TERNARY HOM-NAMBU-LIE ALGEBRAS INDUCED BY HOM-LIE ALGEBRAS

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ABSTRACT. The purpose of this paper is to investigate ternary multiplications constructed from a binary multiplication, linear twisting maps and a trace function. We provide a construction of ternary Hom-Nambu and Hom-Nambu-Lie algebras starting from a binary multiplication of a Hom-Lie algebra and a trace function satisfying certain compatibility conditions involving twisting maps. We show that mutual position of kernels of twisting maps and the trace play important role in this context, and provide examples of Hom-Nambu-Lie algebras obtained using this construction.

1. INTRODUCTION

The n-ary algebraic structures and in particular ternary algebraic structures appear naturally in various domains of theoretical and mathematical physics as for example Nambu mechanics [28, 29]. Further recent motivation to study n-ary operations comes from string theory and M-branes involving naturally an algebra with a ternary operation, called Bagger-Lambert algebra [3]. For other physical applications see [7, 17, 18, 19, 20]. This provides additional motivation for development of mathematical concepts such as Leibniz n-ary algebras [6, 14].

In [4], generalizations of n-ary algebras of Lie type and associative type by twisting the identities using linear maps have been introduced. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type including n-ary Nambu algebras, n-ary Nambu-Lie algebras and n-ary Lie algebras, and n-ary algebras of associative type including n-ary totally associative and n-ary partially associative algebras.

The general Hom-algebra structures arose first in connection to quasi-deformation and discretizations of Lie algebras of vector fields. These quasi-deformations lead to quasi-Lie algebras, a generalized Lie algebra structure in which the skew-symmetry and Jacobi conditions are twisted. The first examples were concerned with q-deformations of the Witt and Virasoro algebras (see for example [1, 8, 10, 11, 9, 12, 13, 16]). Motivated by these and new examples arising as applications of the general quasi-deformation construction of [15, 21, 22] on the one hand, and the desire to be able to treat within the same framework such well-known generalizations

2000 *Mathematics Subject Classification.* 17A30,17A40,17A42,17D99.

Key words and phrases. Triple commutator brackets, ternary Nambu-Lie algebra, ternary Hom-Nambu-Lie algebra.

This work was partially supported by the Crafoord Foundation, The Swedish Foundation for International Cooperation in Research and Higher Education (STINT), The Swedish Research Council, The Royal Swedish Academy of Sciences, The Royal Physiographic Society in Lund, and The SIDA Foundation. We also would like to thank the Royal Institute of Technology for support and hospitality in connection to visits there while working on this project.

of Lie algebras as the color and super Lie algebras on the other hand, quasi-Lie algebras and subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras were introduced in [15, 21, 22, 23]. In the subclass of Hom-Lie algebras skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as for Lie algebras, reducing to ordinary Lie algebras when the linear twisting map is the identity map. Hom-associative algebras replacing associative algebras in the context of Hom-Lie algebras and also more general classes of Hom-Lie admissible algebras, G-Hom-associative algebras, where introduced in [24]. The first steps in the construction of universal enveloping algebras for Hom-Lie algebras have been made in [30]. Formal deformations and elements of (co-)homology for Hom-Lie algebras have been studied in [27, 31], whereas dual structures such as Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras appeared first in [25, 26] and further investigated in [5, 32].

This paper is organized as follows. In Section 2 we review basic concepts of Hom-Lie, ternary Hom-Nambu and ternary Hom-Nambu-Lie algebras. We also recall the method of composition with endomorphism for the construction of Hom-Lie, ternary Hom-Nambu and ternary Hom-Nambu-Lie algebras from Lie, Nambu and Nambu-Lie algebras. In Section 3 we provide a construction procedure of ternary Hom-Nambu and Hom-Nambu-Lie algebras starting from a binary bracket of a Hom-Lie algebra and a trace function satisfying certain compatibility conditions involving the twisting maps. To this end, we use the ternary bracket introduced in [2]. In Section 4, we investigate how restrictive the compatibility conditions are. The mutual position of kernels of twisting maps and the trace play an important role in this context. Finally, in Section 5, we provide examples of Hom-Nambu-Lie algebras obtained using constructions presented in the paper.

2. Ternary Nambu-Lie algebras

Let us first recall some basic facts about Hom-algebras. A Hom-algebra structure is a multiplication on a vector space, which is twisted by a linear map. In what follows, all vector spaces will be defined over a field $\mathbb{K}$ of characteristic 0, and V will always denote such a vector space.

The notion of a Hom-Lie algebra was initially motivated by examples of deformed Lie algebras coming from twisted discretizations of vector fields [15, 21, 22]. We will follow notation conventions of [24].

Definition 2.1. A *Hom-Lie algebra* is a triple $(V, [\cdot,\cdot], \alpha)$ where $[\cdot,\cdot] : V \times V \to V$ is a bilinear map and $\alpha : V \to V$ a linear map satisfying

$$\begin{aligned} [x,y] &= -[y,x], \quad \text{(skew-symmetry)} \\ \circlearrowleft_{x,y,z} [\alpha(x),[y,z]] &= 0 \quad \text{(Hom-Jacobi condition)} \end{aligned}$$

for all x, y, z from V, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutations of x, y, z.

Analogously, one can introduce Hom-algebra equivalents of n-ary algebras [4].

Definition 2.2. A *ternary Hom-Nambu algebra* is a triple $\big(V, [\cdot,\cdot,\cdot], \tilde{\alpha}\big)$, consisting of a vector space V, a trilinear map $[\cdot,\cdot,\cdot] : V \times V \times V \to V$ and a pair of linear maps $\tilde{\alpha} = (\alpha_1, \alpha_2)$ satisfying

$$\begin{aligned} [\alpha_1(x_1),\alpha_2(x_2),[x_3,x_4,x_5]] &= [[x_1,x_2,x_3],\alpha_1(x_4),\alpha_2(x_5)] \\ &\quad + [\alpha_1(x_3),[x_1,x_2,x_4],\alpha_2(x_5)] + [\alpha_1(x_3),\alpha_2(x_4),[x_1,x_2,x_5]]. \end{aligned} \tag{2.1}$$

The identity (2.1) is called ternary Hom-Nambu identity.

Remark 2.3. Let $(V,[\cdot,\cdot,\cdot],\tilde{\alpha})$ be a ternary Hom-Nambu algebra where $\tilde{\alpha}=(\alpha_1,\alpha_2)$. Let $x=(x_1,x_2)\in V\times V$, $\tilde{\alpha}(x)=(\alpha_1(x_1),\alpha_2(x_2))\in V\times V$ and $y\in V$. Let L_x be a linear map on V, defined by

$$L_x(y)=[x_1,x_2,y].$$

Then the Hom-Nambu identity writes

$$\begin{aligned}L_{\tilde{\alpha}(x)}([x_3,x_4,x_5]) =&[L_x(x_3),\alpha_1(x_4),\alpha_2(x_5)]+[\alpha_1(x_3),L_x(x_4),\alpha_2(x_5)]\\&+[\alpha_1(x_3),\alpha_2(x_4),L_x(x_5)].\end{aligned}$$

Remark 2.4. When the maps $(\alpha_i)_{i=1,2}$ are all identity maps, one recovers the classical ternary Nambu algebras. The identity obtained in this special case of classical n-ary Nambu algebra is known also as the fundamental identity or Filippov identity [14, 28, 29].

Definition 2.5. A ternary Hom-Nambu algebra $\big(V,[\cdot,\cdot,\cdot],(\alpha_1,\alpha_2)\big)$ is called a *ternary Hom-Nambu-Lie algebra* if the bracket is skew-symmetric, that is

$$[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}]=Sgn(\sigma)[x_1,x_2,x_3],\quad \forall\,\sigma\in\mathcal{S}_3 \text{ and } \forall\,x_1,x_2,x_3\in V \tag{2.2}$$

where $\mathcal{S}_3$ stands for the permutation group on 3 elements.

The morphisms of ternary Hom-Nambu algebras are defined in the natural way. It should be pointed out however that the morphisms should intertwine not only the ternary products but also the twisting linear maps. Let $(V,[\cdot,\cdot,\cdot],\tilde{\alpha})$ and $(V',[\cdot,\cdot,\cdot]',\tilde{\alpha}')$ be two n-ary Hom-Nambu algebras where $\tilde{\alpha}=(\alpha_1,\alpha_2)$ and $\tilde{\alpha}'=(\alpha'_1,\alpha'_2)$. A linear map $\rho:V\to V'$ is a ternary Hom-Nambu algebra morphism if it satisfies

$$\begin{aligned}\rho([x_1,x_2,x_3]) &= [\rho(x_1),\rho(x_2),\rho(x_3)]'\\ \rho\circ\alpha_i &= \alpha'_i\circ\rho \quad \text{for } i=1,2.\end{aligned}$$

The following theorem, given in [4] for n-ary algebras of Lie type, provides a way to construct ternary Hom-Nambu algebras (resp. n-ary Hom-Nambu-Lie algebras) starting from ternary Nambu algebra (resp. n-ary Nambu-Lie algebra) and an algebra endomorphism.

Theorem 2.6 ([4])**.** *Let $(V,[\cdot,\cdot,\cdot])$ be a ternary Nambu algebra (resp. ternary Nambu-Lie algebra) and let $\rho:V\to V$ be a ternary Nambu (resp. ternary Nambu-Lie) algebra endomorphism. If we set $\tilde{\rho}=(\rho,\rho)$, then $(V,\rho\circ[\cdot,\cdot,\cdot],\tilde{\rho})$ is a ternary Hom-Nambu algebra (resp. ternary Hom-Nambu-Lie algebra).*

Moreover, suppose that $(V',[\cdot,\cdot,\cdot]')$ is another ternary Nambu algebra (resp. ternary Nambu-Lie algebra) and $\rho':V'\to V'$ is a ternary Nambu (resp. ternary Nambu-Lie) algebra endomorphism. If $f:V\to V'$ is a ternary Nambu algebra morphism (resp. ternary Nambu-Lie algebra morphism) that satisfies $f\circ\rho=\rho'\circ f$ then

$$f:(V,\rho\circ[\cdot,\cdot,\cdot],\tilde{\rho})\longrightarrow(V',\rho'\circ[\cdot,\cdot,\cdot]',\tilde{\rho}')$$

is a ternary Hom-Nambu algebra morphism (resp. ternary Hom-Nambu-Lie algebra morphism).

Example 2.7. *An algebra V consisting of polynomials or possibly of other differentiable functions in 3 variables x_1, x_2, x_3, equipped with well-defined bracket multiplication given by the functional jacobian $J(f) = (\frac{\partial f_i}{\partial x_j})_{1\leq i,j\leq 3}$:*

$$[f_1, f_2, f_3] = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}, \tag{2.3}$$

is a ternary Nambu-Lie algebra. By considering a ternary Nambu-Lie algebra endomorphism of such algebra, we construct a ternary Hom-Nambu-Lie algebra. Let $\gamma(x_1, x_2, x_3)$ be a polynomial or a more general differentiable transformation of three variables, mapping elements of V to elements of V under composition $f \mapsto f \circ \gamma$, and such that $\det J(\gamma) = 1$. Let $\rho_\gamma : V \mapsto V$ be the composition transformation defined by $f \mapsto f \circ \gamma$ for any $f \in V$. By the general chain rule for composition of transformations of several variables,

$$J(\rho_\gamma(f)) = J(f \circ \gamma) = (J(f) \circ \gamma)J(\gamma) = \rho_\gamma(J(f))J(\gamma),$$
$$\det J(\rho_\gamma(f)) = \det(J(f) \circ \gamma) \det J(\gamma) = \det \rho_\gamma(J(f)) \det J(\gamma).$$

Hence, for any transformation γ with $\det J(\gamma) = 1$, the composition transformation ρ_γ defines an endomorphism of the ternary Nambu-Lie algebra with ternary product (2.3). Therefore, by Theorem 2.6, for any such transformation γ, the triple

$$(V, \rho_\gamma \circ [\cdot, \cdot, \cdot], (\rho_\gamma, \rho_\gamma))$$

is a ternary Hom-Nambu-Lie algebra.

3. Hom-Nambu-Lie algebras induced by Hom-Lie algebras

In this Section we provide a construction procedure of ternary Hom-Nambu and Hom-Nambu-Lie algebras starting from a binary bracket of a Hom-Lie algebra and a trace function satisfying certain compatibility conditions involving the twisting maps. To this end, we use the ternary bracket introduced in [2].

Definition 3.1. Let $(V, [\cdot, \cdot])$ be a binary algebra and let $\tau : V \to \mathbb{K}$ be a linear map. The trilinear map $[\cdot, \cdot, \cdot]_\tau : V \times V \times V \to V$ is defined as

$$[x, y, z]_\tau = \tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y]. \tag{3.1}$$

Lemma 3.2. *If the bilinear multiplication $[\cdot, \cdot]$ in Definition 3.1 is skew-symmetric, then the trilinear map $[\cdot, \cdot, \cdot]_\tau$ is skew-symmetric as well.*

Proof. The permutation group S_3 is generated by the two transpositions of neighboring indexes $(1, 2)$ and $(2, 3)$ and Sgn is multiplicative functional. Thus the proof of skew-symmetry (2.2) will be completed if it is done for $\sigma \in \{(1, 2), (2, 3)\}$. This is proved using skew-symmetry of the bilinear $[\cdot, \cdot]$ multiplication as follows. If $\sigma = (1, 2)$, then

$$\begin{aligned}[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] &= [x_2, x_1, x_3] = \tau(x_2)[x_1, x_3] + \tau(x_1)[x_3, x_2] + \tau(x_3)[x_2, x_1] = \\ &= -\tau(x_1)[x_2, x_3] - \tau(x_2)[x_3, x_1] - \tau(x_3)[x_1, x_2] = \\ &= -[x_1, x_2, x_3] = Sgn(\sigma)[x_1, x_2, x_3].\end{aligned}$$

If $\sigma = (2,3)$, then

$$\begin{aligned}
[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] &= [x_1, x_3, x_2] = \tau(x_1)[x_3, x_2] + \tau(x_3)[x_2, x_1] + \tau(x_2)[x_1, x_3] = \\
&= -\tau(x_1)[x_2, x_3] - \tau(x_2)[x_3, x_1] - \tau(x_3)[x_1, x_2] = \\
&= -[x_1, x_2, x_3] = Sgn(\sigma)[x_1, x_2, x_3]. \quad \square
\end{aligned}$$

If $\tau : V \to \mathbb{K}$ is a linear map such that $\tau([x,y]) = 0$ for all $x, y \in V$, then we call τ a *trace function on* $(V, [\cdot,\cdot])$. It follows immediately that $\tau([x,y,z]_\tau) = 0$ for all $x, y, z \in V$ if τ is a trace function.

Theorem 3.3. *Let $(V, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra and $\beta : V \to V$ be a linear map. Furthermore, assume that τ is a trace function on V fulfilling*

$$\tau\big(\alpha(x)\big)\tau(y) = \tau(x)\tau\big(\alpha(y)\big) \tag{3.2}$$

$$\tau\big(\beta(x)\big)\tau(y) = \tau(x)\tau\big(\beta(y)\big) \tag{3.3}$$

$$\tau\big(\alpha(x)\big)\beta(y) = \tau\big(\beta(x)\big)\alpha(y) \tag{3.4}$$

for all $x, y \in V$. Then $(V, [\cdot,\cdot,\cdot]_\tau, (\alpha, \beta))$ is a Hom-Nambu-Lie algebra, and we say that it is induced by $(V, [\cdot,\cdot], \alpha)$.

Proof. Since $[\cdot,\cdot,\cdot]_\tau$ is skew-symmetric and trilinear by construction, one only has to prove that the Hom-Nambu identity is fulfilled. Expanding the Hom-Nambu identity

$$\begin{aligned}
[\alpha(x), \beta(y), [z,u,v]] = [[x,y,z], \alpha(u), \beta(v)] + [\alpha(z), [x,y,u], \beta(v)] \\
+ [\alpha(z), \beta(u), [x,y,v]]
\end{aligned}$$

gives 24 different terms. Six of these can be grouped into three pairs as follows

$$\begin{aligned}
&\big[\beta(v), [x,y]\big]\Big(\tau\big(\alpha(u)\big)\tau(z) - \tau\big(\alpha(z)\big)\tau(u)\Big) \\
&\big[\tau\big(\alpha(z)\big)\tau(v)\beta(u) - \tau\big(\beta(v)\big)\tau(z)\alpha(u), [x,y]\big] \\
&\big[\alpha(z), [x,y]\big]\Big(\tau\big(\beta(v)\big)\tau(u) - \tau\big(\beta(u)\big)\tau(v)\Big),
\end{aligned}$$

which all vanish separately by (3.2), (3.3) and (3.4). The remaining 18 terms can be grouped in to six triples of the type

$$\tau\big(\alpha(x)\big)\tau(u)\big[\beta(y), [z,v]\big] + \tau\big(\alpha(u)\big)\tau(x)\big[\beta(v), [y,z]\big] + \tau\big(\beta(u)\big)\tau(x)\big[\alpha(z), [v,y]\big].$$

By using (3.4) one can rewrite this term as

$$\tau\big(\beta(x)\big)\tau(u)\big[\alpha(y), [z,v]\big] + \tau\big(\beta(u)\big)\tau(x)\big[\alpha(v), [y,z]\big] + \tau\big(\beta(u)\big)\tau(x)\big[\alpha(z), [v,y]\big],$$

and by using (3.3) and the Hom-Jacobi identity one sees that this term vanishes. The remaining five triples of terms can be shown to vanish in an analogous way. Hence, the Hom-Nambu identity is satisfied. $\square$

Remark 3.4. If we choose $\beta = \alpha$ in Theorem 3.3, conditions (3.2) – (3.4) reduce to the single relation

$$\tau\big(\alpha(x)\big)\tau(y) = \tau(x)\tau\big(\alpha(y)\big).$$

Choosing α and β to be identity maps in Theorem 3.3 one obtains the result in [2].

Corollary 3.5. *Let $(V, [\cdot,\cdot])$ be a Lie algebra and $\tau : V \to \mathbb{K}$ be a trace function on V. Then $(V, [\cdot,\cdot,\cdot]_\tau)$ is a Nambu-Lie algebra.*

Relation (3.4) effectively allows for the interchange of α and β in equations involving τ. Therefore, even though β is only assumed to be a linear map, relation (3.4) induces a Hom-Jacobi identity for $(V,[\cdot,\cdot])$ with respect to β in many cases.

Proposition 3.6. *Let $(V,[\cdot,\cdot],\alpha)$ be a Hom-Lie algebra and let $\tau : V \to \mathbb{K}$ and $\beta : V \to V$ be linear maps satisfying $\tau\big(\alpha(x)\big)\beta(y) = \tau\big(\beta(x)\big)\alpha(y)$ for all $x, y \in V$, and such that there exists an element $v \in V$ with $\tau\big(\alpha(v)\big) \neq 0$. Then $(V,[\cdot,\cdot],\beta)$ is a Hom-Lie algebra.*

Proof. Multiplying the Hom-Jacobi identity with $\tau\big(\beta(v)\big)$ gives

$$\tau\big(\beta(v)\big)\Big(\big[\alpha(x),[y,z]\big] + \big[\alpha(y),[z,x]\big] + \big[\alpha(z),[x,y]\big]\Big) = 0,$$

and by using the relation between α, β and τ in all three terms one obtains

$$\tau\big(\alpha(v)\big)\Big(\big[\beta(x),[y,z]\big] + \big[\beta(y),[z,x]\big] + \big[\beta(z),[x,y]\big]\Big) = 0.$$

By assumption $\tau\big(\alpha(v)\big) \neq 0$, which reduces the above identity to the Hom-Jacobi identity for β. Since x, y, z were chosen to be arbitrary, this proves that $(V,[\cdot,\cdot],\beta)$ is a Hom-Lie algebra. □

4. Properties of the compatibility conditions

When inducing a Hom-Nambu-Lie algebra via Theorem 3.3, one might ask how restrictive the α, β, τ-compatibility conditions (3.2) – (3.4) are? For instance, given a Hom-Lie algebra $(V,[\cdot,\cdot],\alpha)$, how much freedom does one have to choose β? It turns out that generically β has to be proportional to α, except in the case when the images of α and β lie in the kernel of τ (see Proposition 4.6).

In the following we shall study consequences of the α, β, τ-compatibility conditions by studying the kernels of α, β and τ.

Definition 4.1. Let V be a vector space, α and β linear maps $V \to V$ and τ a trace function on V. We say that the triple (α, β, τ) is *compatible on* V if conditions (3.2) – (3.4) hold. Moreover, if $\ker\tau \neq \{0\}$ and $\ker\tau \neq V$ then we call the triple *nondegenerate*.

Proposition 4.2. *Let $\mathcal{A} = (V,[\cdot,\cdot,\cdot]_\tau,(\alpha,\beta))$ be a Hom-Nambu-Lie algebra induced by $(V,[\cdot,\cdot],\alpha)$. If $\ker\tau = \{0\}$ or $\ker\tau = V$ then $\mathcal{A}$ is abelian.*

Proof. First, assume that $\ker\tau = \{0\}$. Since $\tau([x,y]) = 0$ for all $x, y \in V$ it follows that $[x,y] = 0$ for all $x, y \in V$. By the definition of $[\cdot,\cdot,\cdot]_\tau$ this implies that $[x,y,z] = 0$ for all $x, y, z \in V$.

Now, assume that $\ker\tau = V$. This directly implies (see (3.1)) that $[x,y,z] = 0$ for all $x, y, z \in V$. □

By an ideal of a Hom-Lie algebra $(V,[\cdot,\cdot],\alpha)$ we mean a subset $I \subseteq V$ such that $[I,V] \subseteq I$, and we say that a Hom-Lie algebra is *simple* if it has no ideals other than $\{0\}$ and V. Since the kernel of a trace function is always an ideal, one obtains the following corollary to Proposition 4.2.

Corollary 4.3. *A Hom-Nambu-Lie algebra induced by a simple Hom-Lie algebra is abelian.*

Proof. Assume that the induced Hom-Nambu-Lie algebra is not abelian. Then, by Proposition 4.2, the kernel of τ is neither $\{0\}$ nor V. This implies that $\ker\tau$ is a non-trivial ideal of the Hom-Lie algebra, which contradicts that it is assumed to be simple. □

In Proposition 4.2 we noted that if the kernel of τ is either the complete vector space or $\{0\}$, then the induced Hom-Nambu-Lie algebra will be abelian, and therefore we shall focus on nondegenerate triples in the following. To fix notation, we introduce $K = \ker\tau$ and U such that $U = V\backslash K$. For a nondegenerate compatible triple, U and K are always different from $\{0\}$.

Lemma 4.4. *Let (α, β, τ) be a nondegenerate compatible triple on V. Then it holds that $\alpha(K) \subseteq K$ and $\beta(K) \subseteq K$.*

Proof. Since the triple is assumed to be nondegenerate, one can find an element $y \in V$ such that $\tau(y) \neq 0$. For any $x \in K$, equations (3.2) and (3.3) imply that $\tau\big(\alpha(x)\big) = 0$ and $\tau\big(\beta(x)\big) = 0$. Hence, α and β map K into K. □

Lemma 4.5. *Let (α, β, τ) be a nondegenerate compatible triple on V and assume that there exists an element $u \in U$ such that $\alpha(u) \in K$ (or $\beta(u) \in K$). Then $\alpha(U) \subseteq K$ (or $\beta(U) \subseteq K$).*

Proof. For a general $x \in V$ it follows from (3.2) (on u and x) that $\tau(\alpha(x)) = 0$, since $u \in U$ and $\alpha(u) \in K$. An identical argument goes through for β by using (3.3). □

These results allow us to split the problem into four possible cases:

(C1) $\alpha(U) \subseteq U$ and $\beta(U) \subseteq U$

(C2) $\alpha(U) \subseteq K$ and $\beta(U) \subseteq K$

(C3) $\alpha(U) \subseteq U$ and $\beta(U) \subseteq K$

(C4) $\alpha(U) \subseteq K$ and $\beta(U) \subseteq U$

Clearly, in case (C2) the compatibility conditions will be identically satisfied since $\tau\big(\alpha(x)\big) = \tau\big(\beta(x)\big) = 0$ for all $x \in V$. In the other cases, the next proposition shows that one does not have any freedom to choose α and β independently.

Proposition 4.6. *Let (α, β, τ) be a nondegenerate compatible triple on V. Then, refering to the cases in (C1) – (C4), the following holds:*

(C1) $\exists\, \lambda \in \mathbb{K}\backslash\{0\} : \ \beta = \lambda\alpha$

(C3) $\beta \equiv 0$

(C4) $\alpha \equiv 0$

Proof. Case (C1): If one chooses $u \in U$ and $x \in V$, relation (3.4) gives

$$\beta(x) = \frac{\tau\big(\beta(u)\big)}{\tau\big(\alpha(u)\big)}\alpha(x),$$

where $\tau\big(\beta(u)\big) \neq 0$ by assumption. Case (C3): By choosing $u \in U$ and $x \in V$, relation (3.4) gives $\beta(x) = 0$. Case (C4) is proven in the same way. □

5. Examples

In this section we provide several examples of ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras by means of the method described in Theorem 3.3. From the results in the previous section (see Proposition 4.6), there are two (non-trivial) possibilities for α and β. Either $\beta = \lambda\alpha$ or the images of α and β are in the kernel of τ, in which case it is possible to have $\beta \neq \lambda\alpha$. We provide examples in both cases.

Example 5.1. *In our first example, we let V be the vector space of $n \times n$ matrices, and $\alpha : V \to V$ acts through conjugation by an invertible matrix s, i.e. $\alpha(x) = s^{-1}xs$. Then $(V, \alpha \circ [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra. For matrices, any trace function is proportional to the matrix trace, so we let $\tau(x) = \mathrm{tr}(x)$. If we want to choose a $\beta \neq 0$, it follows from Proposition 4.6 that β has to be proportional to α, i.e. $\beta = \lambda\alpha$ for some $\lambda \neq 0$. Since $\mathrm{tr}\big(\alpha(x)\big) = \mathrm{tr}(x)$ it is clear that $(\alpha, \lambda\alpha, \mathrm{tr})$ is a nondegenerate compatible triple on V, which implies, by Theorem 3.3, that $\big(V, [\cdot,\cdot,\cdot]_{\mathrm{tr}}, (\alpha, \lambda\alpha)\big)$ is a Hom-Nambu-Lie algebra induced by $(V, \alpha \circ [\cdot,\cdot], \alpha)$.*

Example 5.2. *Let us start with the vector space V spanned by $\{x_1, x_2, x_3, x_4\}$ with a skew-symmetric bilinear map defined through*

$$[x_i, x_j] = a_{ij}x_3 + b_{ij}x_4$$

where a_{ij} and b_{ij} are antisymmetric 4×4 matrices. Defining

$$\begin{aligned} &\alpha(x_i) = x_3 \qquad &&\beta(x_i) = x_4 \qquad &&i = 1, \dots, 4 \\ &\tau(x_1) = \gamma_1 \qquad &&\tau(x_2) = \gamma_2 \qquad &&\tau(x_3) = \tau(x_4) = 0, \end{aligned}$$

one immediately observes that τ is a trace function, $\mathrm{im}\,\alpha \subseteq \ker\tau$, $\mathrm{im}\,\beta \subseteq \ker\tau$, and $\beta \neq \lambda\alpha$. Furthermore, $(V, [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra provided

$$\begin{aligned} b_{13} &= b_{12} + b_{23} \\ b_{14} &= b_{12} + b_{23} + b_{34} \\ b_{24} &= b_{23} + b_{34}. \end{aligned}$$

The four independent ternary brackets of the induced Hom-Nambu-Lie algebra can be written as

$$\begin{aligned} [x_1, x_2, x_3] &= \big(\gamma_1 a_{23} - \gamma_2 a_{13}\big)x_3 + \big(\gamma_1 b_{23} - \gamma_2(b_{12} + b_{23})\big)x_4 \\ [x_1, x_2, x_4] &= \big(\gamma_1 a_{24} - \gamma_2 a_{14}\big)x_3 + \big(\gamma_1(b_{23} + b_{34}) - \gamma_2(b_{12} + b_{23} + b_{34})\big)x_4 \\ [x_1, x_3, x_4] &= (\gamma_1 a_{34})x_3 + (\gamma_1 b_{34})x_4 \\ [x_2, x_3, x_4] &= (\gamma_2 a_{34})x_3 + (\gamma_2 b_{34})x_4. \end{aligned}$$

For instance, choosing $\gamma_1 = \gamma_2 = 1$ and $a_{i<j} = 1$, one obtains the Hom-Nambu-Lie algebra $\big(\langle x_1, x_2, x_3, x_4\rangle, [\cdot,\cdot,\cdot], (\alpha, \beta)\big)$ defined by

$$\begin{aligned} [x_1, x_2, x_3] &= -b_{12}x_4 \\ [x_1, x_2, x_4] &= -b_{34}x_4 \\ [x_1, x_3, x_4] &= x_3 + b_{34}x_4 \\ [x_2, x_3, x_4] &= x_3 + b_{34}x_4 \end{aligned}$$

together with $\alpha(x_i) = x_3$ and $\beta(x_i) = x_4$.

Example 5.3. *We consider the 3-dimensional Hom-Lie algebra defined with respect to a basis* $\{x_1, x_2, x_3\}$ *by*

$$\begin{aligned} [x_1, x_2] &= a_1 x_2 - \frac{a_2 a_4}{a_3} x_3, \\ [x_1, x_3] &= -\frac{a_1 a_3}{a_4} x_2 + a_2 x_3, \\ [x_2, x_3] &= a_3 x_2 + a_4 x_3, \end{aligned}$$

where a_1, a_2, a_3, a_4 *are parameters in* $\mathbb{K}$ *and* $a_3, a_4 \neq 0$. *The map* α *is defined by*

$$\alpha(x_1) = p x_1$$
$$\alpha(x_2) = q x_3$$
$$\alpha(x_3) = q x_4,$$

for any $p, q \in \mathbb{K}$. *We define a trace function as*

$$\tau(x_1) = t,\ \tau(x_2) = 0,\ \tau(x_3) = 0,$$

for any $t \in \mathbb{K}$.

If $p \neq 0$, *we let* β *be the linear map defined by*

$$\beta(x_1) = r x_1$$
$$\beta(x_2) = \frac{qr}{p} x_2$$
$$\beta(x_3) = s x_3,$$

for any $r, s \in \mathbb{K}$. *The previous data satisfies the conditions* (3.2), (3.3) *and* (3.4). *Then, according to Theorem 3.3, we obtain a ternary Hom-Nambu-Lie algebra defined by*

$$[x_1, x_2, x_3] = t(a_3 x_2 + a_4 x_3).$$

If $p = 0$ *then one may consider a map* β *of the form*

$$\beta(x_1) = 0$$
$$\beta(x_2) = r_1 x_1 + r_2 x_2 + r_3 x_3$$
$$\beta(x_3) = r_4 x_1 + r_5 x_2 + r_6 x_3.$$

for any $r_1, r_2, r_3, r_4, r_5, r_6 \in \mathbb{K}$. *The ternary bracket is the same as for* $p \neq 0$ *case.*

Example 5.4. *We consider the 3-dimensional Hom-Lie algebra defined with respect to a basis* $\{x_1, x_2, x_3\}$ *by*

$$\begin{aligned} [x_1, x_2] &= -a_1 x_2 + a_2 x_3, \\ [x_1, x_3] &= a_3 x_2 + a_1 x_3, \\ [x_2, x_3] &= a_4 x_2 + a_5 x_3, \end{aligned}$$

where a_1, a_2, a_3, a_4, a_5 *are parameters in* $\mathbb{K}$. *The map* α *is defined by*

$$\alpha(x_1) = 0$$
$$\alpha(x_2) = q x_3$$
$$\alpha(x_3) = q x_4,$$

for any $q \in \mathbb{K}$, *and we define a trace function as*

$$\tau(x_1) = t,\ \tau(x_2) = 0,\ \tau(x_3) = 0,$$

for any $t \in \mathbb{K}$.

Let β be the linear map defined by

$$\beta(x_1) = 0$$
$$\beta(x_2) = r_1x_1 + r_2x_2 + r_3x_3$$
$$\beta(x_3) = r_4x_1 + r_5x_2 + r_6x_3,$$

for any $r_1, r_2, r_3, r_4r_5, r_6 \in \mathbb{K}$. The previous data satisfies the condition (3.2)*,* (3.3) *and* (3.4)*. Then, according to Theorem 3.3, we obtain a ternary Hom-Nambu-Lie algebra defined by*

$$[x_1, x_2, x_3] = t(a_4x_2 + a_5x_3).$$

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