structure of the spacetime geometry probed by photons to first order quantum corrections only hinges on the induced area metric. In homogeneous and isotropic spacetimes one may still employ standard metric techniques to study the resulting causal structure. But for less symmetric situations, one is thrown back to the use of the area metric geometry presented in this paper, and indeed the causal structure defined by it. We just mention that the causal structure of spacetime seen by photons in the early universe is changed compared to the one defined by the bare metric to such an extent that if there are no otherwise compensating effects, the cosmological horizon problem simply does not pose itself.

The Drummond-Hathrell result indicates that area metrics may play a more fundamental role in the structure of spacetime. After all, most of what we know about the large scale structure of spacetime, we know from astrophysical observations of photons reaching us. However, at first sight one seems compelled to discard the idea that the spacetime geometry itself could be generically area metric instead of metric. Too much of our understanding of standard physics seems to hinge on the very concept of a metric spacetime structure. In particle physics [2], the Poincare group associated with a Lorentzian metric conveniently restricts the admissible types of matter fields and their dynamics, as was first pointed out by Wigner [3]. Second, dynamics for Lorentzian metrics with a well-posed initial value problem almost inevitably [4] are those of Einstein-Hilbert theory, with the well-known physical implications [5]: the big bang singularity, precession of planetary orbits, gravitational lensing and an expanding universe. Together, remarkably much of what we infer about the structure of spacetime, its matter contents, and indeed their interplay, hinges on the presumed Lorentzian spacetime structure.

However, particularly over the last decade, disturbingly robust and diverse observational evidence has been accumulated that there is something significant we currently do not understand about the matter contents of the universe, gravitational dynamics, or both. For instance, in order to explain the observed late-time accelerated expansion of the universe [6] and at the same time the data collected from the lensing of light through galaxies [7], one would have to assume that a spectacular 74% of energy and 22% of matter in the universe are of entirely unknown origin, and do not interact

in any other conceivable way than gravitationally [8].

But while the existence of such vast amounts of dark energy and dark matter may indeed be the correct conclusion to be drawn from the observational data, this seems not an uncontestably plausible or compelling conclusion. Much less so because there are a number of further anomalies in gravitational physics, such as the flattened galaxy rotation curves, the anomalous accelerations of Pioneer 10 and 11, the fly-by anomaly, and others [9]. In summary, there is an increasing list of discrepancies between observation and theory, which in some cases hint at new particle physics [10], in other cases at new gravitational physics [11], and one may well speculate that some hint at both [12].

Now on the one hand, it may be the case that all of these anomalies are mutually independent, and require a different resolution each. On the other hand, and this is the line of thought we want to pursue here, one would expect both, new gravitational and new particle physics, if the geometry of spacetime turned out to be different from that of a Lorentzian metric manifold. Such a generalized spacetime geometry would have to be sufficiently general to capture various of the anomalies currently escaping explanation, while at the same time providing feasible spacetime backgrounds for particle physics. Area metric manifolds [13] present a promising candidate for a refinement of Lorentzian geometry addressing these issues. In particular, one can write down a refinement of the Einstein-Hilbert action, such that the gravitational field is encoded in an area metric. A thorough investigation of a radiation-dominated early universe displays no difference to the Einsteinian one, but there is an accelerating solution for the late matter-dominated universe. The results of the present paper underly a currently conducted study of the representations of the hypersurface deformation algebra in admissible area metric spacetimes, in fashion of the seminal work of Hojman, Kuchař and Teitelboim for Lorentzian manifolds. Thus it would be possible to identify all well-posed area metric gravity theories in a constructive way.

Outline. In section 2, we start by reviewing the most important aspects of area metric geometry, in so far as they play a role in the present work. To get a feeling for the way in which area metric geometry presents a refinement of metric geometry, we in particular investigate the mathematical properties of low-dimensional area metrics. Provided with the mathematical definitions, and supported by the obtained physical intuition for area metric

geometry, we then turn to one of the key points of this article: we employ Maxwell theory to define a causal structure on area metric manifolds. With the result of this construction, we are then equipped to present the central definitions of weakly and strongly hyperbolic area metric spacetimes. These provide an *analytic* characterization of area metric manifolds that present viable spacetime structures. Relevant global causality conditions for area metric manifolds can then be imposed in addition, and we observe that celebrated theorems, such as the equivalence of the Alexandrov topology with the underlying manifold topology, directly extend to area metric spacetimes.

Equipped with the basic definition and causality properties of area metric manifolds, we highlight a number of physical applications of area metric geometry in section 3. We spell out in detail how area metrics encode all material properties of non-dissipative linear optical media, and explain how good causality restricts the class of possible crystals. Most interestingly, the classification of area metric backgrounds achieved in later section clearly becomes a classification of optical crystals. As a second application we consider the propagation of photons on metric spacetime as calculated by Drummond and Hathrell by taking into account quantum effects. The resulting effective action is interpreted using area geometric methods. The causality discussion of this paper nicely clarifies some issues of photon propagation that were not fully resolved before. Finally we consider area metric manifolds in their own right as gravitational backgrounds that allow the same complexity as do optical crystals. We review some results on area metric gravity, and provide an outlook on how the new results on causality will deepen the understanding of the theory.

In section 4, we turn to an *algebraic* classification of four-dimensional area metric manifolds. We obtain a complete overview of all possible area metric manifolds and provide, as a corollary to our classification theorem, a list of area metric normal forms. The provision of such normal forms, together with our detailed study of the respective algebras describing the involved gauge ambiguity, constitutes an immensely useful calculational tool, very much like in the familiar case of pseudo-Riemannian metrics.

We combine our findings on the analytic characterization of the causal properties of area metric spacetimes with the algebraic classification of four-dimensional area metrics in section 5. This culminates in the proof that a large number of algebraic classes do not present area metric spacetimes. An even stronger version of this theorem can be obtained by focusing on phenomenologically important cases of highly symmetric area metric spacetimes.

In a conclusion, we finally place our results and the methods employed

in this article in a wider context, emphasize what has been achieved and where the limitations of the current study lie.

2. Area metric geometry and causal structure

The central aspects of area metric geometry, as far as they play a role for the developments in this article, are presented and discussed in this section. As an immediate physical question the causal structure of Maxwell theory on generic area metric manifolds is studied in some detail. These constructions culminate in the definition of strongly hyperbolic area metric spacetimes and present the first technical pillar of this article.

2.1. Area metric manifolds

We start with the fundamental definitions of area metric geometry [14] in d dimensions which presents a generalization of metric geometry.

Definition 2.1. *An area metric manifold (M, G) is a smooth d -dimensional manifold M equipped with a fourth-rank covariant tensor field G with the following symmetry and invertibility properties at each point p of M :*

- (i) $G(X, Y, A, B) = G(A, B, X, Y)$ for all X, Y, A, B in $T_p M$
- (ii) $G(X, Y, A, B) = -G(Y, X, A, B)$ for all X, Y, A, B in $T_p M$
- (iii) For each p of M and X, Y, A, B in $T_p M$ the map $\hat{G} : \Lambda^2 T_p M \rightarrow \Lambda^2 T_p^* M$, defined through $\hat{G}(X \wedge Y)(A \wedge B) := G(X, Y, A, B)$ by linear continuation, is invertible. Its inverse then defines a fourth-rank contravariant tensor field G^{-1} called the inverse area metric.

Here $\Lambda^2 T_p M = T_p M \wedge T_p M$ denotes the space of all contravariant antisymmetric tensors of rank two and we will drop the hat on G where no confusion arises.

Given a basis $\{e_a\}$ on $T_p M$, the symmetry conditions can be written in terms of the components $G(e_a, e_b, e_c, e_d) = G_{abcd}$ of the area metric:

$$G_{abcd} = G_{cdab} = -G_{bacd}. \quad (1)$$

Due to these symmetries, the indices of G may be combined to antisymmetric Petrov pairs $[ab]$ such that G can be represented by a symmetric square matrix of dimension $D = d(d-1)/2$. More precisely, we introduce Petrov indices $A = 1, \dots, d(d-1)/2$ for every antisymmetric pair of small indices $[ab]$. The Petrov indices can be calculated as follows: without loss of generality we assume $a < b$ and calculate the Petrov index A in terms of

a and b as $A = (a(2d - 3) - a^2)/2 + b$. If it is not clear from the indices that we use the Petrov notation of an object Γ we write $Petrov(\Gamma)$. In four dimensions for instance, which is the case of direct physical interest, we have index pairs [01], [02], [03], [12], [31], [23] with the corresponding Petrov indices $A = 1, \dots, 6$. The independent components of an area metric G in four dimensions may hence be arranged as the 6×6 Petrov matrix

$$Petrov(G) = \begin{bmatrix} G_{0101} & G_{0102} & G_{0103} & G_{0112} & G_{0131} & G_{0123} \\ & G_{0202} & G_{0203} & G_{0212} & G_{0231} & G_{0223} \\ & \ddots & & G_{0303} & G_{0312} & G_{0331} & G_{0323} \\ & & \ddots & & G_{1212} & G_{1231} & G_{1223} \\ & & & \ddots & & G_{3131} & G_{3123} \\ & & & & \ddots & & G_{2323} \end{bmatrix} \quad (2)$$

with the components under the diagonal filled by symmetry.

Some care has to be taken when extending the summation convention to Petrov indices. Since the summation over Petrov indices A essentially corresponds to a sum over *ordered* antisymmetric pairs of tangent space indices, we need to multiply by a factor of $1/2$ when resolving a contraction over Petrov indices in terms of a double contraction over an *unordered* pair of tangent space indices: $X^A \Omega_A = 1/2 X^{ab} \Omega_{ab}$ for antisymmetric tensor fields X and Ω of valence $(2, 0)$ and $(0, 2)$, respectively.

The invertibility requirement (iii) implies that the Petrov matrix $Petrov(G)$ representing G is non-degenerate and $Petrov(G^{-1}) = Petrov(G)^{-1}$. By direct calculation, we obtain that the components of the inverse area metric G^{-1} satisfy the identity

$$(G^{-1})^{abmn} G_{mncd} = 4 \delta_c^{[a} \delta_d^{b]}. \quad (3)$$

where the factor of 4 arises due to the use of the above described summation convention and weighted antisymmetrization

Area metric geometry is a refinement of metric geometry, insofar as every pseudo-Riemannian manifold is an area metric manifold, but not all area metric manifolds are induced from a metric one. Nevertheless, it is sometimes interesting to discuss the following special type of area metrics:

Definition 2.2. *An area metric G is said to be metric-induced if there exists a metric g such that*

$$G(X, Y, A, B) = g(X, A)g(Y, B) - g(X, B)g(Y, A). \quad (4)$$

For an area metric G_g induced in this fashion we have that $G_g(X, Y, X, Y) = g(X, X)g(Y, Y) \sin^2[\angle(X, Y)]$ is the squared area of the parallelogram spanned by vectors X, Y as measured in the underlying metric geometry. For later use we write the metric-induced area metric in components:

$$G_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc} \quad (5)$$

and by virtue of equation (3) we have

$$(G^{-1})^{abcd} = g^{ac}g^{bd} - g^{ad}g^{bc}. \quad (6)$$

Note that for metric-induced and generic area metrics alike, any pair of $SL(2, \mathbb{R})$ -related parallelograms (X, Y) and (\tilde{X}, \tilde{Y}) , i.e. $\tilde{X} = aX + bY$ and $\tilde{Y} = cX + dY$ with $ad - bc = 1$, have identical areas as measured by the area metric, $G(X, Y, X, Y) = G(\tilde{X}, \tilde{Y}, \tilde{X}, \tilde{Y})$. Thus an area metric does not distinguish parallelograms (X, Y) that describe the same oriented area $X \wedge Y$. This property, together with the reproduction of the familiar notion of area in the metric-induced case, justifies to call G an area metric.

It should be noted that a generic area metric contains more algebraic degrees of freedom than a metric, starting from dimension four. This can be seen by counting the independent components of the symmetric $D \times D$ Petrov matrix representing the area metric, which amounts to $D(D+1)/2$ independent real numbers. The invertibility requirement does not further reduce this number since it is an open condition. Thus area metrics in dimensions 2, 3, 4 and 5 have 1, 6, 21 and 55 independent components, respectively.

An area metric G naturally gives rise to a scalar density $|\det(\text{Petrov}(G))|^{1/(2d-2)}$ of weight +1. That $\det(\text{Petrov}(G))$ transforms as a density of weight $2d-2$ under a change of frame on the underlying d -dimensional manifold,

$$\det(\text{Petrov}(G_{mnpq} T^m_a T^n_b T^p_c T^q_d)) = \det(T^a_b)^{2d-2} \det(\text{Petrov}(G)), \quad (7)$$

for a transformation matrix T^a_b , follows from the identity

$$\det(\text{Petrov}(T^a_c T^b_d)) = (\det(T^a_b))^{d-1}, \quad (8)$$

which deserves a

Proof [15]. Consider a d -dimensional vector space V and an automorphism $T : V \rightarrow V$. We define the induced endomorphism $T \wedge T : V \wedge V \rightarrow V \wedge V$ on the induced $d(d-1)/2$ -dimensional vector space $V \wedge V$ as $(T \wedge T)(v \wedge w) = T(v) \wedge T(w)$ for vectors $v, w \in V$. Choose an arbitrary vector $e_1 \in V$ and first assume that

$$e_1 \quad \text{and} \quad e_{i+1} := T(e_i), \quad (9)$$

for $i = 1, \dots, d-1$ defines a basis for V . The case in which this assumption does not immediately hold is discussed further below. Clearly $T(e_d) = \sum_{i=1}^d c_i e_i$ for coefficients c_i , so that in the basis $\{e_a\}$, the $d \times d$ matrix representing T takes the form

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & c_1 \\ 1 & 0 & 0 & 0 & c_2 \\ 0 & \ddots & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & c_d \end{bmatrix}, \quad (10)$$

such that one recognizes that $\det(T) = (-1)^{d-1} c_1$ and so we have $(\det(T))^{d-1} = (-1)^{d-1} c_1^{d-1}$. Now do also construct the induced basis $\{e_a \wedge e_b\}$, with $a < b$, on $V \wedge V$, and choose the order

$$e_1 \wedge e_2, \dots, e_1 \wedge e_d, e_2 \wedge e_3, \dots, e_2 \wedge e_d, \dots, e_{d-1} \wedge e_d. \quad (11)$$

Using the definition of $T \wedge T$, we may now calculate the $d(d-1)/2$ -dimensional square matrix representing $T \wedge T$ in this basis. In four dimensions for instance, the 6×6 matrix representing $T \wedge T$ takes the form

$$T \wedge T = \begin{bmatrix} 0 & 0 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_1 \\ 1 & 0 & c_2 & 0 & -c_2 & 0 \\ 0 & 1 & c_4 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 1 & c_4 & -c_3 \end{bmatrix}. \quad (12)$$

We may then calculate the determinant $\det(T \wedge T)$ by recursively expanding all required minors with respect to their first rows, say. Together with the choice of basis we made, this implies that after $d-1$ steps the remaining minor to calculate is the determinant of the $(d-1)(d-2)/2$ -dimensional unit matrix. The result of this calculation is $\det(T \wedge T) = (-1)^{(d-1)(d^2-2d+6)/3} (-c_1)^{d-1}$. Since by construction the exponent $(d-1)(d^2-2d+6)/3$ is an integer and its divisibility by 2 is not affected by multiplication by 3, it is always an even integer. Thus we arrive at $\det(T \wedge T) = (-1)^{d-1} (c_1)^{d-1}$, which under the assumption that (9) already defines a basis for V concludes the proof. It remains to show that if the first $k < d$ basis vectors form an invariant subspace of V , i.e. $T(e_k) = \sum_{i=1}^k c_i e_i$ for some $k < d$, the identity (8) still holds. In this case we have to choose another

arbitrary vector $e_{k+1} \in V$ that is linearly independent from the e_i with $i \leq k$ to construct the next basis vectors according to (9). Repeat this procedure until a complete basis is found. Then the matrices representing T and $T \wedge T$ decompose into block-diagonal form and the determinant is separately taken over every block in the same fashion as shown above. This yields the same result, and completes the proof.

Employing the density $|\det(\text{Petrov}(G))|^{1/(2d-2)}$, we can define a volume form ω_G on an area metric manifold:

Definition 2.3. *An area metric manifold (M, G) carries a canonical volume form ω_G , defined by*

$$\omega_{G a_1 \dots a_d} = |\det(\text{Petrov}(G))|^{1/(2d-2)} \epsilon_{a_1 \dots a_d}, \quad (13)$$

where ϵ is the Levi-Civita tensor density normalized such that $\epsilon_{0 \dots d-1} = 1$.

The volume form plays an essential role in our algebraic classification of four-dimensional area metrics, as we will see in section 3.

Having introduced the very basic notions of area metric geometry we analyse low-dimensional area metric manifolds in some detail, in the next section. Apart from conveying some further intuition for area metrics, we will discuss some particular properties of area metrics in four dimensions, which presents the case of most immediate physical interest for this article.

2.2. Low dimensional area metric manifolds

The study of low-dimensional cases of area metric manifolds reveals two insights. On the one hand it illustrates in what sense area metrics are a refinement of metric geometry. On the other hand we will see that in four dimensions area metrics play a very special role indeed.

$d=1$: There are no area metrics in one dimension. For from the symmetries of the area metric tensor G it is clear that there is no non-vanishing component of the area metric in only one dimension. Thus no such G can be invertible.

$d=2$: In two dimensions an area metric G is entirely determined by a scalar density $\tilde{\Phi} = G_{abcd} \epsilon^{ab} \epsilon^{cd} / 4$ of weight $+2$ by virtue of

$$G_{abcd} = \tilde{\Phi} (\epsilon_{ac} \epsilon_{bd} - \epsilon_{ad} \epsilon_{bc}), \quad (14)$$

where ϵ_{ab} denotes the components of the totally antisymmetric tensor density. This can be seen by contracting both sides with $\epsilon^{ab} \epsilon^{cd}$. The

only remaining component of the area metric tensor is $G_{0101} = \tilde{\Phi}$ and all other unrelated components vanish. Thus in two dimensions, an area metric is not a refinement of a metric, but rather a coarser structure. In fact, area metric geometry in two dimensions is symplectic geometry [16] with the symplectic form $\tilde{\Phi}^{1/2}\epsilon$.

$d=3$: An area metric in three dimensions has six independent components, just like a metric. This is more than a coincidence. We can even show that every area metric G in three dimensions is metric-induced, with the inducing metric

$$g_{ab} = \frac{1}{8} \omega_G^{ijk} \omega_G^{pqr} G_{arij} G_{pqkb}. \quad (15)$$

Indeed, one easily verifies that $g_{ab} = g_{ba}$ and

$$0 \neq \det(\text{Petrov}(G)) = \det(\text{Petrov}(g_{a[c}g_{d]b})) = (\det g)^{d-1}, \quad (16)$$

again using the identity (8), proves that g is indeed a metric. What remains to be shown is that the area metric G is in fact induced by this metric g . For that purpose we write the area metric in Petrov notation

$$\text{Petrov}(G) = \begin{bmatrix} G_{0101} & G_{0102} & G_{0112} \\ G_{0102} & G_{0202} & G_{0212} \\ G_{0112} & G_{0212} & G_{1212} \end{bmatrix} \quad (17)$$

and the components of the inverse area metric volume form ω_G read

$$\omega_G^{ijk} = |\det(\text{Petrov}(G))|^{-1/4} \epsilon^{ijk}, \quad (18)$$

where the determinant is taken over the matrix (17). We now show the proposition for the component G_{0101} of the area metric. We need to calculate the components g_{00} , g_{11} and g_{01} . According to equation (15) these are

$$\begin{aligned} g_{00} &= |\det(\text{Petrov}(G))|^{-1/2} (G_{0101}G_{0202} - G_{0102}G_{0102}), \\ g_{11} &= |\det(\text{Petrov}(G))|^{-1/2} (G_{0101}G_{1212} - G_{0112}G_{0112}), \\ g_{01} &= |\det(\text{Petrov}(G))|^{-1/2} (G_{0212}G_{0101} - G_{0102}G_{0112}). \end{aligned}$$

Inserting this into equation (5) and using the determinant of the matrix (17) proofs the equality

$$g_{00}g_{11} - (g_{01})^2 = G_{0101}. \quad (19)$$

Repeating this calculation for the other components of G completes the proof. This means area metric geometry in three dimensions is metric geometry, and vice versa. Thus the three-dimensional area metric geometry may be viewed as metric or area metric, with no way to distinguish one from the other. This result is implicit in Cartan's treatise [17]

$d=4$: In four dimensions, an area metric has 21 independent components, whereas a metric has only 10. Thus an area metric contains more algebraic degrees of freedom than a metric. It is intuitively clear that using a $GL(4)$ transformation, at most 16 of the 21 parameters of the area metric at a point can be brought to zero. A generic area metric can therefore be expected to locally determine up to five $GL(4)$ -scalars. That this is indeed the case will be an essential result in the algebraic classification in section 3. This classification will rely on the remarkable feature that in four dimensions, the canonical volume form defined by (13) is an area metric in its own right.

One may justifiably wonder whether one could consider even more refined structures than an area metric, such as a 3-volume metric $V_{[abc][def]}$, a 4-volume metric, and so on. However, while a 3-volume metric would indeed be a refinement of area metric geometry on manifolds of dimension six or higher, one easily verifies that in dimension four, a 3-volume geometry is actually coarser than an area metric geometry. Essentially this is clear by dualizing the antisymmetric triple $[abc]$ using the volume form. Similarly for higher forms in higher dimensions. In this sense area metric geometry is the most refined geometric structure in the above sequence, when we consider the physically immediately relevant case of four dimensions.

The four examples presented here shall be sufficient to get a feeling for how area metric geometry differs from metric geometry. We now start addressing the crucial issue of the causal structure defined by such backgrounds, which will finally lead to analytic criteria underpinning our definition of area metric spacetimes.

2.3. Causal structure of area metric manifolds

In the following we want to study the causal structure of four-dimensional area metric manifolds. To this end we must carefully consider what causality means and how statements about the causal structure of area metric manifolds can be deduced from first principles. It is very important to understand that causality is not an intrinsic property of an underlying background geometry, but rather the effect of an interplay between fundamental matter

propagating on the manifold and the geometric properties of the latter [18]. That is, the analysis of the causal structure of any manifold is deeply related to the causal analysis and well-posedness of the field equations that govern the evolution of some particular matter field on the manifold.

In the familiar vacuum Maxwell electrodynamics on a metric manifold, the causal structure is encoded in the Lorentzian cones [19]. Remarkably, open convex cones also arise naturally as the causal structure of electrodynamics on area metric manifolds. To make this statement precise is the purpose of this and the following section.

Maxwell electrodynamics on a generic four-dimensional area metric manifold (M, G) [13] has the action

$$S[A] = -\frac{1}{8} \int d^4x |\det(\text{Petrov}(G))|^{1/6} F_{ab} F_{cd} G^{abcd}. \quad (20)$$

The definition of the field strength $F = dA$ in terms of a gauge potential A and variation of the above action with respect to the latter lead to the electromagnetic equations of motion for the field strength F and induction H ,

$$dF = 0 \quad \text{and} \quad dH = 0, \quad (21)$$

with the electromagnetic induction H being related to the field strength F through

$$H_{ab} = -\frac{1}{4} |\det(\text{Petrov}(G))|^{1/6} \epsilon_{abmn} G^{mnpq} F_{pq}. \quad (22)$$

The causal structure of Maxwell theory on an area metric manifold is of course fully contained in the field equations (21), which may be written in components as

$$(\omega_G^{-1})^{abcd} \partial_b F_{cd} = 0, \quad |\det(\text{Petrov}(G))|^{-1/6} \partial_b (|\det(\text{Petrov}(G))|^{1/6} G^{abcd} F_{cd}) = 0, \quad (23)$$

with the inverse area metric volume form ω_G^{-1} defined according to (3), which is applicable since in four dimensions the volume form (13) is itself an area metric.

For a complete description of the initial value problem of Maxwell electrodynamics we further have to specify initial data. We introduce coordinates $x^\alpha = (t, x^\alpha)$ such that our initial data surface Σ is described by $t = 0$ and we define the electric and magnetic fields as $E_\alpha = F(\partial_t, \partial_\alpha)$ and $B^\alpha = \omega_G^{-1}(dt, dx^\alpha, F)$, respectively. Now observe that the system (23) provides eight equations for six fields (E_α, B^α) , however, two of these eight

equations are constraint equations. Indeed, in the chosen coordinates, the t -components of the two equations (23) do not contain any time derivatives:

$$C_1 = (\omega_G^{-1})^{0bcd} \partial_b F_{cd} = 0, \quad (24)$$

$$C_2 = |\det(\text{Petrov}(G))|^{-1/6} \partial_b (|\det(\text{Petrov}(G))|^{1/6} G^{0bcd} F_{cd}) = 0. \quad (25)$$

Thus they constrain the initial data one may provide for the fields (E_α, B^α) . Using the remaining evolution equations one finds that

$$\partial_t C_{1,2} = -C_{1,2} \partial_t \ln |\det(\text{Petrov}(G))|^{1/6}, \quad (26)$$

so that the constraints are preserved under evolution in time.

The evolution equations themselves now are of the general form

$$A^b{}_N{}^M \partial_b u^N + B^M{}_N u^N = 0, \quad (27)$$

where $u^N = (E_\alpha, B^\alpha)$ and the four 6×6 matrices A^b

$$A^0 = \begin{bmatrix} G^{0\mu 0\nu} & 0 \\ 0 & \delta_\nu^\mu \end{bmatrix}, \quad A^\alpha = \begin{bmatrix} -2G^{0(\mu\nu)\alpha} & -\frac{1}{2}(\omega_G)_{0\nu\gamma\delta} G^{\gamma\delta\mu\alpha} \\ (\omega_G^{-1})^{0\mu\nu\alpha} & 0 \end{bmatrix}. \quad (28)$$

From the theory of partial differential equations [20], it is known that the local causal behaviour of such a system of differential equations is encoded in the so-called characteristic polynomial $P(k) = \det(A^b k_b)$ defined over transversal ($k_0 \neq 0$) covectors k . Casting this expression into manifestly covariant form (conveniently rescaling k_0 to be unity), one finds for a four-dimensional area metric manifold

$$P(k) = -|\det(\text{Petrov}(G))|^{-1/3} \mathcal{G}(k, k, k, k), \quad (29)$$

where the quartic Fresnel polynomial $\mathcal{G}(k, k, k, k)$ is defined as

$$\mathcal{G}(k, k, k, k) = -\frac{1}{24} (\omega_G)_{mnpq} (\omega_G)_{rstu} G^{mnr(a} G^{b|ps|c} G^{d)qtu} k_a k_b k_c k_d. \quad (30)$$

The tensor \mathcal{G} and its physical interpretation has been first obtained by Rubilar [21] in the context of pre-metric electrodynamics [22], by studying the propagation of electromagnetic field discontinuities. Our derivation here is a complementary one, which we choose since it directly leads to the related causality theory.

Furthermore, the theory of partial differential equations shows that a necessary condition for electromagnetic fields to propagate through an area

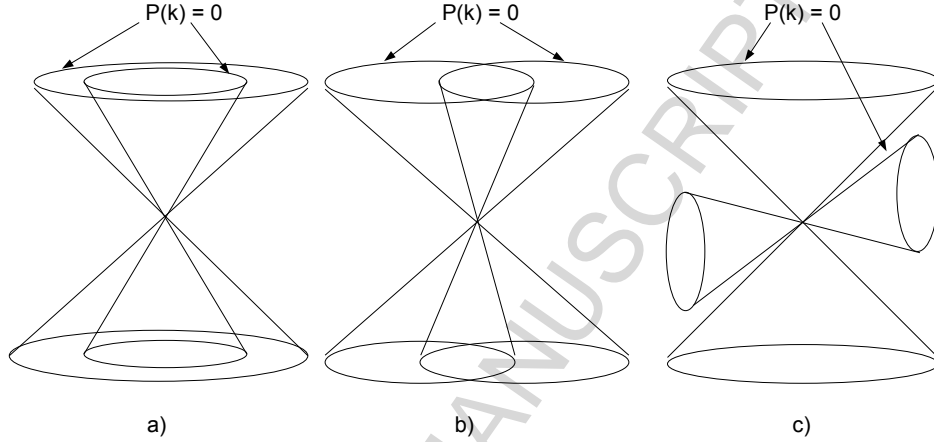


Figure 1: some quartic null cones in cotangent space

metric manifold at all, is that the characteristic polynomial (29) admits non-vanishing null covectors, $P(k) = 0$. Using the definition of the characteristic polynomial this condition reduces to the Fresnel equation

$$\mathcal{G}(k, k, k, k) = 0. \quad (31)$$

From the linearity of the Fresnel polynomial, it follows that the null covectors constitute a cone L_p in each cotangent space T_p^*M , i.e. a subset $L_p \subset T_p^*M$ such that $\lambda L_p \subseteq L_p$ for any real positive λ . Physically speaking, this statement on the admissible wave covectors k is one on the geometric-optical limit of Maxwell theory.

For a metric-induced area metric (5), the quartic Fresnel equation (31) factorizes to the bi-quadratic equation $(g^{ab}k_a k_b)^2 = 0$, which in turn reproduces the familiar notion of covector null cones in Lorentzian geometry. However, the generic case of an area metric manifold leads to more elaborate local null structures (see figure 1 for examples). Going beyond the geometric-optical limit of Maxwell theory one observes that the polarization of light determines which sheet of the surface in cotangent space defined by the quartic condition (31) is chosen [23].

2.4. Convex causal future cones and their duals

Once we ensured that the field propagates at all, we may turn to the question of well-posedness of the initial value problem for Maxwell theory.

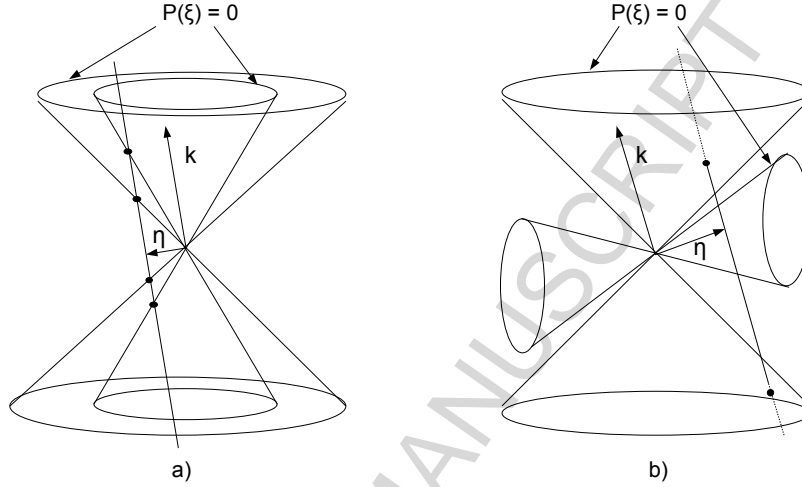


Figure 2: Illustration of a) a timelike covector k and b) a covector k that cannot be timelike

To this end we need to ensure that there are initial data surfaces Σ , which can only in the case if there are covectors k normal to Σ , i.e. $k(\Sigma) = 0$, and for which $P(k) \neq 0$ and $P(\eta - \lambda k)$ has only real roots λ for any covector η . Any such covector k on a four-dimensional area metric manifold is called timelike. Geometrically, a covector k is timelike if *any* line in the direction of k intersects the surface of null covectors four times. This is illustrated in figure 2. We see that timelike covectors exist in the example from figure 1a but there are no timelike covectors in the example from figure 1c.

With the help of the characteristic polynomial $P(k)$ it is also possible to distinguish between future and past with respect to a given time orientation, which is chosen in terms of an everywhere timelike covector field τ : we define the future timelike covector cone C_p^* at a point $p \in M$ as the set of all covectors $\xi \in T_p^*M$ such that the roots λ of $P(\eta - \lambda\xi)$ for any covector $\eta \in T_p^*M$ are positive with respect to the time orientation τ . It can be shown that the future timelike covector cone is a convex cone [24], [25], i.e. for any covector $v \in C_p^*$ we have $\lambda v \in C_p^*$ for any $\lambda \in \mathbb{R}^+$ and for any two covectors $v, w \in C_p^*$ it is true that $v + w \in C_p^*$. Furthermore, C_p^* is open. These two properties and the fact that any two covectors $v, w \in C_p^*$ satisfy the inverse triangle inequality $P(v + w) \geq P(v) + P(w)$ [26] render the concept of the future timelike covector cone on an area metric manifold a true generalization of Lorentzian geometry.

It should be noted that once one has identified future timelike covectors

it is also possible to define future timelike vectors, and to relate them in a one-to-one fashion. To this end, we define the cone C_p of future timelike vectors at a point p , which itself is open and convex, as the set $C_p = \{v \in T_p M \mid k(v) \geq 0 \text{ for all } k \in C_p^*\}$, which is the dual cone to C_p^* . Note that a general feature of the relation between a cone and its dual is that given two cones C_1^* and C_2^* with $C_1^* \subset C_2^*$ we have $C_2 \subset C_1$ for the dual cones C_1 and C_2 . Thus there is an inversion of the inclusion relation when considering two cones and their respective dual cones.

It also turns out that the duality map between future timelike covectors and future timelike vectors is given in terms of the Fresnel tensor \mathcal{G} , whose components may be computed by differentiation of the Fresnel polynomial (30) with respect to the components of the covector k .

Theorem 2.1. *Let (M, G) be a four-dimensional area metric manifold. Let $C_p^* \subset T_p^* M$ be the future timelike covector cone at a point $p \in M$. Then for any $k \in C_p^*$ there is a bijection to the cone C_p of future timelike vectors,*

$$C_p^* \rightarrow C_p, \quad k \mapsto \mathcal{G}(k, k, k, \cdot). \quad (32)$$

Proof. The proof uses the fact that the mapping $C_p^* \rightarrow \mathbb{R}$ defined through $\tau \mapsto -\ln \mathcal{G}(\tau, \tau, \tau, \tau)$ is the so-called self-concordant barrier functional. According to [27], see theorem 27 of [25], it follows that the mapping $C_p^* \rightarrow C_p$ defined as $k \mapsto D \ln \mathcal{G}(k, k, k, k)$ is a bijection. Now it is easily checked that

$$(D \ln \mathcal{G}(k, k, k, k))(\tau) = \left. \frac{d}{dt} \right|_{t=0} \ln \mathcal{G}(k+t\tau, k+t\tau, k+t\tau, k+t\tau) = 4 \frac{\mathcal{G}(k, k, k, \tau)}{\mathcal{G}(k, k, k, k)} \quad (33)$$

for any $\tau \in C_p^*$. Since the denominator never vanishes on the future timelike covector cone C_p^* , one finds that also $k \mapsto \mathcal{G}(k, k, k, \cdot)$ is a bijection $C_p^* \rightarrow C_p$, which concludes the proof.

Thus Theorem 2.1 describes the area metric spacetime analogue of raising and lowering indices on timelike vectors and covectors in Lorentzian geometry. It will also be useful to have the

Definition 2.4. *Let (M, G) be an area metric spacetime. A vector $X \in T_p M$ is called a future causal vector if X lies in the closure $\overline{C_p}$ of the future timelike vector cone C_p .*

It should be noted that vectors X lying on the boundary $\partial \overline{C_p}$ of the closure $\overline{C_p}$ are indeed null vectors, but (in contrast to the special case of

Lorentzian metric manifolds) not every null vector lies on this boundary. We will have opportunity to return to this definitions when we discuss the global causal structure of area metric spacetimes. Note that (given a time orientation) past timelike vectors and past causal vectors may be defined accordingly.

2.5. Area metric spacetimes

We now turn to the definitions of area metric spacetimes, which include conditions of various strength, beyond the mere area metric manifold structure.

Definition 2.5. *Let (M, G) be a four-dimensional area metric manifold. We call (M, G) a weakly hyperbolic area metric spacetime if there exists a time orientation in terms of an everywhere timelike covector field τ .*

The so defined weakly hyperbolic area metric spacetimes are only necessary, but not sufficient to ensure a well-posed initial value problem for Maxwell theory described by the action (20). Indeed, it will be instructive to formulate a notion of strongly hyperbolic area metric spacetimes, which in fact guarantees that the initial value problem for Maxwell theory is well posed, at least locally. To this end, we rewrite the first order PDE system (27) in the form

$$\partial_0 u^M = C^{\alpha M}{}_N \partial_\alpha u^N + D^M{}_N u^N, \quad (34)$$

where $C^\alpha = -(A^0)^{-1} A^\alpha$ and $D = -(A^0)^{-1} B$. Now, consider the matrix $C(k) = C^\alpha k_\alpha$, where k_α are the components of some purely spatial covector k . The matrix $C(k)$ plays a key role in the definition of strongly hyperbolic area metric spacetimes.

Definition 2.6. *Let (M, G) be a four-dimensional area metric manifold. We call (M, G) a strongly hyperbolic area metric spacetime if*

1. *the matrix $C(k) = C^\alpha k_\alpha$ is diagonalizable for all spatial covectors k and has only real eigenvalues λ_i and*
2. *the diagonalisation of $C(k)$ is well-conditioned: if $S(k)C(k)S(k)^{-1} = \text{diag}(\lambda_i)$ then $\sup_{k \in S^2} \|S(k)^{-1}\| \|S(k)\| < \infty$.*

The first requirement simply reformulates the weak hyperbolicity requirement of Definition 2.5, but the second requirement in this definition deserves some further comment. A solution of equation (34) may be obtained by performing a Fourier transformation on the spatial components to

get rid of the spatial derivatives. The resulting ordinary differential equation does only contain time derivatives of the transformed fields \hat{u}_N and may be solved in standard fashion. To obtain the solution of the original system (34) we need to perform an inverse Fourier transformation of the fields \hat{u}_N . The second condition in the definition ensures that the \hat{u}_N are square integrable which means that the inverse Fourier transformation is indeed possible.

It can now be shown that a strongly hyperbolic area metric spacetime renders the initial value problem for Maxwell theory locally well-posed [20]. This proposition only holds locally since we were investigating the field equations (34) in a sufficiently small neighborhood of a point $p \in M$ such that the coefficients in (34) may be assumed constant, but this suffices for our purposes. In the next section, we develop an algebraic classification of area metric manifolds which will be related, in section 5, to the purely analytic characterisation of area metric spacetimes we gave here in Definitions 2.5 and 2.6.

Global causality conditions for area metric spacetimes may now be imposed, exactly following the lines known from Lorentzian geometry [28], employing the technical machinery developed above. Recall that future timelike vectors are vectors X that lie in the future timelike vector cone C_p while future causal vectors lie in the topological closure $\overline{C_p}$ of C_p . A curve $\gamma : I \subseteq \mathbb{R} \rightarrow M$ is said to be timelike (causal) if its tangent vectors at every point $p \in M$ are timelike (causal). We call an area metric spacetime *chronological* if it does not contain any closed timelike curves. Physically speaking, this means that no one is able to meet himself in the past. The proof of the theorem that any compact area metric spacetime contains closed timelike curves and thus fails to be chronological follows the same lines as the proof in the analogous theorem in Lorentzian geometry. An area metric spacetime (M, G) is called causal if there exist no closed causal curves. This definition is a little more restrictive than (M, G) being chronological. Again physically speaking, this means no one is able to communicate with himself in the past. One may then also define the *chronological future* of a point $p \in M$ as the set $I^+(p)$ of all points q for which there exist a future directed timelike curve γ from p to q . It may be shown that $I^+(p)$ is an open set. The *causal future* $J^+(p)$ of a point p is defined analogously with the only difference that the curve connecting p and q has to be causal. In contrast to $I^+(p)$, the set $J^+(p)$ is neither open nor closed. Accordingly, the chronological past $I^-(p)$ and the causal past $J^-(p)$ of a point p may be defined.

With the above notions, one may define even stronger causal requirements for area metric spacetimes by extending the attention to entire neighborhoods $U(p)$ of some point p . An area metric spacetime is called *strongly*

causal if no causal curve that leaves a sufficiently small neighborhood $U(p)$ of a point p ever returns. Remarkably, one may prove that for a strongly causal area metric spacetime (M, G) , the Alexandrov topology (which is generated by all diamonds of the form $I^+(p) \cap I^-(q)$ with $p, q \in M$) agrees with the given topology of the manifold M . Again the proof follows the same line of arguments as the proof of the analogous proposition in Lorentzian geometry.

Clearly, we were only able to touch on the very basics of the global causality theory for area metric spacetimes here. As in the case of Lorentzian manifolds, it should be interesting to push this study further, to see which of the other well-known metric theorems directly extend to the area metric case, and what further physical conclusions can be drawn.

3. Applications in physics

Equipped with the basic notions of area metric geometry, we now pause the formal developments of this paper to present three physical applications where area metric manifolds emerge naturally. More precisely, we consider crystal optics, the propagation of photons in gravitational vacua, and gravitational dynamics for an area metric manifold.

The fundamental role played by area metrics in these applications, however, raises a pressing issue. As we discussed in the previous section, an area metric manifold needs to satisfy hyperbolicity conditions in order to not prohibit a causal propagation of light. But it is difficult to directly apply the causality criterion provided by strong hyperbolicity. This problem is remedied by sections 4 and 5, where we identify entire algebraic classes that cannot possibly serve as physically consistent backgrounds. But now we turn to the concrete physical applications where these results will have a concrete impact.

3.1. Crystal optics

Maxwell electrodynamics in linear optical crystals is governed by the following constitutive relations between the electromagnetic field strength vectors \mathbf{E}, \mathbf{B} and the electromagnetic induction vector densities \mathbf{D}, \mathbf{H} ,

$$\mathbf{D} = \varepsilon \mathbf{E} + \alpha \mathbf{B}, \quad (35)$$

$$\mathbf{H} = \beta \mathbf{E} + \mu^{-1} \mathbf{B}. \quad (36)$$

The matrices ε and μ encode the permittivity and permeability of the material, while the matrices α and β are responsible for magneto-electric effects.

If the observer measuring these relations is described by a frame e_0, e_1, e_2, e_3 , the fields and inductions, and the constitutive relation, give rise to a space-time two-form F_{ab} , a bivector density $H^{ab} = H^{[ab]}$, and a tensor density $\chi^{abcd} = \chi^{[ab][cd]}$. In Petrov notation this simply amounts to

$$Petrov(F) = \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix}, \quad Petrov(H) = \begin{bmatrix} \mathbf{D} \\ \mathbf{H} \end{bmatrix}, \quad (37)$$

$$Petrov(\chi) = \begin{bmatrix} \varepsilon & \alpha \\ \beta & \mu^{-1} \end{bmatrix}. \quad (38)$$

With these identifications, the covariant form of the constitutive relation reads

$$H^{ab} = \frac{1}{2} \chi^{abcd} F_{cd}. \quad (39)$$

Due to its two Petrov indices, each of which may take six values, the tensor density χ in this relation has 36 algebraic components; these decompose further as $36 = 21 \oplus 15$ into symmetric and antisymmetric representations. The antisymmetric 15 has been termed the skewon part, e.g. in [29], but cannot be obtained from an action for the electromagnetic field. Indeed, in order to produce a linear relation of the type (39), the action $\int d^4x \mathcal{L}$ must feature a Lagrangian scalar density quadratic in F ,

$$\mathcal{L} = \frac{1}{4} \chi^{abcd} F_{ab} F_{cd}. \quad (40)$$

Here the 15 representation in χ does not contribute. The standard definition $H^{ab} = \partial \mathcal{L} / \partial F_{ab}$ gives (39), and variation yields the Maxwell equations.

Making use of an action means restricting attention to the important class of optical media governed by non-dissipative, reversible electromagnetic processes. The reason for this is that the diffeomorphism invariance of the action results in an equation of energy momentum conservation. The symmetries of χ in the remaining 21 representation are now precisely the symmetries of an area metric. In this case we have symmetric ε , μ and $\beta = \alpha^T$. A comparison of the Lagrangian (40) with that of electrodynamics on area metric backgrounds in (20) identifies

$$\chi^{abcd} = |\det(Petrov(G))|^{1/6} G^{abcd}. \quad (41)$$

So all non-dissipative linear optical media are fully described by area metric backgrounds, and the relevant electrodynamic equations are those of area metric electrodynamics.

Let us return to the equations (35) and (36). Within the experimental bounds of accuracy our measurements in vacuum tell us that $\alpha = 0$, $\beta = 0$, and $\varepsilon = \varepsilon_0 \mathbb{1}$, $\mu = \mu_0 \mathbb{1}$. Hence, in Heaviside units and with respect to an observer's frame,

$$Petrov(\chi) = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix}. \quad (42)$$

This form of χ corresponds to the simple case of a metric-induced area metric, as becomes clear by inserting definition (6) into (41) and using a metric orthonormal frame.

But there exist a whole number of non-dissipative optical media with anisotropic ε , μ , and also with magnetoelectric effects mediated by α , see references in [30]. As backgrounds for electrodynamics, these require a full area metric description. The causality analysis of this paper tells us from first principles which crystal backgrounds may appear in Nature, by placing restrictions on the allowed relations χ , and hence on ε , μ , and α . The algebraic classification of area metrics performed in the following sections thus becomes a geometric classification of optical materials. Through relation (41) each optical material will be associated with a unique normal form of its area metric; this completes the program of [31].

3.2. Photons in curved spacetime

Drummond and Hathrell calculated the one-loop effective action for photon propagation on a curved metric background, by taking into account the production of virtual electron-positron pairs in the framework of quantum electrodynamics [1]. In a gravitational vacuum, i.e., where the Lorentzian spacetime is Ricci flat, they obtain the effective electromagnetic action

$$W[A] \sim \int d^4x \sqrt{-g} (g^{a[c} g^{d]b} + \lambda C^{abcd}) F_{ab} F_{cd} + O(\lambda^2), \quad (43)$$

where C denotes the Weyl curvature tensor of g and $\lambda = \alpha/(90\pi m^2) \cong 3,85 \text{ fm}^2$ (with electron mass m and fine structure constant α) is the characteristic scale of the interaction.

The field equations arising from the effective action above are modified at $O(\lambda)$. It is important to realize that this affects their geometric optical limit, and hence the spacetime trajectory of photons. Precisely how photon trajectories change is easily answered for conformally flat spacetimes where the Weyl tensor vanishes. Then the action and field equations are reduced to the metric case, so that photons propagate on null geodesics. The same question is more difficult to answer if the Weyl tensor is non-zero; now the

propagation is non-trivially modified. This is in stark contrast to the view that the Lorentzian cone of the metric has a physical meaning independent of the one defined by the propagation of light, which leads to a number of unexpected subtle results [32, 33].

These effects are nicely explained from the area metric perspective. We observe that, to leading order in λ , one may view the Drummond-Hathrell effective action as Maxwell theory on an area metric manifold with area metric $G_{DH} = g^{a[c}g^{d]b} + \lambda C^{abcd}$. In other words, the first order quantum corrections of Maxwell theory in vacuum on a curved metric spacetime may be absorbed into an area metric structure [34]. The causality discussion of the previous section then identifies the causal cones relevant for the propagation of fields, and clarifies why superluminal propagation with respect to the metric must be expected. The metric simply is no longer the relevant structure to describe the quantum corrected background. As discussed in detail in [23] photons now propagate along null Finsler geodesics with respect to a Finsler norm given by the Fresnel tensor of the area metric.

Already Drummond and Hathrell [1] remarked on the fact that the Weyl curvature modification in their effective action would lead to an opening of the light cones in the early Universe, which addresses the horizon problem by potentially creating much larger causally connected regions before the surface of last scattering. This calculation was possible because of the high degree of symmetry in Robertson-Walker cosmologies. The Weyl curvature effect then can be absorbed into a conformal factor of the spacetime metric, and the relevant cones could be understood in standard fashion.

The technology presented in this paper now allows for an understanding of the general causal cones in arbitrarily modified area metric backgrounds. In the case of Drummond-Hathrell electrodynamics this means the change of causal cones can be discussed on arbitrary backgrounds; for instance for inhomogeneous cosmologies.

3.3. Refinement of Einstein-Hilbert gravity

From the perspective of a metric vacuum spacetime, crystal optics appears as a complex phenomenon, and we have seen that its description requires the more refined geometry and causality of area metrics. Interestingly, the results of Drummond and Hathrell on photon propagation tell us that also the notion of a metric vacuum must be modified if quantum effects are taken into account. This may be seen as evidence for a more complex structure of the spacetime vacuum: area geometry which only reduces to metric geometry in simple physical situations. In other words, one might won-

der, whether spacetime is as complicated as a crystal, whether gravitational lenses exist with the properties of optical lenses.

Taking this point of view serious requires a dynamical theory for area metrics in their own right: area metric gravity. One would like to write down second order differential equations for area metrics that in the metric case reduce to Einstein's equations. Moreover, the equations should not involve any tunable scales that could be used to render departures from Einstein-Hilbert gravity so small that the theory could no longer be predictive.

Based on the construction of an area metric curvature scalar \mathcal{R}_G , we have developed such a theory in [14], where we refer the reader for full detail. The action takes the simple form

$$S[G] = \int d^4x |\det(\text{Petrov}(G))|^{1/6} \mathcal{R}_G, \quad (44)$$

and reduces to the Einstein-Hilbert action in case the area metric is metric-induced. All requirements on the theory as formulated above are met in applications to cosmology and spherical symmetry (the solar system), and these applications lead to potentially very promising results [12, 35, 36]. In particular, an area metric G with Robertson-Walker symmetries is metric-induced by a Robertson-Walker metric, but contains an additional cosmological scalar field that controls the totally antisymmetric part $G_{[abcd]}$. We have shown in [12] that this scalar field modifies the standard Friedmann and acceleration equation of the Universe, and may indeed provide an explanation for the accelerating late Universe.

The relevance of the results of this paper on the causal structure of area metric spacetime will be the fundamental basis for a deeper understanding of area metric gravity. This applies not only to the mathematically precise definition of area metric observers, but also to the gravity dynamics. The causal structure is the key ingredient in a rederivation of area metric gravity from the deformation algebra of hypersurfaces in area metric spacetime, in analogy to the metric calculation of [4].

We now resume the formal developments of this paper.

4. Algebraic classification of four-dimensional area metrics

In this section we present the algebraic classification of area metrics in four dimensions. Besides our discussion of causal cones, this classification constitutes the second technical pillar on which the results to be derived in section 5 rest. In essence, the here obtained classification amounts to 46 continuous families of distinct algebraic classes of four-dimensional area

metrics. These may be conveniently grouped into 23 metaclasses, which we explicitly display at the end of this section, and which play an important role in deciding which algebraic classes of area metrics can constitute spacetimes (or physically valid optical backgrounds).

4.1. Formulation of the problem

We first need to decide according to which criterion we want to classify area metrics. Since we are interested in the area metric data that are independent of a choice of frame, we choose to locally identify area metrics that contain the same information up to a change of the local frame. Therefore we will classify area metrics according to the following

Definition 4.1. *We call two area metrics G and H on the same d -dimensional manifold M strongly equivalent, $G \sim H$, if for every point $p \in M$ there exists a $GL(d)$ -transformation t such that*

$$G_{abcd} = t^m_a t^n_b t^p_c t^q_d H_{mnpq}. \quad (45)$$

Clearly, the relation \sim is an equivalence relation. The problem of classification can now be stated as follows: identify the algebraic classes of area metrics as the equivalence classes with respect to the equivalence relation \sim . In other words, two area metrics G and G^* which cannot be pointwise related by a change of frame, according to (45), belong to different algebraic classes.

Once the equivalence classes are identified, it is convenient to pick a particularly simple representative of each algebraic class, which we will refer to as the normal form of this class.

In two dimensions, area metric geometry is essentially symplectic geometry, as we have seen in section 2. Therefore the classification of area metrics in two dimensions is obtained by virtue of Darboux's theorem for symplectic vector spaces [16]. It states that up to frame transformations, there is only one symplectic form. In three dimensions, area metric geometry is essentially metric geometry. Consequently, the classification of area metrics in three dimensions can be carried out with the help of Sylvester's theorem [37] for symmetric bilinear forms. From the fact that one needs to employ rather different classification theorems in two and three dimensions, namely Darboux's theorem on the one hand and Sylvester's theorem on the other hand, one may expect that yet another theorem must be found to classify area metrics in four dimensions. In this section, we will show that this is indeed the case. The classification in dimensions higher than four currently remains an open problem.

4.2. *Weak classification of area metrics*

As a first step towards the classification of four-dimensional area metrics with respect to the equivalence relation \sim , we will first consider a weak classification of area metrics in arbitrary dimensions d . We will then use the insights of these considerations to solve our original classification problem.

From the symmetries of the area metric G , it is immediately clear that equation (45) takes the form

$$G_{abcd} = t^{[m}_a t^{n]}_b t^{[p}_c t^{q]}_d H_{mnpq}. \quad (46)$$

Let us now use Petrov notation to write equation (46) in the form

$$G_{AB} = T^M_A T^N_B H_{MN}, \quad (47)$$

where T^M_A is the Petrov matrix associated to the tensor $2t^{[m}_a t^{n]}_b$,

$$T = \text{Petrov}(2t^{[m}_a t^{n]}_b). \quad (48)$$

Although we introduced T as a transformation induced by $t \in GL(d)$, one may also read (47) as an equation for an arbitrary $T \in GL(d(d-1)/2)$. It is clear that such a T is generally not induced from a $t \in GL(d)$. Hence the requirement that two area metrics be related by a $GL(d(d-1)/2)$ transformation as in (47) is weaker than the requirement (45). This gives rise to the following definition:

Definition 4.2. *We call two area metrics G and H on the same d -dimensional manifold M weakly equivalent, $G \approx H$, if for every point $p \in M$ there exists a $GL(d(d-1)/2)$ -transformation T such that equation (47) holds.*

Obviously, the relation \approx is also an equivalence relation with respect to which we may classify area metrics. It is also clear that two area metrics G and G^* that are strongly equivalent are automatically weakly equivalent since any $t \in GL(d)$ induces a $T \in GL(d(d-1)/2)$ as we have seen above. However, the converse does not hold. Classification of area metrics with respect to strong equivalence \sim rather amounts to weak classification under the constraint of picking only those $T \in GL(d(d-1)/2)$ that are indeed induced by a $t \in GL(d)$. Finding a condition in four dimensions that ensures that T is of the form (48) is the task solved in the next section.

The classification of d -dimensional area metrics with respect to the weak equivalence relation \approx itself can be achieved easily by applying Sylvester's theorem, since the area metric in equation (47) defines a symmetric bilinear

form on $\mathbb{R}^{d(d-1)/2}$. Sylvester's theorem states that the $GL(d(d-1)/2)$ -signature of such an inner product is the only frame-independent information. We can therefore classify area metrics in d dimensions according to their $GL(d(d-1)/2)$ -signature. This amounts to $d(d-1)/2 + 1$ possible algebraic classes.

4.3. Strong classification of area metrics in four dimensions

We may now formulate a condition for a transformation $T \in GL(6)$ to be induced by a $GL(4)$ transformation to refine the weak classification to the algebraic classification we actually aimed for. In four dimensions, there indeed is such a condition, using the fact that the canonical area metric volume form (13) which in Petrov form reads

$$Petrov(\omega_G) = |\det(Petrov(G))|^{1/6} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (49)$$

is an area metric in its own right. With this in mind, we have the following

Theorem 4.1. *Let G and H be two area metrics on an orientable four-dimensional manifold M . If the weak equivalences $G \approx H$ and $\omega_G \approx \omega_H$ hold simultaneously with the same $GL(6)$ transformation, then we have either the strong equivalence $G \sim H$ or $G \sim \Sigma^t H \Sigma$, where the components Σ_{cd}^{ab} of the endomorphism Σ are numerically the same as ϵ_{abcd} with $\epsilon_{0123} = 1$.*

Proof. With the help of the inverse identification of the capital Petrov indices with antisymmetric pairs of indices $[ab]$ over some given frame $\{e_a\}$ of \mathbb{R}^4 , the weak equivalences may be expressed as

$$G_{abcd} = \frac{1}{4} T_{ab}^{mn} T_{cd}^{pq} H_{mnpq}, \quad 4|\det(Petrov(GH^{-1}))|^{1/6} \epsilon_{abcd} = T_{ab}^{mn} T_{cd}^{pq} \epsilon_{mnpq}. \quad (50)$$

The second condition can be read as a restriction on the six bivectors $\{T_{01}^{ab}, T_{02}^{ab}, T_{03}^{ab}, T_{12}^{ab}, T_{31}^{ab}, T_{23}^{ab}\}$: we must have vanishing $T_{01} \wedge T_{02}$, $T_{02} \wedge T_{03}$ and $T_{03} \wedge T_{01}$ and all T_{ab} must be wedge products of two vectors which is equivalent to have vanishing $T_{ab} \wedge T_{ab}$ (no sum). The first three conditions can be solved in two inequivalent ways: either T_{01} , T_{02} and T_{03} have a direction in common, or they pairwise intersect. But this precisely corresponds to either

$T_{cd}^{ab} = t^{[a} t^b]_d$ or $T_{cd}^{ab} = t^{[m} t^n]_d \Sigma_{mn}^{ab}$ in terms of some $GL(4)$ transformation with $\det(t) > 0$ [38], [39]. Since M is orientable we can consistently restrict our attention to this case, which concludes the proof.

The result of this theorem deserves some further comments. The correspondence between the weak and the strong equivalence involves an ambiguity in the order of frame on \mathbb{R}^6 by means of the Σ -symbols. This only implies that once we have found suitable normal forms of the simultaneous weak classification of G and ω_G , we must be careful with the interpretation of the Petrov matrix representing the normal forms. We will return to that point later.

Fortunately, the remaining step in obtaining the desired strong classification of the area metric G now reduces to the problem of the simultaneous weak classification of the area metric G and its associated area metric volume form ω_G . The solution to this problem is known, and we cite the relevant theorem without proof [40] in a form that is congenial for our purpose.

Theorem 4.2. *Let G and ω_G be two symmetric bilinear forms on \mathbb{R}^6 . Then there exists a basis on \mathbb{R}^6 such that the matrices that represent G and ω_G take the following block diagonal form:*

$$\begin{aligned} \text{Petrov}(G) &= R_1 \oplus \dots \oplus R_m \oplus C_{m+1} \oplus \dots \oplus C_n, \\ \text{Petrov}(\omega_G) &= \epsilon_1 P_1 \oplus \dots \oplus \epsilon_m P_m \oplus P_{m+1} \oplus \dots \oplus P_n, \end{aligned}$$

where blocks with the same index have equal size and the matrices representing the blocks R_p are of the form

$$R_p(\lambda_p) = \begin{bmatrix} 0 & \dots & 0 & \lambda_p \\ \vdots & & \ddots & 1 \\ 0 & \ddots & \ddots & 0 \\ \lambda_p & 1 & 0 & 0 \end{bmatrix}$$

with real numbers λ_p (which incidentally correspond to the (real) eigenvalues of the endomorphism $J = \omega_G^{-1}G$), whereas the C_q take the form

$$C_q(\sigma_q, \tau_q) = \begin{bmatrix} 0 & 0 & 0 & 0 & -\tau_q & \sigma_q \\ 0 & 0 & 0 & 0 & \sigma_q & \tau_q \\ 0 & 0 & & \ddots & 0 & 1 \\ 0 & 0 & \ddots & & 1 & 0 \\ -\tau_q & \sigma_q & 0 & 1 & 0 & 0 \\ \sigma_q & \tau_q & 1 & 0 & 0 & 0 \end{bmatrix}.$$

with real numbers σ_q and τ_q (corresponding to the (complex) eigenvalues $\sigma \pm i\tau$ of J) with $\tau_q > 0$. Finally, we have the signs $\epsilon_j = \pm 1$ and

$$P_j = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \ddots & 0 \\ 0 & \ddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

This theorem now applies to our problem as follows. Given an area metric G and its associated volume form ω_G in Petrov notation we use the theorem to bring G and ω_G to the stated simultaneous normal forms by a $GL(6)$ transformation. After that we still need to apply a further change of basis to bring ω_G to the form of equation (49) so that we can apply our Theorem 4.1. Such a change of basis on \mathbb{R}^6 is always possible if the matrix provided by the theorem that represents ω_G has $GL(6)$ signature $(3, 3)$, since we know that the signature is the only local frame-independent information for such a symmetric bilinear form. The change of basis has to be simultaneously applied to the area metric G , and the resulting matrix then is the desired normal form of the area metric. Thus all $GL(6)$ -inequivalent pairs (G, ω_G) appearing in Theorem 4.2 for which ω_G has $GL(6)$ -signature $(3, 3)$ represent a different algebraic class of the area metric G . Distinguishing the different sign characteristics of the blocks in ω_G , a simple counting reveals that there are 46 distinct algebraic classes.

There is another subtlety in the application of the theorem. To make sure that the obtained pair of normal forms (G, ω_G) is a pair of an area metric and its associated volume form we need to require $|\det(\text{Petrov}(G))| = -\det(\text{Petrov}(\omega_G)) = 1$ which follows directly from the definition of the volume form. This requirement constraints the scalars in the normal form of the area metric G . The theorem itself determines up to six scalars and the extra condition on $\det(\text{Petrov}(G))$ renders only five of them independent. This confirms our intuitive claim from section 2 where we suspected a four-dimensional area metric to determine up to five $GL(4)$ -scalars.

Before we continue, we illustrate the above-described procedure for obtaining the normal form of a given area metric G .

Example. Consider the following possible pair of matrices provided by Theorem 4.2 that represent two bilinear forms G and ω_G

$$Petrov(\omega_G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (51)$$

$$Petrov(G) = \begin{bmatrix} -\tau_1 & \sigma_1 & 0 & 0 & 0 & 0 \\ \sigma_1 & \tau_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau_2 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 & \tau_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tau_3 & \sigma_3 \\ 0 & 0 & 0 & 0 & \sigma_3 & \tau_3 \end{bmatrix}. \quad (52)$$

To recover ω_G in the form (49) (which is possible in the first place since $Petrov(\omega_G)$ indeed has signature $(3, 3)$), we interchange the first and fifth basis vector on \mathbb{R}^6 which amounts to a change of the first and fifth row and column in $Petrov(\omega_G)$ and $Petrov(G)$. Then we also exchange the second and fifth basis vector. The matrix representing G then takes the normal form

$$Petrov(G) = \begin{bmatrix} -\tau_1 & 0 & 0 & 0 & 0 & \sigma_1 \\ 0 & -\tau_3 & 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & -\tau_2 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 & \tau_2 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 & \tau_3 & 0 \\ \sigma_1 & 0 & 0 & 0 & 0 & \tau_1 \end{bmatrix}. \quad (53)$$

We have to keep in mind that the six scalars in $Petrov(G)$ have to satisfy the condition $|\det(Petrov(G))| = 1$.

We should emphasize an important point. The scalars in $Petrov(G)$ locally determine the area metric completely. Actually every different set of scalars in any of the 46 distinct algebraic classes provided by Theorem 4.2 determines a separate algebraic class for area metrics. In other words, there are infinitely many algebraic classes of four-dimensional area metrics.

Having reduced the Petrov matrix $Petrov(G)$ of a given area metric G to its normal form according to Theorem 4.2 we can apply Theorem 4.1 to find the actual normal forms of the area metric G with respect to \sim . We only need to be careful with the mentioned ambiguity by means of the

Σ -symbols in Theorem 4.1 when we identify the entries of $Petrov(G)$ with components G_{abcd} of the area metric G . Both G_{abcd} and $G_{mnpq}\Sigma_{ab}^{mn}\Sigma_{cd}^{pq}$ are the components of distinct normal forms of the area metric G .

It proves useful to group the infinitely many normal forms in four dimensions into coarser classes labeled by the Segré type of the endomorphism J appearing in Theorem 4.2. We introduce these metaclasses in the following section and then display the resulting families of normal forms.

4.4. Metaclasses and normal forms

A convenient way to group the possible normal forms of the area metric G is a division into metaclasses labeled by the Segré type [41] of the endomorphism J defined in Theorem 4.2. The Segré types of the endomorphism J only take into account the size of the Jordan blocks [37] in J , and whether the eigenvalues of the corresponding block are complex or real. That is, a Segré type is given by a symbol $[A\bar{A} \dots BCD]$ where A, B, C, D are positive integers. If an integer A in the label is followed by \bar{A} , the endomorphism J contains a Jordan block of size A with a complex eigenvalue and a block with the same size and the complex conjugate eigenvalue. Otherwise the endomorphism contains a real Jordan Block of size B, C and D . For example, the normal form $Petrov(G)$ in (53) is of Segré type $[1\bar{1} \ 1\bar{1} \ 1\bar{1}]$ because the corresponding endomorphism J has six distinct complex eigenvalues where three of them are simply the complex conjugates of the other three, and the Jordan block for every eigenvalue has size one.

The metaclasses of area metrics, labeled as defined by the various Segré types, disregard both the signs ϵ_j as they appear in Theorem 4.2, and the actual eigenvalues of the endomorphism J . This gives rise to 23 different metaclasses of area metrics:

- three metaclasses where the Jordan blocks of the corresponding endomorphism J only have complex eigenvalues $\sigma_i \pm i\tau_i$

$$I = [1\bar{1} \ 1\bar{1} \ 1\bar{1}], \quad II = [1\bar{1} \ 2\bar{2}], \quad III = [3\bar{3}],$$

- four metaclasses with real Jordan blocks in J of at most size one

$$IV = [1\bar{1} \ 1\bar{1} \ 11], \quad V = [2\bar{2} \ 11], \quad VI = [1\bar{1} \ 1111], \quad VII = [111111]$$

- 16 metaclasses with at least one real Jordan block in J of size greater or equal to two.

These metaclasses prove very useful. Indeed, in the next section we will present a powerful theorem that renders the 16 metaclasses VIII-XXIII (which feature real Jordan blocks of size greater or equal to two) as unphysical since these do not define strongly hyperbolic area metric spacetimes as defined in the previous section.

Now that we have introduced the metaclasses of area metrics labeled by the Segré types of the endomorphism J , we may present a complete list of normal forms of these metaclasses.

Theorem 4.3. *Let (M, G) be a four-dimensional area metric manifold. Then at each point $p \in M$ there exists a frame $\{e_a\}$ in which the Petrov matrix $Petrov(G)$ of G takes one of the following forms.*

$metaclass\ I\ [1\bar{1}\ 1\bar{1}\ 1\bar{1}]$	$metaclass\ II\ [2\bar{2}\ 1\bar{1}]$
$\begin{bmatrix} -\tau_1 & 0 & 0 & 0 & 0 & \sigma_1 \\ 0 & -\tau_3 & 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & -\tau_2 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 & \tau_2 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 & \tau_3 & 0 \\ \sigma_1 & 0 & 0 & 0 & 0 & \tau_1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 & -\tau_1 & \sigma_1 \\ 0 & 0 & 0 & 0 & \sigma_1 & \tau_1 \\ 0 & 0 & -\tau_2 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 & \tau_2 & 0 & 0 \\ -\tau_1 & \sigma_1 & 0 & 0 & 0 & 1 \\ \sigma_1 & \tau_1 & 0 & 0 & 1 & 0 \end{bmatrix}$
$metaclass\ III\ [3\bar{3}]$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & -\tau_1 & \sigma_1 \\ 0 & 0 & 0 & 0 & \sigma_1 & \tau_1 \\ 0 & 0 & -\tau_1 & \sigma_1 & 0 & 1 \\ 0 & 0 & \sigma_1 & \tau_1 & 1 & 0 \\ -\tau_1 & \sigma_1 & 0 & 1 & 0 & 0 \\ \sigma_1 & \tau_1 & 1 & 0 & 0 & 0 \end{bmatrix}$	
$metaclass\ IV\ [1\bar{1}\ 1\bar{1}\ 11]$	$metaclass\ V\ [2\bar{2}\ 11]$
$\begin{bmatrix} -\tau_1 & 0 & 0 & 0 & 0 & \sigma_1 \\ 0 & -\tau_2 & 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_1 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & \tau_2 & 0 \\ \sigma_1 & 0 & 0 & 0 & 0 & \tau_1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 & -\tau_1 & \sigma_1 \\ 0 & 0 & 0 & 0 & \sigma_1 & \tau_1 \\ 0 & 0 & \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_1 & 0 & 0 \\ -\tau_1 & \sigma_1 & 0 & 0 & 0 & 1 \\ \sigma_1 & \tau_1 & 0 & 0 & 1 & 0 \end{bmatrix}$

metaclass VI [$1\bar{1} 11 11$]

$$\begin{bmatrix} -\tau_1 & 0 & 0 & 0 & 0 & \sigma_1 \\ 0 & \lambda_3 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_1 & 0 & 0 \\ 0 & \lambda_4 & 0 & 0 & \lambda_3 & 0 \\ \sigma_1 & 0 & 0 & 0 & 0 & \tau_1 \end{bmatrix}$$

metaclass VII [$11 11 11$]

$$\begin{bmatrix} \lambda_5 & 0 & 0 & 0 & 0 & \lambda_6 \\ 0 & \lambda_3 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_1 & 0 & 0 \\ 0 & \lambda_4 & 0 & 0 & \lambda_3 & 0 \\ \lambda_6 & 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}$$

The remaining 16 metaclasses involve a choice of signs ϵ_i that take the values ± 1 . Any combination of these signs denotes a different algebraic class.

metaclass VIII [6]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & \epsilon_1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ \lambda_1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

metaclass IX [42]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & \epsilon_1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & \epsilon_2 & 0 \\ \lambda_1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

metaclass X [$4 1\bar{1}$]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & -\tau_1 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_1 & \tau_1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & \epsilon_1 & 0 \\ \lambda_1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

metaclass XI [411]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_3 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & \epsilon_1 & 0 \\ \lambda_1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

metaclass XII [$2\bar{2} 2$]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \tau_1 & \sigma_1 \\ 0 & 0 & 0 & 0 & \sigma_1 & \tau_1 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & \epsilon_1 & 0 & 0 \\ -\tau_1 & \sigma_1 & 0 & 0 & 0 & 1 \\ \sigma_1 & \tau_1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

metaclass XIII [222]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & \epsilon_2 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_3 & \epsilon_3 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & \epsilon_1 \end{bmatrix}$$

metaclass XIV [$22 1\bar{1}$]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & \epsilon_2 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & -\tau_1 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_1 & \tau_1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & \epsilon_1 \end{bmatrix}$$

metaclass XV [2211]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & \epsilon_2 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_4 & \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & \epsilon_1 \end{bmatrix}$$

metaclass XVI [$2 1\bar{1} 1\bar{1}$]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & -\tau_2 & 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & -\tau_1 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_1 & \tau_1 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & \tau_2 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & \epsilon_1 \end{bmatrix}$$

metaclass XVII [$211 1\bar{1}$]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & -\tau_2 & 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & \lambda_2 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_2 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & \tau_2 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & \epsilon_1 \end{bmatrix}$$

metaclass XVIII [21111]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & \lambda_4 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & \lambda_2 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_2 & 0 & 0 \\ 0 & \lambda_5 & 0 & 0 & \lambda_4 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & \epsilon_1 \end{bmatrix}$$

metaclass XIX [51]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & \frac{\epsilon_1}{2}(\lambda_1 - \lambda_2) & \frac{1}{2}(\lambda_1 + \lambda_2) & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{2}(\lambda_1 + \lambda_2) & \frac{\epsilon_1}{2}(\lambda_1 - \lambda_2) & \frac{\epsilon_1}{\sqrt{2}} & 0 \\ 0 & \lambda_1 & \frac{1}{\sqrt{2}} & \frac{\epsilon_1}{\sqrt{2}} & 0 & 0 \\ \lambda_1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

metaclass XX [33]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ 0 & \frac{1}{2}(\lambda_1 - \lambda_2) & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2}(\lambda_1 + \lambda_2) & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \lambda_2 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2}(\lambda_1 + \lambda_2) & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2}(\lambda_1 - \lambda_2) & \frac{1}{\sqrt{2}} \\ \lambda_1 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

metaclass XXI [321]

$$\begin{bmatrix} \epsilon_2 & 0 & 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & \frac{\epsilon_1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \lambda_1 & 0 \\ 0 & \frac{\epsilon_1}{\sqrt{2}} & \frac{\epsilon_1}{2}(\lambda_1 - \lambda_3) & \frac{1}{2}(\lambda_1 + \lambda_3) & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2}(\lambda_1 + \lambda_3) & \frac{\epsilon_1}{2}(\lambda_1 - \lambda_3) & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

metaclass XXII [31 11̄]

$$\begin{bmatrix} -\tau_1 & 0 & 0 & 0 & 0 & \sigma_1 \\ 0 & 0 & \frac{\epsilon_1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \lambda_1 & 0 \\ 0 & \frac{\epsilon_1}{\sqrt{2}} & \frac{\epsilon_1}{2}(\lambda_1 - \lambda_2) & \frac{1}{2}(\lambda_1 + \lambda_2) & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2}(\lambda_1 + \lambda_2) & \frac{\epsilon_1}{2}(\lambda_1 - \lambda_2) & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ \sigma_1 & 0 & 0 & 0 & 0 & \tau_1 \end{bmatrix}$$

metaclass XXIII [3111]

$$\begin{bmatrix} \lambda_3 & 0 & 0 & 0 & 0 & \lambda_4 \\ 0 & \frac{\epsilon_1}{2}(\lambda_1 - \lambda_2) & \frac{\epsilon_1}{\sqrt{2}} & 0 & \frac{1}{2}(\lambda_1 + \lambda_2) & 0 \\ 0 & \frac{\epsilon_1}{\sqrt{2}} & 0 & \lambda_1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(\lambda_1 + \lambda_2) & \frac{1}{\sqrt{2}} & 0 & \frac{\epsilon_1}{2}(\lambda_1 - \lambda_2) & 0 \\ \lambda_4 & 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

With the above theorem guaranteeing the existence of a frame such that an area metric takes one of the listed 23 normal forms, it only remains to study whether such a frame is uniquely determined or, if not, how it is related to other such frames. This question is addressed by the following theorem, already anticipating the result from the next section that the metaclasses VIII to XXIII must be discarded as physically not viable area metric spacetimes, see section 4.1.

Theorem 4.4. *Let (M, G) be a four-dimensional area metric manifold with an area metric falling into one of the metaclasses I to VII. Then the frame in which the Petrov matrix $\text{Petrov}(G)$ representing G takes the form displayed in theorem 4.3 is determined up to a transformation obtained by exponentiation of the algebras presented in table 1. The elements of the possible gauge algebra of metaclass II area metrics are of the form*

$$\omega_1^{II} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \omega_2^{II} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (54)$$

metaclass	local gauge algebra...	...in presence of degeneracies
I	$o(1, 1) \oplus o(2)$ $o(1, 3)$	w.l.o.g. $\sigma_1 = \sigma_2, \tau_1 = \tau_2$ all $\sigma_i = \sigma_j, \tau_i = \tau_j$
II	see generators (54)	$\tau_1 = \tau_2$ and $\sigma_1 = \sigma_2$
III	no symmetries	
IV	$o(2)$ $o(1, 1) \oplus o(2)$ $o(1, 1) \oplus o(2) \oplus o(2) \oplus o(2)$	$\tau_i = \lambda_1$ and $\sigma_i = \lambda_2, i = 1 \vee i = 2$ $\tau_1 = \tau_2$ and $\sigma_1 = \sigma_2$ $\tau_1 = \tau_2 = \lambda_1$ and $\sigma_1 = \sigma_2 = \lambda_2$
V	no symmetries	
VI	$o(2)$ $o(2) \oplus o(2)$ $o(2) \oplus o(2) \oplus o(2) \oplus o(2)$	$\tau_1 = \lambda_1$ and $\sigma_1 = \lambda_2$ or $\tau_1 = \lambda_3$ and $\sigma_1 = \lambda_4$ $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$ $\tau_1 = \lambda_1 = \lambda_3$ and $\sigma_1 = \lambda_2 = \lambda_4$
VII	$o(2) \oplus o(2)$ $o(4)$	w.l.o.g. $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$ $\lambda_1 = \lambda_3 = \lambda_5$ and $\lambda_2 = \lambda_4 = \lambda_6$

Table 1: Local gauge algebras for metaclass I-VII area metrics

Proof. The requirement that the Petrov matrix $Petrov(G)$ representing G stay invariant under a change of the local frame can be expressed in a given basis $\{e_a\}$ as

$$t^m_a t^n_b t^p_c t^q_d G_{mnpq} = G_{abcd}. \quad (55)$$

Without loss of generality, we may assume that G is given in normal form, and anticipating the unphysicality of metaclasses VIII-XXIII shown in section 5.1, we restrict attention to metaclasses I to VII. Focusing on the connected component of the identity of the invariance group, we consider infinitesimal transformations of the form $t^a_b = \delta^a_b + h\omega^a_b$ with infinitesimally small h and generators ω^a_b . Equation (55) then reads

$$0 = \omega^m_a G_{mbcd} + \omega^m_b G_{amcd} + \omega^m_c G_{abmd} + \omega^m_d G_{abcm}. \quad (56)$$

These are 21 equations for the sixteen components ω^a_b of the generator. This system can now be analyzed for the various metaclasses I to VII. We illustrate the procedure of calculating the generators ω for metaclass I area metrics (53).

For a metaclass I area metric G in normal form (53) the 21 equations (56) decouple into two sets of equations, nine for the diagonal elements of the generators ω , and 12 for the off-diagonal elements. The nine coupled homo-

geneous equations for the diagonal elements ω^i_i are

$$\begin{aligned}\sigma_1(\omega^0_0 + \omega^1_1 + \omega^2_2 + \omega^3_3) &= 0, \\ \sigma_2(\omega^0_0 + \omega^1_1 + \omega^2_2 + \omega^3_3) &= 0, \\ \sigma_3(\omega^0_0 + \omega^1_1 + \omega^2_2 + \omega^3_3) &= 0, \\ \tau_1(\omega^0_0 + \omega^1_1) &= 0, \quad \tau_1(\omega^2_2 + \omega^3_3) = 0, \\ \tau_2(\omega^0_0 + \omega^3_3) &= 0, \quad \tau_2(\omega^1_1 + \omega^2_2) = 0, \\ \tau_3(\omega^0_0 + \omega^2_2) &= 0, \quad \tau_3(\omega^1_1 + \omega^3_3) = 0.\end{aligned}$$

If the area metric is non-degenerate, i.e. $\sigma_n \neq 0$ and $\tau_n \neq 0$, only four of these nine equations are independent, and then imply that all diagonal elements vanish.

Further there are 12 coupled homogeneous equations for the 12 unknown off-diagonal elements ω^a_b . We may write these equations as a matrix equation $A \cdot x = 0$ for the vector

$$x = (\omega^0_1, \omega^1_0, \omega^0_2, \omega^2_0, \omega^0_3, \omega^3_0, \omega^1_2, \omega^2_1, \omega^1_3, \omega^3_1, \omega^2_3, \omega^3_2)$$

and the matrix

$$A = \begin{bmatrix} \tau_2 & -\tau_3 & & & & & & & & & & & \sigma_{32} \\ \tau_3 & -\tau_2 & & & & & & & & & & & \sigma_{32} \\ & & \tau_2 & -\tau_1 & & & & & & & & \sigma_{21} & \\ & & \tau_1 & -\tau_2 & & & & & & & & \sigma_{21} & \\ & & & & \tau_3 & -\tau_1 & \sigma_{13} & & & & & & \\ & & & & \tau_1 & -\tau_3 & & \sigma_{13} & & & & & \\ & & & & \sigma_{31} & & \tau_3 & \tau_1 & & & & & \\ & & & & & & \sigma_{13} & \tau_1 & \tau_3 & & & & \\ & & & & & & & & & \tau_2 & \tau_1 & & \\ & & & & & & & & & \sigma_{12} & & & \\ & & & & & & & & & \sigma_{21} & & & \\ \sigma_{23} & & & & & & & & & & & & \tau_2 & \tau_3 \\ & & & & & & & & & & & & \tau_3 & \tau_2 \\ & & & & & & & & & & & & \sigma_{32} & \end{bmatrix},$$

where we used the shorthand $\sigma_{ij} = \sigma_i - \sigma_j$. The symmetry generators are obviously the non-trivial solutions of the system $A \cdot x = 0$. Such solutions do only exist if $\det(A) = 0$. Now we observe that if for all pairs $(\sigma_i, \tau_i) \neq (\sigma_j, \tau_j)$ for $i \neq j$ the matrix A has full rank and thus $\omega^i_j = 0$ for all i, j . In this case the area metric has no gauge symmetries at all.

Let us now have $\sigma_1 = \sigma_2$ and $\tau_1 = \tau_2$. Then we have non-trivial solutions

for ω^2_0 , ω^0_2 , ω^3_1 and ω^1_3 . We may write the two resulting generators ω_1 and ω_2 in matrix form,

$$\omega_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (57)$$

In like fashion we find the generators ω_3 and ω_4 if $\sigma_1 = \sigma_3$ and $\tau_1 = \tau_3$:

$$\omega_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \omega_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (58)$$

Finally if $\sigma_2 = \sigma_3$ and $\tau_2 = \tau_3$ we obtain the generators ω_5 and ω_6 :

$$\omega_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \omega_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (59)$$

We may identify the generators ω_1 , ω_3 and ω_5 as boost generators whereas ω_2 , ω_4 and ω_6 describe spatial rotations if e_0 is timelike. That this is indeed the case one can verify following the construction presented in section 2.4. In the three cases presented above the area metric has the local gauge group $o(1, 1) \oplus o(2)$.

It is clear now that if $(\sigma_1, \tau_1) = (\sigma_2, \tau_2) = (\sigma_3, \tau_3)$, the area metric features full Lorentz symmetry and the local gauge algebra is $o(1, 3)$.

Exactly along the same lines, one calculates the symmetry generators for the other metaclasses, depending on the possible degeneracies. This concludes the proof.

Finally, we remark that a direct calculation shows that an area metric that is induced by a metric with Lorentzian signature lies in metaclass I. This is not surprising since metaclass I area metrics are the only ones where the Lorentz group is one of the possible gauge groups. Similarly, one finds that area metrics induced by a Riemannian metric lie in class VII.

With the results of this section at hand, we now afford a complete algebraic overview over four-dimensional area metrics. The question whether this classification can be employed in deciding if an area metric manifold defines a spacetime is answered in the affirmative in the following section.

5. Algebraic criteria for hyperbolicity

As an application of the algebraic classification of four-dimensional area metrics, we discuss the various metaclasses with respect to their physical viability. The developments in the present section draw heavily on the causal structure of area metric spacetimes developed in section 2, and the algebraic classification obtained in section 3. In particular, we prove a theorem that excludes 16 of the 23 metaclasses of four-dimensional area metrics as viable spacetimes. An even stronger exclusion theorem is then obtained for spherically symmetric area metric spacetimes.

5.1. Metaclasses containing no spacetimes

With the explicit normal forms of an area metric at hand, we can discuss the families of area metrics contained in the various metaclasses with respect to their physical viability. We would like to know which area metrics provide possible spacetime backgrounds for dynamical systems such as Maxwell electrodynamics. The crucial ingredients to solve this question have been reviewed in section 2 and 4. We now present a rather powerful theorem that excludes 16 of the 23 metaclasses as feasible backgrounds for physical theories since they do not provide strongly hyperbolic area metric spacetimes.

Theorem 5.1. *Let (M, G) be a four-dimensional area metric manifold of metaclass VIII to XXIII. Then the Cauchy problem for Maxwell electrodynamics is not well-posed.*

This theorem can be proven with the help of two lemmata. First note that $J^{-1} = \omega_G G^{-1}$ has the same Segré-classification as J .

Lemma 5.1. *Let (M, G) be a four-dimensional area metric manifold of metaclass VIII to XXIII. Then there exists a plane $\theta^1 \wedge \theta^2$ of null covectors.*

Proof. The endomorphism J^{-1} has a real Jordan block of at least dimension two with eigenvalue λ . Then there exist $\Omega_1, \Omega_2 \in \Lambda^2 T_p^* M$ with $J^{-1}\Omega_1 = \lambda\Omega_1$ and $J^{-1}\Omega_2 = \Omega_1 + \lambda\Omega_2$. Now J^{-1} is symmetric with respect to the bilinear form ω_G^{-1} . Expanding $\omega_G^{-1}(J^{-1}\Omega_1, \Omega_2) = \omega_G^{-1}(\Omega_1, J^{-1}\Omega_2)$ shows that $\omega_G^{-1}(\Omega_1, \Omega_2) = 0$, and hence Ω_1 is simple, i.e. there exist covectors θ_1 and θ_2 such that $\Omega_1 = \theta_1 \wedge \theta_2$. To show that this is a null plane consider $\xi \in \langle \theta^1, \theta^2 \rangle$. Then we have $\xi \wedge J^{-1}(\theta^1 \wedge \theta^2) = 0$. Using the definition of J^{-1} this may be rewritten as $G^{-1}(\xi, \cdot, \theta^1, \theta^2) = 0$. This condition implies

$\text{rk } G^{-1}(\xi, \cdot, \xi, \cdot) < 3$. By definition of the Fresnel polynomial (30) it follows that all $\xi \in \langle \theta^1, \theta^2 \rangle$ are null covectors: $\mathcal{G}(\xi, \xi, \xi, \xi) = 0$, which concludes the proof.

We will use the result of this lemma in the proof of

Lemma 5.2. *Let (M, G) be a four-dimensional area metric manifold of metaclass VIII to XXIII. Then for every time space splitting there always exists a spatial covector k such that the matrix $C(k) = C^\alpha k_\alpha$ defined in (34) is not diagonalizable.*

Proof. Choose an arbitrary time component t with corresponding initial data surface of gradient dt . Without loss of generality assume that $\mathcal{G}(dt, dt, dt, dt) \neq 0$.

To show that there exists a covector k such that the matrix $C(k)$ is not diagonalizable we compare the geometric and algebraic multiplicities of the zero eigenvalues of $C(k)$.

We choose the covector basis $(\theta^0 = dt, \theta^1, \theta^2, \theta^3)$ where $\theta^1 \wedge \theta^2$ is the discussed null plane. The eigenvalues λ of the matrix $C(k)$ can be calculated according to

$$\det(\lambda \mathbf{1} - C^\alpha k_\alpha) = \lambda^2 \mathcal{G}(\lambda dt + k_\alpha dx^\alpha, \dots, \lambda dt + k_\alpha dx^\alpha) = 0 \quad (60)$$

for any spatial covector k . We now examine the particular covector $\theta^{\bar{1}} = \theta^1 + a\theta^2$ for some $a \neq 0$. Then equation (60) implies together with

$$\mathcal{G}(\lambda dt + \theta^{\bar{1}}, \dots, \lambda dt + \theta^{\bar{1}}) = \lambda^4 \mathcal{G}^{0000} + 4\lambda^3 \mathcal{G}^{000\bar{1}} + 6\lambda^2 \mathcal{G}^{00\bar{1}\bar{1}}, \quad (61)$$

that at least four of the six eigenvalues of the matrix $C(\theta^{\bar{1}})$ are zero. Here we used that $\mathcal{G}^{\bar{1}\bar{1}\bar{1}\bar{1}}$ vanishes since $\theta^{\bar{1}}$ is a null covector and $\mathcal{G}^{0\bar{1}\bar{1}\bar{1}} = 0$ since $G^{-1}(\xi, \cdot, \theta^1, \theta^2) = 0$ for any $\xi \in \theta^1 \wedge \theta^2$ as we have seen in the proof of lemma 5.1. For the geometric multiplicity of the zero eigenvalues we have to find the number of eigenvectors $(u, v)^t$ corresponding to the zero eigenvalues of $C(\theta^{\bar{1}})$. Finding the eigenvectors of $C(\theta^{\bar{1}})$ corresponding to the zero eigenvalues is equivalent to solving the system $Ru = 0$ and $Pu + Qv = 0$ with matrices

$$R = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & -1 \\ -a & 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ G^{2331} + aG^{2332} & G^{3131} + aG^{3132} & 0 \end{bmatrix}, \quad (62)$$

$$Q = \begin{bmatrix} 0 & 0 & G^{0131} + 2aG^{0(13)2} \\ 0 & 0 & 2G^{0(23)1} + aG^{0232} \\ G^{0131} + 2aG^{0(13)2} & 2G^{0(23)1} + aG^{0232} & 2G^{0331} + 2aG^{0332} \end{bmatrix}. \quad (63)$$

for vectors u and v . We now observe that the choice of $\theta^{\bar{1}}$ never gives rise to four eigenvectors, unless $u \in \langle u_0 = (1, a, 0)^t \rangle$, $Pu_0 = 0$ and $Q = 0$. If the area metric is such that this cannot happen the proof is already complete. An additional step is only needed for area metrics with vanishing G^{2331} , G^{2332} , G^{3131} , G^{0131} , G^{0232} and $G^{0231} = G^{0123}$. In this case we change the spatial covector to $\theta^{\bar{1}} = \theta^1 + b\theta^3$. Along similar lines it can now be shown that the geometric multiplicity is always lower than the algebraic multiplicity. Hence, there always exists a spatial covector k such that the matrix $C(k)$ is not diagonalizable. Since the coordinate choice of time t was arbitrary this completes the proof.

From these two lemmata we immediately see that the proposition of theorem 5.1 holds. This theorem is quite a strong restriction on physically viable area metric backgrounds. That means we can restrict our further analysis of the normal forms to the physically viable metaclasses I to VII. Within these classes there may still be area metrics that do not admit a well-posed initial value problem for Maxwell electrodynamics. Since we have not been able to prove general theorems on metaclasses I to VII, their hyperbolicity properties have to be decided case by case. In the following section we present such a treatment for the case of spherically symmetric area metric manifolds.

5.2. Highly symmetric area metric spacetimes

Invariance of an area metric tensor field G under its flow along some vector field K identifies a symmetry of the area metric manifold and is conveniently formulated as a vanishing Lie derivative $\mathcal{L}_K G$, in complete analogy to pseudo-Riemannian geometry. Under the assumption of sufficiently high symmetry, we now further refine our study of the hyperbolicity properties of classes I to VII, on which the theorem proven in the previous chapter makes no statement.

In particular, we now examine spherically symmetric area metric spacetimes in some detail and comment on the simpler case of homogeneous and isotropic symmetry. We will see that the symmetries alone do not yet determine a unique metaclass. However, requiring that the area metric manifold is strongly hyperbolic will reveal that only metaclass I is physically viable.

To make these statements precise, note that the inverse G^{-1} of some area metric tensor lies in the same metaclass as G , and we make the

Definition 5.1. *An area metric manifold (M, G) is called spherically symmetric if the area metric G possesses three Killing vector fields spanning an $so(3)$ algebra such that the orbit of any point under the corresponding isometries is topologically a two-sphere.*

A slight modification of the calculation in [23] now reveals the canonical form of the inverse G^{-1} of a spherically symmetric area metric G . In a suitable local covector frame $\{\theta^a\}$, the Petrov matrix $Petrov(G^{-1})$ takes the form

$$Petrov(G^{-1}) = \begin{bmatrix} \xi & 0 & 0 & 0 & 0 & 2\sigma + \tau \\ 0 & \epsilon_2 & 0 & 0 & -\sigma + \tau & 0 \\ 0 & 0 & \epsilon_2 & -\sigma + \tau & 0 & 0 \\ 0 & 0 & -\sigma + \tau & \epsilon_1 & 0 & 0 \\ 0 & -\sigma + \tau & 0 & 0 & \epsilon_1 & 0 \\ 2\sigma + \tau & 0 & 0 & 0 & 0 & \epsilon_3^2 \end{bmatrix}, \quad (64)$$

where ξ , σ and τ are functions of t and r , and the ϵ_1 , ϵ_2 , ϵ_3 are signs of possible values $0, \pm 1$. From the exclusion theorem 4.1 it is clear that not every combination of signs $\epsilon_1, \epsilon_2, \epsilon_3$ can possibly give rise to an area metric spacetime. The allowed combinations of signs are summarized in table 2.

ϵ_1	± 1	± 1	± 1	± 1	0
ϵ_2	∓ 1	∓ 1	± 1	± 1	0
ϵ_3^2	1	1	1	1	1
sign(ξ)	-	+	-	+	+
metaclass	I	IV	VI	VII	VII

Table 2: possible combination of signs for spherically symmetric area metrics

We now calculate the Fresnel polynomial $\mathcal{G}(k, k, k, k)$ for some covector k with components k_i , $i = t, r, \theta, \phi$. Up to a power of $\det(Petrov(G))$ we obtain

$$\mathcal{G}(k, k, k, k) \sim \xi u^2 + (\epsilon_1 \epsilon_2 + \xi \epsilon_3^2 - 9\sigma^2)uv + \epsilon_1 \epsilon_2 \epsilon_3^2 v^2, \quad (65)$$

where $u = \epsilon_2 k_t^2 + \epsilon_1 k_r^2$ and $v = k_\theta^2 + k_\phi^2$. Now observe that there can not be future timelike covectors k if $\xi = 0$, since then $\mathcal{G}(k, k, k, k) \sim v$. If $\xi \neq 0$ we may factorize the Fresnel polynomial into two real quadratic factors, $\mathcal{G}(k, k, k, k) \sim (u + \zeta^+ v)(u + \zeta^- v)$ with

$$\zeta^\pm = \frac{1}{2\xi} \left(\epsilon_1 \epsilon_2 + \xi - 9\sigma^2 \pm \sqrt{(\epsilon_1 \epsilon_2 + \xi - 9\sigma^2)^2 - 4\epsilon_1 \epsilon_2 \xi} \right). \quad (66)$$

The Fresnel polynomial now has the form $\mathcal{G}(k, k, k, k) \sim ((g^+)^{ab}k_a k_b)((g^-)^{ab}k_a k_b)$ for two inverse metrics $g^\pm = \text{diag}(\epsilon_2, \epsilon_1, \zeta^\pm, \zeta^\pm)$. For a weakly hyperbolic area metric spacetime both g^+ and g^- need to have Lorentzian signature which requires that ϵ_1 and ϵ_2 have opposite sign. According to table 2 this rules out the metaclasses VI and VII. Without loss of generality we assume $\epsilon_1 = 1$ and $\epsilon_2 = -1$. One may now calculate the future timelike covector cone C_p^* for the metaclasses I and IV. It turns out that the future timelike covector cone of metaclass IV is empty while C_p^* for metaclass I is

$$C_p^* = \{k \in T_p^*M \mid -k_t^2 + k_r^2 + \zeta^-(k_\theta^2 + k_\phi^2) < 0\}. \quad (67)$$

Thus we see that spherically symmetric area metrics spacetimes do only exist in metaclass I. The same result is obtained for homogeneous and isotropic manifolds in four dimensions [14]. With these insights we conclude our demonstration of the various ways in which the algebraic classification of area metrics can be employed in order to decide on the hyperbolicity properties of area metric manifolds, and thus their ability to serve as a refined spacetime structure.

6. Conclusions

The key achievement of the present work is the identification of those four-dimensional area metric manifolds that qualify as viable spacetimes or optical backgrounds. These are distinguished by enabling causal evolution for classical matter fields in general, and at the very minimum for Maxwell theory. Indeed, remarkably much can be learnt from the application of standard constructions within the theory of partial differential equations to abelian gauge fields on an area metric manifold. The entire causal structure of an area metric manifold is revealed this way.

In this context, the central insight consists in the observation that the naturally emerging future timelike cones are open and convex, and that their topological closure defines causal vectors. Based on these notions, we were able to provide analytical definitions for weakly and strongly hyperbolic area metric spacetimes, such that the known theorems concerning the well-definition of initial value problems directly extend from the familiar special case of metric manifolds. Indeed, we were able to rigorously develop all concepts needed to address global causality conditions, leading for instance to the area metric version of the equivalence of the Alexandrov topology with that of the underlying manifold whenever the area metric spacetime is strongly causal.

The second major technical part of this article, namely the complete algebraic classification of four-dimensional area metric manifolds, was prompted by the desire to obtain simple *algebraic* criteria for the above *analytic* characterization of strongly hyperbolic area metric spacetimes. Since four-dimensional area metric manifolds contain more algebraic degrees of freedom at each point than could possibly be trivialized by a change of the local frame, the algebraic classification results in continuous families of normal forms. Grouping these into 23 metaclasses allows to prove a remarkable theorem, linking our analytical conditions for a strongly hyperbolic area metric spacetime to the obtained algebraic classification: 16 of the 23 metaclasses of area metrics cannot provide strongly hyperbolic spacetime geometries.

It is important to note that the algebraic classification of area metrics directly translates into a classification of all non-dissipative linear optical media, as we have argued in our application section. This result should be of great experimental value. In this context the requirement of good causality simply excludes all classes that cannot exist.

We wish to emphasize that currently we have comparatively little to say about the hyperbolicity properties of the remaining seven algebraic metaclasses, unless further assumptions, such as the existence of sufficiently many Killing symmetries, are made. That this does not necessarily present a problem in practice, we demonstrated by scrutinizing spherically symmetric area metric spacetimes as a concrete example of phenomenological interest. Here we were indeed able to give a full algebraic classification of all strongly hyperbolic spherically symmetric area metric spacetimes. The same holds for homogeneous and isotropic area metric spacetimes. In both cases, strong hyperbolicity is equivalent to the respective area metrics being of algebraic metaclass I.

Let us briefly muse on what we have learnt beyond the immediate technical details when studying the physical viability of an area metric spacetime structure.

First and foremost, the questions discussed here for the particular case of area metric manifolds must be posed for any candidate geometry aspiring to replace the Lorentzian spacetime structure underlying general relativity and our current fundamental theories of matter. That indeed area metric geometry passes key criteria one must expect a spacetime geometry to satisfy provides further evidence toward the viability of the area metric hypothesis at a fundamental level.

Second, the treatment given here immediately includes the corresponding findings in the metric case, in which all constructions recover what is often merely postulated, but rarely emphasized to be intimately linked to other

assumptions made in the theory. A case in point is the physically somewhat incomplete (though mathematically elegant) discussion of the causal structure of spacetime purely in terms of the geometry, but without pointing out the relation to (and indeed logical origin in) the hyperbolicity properties of distinguished matter fields. Thus, what might appear to be a more intricate treatment of these questions in area metric geometry actually only highlights the conceptual steps to be followed also in the familiar metric case.

Third, area metric spacetimes provide a now well-understood example for a geometry where local Lorentz invariance may be gradually broken (see section 4.4) while, and this is a mathematically and physically important point, their causal structure is still given by convex cones. It is this latter property, which ultimately renders for example the decay of massless particles into massive ones kinematically impossible.

Naturally, the developments of the present article are of greatest value particularly for the further pursuit of the programme to study the physical implications of an area metric spacetime structure. This holds especially with regard to area metric gravity theory which can be formulated as a refinement of metric Einstein-Hilbert gravity as reviewed in the application section. The future cones, for instance, are of central relevance in defining local observers, and thus for the extraction of precise physical predictions from the theory. The normal forms, and particularly those that could be identified as providing strongly hyperbolic backgrounds, are of obvious value for actual calculations, and useful for a number of constructions that are not possible for non-hyperbolic area metric spacetimes. Work that has been enabled by these and other results obtained in this article is currently under way and should eventually lead to a much deeper understanding of area metric gravity on a fundamental level.

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References

References

- [1] I. T. Drummond, S. J. Hathrell, QED Vacuum Polarization in a Background Gravitational Field and Its Effect on the Velocity of Photons, *Phys. Rev. D* 22 (1980) 343. doi:10.1103/PhysRevD.22.343.
- [2] P. Ramond, *Field theory: A modern primer*, 2nd Edition, *Frontiers in Physics*, Perseus, 1997.
- [3] E. P. Wigner, On unitary representations of the inhomogenous Lorentz group, *Ann. Math.* 40 (1939) 149–204.
- [4] S. Hojman, K. Kuchar, C. Teitelboim, *Geometrodynamics Regained*, *Ann. Phys. (N.Y.)* 96 (1976) 88.
- [5] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, 2nd Edition, W. H. Freeman, 1973.
- [6] D. N. Spergel, et al., Wilkinson Microwave Anisotropy Probe (WMAP) three year results: Implications for cosmology, *Astrophys. J. Suppl.* 170 (2007) 377. arXiv:astro-ph/0603449, doi:10.1086/513700.
- [7] R. A. Knop, et al., New Constraints on Ω_M , Ω_Λ , and w from an Independent Set of Eleven High-Redshift Supernovae Observed with HST, *Astrophys. J.* 598 (2003) 102. arXiv:astro-ph/0309368, doi:10.1086/378560.
- [8] V. Sahni, Dark matter and dark energy, *Lect. Notes Phys.* 653 (2004) 141–180. arXiv:astro-ph/0403324.
- [9] C. Laemmerzahl, O. Preuss, H. Dittus, Is the physics within the Solar system really understood? arXiv:gr-qc/0604052.
- [10] A. Boyarsky, A. Neronov, O. Ruchayskiy, M. Shaposhnikov, Constraints on sterile neutrino as a dark matter candidate from the diffuse X-ray background, *Mon. Not. Roy. Astron. Soc.* 370 (2006) 213–218. arXiv:astro-ph/0512509, doi:10.1111/j.1365-2966.2006.10458.x.
- [11] J. D. Bekenstein, Relativistic gravitation theory for the MOND paradigm, *Phys. Rev. D* 70 (2004) 083509. arXiv:astro-ph/0403694, doi:10.1103/PhysRevD.70.083509.

- [12] R. Punzi, F. P. Schuller, M. N. R. Wohlfarth, Geometry for the accelerating universe, *Phys. Rev. D* 76 (2007) 101501. arXiv:hep-th/0612133, doi:10.1103/PhysRevD.76.101501.
- [13] F. P. Schuller, M. N. R. Wohlfarth, Geometry of manifolds with area metric, *Nucl. Phys. B* 747 (2006) 398–422. arXiv:hep-th/0508170, doi:10.1016/j.nuclphysb.2006.04.019.
- [14] R. Punzi, F. P. Schuller, M. N. R. Wohlfarth, Area metric gravity and accelerating cosmology, *JHEP* 02 (2007) 030. arXiv:hep-th/0612141.
- [15] H.-P. Tuschik, private communication.
- [16] A. Cannas Da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, Springer, 2001.
- [17] E. Cartan, *Les espace mtriques fonds sur la notion d’aire*, Hermann, Paris, 1933.
- [18] C. Laemmerzahl, The geometry of matter fields, in: V. de Sabbata, J. Audretsch (Eds.), *Quantum mechanics in curved spacetime*.
- [19] R. M. Wald, *General Relativity*, 1st Edition, The University of Chicago Press, Chicago and London, 1984.
- [20] S. Benzoni-Gavage, D. Serre, *Multi-dimensional Hyperbolic Partial Differential Equations: First-order Systems and Applications*, Oxford University Press, 2007.
- [21] G. F. Rubilar, Linear pre-metric electrodynamics and deduction of the light cone, *Annalen Phys.* 11 (2002) 717–782. arXiv:0706.2193.
- [22] F. W. Hehl, Y. N. Obukhov, *Foundations of classical electrodynamics*, Birkhäuser, Basel, 2003.
- [23] R. Punzi, M. N. R. Wohlfarth, F. P. Schuller, Propagation of light in area metric backgrounds, *Class. Quant. Grav.* 26 (2009) 035024. arXiv:0711.3771, doi:10.1088/0264-9381/26/3/035024.
- [24] L. Gårding, Linear hyperbolic equations with constant coefficients, *Acta Math.* 85 (1951) 1–62.
- [25] J. Renegar, Hyperbolic Programs, and Their Derivative Relaxations, *Foundations of Computational Mathematics* 6 (2006) 59.

- [26] L. Gårding, An inequality for hyperbolic polynomials, *J. Math.* 8 (1959) 957–965.
- [27] J. Renegar, *A mathematical view of interior-point methods in convex optimization*, SIAM.
- [28] J. K. Beem, P. E. Ehrlich, K. L. Easley, *Global Lorentzian Geometry*, 2nd Edition, Marcel Dekker, 1996.
- [29] F. W. Hehl, Yu. N. Obukhov, G. F. Rubilar and M. Blagojevic, On the theory of the skewon field: From electrodynamics to gravity, *Phys. Lett. A* 347 (2005) 14 [arXiv:gr-qc/0506042].
- [30] F. W. Hehl, Y. N. Obukhov, J. P. Rivera and H. Schmid, Relativistic analysis of magnetoelectric crystals: extracting a new 4-dimensional P odd and T odd pseudoscalar from Cr_2O_3 data, arXiv:0708.2069 [gr-qc].
- [31] A. Janner, Looking for a relativistic crystal optics, *Ferroelectrics* 161 (1994) 191.
- [32] T. J. Hollowood and G. M. Shore, The Causal Structure of QED in Curved Spacetime: Analyticity and the Refractive Index, *JHEP* 0812 (2008) 091 [arXiv:0806.1019 [hep-th]].
- [33] T. J. Hollowood and G. M. Shore, The Refractive Index of Curved Spacetime: the Fate of Causality in QED, *Nucl. Phys. B* 795 (2008) 138 [arXiv:0707.2303 [hep-th]].
- [34] R. Punzi, F. P. Schuller, M. N. R. Wohlfarth, Light clocks in strong gravitational fields arXiv:0902.1811.
- [35] F. P. Schuller and M. N. R. Wohlfarth, Radiation-dominated area metric cosmology, *JCAP* **0712** (2007) 013 [arXiv:0705.4656 [hep-th]].
- [36] R. Punzi, F. P. Schuller, M. N. R. Wohlfarth, Brans-Dicke geometry, *Phys. Lett. B* 670 (2008) 161–164. arXiv:0804.4067, doi:10.1016/j.physletb.2008.10.046.
- [37] S. Lang, *Linear Algebra*, Springer, 2004.
- [38] M. P. Reisenberger, New constraints for canonical general relativity, *Nucl. Phys. B* 457 (1995) 643–687. arXiv:gr-qc/9505044, doi:10.1016/0550-3213(95)00448-3.

- [39] R. De Pietri, L. Freidel, $so(4)$ Plebanski Action and Relativistic Spin Foam Model, *Class. Quant. Grav.* 16 (1999) 2187–2196. arXiv:gr-qc/9804071, doi:10.1088/0264-9381/16/7/303.
- [40] I. Gohberg, P. Lancaster, L. Rodman, *Indefinite Linear Algebra and Applications*, Birkhuser, 2005.
- [41] G. S. Hall, *Symmetries and curvature structure in general relativity*, World Scientific, 2004.