DISCRETE CURVATURE AND THE GAUSS–BONNET THEOREM

JOAKIM ARNLIND, JENS HOPPE, AND GERHARD HUISKEN

Abstract. For matrix analogues of embedded surfaces we define discrete curvatures and Euler characteristics, and a non-commutative Gauss–Bonnet theorem is shown to follow. We derive simple expressions for the discrete Gauss curvature in terms of matrices representing the embedding coordinates, and provide a large class of explicit examples illustrating the new notions.

1. Introduction

A particular way of discretizing surfaces by replacing functions by matrices has for a long time been used in physics to obtain a quantum theory of surfaces (membranes) moving in Minkowski space, sweeping out 3-manifolds of vanishing mean curvature [Hop82]. The discretization, sometimes called "Matrix Regularization", is of independent mathematical interest and contains many interesting structures. One of the main features of the correspondence between functions on the surfaces and matrices is that the Poisson bracket of two functions becomes the commutator of two matrices. This allows for an easy construction of discrete analogues of any expression involving products and Poisson brackets of functions. In a recent paper [AHH10], the geometry of surfaces embedded in Riemannian manifolds has been expressed in terms of Poisson brackets of the embedding coordinates. Using these formulas, one can define discretizations of the Gaussian curvature and the Euler characteristic, and it is immediate to prove a discrete Gauss-Bonnet theorem (see Theorem 3.10).

Apart from being interesting in their own right, these discrete concepts might also help to solve questions related to the regularization in the above mentioned Membrane Theory. For instance, solving the equations of motion in Membrane Theory yields matrices corresponding to a discrete surface. As solutions corresponding to surfaces of arbitrary topology exist, one would like to be able to determine the geometry from the matrices in some way. In Theorem 3.11 we provide formulas for computing the discrete curvature and the discrete Euler characteristic given the matrix analogues of the embedding coordinates (which are the solutions to the equations of motion in Membrane Theory). Thus, in the limit of large matrices one may determine the Euler characteristic, and hence the topology, of the surface.

2. Surface geometry and Poisson brackets

Let us recall some of the results obtained in [AHH10]. Consider a surface Σ embedded in a Riemannian manifold M, of dimension $m = 2 + p$, via the coordinates $x^1(u^1, u^2), \ldots, x^m(u^1, u^2)$, where u^1, u^2 are local coordinates on Σ. Furthermore, let $n_A^1(u^1, u^2), \ldots, n_A^m(u^1, u^2)$ for $A = 1, \ldots, p$ denote the components of p orthonormal vectors N_A normal to the surface at each point. Indices i, j, k, l will run from

1 to m and indices a, b will run from 1 to 2. The metric of M is denoted by $\bar{g}_{ij}$, the Christoffel symbols by $\bar{\Gamma}^i_{jk}$ and the covariant derivative by $\bar{\nabla}$. Regarded as a subspace of TM, the tangent space $T\Sigma$ is spanned by the vectors $e_a = (\partial_a x^i)\partial_i$.

Letting $\rho(u^1, u^2)$ be an arbitrary non-vanishing density on Σ, one defines a Poisson bracket on $C^\infty(\Sigma)$ by setting

$$\{f, h\} = \frac{1}{\rho}\varepsilon^{ab}\big(\partial_a f\big)\big(\partial_b h\big), \tag{2.1}$$

where ε^{ab} is antisymmetric with $\varepsilon^{12} = 1$ and $\partial_a = \frac{\partial}{\partial u^a}$. With this bracket we define the tensors

$$\mathcal{P}^{ij} = \{x^i, x^j\} \tag{2.2}$$

$$\mathcal{S}^{ij}_A = \frac{1}{\rho}\varepsilon^{ab}\big(\partial_a x^i\big)\big(\bar{\nabla}_b N_A\big)^j = \{x^i, n^j_A\} + \{x^i, x^k\}\bar{\Gamma}^j_{kl}n^l_A, \tag{2.3}$$

and one can also consider them as maps $TM \to TM$ by lowering the second index with the ambient metric $\bar{g}$, i.e.

$$\mathcal{P}(X) = \mathcal{P}^{ik}\bar{g}_{kj}X^j\partial_i \tag{2.4}$$

$$\mathcal{S}_A(X) = \mathcal{S}^{ik}_A\bar{g}_{kj}X^j\partial_i. \tag{2.5}$$

With these definitions, one finds that

$$\operatorname{Tr}\mathcal{S}^2_A \equiv \big(\mathcal{S}_A\big)^i_j\big(\mathcal{S}_A\big)^j_i = -\frac{2}{\rho^2}\det(h_{A,ab}) \tag{2.6}$$

$$\operatorname{Tr}\mathcal{P}^2 \equiv \mathcal{P}^i_j\mathcal{P}^j_i = -2\frac{g}{\rho^2}, \tag{2.7}$$

where $g = \det\big(\bar{g}(e_a, e_b)\big)$ is the determinant of the induced metric on Σ, and $h_{A,ab}$ is the second fundamental form corresponding to the normal vector N_A.

In the main part of this article we will make use of the following result:

Theorem 2.1 ([AHH10]). *Let K denote the Gaussian curvature of Σ. Then*

$$K = \frac{1}{g}\bar{g}\big(\bar{R}(e_1, e_2)e_2, e_1\big) - \frac{\rho^2}{2g}\sum_{A=1}^{p}\operatorname{Tr}\mathcal{S}^2_A, \tag{2.8}$$

where $\bar{R}$ is the curvature tensor of M.

3. Matrix regularizations and discrete curvatures

In the following, we shall assume that Σ is a compact closed orientable surface. Let us first define what is meant by a "matrix regularization", and then show some of its properties.

Definition 3.1. Let $N_1, N_2, \ldots$ be a strictly increasing sequence of positive integers, let $\{T^\alpha\}$ for $\alpha = 1, 2, \ldots$ be linear maps from $C^\infty(\Sigma, \mathbb{R})$ to hermitian $N_\alpha \times N_\alpha$ matrices and let $\hbar(N)$ be a real-valued strictly positive decreasing function such that $\lim_{N\to\infty} N\hbar(N) < \infty$. Furthermore, let ω be a symplectic form on Σ and let $\{\cdot, \cdot\}$ denote the Poisson bracket induced by ω. If $\{T^\alpha\}$ has the following properties for all $f, h \in C^\infty(\Sigma)$

$$\lim_{\alpha\to\infty} ||T^\alpha(f)|| < \infty, \tag{3.1}$$

$$\lim_{\alpha\to\infty} ||T^\alpha(fh) - T^\alpha(f)T^\alpha(h)|| = 0, \tag{3.2}$$

$$\lim_{\alpha\to\infty} \left|\left| \frac{1}{i\hbar_\alpha}\left[T^\alpha(f), T^\alpha(h)\right] - T^\alpha\big(\{f,h\}\big)\right|\right| = 0, \tag{3.3}$$

$$\lim_{\alpha\to\infty} 2\pi\hbar_\alpha \operatorname{Tr} T^\alpha(f) = \int_\Sigma f\omega, \tag{3.4}$$

where $||\cdot||$ denotes the operator norm and $\hbar_\alpha = \hbar(N_\alpha)$, then we call the pair $(T^\alpha, \hbar)$ a *matrix regularization of* (Σ, ω).

Given local coordinates u^1, u^2 on Σ, we write $\omega = \rho(u^1,u^2)du^1 \wedge du^2$, and it is easy to see that the induced Poisson bracket becomes

$$\{f,h\} = \frac{1}{\rho}\varepsilon^{ab}\big(\partial_a f\big)\big(\partial_b h\big).$$

Definition 3.2. If $\hat{f}_1, \hat{f}_2, \ldots$ is a sequence of matrices such that $\hat{f}_\alpha$ has dimension N_α and if it holds that

$$\lim_{\alpha\to\infty} \left|\left|\hat{f}_\alpha - T^\alpha(f)\right|\right| = 0, \tag{3.5}$$

then we say that the sequence *converges to the function* f.

Definition 3.3. A matrix regularization $(T^\alpha, \hbar)$ is called *unital* if

$$\lim_{\alpha\to\infty} ||\mathbb{1}_{N_\alpha} - T^\alpha(1)|| = 0. \tag{3.6}$$

Remark 3.4. Although unital matrix regularizations seem natural, and all our examples fall into this category, it is easy to construct examples of non-unital matrix regularizations. Namely, let $(T^\alpha, \hbar)$ be a matrix regularization and consider the map $\tilde{T}^\alpha$ defined by

$$\tilde{T}^\alpha(f) = \begin{pmatrix} & & 0 \\ & T^\alpha(f) & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Then $(\tilde{T}^\alpha, \hbar)$ is a matrix regularization which is not unital, since

$$\lim_{\alpha\to\infty} \left|\left|\tilde{T}^\alpha(1) - \mathbb{1}_{N_\alpha+1}\right|\right| \geq 1.$$

Proposition 3.5. *Let $(T^\alpha, \hbar)$ be a unital matrix regularization. Then*

$$\lim_{\alpha\to\infty} 2\pi N_\alpha \hbar_\alpha = \int_\Sigma \omega. \tag{3.7}$$

Proof. Let us use formula (3.4) with $f = 1$.

$$\begin{aligned}\int_\Sigma \omega &= \lim_{\alpha\to\infty} 2\pi\hbar_\alpha \operatorname{Tr} T^\alpha(1) = \lim_{\alpha\to\infty} 2\pi\hbar_\alpha \operatorname{Tr}\left[T^\alpha(1) + \mathbb{1}_{N_\alpha} - \mathbb{1}_{N_\alpha}\right] \\ &= \lim_{\alpha\to\infty}\Big(2\pi\hbar_\alpha N_\alpha + 2\pi\hbar_\alpha \operatorname{Tr}(T^\alpha(1) - \mathbb{1}_{N_\alpha})\Big) = \lim_{\alpha\to\infty} 2\pi\hbar_\alpha N_\alpha\end{aligned}$$

since

$$\lim_{\alpha\to\infty}|2\pi\hbar_\alpha\operatorname{Tr}(T^\alpha(1)-\mathbb{1}_{N_\alpha})|\leq\lim_{\alpha\to\infty}2\pi\hbar_\alpha N_\alpha\,||T^\alpha(1)-\mathbb{1}_{N_\alpha}||=0,$$

due to the fact that the matrix regularization is unital. □

Proposition 3.6. *Let $(T^\alpha,\hbar)$ be a matrix regularization of (Σ,ω) and let $\{\hat{f}_k^\alpha\}$ be a sequence of matrices converging to $f_k\in C^\infty(\Sigma)$ for $k=1,\ldots,n$. Then $\{a_1\hat{f}_1^\alpha+\cdots+a_n\hat{f}_n^\alpha\}$ converges to $a_1f_1+\cdots+a_nf_n$ for any $a_1,\ldots,a_n\in\mathbb{R}$ and*

$$\lim_{\alpha\to\infty}\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right|\right|\leq\prod_{k=1}^n\lim_{\alpha\to\infty}||T^\alpha(f_k)|| \tag{3.8}$$

$$\lim_{\alpha\to\infty}\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha-T^\alpha(f_1\cdots f_n)\right|\right|=0 \tag{3.9}$$

$$\lim_{\alpha\to\infty}2\pi\hbar_\alpha\operatorname{Tr}\left(\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right)=\int_\Sigma f_1\cdots f_n\omega. \tag{3.10}$$

Proof. The first statement about $\{a_1\hat{f}_1^\alpha+\cdots a_n\hat{f}_n^\alpha\}$ follows directly from the linearity of T^α. Let us prove (3.8) by induction on n. Thus, we assume that (3.8) holds and compute

$$\begin{aligned}\lim_{\alpha\to\infty}\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_{n+1}^\alpha\right|\right|&\leq\lim_{\alpha\to\infty}\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right|\right|\left|\left|\hat{f}_{n+1}^\alpha-T^\alpha(f_{n+1})+T^\alpha(f_{n+1})\right|\right|\\&\leq\lim_{\alpha\to\infty}\Big(\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right|\right|\left|\left|\hat{f}_{n+1}^\alpha-T^\alpha(f_{n+1})\right|\right|+\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right|\right|||T^\alpha(f_{n+1})||\Big)\\&=\lim_{\alpha\to\infty}\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right|\right|||T^\alpha(f_{n+1})||\leq\prod_{k=1}^{n+1}\lim_{\alpha\to\infty}||T^\alpha(f_k)||\,.\end{aligned}$$

To prove (3.9) we again proceed by induction and assume that (3.9) holds for any given n, and then compute

$$\begin{aligned}&\lim_{\alpha\to\infty}\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_{n+1}^\alpha-T^\alpha(f_1\cdots f_{n+1})\right|\right|\\&\leq\lim_{\alpha\to\infty}\Big(\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right|\right|\left|\left|\hat{f}_{n+1}^\alpha-T^\alpha(f_{n+1})\right|\right|+\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha T^\alpha(f_{n+1})-T^\alpha(f_1\cdots f_{n+1})\right|\right|\Big)\\&\leq\lim_{\alpha\to\infty}\Big(\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right|\right|\left|\left|\hat{f}_{n+1}^\alpha-T^\alpha(f_{n+1})\right|\right|+\Big|\Big|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha T^\alpha(f_{n+1})\\&\qquad-T^\alpha(f_1\cdots f_{n+1})-T^\alpha(f_1\cdots f_n)T^\alpha(f_{n+1})+T^\alpha(f_1\cdots f_n)T^\alpha(f_{n+1})\Big|\Big|\Big)\\&\leq\lim_{\alpha\to\infty}\Big(\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right|\right|\left|\left|\hat{f}_{n+1}^\alpha-T^\alpha(f_{n+1})\right|\right|+||T^\alpha(f_{n+1})||\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha-T^\alpha(f_1\cdots f_n)\right|\right|\\&+||T^\alpha(f_1\cdots f_n)T^\alpha(f_{n+1})-T^\alpha(f_1\cdots f_{n+1})||\Big)=0.\end{aligned}$$

Finally, we prove the trace formula:

$$\begin{aligned}&\lim_{\alpha\to\infty}2\pi\hbar_\alpha\operatorname{Tr}\left(\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha\right)\\&\qquad=\lim_{\alpha\to\infty}2\pi\hbar_\alpha\Big[\operatorname{Tr}T^\alpha(f_1\cdots f_n)+\operatorname{Tr}\left(\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha-T^\alpha(f_1\cdots f_n)\right)\Big]\\&\qquad=\int_\Sigma f_1\cdots f_n\omega,\end{aligned}$$

since

$$\lim_{\alpha\to\infty}\left|\hbar_\alpha \operatorname{Tr}\left(\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha - T^\alpha(f_1\cdots f_n)\right)\right| \leq \lim_{\alpha\to\infty}\hbar_\alpha N_\alpha\left|\left|\hat{f}_1^\alpha\cdots\hat{f}_n^\alpha - T^\alpha(f_1\cdots f_n)\right|\right| = 0$$

by formula (3.9). □

The above result allows one to easily construct sequences of matrices converging to any sum of products of functions and Poisson brackets. Namley, simply substitute for every factor in every term of the sum, a sequence converging to that function, where Poisson brackets of functions may be replaced by commutators of matrices. Proposition 3.6 then guarantees that the matrix sequence obtained in this way converges to the sum of the products of the corresponding functions.

Proposition 3.7. *Let $(T^\alpha,\hbar)$ be a matrix regularization and assume that $\{\hat{f}_\alpha\}$ converges to f. Then $\{\hat{f}_\alpha^\dagger\}$ converges to f.*

Proof. Due to the fact that $||A|| = ||A^\dagger||$ one sees that

$$\lim_{\alpha\to\infty}\left|\left|\hat{f}_\alpha^\dagger - T^\alpha(f)\right|\right| = \lim_{\alpha\to\infty}\left|\left|\left(\hat{f}_\alpha - T^\alpha(f)\right)^\dagger\right|\right| = \lim_{\alpha\to\infty}\left|\left|\hat{f}_\alpha - T^\alpha(f)\right|\right| = 0,$$

since $\{\hat{f}_\alpha\}$ converges to f. □

If the matrix regularization is unital, one can relate the matrix sequence converging to the function $1/f$, to the inverse of a sequence converging to f.

Proposition 3.8. *Let $(T^\alpha,\hbar)$ be a unital matrix regularization and assume that f is a nowhere vanishing function and that $\{\hat{f}_\alpha\}$ converges to f. If $\hat{f}_\alpha^{-1}$ exists and $||\hat{f}_\alpha^{-1}||$ is uniformly bounded for all α, then $\{\hat{f}_\alpha^{-1}\}$ converges to $1/f$.*

Proof. One calculates

$$\begin{aligned}\lim_{\alpha\to\infty}\left|\left|\hat{f}_\alpha^{-1} - T^\alpha(1/f)\right|\right| &\leq \lim_{\alpha\to\infty}\left|\left|\hat{f}_\alpha^{-1}\right|\right|\left|\left|\mathbb{1}_{N_\alpha} - \hat{f}_\alpha T^\alpha(1/f)\right|\right|\\ &= \lim_{\alpha\to\infty}\left|\left|\hat{f}_\alpha^{-1}\right|\right|\left|\left|\mathbb{1}_{N_\alpha} - \hat{f}_\alpha T^\alpha(1/f) + T^\alpha(1) - T^\alpha(1)\right|\right|\\ &\leq \lim_{\alpha\to\infty}\left|\left|\hat{f}_\alpha^{-1}\right|\right|\left(||\mathbb{1}_{N_\alpha} - T^\alpha(1)|| + \left|\left|\hat{f}_\alpha T^\alpha(1/f) - T^\alpha(1)\right|\right|\right)\\ &= 0,\end{aligned}$$

since the matrix regularization is unital and $||\hat{f}_\alpha^{-1}||$ is assumed to be uniformly bounded. □

Recall that Σ is embedded in a $m = 2+p$ dimensional manifold M via the embedding coordinates $x^1(u^1,u^2),\ldots,x^m(u^1,u^2)$, and that p orthonormal normal vectors are given with components n_A^i. By $\{X_\alpha^i\}$ and $\{N_{A,\alpha}^i\}$ we will denote arbitrary sequences converging to x^i and n_A^i respectively. Moreover, given the metric $\bar{g}_{ij}$ and the Christoffel symbols $\bar{\Gamma}_{jk}^i$ of M, we let $\{\hat{G}_{ij,\alpha}\}$ and $\{\hat{\Gamma}_{jk,\alpha}^i\}$ denote sequences converging to $\bar{g}_{ij}$ and Γ_{jk}^i respectively. To avoid excess of notation, we suppress the index α whenever all matrices are considered at a fixed (but arbitrary) α.

In analogy with (2.3) we define

$$(\hat{\mathcal{S}}_A)_k^j = \frac{1}{i\hbar}[X^j, N_A^{k'}]\hat{G}_{k'k} + \frac{1}{i\hbar}[X^j, X^l]\hat{\Gamma}_{lm}^{k'}N_A^m\hat{G}_{k'k}, \tag{3.11}$$

and

$$\widehat{\operatorname{tr}}\,\hat{\mathcal{S}}_A^2 = \big((\hat{\mathcal{S}}_A)^j_k\big)^\dagger(\hat{\mathcal{S}}_A)^k_j. \tag{3.12}$$

Let g_{ab} be the induced metric on Σ, and g its determinant. We set

$$\gamma = \frac{\sqrt{g}}{\rho}, \tag{3.13}$$

and denote by $\{\hat{\gamma}_\alpha\}$ an arbitrary sequence of invertible matrices converging to γ. By defining

$$\hat{\mathcal{P}}^j_k = \frac{1}{i\hbar}[X^j, X^l]\hat{G}_{lk},$$

it follows from (2.7) that

$$-\frac{1}{2}(\hat{\mathcal{P}}^i_k)^\dagger\hat{\mathcal{P}}^k_i = \frac{1}{2\hbar^2}\hat{G}^\dagger_{jk}[X^i, X^j][X^k, X^l]\hat{G}_{li} \tag{3.14}$$

converges to γ^2.

If the embedding space is $\mathbb{R}^m$, the above formulas reduce to

$$\widehat{\operatorname{tr}}\,\hat{\mathcal{S}}_A^2 = -\frac{1}{\hbar^2}\sum_{i,j=1}^m [X^i, N^j_A][X^j, N^i_A], \tag{3.15}$$

$$-\frac{1}{2}(\hat{\mathcal{P}}^i_k)^\dagger\hat{\mathcal{P}}^k_i = -\frac{1}{\hbar^2}\sum_{i<j}^m [X^i, X^j]^2, \tag{3.16}$$

and in $\mathbb{R}^3$ one obtains

$$\widehat{\operatorname{tr}}\,\hat{\mathcal{S}}^2 = \frac{1}{4\hbar^4}\sum^3 \varepsilon_{jkl}\varepsilon_{ik'l'}(\hat{\gamma}^\dagger)^{-1}\big[X^i,[X^k,X^l]\big]\big[X^j,[X^{k'},X^{l'}]\big]\hat{\gamma}^{-1}, \tag{3.17}$$

since

$$n^i = \frac{1}{2\gamma}\varepsilon^i_{jk}\{x^j, x^k\}, \tag{3.18}$$

defines a unit normal vector to the surface (cp. [AHH10], where (3.17) is also given for arbitrary codimension).

We are now ready to define and present formulas for the discrete curvature in a matrix regularization of Σ.

Definition 3.9. Let $(T^\alpha, \hbar)$ be a matrix regularization of (Σ, ω) and let K be the Gaussian curvature of Σ. A *Discrete Curvature of* Σ is a matrix sequence $\{\hat{K}_1, \hat{K}_2, \hat{K}_3, \ldots\}$ converging to K, and a *Discrete Euler Characteristic of* Σ is a sequence $\{\hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_3, \ldots\}$ such that $\lim_{\alpha\to\infty}\hat{\chi}_\alpha = \chi$.

From the classical Gauss-Bonnet theorem, it is immediate to derive a discrete analogue for matrix regularizations.

Theorem 3.10. *Let $(T^\alpha, \hbar)$ be a matrix regularization of (Σ, ω), and let $\{\hat{K}_1, \hat{K}_2, \ldots\}$ be a discrete curvature of Σ. Then the sequence $\hat{\chi}_1, \hat{\chi}_2, \ldots$ defined by*

$$\hat{\chi}_\alpha = \hbar_\alpha \operatorname{Tr}\Big[\hat{\gamma}_\alpha\hat{K}_\alpha\Big], \tag{3.19}$$

is a discrete Euler characteristic of Σ.

Proof. To prove the statement, we compute $\lim_{\alpha\to\infty}\hat{\chi}_\alpha$ and show that it is equal to $\chi(\Sigma)$. Thus

$$\lim_{\alpha\to\infty}\hat{\chi}_\alpha = \lim_{\alpha\to\infty}\frac{1}{2\pi}2\pi\hbar_\alpha \operatorname{Tr}\left[\hat{\gamma}_\alpha\hat{K}_\alpha\right],$$

and by using Proposition 3.6 we can write

$$\lim_{\alpha\to\infty}\hat{\chi}_\alpha = \frac{1}{2\pi}\int_\Sigma K\frac{\sqrt{g}}{\rho}\omega = \frac{1}{2\pi}\int_\Sigma K\frac{\sqrt{g}}{\rho}\rho du dv = \frac{1}{2\pi}\int_\Sigma K\sqrt{g}du dv = \chi(\Sigma),$$

where the last equality is the classical Gauss-Bonnet theorem. □

Theorem 3.11. *Let $(T^\alpha,\hbar)$ be a unital matrix regularization of (Σ,ω) and let $\{\bar{K}^\alpha_{12}\}$ be a matrix sequence converging to $\bar{g}\big(\bar{R}(e_1,e_2)e_2,e_1\big)/g$ (the sectional curvature of $T\Sigma$ in M). Then the sequence $\hat{K}$ of matrices defined by*

$$\hat{K} = \bar{K}_{12} - \frac{1}{2}\sum_{A=1}^{p}(\hat{\gamma}^\dagger)^{-1}\big(\widehat{\operatorname{tr}}\,\hat{\mathcal{S}}_A^2\big)\hat{\gamma}^{-1} \tag{3.20}$$

is a discrete curvature of Σ. Thus, a discrete Euler characteristic is given by

$$\hat{\chi} = \hbar\operatorname{Tr}\big(\hat{\gamma}\bar{K}_{12}\big) - \frac{\hbar}{2}\sum_{A=1}^{p}\operatorname{Tr}\left[\hat{\gamma}^{-1}\widehat{\operatorname{tr}}\,\hat{\mathcal{S}}_A^2\right]. \tag{3.21}$$

Proof. By using the way of constructing matrix sequences given through Proposition 3.6, the result follows immediately from Theorem 2.1. □

Note that if $\rho=\sqrt{g}$, then $\gamma=1$ which implies that one can choose $\hat{\gamma}_\alpha = \mathbb{1}_{N_\alpha}$ when the matrix regularization is unital.

4. Two simple examples

4.1. The round fuzzy sphere. For the sphere embedded in $\mathbb{R}^3$ as

$$\vec{x} = (x^1,x^2,x^3) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta,\cos\theta) \tag{4.1}$$

with the induced metric

$$(g_{ab}) = \begin{pmatrix}1 & 0\\ 0 & \sin^2\theta\end{pmatrix}, \tag{4.2}$$

it is well known that one can construct a matrix regularization from representations of $su(2)$. Namely, let S_1,S_2,S_3 be hermitian $N\times N$ matrices such that $[S^j,S^k] = i\epsilon^{jk}{}_l S^l$, $(S^1)^2+(S^2)^2+(S^3)^2 = (N^2-1)/4$, and define

$$X^i = \frac{2}{\sqrt{N^2-1}}S^i. \tag{4.3}$$

Then there exists a map $T^{(N)}$ (which can be defined through expansion in spherical harmonics) such that $T^{(N)}(x^i) = X^i$ and $(T^{(N)},\hbar = 2/\sqrt{N^2-1})$ is a unital matrix regularization of $(S^2,\sqrt{g}d\theta\wedge d\varphi)$ [Hop82]. A unit normal of the sphere in $\mathbb{R}^3$ is given by $N\in T\mathbb{R}^3$ with $N = x^i\partial_i$, which gives $N^i = X^i$, and one can compute the discrete curvature as

$$\hat{K}_N = -\frac{1}{\hbar^2}\sum_{i<j=1}^{m}\operatorname{Tr}[X^i,X^j]^2 = \mathbb{1}_N \tag{4.4}$$

which gives the discrete Euler characteristic

$$\hat{\chi}_N = \hbar \operatorname{Tr} \hat{K}_N = \hbar N = \frac{2N}{\sqrt{N^2-1}}, \tag{4.5}$$

converging to 2 as $N \to \infty$.

4.2. **The fuzzy Clifford torus.** The Clifford torus in S^3 can be regarded as embedded in $\mathbb{R}^4$ through

$$\vec{x} = (x^1, x^2, x^3, x^4) = \frac{1}{\sqrt{2}}(\cos\varphi_1, \sin\varphi_1, \cos\varphi_2, \sin\varphi_2),$$

with the induced metric

$$(g_{ab}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and two orthonormal vectors, normal to the tangent plane of the surface in $T\mathbb{R}^4$, can be written as

$$N_\pm = x^1\partial_1 + x^2\partial_2 \pm x^3\partial_3 \pm x^4\partial_4.$$

To construct a matrix regularization for the Clifford torus, one considers the $N \times N$ matrices g and h with non-zero elements

$$\begin{aligned} &g_{kk} = \omega^{k-1} \quad \text{for } k = 1, \dots, N \\ &h_{k,k+1} = 1 \quad \text{for } k = 1, \dots, N-1 \\ &h_{N,1} = 1, \end{aligned}$$

where $\omega = \exp(i2\theta)$ and $\theta = \pi/N$. These matrices satisfy the relation $hg = \omega gh$. The map $T^{(N)}$ is then defined on the Fourier modes

$$Y_{\vec{m}} = e^{i\vec{m}\cdot\vec{\varphi}} = e^{im_1\varphi_1 + im_2\varphi_2}$$

as

$$T^{(N)}(Y_{\vec{m}}) = \omega^{\frac{1}{2}m_1 m_2} g^{m_1} h^{m_2},$$

and the pair $(T^{(N)}, \hbar = \sin\theta)$ is a unital matrix regularization of the Clifford torus with respect to $\sqrt{g}d\varphi_1 \wedge d\varphi_2$ [FFZ89, Hop89]. Thus, using this map one finds that

$$\begin{aligned} X^1 &= T(x^1) = \frac{1}{\sqrt{2}}T(\cos\varphi_1) = \frac{1}{2\sqrt{2}}(g^\dagger + g) \\ X^2 &= T(x^2) = \frac{1}{\sqrt{2}}T(\sin\varphi_1) = \frac{i}{2\sqrt{2}}(g^\dagger - g) \\ X^3 &= T(x^3) = \frac{1}{\sqrt{2}}T(\cos\varphi_2) = \frac{1}{2\sqrt{2}}(h^\dagger + h) \\ X^4 &= T(x^4) = \frac{1}{\sqrt{2}}T(\sin\varphi_2) = \frac{i}{2\sqrt{2}}(h^\dagger - h) \end{aligned}$$

which implies that $N_\pm^1 = X^1$, $N_\pm^2 = X^2$, $N_\pm^3 = \pm X^3$ and $N_\pm^4 = \pm X^4$. By a straightforward computation one obtains

$$-\frac{1}{\hbar^2}\sum_{i,j=1}^{4}[X^i, X^j]^2 = 2\mathbb{1}$$

and therefore

$$\frac{1}{2\hbar^2}\sum_{i,j=1}^{4}[X^i,N_+^j][X^j,N_+^i] = -\frac{1}{2\hbar^2}\sum_{i,j=1}^{4}[X^i,X^j]^2 = \mathbb{1},$$

and since $[X^1,X^2]=[X^3,X^4]=0$ it follows that

$$\frac{1}{2\hbar^2}\sum_{i,j=1}^{4}[X^i,N_-^j][X^j,N_-^i] = \frac{1}{2\hbar^2}\sum_{i,j=1}^{4}[X^i,X^j]^2 = -\mathbb{1}.$$

This implies that the discrete curvature vanishes, i.e.

$$\hat{K}_N = \frac{1}{2\hbar^2}\sum_{i,j=1}^{4}[X^i,N_+^j][X^j,N_+^i] + \frac{1}{2\hbar^2}\sum_{i,j=1}^{4}[X^i,N_-^j][X^j,N_-^i] = \mathbb{1}-\mathbb{1} = 0,$$

which immediately gives $\hat{\chi}_N = 0$.

5. Axially symmetric surfaces in $\mathbb{R}^3$

Recall the classical description of general axially symmetric surfaces:

$$\begin{aligned}\vec{x} &= \big(f(u)\cos v, f(u)\sin v, h(u)\big)\\ \vec{n} &= \frac{\pm 1}{\sqrt{h'(u)^2+f'(u)^2}}\big(h'(u)\cos v, h'(u)\sin v, -f'(u)\big),\end{aligned} \tag{5.1}$$

which implies

$$\big(g_{ab}\big) = \begin{pmatrix} f'^2+h'^2 & 0\\ 0 & f^2\end{pmatrix} \qquad \big(h_{ab}\big) = \frac{\pm 1}{\sqrt{h'^2+f'^2}}\begin{pmatrix} h'f''-h''f' & 0\\ 0 & -fh'\end{pmatrix},$$

where h_{ab} are the components of the second fundamental form. The Euler characteristic can be computed as

$$\chi = \frac{1}{2\pi}\int K\sqrt{g} = -\int_{u_-}^{u_+}\frac{h'\big(h'f''-h''f'\big)}{\big(f'^2+h'^2\big)^{3/2}}du = -\frac{f'}{\sqrt{f'^2+h'^2}}\Bigg|_{u_-}^{u_+}, \tag{5.2}$$

which is equal to zero for tori (due to periodicity) and equal to $+2$ for spherical surfaces (due to $f'(u_\pm)=\mp\infty$).

While a general procedure for constructing matrix analogues of surfaces embedded in $\mathbb{R}^3$ was obtained in [ABH+09b, ABH+09a] (cp. also [Arn08b]), let us restrict now to $h(u)=u=z$, hence describe the axially symmetric surface Σ as a level set, $C=0$, of

$$C(\vec{x}) = \frac{1}{2}\big(x^2+y^2-f^2(z)\big), \tag{5.3}$$

to carry out the construction in detail, and make the resulting formulas explicit. Defining

$$\{F(\vec{x}),G(\vec{x})\}_{\mathbb{R}^3} = \nabla C\cdot\big(\nabla F\times\nabla G\big), \tag{5.4}$$

one has

$$\{x,y\} = -ff'(z), \quad \{y,z\} = x, \quad \{z,x\} = y, \tag{5.5}$$

respectively

$$[X,Y] = i\hbar ff'(Z), \quad [Y,Z] = i\hbar X, \quad [Z,X] = i\hbar Y \tag{5.6}$$

for the "quantized" ("non-commutative") surface. In terms of the parametrization given in (5.1), the above Poisson bracket is equivalent to

$$\{F(u,v),G(u,v)\} = \varepsilon^{ab}\big(\partial_a F\big)\big(\partial_b G\big) \tag{5.7}$$

where $\partial_1 = \partial_v$ and $\partial_2 = \partial_u$. By finding matrices of increasing dimension satisfying (5.6), one can construct a map T^α having the properties (3.2) and (3.3) of a matrix regularization restricted to polynomial functions in x, y, z (cp. [Arn08a]).

For the round 2-sphere, $f(z) = 1 - z^2$, (5.6) gives the Lie algebra $su(2)$, and its celebrated irreducible representations satisfy

$$X^2 + Y^2 + Z^2 = \mathbb{1} \quad \text{if} \quad \hbar = \frac{2}{\sqrt{N^2-1}}. \tag{5.8}$$

When f is arbitrary, one can still find finite dimensional representations of (5.6) as follows: rewrite (5.6) as

$$[Z,W] = \hbar W \tag{5.9}$$

$$[W,W^\dagger] = -2\hbar f f'(Z) \tag{5.10}$$

implying that $z_i - z_j = \hbar$ whenever $W_{ij} \neq 0$ and Z diagonal. Assuming $W = X + iY$ with non-zero matrix elements $W_{k,k+1} = w_k$ for $k = 1,\ldots,N-1$, one thus obtains (with $w_0 = w_N = 0$)

$$\begin{aligned} Z_{kk} &= \frac{\hbar}{2}\big(N+1-2k\big)\\ w_k^2 - w_{k-1}^2 &= -2\hbar f f'\big(\hbar(N+1-2k)/2\big) \equiv Q_k, \end{aligned}$$

which implies that

$$w_k^2 = \sum_{l=1}^{k} Q_l$$

and the only non-trivial problem is to find the analogue of (5.8). To this end, define

$$\hat{f}^2 = X^2 + Y^2 = \frac{1}{2}\big(WW^\dagger + W^\dagger W\big), \tag{5.11}$$

with W given as above. As Z has pairwise different eigenvalues, the diagonal matrix given in (5.11) can be thought of as a function of Z; hence as $\hat{f}^2(Z)$. It then trivially holds that

$$\hat{C} = X^2 + Y^2 - \hat{f}^2(Z) = 0, \tag{5.12}$$

for the representation defined above. The quantization of $\hbar$ comes through the requirement that $\hat{f}^2$ should correspond to f^2. While for the *round* 2-sphere $\hat{f}^2$ equals f^2, provided $\hbar$ is chosen as in (5.8), it is easy to see that in general they can not coincide, as

$$\begin{aligned} [X^2+Y^2-f(Z)^2, W] &= [(WW^\dagger + W^\dagger W)/2 - f(Z)^2, W]\\ &= \frac{1}{2}W[W^\dagger,W] + \frac{1}{2}[W^\dagger,W]W - f(Z)[f(Z),W] - [f(Z),W]f(Z)\\ &= \cdots = f(Z)\big(\hbar f'(Z)W - [f(Z),W]\big) + \big(\hbar f'(Z)W - [f(Z),W]\big)f(Z) \end{aligned}$$

with off-diagonal elements

$$\big(f(z_k) + f(z_{k-1})\big)\big(\hbar f'(z_k) - (f(z_k) - f(z_{k-1}))\big)$$

that are in general non-zero (hence $X^2+Y^2+f^2(Z)$ is usually not even a Casimir, except in leading order).

How it *does* work is perhaps best illustrated by a non-trivial example, $f(z) = 1-z^4$:

$$\begin{aligned} w_k^2 &= \frac{\hbar^4}{2}\Big((N+1)^3k - 3(N+1)^2k(k+1)+ \\ &\qquad 2(N+1)k(k+1)(2k+1) - 2k^2(k+1)^2\Big) \\ \hat{f}_k^2 &= \frac{1}{2}(w_k^2+w_{k-1}^2) = \frac{\hbar^4}{4}\Big((N+1)^3(2k-1) - 6(N+1)^2k^2 \\ &\qquad + 4(N+1)k(2k^2+1) - 4k^2(k^2+1)\Big) \end{aligned} \tag{5.13}$$

(note that $w_0^2 = w_N^2 = 0$ is explicit in (5.13)) so that

$$\big(X^2+Y^2+Z^4\big)_{kk} = \hbar^4\left[\frac{(N+1)^4}{16} - \frac{(N+1)^3}{4} + k(N+1) - k^2\right]. \tag{5.14}$$

Expressing the last two terms via Z^2 (note that the cancellation of k^3 and k^4 terms shows the absence of Z^3 and higher corrections) one finds

$$\begin{aligned} X^2+Y^2+Z^4+\hbar^2Z^2 &= \hbar^4\frac{(N+1)^2}{16}\Big((N+1)^2 - 4(N+1) + 4\Big)\mathbb{1} \\ &= \hbar^4\frac{(N^2-1)^2}{16}\mathbb{1}, \end{aligned}$$

which equals $\mathbb{1}$ if $\hbar$ is chosen as $2/\sqrt{N^2-1}$. Note that this is the *same* expression for $\hbar$ then for the round sphere, $f^2 = 1-z^2$ (cp. (5.8)).

A more elegant way to derive the quantum Casimir (cp. also [Roc91, GPS09])

$$Q = X^2+Y^2+Z^4+\hbar^2Z^2 \tag{5.15}$$

is to calculate

$$\begin{aligned} [X^2+Y^2+Z^4, W] &= [(WW^\dagger + W^\dagger W)/2 + Z^4, W] \\ &= \cdots = \hbar^2[W, Z^2], \end{aligned}$$

which determines the terms proportional to $\hbar$ in the Casimir.

Due to the general formula

$$\hat{K} = -\frac{1}{8\hbar^4}\varepsilon_{jkl}\varepsilon_{ipq}(\hat{\gamma}^\dagger)^{-2}\big[X^i,[X^k,X^l]\big]\big[X^j,[X^p,X^q]\big]\hat{\gamma}^{-2} \tag{5.16}$$

one obtains, for the axially symmetric surfaces discussed above,

$$\hat{K} = \hat{\gamma}^{-2}\left((ff')^2(Z) + \frac{1}{2\hbar}[W, ff'(Z)]W^\dagger + \frac{1}{2\hbar}W^\dagger[W, ff'(Z)]\right)\hat{\gamma}^{-2} \tag{5.17}$$

with

$$\hat{\gamma}^2 = \frac{1}{2}\big(WW^\dagger + W^\dagger W\big) + (ff')^2(Z) = f(Z)^2\big(f'(Z)^2+\mathbb{1}\big) + O(\hbar), \tag{5.18}$$

giving

$$\hat{K} = -\big(f'(Z)^2+\mathbb{1}\big)^{-2}f(Z)^{-1}f''(Z) + O(\hbar) \tag{5.19}$$

and for $f(z)^2 = 1 - z^4$ one has

$$\hat{K} = \big(4Z^6 + \mathbb{1} - Z^4\big)^{-2}\big(6Z^2 - 2Z^6\big) + O(\hbar) \tag{5.20}$$

$$\hat{\gamma}^2 = \mathbb{1} - Z^4 + 4Z^6 + O(\hbar). \tag{5.21}$$

Note that (cp. (5.9)) $z_j - z_{j-1} = \hbar$ for arbitrary f, and that (due to the axial symmetry) $\hat{K}$ and $\hat{\gamma}^2$ are *diagonal* matrices, so that

$$\hat{\chi} = \hbar\,\mathrm{Tr}\left(\sqrt{\hat{\gamma}^2}\hat{K}\right),$$

in this case simply being a Riemann sum approximation of $\int K\sqrt{g}$, indeed converges to 2, the Euler characteristic of spherical surfaces.

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Max Planck Institute for Gravitational Physics (AEI), Am Mühlenberg 1, D-14476 Golm, Germany.
E-mail address: `joakim.arnlind@aei.mpg.de`

Kungliga Tekniska Högskolan, 10044 Stockholm, Sweden.
E-mail address: `hoppe@math.kth.se`

Max Planck Institute for Gravitational Physics (AEI), Am Mühlenberg 1, D-14476 Golm, Germany.
E-mail address: `gerhard.huisken@aei.mpg.de`

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