Universal properties of distorted Kerr-Newman black holes

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Abstract. We discuss universal properties of axisymmetric and stationary configurations consisting of a central black hole and surrounding matter in Einstein-Maxwell theory. In particular, we find that certain physical equations and inequalities (involving angular momentum, electric charge and horizon area) are not restricted to the Kerr-Newman solution but can be generalized to the situation where the black hole is distorted by an arbitrary axisymmetric and stationary surrounding matter distribution.

1. Introduction

One of the most important exact solutions in General Relativity is the Kerr metric describing the spacetime around axisymmetric and stationary black holes in vacuum. This solution is particularly interesting because it is generally expected that it will be approached asymptotically by any collapsing matter distribution – provided that all the matter finally falls into the black hole. It is, however, conceivable that for a long time period a substantial portion of the matter distribution moves around the black hole in a quasi-stationary state. Such a configuration can be approximated well by an equilibrium system consisting of a central black hole and surrounding matter, e.g., a fluid ring or disk (describing a black hole with an accretion disk or a galaxy with a central black hole).

Little is known about how such surrounding matter influences the central black hole. In this essay we discuss several *universal* properties that black holes possess independent of the nature of their environment. Our investigations apply to the more general case in which additional electromagnetic fields are considered, i.e. we study "distorted Kerr-Newman black holes" within Einstein-Maxwell theory.

2. A universal inequality for sub-extremal black holes

It is well-known that the three-parameter Kerr-Newman family of solutions describes single rotating, electrically charged, axisymmetric and stationary black holes in electrovacuum. For the parameters one can choose, e.g., the mass M, the angular momentum J, and the electric charge Q of the black hole[1]. For a solitary black hole, which is characterized by the existence of an event horizon, the relation

$$\left(\frac{J}{M}\right)^2 + Q^2 \leq M^2 \tag{1}$$

must be satisfied. Equivalently, expressing M through J, Q, and the area A of the event horizon, we find that the three parameters J, Q, and A have to obey

$$(8\pi J)^2 + (4\pi Q^2)^2 \leq A^2 \,. \tag{2}$$

Now, a Kerr-Newman black hole with equality in (1) (or (2)) is called *extremal*, otherwise *sub-extremal*.

A different definition of sub-extremality of black holes refers to the existence of trapped surfaces in every sufficiently small interior neighborhood of their event horizons [5]. In the Kerr-Newman family, both notions of sub-extremality turn out to be equivalent. However, the concept of trapped surfaces is more general and applies to arbitrary axisymmetric and stationary black holes. Hence it is appropriate to use the corresponding definition of sub-extremality, and we will do so in the following.

In [8, 10], we have investigated such sub-extremal stationary and axisymmetric black holes surrounded by matter in full Einstein-Maxwell theory and found that inequality (2) strictly holds:

$$(8\pi J)^2 + (4\pi Q^2)^2 < A^2. \tag{3}$$

Through this result we generalize the link between the two notions of sub-extremality to the realm of distorted black holes, i.e. we establish a universal relation between the geometric concept of existence of trapped surfaces and the inequality to be obeyed by the physical black hole parameters J, Q, and A. [2]

For the proof of (3) for sub-extremal distorted black holes it suffices to study the Einstein-Maxwell equations in an electrovacuum vicinity of the event horizon. Using these equations, it is possible to reformulate (3) in terms of a variational problem: An appropriate functional depending on the horizon values of three metric and electromagnetic potentials must always be greater than or equal to 1. As shown

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[1] In the special case of the Kerr-Newman metric, these quantities can be read off from the asymptotic behavior of the metric coefficients.

[2] Note that these quantities can be determined locally on the event horizon. We further remark that the definition of J (in contrast to that of Q and A) involves a choice: The total angular momentum of the spacetime is composed of matter, electromagnetic field, and black hole contributions. While clearly the matter part should be excluded from the definition of J, both a Komar integral and an appropriate electromagnetic event horizon integral is taken into account for our notion of the black hole's local angular momentum (cf. [2] for a more thorough discussion).

in [8, 10], this variational problem can be analyzed with methods from the calculus of variations.

A fascinating application of (3) is the solution of an old problem in General Relativity, the *balance problem* for two black holes: Is it possible that two aligned axisymmetric and stationary black holes are in equilibrium, i.e. can the spin-spin repulsion of two rotating black holes compensate their gravitational attraction? The application of a particular soliton method (the *inverse scattering method*)[3], provides a proof of the non-existence of such equilibrium situations, showing that at least one of the two black holes in this ensemble would violate inequality (3), cf. [14].

3. The interior of the black hole

Having discussed universal properties on a black hole's event horizon, we will now direct our attention to its interior. Again, we were able to generalize a well-known feature of Kerr-Newman solutions, namely the existence of a second horizon inside the black hole – the *inner Cauchy horizon* – which is defined as the future boundary of the domain of dependence of the event horizon. Since predictability breaks down beyond the Cauchy horizon, its stability is an important issue. Based on an argument by R. Penrose [15], it was assumed that generic perturbations grow infinitely near the Cauchy horizon. However, for purely axisymmetric and stationary (arbitrarily large) perturbations, i.e. for our situation of a central black hole with surrounding matter, a regular inner Cauchy horizon can be shown to exist provided the angular momentum J and the charge Q of the black hole do not vanish simultaneously. Moreover, the region between event horizon and Cauchy horizon is then completely regular.[4] In contrast, the Cauchy horizon becomes singular and approaches a scalar curvature singularity in the limit $J \to 0$, $Q \to 0$ (precisely as in the transition from a Kerr-Newmann black hole to a Schwarzschild black hole).

Remarkably, if the inner Cauchy horizon exists (i.e. if J and Q do not vanish simultaneously), then the area A_{CH} of the Cauchy horizon and the event horizon area A are related via

$$(8\pi J)^2 + (4\pi Q^2)^2 = A_{\mathrm{CH}}A. \tag{4}$$

Note that the left hand side of (4) is the same as in (3).

The proof of these statements [3, 4, 9] utilizes again soliton methods. From an analysis of the linear matrix problem (see footnote 3) on a closed path (along the event horizon, the Cauchy horizon and the two parts of the symmetry axis, which connect both horizons), it is possible to deduce the existence of a Cauchy horizon. Moreover, explicit formulas for the metric and electromagnetic potentials on the Cauchy horizon

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[3] Soliton methods are based on the existence of a *linear* matrix problem which is equivalent to the *nonlinear* field equations via its integrability condition [12]. (See [13] for a sophisticated introduction to soliton methods in General Relativity.)

[4] A particular exact solution for the interior black hole region is discussed in [1].

emerge, and these are used to show (4). The second essential ingredient of the proof is a theorem by P. Chruściel [7].

Finally, we note that the interior spacetime region of axisymmetric and stationary black holes is closely related to Gowdy spacetimes (cosmological models with two spacelike Killing fields), making the above statements applicable in this cosmological context as well. They also imply the analogous existence and regularity statements for Cauchy horizons [11].

4. Configurations with degenerate black holes

In the degenerate limit of converging Cauchy and event horizons (expressed by vanishing surface gravity κ), we obtain equality in equation (2), cf. [2]:

$$(8\pi J)^2 + (4\pi Q^2)^2 = A^2. \tag{5}$$

Consistently, equation (5) also follows from equation (4) as the areas A and A_{CH} coincide in the degenerate limit.

As a consequence of (5), the characterizing relation

$$\left(\frac{J}{M}\right)^2 + Q^2 = M_{CR}^2 \tag{6}$$

of the extremal Kerr-Newman solution continues to hold in presence of surrounding matter – in accordance with the fact that Kerr-Newman black holes are degenerate precisely if they are extremal. (In this general situation, M_{CR} denotes the Christodoulou-Ruffini mass [6] of the black hole which agrees with the ADM mass in the Kerr-Newman setting.)

5. Discussion

We have demonstrated that the following statements about the Kerr-Newman black hole are more generally valid for any axisymmetric and stationary black hole with surrounding matter in Einstein-Maxwell theory:

- Physically relevant black holes are expected to be sub-extremal, where sub-extremality is defined by the existence of trapped surfaces in every sufficiently small interior neighborhood of the event horizon. Angular momentum J and electric charge Q of a sub-extremal black hole are bounded in terms of its horizon area A: $(8\pi J)^2 + (4\pi Q^2)^2 < A^2$.
- The black hole possesses a second horizon – an inner Cauchy horizon – if and only if J and Q do not vanish simultaneously. The Cauchy horizon is regular whenever it exists.
- The spacetime region between event horizon and Cauchy horizon is completely regular.
- The horizon areas satisfy the universal relation $(8\pi J)^2 + (4\pi Q^2)^2 = A_{\mathrm{CH}} A$.

- The relations $(8\pi J)^2 + (4\pi Q^2)^2 = A^2$ and $\left(\frac{J}{M}\right)^2 + Q^2 = M_{CR}^2$ hold for degenerate black holes (M_{CR} : Christodoulou-Ruffini mass). In that sense, degenerate black holes could be named *extremal*.

It would be interesting to identify further universal properties of the black hole configurations in question, e.g. black holes in Yang-Mills theory or in higher dimensions.

Acknowledgments

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