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Ternary Hom–Nambu–Lie algebras induced by Hom–Lie algebras

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The need to consider n-ary algebraic structures, generalizing Lie and Poisson algebras, has become increasingly important in physics, and it should therefore be of interest to study the mathematical concepts related to n-ary algebras. The purpose of this paper is to investigate ternary multiplications (as deformations of n-Lie structures) constructed from the binary multiplication of a Hom–Lie algebra, a linear twisting map, and a trace function satisfying certain compatibility conditions. We show that the relation between the kernels of the twisting maps and the trace function plays an important role in this context and provide examples of Hom–Nambu–Lie algebras obtained using this construction. © *2010 American Institute of Physics.* [doi:10.1063/1.3359004]

I. INTRODUCTION

Lie algebras and Poisson algebras have played an extremely important role in physics for a long time. Their generalizations, known as n-Lie algebras and "Nambu algebras",[32,15,33] also arise naturally in physics and have recently been studied in the context of "M-branes".[6,17] It turns out that in the dynamic study of strings and M-branes, an algebra with ternary multiplication called Bagger–Lambert algebra appears naturally. It was used in Ref. 6 as one of the main ingredients in the construction of a new type of supersymmetric gauge theory that is consistent with all the symmetries expected of a multiple M2-brane theory: 16 supersymmetries, conformal invariance, and an SO(8) R-symmetry that acts on the eight transverse scalars. Hundreds of papers are dedicated to Bagger–Lambert algebra by now. Other applications of Nambu algebras to M-branes, quantization of Nambu mechanics, volume preserving diffeomorphisms, integrable systems, and related generalization of Lax equation have been considered in Ref. 17.

A long-standing problem related to Nambu algebras is their quantization. For Poisson algebras, the problem of finding an operator algebra where the commutator Lie algebra corresponds to the Poisson algebra is a well-studied problem. For higher order algebras, much less is known and the corresponding problem seems to be hard. A Nambu–Lie algebra is defined, in general, by an n-ary multilinear multiplication, which is skew symmetric (see (2.4) for $n=3$), and satisfies an identity extending the Jacobi identity for the Lie algebras. For $n=3$, this identity is

$$[x_1,x_2,[x_3,x_4,x_5]]=[[x_1,x_2,x_3],x_4,x_5]+[x_3,[x_1,x_2,x_4],x_5]+[x_3,x_4,[x_1,x_2,x_5]].$$

In Nambu–Lie algebras, the additional freedom in comparison to Lie algebras is mainly limited to extra arguments in the multilinear multiplication. The identities of Nambu–Lie algebras also closely resemble the identities for Lie algebras. As a result, there is a close similarity between Lie

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algebras and Nambu–Lie algebras in their appearances in connection to other algebraic and analytic structures and in the extent of their applicability. Thus, it is not surprising that it becomes unclear how to associate in meaningful ways ordinary Nambu–Lie algebras with the important in physics generalizations and quantum deformations of Lie algebras when typically the ordinary skew symmetry and Jacobi identities of Lie algebras are violated. However, if the class of Nambu–Lie algebras is extended with enough extra structure beyond just adding more arguments in multilinear multiplication, the natural ways of association of such multilinear algebraic structures with generalizations and quantum deformations of Lie algebras may become feasible.[3] Hom–Nambu–Lie algebras are defined by a similar but more general identity than that of Nambu–Lie algebras involving some additional linear maps (see Definition 2.7). These linear maps, twisting or deforming the main identities, introduce substantial new freedom in the structure, allowing one to consider Hom–Nambu–Lie algebras as deformations of Nambu–Lie algebras (n-Lie algebras). The extra freedom built into the structure of Hom–Nambu–Lie algebras may provide a path to quantization beyond what is possible for ordinary Nambu–Lie algebras. All this gives also important motivation for investigation of mathematical concepts and structures such as Leibniz n-ary algebras[8,15] and their modifications and extensions, as well as Hom-algebra extensions of Poisson algebras[31] (see Definition 2.2). For discussion of physical applications of these and related algebraic structures to models for elementary particles and unification problems for interactions, see Refs. 1, 19–21, and 34.

The main result of this paper is contained in Theorem 3.3 where ternary Hom–Nambu–Lie algebras are constructed from a Hom–Lie algebra over a vector space V, a linear map $\beta: V \to V$, and a generalized trace function $\rho: V \to \mathbb{K}$, satisfying certain compatibility conditions.

In Ref. 5, generalizations of n-ary algebras of Lie type and associative type by twisting the identities using linear maps have been introduced. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type including n-ary Nambu algebras, n-ary Nambu–Lie algebras and n-ary Lie algebras, and n-ary algebras of associative type including n-ary totally associative and n-ary partially associative algebras.

The general Hom-algebra structures arose first in connection to quasi-deformation and discretizations of Lie algebras of vector fields. These quasi-deformations lead to quasi-Lie algebras, a generalized Lie algebra structure in which the skew symmetry and Jacobi conditions are twisted. The first examples were concerned with q-deformations of the Witt and Virasoro algebras (see, for example, Refs. 2, 9, 11, 12, 10, 13, 14, 18, 27, 25, and 26). Motivated by these and new examples arising as applications of the general quasi-deformation construction of Refs. 16, 22, and 24, on the one hand, and the desire to be able to treat within the same framework such well-known generalizations of Lie algebras as the color and super-Lie algebras on the other hand, quasi-Lie algebras and subclasses of quasi-Hom–Lie algebras and Hom–Lie algebras were introduced in Refs. 16, 22, 24, and 23. In the subclass of Hom–Lie algebras, skew symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as for Lie algebras, reducing to ordinary Lie algebras when the linear twisting map is the identity map. Hom-associative algebras replacing associative algebras in the context of Hom–Lie algebras and also more general classes of Hom–Lie admissible algebras, G-Hom-associative algebras, were introduced in Ref. 28. The first steps in the construction of universal enveloping algebras for Hom–Lie algebras have been made in Ref. 35. Formal deformations and elements of (co)homology for Hom–Lie algebras have been studied in Refs. 31 and 36, whereas dual structures such as Hom-coalgebras, Hom-bialgebras, and Hom–Hopf algebras appeared first in Refs. 29 and 30 and further investigated in Refs. 7 and 37.

This paper is organized as follows. In Sec. II we review basic concepts of Hom-Lie, ternary Hom–Nambu, and ternary Hom–Nambu–Lie algebras. We also recall the method of composition with endomorphism for the construction of Hom–Lie, ternary Hom–Nambu, and ternary Hom–Nambu–Lie algebras from Lie, Nambu, and Nambu–Lie algebras. In Sec. III we provide a construction procedure of ternary Hom–Nambu and Hom–Nambu–Lie algebras starting from a binary bracket of a Hom–Lie algebra and a trace function satisfying certain compatibility conditions involving the twisting maps. To this end, we use the ternary bracket introduced in Ref. 4. In Sec.

IV, we investigate how restrictive the compatibility conditions are. The mutual position of kernels of twisting maps and the trace play an important role in this context. Finally, in Sec. V, we provide examples of Hom–Nambu–Lie algebras obtained using constructions presented in the paper.

II. TERNARY NAMBU–LIE ALGEBRAS

Let us first recall some basic facts about Hom algebras. A Hom-algebra structure is a multiplication on a vector space, which is twisted by a linear map. In what follows, all vector spaces will be defined over a field $\mathbb{K}$ of characteristic 0, and V will always denote such a vector space.

The notion of a Hom–Lie algebra was initially motivated by examples of deformed Lie algebras coming from twisted discretizations of vector fields.[16,22,24] We will follow notation conventions in Ref. 28.

Definition 2.1: A *Hom–Lie algebra* is a triple $(V,[\cdot,\cdot],\alpha)$, where $[\cdot,\cdot]:V\times V\to V$ is a bilinear map and $\alpha:V\to V$ a linear map satisfying

$$[x,y]=-[y,x] \quad (\text{skew symmetry}),$$

$$\circlearrowleft_{x,y,z}[\alpha(x),[y,z]]=0 \quad (\text{Hom-Jacobi condition})$$

for all x,y,z from V, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutations of x,y,z.

In connection to Hom–Lie algebras, a natural notion of Hom–Poisson algebra generalizing Poisson algebra emerges naturally in the deformation theory of Hom–Lie algebras where it plays the same role as Poisson algebras in the context of formal deformations of Lie algebras and deformation quantization.[31]

Definition 2.2: A *Hom–Poisson algebra* is a quadruple $(V,\mu,\{\cdot,\cdot\},\alpha)$ consisting of a vector space V, bilinear maps $\mu:V\times V\to V$, and $\{\cdot,\cdot\}:V\times V\to V$, and a linear map $\alpha:V\to V$ satisfying following:

(1) (V,μ,α) is a commutative Hom-associative algebra;
(2) $(V,\{\cdot,\cdot\},\alpha)$ is a Hom–Lie algebra; and
(3) for all x,y,z in V,

$$\{\alpha(x),\mu(y,z)\}=\mu(\alpha(y),\{x,z\})+\mu(\alpha(z),\{x,y\}). \tag{2.1}$$

Remark 2.3: The notion of Hom-associative algebra generalizing associative algebras to a situation where associativity law is twisted by a linear map was introduced in Ref. 28. The twisted associativity identity is written as

$$\mu(\alpha(x),\mu(y,z))=\mu(\mu(x,y),\alpha(z)). \tag{2.2}$$

where μ is the multiplication and α is a linear map. It was shown in Ref. 28 that the commutator product, defined using the multiplication in a Hom-associative algebra, leads naturally to a Hom–Lie algebra.

Analogously, one can introduce Hom-algebra equivalents of n-ary algebras of Lie type.[5]

Definition 2.4: A *ternary Hom–Nambu algebra* is a triple $(V,[\cdot,\cdot,\cdot],\tilde{\alpha})$, consisting of a vector space V, a trilinear map $[\cdot,\cdot,\cdot]:V\times V\times V\to V$, and a pair of linear maps $\tilde{\alpha}=(\alpha_1,\alpha_2)$ satisfying

$$\begin{aligned}[\alpha_1(x_1),\alpha_2(x_2),[x_3,x_4,x_5]]&=[[x_1,x_2,x_3],\alpha_1(x_4),\alpha_2(x_5)]+[\alpha_1(x_3),[x_1,x_2,x_4],\alpha_2(x_5)]\\&\quad+[\alpha_1(x_3),\alpha_2(x_4),[x_1,x_2,x_5]].\end{aligned} \tag{2.3}$$

The identity (2.3) is called ternary Hom–Nambu identity.

Remark 2.5: Let $(V,[\cdot,\cdot,\cdot],\tilde{\alpha})$ be a ternary Hom–Nambu algebra where $\tilde{\alpha}=(\alpha_1,\alpha_2)$. Let $x=(x_1,x_2)\in V\times V$, $\tilde{\alpha}(x)=(\alpha_1(x_1),\alpha_2(x_2))\in V\times V$, and $y\in V$. Let L_x be a linear map on V, defined by

$$L_x(y) = [x_1, x_2, y].$$

Then the Hom–Nambu identity is written as

$$L_{\tilde{\alpha}(x)}([x_3, x_4, x_5]) = [L_x(x_3), \alpha_1(x_4), \alpha_2(x_5)] + [\alpha_1(x_3), L_x(x_4), \alpha_2(x_5)] + [\alpha_1(x_3), \alpha_2(x_4), L_x(x_5)].$$

Remark 2.6: When the maps $(\alpha_i)_{i=1,2}$ are all identity maps, one recovers the classical ternary Nambu algebras. The identity obtained in this special case of classical n-ary Nambu algebra is known also as the fundamental identity or Filippov identity.[15,32,33]

Definition 2.7: A ternary Hom–Nambu algebra $(V, [\cdot, \cdot, \cdot], (\alpha_1, \alpha_2))$ is called a *ternary Hom–Nambu–Lie algebra* if the bracket is skew symmetric, that is,

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = \mathrm{Sgn}(\sigma)[x_1, x_2, x_3], \quad \forall\, \sigma \in \mathcal{S}_3 \quad \text{and} \quad \forall\, x_1, x_2, x_3 \in V, \tag{2.4}$$

where $\mathcal{S}_3$ stands for the permutation group on three elements.

The morphisms of ternary Hom–Nambu algebras are defined in the natural way. It should be pointed out, however, that the morphisms should intertwine not only the ternary products but also the twisting linear maps. Let $(V, [\cdot, \cdot, \cdot], \tilde{\alpha})$ and $(V', [\cdot, \cdot, \cdot]', \tilde{\alpha}')$ be two n-ary Hom–Nambu algebras, where $\tilde{\alpha} = (\alpha_1, \alpha_2)$ and $\tilde{\alpha}' = (\alpha_1', \alpha_2')$. A linear map $\rho: V \to V'$ is a ternary Hom–Nambu algebra morphism if it satisfies

$$\rho([x_1, x_2, x_3]) = [\rho(x_1), \rho(x_2), \rho(x_3)]',$$

$$\rho \circ \alpha_i = \alpha_i' \circ \rho \quad \text{for } \ i = 1, 2.$$

The following theorem, given in Ref. 5 for n-ary algebras of Lie type, provides a way to construct ternary Hom–Nambu algebras (n-ary Hom–Nambu–Lie algebras) starting from ternary Nambu algebra (n-ary Nambu–Lie algebra) and an algebra endomorphism.

Theorem 2.8: *(Ref. 5) Let $(V, [\cdot, \cdot, \cdot])$ be a ternary Nambu algebra (ternary Nambu–Lie algebra) and let $\rho: V \to V$ be a ternary Nambu (ternary Nambu–Lie) algebra endomorphism. If we set $\tilde{\rho} = (\rho, \rho)$, then $(V, \rho \circ [\cdot, \cdot, \cdot], \tilde{\rho})$ is a ternary Hom–Nambu algebra (ternary Hom–Nambu–Lie algebra).*

Moreover, suppose that $(V', [\cdot, \cdot, \cdot]')$ is another ternary Nambu algebra (ternary Nambu–Lie algebra) and $\rho': V' \to V'$ is a ternary Nambu (ternary Nambu–Lie) algebra endomorphism. If $f: V \to V'$ is a ternary Nambu algebra morphism (ternary Nambu–Lie algebra morphism) that satisfies $f \circ \rho = \rho' \circ f$, then

$$f:(V, \rho \circ [\cdot, \cdot, \cdot], \tilde{\rho}) \to (V', \rho' \circ [\cdot, \cdot, \cdot]', \tilde{\rho}')$$

is a ternary Hom–Nambu algebra morphism (ternary Hom–Nambu–Lie algebra morphism).

Example 2.9: An algebra V consisting of polynomials or possibly of other differentiable functions in three variables x_1, x_2, and x_3, equipped with well-defined bracket multiplication given by the functional Jacobian $J(f) = (\partial f_i / \partial x_j)_{1 \le i,j \le 3}$,

$$[f_1, f_2, f_3] = \det\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \dfrac{\partial f_1}{\partial x_3} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \dfrac{\partial f_2}{\partial x_3} \\ \dfrac{\partial f_3}{\partial x_1} & \dfrac{\partial f_3}{\partial x_2} & \dfrac{\partial f_3}{\partial x_3} \end{pmatrix}, \tag{2.5}$$

is a ternary Nambu–Lie algebra. By considering a ternary Nambu–Lie algebra endomorphism of such algebra, we construct a ternary Hom–Nambu–Lie algebra. Let $\gamma(x_1, x_2, x_3)$ be a polynomial or a more general differentiable transformation of three variables, mapping elements of V to elements of V under composition $f \mapsto f \circ \gamma$ and such that $\det J(\gamma) = 1$. *Let $\rho_\gamma: V \mapsto V$ be the compo-*

sition transformation defined by $f \mapsto f \circ \gamma$ *for any* $f \in V$. *By the general chain rule for composition of transformations of several variables,*

$$J(\rho_\gamma(f)) = J(f \circ \gamma) = (J(f) \circ \gamma)J(\gamma) = \rho_\gamma(J(f))J(\gamma),$$

$$\det J(\rho_\gamma(f)) = \det(J(f) \circ \gamma)\det J(\gamma) = \det \rho_\gamma(J(f))\det J(\gamma).$$

Hence, for any transformation γ *with* $\det J(\gamma)=1$, *the composition transformation* ρ_γ *defines an endomorphism of the ternary Nambu–Lie algebra with ternary product (2.5). Therefore, by Theorem 2.8, for any such transformation* γ, *the triple*

$$(V, \rho_\gamma \circ [\cdot,\cdot,\cdot], (\rho_\gamma, \rho_\gamma))$$

is a ternary Hom–Nambu–Lie algebra.

III. HOM–NAMBU–LIE ALGEBRAS INDUCED BY HOM–LIE ALGEBRAS

In this section we provide a construction procedure of ternary Hom–Nambu and Hom–Nambu–Lie algebras starting from a binary bracket of a Hom–Lie algebra and a trace function satisfying certain compatibility conditions involving the twisting maps. To this end, we use the ternary bracket introduced in Ref. 4.

Definition 3.1: Let $(V,[\cdot,\cdot])$ be a binary algebra and let $\tau: V \to \mathbb{K}$ be a linear map. The trilinear map $[\cdot,\cdot,\cdot]_\tau: V \times V \times V \to V$ is defined as

$$[x,y,z]_\tau = \tau(x)[y,z] + \tau(y)[z,x] + \tau(z)[x,y]. \tag{3.1}$$

Lemma 3.2: If the bilinear multiplication $[\cdot,\cdot]$ *in Definition 3.1 is skew symmetric, then the trilinear map* $[\cdot,\cdot,\cdot]_\tau$ *is skew symmetric as well.*

Proof: The permutation group S_3 is generated by the two transpositions of neighboring indices (1,2) and (2,3) and Sgn is multiplicative functional. Thus the proof of skew-symmetry (2.4) will be completed if it is done for $\sigma \in \{(1,2),(2,3)\}$. This is proved using skew symmetry of the bilinear $[\cdot,\cdot]$ as follows. If $\sigma=(1,2)$, then

$$\begin{aligned}[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}] &= [x_2,x_1,x_3] = \tau(x_2)[x_1,x_3] + \tau(x_1)[x_3,x_2] + \tau(x_3)[x_2,x_1] \\ &= -\tau(x_1)[x_2,x_3] - \tau(x_2)[x_3,x_1] - \tau(x_3)[x_1,x_2] = -[x_1,x_2,x_3] = \mathrm{Sgn}(\sigma)[x_1,x_2,x_3].\end{aligned}$$

If $\sigma=(2,3)$, then

$$\begin{aligned}[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}] &= [x_1,x_3,x_2] = \tau(x_1)[x_3,x_2] + \tau(x_3)[x_2,x_1] + \tau(x_2)[x_1,x_3] \\ &= -\tau(x_1)[x_2,x_3] - \tau(x_2)[x_3,x_1] - \tau(x_3)[x_1,x_2] = -[x_1,x_2,x_3] = \mathrm{Sgn}(\sigma)[x_1,x_2,x_3].\end{aligned}$$

□

If $\tau: V \to \mathbb{K}$ is a linear map such that $\tau([x,y])=0$ for all $x, y \in V$, then we call τ a *trace function on* $(V,[\cdot,\cdot])$. It follows immediately that $\tau([x,y,z]_\tau)=0$ for all $x,y,z \in V$ if τ is a trace function.

Theorem 3.3: *Let* $(V,[\cdot,\cdot],\alpha)$ *be a Hom–Lie algebra and* $\beta: V \to V$ *be a linear map. Furthermore, assume that* τ *is a trace function on V fulfilling*

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y)), \tag{3.2}$$

$$\tau(\beta(x))\tau(y) = \tau(x)\tau(\beta(y)), \tag{3.3}$$

$$\tau(\alpha(x))\beta(y) = \tau(\beta(x))\alpha(y) \tag{3.4}$$

for all $x,y \in V$. *Then* $(V,[\cdot,\cdot,\cdot]_\tau,(\alpha,\beta))$ *is a Hom–Nambu–Lie algebra, and we say that it is induced by* $(V,[\cdot,\cdot],\alpha)$.

Proof: Since $[\cdot,\cdot,\cdot]_\tau$ is skew symmetric and trilinear by construction, one only has to prove that the Hom–Nambu identity is fulfilled. Expanding the Hom–Nambu identity

$$[\alpha(x),\beta(y),[z,u,v]] = [[x,y,z],\alpha(u),\beta(v)] + [\alpha(z),[x,y,u],\beta(v)] + [\alpha(z),\beta(u),[x,y,v]]$$

gives 24 different terms. Six of these can be grouped into three pairs as follows:

$$[\beta(v),[x,y]](\tau(\alpha(u))\tau(z) - \tau(\alpha(z))\tau(u)),$$

$$[\tau(\alpha(z))\tau(v)\beta(u) - \tau(\beta(v))\tau(z)\alpha(u),[x,y]],$$

$$[\alpha(z),[x,y]](\tau(\beta(v))\tau(u) - \tau(\beta(u))\tau(v)),$$

which all vanish separately by (3.2)–(3.4). The remaining 18 terms can be grouped into six triples of the type

$$\tau(\alpha(x))\tau(u)[\beta(y),[z,v]] + \tau(\alpha(u))\tau(x)[\beta(v),[y,z]] + \tau(\beta(u))\tau(x)[\alpha(z),[v,y]].$$

By using (3.4), one can rewrite this term as

$$\tau(\beta(x))\tau(u)[\alpha(y),[z,v]] + \tau(\beta(u))\tau(x)[\alpha(v),[y,z]] + \tau(\beta(u))\tau(x)[\alpha(z),[v,y]],$$

and by using (3.3) and the Hom–Jacobi identity, one sees that this term vanishes. The remaining five triples of terms can be shown to vanish in an analogous way. Hence, the Hom–Nambu identity is satisfied. □

Remark 3.4: If we choose $\beta=\alpha$ in Theorem 3.3, conditions (3.2)–(3.4) reduce to the single relation

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y)).$$

Choosing α and β to be identity maps in Theorem 3.3, one obtains the result in Ref. 4.

Corollary 3.5: Let $(V,[\cdot,\cdot])$ be a Lie algebra and $\tau: V\to\mathbb{K}$ be a trace function on V. Then $(V,[\cdot,\cdot,\cdot]_\tau)$ is a Nambu–Lie algebra.

Relation (3.4) effectively allows for the interchange of α and β in equations involving τ. Therefore, even though β is only assumed to be a linear map, relation (3.4) induces a Hom–Jacobi identity for $(V,[\cdot,\cdot])$ with respect to β in many cases.

Proposition 3.6: Let $(V,[\cdot,\cdot],\alpha)$ be a Hom–Lie algebra and let $\tau: V\to\mathbb{K}$ and $\beta: V\to V$ be linear maps satisfying $\tau(\alpha(x))\beta(y)=\tau(\beta(x))\alpha(y)$ for all $x,y\in V$, and such that there exists an element $v\in V$ with $\tau(\alpha(v))\neq 0$. Then $(V,[\cdot,\cdot],\beta)$ is a Hom–Lie algebra.

Proof: Multiplying the Hom–Jacobi identity with $\tau(\beta(v))$ gives

$$\tau(\beta(v))([\alpha(x),[y,z]] + [\alpha(y),[z,x]] + [\alpha(z),[x,y]]) = 0,$$

and by using the relation between α, β, and τ in all three terms one obtains

$$\tau(\alpha(v))([\beta(x),[y,z]] + [\beta(y),[z,x]] + [\beta(z),[x,y]]) = 0.$$

By assumption, $\tau(\alpha(v))\neq 0$, which reduces the above identity to the Hom–Jacobi identity for β. Since x,y,z were chosen to be arbitrary, this proves that $(V,[\cdot,\cdot],\beta)$ is a Hom–Lie algebra. □

IV. PROPERTIES OF THE COMPATIBILITY CONDITIONS

When inducing a Hom–Nambu–Lie algebra via Theorem 3.3, one might ask how restrictive the α,β,τ-compatibility conditions (3.2)–(3.4) are? For instance, given a Hom–Lie algebra $(V,[\cdot,\cdot],\alpha)$, how much freedom does one have to choose β? It turns out that generically β has to be proportional to α, except in the case when the images of α and β lie in the kernel of τ (see Proposition 4.6).

In the following, we shall study consequences of the α,β,τ-compatibility conditions by studying the kernels of α, β, and τ.

Definition 4.1: Let V be a vector space, α and β linear maps $V\to V$, and τ a trace function on V. We say that the triple (α,β,τ) is *compatible on* V if conditions (3.2)–(3.4) hold. Moreover, if $\ker\tau\neq\{0\}$ and $\ker\tau\neq V$, then we call the triple *nondegenerate*.

Proposition 4.2: Let $\mathcal{A}=(V,[\cdot,\cdot,\cdot]_\tau,(\alpha,\beta))$ be a Hom–Nambu–Lie algebra induced by $(V,[\cdot,\cdot],\alpha)$. If $\ker\tau=\{0\}$ or $\ker\tau=V$, then $\mathcal{A}$ is Abelian.

Proof: First, assume that $\ker\tau=\{0\}$. Since $\tau([x,y])=0$ for all $x,y\in V$ it follows that $[x,y]=0$ for all $x,y\in V$. By the definition of $[\cdot,\cdot,\cdot]_\tau$, this implies that $[x,y,z]=0$ for all $x,y,z\in V$.

Now, assume that $\ker\tau=V$. This directly implies (see (3.1)) that $[x,y,z]=0$ for all $x,y,z\in V$. □

By an ideal of a Hom–Lie algebra $(V,[\cdot,\cdot],\alpha)$ we mean a subset $I\subseteq V$ such that $[I,V]\subseteq I$, and we say that a Hom–Lie algebra is *simple* if it has no ideals other than $\{0\}$ and V. Since the kernel of a trace function is always an ideal, one obtains the following corollary to Proposition 4.2.

Corollary 4.3: A Hom–Nambu–Lie algebra induced by a simple Hom–Lie algebra is Abelian.

Proof: Assume that the induced Hom–Nambu–Lie algebra is not Abelian. Then, by Proposition 4.2, the kernel of τ is neither $\{0\}$ nor V. This implies that $\ker\tau$ is a nontrivial ideal of the Hom–Lie algebra, which contradicts that it is assumed to be simple. □

In Proposition 4.2 we noted that if the kernel of τ is either the complete vector space or $\{0\}$, then the induced Hom–Nambu–Lie algebra will be Abelian, and therefore we shall focus on nondegenerate triples in the following. To fix notation, we introduce $K=\ker\tau$ and U such that $U=V\setminus K$. For a nondegenerate compatible triple, U and K are always different from $\{0\}$.

Lemma 4.4: Let (α,β,τ) be a nondegenerate compatible triple on V. Then it holds that $\alpha(K)\subseteq K$ and $\beta(K)\subseteq K$.

Proof: Since the triple is assumed to be nondegenerate, one can find an element $y\in V$ such that $\tau(y)\neq 0$. For any $x\in K$, Eqs. (3.2) and (3.3) imply that $\tau(\alpha(x))=0$ and $\tau(\beta(x))=0$. Hence, α and β map K into K. □

Lemma 4.5: Let (α,β,τ) be a nondegenerate compatible triple on V and assume that there exists an element $u\in U$ such that $\alpha(u)\in K$ (or $\beta(u)\in K$). Then $\alpha(U)\subseteq K$ (or $\beta(U)\subseteq K$).

Proof: For a general $x\in V$, it follows from (3.2) (on u and x) that $\tau(\alpha(x))=0$ since $u\in U$ and $\alpha(u)\in K$. An identical argument goes through for β by using (3.3). □

These results allow us to split the problem into four possible cases,

$$\text{(C1)}\quad \alpha(U)\subseteq U \quad\text{and}\quad \beta(U)\subseteq U,$$

$$\text{(C2)}\quad \alpha(U)\subseteq K \quad\text{and}\quad \beta(U)\subseteq K,$$

$$\text{(C3)}\quad \alpha(U)\subseteq U \quad\text{and}\quad \beta(U)\subseteq K,$$

$$\text{(C4)}\quad \alpha(U)\subseteq K \quad\text{and}\quad \beta(U)\subseteq U.$$

Clearly, in Case (C2) the compatibility conditions will be identically satisfied since

$$\tau(\alpha(x))=\tau(\beta(x))=0 \quad\text{for all}\quad x\in V.$$

In the other cases, the next proposition shows that one does not have any freedom to choose α and β independently.

Proposition 4.6: Let (α,β,τ) be a nondegenerate compatible triple on V. Then, referring to the cases in (C1)–(C4), the following holds:

$$\text{(C1)}\quad \exists\,\lambda\in\mathbb{K}\setminus\{0\}:\beta=\lambda\alpha,$$

$$\text{(C3)}\quad \beta\equiv 0,$$

$$\text{(C4)}\quad \alpha\equiv 0.$$

Proof: Case (C1): if one chooses $u \in U$ and $x \in V$, relation (3.4) gives

$$\beta(x) = \frac{\tau(\beta(u))}{\tau(\alpha(u))}\alpha(x),$$

where $\tau(\beta(u)) \neq 0$ by assumption. Case (C3): by choosing $u \in U$ and $x \in V$, relation (3.4) gives $\beta(x)=0$. Case (C4) is proven in the same way. □

V. EXAMPLES

In this section we provide several examples of ternary Hom–Nambu–Lie algebras induced by Hom–Lie algebras by means of the method described in Theorem 3.3. From the results in Sec. IV (see Proposition 4.6), there are two (nontrivial) possibilities for α and β. Either $\beta=\lambda\alpha$ or the images of α and β are in the kernel of τ, in which case it is possible to have $\beta \neq \lambda\alpha$. We provide examples in both cases.

Example 5.1: In our first example, V is the vector space of $n\times n$ matrices, and $\alpha: V \to V$ acts as conjugation by an invertible matrix s, i.e., $\alpha(x)=s^{-1}xs$. Then $(V,\alpha\circ[\cdot,\cdot],\alpha)$ is a Hom–Lie algebra. For matrices, any trace function is proportional to the matrix trace, so we let $\tau(x)=\mathrm{tr}(x)$. If we want to choose a $\beta\neq 0$, it follows from Proposition 4.6 that β has to be proportional to α, i.e., $\beta=\lambda\alpha$ for some $\lambda\neq 0$. Since $\mathrm{tr}(\alpha(x))=\mathrm{tr}(x)$, it is clear that $(\alpha,\lambda\alpha,\mathrm{tr})$ is a nondegenerate compatible triple on V, which implies, by Theorem 3.3, that $(V,[\cdot,\cdot,\cdot]_{\mathrm{tr}},(\alpha,\lambda\alpha))$ is a Hom–Nambu–Lie algebra induced by $(V,\alpha\circ[\cdot,\cdot],\alpha)$.

Example 5.2: Let us start with the vector space V spanned by $\{x_1,x_2,x_3,x_4\}$ with a skew-symmetric bilinear map defined through

$$[x_i,x_j] = a_{ij}x_3 + b_{ij}x_4,$$

where a_{ij} and b_{ij} are antisymmetric 4×4 matrices. Defining

$$\alpha(x_i) = x_3, \quad \beta(x_i) = x_4, \quad i = 1, \ldots ,4,$$

$$\tau(x_1) = \gamma_1, \quad \tau(x_2) = \gamma_2, \quad \tau(x_3) = \tau(x_4) = 0,$$

one immediately observes that τ is a trace function, $\mathrm{im}\,\alpha \subseteq \ker\tau$, $\mathrm{im}\,\beta\subseteq\ker\tau$, and $\beta\neq\lambda\alpha$. Furthermore, $(V,[\cdot,\cdot],\alpha)$ is a Hom–Lie algebra provided that

$$b_{13} = b_{12} + b_{23},$$

$$b_{14} = b_{12} + b_{23} + b_{34},$$

$$b_{24} = b_{23} + b_{34}.$$

The four independent ternary brackets of the induced Hom–Nambu–Lie algebra can be written as

$$[x_1,x_2,x_3] = (\gamma_1 a_{23} - \gamma_2 a_{13})x_3 + (\gamma_1 b_{23} - \gamma_2(b_{12} + b_{23}))x_4,$$

$$[x_1,x_2,x_4] = (\gamma_1 a_{24} - \gamma_2 a_{14})x_3 + (\gamma_1(b_{23} + b_{34}) - \gamma_2(b_{12} + b_{23} + b_{34}))x_4,$$

$$[x_1,x_3,x_4] = (\gamma_1 a_{34})x_3 + (\gamma_1 b_{34})x_4,$$

$$[x_2,x_3,x_4] = (\gamma_2 a_{34})x_3 + (\gamma_2 b_{34})x_4.$$

For instance, choosing $\gamma_1=\gamma_2=1$ and $a_{i<j}=1$, one obtains the Hom–Nambu–Lie algebra $(\mathrm{Span}(x_1,x_2,x_3,x_4),[\cdot,\cdot,\cdot],(\alpha,\beta))$ defined by

$$[x_1,x_2,x_3] = -b_{12}x_4,$$

$$[x_1,x_2,x_4] = -b_{34}x_4,$$

$$[x_1,x_3,x_4] = x_3 + b_{34}x_4,$$

$$[x_2,x_3,x_4] = x_3 + b_{34}x_4,$$

together with $\alpha(x_i)=x_3$ *and* $\beta(x_i)=x_4$.

Example 5.3: We consider the three-dimensional Hom–Lie algebra defined with respect to a basis $\{x_1,x_2,x_3\}$ *by*

$$[x_1,x_2] = a_1x_2 - \frac{a_2a_4}{a_3}x_3,$$

$$[x_1,x_3] = -\frac{a_1a_3}{a_4}x_2 + a_2x_3,$$

$$[x_2,x_3] = a_3x_2 + a_4x_3,$$

where a_1 , a_2 , a_3 , *and* a_4 *are parameters in* $\mathbb{K}$ *and* $a_3,a_4 \neq 0$. *The map* α *is defined by*

$$\alpha(x_1) = px_1,$$

$$\alpha(x_2) = qx_3,$$

$$\alpha(x_3) = qx_4$$

for any $p,q \in \mathbb{K}$. *We define a trace function as*

$$\tau(x_1) = t, \quad \tau(x_2) = 0, \quad \tau(x_3) = 0$$

for any $t \in \mathbb{K}$.

If $p \neq 0$, *we let* β *be the linear map defined by*

$$\beta(x_1) = rx_1,$$

$$\beta(x_2) = \frac{qr}{p}x_2,$$

$$\beta(x_3) = sx_3$$

for any $r,s \in \mathbb{K}$. *Then conditions (3.2)–(3.4) are satisfied. Thus, according to Theorem 3.3, we obtain a ternary Hom–Nambu–Lie algebra defined by*

$$[x_1,x_2,x_3] = t(a_3x_2 + a_4x_3).$$

If $p=0$, *then one may consider a map* β *of the form*

$$\beta(x_1) = 0,$$

$$\beta(x_2) = r_1x_1 + r_2x_2 + r_3x_3,$$

$$\beta(x_3) = r_4x_1 + r_5x_2 + r_6x_3$$

for any $r_1, r_2, r_3, r_4, r_5, r_6 \in \mathbb{K}$. The ternary bracket is the same as for $p \neq 0$ case.

Example 5.4: We consider the three-dimensional Hom–Lie algebra defined with respect to a basis $\{x_1, x_2, x_3\}$ by

$$[x_1, x_2] = -a_1x_2 + a_2x_3,$$

$$[x_1, x_3] = a_3x_2 + a_1x_3,$$

$$[x_2, x_3] = a_4x_2 + a_5x_3,$$

where a_1, a_2, a_3, a_4, and *a_5 are parameters in $\mathbb{K}$. The map α is defined by*

$$\alpha(x_1) = 0,$$

$$\alpha(x_2) = qx_3,$$

$$\alpha(x_3) = qx_4$$

for any $q \in \mathbb{K}$, and we define a trace function as

$$\tau(x_1) = t, \quad \tau(x_2) = 0, \quad \tau(x_3) = 0$$

for any $t \in \mathbb{K}$.

Let β be the linear map defined by

$$\beta(x_1) = 0,$$

$$\beta(x_2) = r_1x_1 + r_2x_2 + r_3x_3,$$

$$\beta(x_3) = r_4x_1 + r_5x_2 + r_6x_3$$

for any $r_1, r_2, r_3, r_4r_5, r_6 \in \mathbb{K}$. The previous data satisfy conditions (3.2)–(3.4). Then, according to Theorem 3.3, we obtain a ternary Hom–Nambu–Lie algebra defined by $[x_1, x_2, x_3] = t(a_4x_2 + a_5x_3)$.

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