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Universal Mass Texture, CP violation and Quark-Lepton Complementarity

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The measurements of the neutrino and quark mixing angles satisfy the empirical relations called Quark-Lepton Complementarity (QLC). These empirical relations suggest the existence of a correlation between the mixing matrices of leptons and quarks. In this work, we examine the possibility that this correlation between the mixing angles of quarks and leptons originates in the similar hierarchy of quarks and charged lepton masses and the seesaw mechanism type I, that gives mass to the Majorana neutrinos. We assume that the similar mass hierarchies of charged lepton and quark masses allows us to represent all the mass matrices of Dirac fermions in terms of a texture with four zeroes.

Keywords: Flavour symmetries; Quark and lepton masses and mixings; Neutrino masses and mixings

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I. INTRODUCTION

The neutrino oscillations between different flavour states were measured in a series of experiments with atmospheric neutrinos [1], solar neutrinos [2], neutrinos produced in nuclear reactors [3] and accelerators [4]. As a result of the global combined analysis including all dominant and subdominant oscillation effects, the difference of the squared neutrino masses and the mixing angles in the lepton mixing matrix,$U_{_{PMNS}}$, were determined at 1σ (3σ) confidence level [5]:

$$\Delta m^2_{21} = 7.67^{+0.22}_{-0.21}\left(^{+0.67}_{-0.61}\right)\times 10^{-5}\ \text{eV}^2,$$

$$\Delta m^2_{31} = \begin{cases} -2.37 \pm 0.15\left(^{+0.43}_{-0.46}\right)\times 10^{-3}\ \text{eV}^2, \\ \quad (m_{\nu_2} > m_{\nu_1} > m_{\nu_3}). \\ \\ +2.46 \pm 0.15\left(^{+0.47}_{-0.42}\right)\times 10^{-3}\ \text{eV}^2, \\ \quad (m_{\nu_3} > m_{\nu_2} > m_{\nu_1}). \end{cases} \quad (1)$$

$$\theta^l_{12} = 34.5^o \pm 1.4\left(^{+4.8}_{-4.0}\right),\ \theta^l_{23} = 42.3^{o}{}^{+5.1}_{-3.3}\left(^{+11.3}_{-7.7}\right),$$
$$\theta^l_{13} = 0.0^{o}{}^{+7.9}_{-0.0}\left(^{+12.9}_{-0.0}\right). \quad (2)$$

Thus, values of the magnitudes of all nine elements of the lepton mixing matrix, $U_{_{PMNS}}$, at 90% CL, are:

$$U_{_{PMNS}} = \begin{pmatrix} 0.80 \to 0.84 & 0.53 \to 0.60 & 0.00 \to 0.17 \\ 0.29 \to 0.52 & 0.51 \to 0.69 & 0.61 \to 0.76 \\ 0.26 \to 0.50 & 0.46 \to 0.66 & 0.64 \to 0.79 \end{pmatrix}. \quad (3)$$

The CHOOZ experiment [6] determined an upper bound for the θ^l_{13} mixing angle. The latest analyses give the following best values: [7, 8]:

$$\theta^l_{13} = -0.07^{+0.18}_{-0.11} \quad (4)$$

and (at $1\sigma(3\sigma)$)

$$\theta^l_{13} = 5.6^{+3.0}_{-2.7}\,(\leq 12.5)^o,\ \theta^l_{13} = 5.1^{+3.0}_{-3.3}\,(\leq 12.0)^o, \quad (5)$$

see also [9]. On the other hand, in the last years extensive research has been done in the precise determination of the values of the $V_{_{CKM}}$ quark mixing matrix elements. The most precise fit results for the values of the magnitudes of all nine CKM elements are [10]

$$V_{_{CKM}} = \begin{pmatrix} 0.97419 \pm .00022 & 0.2257 \pm .0010 & 0.00359 \pm .00016 \\ 0.2256 \pm .0010 & 0.97334 \pm .00023 & 0.0415^{+.0010}_{-.0011} \\ 0.00874^{+.00026}_{-.00037} & 0.0407 \pm .0010 & 0.999133^{+.000044}_{-.000043} \end{pmatrix} \quad (6)$$

and the Jarlskog invariant is

$$J^q = \left(3.05^{+0.19}_{-0.20}\right)\times 10^{-5}. \quad (7)$$

We also have the three angles of the unitarity triangle with the following reported best values [10]:

$$\alpha = \left(88^{+6}_{-5}\right)^o,\ \beta = (21.46 \pm 0.71)^o,\ \gamma = \left(77^{+30}_{-32}\right)^o. \quad (8)$$

Each of the elements of the $V_{_{CKM}}$ matrix can be extracted from a large number of decays and, for the purpose of our analysis, will be considered as independent. Hence, current knowledge of the mixing angles for the quark sector can be summarized at 1σ as [10]:

$$\sin\theta^q_{12} = 0.2257 \pm 0.001,\ \sin\theta^q_{23} = 0.0415^{+0.001}_{-0.0011},$$
$$\sin\theta^q_{13} = 0.00359 \pm 0.00016. \quad (9)$$

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The solar mixing angle θ^l_{12} and the correponding angle in the quark sector, the Cabibbo angle θ^q_{12}, satisfy an interesting and intriguing numerical relation (at 90% confidence level) [11],

$$\theta^l_{12} + \theta^q_{12} \approx 45^o + 2.5^o \pm 1.5^o, \qquad (10)$$

see also [12]. The equation (10) relates the 1-2 mixing angles in the quark and lepton sectors, it is commonly known as Quark-Lepton Complementarity relation (QLC) and, if not accidental, it could imply a quark-lepton symmetry.
A second QLC relation between the atmospheric and 2-3 mixing angles, is also satisfied [13],

$$\theta^l_{23} + \theta^q_{23} = \left(44.67^{+5.1}_{-3.3}\right)^o . \qquad (11)$$

However, this is not as interesting as (10) because θ^q_{23} is only about two degrees, and the corresponding QLC relation would be satisfied, within the errors, even if the angle θ^q_{23} had been zero,as long as θ^l_{23} is close to the maximal value $\pi/4$.
A third possible QLC relation is not realized at all, or at least not realized in the same way, since it is less than ten degrees [13].

$$\theta^l_{13} + \theta^q_{13} < 8.1^o. \qquad (12)$$

The equations (10)-(12) are known as the extended Quark Lepton Complementarity, for a review see [14]. The extended QLC relations could imply a quark-lepton symmetry [14] or a quark lepton unification [15].
A systematic numerical exploration of all CP conserving textures of the neutrino mass matrix compatible the QLC relations and the experimental information on neutrino mixing is given in [16]

The neutrino oscillations do not provide information about either the absolute mass scale or if neutrinos are Dirac or Majorana particles [17]. Thus, one of the most fundamental problems of the neutrinos physics is the question of the nature of massive neutrinos. A direct way to reveal the nature of massive neutrinos is to investigate processes in which the total lepton number is not conserved [18]. The matrix elements for these processes are proportional to the effective Majorana neutrino masses, which are defined as

$$\langle m_{ll} \rangle \equiv \sum_{j=1}^{3} m_{\nu_j} U^2_{lj}, \qquad l = e, \mu, \tau, \qquad (13)$$

where m_{ν_j} is the neutrino Majorana mass and U_{lj} are the elements of the lepton mixing matrix.

In this work, we will focus our attention on understanding the nature of the QLC relation, and finding possible values for the effective Majorana neutrino masses. Thus, we made a unified treatment of quarks and leptons, where we assume that the charged lepton and quark mass matrices have the same generic form with a four zeroes texture from a universal S_3 flavour symmetry and its sequential explicit breaking.

II. UNIVERSAL MASS MATRIX WITH A FOUR ZEROES TEXTURE

In particle physics, the imposition of a flavour symmetry has been successful in reducing the number of parameters of the Standard Model. Recent flavour symmetry models are reviewed in [19], see also the references therein. In particular, a permutational S_3 flavour symmetry and its sequential explicit breaking allows us to represent the mass matrices in a generic form with a texture with four zeroes [20]:

$$\mathbf{M_i^{(F)}} = \begin{pmatrix} 0 & A_i & 0 \\ A_i^* & B_i & C_i \\ 0 & C_i & D_i \end{pmatrix}, \quad i = u, d, l, \nu_{_D}. \qquad (14)$$

Some reasons to propose the validity of a texture with four zeroes as a universal mass texture for all Dirac fermions in the theory are the following:

1. The idea of S_3 flavour symmetry and its explicit breaking has been succesfully realized as a mass matrix with four texture zeroes in the quark sector to interpret the strong mass hierarchy of up and down type quarks [21].

2. The quark mixing angles and the CP violating phase, appearing in the $V_{_{CKM}}$ mixing matrix, were computed as explicit, exact functions of the four quark mass ratios $(m_u/m_t, m_c/m_t, m_d/m_b, m_s/m_b)$, one symmetry breaking parameter $Z^{1/2} = \left(\frac{81}{32}\right)^{1/2}$ and one CP violating phase $\phi_{_{u-d}} = 90^o$, in good agreement with the experimetal data ten years ago when the first fitting was made [20]. This agreement improved as the precision of the experimental data improved and, now, it is very good [10].

3. Since the mass spectrum of the charged leptons exhibits a hierarchy similar to the quark's one, it would be natural to consider the same S_3 symmetry and its explicit breaking to justify the use of the same texture for the charged lepton mass matrix.

4. As for the Dirac neutrinos, we have no direct information about the absolute values or the relative values of the neutrino masses, but the mass matrix with a four zeroes texture can be obtained from an $SO(10)$ neutrino model which describes these data well [22]. Furthermore, from supersymmetry arguments, it would be sensible to assume that the Dirac neutrinos have a mass hierarchy similar to that of the u-quarks and it would be natural to take for the Dirac neutrino mass matrix also a matrix with a texture with four zeroes.

The Hermitian mass matrix (14) may be written in terms of a real symmetric matrix $\bar{M}_i$ and a diagonal matrix of phases $P_i \equiv \mathrm{diag}\left[1, e^{i\phi_i}, e^{i\phi_i}\right]$ as follows:

$$M_i^{(F)} = P_i^\dagger \bar{M}_i P_i \, . \qquad (15)$$

The real symetric matrix $\bar{M}_i$ may be brought to diagonal form by means of an orthogonal transformation,

$$\bar{M}_i = \mathbf{O}_i \mathrm{diag}\left\{m_{i1}, m_{i2}, m_{i3}\right\} \mathbf{O}_i^T, \qquad (16)$$

where the m's are the eigenvalues of $M_i^{(F)}$ and $\mathbf{O}_i$ is a real orthogonal matrix. Now computing the invariants of the real symetric matrix $\bar{M}_i$, tr$\left\{\bar{M}_i\right\}$, tr$\left\{\bar{M}_i^2\right\}$ and det$\left\{\bar{M}_i\right\}$,we may express the parameters A_i, B_i, C_i and D_i occuring in (14) in terms of the mass eigenvalues, in this way, we get that the $\bar{M}_i$ matrix, reparametrized in terms of its eigenvalues ($i = u, d, l, \nu_D$) is

$$\bar{M}_i = \begin{pmatrix} 0 & \sqrt{\frac{\widetilde{m}_{i1}\widetilde{m}_{i2}}{1-\delta_i}} & 0 \\ \sqrt{\frac{\widetilde{m}_{i1}\widetilde{m}_{i2}}{1-\delta_i}} & \widetilde{m}_{i1} - \widetilde{m}_{i2} + \delta_i & \sqrt{\frac{\delta_i}{(1-\delta_i)} f_{i1} f_{i2}} \\ 0 & \sqrt{\frac{\delta_i}{(1-\delta_i)} f_{i1} f_{i2}} & 1 - \delta_i \end{pmatrix}, \qquad (17)$$

where $\widetilde{m}_{i1} = \frac{m_{i1}}{m_{i3}}$, $\widetilde{m}_{i2} = \frac{|m_{i2}|}{m_{i3}}$,

$$f_{i1} = 1 - \widetilde{m}_{i1} - \delta_i, \quad f_{i2} = 1 + \widetilde{m}_{i2} - \delta_i. \qquad (18)$$

The small parameters δ_i are also functions of the mass ratios and the flavour symmetry breaking parameter $Z^{1/2}$ [20],

$$\delta_i = \frac{Z_i}{Z_i + 1} \frac{\left(\widetilde{m}_{i2} - \widetilde{m}_{i1}\right)^2}{W_i\left(Z\right)} \qquad (19)$$

where

$$\begin{aligned} W_i\left(Z\right) &= \left[p^3 + 2q^2 + 2q\sqrt{p^3 + q^2}\right]^{\frac{1}{3}} - |p| + \\ &+ \left[p^3 + 2q^2 - 2q\sqrt{p^3 + q^2}\right]^{\frac{1}{3}} + \\ &+ \tfrac{1}{9}\left(Z_i\left(2\left(\widetilde{m}_{i2} - \widetilde{m}_{i1}\right) + 1\right) + \left(\widetilde{m}_{i2} - \widetilde{m}_{i1}\right) + 2\right)^2 \\ &- \tfrac{1}{3}\left(\left[q + \sqrt{p^3 + q^2}\right]^{\frac{1}{3}} + \left[q - \sqrt{p^3 + q^2}\right]^{\frac{1}{3}}\right) \times \\ &\times \left(Z_i\left(2\left(\widetilde{m}_{i2} - \widetilde{m}_{i1}\right) + 1\right) + \left(\widetilde{m}_{i2} - \widetilde{m}_{i1}\right) + 2\right) \end{aligned} \qquad (20)$$

with

$$\begin{aligned} p &= -\tfrac{1}{3}\tfrac{Z_i}{Z_i+1}\left(Z_i\left(2\left(\widetilde{m}_{i2} - \widetilde{m}_{i1}\right) + 1\right) + \widetilde{m}_{i2} - \right. \\ &\left.\widetilde{m}_{i1} + 2\right)^2 + \tfrac{1}{Z_i+1}\left[Z_i\left(\widetilde{m}_{i2} - \widetilde{m}_{i1}\right)\left(\widetilde{m}_{i2} - \widetilde{m}_{i1} + \right.\right. \\ &\left.\left. + 2\right)\left(1 + \widetilde{m}_{i2}\right)\left(1 - \widetilde{m}_{i1}\right)\right], \end{aligned} \qquad (21)$$

$$\begin{aligned} q &= -\tfrac{1}{27}\tfrac{1}{(Z_i+1)^3}\left(Z_i\left(2\left(\widetilde{m}_{i2} - \widetilde{m}_{i1}\right) + 1\right) + \widetilde{m}_{i2} - \right. \\ &\left.\widetilde{m}_{i1} + 2\right)^3 + \tfrac{1}{6}\tfrac{1}{(Z_i+1)^2}\left[Z_i\left(\widetilde{m}_{i2} - \widetilde{m}_{i1}\right)\left(\widetilde{m}_{i2} - \right.\right. \\ &\left.\left. - \widetilde{m}_{i1} + 2\right)\left(1 + \widetilde{m}_{i2}\right)\left(1 - \widetilde{m}_{i1}\right)\right]\left(Z_i\left(2\left(\widetilde{m}_{i2} - \right.\right.\right. \\ &\left.\left.\left. - \widetilde{m}_{i1}\right) + 1\right) + \widetilde{m}_{i2} - \widetilde{m}_{i1} + 2\right). \end{aligned} \qquad (22)$$

The values allowed for the parameters δ_i are in the following range $0 < \delta_i < 1 - \widetilde{m}_{i1}$.

The entries in the real orthogonal matrix, $\mathbf{O}$, are also expressed in terms of the eigenvalues of the mass matrix (14) as

$$\mathbf{O_i} = \begin{pmatrix} \left[\frac{\widetilde{m}_{i2} f_{i1}}{D_{i1}}\right]^{\frac{1}{2}} & -\left[\frac{\widetilde{m}_{i1} f_{i2}}{D_{i2}}\right]^{\frac{1}{2}} & \left[\frac{\widetilde{m}_{i1}\widetilde{m}_{i2}\delta_i}{D_{i3}}\right]^{\frac{1}{2}} \\ \left[\frac{\widetilde{m}_{i1}(1-\delta_i) f_{i1}}{D_{i1}}\right]^{\frac{1}{2}} & \left[\frac{\widetilde{m}_{i2}(1-\delta_i) f_{i2}}{D_{i2}}\right]^{\frac{1}{2}} & \left[\frac{(1-\delta_i)\delta_i}{D_{i3}}\right]^{\frac{1}{2}} \\ -\left[\frac{\widetilde{m}_{i1} f_{i2}\delta_i}{D_{i1}}\right]^{\frac{1}{2}} & -\left[\frac{\widetilde{m}_{i2} f_{i1}\delta_i}{D_{i2}}\right]^{\frac{1}{2}} & \left[\frac{f_{i1} f_{i2}}{D_{i3}}\right]^{\frac{1}{2}} \end{pmatrix}, \qquad (23)$$

where,

$$D_{i1} = (1 - \delta_i)(\widetilde{m}_{i1} + \widetilde{m}_{i2})(1 - \widetilde{m}_{i1}), \qquad (24)$$

$$D_{i2} = (1 - \delta_i)(\widetilde{m}_{i1} + \widetilde{m}_{i2})(1 + \widetilde{m}_{i2}), \qquad (25)$$

$$D_{i3} = (1 - \delta_i)(1 - \widetilde{m}_{i1})(1 + \widetilde{m}_{i2}). \qquad (26)$$

III. SEESAW MECHANISM AND PHASES OF THE RIGHT HANDED NEUTRINO MASS MATRIX

The left handed Majorana neutrinos naturally acquire their small masses through an effective type I seesaw mechanism of the form

$$M_{\nu_L} = M_{\nu_D} M_{\nu_R}^{-1} M_{\nu_D}^T, \qquad (27)$$

where M_{ν_D} and M_{ν_R} denote the Dirac and right handed Majorana neutrinos mass matrices, respectively. The symmetry of the mass matrix of the left handed Majorana neutrinos, $M_{\nu_L} = M_{\nu_L}^T$, and the seesaw mechanism of type I, eq. (27), fix the form of the right handed Majorana neutrinos mass matrix, M_{ν_R}, which has to be nonsingular and symmetric. Further restrictions on M_{ν_R}, follow from requiring that M_{ν_L} also had a texture with four zeroes, as will be shown below. From eq. (27), the seesaw mechanism may be written in a more explicit form as:

$$M_{\nu_L} = \frac{1}{\det\left(M_{\nu_R}\right)} M_{\nu_D} \mathrm{adj}\left(M_{\nu_R}\right) M_{\nu_D}^T, \qquad (28)$$

where det(M_{ν_R}) and adj$\left(M_{\nu_R}\right)$ are the determinant and adjugate matrix of M_{ν_R}, respectively. Calling C_{nm}, $m, n = 1, 2, 3$, the cofactors of the M_{ν_R} matrix, eq. (28) may be written as

$$M_{\nu_L} = \frac{1}{\det\left(M_{\nu_R}\right)} \begin{pmatrix} f_{\nu_L} & a_{\nu_L} & e_{\nu_L} \\ a_{\nu_L} & b_{\nu_L} & c_{\nu_L} \\ e_{\nu_L} & c_{\nu_L} & d_{\nu_L} \end{pmatrix}, \qquad (29)$$

where

$$\det(M_{\nu_R}) = f_{\nu_R} C_{11} - a_{\nu_R} C_{12} + e_{\nu_R} C_{13} \qquad (30)$$

and

$$\begin{aligned} f_{\nu_L} &= C_{22} a_{\nu_D}^2, \\ a_{\nu_L} &= -C_{12}|a_{\nu_D}| + C_{22} a_{\nu_D} b_{\nu_D} - C_{23} a_{\nu_D} c_{\nu_D}, \\ b_{\nu_L} &= C_{11} a_{\nu_D}^{*2} + C_{22} b_{\nu_D}^2 + C_{33} c_{\nu_D}^2 - 2c_{12} a_{\nu_D}^* b_{\nu_D} \\ &\quad + 2C_{13} a_{\nu_D}^* c_{\nu_D} - 2C_{23} b_{\nu_D} c_{\nu_D}, \\ e_{\nu_L} &= C_{22} a_{\nu_D} c_{\nu_D} - C_{23} a_{\nu_D} d_{\nu_D}, \\ c_{\nu_L} &= C_{13} a_{\nu_D}^* d_{\nu_D} - C_{12} a_{\nu_D}^* c_{\nu_D} + C_{22} b_{\nu_D} c_{\nu_D} \\ &\quad - C_{23}\left(b_{\nu_D} d_{\nu_D} + c_{\nu_D}^2\right) + C_{33} c_{\nu_D} d_{\nu_D}, \\ d_{\nu_L} &= C_{22} c_{\nu_D}^2 - 2C_{23} c_{\nu_D} d_{\nu_D} + C_{33} d_{\nu_D}^2, \end{aligned} \qquad (31)$$

From eqs. (29) and (31), the mass matrix of the left handed Majorana neutrino will have the same universal texture with four zeroes of the Dirac mass matrices when conditions $C_{22} = C_{23} = 0$ are satisfied. These last conditions are equivalent to

$$f_{\nu_R} d_{\nu_R} = e_{\nu_R}^2, \qquad f_{\nu_R} c_{\nu_R} = a_{\nu_R} e_{\nu_R}, \tag{32}$$

Thus we obtain the relation

$$\frac{a_{\nu_R}}{c_{\nu_R}} = \frac{e_{\nu_R}}{d_{\nu_R}}. \tag{33}$$

For non vanishing $\det(M_{\nu_R})$, these relation are satisfied, if and only if

$$e_{\nu_R} = 0 \quad \text{and} \quad f_{\nu_R} = 0. \tag{34}$$

If we extended the meaning of a texture with four zeroes, defined in (14), to include the symmetric mass matrix of the right handed Majorana neutrinos, M_{ν_R} [23], which is non-Hermitian, we could say that the texture with four zeroes is invariant under the action of the seesaw mechanism of type I [13, 23, 24].
It may also be noticed that, if we set $b_{\nu_R} = 0$ or/and $c_{\nu_R} = 0$, the resulting expression for M_{ν_L} still has a texture with four zeroes. Therefore, M_{ν_L} may also have a texture with four zeroes when M_{ν_R} has a texture with four or six zeroes (sometimes called a Fritzsch Texture).

Let us further assume that the phases in the entries of the M_{ν_R} may be factorized out as

$$M_{\nu_R} = R \bar{M}_{\nu_R} R, \tag{35}$$

where

$$\bar{M}_{\nu_R} = \begin{pmatrix} 0 & a_{\nu_R} & 0 \\ a_{\nu_R} & |b_{\nu_R}| & |c_{\nu_R}| \\ 0 & |c_{\nu_R}| & d_{\nu_R} \end{pmatrix}, \tag{36}$$

and $R \equiv \mathrm{diag}\left[e^{-i\phi_c}, e^{i\phi_c}, 1\right]$ with $\phi_c \equiv \arg\left\{c_{\nu_R}\right\}$.
Then, the type I seesaw mechanism takes the form:

$$M_{\nu_L}^{(F)} = P_D^\dagger \bar{M}_{\nu_D} P_D R^\dagger \bar{M}_{\nu_R}^{-1} R^\dagger P_D \bar{M}_{\nu_D} P_D^\dagger, \tag{37}$$

and the mass matrix of the left handed neutrinos has the following four texture zeroes [30]:

$$M_{\nu_L} = \begin{pmatrix} 0 & a_{\nu_L} & 0 \\ a_{\nu_L} & b_{\nu_L} & c_{\nu_L} \\ 0 & c_{\nu_L} & d_{\nu_L} \end{pmatrix}, \tag{38}$$

where

$$\begin{aligned} a_{\nu_L} &= \frac{|a_{\nu_D}|^2}{a_{\nu_R}}, \\ b_{\nu_L} &= \frac{c_{\nu_D}^2}{d_{\nu_R}} + \frac{|c_{\nu_R}|^2 - |b_{\nu_R}| d_{\nu_R}}{d_{\nu_R}} \frac{|a_{\nu_D}|^2}{a_{\nu_R}^2} e^{i2\left(\phi_c - \phi_{\nu_D}\right)} \\ &\quad + 2\frac{|a_{\nu_D}|}{|a_{\nu_R}|}\left(b_{\nu_D} e^{-i\phi_{\nu_D}} - \frac{c_{\nu_D}|c_{\nu_R}|}{d_{\nu_R}} e^{i\left(\phi_c - \phi_{\nu_D}\right)}\right), \\ c_{\nu_L} &= \frac{c_{\nu_D} d_{\nu_D}}{d_{\nu_R}} + \\ &\quad + \frac{|a_{\nu_D}|}{|a_{\nu_R}|}\left(c_{\nu_D} e^{-i\phi_{\nu_D}} - \frac{|c_{\nu_R}| d_{\nu_D}}{d_{\nu_R}} e^{i\left(\phi_c - \phi_{\nu_D}\right)}\right), \\ d_{\nu_L} &= \frac{d_{\nu_D}^2}{d_{\nu_R}}. \end{aligned} \tag{39}$$

Now, to diagonalize the left handed Majorana neutrino mass matrix M_{ν_L} by means of a unitary matrix, we need to construct the hermitian matrices $M_{\nu_L} M_{\nu_L}^\dagger$ and $M_{\nu_L}^\dagger M_{\nu_L}$, which can be diagonalized with unitary matrices through of the following transformations:

$$\begin{aligned} U_R^\dagger M_{\nu_L}^\dagger M_{\nu_L} U_R &= \mathrm{diag}\left[\left|m_{\nu_1}^s\right|^2, \left|m_{\nu_2}^s\right|^2, \left|m_{\nu_3}^s\right|^2\right], \\ U_L^\dagger M_{\nu_L} M_{\nu_L}^\dagger U_L &= \mathrm{diag}\left[\left|m_{\nu_1}^s\right|^2, \left|m_{\nu_2}^s\right|^2, \left|m_{\nu_3}^s\right|^2\right], \end{aligned} \tag{40}$$

where the $m_{\nu_j}^s$ $(j = 1, 2, 3)$ are the singular values of the M_{ν_L} matrix. Thus, with the help of the symmetry of the matrix (38) and the transformations (40), the left handed Majorana neutrino mass matrix, M_{ν_L}, can be diagonalized by a unitary matrix through the transformation

$$U_\nu^\dagger M_{\nu_L} U_\nu^* = \mathrm{diag}\left[\left|m_{\nu_1}^s\right|, \left|m_{\nu_2}^s\right|, \left|m_{\nu_3}^s\right|\right], \tag{41}$$

where $U_\nu \equiv U_L \mathcal{K}$ and $\mathcal{K} \equiv \mathrm{diag}\left[e^{i\eta_1/2}, e^{i\eta_2/2}, e^{i\eta_3/2}\right]$ is the diagonal matrix of the Majorana phases.
From the previous analysis, the matrix M_{ν_L} has only two phases, which are

$$\phi_1 \equiv \arg\left\{b_{\nu_L}\right\} \quad \text{and} \quad \phi_2 \equiv \arg\left\{c_{\nu_L}\right\}. \tag{42}$$

In the particular case, when $\phi_1 = 2\phi_2$, the analysis simplifies since the phases in M_{ν_L} may be factorized out[31] and the following relationship is fulfilled:

$$\tan\phi_1 = \frac{2\Im m\, c_{\nu_L} \Re e\, c_{\nu_L}}{\left(\Re e\, c_{\nu_L}\right)^2 - \left(\Im m\, c_{\nu_L}\right)^2}. \tag{43}$$

Then, the left handed Majorana neutrino mass matrix may be written as follows

$$M_{\nu_L} = Q \bar{M}_{\nu_L} Q, \tag{44}$$

where $Q \equiv \mathrm{diag}\left[e^{-i\phi_2}, e^{i\phi_2}, 1\right]$ is a diagonal matrix of phases and $\bar{M}_{\nu_L}$ is a real symetric matrix. Now, the matrix M_{ν_L}, can be diagonalized by a unitary matrix through the transformation

$$U_\nu^\dagger M_{\nu_L} U_\nu^* = \mathrm{diag}\left[m_{\nu_1}, m_{\nu_2}, m_{\nu_3}\right]; \tag{45}$$

where m_{ν_j} $(j = 1, 2, 3)$ are the eigenvalues of the matrix M_{ν_L}, and the unitary matrix is $U_\nu \equiv Q\mathbf{O}_\nu\mathcal{K}$ where $\mathbf{O}_\nu$ is the orthogonal real matrix (23), that diagonalizes the real symetric matrix $\bar{M}_{\nu_L}$.

It is also important to mention that when the Hermitian matrix with a texture with four zeroes defined in eq. (14), is taken as a universal mass texture for all fermions [13], the phases of all entries in the right handed Majorana neutrino mass matrix are fixed at the numerical value of $\phi_{\nu_R} = n\pi$. Thus, the right handed Majorana neutrinos mass matrix is real and symmetric and has the texture with four zeroes shown in (14). In the more general case in which the Dirac and right handed neutrino mass matrices are represented by Hermitian matrices that can be written in polar form as $A = P^\dagger \bar{A} P$, where P is a diagonal matrix of phases and $\bar{A}$ is a real symmetric matrix, the symmetry of the left handed Majorana neutrino mass matrix also fixes all phases in the mass matrix of the right handed neutrinos at the numerical value $\phi_{\nu_R} = n\pi$. Then, the only undetermined phases in the mass matrix of the left handed Majorana neutrinos M_{ν_L} are the phases ϕ_{ν_D}, coming from the mass matrix of the Dirac neutrinos.

IV. MIXING MATRICES

The quark and lepton flavour mixing matrices, U_{PMNS} and V_{CKM}, arise from the mismatch between diagonalization of the mass matrices of u and d type quarks [10] and the diagonalization of the mass matrices of charged leptons and left handed neutrinos [25] respectively,

$$U_{PMNS} = U_l^\dagger U_\nu, \quad V_{CKM} = U_u U_d^\dagger. \tag{46}$$

Therefore, in order to obtain the unitary matrices appearing in (46) and get predictions for the flavour mixing angles and CP violating phases, we should specify the mass matrices.

In the quark sector, the unitarity of V_{CKM} leads to the relations $\sum_i V_{ij}V_{ik}^* = \delta_{jk}$ and $\sum_j V_{ij}V_{kj}^* = \delta_{ik}$. The vanishing combinations can be represented as triangles in a complex plane. The area of all triangles is equal to half of the Jarlskog invariant, J_q [26], which is a rephasing invariant measure of CP violation. The term unitarity triangle is usually reserved for the tringle obtained from the relation $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$. In this case de Jarlskog invariant is

$$J_q = \Im m \left[V_{us}V_{cs}^*V_{ub}^*V_{cb} \right], \tag{47}$$

and the inner angles of the unitarity triangle are

$$\begin{aligned} &\alpha \equiv \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right), \quad \beta \equiv \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \\ &\gamma \equiv \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right). \end{aligned} \tag{48}$$

For the lepton sector, when the left handed neutrinos are Majorana particles, the mixing matrix is defined as [27] $U_{PMNS} = U_l^\dagger U_L K$ where $K \equiv \mathrm{diag}\left[1, e^{i\beta_1}, e^{i\beta_2}\right]$ is the diagonal matrix of the Majorana CP violating phases. Also in the case of three neutrino mixing there are three CP violation rephasing invariants [25], associated with the three CP violating phases present in the U_{PMNS} matrix. The rephasing invariant related to the Dirac phase, analogous to the Jarlskog invariant in the quark sector, is given by:

$$J_l \equiv \Im m \left[U_{e1}^* U_{\mu 3}^* U_{e3} U_{\mu 1} \right]. \tag{49}$$

The rephasing invariant J_l controls the magnitude of CP violation effects in neutrino oscillations and is a directly observable quantity. The other two rephasing invariants associated with the two Majorana phases in the U_{PMNS} matrix, can be chosen as:

$$S_1 \equiv \Im m \left[U_{e1}U_{e3}^* \right], \quad S_2 \equiv \Im m \left[U_{e2}U_{e3}^* \right]. \tag{50}$$

These rephasing invariants are not uniquely defined, but the ones shown in the eqs (49) and (50) are relevant for the definition of the effective Majorana neutrino mass, m_{ee}, in the neutrinoless double beta decay.

A. Mixing Matrices as Functions of the Fermion Masses

The unitary matrices U_i $(i = u, d)$ occurring in the definition of V_{CKM}, eq.(46), may be written in polar form as $U_i = \mathbf{O}_i^T P_i$. In this expresion, P_i is the diagonal matrix of phases appearing in the four texutre zeroes mass matrix (15). Then, from (46), the quark mixing matrix takes the form

$$V_{CKM}^{th} = \mathbf{O_u}^T P^{(u-d)} \mathbf{O}_d, \tag{51}$$

where $P^{(u-d)} = \mathrm{diag}\left[1, e^{i\phi}, e^{i\phi}\right]$ with $\phi = \phi_u - \phi_d$, and $\mathbf{O}_i$, are the real orthogonal matrices (23) that diagonalize the real symmetric mass matrices $\bar{M}_i^{(F)}$.

A similar analysis shows that U_{PMNS} may also be written as $U_{PMNS} = U_l^\dagger U_\nu$, with $U_j = P_j \mathbf{O_j}$ $(j = \nu, l)$, this matrix takes the form

$$U_{PMNS}^{th} = \mathbf{O}_l^T P^{(\nu-l)} \mathbf{O}_\nu K, \tag{52}$$

where $P^{(\nu-l)} = \mathrm{diag}\left[1, e^{i\Phi_1}, e^{i\Phi_2}\right]$ is the diagonal matrix of the Dirac phases, with $\Phi_1 = 2\phi_2 - \phi_l$ and $\Phi_2 = \phi_2 - \phi_l$. The real orthogonal matrices $\mathbf{O}_j$ are defined in (23).

Substitution of the expressions (18)-(26) in the unitary matices (51) and (52) allows us to express the mixing matrices V_{CKM}^{th} and U_{PMNS}^{th} as explicit functions of the masses of quarks and leptons. For the elements of the V_{CKM} mixing matrix, we obtained the same theoretical expressions given by Mondragón and Rodríguez-Jauregui [20]:

$$V_{CKM}^{th} = \begin{pmatrix} V_{ud}^{th} & V_{us}^{th} & V_{ub}^{th} \\ V_{cd}^{th} & V_{cs}^{th} & V_{cb}^{th} \\ V_{td}^{th} & V_{ts}^{th} & V_{tb}^{th} \end{pmatrix}, \tag{53}$$

where

$$
\begin{aligned}
V_{ud}^{th} &= \sqrt{\frac{\widetilde{m}_c\widetilde{m}_s f_{u1} f_{d1}}{D_{u1}D_{d1}}} + \sqrt{\frac{\widetilde{m}_u\widetilde{m}_d}{D_{u1}D_{d1}}}\left(\sqrt{(1-\delta_u)(1-\delta_d) f_{u1}f_{d1}} + \sqrt{\delta_u\delta_d f_{u2}f_{d2}}\right)e^{i\phi},\\
V_{us}^{th} &= -\sqrt{\frac{\widetilde{m}_c\widetilde{m}_d f_{u1} f_{d2}}{D_{u1}D_{d2}}} + \sqrt{\frac{\widetilde{m}_u\widetilde{m}_s}{D_{u1}D_{d2}}}\left(\sqrt{(1-\delta_u)(1-\delta_d) f_{u1}f_{d2}} + \sqrt{\delta_u\delta_d f_{u2}f_{d1}}\right)e^{i\phi},\\
V_{ub}^{th} &= \sqrt{\frac{\widetilde{m}_c\widetilde{m}_d\widetilde{m}_s \delta_d f_{u1}}{D_{u1}D_{d3}}} + \sqrt{\frac{\widetilde{m}_u}{D_{u1}D_{d3}}}\left(\sqrt{(1-\delta_u)(1-\delta_d)\delta_d f_{u1}} - \sqrt{\delta_u f_{u2}f_{d1}f_{d2}}\right)e^{i\phi},\\
V_{cd}^{th} &= -\sqrt{\frac{\widetilde{m}_u\widetilde{m}_s f_{u2} f_{d1}}{D_{u2}D_{d1}}} + \sqrt{\frac{\widetilde{m}_c\widetilde{m}_d}{D_{u2}D_{d1}}}\left(\sqrt{(1-\delta_u)(1-\delta_d) f_{u2}f_{d1}} + \sqrt{\delta_u\delta_d f_{u1}f_{d2}}\right)e^{i\phi},\\
V_{cs}^{th} &= \sqrt{\frac{\widetilde{m}_u\widetilde{m}_d f_{u2} f_{d2}}{D_{u2}D_{d2}}} + \sqrt{\frac{\widetilde{m}_c\widetilde{m}_s}{D_{u2}D_{d2}}}\left(\sqrt{(1-\delta_u)(1-\delta_d) f_{u2}f_{d2}} + \sqrt{\delta_u\delta_d f_{u1}f_{d1}}\right)e^{i\phi},\\
V_{cb}^{th} &= -\sqrt{\frac{\widetilde{m}_u\widetilde{m}_d\widetilde{m}_s \delta_d f_{u2}}{D_{u2}D_{d3}}} + \sqrt{\frac{\widetilde{m}_c}{D_{u2}D_{d3}}}\left(\sqrt{(1-\delta_u)(1-\delta_d)\delta_d f_{u2}} - \sqrt{\delta_u f_{u1}f_{d1}f_{d2}}\right)e^{i\phi},\\
V_{td}^{th} &= \sqrt{\frac{\widetilde{m}_u\widetilde{m}_c\widetilde{m}_s \delta_u f_{d1}}{D_{u3}D_{d1}}} + \sqrt{\frac{\widetilde{m}_d}{D_{u3}D_{d1}}}\left(\sqrt{\delta_u(1-\delta_u)(1-\delta_d) f_{d1}} - \sqrt{\delta_d f_{u1}f_{u2}f_{d2}}\right)e^{i\phi},\\
V_{ts}^{th} &= -\sqrt{\frac{\widetilde{m}_u\widetilde{m}_c\widetilde{m}_d \delta_u f_{d2}}{D_{u3}D_{d2}}} + \sqrt{\frac{\widetilde{m}_s}{D_{u3}D_{d2}}}\left(\sqrt{\delta_u(1-\delta_u)(1-\delta_d) f_{d2}} - \sqrt{\delta_d f_{u1}f_{u2}f_{d1}}\right)e^{i\phi},\\
V_{tb}^{th} &= \sqrt{\frac{\widetilde{m}_u\widetilde{m}_c\widetilde{m}_d\widetilde{m}_s \delta_u\delta_d}{D_{u3}D_{d3}}} + \left(\sqrt{\frac{f_{u1}f_{u2}f_{d1}f_{d2}}{D_{u3}D_{d3}}} + \sqrt{\frac{\delta_u\delta_d(1-\delta_u)(1-\delta_d)}{D_{u3}D_{d3}}}\right)e^{i\phi}.
\end{aligned}
\tag{54}
$$

Now, with the help of the equations (23) and (52), we obtain the theoretical expresion of the elements of the lepton mixing matrix, $U_{_{PMNS}}^{th}$. This expresions have the following form:

$$
U_{_{PMNS}}^{th} = \begin{pmatrix} U_{e1}^{th} & U_{e2}^{th}e^{i\beta_1} & U_{e3}^{th}e^{i\beta_2} \\ U_{\mu1}^{th} & U_{\mu2}^{th}e^{i\beta_1} & U_{\mu3}^{th}e^{i\beta_2} \\ U_{\tau1}^{th} & U_{\tau2}^{th}e^{i\beta_1} & U_{\tau3}^{th}e^{i\beta_2} \end{pmatrix} \tag{55}
$$

where

$$
\begin{aligned}
U_{e1}^{th} &= \sqrt{\frac{\widetilde{m}_\mu\widetilde{m}_{\nu_2} f_{l1} f_{\nu1}}{D_{l1}D_{\nu1}}} + \sqrt{\frac{\widetilde{m}_e\widetilde{m}_{\nu_1}}{D_{l1}D_{\nu1}}}\left(\sqrt{(1-\delta_l)(1-\delta_\nu) f_{l1}f_{\nu1}}e^{i\Phi_1} + \sqrt{\delta_l\delta_\nu f_{l2}f_{\nu2}}e^{i\Phi_2}\right),\\
U_{e2}^{th} &= -\sqrt{\frac{\widetilde{m}_\mu\widetilde{m}_{\nu_1} f_{l1} f_{\nu2}}{D_{l1}D_{\nu2}}} + \sqrt{\frac{\widetilde{m}_e\widetilde{m}_{\nu_2}}{D_{l1}D_{\nu2}}}\left(\sqrt{(1-\delta_l)(1-\delta_\nu) f_{l1}f_{\nu2}}e^{i\Phi_1} + \sqrt{\delta_l\delta_\nu f_{l2}f_{\nu1}}e^{i\Phi_2}\right),\\
U_{e3}^{th} &= \sqrt{\frac{\widetilde{m}_\mu\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2} \delta_\nu f_{l1}}{D_{l1}D_{\nu3}}} + \sqrt{\frac{\widetilde{m}_e}{D_{l1}D_{\nu3}}}\left(\sqrt{\delta_\nu(1-\delta_l)(1-\delta_\nu) f_{l1}}e^{i\Phi_1} - \sqrt{\delta_e f_{l2}f_{\nu1}f_{\nu2}}e^{i\Phi_2}\right),\\
U_{\mu1}^{th} &= -\sqrt{\frac{\widetilde{m}_e\widetilde{m}_{\nu_2} f_{l2} f_{\nu1}}{D_{l2}D_{\nu1}}} + \sqrt{\frac{\widetilde{m}_\mu\widetilde{m}_{\nu_1}}{D_{l2}D_{\nu1}}}\left(\sqrt{(1-\delta_l)(1-\delta_\nu) f_{l2}f_{\nu1}}e^{i\Phi_1} + \sqrt{\delta_l\delta_\nu f_{l1}f_{\nu2}}e^{i\Phi_2}\right),\\
U_{\mu2}^{th} &= \sqrt{\frac{\widetilde{m}_e\widetilde{m}_{\nu_1} f_{l2} f_{\nu2}}{D_{l2}D_{\nu2}}} + \sqrt{\frac{\widetilde{m}_\mu\widetilde{m}_{\nu_2}}{D_{l2}D_{\nu2}}}\left(\sqrt{(1-\delta_l)(1-\delta_\nu) f_{l2}f_{\nu2}}e^{i\Phi_1} + \sqrt{\delta_l\delta_\nu f_{l1}f_{\nu1}}e^{i\Phi_2}\right),\\
U_{\mu3}^{th} &= -\sqrt{\frac{\widetilde{m}_e\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2} \delta_\nu f_{l2}}{D_{l2}D_{\nu3}}} + \sqrt{\frac{\widetilde{m}_\mu}{D_{l2}D_{\nu3}}}\left(\sqrt{\delta_\nu(1-\delta_l)(1-\delta_\nu) f_{l2}}e^{i\Phi_1} - \sqrt{\delta_l f_{l1}f_{\nu1}f_{\nu2}}e^{i\Phi_2}\right),\\
U_{\tau1}^{th} &= \sqrt{\frac{\widetilde{m}_e\widetilde{m}_\mu\widetilde{m}_{\nu_2} \delta_l f_{\nu1}}{D_{l3}D_{\nu1}}} + \sqrt{\frac{\widetilde{m}_{\nu_1}}{D_{l3}D_{\nu1}}}\left(\sqrt{\delta_l(1-\delta_l)(1-\delta_\nu) f_{\nu1}}e^{i\Phi_1} - \sqrt{\delta_\nu f_{l1}f_{l2}f_{\nu2}}e^{i\Phi_2}\right),\\
U_{\tau2}^{th} &= -\sqrt{\frac{\widetilde{m}_e\widetilde{m}_\mu\widetilde{m}_{\nu_1} \delta_l f_{\nu2}}{D_{l3}D_{\nu2}}} + \sqrt{\frac{\widetilde{m}_{\nu_2}}{D_{l3}D_{\nu2}}}\left(\sqrt{\delta_l(1-\delta_l)(1-\delta_\nu) f_{\nu2}}e^{i\Phi_1} - \sqrt{\delta_\nu f_{l1}f_{l2}f_{\nu1}}e^{i\Phi_2}\right),\\
U_{\tau3}^{th} &= \sqrt{\frac{\widetilde{m}_e\widetilde{m}_\mu\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2} \delta_l\delta_\nu}{D_{l3}D_{\nu3}}} + \sqrt{\frac{\delta_l\delta_\nu(1-\delta_l)(1-\delta_\nu)}{D_{l3}D_{\nu3}}}e^{i\Phi_1} + \sqrt{\frac{f_{l1}f_{l2}f_{\nu1}f_{\nu2}}{D_{l3}D_{\nu3}}}e^{i\Phi_2}.
\end{aligned}
\tag{56}
$$

in these expresions the $\widetilde{m}$'s, f's and D's are defined in (18) and (24)-(26), respectively.

B. The χ^2 fit for the Quark Mixing Matrix

We made a χ^2 fit of the exact theoretical expressions for the modulii of the entries of the quark mixing matrix $|(V_{_{CKM}}^{th})_{ij}|$ and the inner angles of the unitarity triangle α^{th}, β^{th} and γ^{th} to the experimental values given by Amsler [10].In this fit, we computed the modulii of the entries of the quark mixing matrix and the inner angles of the unitarity triangle from the theoretical expresion (54) with the following numerical values of the quark mass ratio

$$
\begin{aligned}
\widetilde{m}_u &= 2.5469\times10^{-5}, \quad \widetilde{m}_c = 3.9918\times10^{-3},\\
\widetilde{m}_d &= 1.5261\times10^{-3}, \quad \widetilde{m}_s = 3.2319\times10^{-2}.
\end{aligned}
\tag{57}
$$

The resulting best values of the parameters δ_u and δ_d are

$$\delta_u = 3.829\times10^{-3}, \quad \delta_d = 4.08\times10^{-4} \tag{58}$$

and the Dirac CP violating phase $\phi = 90^o$.
The best values for the modulii of the entries of the CKM mixing matrix are given in the following expresion

$$\left|V^{th}_{CKM}\right| = \begin{pmatrix} 0.97421 & 0.22560 & 0.003369 \\ 0.22545 & 0.97335 & 0.041736 \\ 0.008754 & 0.04094 & 0.99912 \end{pmatrix} \tag{59}$$

and

$$\alpha^{th} = 91.24^o, \quad \beta^{th} = 20.41^o, \quad \gamma^{th} = 68.33^o. \tag{60}$$

The Jarlskog invariant takes the value $J^{th}_q = 2.9\times10^{-5}$, all these results are in very good agreement with the experimental values.

C. The χ^2 fit for the Lepton Mixing Matrix

In the case of the lepton mixing matrix, we made a χ^2 fit of the theoretical expressions for the modulii of the entries of the lepton mixing matrix $|(U^{th}_{PMNS})_{ij}|$ given in eq. (56) to the values extracted from experiment as given by Gonzalez-Garcia [5] and quoted in (3).The computation was made using the following values for the charged leptonmasses [10]:

$$\begin{aligned} &m_e = 0.5109\text{MeV}, \ \ m_\mu = 105.685\text{MeV}, \\ &m_\tau = 1776.99\text{GeV}, \end{aligned} \tag{61}$$

we took for the masses of the left handed Majorana neutrinos a normal hierarchy. This allows us to write the left handed Majorana neutrinos mass ratios in terms of the neutrino squared mass differences and the neutrino mass m_{ν_3} in the following form:

$$\widetilde{m}_{\nu_1} = \sqrt{1 - \frac{\left(\Delta m^2_{32} + \Delta m^2_{21}\right)}{m^2_{\nu_3}}}, \ \widetilde{m}_{\nu_2} = \sqrt{1 - \frac{\Delta m^2_{32}}{m^2_{\nu_3}}}. \tag{62}$$

The neutrino squared mass differences were obtained from the experimental data on neutrino oscillations given in Gonzalez-Garcia [5] and we left the mass m_{ν_3} as a free parameter of the χ^2 fit. From the best values obtained for m_{ν_3} and the experimental values of the Δm^2_{32} and Δm^2_{21}, we obtained the following best values for the neutrino masses

$$\begin{aligned} &m_{\nu_1} = 2.7\times10^{-3}\text{eV}, \quad m_{\nu_2} = 9.1\times10^{-3}\text{eV}, \\ &m_{\nu_3} = 4.7\times10^{-2}\text{eV}. \end{aligned} \tag{63}$$

The resulting best values of the parameters δ_e and δ_ν are

$$\delta_e = 0.06, \qquad \delta_\nu = 0.522, \tag{64}$$

and the best values of the Dirac CP violating phases are $\Phi_1 = 0$ and $\Phi_2 = 90^o$. The best values for the modulii of the entries of the $PMNS$ mixing matrix are given in the following expresion

$$\left|U^{th}_{PMNS}\right| = \begin{pmatrix} 0.820421 & 0.568408 & 0.061817 \\ 0.385027 & 0.613436 & 0.689529 \\ 0.422689 & 0.548277 & 0.721615 \end{pmatrix}. \tag{65}$$

The value of the rephasing invariant related to the Dirac phase is

$$J^{th}_l = 8.8\times10^{-3}. \tag{66}$$

Since we have no experimental information about the Majorana phases β_1 and β_2, the other two rephasing invariants associated with the two Majorana phases in the U_{PMNS} matrix, could not be determined from the experiment values. Therefore, in order to make a numerical estimate, we maximized the rephasing invariants S_1 and S_2, thus obtaining a numerical value for the Majorana phases β_1 and β_2. Then, the maximum values of the rephasing invariants, eq(50), are:

$$S^{max}_1 = -4.9\times10^{-2}, \quad S^{max}_2 = 3.4\times10^{-2}, \tag{67}$$

with $\beta_1 = -1.4^o$ and $\beta_2 = 77^o$. In this analysis, the minimum value of the χ^2, corresponding to the best fit is $\chi^2 = 0.288$, and all the numerical results of the fit are in very good agreement with the values of the moduli of the entries in the matrix U_{PMNS} as given in Gonzalez-Garcia [5].

V. THE MIXING ANGLES

In the standard PDG parametrization, the entries in the quark and lepton mixing matrices are parametrized in terms of the mixing angles and phases. Thus, the modulii of the entries of the quark (lepton) mixing matrix $V_{CKM}(U_{PMNS})$ are related to the mixing angles through the expressions:

$$\begin{aligned} \sin^2\theta^{q(l)}_{12} &= \tfrac{|V_{us}(U_{e2})|^2}{1-|V_{ub}(U_{e3})|^2}, \\ \sin^2\theta^{q(l)}_{23} &= \tfrac{|V_{cb}(U_{\mu3})|^2}{1-|V_{ub}(U_{e3})|^2}, \\ \sin^2\theta^{q(l)}_{13} &= |V_{ub}\left(U_{e3}\right)|^2. \end{aligned} \tag{68}$$

The theoretical expression for the quark mixing angles as functions of the quark mass ratios are readily obtained when the theoretical expressions for the modulii of the entries in the CKM mixing matrix, given in eqs. (54) and (24)-(24),are substituted for $|V_{ij}|$ in the right hand side of eqs.(68). In this way,and keeping only the leading order terms, we get :

$$\sin^2\theta^q_{12} \approx \frac{\frac{\widetilde{m}_d}{\widetilde{m}_s} + \frac{\widetilde{m}_u}{\widetilde{m}_c} - 2\sqrt{\frac{\widetilde{m}_u}{\widetilde{m}_c}\frac{\widetilde{m}_d}{\widetilde{m}_s}}\cos\phi}{\left(1+\frac{\widetilde{m}_u}{\widetilde{m}_c}\right)\left(1+\frac{\widetilde{m}_d}{\widetilde{m}_s}\right)}, \tag{69}$$

$$\sin^2\theta^q_{23} \approx \frac{\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)^2}{\left(1+\frac{\widetilde{m}_u}{\widetilde{m}_c}\right)}, \tag{70}$$

$$\sin^2\theta^q_{13} \approx \frac{\frac{\widetilde{m}_u}{\widetilde{m}_c}\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)^2}{\left(1+\frac{\widetilde{m}_u}{\widetilde{m}_c}\right)}. \tag{71}$$

Now, the numerical values of the quark mixing angles may be computed from eq.(54) and the numerical values of the parameters δ_u and δ_d,eq. (58), and the CP violating phase $\phi = 90^o$ obtained from χ^2 fit of $\left|V^{th}_{CKM}\right|$ to the experimentally determined values $\left|V^{exp}_{CKM}\right|$. In this way we obtain

$$\theta^q_{12} = 13^o, \quad \theta^q_{23} = 2.38^o, \quad \theta^q_{13} = 0.19^o, \tag{72}$$

in very good agreement with the latest analysis of the experimental data [28], see (9).
The numerical values of the leptonic mixing angles are computed in a similar fashion. The theoretical expressions for the lepton mixing angles as funtion of the charged lepton and neutrino mass ratios are obtained from eqs (68) when the theoretical expressions for the modulii of the entries in the $PMNS$ mixing matrix, given in eqs.(56) and eqs. (24)-(24), are substituted for $|U_{ij}|$ in the right hand side of eqs.(68). If we keep only the leading terms, we obtain:

$$\begin{aligned}\sin^2\theta^l_{12} \approx & \frac{f_{\nu 2}}{\left(1+\widetilde{m}_{\nu_2}\right)(1-\delta_\nu)}\times \\ & \times\left\{\frac{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)+2\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)}\cos\Phi_1}{\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)}\right\},\end{aligned} \tag{73}$$

$$\sin^2\theta^l_{23} \approx \frac{\delta_\nu+\delta_e f_{\nu 2}-\sqrt{\delta_\nu\delta_e f_{\nu 2}}\cos\left(\Phi_1-\Phi_2\right)}{\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)\left(1+\widetilde{m}_{\nu_2}\right)}, \tag{74}$$

$$\begin{aligned}\sin^2\theta^l_{13} \approx & \frac{\delta_\nu}{\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)\left(1+\widetilde{m}_{\nu_2}\right)}\left\{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}+\frac{\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}}{(1-\delta_\nu)}-\right. \\ & \left.-2\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\frac{\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}}{(1-\delta_\nu)}}\cos\Phi_1\right\}.\end{aligned} \tag{75}$$

The expressions quoted above are written in terms of the ratios of the lepton masses. When the well known values of the charged lepton masses, the values of the neutrino masses, eq.(63), the values of the delta parameters eq. (64) and the values of the Dirac CP violating phases obtained from χ^2 fit in the lepton sector, are inserted in eqs. (73)-(75), we obtain the following numerical values for the mixing angles

$$\theta^l_{12} = 34.7^o, \quad \theta^l_{23} = 43.6^o, \quad \theta^l_{13} = 3.5^o, \tag{76}$$

which are in very good agreement with the latest experimental data [5, 8].

VI. QUARK-LEPTON COMPLEMENTARITY

We may now address the question of the meaning of the quark-lepton complementarity relations as expressed in eq. (10)-(12). The relations between mixing angles and the moduli of the entries of the mixing matrices given in eqs. (68), allow us to write the Quark Lepton Complementarity relations in the following form; the first QLC relation, between the Cabibbo angle and the solar angle is

$$\tan\left(\theta^q_{12}+\theta^l_{12}\right) = 1+\Delta^{th}_{12}, \tag{77}$$

where

$$\Delta^{th}_{12} = \frac{\left|V^{th}_{us}\right|\left(\left|U^{th}_{e1}\right|+\left|U^{th}_{e2}\right|\right)+\left|V^{th}_{ud}\right|\left(\left|U^{th}_{e2}\right|-\left|U^{th}_{e1}\right|\right)}{\left|U^{th}_{e1}\right|\left|V^{th}_{ud}\right|-\left|U^{th}_{e2}\right|\left|V^{th}_{us}\right|}. \tag{78}$$

The second QLC relation, between the atmospheric angle and the 2-3 mixing angle of the quarks is

$$\tan\left(\theta^q_{23}+\theta^l_{23}\right) = 1+\Delta^{th}_{23}, \tag{79}$$

where

$$\begin{aligned}\Delta^{th}_{23} = & \frac{\left|V^{th}_{cb}\right|\sqrt{1-\left|U^{th}_{e3}\right|^2-\left|U^{th}_{\mu 3}\right|^2}+\left|U^{th}_{\mu 3}\right|\left|V^{th}_{cb}\right|+}{\sqrt{1-\left|V^{th}_{ub}\right|^2-\left|V^{th}_{cb}\right|^2}\sqrt{1-\left|U^{th}_{e3}\right|^2-\left|U^{th}_{\mu 3}\right|^2}-}\times \\ & \times\frac{+\sqrt{1-\left|V^{th}_{ub}\right|^2-\left|V^{th}_{cb}\right|^2}\left(\left|U^{th}_{\mu 3}\right|-\sqrt{1-\left|U^{th}_{e3}\right|^2-\left|U^{th}_{\mu 3}\right|^2}\right)}{-\left|V^{th}_{cb}\right|\left|U^{th}_{\mu 3}\right|}.\end{aligned} \tag{80}$$

The last QLC relation, between the 1-3 mixing angles of the quarks and the leptons, is

$$\tan\left(\theta^q_{13}+\theta^l_{13}\right) = \frac{\left|V^{th}_{ub}\right|\sqrt{1-\left|U^{th}_{e3}\right|}+\left|U^{th}_{e3}\right|\sqrt{1-\left|V^{th}_{ub}\right|}}{\sqrt{1-\left|V^{th}_{ub}\right|}\sqrt{1-\left|U^{th}_{e3}\right|}-\left|U^{th}_{e3}\right|\left|V^{th}_{ub}\right|} \tag{81}$$

The substitution of expresions (54) and (56) for the modulii of the elements of the mixing matrices V_{CKM} and U^{th}_{PMNS}, allows us express the Quark Lepton Complementarity relations in term of the mass ratios of quarks and leptons.
Then, the (77)-(81) take the follwing form:

$$\tan\left(\theta^q_{12}+\theta^l_{12}\right) = 1+\Delta^{th}_{12}, \tag{82}$$

where

$$\begin{aligned}\Delta^{th}_{12} \approx & \frac{\left(\frac{\widetilde{m}_d}{\widetilde{m}_s}+\frac{\widetilde{m}_u}{\widetilde{m}_c}\right)^{\frac{1}{2}}\left[(1-\delta_\nu)\left(1+\widetilde{m}_{\nu_2}\right)\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)-\right.}{\left(1+\frac{\widetilde{m}_d}{\widetilde{m}_s}\frac{\widetilde{m}_u}{\widetilde{m}_c}\right)^{\frac{1}{2}}\left[(1-\delta_\nu)\left(1+\widetilde{m}_{\nu_2}\right)\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)-\right.}\times \\ & \times\frac{\left.-f_{\nu 2}\left(\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}}+\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)}\right)^2\right]^{\frac{1}{2}}+\left(\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}}+\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)}\right)}{\left.-f_{\nu 2}\left(\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}}+\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)}\right)^2\right]^{\frac{1}{2}}+\sqrt{\frac{\widetilde{m}_d}{\widetilde{m}_s}+\frac{\widetilde{m}_u}{\widetilde{m}_c}f_{\nu 2}}}\times \\ & \times\frac{\sqrt{f_{\nu 2}}\left[\left(\frac{\widetilde{m}_d}{\widetilde{m}_s}+\frac{\widetilde{m}_u}{\widetilde{m}_c}\right)^{\frac{1}{2}}+\left(1+\frac{\widetilde{m}_d}{\widetilde{m}_s}\frac{\widetilde{m}_u}{\widetilde{m}_c}\right)^{\frac{1}{2}}\right]}{\left(\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}}+\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)}\right)}.\end{aligned} \tag{83}$$

Similarly,

$$\tan\left(\theta^q_{23}+\theta^l_{23}\right) = 1+\Delta^{th}_{23}, \tag{84}$$

where

$$\Delta_{23}^{th} \approx \frac{\left(\left[\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)\left(1+\widetilde{m}_{\nu_2}\right)-\delta_\nu-\delta_e f_{\nu 2}\right]^{\frac{1}{2}}+\sqrt{\delta_\nu+\delta_e f_{\nu 2}}\right)}{-\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)\sqrt{\delta_\nu+\delta_e f_{\nu 2}}+} \times$$
$$\times \frac{\left(\sqrt{1+\frac{\widetilde{m}_u}{\widetilde{m}_c}-\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)^2}+\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)\right)}{+\left[\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)\left(1+\widetilde{m}_{\nu_2}\right)-\delta_\nu-\delta_e f_{\nu 2}\right]^{\frac{1}{2}}\sqrt{1+\frac{\widetilde{m}_u}{\widetilde{m}_c}-\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)^2}}. \tag{85}$$

Also,

$$\tan\left(\theta_{13}^q+\theta_{13}^l\right) \approx \frac{\sqrt{\frac{\widetilde{m}_u}{\widetilde{m}_c}}\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)}{\sqrt{1+\frac{\widetilde{m}_u}{\widetilde{m}_c}-\frac{\widetilde{m}_u}{\widetilde{m}_c}\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)^2}} \times$$
$$\times \frac{\left[\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)\left(1+\widetilde{m}_{\nu_2}\right)-\delta_\nu\left(\sqrt{\frac{\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}}\right)^2\right]^{\frac{1}{2}}+}{\left[\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)\left(1+\widetilde{m}_{\nu_2}\right)-\delta_\nu\left(\sqrt{\frac{\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}}\right)^2\right]^{\frac{1}{2}}-} \times \tag{86}$$
$$\times \frac{+\sqrt{\delta_\nu}\left(\sqrt{\frac{\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}}\right)\sqrt{1+\frac{\widetilde{m}_u}{\widetilde{m}_c}-\frac{\widetilde{m}_u}{\widetilde{m}_c}\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)^2}}{-\sqrt{\frac{\widetilde{m}_u}{\widetilde{m}_c}}\left(\sqrt{\delta_u}-\sqrt{\delta_d}\right)\sqrt{\delta_\nu}\left(\sqrt{\frac{\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}}\right)}.$$

After substitution of the numerical values of the mass ratios of quarks and leptons in eqs. (83)-(86), we obtain,

$$\theta_{12}^q+\theta_{12}^l = 45^o + 2.7^o. \tag{87}$$

$$\theta_{23}^q+\theta_{23}^l = 45^o + 1^o, \tag{88}$$

$$\theta_{13}^q+\theta_{13}^l = 3.7^o. \tag{89}$$

The equations (82) and (83) are obtained from an exact analytical expression for $\tan\left(\theta_{12}^q+\theta_{12}^l\right)$ as a funtion of the absolute values of the entries in the mixing matrices V_{CKM}^{th} and U_{PMNS}^{th}, eqs (77). In eqs. (54) and (56), the elements of the mixing matrices V_{CKM}^{th} and U_{PMNS}^{th} are given as exact explicit analytical funtions of the quark and lepton mass ratios. Let us stress that these expresions are exact and valid for any possible values of the quark and lepton mass ratios. Therefore, the smallness of the term Δ_{12}^{th} is only due to the smallness of the ratios $\widetilde{m}_d/\widetilde{m}_s$, $\widetilde{m}_u/\widetilde{m}_c$ and $\widetilde{m}_e/\widetilde{m}_\mu$.
We may conclude that the Quark-Lepton Complementarity as expresed in (82) is not a numerical coincidence, it is the result of the combined effect of two factors:

1. The strong mass hierarchy of the Dirac fermions which produces small and very small mass ratios of u and d-type quarks and charged leptons. The quark mass hierarchy is then reflected in a similar hierarchy of small and very small quark mixing angles.

2. The normal seesaw mechanism gives very small masses to the left handed Majorana neutrinos with relatively large values of the neutrino mass ratio m_{ν_1}/m_{ν_2} and allows for large θ_{12}^l and θ_{23}^l mixing angles.

VII. THE EFFECTIVE MAJORANA MASSES

The square of the magnitudes of the effective Majorana neutrino masses, eq.(13), are

$$\begin{aligned}|\langle m_{ll}\rangle|^2 = \textstyle\sum_{j=1}^3 m_{\nu_j}^2 \left|U_{lj}\right|^4 + 2\sum_{j<k}^3 m_{\nu_j} m_{\nu_k} \times \\ \times \left|U_{lj}\right|^2 \left|U_{lk}\right|^2 \cos 2\left(w_{lj}-w_{lk}\right),\end{aligned} \tag{90}$$

where $w_{lj}=\arg\{U_{lj}\}$; this term includes phases of both types, Dirac and Majorana.

The theoretical expression for the squared magnitud of the effective Majorana neutrino mass of electron neutrino, written in terms of the ratios of the quark masses, is:

$$\begin{aligned}|\langle m_{ee}\rangle|^2 \approx &\frac{1}{\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)^2\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)^2}\left\{m_{\nu_1}^2\left(1-\right.\right.\\
&\left.-4\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}(1-\delta_\nu)}\right)+\frac{m_{\nu_2}^2 f_{\nu 2}^2}{\left(1+\widetilde{m}_{\nu_2}\right)^2(1-\delta_\nu)^2}\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\left(\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right.\\
&\left.+4\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}(1-\delta_\nu)}+6\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)\right)\\
&+2\frac{m_{\nu_1}m_{\nu_3}\delta_\nu}{(1+\widetilde{m}_{\nu_2})}\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)\left(\sqrt{\frac{\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}}\right)^2\times\\
&\times\cos 2(w_{e1}-w_{e3})+2\frac{m_{\nu_1}m_{\nu_2}f_{\nu 2}}{(1+\widetilde{m}_{\nu_2})(1-\delta_\nu)}\left(\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right.\\
&\left.+2\left(1-\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)}\right)\cos 2(w_{e1}-w_{e2})\\
&+2\frac{m_{\nu_2}m_{\nu_3}f_{\nu 2}\delta_\nu}{\left(1+\widetilde{m}_{\nu_2}\right)^2(1-\delta_\nu)^2}\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)\left(2\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}\right.\\
&\left.\left.+\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}(1-\delta_\nu)}\right)\cos 2(w_{e2}-w_{e3})\right\}\end{aligned} \tag{91}$$

where $w_{e2}=\beta_1$ and

$$w_{e1} = \arctan\left\{-\frac{\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\frac{\widetilde{m}_e}{\widetilde{m}_\mu}}\delta_e\delta_\nu f_{\nu 2}}{\sqrt{(1-\delta_\nu)}+\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\frac{\widetilde{m}_e}{\widetilde{m}_\mu}}(1-\delta_\nu)}\right\}, \tag{92}$$

$$w_{e3} \approx \arctan\left\{\frac{\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\delta_e f_{\nu 2}(1-\delta_\nu)}+}{-\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\delta_e f_{\nu 2}(1-\delta_\nu)}\tan\beta_2+}\right.$$
$$\left.\frac{+\sqrt{\delta_\nu}\left(\sqrt{\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}}-\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)}\right)\tan\beta_2}{+\sqrt{\delta_\nu}\left(\sqrt{\widetilde{m}_{\nu_1}\widetilde{m}_{\nu_2}}-\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}(1-\delta_\nu)}\right)}\right\}. \tag{93}$$

In a similar way, the theoretical expression for the squared magnitud of the effective Majorana neutrino mass of the muon neutrino is:

$$
\begin{aligned}
&|\langle m_{\mu\mu}\rangle|^2 \approx \frac{1}{\left(1+\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\right)^2\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)^2\left(1+\widetilde{m}_{\nu_2}\right)}\left\{\frac{m_{\nu_3}^2}{\left(1+\widetilde{m}_{\nu_2}\right)}\right.\\
&\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)^2\left(\delta_\nu+2\delta_e f_{\nu 2}\right)+\frac{m_{\nu_2}^2}{\left(1+\widetilde{m}_{\nu_2}\right)\left(1-\delta_\nu\right)}\left(1-\delta_\nu\right.\\
&\left.-4\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\left(1-\delta_\nu\right)}+6\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)+2m_{\nu_1}m_{\nu_2}f_{\nu 2}\\
&\left(\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\left(1-\delta_\nu\right)+2\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\left(1-\delta_\nu\right)}\left(1-\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)\right)\\
&\cos 2(w_{\mu 1}-w_{\mu 2})+2m_{\nu_1}m_{\nu_3}\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)\left(2\delta_\nu\times\right.\\
&\left.\times\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\left(1-\delta_\nu\right)}+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\left(1-\delta_\nu\right)\left(\delta_\nu+\delta_e f_{\nu 2}\right)\right)\\
&\cos 2(w_{\mu 1}-w_{\mu 3})+2\frac{m_{\nu_2}m_{\nu_3}f_{\nu 2}}{\left(1+\widetilde{m}_{\nu_2}\right)\left(1-\delta_\nu\right)}\left(1+\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\right)\\
&\left(\left(1-\delta_\nu\right)\left(\delta_\nu+\delta_e f_{\nu 2}\right)-2\delta_\nu\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\left(1-\delta_\nu\right)}\right)\\
&\left.\cos 2(w_{\mu 2}-w_{\mu 3})\right\}
\end{aligned}
\tag{94}
$$

where

$$
w_{\mu 1}=\arctan\left\{\frac{\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\delta_e\delta_\nu f_{\nu 2}}}{\sqrt{\frac{\widetilde{m}_e}{\widetilde{m}_\mu}\left(1-\delta_\nu\right)}+\sqrt{\frac{\widetilde{m}_{\nu_1}}{\widetilde{m}_{\nu_2}}\left(1-\delta_\nu\right)}}\right\}, \tag{95}
$$

and

$$
w_{\mu 2}=\arctan\left\{\frac{\sqrt{f_{\nu 2}}\tan\beta_1+\sqrt{\delta_e\delta_\nu}}{\sqrt{f_{\nu 2}}-\sqrt{\delta_e\delta_\nu}\tan\beta_1}\right\}, \tag{96}
$$

$$
w_{\mu 3}=\arctan\left\{\frac{\tan\beta_2-\sqrt{f_{\nu 2}}}{1+\sqrt{f_{\nu 2}}\tan\beta_2}\right\}. \tag{97}
$$

From these expresions and the numerical values of the neutrinos masses given in (63), we obtain the following numerical value of the effective Majorana neutrino masses

$$
|\langle m_{ee}\rangle|\approx 4.6\times 10^{-3}\ \mathrm{eV}, \qquad |\langle m_{\mu\mu}\rangle|\approx 2.1\times 10^{-2}\ \mathrm{eV}. \tag{98}
$$

These numerical values are consistent with the very small experimentally determined upper bounds for the reactor neutrino mixing angle θ_{13}^l [29].

VIII. CONCLUSIONS

In this communication, we outlined a unified treatment of masses and mixing of quarks and leptons in which the left handed Majorana neutrinos acquire their masses via the type I seesaw mechanism, and the mass matrices of all Dirac fermions have a similar texture with four zeroes and a normal hierarchy. Then, the mass matrix of the left handed Majorana neutrinos has also a texture with four zeros. In this scheme, we derived exact, explicit expressions for the Cabibbo (θ_{12}^q) and solar (θ_{12}^l) mixing angles as functions of the quark and lepton masses. The Quark-Lepton Complementarity relation takes the form,

$$
\theta_{12}^q+\theta_{12}^l=45^o+\delta_{12}^{th}. \tag{99}
$$

The correction term, δ_{12}, is an explicit function of the ratios of quark and lepton masses, given in eq. (83), which reproduces the experimentally determined value, $\delta_{12}^{exp}\approx 2.7^o$, when the numerical values of the quark and lepton masses are substituted in (83).

Three essential ingredients are needed to explain the correlations implicit in the small numerical value of δ_{12}^{th}:

1. The strong hierarchy in the mass spectra of the quarks and charged leptons, realized in our scheme through the explicit breaking of the S_3 flavour symmetry in the texture with four zeroes for mass matrices, explains the resulting small or very small quark mixing angles, the very small charged lepton mass ratios explain the very small value of θ_{13}^l.

2. The normal seesaw mechanism that gives very small masses to the left handed Majorana neutrinos with relatively large values of the neutrino mass ratio m_{ν_1}/m_{ν_2} and allows for large θ_{12}^l and θ_{23}^l mixing angles.

3. The assumption of a normal hierarchy for the masses of the Majorana neutrinos.

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