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SUPERCONFORMAL INDICES OF $\mathcal{N}=4$ DUAL FIELD THEORIES

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ABSTRACT. Using the superconformal indices techniques, we construct new interesting conjectures for the elliptic hypergeometric integral identities based on the exceptional groups G_2 and F_4. These identities arise from S-dualities for $\mathcal{N}=4$ supersymmetric Yang-Mills theories with G_2 and F_4 gauge groups.

1. INTRODUCTION

The question of strong-weak duality of $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory is a quite old area of research [1, 2, 3]. This duality (called also S-duality) states the equivalence of the theory with an (electric) gauge group G_c with a similar theory with a (magnetic) gauge group $G_c^\vee$ and the inverse coupling constant. If one introduces the coupling constant as

$$\tau \ = \ \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}, \tag{1}$$

then the S-duality transformation of the theory maps τ for the simply-laced gauge group[1] to the coupling constant $-1/\tau$,

$$S: \ \tau \to -\frac{1}{\tau}. \tag{2}$$

Together with the symmetry transformation

$$T: \ \tau \to \tau + 1 \tag{3}$$

the strong-weak duality becomes equivalent to the $SL(2,\mathbb{Z})$ group of transformations

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad ad-bc=1, \ a,b,c,d \in \mathbb{Z}. \tag{4}$$

For $\mathcal{N}=4$ SYM theories with non-simply laced gauge groups one has the following realization of S-duality

$$\widetilde{S}: \ \tau \to -\frac{1}{m\tau}, \tag{5}$$

where m is the ratio of the lengths-squared of long and short roots of the corresponding root system ($m=2$ for $SO(2N+1), SP(2N), F_4$ and $m=3$ for G_2). In [4] $\mathcal{N}=4$ theories with G_2 and F_4 gauge groups were analyzed from the algebraic point of view and the action of S-duality on the moduli space was described. Here we would like to provide another approach for testing validity of this conjectural duality between field theories.

[1] A simply laced group is a Lie group whose Dynkin diagram contains only simple links, and therefore all nonzero roots of the corresponding Lie algebra have the same length. These groups are $SU(N), SO(2N), E_6, E_7$ and E_8.

For this purpose we use the technique based on the calculation of the superconformal indices for $\mathcal{N}=4$ theories suggested by Kinney et al in [5] (for the definition of indices in $\mathcal{N}=1$ theories, see [6, 7]). $\mathcal{N}=4$ SYM theory has the $PSU(2,2|4)$ space-time symmetry group which is generated by $J_a, \overline{J}_a$, $a=1,2,3$, representing $SU(2)$ subgroups (Lorentz rotations), $P_\mu, Q_{i,\alpha}, \overline{Q}_{i,\dot\alpha}$ (supertranslations) with $i=1,2,3,4$ and $\alpha,\dot\alpha=1,2$; $K_\mu, S^{i,\alpha}, \overline{S}^{i,\dot\alpha}$ (special superconformal transformations), H (dilations) with the state eigenvalues given by conformal dimensions, and R_1, R_2, R_3 ($SU(4)_R$ R-symmetry subgroup) [8]. For a distinguished pair of supercharges, say, $Q\equiv Q_{1,1}$ and $Q^\dagger \equiv S^{1,1}$, one has

$$\{Q,Q^\dagger\} = H - 2J_3 - 2\sum_{k=1}^{3}\left(1-\frac{k}{4}\right)R_k \equiv \Delta, \tag{6}$$

and the superconformal index is defined by the matrix integral [5]

$$I(t,y,v,w) = \int_{G_c}[dU]\,\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m}f(t^m,y^m,v^m,w^m)Tr(U^\dagger)^m Tr U^m\right\}, \tag{7}$$

where $[dU]$ is the G_c-invariant measure and $f(t,y,v,w)$ is the so-called single-particle states index. The integrand in (7) is given by the following expression

$$Tr\left((-1)^{\mathrm{F}} t^{2(H+J_3)} y^{2\overline{J}_3} v^{R_2} w^{R_3} e^{-\beta\Delta}\right), \tag{8}$$

where F is the fermion number operator and t,y,v,w are group parameters (chemical potentials). The trace is taken over the states corresponding to zero modes of the operator Δ because relation (6) is preserved by operators in (8) (the contributions from other states cancel together with the dependence on β). All the fields in $\mathcal{N}=4$ supermultiplet lie in the same representation of the gauge group G_c. It means that, in comparison with the superconformal indices in $\mathcal{N}=1,2$ SYM theories, the contribution from the fields will be given by the adjoint representation only. The problem of counting various BPS states in $\mathcal{N}=4$ theories and computation of the related characters was discussed in [9, 10].

The technique based on the calculation of superconformal indices has already found many applications in supersymmetric field theories. In [7] the Seiberg duality for $\mathcal{N}=1$ SYM theories was conjectured to lead to the equality of superconformal indices of dual theories. Later on Dolan and Osborn explicitly confirmed this conjecture for a number of examples in [11]. It appears that superconformal indices are expressed in terms of elliptic hypergeometric integrals whose theory was developed earlier in [12, 13] (see also [14] for a general survey). Moreover, equality of indices in dual theories happened to be equivalent either to exact computability of elliptic beta integrals discovered in [12] or to nontrivial Weyl group symmetry transformations for higher order elliptic hypergeometric functions [13, 15]. In the series of papers [16, 17, 18] we applied this technique for analyzing all previously known Seiberg dualities and suggested many new similar dualities on the basis of known identities for superconformal indices.As a payback to mathematics, it happened that many known dualities lead to new still unproven highly nontrivial relations for elliptic hypergeometric integrals [17].

This line of thoughts was further developed in beautiful papers by Gadde et al [19, 20]. In [19], a fresh identity from [21] describing $W(F_4)$ Weyl group transformation for a particular one dimensional elliptic hypergeometric integral was used for confirming S-duality for $\mathcal{N}=2$ SYM theory with $SU(2)$ gauge group and four hypermultiplets [22, 23] and for ensuring associativity of the operator algebra of 2D theories behind that duality. In [20], using the inversion of an elliptic hypergeometric integral transform from [24], they suggested new mathematical identities following from known dualities [25] for superconformal $\mathcal{N}=2$ SYM theory with $G_c = SU(N)$.

The main purpose of our paper consists in the consideration of S-duality for $\mathcal{N}=4$ SYM theories with G_2 and F_4 gauge groups [1, 4] from the superconformal indices point of view. Such a consideration was performed already by Gadde et al in [19] in the case of $G_c = SP(2N)$ and $G_c^\vee = SO(2N+1)$ groups duality. We give here a new sufficiently strong mathematical argument in favor of the equality of the corresponding indices in all three cases.

2. $\mathcal{N}=4$ SYM theory with G_2 gauge group

Let us start from the S-duality conjecture for $\mathcal{N}=4$ SYM theory with G_2 gauge group following from the considerations of [1], which was made more explicit in [2, 3] and discussed in detail in [4]. We give two forms of the superconformal indices: the short version, where we use definition (8) with the chemical potentials $w=v=1$, and the long one with arbitrary parameters w and v.[2] Then the short "electric" index is

$$I_E \;=\; \kappa_2 \int_{\mathbb{T}^2} \prod_{1\le i<j\le 3} \frac{\Gamma^3(t^2 z_i^{\pm1} z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1} z_j^{\pm1};p,q)} \prod_{j=1}^{2} \frac{dz_j}{2\pi i z_j} \tag{9}$$

and the "magnetic" index is

$$I_M \;=\; \kappa_2 \int_{\mathbb{T}^2} \prod_{1\le i<j\le 3} \frac{\Gamma^3(t^2 (y_i y_j)^{\pm1}, t^2 (y_i y_j^{-1})^{\pm3};p,q)}{\Gamma((y_i y_j)^{\pm1}, (y_i y_j^{-1})^{\pm3};p,q)} \prod_{j=1}^{2} \frac{dy_j}{2\pi i y_j}. \tag{10}$$

Here $\mathbb{T}$ denotes the unit circle with positive orientation and we use conventions $\Gamma(a,b;p,q) \equiv \Gamma(a;p,q)\Gamma(b;p,q)$, $\Gamma(az^{\pm1};p,q) \equiv \Gamma(az;p,q)\Gamma(az^{-1};p,q)$, where

$$\Gamma(z;p,q) = \prod_{i,j=0}^{\infty} \frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j}, \quad |p|,|q|<1,$$

is the elliptic gamma function. The coefficient in front of the integrals is

$$\kappa_2 \;=\; \frac{(p;p)_\infty^2 (q;q)_\infty^2}{2^2 3} \Gamma^6(t^2;p,q)$$

with $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$, and also

$$p \;=\; t^3 y, \qquad q \;=\; \frac{t^3}{y}, \qquad z_1z_2z_3 \;=\; 1, \qquad y_1y_2y_3 \;=\; 1.$$

The constraint $t^6 = pq$ plays the role of balancing condition for the integrals [14]. G_2 has two dimensional maximal torus parametrized by $z_{1,2}$, but it is convenient to introduce the third group variable $z_3 = z_1^{-1} z_2^{-1}$ as described above.

[2] We denote as I_E and I_M superconformal indices with $v=w=1$ and as J_E and J_M the indices for arbitrary v, w.

The group $SU(4)_R$ has different commuting R-symmetry charges, R_2 and R_3, that is why one can introduce two additional chemical potentials v and w into the index. This extended index in the electric theory has the form

$$J_E = \chi_2 \int_{\mathbb{T}^2} \prod_{1\leq i<j\leq 3} \frac{\Gamma(t^2 v z_i^{\pm 1} z_j^{\pm 1}, t^2 \frac{1}{w} z_i^{\pm 1} z_j^{\pm 1}, t^2 \frac{w}{v} z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^{2} \frac{dz_j}{2\pi i z_j}, \quad (11)$$

and for the magnetic theory we obtain

$$J_M = \chi_2 \int_{\mathbb{T}^2} \prod_{1\leq i<j\leq 3} \frac{\Gamma(t^2 v (y_i y_j)^{\pm 1}, t^2 \frac{1}{w} (y_i y_j)^{\pm 1}, t^2 \frac{w}{v} (y_i y_j)^{\pm 1}; p, q)}{\Gamma((y_i y_j)^{\pm 1}, (y_i y_j^{-1})^{\pm 3}; p, q)}$$
$$\times \Gamma(t^2 v (y_i y_j^{-1})^{\pm 3}, t^2 \frac{1}{w} (y_i y_j^{-1})^{\pm 3}, t^2 \frac{w}{v} (y_i y_j^{-1})^{\pm 3}; p, q) \prod_{j=1}^{2} \frac{dy_j}{2\pi i y_j}, \quad (12)$$

where $|t^2 v|, |t^2/w|, |t^2 w/v| < 1$,

$$\chi_2 = \frac{(p;p)_\infty^2 (q;q)_\infty^2}{2^2 3} \Gamma^2(t^2 v, t^2 \frac{1}{w}, t^2 \frac{w}{v}; p, q),$$

and, again, $z_1 z_2 z_3 = 1,\ y_1 y_2 y_3 = 1$.

S-duality for this theory leads thus to a nice conjecture for the equality of elliptic hypergeometric integrals based on the G_2 group:

$$J_E = J_M \quad (13)$$

in the indicated domain of values of parameters. We rewrite this equality as

$$\int_{\mathbb{T}^2} \Delta_E(\underline{z}, v, w) \prod_{j=1}^{2} \frac{dz_j}{2\pi i z_j} = \int_{\mathbb{T}^2} \Delta_M(\underline{y}, v, w) \prod_{j=1}^{2} \frac{dy_j}{2\pi i y_j}, \quad (14)$$

where the kernels $\Delta_E(\underline{z}, v, w)$ and $\Delta_M(\underline{y}, v, w)$ are read from the integrals (11) and (12). Then we compose the function

$$\rho(\underline{z}, \underline{y}, v, w) = \frac{\Delta_E(\underline{z}, v, w)}{\Delta_M(\underline{y}, v, w)}. \quad (15)$$

We have verified that this function represents the so-called totally elliptic hypergeometric term [26, 17]. This is a rather rich mathematical statement giving a strong evidence on the validity of the stated equality of integrals. It means that all the functions

$$h_i^{(z)} = \frac{\rho(\ldots q z_i \ldots, \underline{y}, v, w)}{\rho(\underline{z}, \underline{y}, v, w)}, \quad h_i^{(y)} = \frac{\rho(\underline{z}, \ldots q y_i \ldots, v, w)}{\rho(\underline{z}, \underline{y}, v, w)}, \quad i = 1, 2,$$

$$h^{(v)} = \frac{\rho(\underline{z}, \underline{y}, qv, w)}{\rho(\underline{z}, \underline{y}, v, w)}, \qquad h^{(w)} = \frac{\rho(\underline{z}, \underline{y}, v, qw)}{\rho(\underline{z}, \underline{y}, v, w)},$$

where we assume that $z_3 = z_1^{-1} z_2^{-1},\ y_3 = y_1^{-1}, y_2^{-1}$, are elliptic functions of all their arguments z_i, y_i, v, w and q. For instance,

$$h_i^{(z)}(\ldots p z_k \ldots, \underline{y}, v, w; q; p) = h_i^{(z)}(\underline{z}, \ldots p y_k \ldots, v, w; q; p) = h_i^{(z)}(\underline{z}, \underline{y}, pv, w; q; p)$$
$$= h_i^{(z)}(\underline{z}, \underline{y}, v, pw; q; p) = h_i^{(z)}(\underline{z}, \underline{y}, v, w; p^3 q; p) = h_i^{(z)}(\underline{z}, \underline{y}, v, w; q; p).$$

For further consequences of the total ellipticity and various technical details of such computations, we refer to papers [14, 26, 17].

3. $\mathcal{N}=4$ SYM theory with F_4 gauge group

Now we discuss S-duality for $\mathcal{N}=4$ SYM theory with F_4 gauge group [1, 2, 3, 4]. "Short" versions of the corresponding superconformal indices are

$$\begin{aligned} I_E &= \kappa_4 \int_{\mathbb{T}^4} \prod_{1\le i<j\le 4} \frac{\Gamma^3(t^2 z_i^{\pm 2} z_j^{\pm 2};p,q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2};p,q)} \\ &\times \prod_{j=1}^4 \frac{\Gamma^3(t^2 z_j^{\pm 2};p,q)}{\Gamma(z_i^{\pm 2};p,q)} \frac{\Gamma^3(t^2 z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1};p,q)}{\Gamma(z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1};p,q)} \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} I_M &= \kappa_4 \int_{\mathbb{T}^4} \prod_{1\le i<j\le 4} \frac{\Gamma^3(t^2 y_i^{\pm 1} y_j^{\pm 1};p,q)}{\Gamma(y_i^{\pm 1} y_j^{\pm 1};p,q)} \\ &\times \prod_{j=1}^4 \frac{\Gamma^3(t^2 y_j^{\pm 2};p,q)}{\Gamma(y_i^{\pm 2};p,q)} \frac{\Gamma^3(t^2 y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1};p,q)}{\Gamma(y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1};p,q)} \prod_{j=1}^4 \frac{dy_j}{2\pi i y_j}, \end{aligned} \tag{17}$$

where $t^6 = pq$ and

$$\kappa_4 = \frac{(p;p)_\infty^4 (q;q)_\infty^4}{2^{11} 3^2} \Gamma^{12}(t^2;p,q).$$

The long versions of the superconformal indices have the forms, in electric case,

$$\begin{aligned} J_E &= \chi_4 \int_{\mathbb{T}^4} \prod_{1\le i<j\le 4} \frac{\Gamma(t^2 v z_i^{\pm 2} z_j^{\pm 2}, t^2 \frac{1}{w} z_i^{\pm 2} z_j^{\pm 2}, t^2 \frac{w}{v} z_i^{\pm 2} z_j^{\pm 2};p,q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2};p,q)} \\ &\times \prod_{j=1}^4 \frac{\Gamma(t^2 v z_j^{\pm 2}, t^2 \frac{1}{w} z_j^{\pm 2}, t^2 \frac{w}{v} z_j^{\pm 2};p,q)}{\Gamma(z_i^{\pm 2};p,q)} \\ &\times \frac{\Gamma(t^2 v z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1}, t^2 \frac{1}{w} z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1}, t^2 \frac{w}{v} z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1};p,q)}{\Gamma(z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1};p,q)} \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \end{aligned} \tag{18}$$

and, in the magnetic case,

$$\begin{aligned} J_M &= \chi_4 \int_{\mathbb{T}^4} \prod_{1\le i<j\le 4} \frac{\Gamma(t^2 v y_i^{\pm 1} y_j^{\pm 1}, t^2 \frac{1}{w} y_i^{\pm 1} y_j^{\pm 1}, t^2 \frac{w}{v} y_i^{\pm 1} y_j^{\pm 1};p,q)}{\Gamma(y_i^{\pm 1} y_j^{\pm 1};p,q)} \\ &\times \prod_{j=1}^4 \frac{\Gamma(t^2 v y_j^{\pm 2}, t^2 \frac{1}{w} y_j^{\pm 2}, t^2 \frac{w}{v} y_j^{\pm 2};p,q)}{\Gamma(y_i^{\pm 2};p,q)} \\ &\times \frac{\Gamma(t^2 v y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1}, t^2 \frac{1}{w} y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1}, t^2 \frac{w}{v} y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1};p,q)}{\Gamma(y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1};p,q)} \prod_{j=1}^4 \frac{dy_j}{2\pi i y_j}, \end{aligned} \tag{19}$$

where $|t^2 v|, |t^2/w|, |t^2 w/v| < 1$ and

$$\chi_4 = \frac{(p;p)_\infty^4 (q;q)_\infty^4}{2^{11} 3^2} \Gamma^4(t^2 v, t^2 \frac{1}{w}, t^2 \frac{w}{v};p,q).$$

Again, S-duality leads us to an interesting conjecture for the elliptic hypergeometric integrals based on the F_4 group:

$$J_E = J_M \tag{20}$$

in the indicated domain of values of parameters. We have checked that the ratio of the kernels of integrals J_E and J_M define a totally elliptic hypergeometric term, which gives a nice mathematical test of the equality of integrals. Note that the described integrals are the first examples of multiple elliptic hypergeometric integrals defined for the F_4 root system (in [21] the integrals were defined on the $SU(2)$ group and $W(F_4)$ was emerging as a transformation symmetry in the parameter space).

4. $\mathcal{N}=4$ SYM theories with $SU(N)$ and $SO(2N)$ gauge groups

For completeness, we present also the superconformal indices for $\mathcal{N}=4$ SYM theories with $SU(N)$ and $SO(2N)$ gauge groups, which are S-self-dual [1].

The superconformal index for the $SU(N)$ theory is

$$J_{SU(N)} = \chi_N \int_{\mathbb{T}^{N-1}} \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} \tag{21}$$
$$\times \prod_{1\leq i<j\leq N} \frac{\Gamma(t^2 v z_i^{-1} z_j, t^2 v z_i z_j^{-1}, t^2 \frac{1}{w} z_i^{-1} z_j, t^2 \frac{1}{w} z_i z_j^{-1}, t^2 \frac{w}{v} z_i^{-1} z_j, t^2 \frac{w}{v} z_i z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)},$$

where $\prod_{j=1}^N z_j = 1$, $|t^2 v|, |t^2/w|, |t^2 w/v| < 1$, and

$$\chi_N = \frac{(p;p)_\infty^{N-1}(q;q)_\infty^{N-1}}{N!}\Gamma^{N-1}(t^2 v, t^2\frac{1}{w}, t^2\frac{w}{v}; p, q).$$

Looking at the ratio of the kernel of this integral to itself with different integration variables, one can get the totally elliptic hypergeometric term. However, consequences of this statement are much less informative than in the previous cases (trivial, in some sense).

The superconformal index for the $SO(2N)$ theory is

$$J_{SO(2N)} = \chi_N \int_{\mathbb{T}^N} \prod_{j=1}^{N} \frac{dz_j}{2\pi i z_j} \tag{22}$$
$$\times \prod_{1\leq i<j\leq N} \frac{\Gamma(t^2 v z_i^{\pm1} z_j^{\pm1}, t^2 \frac{1}{w} z_i^{\pm1} z_j^{\pm1}, t^2 \frac{w}{v} z_i^{\pm1} z_j^{\pm1}; p, q)}{\Gamma(z_i^{\pm1} z_j^{\pm1}; p, q)},$$

where $|t^2 v|, |t^2/w|, |t^2 w/v| < 1$ and

$$\chi_N = \frac{(p;p)_\infty^{N}(q;q)_\infty^{N}}{2^{N-1}N!}\Gamma^{N}(t^2 v, t^2\frac{1}{w}, t^2\frac{w}{v}; p, q).$$

The situation with the total ellipticity condition is similar to the one for (21).

5. $\mathcal{N}=4$ SYM $SO(2N+1) \leftrightarrow SP(2N)$ duality integrals

Also for generality, we present formulas for superconformal indices coming from the S-duality for $\mathcal{N}=4$ SYM theories with $SP(2N)$ and $SO(2N+1)$ gauge groups, which were already described by Gadde et al in [19] (and discussed briefly in the simplest case in [17]). The "short" electric superconformal index is

$$I_E = \kappa_N \int_{\mathbb{T}^N} \prod_{1\leq i<j\leq N} \frac{\Gamma^3(t^2 z_i^{\pm1} z_j^{\pm1}; p, q)}{\Gamma(z_i^{\pm1} z_j^{\pm1}; p, q)} \prod_{j=1}^N \frac{\Gamma^3(t^2 z_j^{\pm2}; p, q)}{\Gamma(z_j^{\pm2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}, \tag{23}$$

while the magnetic index is

$$I_M = \kappa_N \int_{\mathbb{T}^N} \prod_{1\leq i<j\leq N} \frac{\Gamma^3(t^2 y_i^{\pm 1} y_j^{\pm 1}; p, q)}{\Gamma(y_i^{\pm 1} y_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\Gamma^3(t^2 y_j^{\pm 1}; p, q)}{\Gamma(y_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{dy_j}{2\pi i y_j}, \quad (24)$$

where

$$\kappa_N = \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!} \Gamma^{3N}(t^2; p, q).$$

The long versions are, in the electric case,

$$\begin{aligned} J_E &= \chi_N \int_{\mathbb{T}^N} \prod_{1\leq i<j\leq N} \frac{\Gamma(t^2 v z_i^{\pm 1} z_j^{\pm 1}, t^2 \frac{1}{w} z_i^{\pm 1} z_j^{\pm 1}, t^2 \frac{w}{v} z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\ &\times \prod_{j=1}^N \frac{\Gamma(t^2 v z_j^{\pm 2}, t^2 \frac{1}{w} z_j^{\pm 2}, t^2 \frac{w}{v} z_j^{\pm 2}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}, \end{aligned} \quad (25)$$

and, in the magnetic case,

$$\begin{aligned} J_M &= \chi_N \int_{\mathbb{T}^N} \prod_{1\leq i<j\leq N} \frac{\Gamma(t^2 v y_i^{\pm 1} y_j^{\pm 1}, t^2 \frac{1}{w} y_i^{\pm 1} y_j^{\pm 1}, t^2 \frac{w}{v} y_i^{\pm 1} y_j^{\pm 1}; p, q)}{\Gamma(y_i^{\pm 1} y_j^{\pm 1}; p, q)} \\ &\times \prod_{j=1}^N \frac{\Gamma(t^2 v y_j^{\pm 1}, t^2 \frac{1}{w} y_j^{\pm 1}, t^2 \frac{w}{v} y_j^{\pm 1}; p, q)}{\Gamma(y_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{dy_j}{2\pi i y_j}, \end{aligned} \quad (26)$$

where $|t^2 v|, |t^2/w|, |t^2 w/v| < 1$ and

$$\chi_N = \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!} \Gamma^N(t^2 v, t^2 \frac{1}{w}, t^2 \frac{w}{v}; p, q).$$

Duality between the SYM theories leads to the conjecture representing a particular elliptic hypergeometric integrals identity: $J_E = J_M$. We have checked that the ratio of kernels of two integrals in this equality defines a totally elliptic hypergeometric term, which an essential mathematical argument in favor of its validity.

6. Conclusion

In this short note we have described superconformal indices for $\mathcal{N} = 4$ SYM theories with all but E_6, E_7, E_8 classical gauge groups and analyzed some of their mathematical properties. In the case of G_2 and F_4 groups equality of indices lead to new elliptic hypergeometric integral identities complementing the collection of non-trivial relations listed in [14, 17, 19, 20]. The full rigorous proof of them would provide another strong evidence on the validity of S-duality between the original field theories. The integrals for the F_4 gauge group are the first examples of multiple elliptic hypergeometric integrals for the root system F_4.

In order to stress once more importance of the technique based on the calculation of superconformal indices in relation to the theory of elliptic hypergeometric integrals and duality questions, we would like to present the integral describing the superconformal index of $\mathcal{N}=2$ SYM theory given in Fig. 9 of the paper [27]:

$$
\begin{aligned}
I_E \ =\ & \frac{(p;p)_\infty^6 (q;q)_\infty^6}{8} \int_{\mathbb{T}} \frac{dx}{2\pi i x} \int_{\mathbb{T}} \frac{dy}{2\pi i y} \int_{\mathbb{T}^2} \prod_{j=1}^2 \frac{dz_j}{2\pi i z_j} \int_{\mathbb{T}} \frac{dr}{2\pi i r} \int_{\mathbb{T}} \frac{dw}{2\pi i w} \\
& \times \frac{\Gamma(t^2 v x^{\pm1};p,q)}{\Gamma(x^{\pm1};p,q)} \frac{\Gamma(t^2 v y^{\pm2};p,q)}{\Gamma(y^{\pm2};p,q)} \frac{\Gamma(t^2 v z_1^{\pm1} z_2^{\pm1};p,q)}{\Gamma(z_1^{\pm1} z_2^{\pm1};p,q)} \frac{\Gamma(t^2 v r^{\pm2};p,q)}{\Gamma(r^{\pm2};p,q)} \\
& \times \frac{\Gamma(t^2 v w^{\pm1};p,q)}{\Gamma(w^{\pm1};p,q)} \Gamma(\frac{t^2}{\sqrt{v}} y^{\pm1}, \frac{t^2}{\sqrt{v}} r^{\pm1}, \frac{t^2}{\sqrt{v}} x^{\pm1} y^{\pm1}, \frac{t^2}{\sqrt{v}} r^{\pm1} w^{\pm1};p,q) \\
& \times \prod_{j=1}^2 \Gamma(y^{\pm1} z_j^{\pm1}, \frac{t^2}{\sqrt{v}} r^{\pm1} z_j^{\pm1};p,q), \qquad (27)
\end{aligned}
$$

where t is the same parameter as before and the parameter v is the chemical potential associated with some combination of the $U(2)_R$-group R-charges. Introducing the variables $\alpha^2 = z_1 z_2,\ \beta^2 = z_1/z_2,\ \gamma^2 = x$ and $\delta^2 = w$, one can rewrite the integral as

$$
\begin{aligned}
I_M \ =\ & \frac{(p;p)_\infty^6 (q;q)_\infty^6}{64} \\
\times & \int_{\mathbb{T}} \frac{d\gamma}{2\pi i \gamma} \int_{\mathbb{T}} \frac{dy}{2\pi i y} \int_{\mathbb{T}} \frac{d\alpha}{2\pi i \alpha} \int_{\mathbb{T}} \frac{d\beta}{2\pi i \beta} \int_{\mathbb{T}} \frac{dr}{2\pi i r} \int_{\mathbb{T}} \frac{d\delta}{2\pi i \delta} \frac{\Gamma(t^2 v \gamma^{\pm2};p,q)}{\Gamma(\gamma^{\pm2};p,q)} \\
\times & \frac{\Gamma(t^2 v y^{\pm2};p,q)}{\Gamma(y^{\pm2};p,q)} \frac{\Gamma(t^2 v \alpha^{\pm2};p,q)}{\Gamma(\alpha^{\pm2};p,q)} \frac{\Gamma(t^2 v \beta^{\pm2};p,q)}{\Gamma(\beta^{\pm2};p,q)} \frac{\Gamma(t^2 v r^{\pm2};p,q)}{\Gamma(r^{\pm2};p,q)} \frac{\Gamma(t^2 v \delta^{\pm2};p,q)}{\Gamma(\delta^{\pm2};p,q)} \\
\times & \Gamma(\frac{t^2}{\sqrt{v}} \gamma^{\pm1}\gamma^{\pm1} y^{\pm1}, \frac{t^2}{\sqrt{v}} \delta^{\pm1}\delta^{\pm1} r^{\pm1};p,q) \Gamma(\frac{t^2}{\sqrt{v}} y^{\pm1}\alpha^{\pm1}\beta^{\pm1}, \frac{t^2}{\sqrt{v}} r^{\pm1}\alpha^{\pm1}\beta^{\pm1};p,q).
\end{aligned}
$$

The equality of indices $I_E = I_M$ can be interpreted as an identity following from a particular example of duality between the $SO(4)\times SP(2)$ $\mathcal{N}=2$ SYM quiver theory and $SU(2)$ $\mathcal{N}=2$ SYM generalized quiver theory. Namely, the electric part is an $SO(3)\times SP(2)\times SO(4)\times SP(2)\times SO(3)$ $\mathcal{N}=2$ SYM quiver and the magnetic one is the $SU(2)^6$ $\mathcal{N}=2$ SYM generalized quiver described in Fig. 9 of [27]. We hope to discuss in more detail $\mathcal{N}=2$ dualities and corresponding superconformal indices in a separate paper.

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